**Theorem 1** *Let $\alpha_1 = \langle T_{1,1}, T_{2,1}, \ldots, T_{m,1}\rangle$ and $\alpha_2 = \langle T_{1,2}, T_{2,2}, \ldots, T_{n,2}\rangle$ be two paths such that their last transitions $T_{m,1}$ and $T_{n,2}$ are parallelisable. Then, $\alpha_1$ and $\alpha_2$ are parallelisable.*

**Proof 1** *Let it not be so. From Definition of parallelisable pairs of paths, we have the following cases:*

*Case 1:* $\alpha_1 \succ \alpha_2$. *From Definition of successor relationship between two paths, there exist at least one set of paths $\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_n}$ and a set of places $p_1 \in {}^\circ\alpha_1$ and $p_{k_m} \in {}^\circ\alpha_{k_m}, 1 \le m \le n$, such that $\langle$ last$(\alpha_2), p_{k_1}\rangle$, $\langle$ last$(\alpha_{k_1}), p_{k_2}\rangle, \ldots, \langle$ last$(\alpha_{k_n}), p_1\rangle \in O \subseteq T \times P$, $n \ge 0$, and none of them is a back edge. Therefore, using the fact that last$(\alpha) \succ$ first$(\alpha)$, for any path $\alpha$, and reading the above sequence of edges backward, we have last$(\alpha_1)$=$T_{m,1} \succ$ first$(\alpha_1) \succ$ last$(\alpha_{k_n}) \succ$ first$(\alpha_{k_n}) \succ$ last$(\alpha_{k_{n-1}}) \succ \ldots \succ$ first$(\alpha_{k_2}) \succ$ last$(\alpha_{k_1}) \succ$ first$(\alpha_{k_1}) \succ$ last$(\alpha_2) = T_{n,2}$. Hence, $T_{m,1} \succ T_{n,2} \Rightarrow T_{n,2} \not\asymp T_{m,1}$ (Contradiction).*

*Case 2:* $\alpha_2 \succ \alpha_1$. *Following the same argument, as for Case 1, by symmetry with $\alpha_1$ and $\alpha_2$ interchanged, we again obtain the refutation of the hypothesis $T_{m,1} \asymp T_{n,2}$.*

*Case 3:* $\exists \alpha_k, \alpha_l, (\alpha_k \neq \alpha_l \wedge \alpha_1 \succeq \alpha_k \wedge \alpha_2 \succeq \alpha_l \wedge {}^\circ\alpha_k \cap {}^\circ\alpha_l \neq \emptyset)$. ${}^\circ\alpha_k \cap {}^\circ\alpha_l \neq \emptyset \Rightarrow \exists t_{i,k} \in \alpha_k, \exists t_{j,l} \in \alpha_l$ *such that ${}^\circ t_{i,k} \cap {}^\circ t_{j,l} \neq \emptyset$. Let the last transitions of the paths $\alpha_k$ and $\alpha_l$ be $t_{r,k}$ and $t_{s,l}$, respectively. Since $\alpha_1 \succeq \alpha_k$, $T_{m,1} \succeq t_{r,k} \succeq t_{i,k}$; (recall that $T_{m,1} = \{t_{m,1}\}$). Similarly, since $\alpha_2 \succeq \alpha_l$, $T_{n,2} \succeq t_{s,l} \succeq t_{j,l}$. Thus, $T_{m,1} \succeq t_{i,k}$, $T_{n,2} \succeq t_{j,l}$ and ${}^\circ t_{i,k} \cap {}^\circ t_{j,l} \neq \emptyset$. Therefore, $T_{m,1}$ is not parallelisable with $T_{n,2}$ (from Definition of parallelisable paths).*

**Theorem 2** *Let $\Pi$ be the set of all paths of a PRES+ model obtained from a set of static cut-points. For any computation $\mu_p$ of an out-port $p$ of the model, there exists a reorganized sequence $\mu_p^r$ of paths of $\Pi$ such that $\mu_p \simeq \mu_p^r$.*

**Construction of a sequence $\mu_p^r$ of (concatenation of) paths from $\mu_p$:** Algorithm 1 (*construct-PathSequence*) describes a recursive function for constructing from a given computation $\mu_p$ and a set $\Pi$ of paths the desired reorganized sequence $\mu_p^r$ of paths of $\Pi$ such that $\mu_p^r \simeq \mu_p$. If $\mu_p$ is not empty, then a path $\alpha$ is selected from $\Pi$ such that last$(\alpha) \cap$ last$(\mu_p) \neq \emptyset$; if all its transitions are found to occur in $\mu_p$, then it is put as the last member in the reorganized sequence; the member transitions of $\alpha$ are deleted from $\mu_p$ examining the latter backward; the transitions in the last member of $\alpha$ are always deleted from $\mu_p$; each of the other transitions of $\alpha$ is deleted from $\mu_p$ only if it does not occur in any other path in $\Pi$. If $|$last$(\mu_P)| > 1$, then each member transition of last$(\mu_p)$ will result in one path which has to be processed separately through above steps. Once all these paths are processed, the last$(\mu_p)$ will get deleted from $\mu_p$. The resulting $\mu_p$ is then reordered recursively; the process terminates when the input $\mu_p$ becomes empty.

