Global Symmetry Breaking:- A perspective on Symmetry Inconsistencies

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1 Abstract

Global Symmetries are inconsistent with Quantum Gravity. Most global symmetries are broken/gauged away at lower energies . These kind of symmetries, even though are derived well out of Quantum Field Theories and General Relativity, it undergoes spontaneous symmetry breaking at low energy levels and remain lacking at unification. Spontaneous Symmetry Breaking(SSB) gives rise to Goldstone modes which signals the presence of massless excitations. A great example of SSB is the Higgs Field. The problem has been approached from a few different perspectives- namely - global symmetries on the string worldsheet, the consequences of symmetry breaking that happens on a conserved Noether's charge and how divergences occur when the vacuum state does not remain invariant and finally a topological perspective to how gauge symmetry is broken and how it leads to the rise of both broken and unbroken generators and mass terms.

Major focus has been devoted towards understanding how a non-triviality approach in the context of conserved Noether current as well as a Holonomy group can give us deep insight on how divergences lead to inconsistencies(this is very apparent in case of Black Holes). It also provides information as to how compactification of higher dimensional gauge group leads to the breakdown of gauge symmetry.

Contents

2 Basic Introduction for Symmetries and Gauge theory

2.1 Symmetries and Gauge theory, an overview

Considering symmetries , in quantum field theory there are two kinds of symmetry, local symmetry and global symmetry.

A global symmetry is a transformation that acts uniformly on all points in spacetime. Mathematically, a global symmetry transformation is represented by an operator U.
U is an unitary operator that commutes with the field operators of the theory. For a scalar field theory, a global symmetry transformation can be expressed as:
$\psi(x) \rightarrow \psi^{'}(x) = U\psi(x)\phi(x)$ where $\phi(x)$ is the field operator at spacetime point x , and U is the global symmetry operator. Global symmetries lead to conservation laws through Noether's theorem which states that for every symmetry there is a conserved quantity, whether it be momentum, angular momentum or energy.

For example, a global $U(1)$ symmetry leads to the conservation of electric charge in QED.

A local symmetry, also known as gauge symmetry, is a transformation that varies from point to point in spacetime. In gauge theories, such as Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD), local symmetries are associated with gauge fields and gauge bosons. Mathematically, a local symmetry transformation is represented by a gauge transformation

$U(x)$ that depends on spacetime coordinates: $\psi(x) \rightarrow \psi^{'}(x) = U(x)\psi(x)$

Here, $\psi(x)$ represents the field operator of the fermion field, and

$U(x)$ is the gauge transformation. Gauge symmetries introduce redundancy in the description of the theory, which manifests as gauge degrees of freedom. Gauge theories are constructed to be invariant under local gauge transformations, leading to the emergence of gauge fields and ensuring that physical observables are independent of the choice of gauge.

2.2 Gauge Theory , an overview

The gauge principle is a fundamental concept in theoretical physics that states that the laws of physics should be invariant under local transformations of a certain group. In the context of gauge theories, such as electromagnetism and the weak and strong nuclear forces, the gauge principle underlies the symmetries and interactions of elementary particles.

Let's consider a complex scalar field $\psi(x)$ as an example. Under a gauge transformation, the field $\psi(x)$ undergoes a local phase transformation:$\psi(x) \rightarrow \psi'(x) = e^{i\alpha(x)}\psi(x)$. Here, $\alpha(x)$ is an arbitrary real-valued function of spacetime x.
The gauge principle demands that the physical predictions of the theory remain unchanged under such local gauge transformations. Mathematically, this can be expressed as: $\mathcal{L}(\psi, \partial_\mu\psi, A_\mu) = \mathcal{L}(\psi', \partial_\mu\psi', A_\mu)$ where the gauge field A_μ is representing the interaction.
To ensure gauge invariance, we introduce a gauge field $A_\mu(x)$, that transforms under gauge transformations such that the gauge-invariant derivative is preserved. This is done by replacing ordinary derivatives with covariant derivatives:

$$D_\mu = \partial_\mu - iqA_\mu$$

where q is a coupling constant associated with the interaction.
Under a gauge transformation, the gauge field A_μ can be transformed under an Unitary transformation, which gives us the covariant derivative as,

$$D_\mu\psi(x) = d_\mu\psi(x) - iqA_\mu\psi(x)$$

Gauge theory ensures the consistency and invariance of physical laws under local transformations, leading to the introduction of gauge fields and the covariant derivatives that preserve gauge invariance. The gauge principle is associated with a gauge symmetry group, such as $U(1)$ for electromagnetism or $SU(2)$ for the weak force. The choice of gauge group depends on the specific theory being considered.

