

The problem can be reframed as finding the minimum of the function

$$C = \sum_{a,b} \frac{1}{2} \times w_{ab} \times (c_a c_b - 1) \quad (a,b) \in \text{Edge Nodes} \\ \& \quad c_a c_b \in \{-1, 1\}$$

Which is equivalent to finding the ground state of the Hamiltonian,

$$\hat{H}_C = \sum_{a,b} \frac{1}{2} \times w_{ab} \times (Z_a \otimes Z_b - 11) \quad (a,b) \in E$$

The QAOA algorithm requires a cost unitary given by,

$$e^{-i\gamma \hat{H}_C} = e^{-i\gamma \sum \frac{1}{2} \times w_{ab} (Z_a \otimes Z_b - 11)}$$

We must show that this is equivalent to the gate implemented in the code by the function `cost_unitary`. We can use the fact none of the individual terms commute with each other to separate the exponential.

$$e^{-i\gamma \hat{H}_C} = e^{-i\gamma \times \text{some constant}} \left( \bigotimes_{a/b} e^{-i\gamma \times \frac{1}{2} \times w_{ab} \times z_a \otimes z_b} \right)$$

$$\equiv \left( \bigotimes_{a/b} e^{-i\gamma \times \frac{1}{2} \times w_{ab} \times z_a \otimes z_b} \right)$$

We removed a global constant. And decomposed this into several two qubit gates. Let us pay attention to just one of these gates, for the sake of simplicity. What happens when this gate acts on an arbitrary qubit?

$$e^{-i\gamma \frac{1}{2} w_{ab} \times z_1 \otimes z_2} (a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle)$$

$$= e^{i\gamma' z_1 z_2} (a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle)$$

$$= a e^{i\gamma'} |00\rangle + b e^{-i\gamma'} |01\rangle + c e^{-i\gamma'} |10\rangle + d e^{i\gamma'} |11\rangle$$

$$\equiv a|00\rangle + b e^{i\gamma''} |01\rangle + c e^{i\gamma''} |10\rangle + d |11\rangle$$

Now our gate implementation is a Controlled phase gate followed by a phase gate on each qubit.

$$\begin{aligned}
 & (a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle) \\
 & \quad \downarrow CP(\lambda) \\
 & a|00\rangle + b|01\rangle + c|10\rangle + d e^{-2\lambda i} |11\rangle \\
 & \quad \downarrow \text{Phase}_1(\lambda) \otimes \text{Phase}_2(\lambda) \\
 & a|00\rangle + e^{i\lambda} b|01\rangle + c e^{i\lambda} |10\rangle + d |11\rangle
 \end{aligned}$$

We can now see the the equivalence of our circuit to the cost unitary, they differ by just a constant which does not affect measurements.