**Proof 2** *($\mu_p^r \simeq \mu_p$): We first prove that Algorithm 1 terminates; this is accomplished in two steps; first, it is shown that each invocation comprising four loops terminate; we next show that there are only finitely many recursive invocations.*

*Termination of the while loop comprising lines 16-18 is obvious; either $i$ becomes less than one or a member $\mu_p.T_i$ is found to contain last$(\alpha')$ (for some $i > 1$). The for loop comprising lines 22-26 iterates only finitely many times because the number of transitions in any member set of a path (and hence $\mu_p'.T_i$) is finite; the while loop comprising lines 15-29 terminates, because in every iteration, it is examined whether the computation $\mu_p'$ contains the last member of $\alpha'$; if so, $\alpha'$ loses this member in line 27 and the next iteration of the loop executes with $\alpha'$ having one member less. Finally, the for loop comprising lines 10-35 terminates because the set last$(\mu_p)$ of transitions (before entering the loop), and hence the set $\Pi_{last(\mu_p)}$ of paths are finite.*

*The second step follows from the fact that in each recursive invocation, $\mu_p$ has one member (namely, its last member) less than the previous invocation (line 40 in the if statement comprising lines 37-41). Hence, if $\mu_p$ has n members, then there are n total invocations ($n-1$ of them being recursive).*

*Now, for proving $\mu_p^r \simeq \mu_p$, let the first parameter $\mu_p$ for the $k^{th}$ invocation be designated as $\mu_p^{(k)}, 1 \leq k \leq n$; the second parameter $\Pi$ remains the same for all invocations; let the value returned by the $k^{th}$ invocation be $\mu_p^{r(k)}$; specifically, $\mu_p = \mu_p^{(1)}$; $\mu_p^{(n-1)}$ comprises just one member and $\mu_p^{(n)} = \langle\rangle$; $\mu_p^{r(n)} = \langle\rangle$ and $\mu_p^{r(1)}$ is the final reordered sequence of paths $\mu_p^r$.*

*We prove $\mu_p^{(n-m)} \simeq \mu_p^{r(n-m)}, 0 \leq m \leq n-1$, by induction on m. Note that specifically for $m = n-1$, $\mu_p^{(n-m)} = \mu_p^{(1)} = \mu_p$ and $\mu_p^{r(n-m)} = \mu_p^{r(1)} = \mu_p^r$ (by line 41 of the first invocation). Hence, the inductive proof would help us establish that $\mu_p^r \simeq \mu_p$.*

Basis $m = 0$: $\mu_p^{(n)} = \langle\rangle = \mu_p^{r(n)}$ *(by line 2 of the $n^{th}$ invocation)*

Induction Hypothesis: *Let for $m = k-1$, $\mu_p^{(n-k+1)} \simeq \mu_p^{r(n-k+1)}$*

Induction step: *Let m be k. Let us assume that*

$\mu_p^{(n-k)} \simeq \mu_p^{(n-k+1)}.\mu_l^{r(n-k)}$ *(Lemma 1 – proved subsequently)*

$\simeq \mu_p^{r(n-k+1)}.\mu_l^{r(n-k)}$ *( by induction hypothesis)*

$\simeq \mu_p^{r(n-k)}$ *(by line 40 (return statement) for the $(n-k)^{th}$ invocation )*

**Lemma 1** $\mu_p^{(n-k)} \simeq \mu_p^{(n-k+1)}.\mu_l^{r(n-k)}$

**Proof 3** *We mould the lemma for the $k^{th}$ invocation directly as*

$$\mu_p^{(k)} \simeq \mu_p^{(k+1)}.\mu_l^{r(k)} \simeq \mu_p^{(k+1)}.\langle\alpha_{1,k}, \alpha_{2,k}, \ldots, \alpha_{s,k}\rangle, 1 \leq k \leq n, \tag{1}$$

*assuming that $\langle\alpha_{1,k}, \alpha_{2,k}, \ldots, \alpha_{s,k}\rangle$ is what is extracted as $\mu_l^{r(k)}$ from $\mu_p^{(k)}$ in line 40 of the kth iteration. Now, by step 6, the last transition of all the paths in the sequence $\mu_l^{r(k)}$ are parallelisable; hence, from Theorem 1, the paths of $\mu_l^{r(k)}$ are parallelisable. We prove that their transitions can be suitably placed in the member sets of $\mu_p^{(k+1)}$ (as larger sets of parallelisable transitions) to get back $\mu_p^{(k)}$.*

*In the $k^{th}$ invocation, $\mu_p^{(k)}$ is the value of $\mu_p$ before entry to the for-loop comprising lines 10-35 and $\mu_p^{(k+1)}$ is the value of $\mu_p$ at the exit of this loop. Since we are speaking about only the $k^{th}$ invocation, we drop the superfix k for clarity. Instead, we depict $\mu_p^{(k)}$ as $\mu_p^-$, $\mu_p^{(k+1)}$ as $\mu_p^+$ and $\mu_l^{r(k)}$ as $\mu_l^r$. So we have to prove that $\mu_p^- \simeq \mu_p^+.\mu_l^r$.*