2.3 Gauge Symmetry

Now, to ensure gauge invariance, the electromagnetic potential A_μ needs to transform under gauge transformations. It transforms as:

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x)$$

for a function $\Lambda(x)$, we will ask only that x dies suitably quickly at spatial infinity . We call this a gauge symmetry. The field strength remains locally invariant. $\psi(x) = e^{i\alpha(x)}\psi(x)$ which remains phase invariant under $\alpha(x)$ rotations.

So what are we to make of this? We have a theory with an infinite number of symmetries, one for each function x. Previously we only encountered symmetries which act the same at all points in spacetime. Noether's theorem told us that these symmetries give rise to conservation laws. Do we now have an infinite number of conservation laws? . The answer is no! Gauge symmetries have a very different interpretation than the global symmetries that we make use of in Noether's theorem. While the latter take a physical state to another physical state with the same properties, the gauge symmetry is to be viewed as a redundancy in our description. That is, two states related by a gauge symmetry are to be identified: they are the same physical state.

As we get deepen our investigation into gauge and global symmetries, we are going to draw analogies from different theories and contexts. The motivation as mentioned in the beginning , is to give a symmetry treatment to Quantum Gravity. A fundamental question which needs exploration with regards to Quantum Gravity is why are global symmetries inconsistent with it, specifically when the theory of General Relativity and QFTs derive well defined consistencies . Let's begin.

3 Global Symmetries on the string worldsheet

The main question here is - Does global symmetries apply everywhere in spacetime ? YES. Then what happens in Quantum Gravity. ?

The most mathematically consistent theory which we have right now, when it comes to the unification of both Quantum mechanics and General Relativity is string theory. Gravity, as described by Einstein's field equations is a consequence of string theory. If we quantize a string in 11 dimensions, Einstein's field equations comes out of it as a consequence. In string theory, most global symmetries are gauged at lower energies. How exactly does it happen ?. We will see now.

The idea that any theory of quantum gravity cannot have global symmetries has a long history [1].
The lack of such ordinary global continuous symmetries is known to be satisfied in all descriptions of quantum gravity. It was shown in the paper [2], to be satisfied in perturbative string theory - global symmetries on the string world-sheet lead to gauge symmetries in spacetime, and there is no way to have global symmetries in spacetime.

Let's look at it mathematically,

from the Polyakov Action on the worldsheet, [3]

$$S = -\frac{1}{4\pi\alpha'} \int \partial^2\sigma\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

where $g = det(g_{\alpha\beta})$, is the determinant of the metric tensor, $g_{\alpha\beta}$ is the world-sheet metric. The world-sheet metric can also be expressed as:- $g_{\alpha\beta} = 2f(\sigma)\partial_\alpha X.\partial_\beta X$

The equation of motion for X^{μ} , which is a bunch of scalar fields X , coupled to $2d$ gravity will be:-

$$\partial_{\alpha}(\sqrt{-g}g^{\alpha\beta}\partial_{\beta}X^{\mu}) = 0$$

The Polyakov action still has the two symmetries of the Nambu-Goto action. -
Poincare Invariance.
- Reparametrization invariance.
1) Poincare Invariance:- This is a global symmetry on the worldsheet.

$$X_{\mu} \rightarrow \Lambda^{\mu}_{\nu}X^{\nu} + c^{\mu}$$

2) Reparameterization invariance:- also known as diffeomorphisms. This is a gauge symmetry on the worldsheet. The fields X^{μ} transform as worldsheet scalars, while $g_{\alpha\beta}$ transforms in the manner appropriate for a 2d metric.[1]

Along with these symmetries, there is a new kind of symmetry called Weyl invariance.

Weyl invariance:- Under this transformation of the scalar field X^{σ} , the metric transforms as equation (1)

$$g_{\alpha\beta}(\sigma) = \Omega^{2}(\sigma).g_{\alpha\beta}(\sigma) \tag{1}$$

Or, infinitesimally, we can also write, $\Omega^{2}(\sigma) = e^{2\phi}$ for small ϕ such that,

$$\delta g_{\alpha\beta}(\sigma) = 2\phi(\sigma)g_{\alpha\beta}(\sigma)$$

It is simple to see that the Polyakov action is invariant under this transformation.
This means that two metrics which are related by a Weyl transformation as given in equation (1) are to be considered as the same physical state.

So, one component for poincare invariance and two components for Reparameterization invariance. So total, three components for the world-sheet metric $g^{\alpha\beta}$. This means that we expect to be able to set any two of the metric components to a value of our choosing. We will choose to make the metric locally conformally flat.

[1]*Note:- Although, if we encounter a dynamical metric, reparameterization invariance wouldn't lead to the diffeomorphisms which lead to gauge redundancy, we need an invariant operator which allows us to integrate over the whole wordsheet . These are called vertex operators.