*Let $\mu_l^r = \langle\alpha_1, \alpha_2, \ldots, \alpha_s\rangle$ before line 37 just after the end of the for-loop comprising lines $10-35$, where s is the cardinality $|\Pi_{last}(\mu_p)|$ before entry to the loop (because any path has only one unit set of transitions as its last member). Thus, the for-loop comprising lines $10-35$ executes s times visiting step 33; let $\mu_p^{-(i)}$, $\mu_p^{+(i)}$ respectively denote the values of $\mu_p$ before and after the $i^{th}$ iteration of the loop. Let $\mu_l^{r(i)}$ be the value of $\mu_l^r$ after the $i^{th}$ execution of the loop. We have the following boundary conditions: $\mu_p^- = \mu_p^{-(1)}, \mu_p^+ = \mu_p^{+(s)}, \mu_l^{r(1)} = \langle\alpha_1\rangle$ and $\mu_l^r = \mu_l^{r(s)} = \langle\alpha_1, \alpha_2, \ldots, \alpha_s\rangle$. The $i^{th}$ iteration of the for-loop comprising lines $10-35$ starts with $\mu_l^{r(i-1)}$ and obtains $\mu_l^{r(i)}, 1 \leq i \leq l$; so let $\mu_l^{r(0)} = \langle\rangle$ be the value of $\mu_l^r$ with which the first execution of the loop takes place.*

**Algorithm 1** SEQUENCE **constructPathSequence** $(\mu_p, \Pi)$

**Inputs:** $\mu_p$: computation of an out-port $p$ and $\Pi$: set of paths
**Outputs:** A sequence of paths equivalent to $\mu_p$.

1: **if** $\mu_p = \langle\rangle$ **then**
2:     **return** $\langle\rangle$;
3: **else**
4:     Let $\mu_p$ be $\langle T_1, T_2, \ldots, T_i, \ldots T_n\rangle$;
5:     Let $\mu_l^r = \langle\rangle$;
    /* a local sub-sequence of paths which, at the `return` statement 40, contains the sequence of paths with their last transitions in $T_n$ */
6:     Let $\Pi_{last(\mu_p)} = \{\alpha \mid last(\alpha) \cap last(\mu_p) \neq \emptyset\}$;
7:     **if** $\Pi_{last(\mu_p)} = \emptyset$ **then**
8:         $\mu_p.T_n = \emptyset$; //Ignore intermediary transitions of paths
9:     **else**
10:         **for** all $\alpha \in \Pi_{last(\mu_p)}$ **do**
11:             $\alpha' = \alpha - last(\alpha)$;
12:             $\mu_p' = \mu_p$; // work on a copy of $\mu_p$
13:             $\mu_p'.T_n = \mu_p'.T_n - last(\alpha)$;
            /* Delete the last transition of $\alpha$; if it occurs in any other paths (as an intermediary transition), then such a path has already been detected. Now detect whether all the remaining transitions of $\alpha$ are available in $\mu_p(\mu_p')$; as a transition is detected, it is deleted from $\mu_p'$ and the copy $\alpha'$ of $\alpha$ only if it does not occur in any other path in $\Pi$. If all the transitions of $\alpha$ do not occur in $\mu_p$, (i.e., $\alpha'$ does not become empty), then $\alpha$ is ignored and the next path from $\Pi_{last(\mu_p)}$ is taken in the next iteration. */
14:             $i \Leftarrow n - 1$; // detection of transitions proceeds backward
15:             **while** $\alpha' \neq \langle\rangle$ **do**
16:                 **while** $(i \geq 1 \wedge \text{last}(\alpha') \nsubseteq \mu_p'.T_i = \emptyset)$ **do**
17:                     $i = i - 1$;
18:                 **end while**
19:                 **if** $i = 0$ **then**
20:                     break;
21:                 **else**
22:                     **for** all $t \in \text{last}(\alpha')$ **do**
23:                         **if** $t$ does not occur in any path in $\Pi - \{\alpha\}$ **then**
24:                             $\mu_p'.T_i \Leftarrow \mu_p'.T_i - \{t\}$;
25:                         **end if**
26:                     **end for**
27:                     $\alpha' = \alpha' - \text{last}(\alpha') \cap \mu_p.T_i$;
28:                 **end if**
29:             **end while**
30:             /* both $\alpha' \neq \langle\rangle$ and $\alpha' = \langle\rangle$ are possible */
31:             **if** $\alpha' = \langle\rangle$ **then**
32:                 append $(\alpha, \mu_l^r)$;
33:                 $\mu_p = \mu_p'$;
34:             **end if**
35:         **end for**
36:     **end if**
37:     **if** original member $\mu_p.T_n$ is not empty **then**
38:         report failure with $\mu_p$
39:     **else**
40:         **return** (**concatenate** (**constructPathSequence**$(\mu_p, \Pi)$, $\mu_l^r$));
41:     **end if**
42: **end if**

*We prove that* $\mu_p^{-(i)} \simeq \mu_p^{+(i)}.\alpha_i, 1 \leq i \leq s$. *If this relation indeed holds, then specifically for* $i = 1$,

*$\mu_p^{-(1)} \simeq \mu_p^{+(1)}.\alpha_1$; for $i = 2$, $\mu_p^{-(2)}(= \mu_p^{+(1)}) \simeq \mu_p^{+(2)}.\alpha_2$. Combining these two, therefore,*

$\mu_p^{-} = \mu_p^{-(1)} \simeq \mu_p^{+(1)}.\alpha_1 \simeq (\mu_p^{+(2)}.\alpha_2).\alpha_1 \simeq \mu_p^{+(2)}.(\alpha_2.\alpha_1) \simeq \mu_p^{+(2)}.(\alpha_1.\alpha_2) \simeq \mu_p^{+(2)}.\mu_l^{r(2)}$.

*Proceeding this way, we have $\mu_p^{-} = \mu_p^{-(1)} \simeq \ldots \simeq \mu_p^{+(l)}.\mu_l^{r(l)} = \mu_p^{+}.\mu_l^{r}$.*

*Now, let $\mu_p^{-(i)} = \langle T_{1,i}, T_{2,i}, \ldots, T_{k_i,i}\rangle$, $\alpha_i = \langle T'_{1,i}, T'_{2,i}, \ldots, T'_{l_i,i}\rangle$ and $\mu_p^{+(i)} = \langle T^+_{1,i}, T^+_{2,i}, \ldots, T^+_{n,i}\rangle$. Note that $\{\alpha_i \mid 1 \le i \le s\} \subseteq \Pi_{last(\mu_p)}$ and unless all the paths are extracted out, $T_{k_i,i}$ does not become empty and hence $\mu_p^{-(i)}, 1 \le i \le s$, do not change in length. For each transition set $T'_{j,i}$ of $\alpha_i$, $1 \le j \le n$, there exists some transition set $T_{k,i}$ of $\mu_p^{-(i)}$, $1 \le k \le k_i$, such that $T'_{j,i} \subseteq T_{k,i}$. Specifically, for $j = l_i$, $T'_{l_i,i} \subseteq T_{k_i,i}$, since $\alpha_i \in \Pi_{last(\mu_p^-)}$ as ensured in step 6. For other values of $j, 1 \le j < l_i$, the while-loop in steps 16-18, identifies proper $T_{k,i}$ in $\mu_p^{-(i)}$ such that $T'_{j,i} \subseteq T_{k,i}$; note that since $\alpha_i$ has figured in $\mu_l^r$, step 32 is surely executed for $\alpha_i$; so $\alpha'$ has been rendered empty ($\langle\rangle$) through execution of step 27 and hence the while-loop in steps 16-18 does not exit with $i = 0$. Now, step 13 and the for-loop in steps 22-26 ensure that $T^+_{k,i} \cup T_{j,i} = T_{k,i}$.*

*Let $T'_{j,i} \subseteq T_{n_j,i}, 1 \le j \le l_i$. So, $T_{k,i} = T^+_{k,i}$, for $k \ne n_j$, for any $j$, $1 \le j \le l_i$.*

$$\begin{aligned}
\mu_p^+(i).\alpha_i &= \langle T^+_{1,i}, T^+_{2,i}, \ldots, T^+_{n,i}\rangle.\langle T'_{1,i}, T'_{2,i}, \ldots, T'_{l_i,i}\rangle \\
&= \langle T_{1,i}, \ldots, (T^+_{n_1,i} \parallel T'_{1,i}), \ldots (T^+_{n_2,i} \parallel T'_{2,i}), \ldots, (T^+_{n_{l_i-1},i} \parallel T'_{l_i,i}), \ldots, (T^+_{n_i,i} \parallel T'_{l_i,i})\rangle \\
&\qquad \text{(by commutativity of independent transitions)} \\
&= \langle T_{1,i}, \ldots, T_{n_1,i}, \ldots, T_{n_2,i}, \ldots, T_{n_{l_i-1},i}, \ldots T_{n,i}\rangle \\
&= \mu_p^{-(i)}
\end{aligned}$$

**Corollary 1** *If $\mu_p^r$ is of the form $\langle \alpha_1, \alpha_2, \ldots, \alpha_k\rangle$, for all $j, 1 \le j \le i-1, \alpha_j \not\succ \alpha_i$.*

**Theorem 3** *A PRES+ model $N_0$ is contained in another PRES+ model $N_1$, denoted as $N_0 \sqsubseteq N_1$, if there exists a finite path cover $\Pi_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ of $N_0$ for which there exists a set $\Psi_1 = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_m\}$ of sets of paths of $N_1$ such that for all $i, 1 \le i \le m$, (i) $\alpha_i \simeq \beta$, for all $\beta \in \Gamma_i$. (ii) For each $\alpha_i, 1 \le i \le m$, each pre-place of $\alpha_i$ has a place-correspondence with some pre-place of $\beta$, where $\beta \in \Gamma_i$, (iii) all the post-places of $\alpha_i$ have correspondence with all the post-places of $\beta \in \Gamma_i$.*

**Proof 4** *Consider any computation $\mu_{0,p}$ of an out-port $p$ of $N_0$. From Theorem 2, corresponding to $\mu_{0,p}$, there exists a reorganized sequence $\mu^r_{0,p} = \langle \alpha^p_1, \alpha^p_2, \ldots, \alpha^p_n\rangle$, say, of not necessarily distinct paths of $N_0$ such that (i) $\alpha^p_j \in \Pi_0$, $1 \le j \le n$, (ii) for each occurrence of a transition $t$ in $\mu_{0,p}$, there exists exactly one path in $\mu^r_{0,p}$ containing that occurrence, (iii) $p \in (\alpha^p_n)^\circ$ and (iv) $\mu_{0,p} \simeq \mu^r_{0,p}$.*

*Let us now construct from the sequence $\mu^r_{0,p}$, a sequence $\mu^r_{1,p'} = \langle \Gamma^{p'}_1, \Gamma^{p'}_2, \ldots, \Gamma^{p'}_n\rangle$ of not necessarily distinct sets of paths of $N_1$, where (i) $\Gamma^{p'}_n = \{\beta_l\}$ and $p' \in \beta_l^\circ$, and for all $j, 1 \le j \le n$, for each $\beta \in \Gamma^{p'}_j$, (ii) $\beta \simeq \alpha^p_j$, and (iii) each pre-place of $\beta$ has correspondence with some pre-place of $\alpha^p_j$. It is required to prove that* (1) *$p' = f_{out}(p)$ and* (2) *there exists a computation $\mu_{1,p'}$ of $N_1$ such that $\mu_{1,p'} \simeq \mu^r_{1,p'}$.*