3.1 Fixing a Gauge

We can use,

$$g^{\alpha\beta} = e^{2\phi}.\eta_{\alpha\beta} \tag{2}$$

to make the metric conformally flat. Choosing a metric of this form is called conformal gauge., where $\phi(\sigma, \tau)$ is some function on the world-sheet. This is equation (2).

We have only used reparameterization invariance to get to equation (2). We still have Weyl transformations to play with. Clearly, we can use these to remove the last independent component of the metric and set $\phi = 0$ such that,

$$g_{\alpha\beta} = \eta_{\alpha\beta}$$

We end up with the flat metric on the worldsheet in Minkowski coordinates.

As mentioned in this case, the metric has three independent components [4] namely,

$$g^{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix}$$

where, $g_{10} = g_{11}$

Reparametrization invariance allows us to choose two of the components of$g$, so that only one independent component remains. But this remaining component can be gauged away by using the invariance of the action under Weyl rescalings.

So in the case of the string there is sufficient symmetry to gauge fix $g_{\alpha\beta}$ completely. Therefore, the resultant metric can be chosen as, so as to end up with a flat worldsheet.

$$g_{\alpha\beta} = \eta_{\alpha\beta} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.2 BRST Operations

After Gauge fixing, we are left with the gauge redundancy and these redundancies need to be handled carefully when we talk about the symmetries. This is where Faddeev- poppov ghosts come in, Ghosts in the string worldsheet arises as a consequence of gauge-fixing the worldsheet diffeomorphism. [See Appendix A]

To deal with these kinds of Gauge redundancies, we introduce BRST quantization. On a very basic level, BRST is an approach to performing anomaly free perturbative calculations in a non-abelian Gauge theory. Its significance is for rigorous canonical quantization of Yang-Mills theory.

The BRST charge for a closed string, in the context of string theory, is a crucial operator used for quantizing the theory and identifying physical states. It's a combination of left-moving and right-moving BRST charges and is nilpotent, meaning its square is zero. Physical states are defined as those that are annihilated by this charge. [5]

The BRST charge, Q_{BRST} acts on both matter and ghost fields. As we mentioned, it will satisfy,

- $Q^2 = 0$, nilpotent

- Physical states must be invariant under BRST transformations, meaning they are annihilated by the BRST charge, $Q_{BRST} = |\psi\rangle = 0$, closed. [2]

- $|\psi\rangle \approx |\psi\rangle + Q_{BRST}|\mathcal{X}\rangle$ (exact)

This defines the BRST cohomology, which picks out the physical states of the string (i.e., those that survive gauge fixing and are unitary).

Now, if you recall from the previous overview section of symmetries and gauge theory, you will see that the global symmetries give rise to Noether currents and their associated conserved charges. Hence, the worldsheet should definitely have conserved current. The continuity equation can be checked by ,

$$\partial_\alpha J^\alpha_{\mu\nu} = 0$$

This corresponds to a worldsheet global symmetry (e.g. $SU(N), SO(N)$, etc.).

3.3 Vertex Operators

The BRST cohomology describes the physical states of the string, but there are still constraints on the physical state $|\psi\rangle$ described by Virasoro algebra.[6] Sometimes , in order

[2]BRST invariant operators that connects both open and close strings

to handle certain issues like singularities in Conformal Field theory, a normal ordering of operators are required . This is done to define operator products consistently.

In our case, there are ambiguities that arise from the constraints on the physical state. Hence, we need to fix the normal ordering of ambiguities. These ambiguities are usually represented by the conformal weight c, $\tilde{c}$ representing both left and right moving sectors in CFT[two independent sets of degrees of freedom that arise when dealing with two-dimensional spacetime]. How do we do this ?

A simple way is to first replace the states with operator insertions on the worldsheet using the state operator map: $|\psi\rangle \rightarrow \mathcal{O}$. [3]

But we have a further requirement on the operators $\mathcal{O}$: gauge invariance.

There are two gauge symmetries as mentioned before: reparameterization invariance and Weyl symmetry. Both restrict the possible states.

Let's start by considering reparameterization invariance. in a theory with a dynamical metric, this does not give rise to a diffeomorphism invariant operator. To make an object that is invariant under reparameterizations of the worldsheet coordinates, we should integrate over the whole worldsheet. Our operator insertions (in conformal gauge) are therefore of the form,

$$V \sim \int \partial^2 z \mathcal{O}$$

Here the $\sim$ sign reflects an overall normalization constant. Integrating over the worldsheet takes care of diffeomorphisms. But what about Weyl symmetries? The measure $\partial^2 z$ has weight (-1, -1) under rescaling.