*The proof of* (1) *is as follows. Since $p' \in \beta_l^\circ$ and $\beta_l \simeq \alpha_n$, from hypothesis* (*iii*) *of the theorem, $p'$ has correspondence with $p$; since the place $p \in P_0$ is an out-port and the place $p' \in P_1$, $p'$ must be an out-port of $N_1$ and $p' = f_{out}(p)$ (because an out-port of $N_0$ has correspondence with exactly one out-port of $N_1$ specifically, its image under the bijection $f_{out}$).*

*For the proof of* (2), *we first give a mechanical construction of $\mu_{1,p'}$ from $\mu^r_{1,p'}$; we then show that they are equivalent; finally, we argue that $\mu_{1,p'}$ is a computation of $p'$ in $N_1$.*

**Algorithm 2** Sequence **parallelizeSeqSetsOfPaths** ( $\mu_p^r$)

**Inputs:** $\mu_p^r$ : a sequence of sets of paths
**Outputs:** $\mu_{\|}$: a sequence of maximally parallelisable sets of transitions of all the paths in $\mu_p^r$.

1: $\Gamma = \text{head}(\mu_p^r)$; $\mu_p^r = \text{tail}(\mu_p^r)$;
2: $\mu_{\|}$ = some path $\beta \in \Gamma$; $\Gamma = \Gamma - \{\beta\}$; // $\beta$ chosen arbitrarily
3: **while** $\mu_p^r \neq \emptyset$ **do**
4:   **if** $\Gamma \neq \emptyset$ **then**
5:     $\Gamma = head(\mu_p^r)$; $\mu_p^r = tali(\mu_p^r)$; // except for the first iteration, if-condition holds
6:   **end if**
7:   **for** each $\beta \in \Gamma$ **do**
8:     Let $c = 1$;
    /* index to the members of $\mu_{\|} - c^{th}$ member is $\mu_{\|,c}$; for each path of $\mu_p^r$, checking has to be from the first member of $\mu_{\|}$.*/
9:     **while** $\beta \neq \emptyset$ **do**
10:       $T_c = \mu_{\|,c}$;
11:       $T_p$ = head ($\beta$);
      /* $T_p$ is the maximally parallelisable set (member) of $\beta$ presently being considered for fusion with $T_c$ */
12:       $\beta = \text{tail}(\beta)$;
13:       **while** $T_p \succ T_c \wedge c \leq \text{length}\ (\mu_{\|})$ **do**
14:         /* $T_p$ succeeds $T_c$ */
15:         $c++$;
16:         $T_c = \mu_{\|,c}$;
17:       **end while**
18:       **if** $c > \text{length}\ (\mu_{\|})$ **then**
19:         /* $T_p$ is found to be parallelisable with none of the members of $\mu_{\|}$; so $T_p \succ T, \forall T \in \mu_{\|}$ concatenate all the members (including $T_p$) of $\beta$ after $\mu_{\|}$ */
20:         $\mu_{\|} \leftarrow$ concatenate $(\mu_{\|}, \beta)$; $\beta = \emptyset$;
21:       **else**
22:         $\mu_{\|,c} = \mu_{\|,c} \cup T_p$; $c++$;
        /* $T_c \asymp T_p$ or $T_c = T_p$ – absorb $T_p$ in $T_c$ */
23:       **end if**
24:     **end while**
25:   **end for**
26: **end while**
27: **return** $\mu_{\|}$;

**Construction of $\mu_{1,p'}$ from $\mu_{1,p'}^r$:**

*Algorithm 2 describes the construction method of $\mu_{1,p'}$ from $\mu_{1,p'}^r$ (and hence will be invoked with its input $\mu_p^r$ instantiated with $\mu_{1,p'}^r$). The parallelized version of the input $\mu_p^r$ is computed in $\mu_{\|}$ which is to be assigned to $\mu_{1,p'}$ on return. In the initialization step (step 1), a working set $\Gamma$ of paths is initialized to the first member of $\mu_p^r$ and the latter is removed from $\mu_p^r$. In step 2, some path $\beta$ is taken from $\Gamma$ and put into $\mu_{\|}$. In the outermost while-loop (steps 3-26), member sets of $\Gamma$ are taken one by one (in steps 4-6) from $\mu_p^r$; for each chosen set, its member paths are taken in the loop comprising steps 7-25; for each chosen path $\beta$, its member sets (of maximally parallelisable transitions) are examined one after another and checked against the members of $\mu_{\|}$ from the beginning for fusion with them to construct larger sets of parallelisable transitions (steps $9-24$). For each chosen set $T_p$ of transitions of $\beta$, one of the following two situations may arise:*

*Case 1 : The member $T_p$ of the chosen path $\beta$ of $\mu_p^r$ is found to succeed all the members in $\mu_{\|}$, i.e., $T_p$ is not parallelisable with any member of $\mu_{\|}$. In this case, all the remaining members (including $T_p$) of $\beta$ is concatenated at the end of $\mu_{\|}$ [Steps 18-20] .*

*Case 2 : The member $T_p$ of $\beta$ is found not to succeed the $c^{th}$ member $\mu_{\|,c}$ of $\mu_{\|}$, i.e., $T_p$ is parallelisable with $\mu_{\|,c}$, as argued later. In this case, $T_p$ is combined (through union) with $\mu_{\|,c}$; the successor transition sets of $\beta$ need to be compared with only the subsequent members of $\mu_{\|}$, i.e., with $\mu_{\|,c+1}$ onwards [Step 22] .*