From here, we can construct a Vertex operator which includes the Noether's current that arises from Global symmetries, represented by equation(3).

$$V \sim \int \partial^2 z J^\alpha \partial X^\mu \bar{\partial} X^\nu \mathcal{O}(c, \tilde{c}) \tag{3}$$

In Equation(3) , the measure $\partial^2 z$ has weight (-1, -1) under rescaling, J^α is the conserved current, X are a bunch of scalar fields , represented on the worldsheet, $\mathcal{O}$ is the operator. ∂X^μ and $\bar{\partial} X^\nu$ gives us two different excitations and if we check using Weyl invariance, we will get the first excited states.[3]

And this operator corresponds to a massless gauge boson. $A^\alpha_{\mu\nu}$ in spacetime.

It's a BRST-closed operator $Q_{BRST} = |\psi\rangle = 0$, proving that the first excited states of the string are massless, thus part of the physical spectrum of the string. Shown below.

As mentioned, ∂X^{μ} and $\bar{\partial} X^{\nu}$ gives us two different excitations, α^{μ}_{-1} and $\tilde{\alpha}^{\nu}_{+1}$. If we consider an open bosonic string , the first excited state which is level 1 gives us,

$$|\psi\rangle = \Lambda^{\mu}\alpha^{\mu}_{-1}|k\rangle$$

where k^{μ} is the momentum. Applying closed BRST condition on the physical state,

$$Q_{BRST}|\psi\rangle = 0 \Rightarrow k^2 = 0$$

$k^2 = 0$ proves that the first excited states of the string are indeed massless.

The gauge symmetry appears from redundancies in the definition of the vertex operator. (From gauge transformation $A_{\mu}(x) \rightarrow A'_{\mu}(x) = A_{\mu}(x) + \partial_{\mu}\Lambda(x)$, we get the gauge symmetry, $\delta A_{\mu} = \partial_{\mu}\Lambda$)
Hence, we can conclude that Every global symmetry current $J^{\alpha}_{\mu\nu}$on the worldsheet leads to a gauge boson in spacetime $\rightarrow$ symmetry is gauged, not global.

4 Continuous Global Symmetry Breaking

The question here is - why do continuous global symmetries break ?
There are multiple reasons why. Perturbative approaches fail when it comes to QCD. In the Hadronization phase of QCD where the coupling constant is strong, perturbative approaches do not work. At the critical point (, around 150-170 MeV , boundary between the confined and the unconfined phases) , we move away from confinement. The chiral symmetry breaks as confinement breaks the hadron composition.
When a system's ground state does not match the underlying symmetry that the theory possesses, global spontaneous symmetry is broken. A great example is the Higgs mechanism , where a $U(1)$ symmetry is spontaneously broken, giving mass to the W and Z bosons. Whenever a continuous global symmetry is spontaneously broken, there will be massless scalar fields. These fields are called Goldstone bosons, which we are going to get into in more detail. But first, let's look at Symmetries that are available to us in classical as well as Quantum Field theories.

4.1 Symmetries in Classical Field Theories

Continuous symmetries are connected to conservation laws by Noether's theorem as we have mentioned previously. Consider an action where $S = \int d^D x \mathcal{L}\phi$, involving fields. Assuming that there's some infinitesmal transformation on the field ϕ, we are going to get $\phi \rightarrow \phi + \epsilon_a \delta\phi_a$, where a can be considered as a set of global parametres.[7]

Under the set of global parametres, the action remains invariant. We need the derivative of the global parametres which will help us recover the invariance under global transformation in case of constant parametres. To recover the equations of motion, we need the derivative of the action given by:-

$$S = \int d^D J_a^\mu \partial_\mu \epsilon_a(x) = 0$$

where the derivative of the global parametres is give by, $\partial_\mu \epsilon_a(x)$. now, under the equation of motion, the continuity equation under Noether's current remains conserved and this is denoted by, $\partial_\mu J_a^\mu = 0$. The Noether charge denoted by Q_a remains constant under time evolution , it is give by $Q_a = \int d^{D-1} x J_a^0$

4.2 Symmetries in QFTs

As suggested by Wigner's theorem, the symmetry of operations of a quantum system is induced by an unitary or an anti-unitary transformation. Therefore, for continuous symmetries , the unitary transformation can be constructed by Noether's charge. Given by , $U = e^{i\epsilon_a Q_a}$ and as mentioned previously, they act on the fields as :- $\phi \rightarrow \acute{\phi} = U \phi U^\dagger$. This explicit form of U is written in terms of creation and annihilation operators. The correlation function can be defined by the path integral, and the path integral can be written as:-

$$\mathcal{Z} = \int \mathcal{D}\phi \epsilon^{iS}$$

We can find out the ward identities which are the quantum counterpart of the classical laws. The correlation function can be defined as :- $\langle X \rangle = \frac{1}{Z} \int \mathcal{D}\phi X e^{is}$, where X is the product of the fields and is denoted by :- $X = \prod_j \phi(x_j)$. The ward identities of the quantum counterpart of Noether's current J_a^μ will be $\langle J_a^\mu(x) X \rangle$.

We can also consider a set of relations among the correlation functions and conservation laws. It can be denoted as:- $\partial_\mu \langle J_a^\mu(x) X \rangle$ and the ward identities would be:-

$$\partial_\mu \langle J_a^\mu(x) X \rangle = -i \sum_j \delta^(D)(x - x_j) \langle \phi(x_1).....\delta\phi_a(x_j)....\phi(x_N) \rangle \tag{4}$$

This will be useful in understanding divergences a little bit later.

Upon integrating both sides over spacetime from equation (4) we will obtain $\delta\langle\phi(x_1).....(x_N)\rangle = 0$. This is the reflection of the symmetry on the correlation function and it is also a non-trivial statement (undergoes transformation) about symmetry as it is computed as:- $\int \mathcal{D}\phi\delta[\phi(x_1).......\phi(x_N)]e^{is} = 0$

As we can see this is time ordered, so there will be equal time commutators for ϵ which tends to 0.

When continuous global symmetries are spontaneously broken, Noether's current even though it is conserved $(\partial_\mu J_a^\mu = 0)$under symmetry transformations, it starts to exhibit divergences in the set of relations . This divergence, leads to massless Goldstone bosons , which are excitations.

4.3 The Non-triviality of Noether's current in symmetry breaking

Noether's current even though it is a physical observable , produces divergences , because the symmetry is still present in the underlying Lagrangian whether or not the vacuum state is invariant. It produces divergences in the set of relations (in a distribution sense) which gives us the ward identities.

For an unbroken symmetry, Noether's current is conserved . Now, suppose this symmetry is broken but vacuum $|0\rangle$ is not invariant under Noether's charge, we will have :-
$Q|0\rangle = \int d^3x J^0(x)|0\rangle \neq 0$

Q for a symmetry group G. The set of relations for the vacuum state is going to be:-
$\langle 0|\partial_\mu J_a^\mu(x)|0\rangle$.

Considering a scalar field ϕ at vacuum , this will be :- $\langle 0|\partial_\mu J_a^\mu(x)\phi(0)|0\rangle$.
From the previous equation(), this will lead to the vacuum term,
$i\delta(x_0))[\partial_\mu J_a^\mu, \phi(0)] + constant$

Now for a broken symmetry, the vacuum state does not remain invariant, so we can say that the commutator will be non zero, $[\partial_\mu J_a^\mu, \phi(0)] \neq 0 \implies \langle 0|\partial_\mu J_a^\mu(x)\phi(0)|0\rangle \neq 0$

If the commutator is non-zero, the current J_a^μ is non-trivial. This is how divergences appear in the set of relations even if current is conserved.

4.3.1 The rise of massless poles

Let's assume that our current term which is J^{μ}_{a} , associated with a broken symmetry appears between two arbitrary states α and β. With respect to an Unitary transformation, this can be represented as[8]:-

$$\langle\beta|J^{\mu}_{a}(x)|\alpha\rangle = e^{iqx}\langle\beta|J^{\mu}_{a}(0)|\alpha\rangle$$

where the difference between the four momentum for the Goldstone bosons and the four momentum for the arbitrary states is denoted by:- $q^{\mu} \equiv p^{\mu}_{\alpha} - p^{\mu}_{\beta}$

Since, we know that the current term is non-trivial (i.e, it has non zero matrix element), it has a pole at $q^2 = 0$. This creates a goldstone boson. The more elaborate proof is given in. The S-matrix element will emit a Goldstone boson of state $|\beta, q\rangle$ of four momentum q in the transition between α and β. Roughly speaking, for each broken symmetry, there will be atleast one Goldstone Boson.
Let's take a look at non-triviality from a mathematical perspective.

5 Symmetry Breaking: A mathematical proof

So far, we have seen that global symmetries can be spontaneously broken and we have done the rigourous mathematical proof behind how massless Gauge bosons arise as a consequence of symmetry breaking from $SU(2) \rightarrow U(1)$. We know that from the property of Global symmetries, Ward identities emerge and from there a path towards symmetry breaking is formulated.
Now, looking at our problem through the lens of non-trivial holonomy and therefore breaking the resultant gauge symmetry , this approach provides us with a deep insight into Wilson loops, boundary conditions as well as compactifications.