**Termination:** *The algorithm terminates because all the three while loops and the for-loop terminate as given below:*

*The outer loop (steps 3-26) terminates because $\mu_p^r$ is finite to start with; (step 1 outside the loop reduces its length by one;) step 5 inside the loop reduces its length by one on every iteration of the loop. The for-loop (steps 7-25) terminates because the set $\Gamma$ contains a finite number of paths and loses the chosen path in each iteration as per the semantics of the for-construct. The loop comprising steps 9-24 terminates because every path $\beta$ in $\mu_p^r$ contains a finite number of sets of transitions and step 12 reduces the length by one in every iteration of the loop; if, however, any of these iterations do visit steps 19-20, then in step 20, $\beta$ becomes empty and hence, this will be the last iteration of the while loop comprising steps 9 to 24. The loop comprising steps 13-17 terminates because at any stage, and hence on entry to the loop, $\mu_{\|}$ has only a finite number of sets of transitions and in every iteration c increases by one; so finally, the second condition $c \leq length(\mu_{\|})$ is bound to become false if the first condition does not become false by then.*

**Proof of $\mu_{1,p'} \simeq \mu^r_{1,p'}$:** *Let the initial value of $\mu_p^r$ (with which the function in Algorithm 2 is invoked), denoted as $\mu_p^r(-1)$, be of the form $\mu_p^r(-1) = \langle \Gamma_1, \Gamma_2, \ldots, \Gamma_n \rangle$, where, for all $i, 1 \leq i \leq n, \Gamma_i = \{\beta_{1,i}, \beta_{2,i}, \ldots, \beta_{t_i,i}\}$. So the outermost while-loop (steps 3-26) executes n times; for the $i-th$ execution of this loop, the inner for-loop executes $t_i$ times; together, there are $t_1 \times t_2 \times \ldots \times t_n = t$, say, iterations in each of which a path $\beta_{j,i}$ is accounted for. The algorithm treats these paths identically without making any distinction among paths from the same set or different sets. Hence we can treat the members of $\mu_p^r$ as a flat sequence of paths of the form $\langle \beta_1, \beta_2, \ldots \beta_t \rangle$. Let $\mu_p^r(i)$ and $\mu_{\|}(i)$ respectively indicate the values of $\mu_p^r$ and $\mu_{\|}$ at step 8 after the $i-th$ path $\beta_i$ in the above sequence has been treated. So, the first time step 8 is executed, the value of $\mu_{\|}$ is $\mu_{\|}(0)$ = the first member $\beta_1$ of $\mu_p^r(-1)$ and $\mu_p^r(0)$ contains all the remaining members $\beta_2, \ldots, \beta_t$ of $\mu_p^r(-1)$. The final value returned by the algorithm (step 27) is $\mu_{\|}(t)$ and $\mu_p^r(t) = \emptyset$ (by negation of the condition of the outermost while loop (steps 3-26)). We have to prove that $\mu_{1,p'} = \mu_{\|}(t) \simeq \mu_p^r(-1) = \mu^r_{1,p'} \simeq \mu^r_{0,p} \simeq \mu_{0,p}$. We prove the invariant*

$$\mu_{\|}(i).\mu_p^r(i) \simeq \mu_p^r(-1), \forall i, 0 \leq i \leq t \ldots\ldots Inv(1) \tag{2}$$

*by induction on i, where the operator $'.'$ stands for concatenation of two sequences. Note that in this invariant, for $i = t$,*

$\mu_{\|}(t).\mu_p^r(t) \simeq \mu_p^r(-1) \Rightarrow \mu_{1,p'}.\emptyset \simeq \mu_p^r \Rightarrow \mu_{1,p'} \simeq \mu^r_{1,p'}$, *which would accomplish the proof as $\mu^r_{1,p'} \simeq \mu^r_{0,p}$ holds because the former has been obtained by* equivalence substitution of each member *in the latter and $\mu^r_{0,p} \simeq \mu_{0,p}$ by Theorem 2.*

**Basis ($i = 0$):** $\mu_{\|}(0).\mu_p^r(0) = \langle \beta_1 \rangle.\langle \beta_2, \ldots, \beta_t \rangle = \langle \beta_1, \beta_2, \ldots, \beta_t \rangle \simeq \mu_p^r(-1)$.

**Induction Hypothesis:** *Let $\mu_{\|}(i).\mu_p^r(i) \simeq \mu_p^r(-1)$, for $i = m-1$.*

**Induction step ($i = m$):** *R.T.P $\mu_{\|}(m).\mu_p^r(m) \simeq \mu_p^r(-1)$. Let the $m^{th}$ path chosen be $\beta_m = \langle T_{1,m}, T_{2,m}, \ldots, T_{l_m,m} \rangle$. Let $\mu_{\|}(m-1) = \langle T_1, T_2, \ldots, T_k \rangle$. For $T_{1,m}(= T_p)$, comparison starts with the first member $T_1 = T_c$ of $\mu_{\|}(m-1)$.*