But first a few fundamental definitions are in order:-

- Fibre-bundle:- A general structure on a space projected onto another space where each "fibre" represent a set of points which can be considered equivalent in some way.

- Priniciple-bundle:- A specific kind of bundle where the fibres are copies of lie groups.

- Holomorphic function:- A Holomorphic function can be defined as a complex valued function of one or more complex variable which is differentiable to the neighbourhood of each point in a complex space C^n. (include diagram).

- Homotopy :- A continuous deformation of one function or geometric object into another.

- Holonomy :- Holonomy is a geometric consequence of the curvature of a connection(a structure which is defined on a fibre bundle) . The holonomy of a connection on a smooth manifold is the extent to which a parallel transport along a closed loop fails to preserve the geometric data which is being transported. (give diagram)

- Wilson Line :- Consider a manifold $\mathcal{M}$, take a point on the manifold x_i and then project into an identity g_i on a $p = 2$, ($p + 1$ dimensional brane) and see how it changes to g_f which are points on a horiontal subspace , considered after projecting . The change happens on a spacetime curve $\tilde{\gamma}$. where, $\gamma : [0, 1] \to \mathcal{M}$ between x_i and x_f.[9] The curve $\tilde{\gamma(0)} = g_i$ and its tangent vectors lie on the horiontal subspace. The fibre bundle represents a gauge theory between two horiontal subspaces H_i and H_f . This gauge theory can be represented by : - $A_\mu(x) = A^\alpha_\mu T^a(x)$. Therefore, the Wilson line can be given by: -

$$g_f(t_f) = \mathcal{W}[x_i, x_f] = \mathcal{P}exp(i \int_{x_i}^{x_f} A_\mu \partial x^\mu)$$

where $\mathcal{P}$ is the path -ordering operator. $\mathcal{W}$ is gauge invariant under local gauge transformations. The trace of a closed Wilson line is called a Wilson loop. Wilson lines are used to describe particles(charges) along gauge groups.

Consider a principle G bundle , given by: - $\mathcal{P}(\mathcal{M}, G)$, which is connected over a manifold $\mathcal{M}$ with a non-trivial Holonomy representation is denoted by :- $\rho = \pi_1(\mathcal{M}) \to G$.

This representation assigns to each Holonomy class of loops on element G. So, in order to describe a Holonomy representation associated with the loop γ as mentioned before, we can define the representation as:-

$$\mathcal{P}([\gamma]) = \mathcal{P}exp(i \oint_\gamma A)$$

where $[\gamma]$ denotes the Homotopy class of loop γ.

An example of a Holonomy group

We have to remember that a loop γ , defines the transformation on the fibre.

Let's consider an $\mathbb{R}$ bundle over $\mathcal{M}$ where $\mathbb{R}^2 = \mathcal{M}$. The connection one-form , which is given by ω and the loop $[\gamma]$ define a map $\rho : \pi_1(\mathcal{M}) \to G$. Take a point h , which belongs on the group G. and consider a set of loops $\pi_1(\mathcal{M})$ which commutes with $\rho(\pi_1(\mathcal{M}))$, such that $\pi_1(\mathcal{M}) \equiv \gamma[0, 1] \to \mathcal{M}$ [9]

If we wish to define a subgroup that reduces to $\mathcal{H}$ from $\mathcal{G}$, we will define the subgroup as:-

$$\mathcal{H} = \{h \in \mathcal{G} | h\rho([\gamma]) = \rho([\gamma])h \forall [\gamma] \in \pi_1(\mathcal{M})\}$$

Now, the holonomy imposes the condition , for a section along the associated bundle where the section corresponds to a $\mathcal{G}$ equivariant function for $g \in \mathcal{G}$. Since $\rho(\gamma)$ is fixed, only the elements $h \in \mathcal{G}$ commuting with $\rho(\gamma)$ preserve the section. Therefore, the structure group reduces to $\mathcal{H}$, which is what we are looking for when we talk about group structure decomposition in symmetry breaking. [check Appendix B]

Considering a Gauge group with $\mathcal{G} = SU(2)$, on a spacetime manifold represented by $\mathcal{M} = \mathcal{S}^1 * \mathbb{R}^3$. $\mathcal{S}^1$ represents a compact spatial structure which can arise from compactification of the extra dimensions.[10]

$SU(2)$ is motivated by the electroweak sector and U can be given by $U = exp(iQ\sigma^3)$ where σ^3 are the generators of $SU(2)$ and Q is a constant angle parametrerization of the Holonomy, which represents the non-trivial field configuration that exists due to the topology of $\mathcal{S}^1$. We are now going to prove how to perform a reduction of the gauge symmetry from $SU(2) \to U(1)$.

5.1 Gauge Symmetry Reduction

The subgroup, as we know $\mathcal{H} \subset SU(2)$, which commutes with U. Now since U is generated by the form $exp(iQ\sigma^3)$, we can say that $U(1)$ subgroup of $SU(2)$ is generated by σ^3 and as a result the group $SU(2)$ reduces to $U(1)$. [The Holonomy group reduces to a subgroup $\mathcal{H}$, then only gauge transformations on H acts non-trivially]

The Gauge field A_μ can be decomposed into broken and unbroken generators and can be given by:- $g = h \oplus m$ where g is the Lie algebra decomposition.

h is the unbroken generators and m is the broken generator. We have to prove that the massless A_μ can be decomposed into a combination of both broken and unbroken generators. Let's see how this can be accomplished.

Aim:- From the Boundary conditions, we are going to derive both broken and unbroken generators. The goal is to compactify the higher dimensions and by fixing a gauge , we can come up with the unbroken generators and the non-trivial holonomy which cannot be gauged away produces the broken generators and eventually the mass term will emerge under mode expansion of the constant gauge. Only the subgroup which can commute with the non-trivial holonomy remains unbroken.

5.1.1 Boundary Condition

As a continuation from the Holonomy group, let's assume that we are compactifying one spatial direction , let's say $x \sim x + \mathcal{L}$, thereby making spacetime topologically $\mathbb{R}^(d-1) * S^1$.

The holonomy along S^1 can be described by the Wilson loop $\Omega = \mathcal{P}exp(i \oint_\gamma A_x \partial x) \in C$.

or along the circumference we can write, $\Omega = \mathcal{P}exp(i \oint_0^L A_x \partial x) \in G$. Now, a field $\phi(x)$ which is charged under the gauge group $\mathcal{G}$ must satisfy:- $\phi(x + \mathcal{L}) = \Omega\phi(x)$. This is a twisted boundary condition, where the twist is governed by the Holonomy. Now, if the Holonomy is trivial i.e, $\Omega = 1$, the field can be said as periodic.

We are interested in $SU(2)$ decomposing into $U(1)$ through compactification on a circle S^1 with a non-trivial Holonomy. Now, since compactification is introduced, we are interested in compactifying a higher dimensional gauge theory which is a very powerful way to realize Spontaneous Symmetry breaking via Holonomy.

5.1.2 Why Higher Dimensions ?

The reason being , we can obtain Effective $4D$ theories because gauge fields are vector fields , A_μ with no extra scalar-like components unless we introduce them by hand (like the Higgs field). But in higher dimensions, gauge fields have extra components. For eg:- in $5D$ [11], the gauge field can be written as $A_\mu(x^\mu, x^5)$ with $M = 0, 1, 2, 3, 5$. [3]

A_5 behaves like a scalar field from a $4D$ perspective. So, x^5 , when it is compactified on a circle S^1, it would be:-

$$\Omega = \mathcal{P}exp(i \int_0^L A_5 \partial x^5)$$

This holonomy cannot be gauged away globally due to the nontrivial topology of S^1. It leads to symmetry breaking: only the subgroup that commutes with Ω remains unbroken. This holonomy cannot be gauged away globally due to the nontrivial topology of S^1. It leads to symmetry breaking: only the subgroup that commutes with Ω remains unbroken. This method of breaking gauge symmetry spontaneously by a Wilson loop W is also known as the Hosotani mechanism.

[3]Note:- It's actually the index for the fifth coordinate, but in many Kaluza - Klein or extra-dimensional setups, the numbering skips 4 to avoid confusion with the Lorentz index range , $\mu = 0, 1, 2, 3$

5.1.3 The Set-Up

We are considering a Gauge theory on $\mathbb{R}^d * S^1$. Let's start by compactifying one spatial dimension, let's say $x^5 \sim x^5 + L$, so the spacetime is $\mathbb{R}^d * S^1$ within the Gauge group $SU(2)$ So, the Holonomy representation can be written as:-

$$\Omega = \mathcal{P}exp(i \int_0^L A_5(x^5)\partial x^5) \in SU(2)$$

5.1.4 Adjoint on scalar S^1

The aim is to compactify $A_5(5D) \to S^1$. Introduce a Wilson loop, choose a Wilson line gauge which is a constant and derive the generators.

Let $\phi(x) \in g$ (adjoint scalar) . Under Gauge transformation, this will be:-

$$\phi(x+L) = g(x+L)\phi(x)g(x+L)^{-1}$$

If the gauge field has a non-trivial Wilson line Ω, the periodicity [12] will be twisted as mentioned in the previous section. This gives us :- $\phi(x+L) = \Omega\phi(x)\Omega^{-1}$. If we choose a gauge where A_5 is constant, the Wilson loop(Holonomy) along the compact direction can be written as:-

$$\Omega(x^\mu) = \mathcal{P}exp(i \int_0^L A_5(x^\mu, x^5)\partial x^5) \in \mathcal{G}$$

Therefore, the Gauge invariant using $\phi(x+L) = g(x+L)\phi(x)g(x+L)^{-1}$ will be $\Omega(x^\mu) \to g(x^\mu)\Omega(x^\mu)g^{-1}(x^\mu)$.