*Now we need to consider the inner while loop comprising steps 9-24, where the members of $\beta_m$, i.e., $T_{j,m}$, $1 \leq j \leq l_m$, are taken one by one and compared with the members of $\mu_{\|}(m-1)$. Note that the inner loop need not always execute $l_m$ times. Let it execute $n_m \leq l_m$ times. Let $\mu_p^r(m-1)(j)$, $1 \leq j \leq n_m$, represent* *the value of $\mu_p^r(m-1)$ after the $j^{th}$ iteration of this loop for the path $\beta_m$. Thus, $\mu_p^r(i-1)(0)$ is the value of $\mu_p^r(i-1)$ at step 8 when no members of $\beta_i$ have yet been considered. Hence, $\mu_p^r(m-1)(0) = \mu_p^r(m-1)$. Also, $\mu_p^r(i-1)(n_i) = \mu_p^r(i)$. Let $\beta_m(j)$ be the value of $\beta_m$ and $\mu_{\|}(m-1)(j)$ be the value of $\mu_{\|}(m-1)$ after the $j^{th}$ execution of the inner while loop (steps 9-24) for the path $\beta_m$. We prove the invariant*

$$\mu_{\|}(m-1).\beta_m \simeq \mu_{\|}(m-1)(j).\beta_m(j), \forall j, 0 \leq j \leq n_m \ldots\ldots Inv(2) \tag{3}$$

*Let us first examine how the Inv (2) helps us accomplish the proof of the induction step of Inv (1). Putting $j = n_m$ in Inv (2),*

$\mu_{\|}(m-1).\beta_m \simeq \mu_{\|}(m-1)(n_m).\beta_m(n_m) = \mu_{\|}(m).\emptyset$

*(since, $\mu_{\|}(m-1)(n_m) = \mu_{\|}(m)$ and $\beta_m(n_m) = \emptyset$ from the termination condition of the loop comprising steps 9-24).*

*Also, $\beta_m.\mu_p^r(m) = \mu_p^r(m-1)$ [when $\beta_m$ is chosen at step 7]. So for the inductive step proof goal,*

$\mu_{\|}(m).\mu_p^r(m) = (\mu_{\|}(m-1).\beta_m).\mu_p^r(m)$

$= \mu_{\|}(m-1).(\beta_m.\mu_p^r(m))$ *[by associativity of concatenation operation '.']*

$= \mu_{\|}(m-1).\mu_p^r(m-1) \simeq \mu_p^r(-1)$ *[by induction hypothesis]*

*We now carry out the inductive proof of Inv (2) by induction on $j$.*

**Basis** ($j = 0$)**:** *The basis case holds because $\mu_{\|}(i-1)(0) = \mu_{\|}(i-1)$ and $\beta_i(0) = \beta_i$.*

**Induction Hypothesis:** *Let the invariant Inv (2) is true for $j = k-1$, i.e.,*
$\mu_{\|}(m-1).\beta_m \simeq \mu_{\|}(m-1)(k-1).\beta_m(k-1)$.

**Induction step** ($j = k$)**:** *R.T.P $\mu_{\|}(m-1).\beta_m \simeq \mu_{\|}(m-1)(k).\beta_m(k)$. Let $\beta_m(k-1) = \langle T_{k,m}, T_{k+1,m}, \ldots, T_{l_m,m}\rangle$. Without loss of generality, let the iterations $1, \ldots, k-1$ of the loop of steps 9-24 did not visit step 20; otherwise, the loop will not be executed $k^{th}$ time. In the $k^{th}$ iteration of the loop, $T_{k,m}$ is compared with some $T_c \in \mu_{\|}(m-1)(k-1)$. We have the following two cases:*

*Case 1: $T_{k,m}$ is found to succeed all the members of $\mu_{\|}(m-1)(k-1)$ from $T_c$ onwards– Hence, $T_{k,m}$ is parallelisable with no members of $\mu_{\|}(m-1)(k)$. In this case, step 20 is executed resulting in concatenation of all the transition sets of $\beta_m(k-1)$ with $\mu_{\|}(m-1)(k-1)$ and $\beta_m(k)$ becomes empty. So, $\mu_{\|}(m-1)(k) = \mu_{\|}(m-1)(k-1).\beta_m(k-1)$;*
*hence, $\mu_{\|}(m-1).\beta_m \simeq \mu_{\|}(m-1)(k-1).\beta_m(k-1)$ [by Induction hypothesis]*
$= \mu_{\|}(m-1)(k).\beta_m(k)$ *(since $\beta_m(k) = \emptyset$)*

*Case 2: $T_{k,m} \nsucc T_c$– This implies $T_{k,m} \asymp T_c$, as argued below. Note that between the two transition sets $T_{k,m}$ and $T_c$, there can be three mutually exclusive relations possible namely, $T_{k,m} \succ T_c, T_c \succ T_{k,m}$ and $T_{k,m} \asymp T_c$. It is given that $T_{k,m} \nsucc T_c$; now, let $T_c \succ T_{k,m}$. The transition set $T_c$ in $\mu_{\|}(m-1)$ is contributed to by paths which precede the path $\beta_m$ in $\mu_p^r$. Hence $T_c$ does not succeed $T_{k,m}$. Therefore, $T_{k,m} \asymp T_c$. Let $\mu_{\|}(m-1) = \langle T_1, T_2, \ldots, T_c, T_{c+1}, \ldots, T_k, \ldots T_{k_{m-1}}\rangle$. For all $s, 1 \leq s \leq k_{m-1} - c$, $T_c \nsucc T_{c+s}$. By an identical reasoning, $T_{k,m}$ does not also succeed $T_{c+s}$ because otherwise $T_{c+s}$ would have preceded in the path $\beta_m$. Therefore, $T_{c+s}.T_{k,m} \simeq T_{k,m}.T_{c+s}$. So, the concatenation $\langle T_{c+1}, \ldots, T_{k_{m-1}}\rangle.\langle T_{k,m}, T_{k+1,m}, \ldots, T_{l_m,m}\rangle$ is computationally equivalent to*
$\langle T_{k,m}, T_{c+1}, \ldots, T_{k_{m-1}}\rangle.\langle T_{k+1,m}, \ldots, T_{l_m,m}\rangle$. *Now,*
$\mu_{\|}(m-1).\beta_m \simeq \mu_{\|}(m-1)(k-1).\beta_m(k-1)$ *[by induction hypothesis]*
$= \langle T_1, T_2, \ldots T_c, T_{c+1}, \ldots T_{k_m-1}\rangle.\langle T_{k,m}, T_{k+1,m}, \ldots, T_{l_m,m}\rangle$
$\simeq \langle T_1, T_2, \ldots T_c, T_{k,m}, T_{c+1}, \ldots T_{k_m-1}\rangle.\langle T_{k+1,m}, \ldots, T_{l_m,m}\rangle$
$\simeq \langle T_1, T_2, \ldots T_c \cup T_{k,m}, T_{c+1}, \ldots T_{k_m-1}\rangle.\beta_m(k)$

*[since, by step 12, $\beta_m(k) = \langle T_{k+1,m}, \ldots, T_{l_m,m}\rangle$]*

$= \mu_{\|}(m-1)(k).\beta_m(k)$ *[by step 22].*

*Note that since $T_{k,m} \succ T_{c-1}$ in $\mu_{||}(m-1)(k-1)$, as identified in the loop steps 13-17, $T_c$ cannot be combined with $T_{c-1}$ through union.*

**Proof of $\mu_{1,p'}$ being a computation:** *Recall that $\mu_{1,p'}$ is obtained from the sequence $\mu^r_{1,p'} = \langle \Gamma_1^{p'}, \Gamma_2^{p'}, \ldots, \Gamma_n^{p'} \rangle = \langle \{\beta_{1,1}, \beta_{2,1}, \ldots, \beta_{l_1,1}\}\{\beta_{1,2}, \beta_{2,2}, \ldots, \beta_{l_2,2}\}, \ldots, \{\beta_n\} \rangle$ of sets of paths of $N_1$, which, in turn, was constructed from the sequence $\mu^r_{0,p} = \langle \alpha_1^p, \alpha_2^p, \ldots, \alpha_n^p \rangle$ of paths of $\Pi_0$ satisfying the following properties: (i) $p' = \beta_n^\circ$, (ii) for all $j, 1 \le j \le n$, for all $k, 1 \le k \le l_j, \beta_{k,j} \simeq \alpha_j^p$, (iii) each of the places in ${}^\circ\Gamma_j^{p'}$ has correspondence with some place in ${}^\circ\alpha_j$ and (iv) all the places in $\Gamma_j^\circ$ have correspondence with with all the places in $\alpha_j^\circ$.*

*Let $\mu_{1,p'}$ be $\langle T_1, T_2, \ldots, T_l \rangle$, where $T_1$ is the first member of $\beta_{1,1}$ (by step 1 and first time execution of steps 7 and 11 of Algorithm 2). By property (iii) above, the places in ${}^\circ\beta_{1,1} \supseteq {}^\circ T_1$ have correspondence with those in ${}^\circ\alpha_1^p \subseteq inP_0$. Since only the input places of $N_1$ have correspondence with the input places of $N_0$, ${}^\circ T_1 \subseteq {}^\circ\beta_{1,1} \subseteq inP_1$. It has already been proved that $p' = f_{out}(p) \in outP_1$. Since Algorithm 2 introduces the transition sets of the paths strictly in order from $\Gamma_1^{p'}$ to $\Gamma_n^{p'}$, $T_l$ is a unit set containing the last transition of $\beta_n$ and hence, $p' \in T_l^\circ$. Now, consider any $T_i \in \mu_{1,p'}$, $1 \le i < l$; $T_{i+1} \succ T_i$ as ensured by the condition $T_p \succ T_c$ associated with the while loop of steps 13-17. For any $i, 1 \le i < l$, let $T_{i+1}^\circ \subseteq P_{M_{i+1}}$ and $T_i^\circ \subseteq P_{M_i}$. It is required to prove that $M_{i+1} = M_i^+$, where $P_{M_i^+} = \{p \mid p \in t^\circ \wedge t \in T_m\} \cup \{p \mid p \in P_M \wedge p \notin^\circ T_m\}$, by first clause of Definition of successor marking. We have the following two cases:*

*Case 1: $p_1 \in T_{i+1}^\circ \subseteq P_{M_{i+1}}$– $p_1 \in T_{i+1}^\circ \Rightarrow \exists t_1 \in T_{i+1}$ such that $p \in t_1^\circ$. Now, $T_{i+1} = T_{M_i}$, the set of enabled transitions for the marking $M_i$. So, $p_1 \in t_1^\circ$ and $t_1 \in T_{i+1} = T_{M_i} \Rightarrow p_1 \in P_{M_i^+}$ by virtue of its being in the first subset of $P_{M_i^+}$.*

*Case 2: $p_1 \notin T_{i+1}^\circ$ but $\in P_{M_{i+1}}$– So, $p_1 \notin T_{i+1}^\circ = T_{M_i}$. Hence, $p_1 \in P_{M_i}$ because $p_1 \in T_{i-k}^\circ$ for some $k \ge 1$. So $p_1 \in P_{M_i^+}$ by virtue of its being in the second subset of $P_{M_i^+}$. Therefore, $M_{i+1} = M_i^+$.*