So, from here we can choose a gauge which is constant, which will give us both the broken and the unbroken generators. From there, we can derive the mass terms.

We can choose A_5 which is a constant and the gauge can be written as:-

$$A_5(x^\mu, x^5) = \frac{\phi(x^\mu)}{L}, \phi \in g \in SU(2)$$

Let's choose the field ϕ which will give us the generators and thereby, the Holonomy.

$$\phi = \alpha\sigma^3 = \frac{\alpha}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \Rightarrow \Omega = exp(iQ_a\sigma^3)$$

where Q_a is the charge.

If we consider the Wilson loop i.e, $\Omega(x^\mu) = exp(i\phi(x^\mu))$,

- The non-local Wilson line operator becomes a local field $\phi(x^\mu)$

- ϕ behaves like a scalar field in 4 Dimensions.

So, we can say that the generator, $\sigma^3 \in SU(2)$ commutes with Ω , while σ^1 and σ^2 do not. Thus,

- σ^3 generates an unbroken $U(1)$

- σ^1 and σ^2 , generate broken direction.

Therefore, $SU(2) \to U(1)$. A_5 basically becomes a scalar. In Wilson line gauge, the entire content of the gauge-invariant Wilson loop is encoded in a constant background value of A_5 . This value cannot be gauged away globally due to the nontrivial topology of S^1 and it can break gauge symmetry.

5.1.5 The mass term

If we perform mode expansion(Fourier) on the Gauge $A_5(x^\mu, x^5)$, we are going to get:-

$$A_5(x^\mu, x^5) = \sum_{n\in\mathbb{Z}} A^n_\mu(x^\mu)e^(2\pi i n x^5/L)$$

The covariant derivative acting on charged modes is shifted by A_5 . The effective mass denoted by m_n becomes -

$$m_n = \frac{2\pi n}{L} + \frac{Q_a}{L}$$

where, $m_n = \frac{2\pi n}{L}$ is the effective $4D$ mass (From Kaluza–Klein derivation)[13].

So, the off-diagonal component becomes massive , while the diagonal component remains unchanged.

- $A^{\pm}_{\mu} \sim \sigma^{1,2}$ becomes massive while,

- $A^3 \sim \sigma^3$ remains unchanged.

which is consistent with what we had formulated previously, $h =$ broken generators, $m =$ unbroken generators.

5.1.6 Field Decomposition

So, the final field decomposition from $g = h \oplus m$ is going to look like:- $SU(2) = U(1) \oplus m$, where $U(1)$ is generated by:- σ^3 and m is generated by :- σ^1 and σ^2 .

Therefore, from here we can conclude that : -

- Fields in $h = U(1)$ - massless gauge bosons (photon-like).

- Fields in m:- massive bosons due to non-trivial holonomy, which cannot be gauged away.

This is structurally more or less the same in the Higgs mechanism, but geometrically we are describing it using the boundary conditions via the Wilson line.

The Wilson line is going to be useful a little bit later.

5.2 Goldstone Excitations, Spontaneous symmetry breaking

Continuing from section (4.2) , we have the correlation function for the current $J^{\mu}_a(x)$ which is $\langle J^{\mu}_a(x) X\rangle$ from the Ward identity $\langle X\rangle$. From equation (4) , if we picked a single field X, $\phi(y)$ then equation (4) would be reduced to:-

$$\partial_\mu \langle J^{\mu}_a(x)\phi(y)\rangle = -i\delta^D(x-y)\langle \delta_a \phi(y)\rangle$$

we can take the fourier transform with respect to x, and it will give us:-

$$\int d^D x e^{ipx} \partial_\mu \langle J^{\mu}_a(x)\phi(y)\rangle = -i \int d^D x e^{ipx} \delta^D(x-y)\langle \delta_a \phi(y)\rangle$$

$$-i \int d^D x e^{ipx} p_\mu \langle J^{\mu}_a(x)\phi(y)\rangle = -ie^{ipy}\langle \delta_a \phi(y)\rangle$$

$$p_\mu \langle J^{\mu}_a(x)\phi(y)\rangle = e^{ipy}\langle \delta_a \phi(y)\rangle$$

$$p_\mu \left\langle J^\mu_a(x) e^{-ipy} \phi(y) \right\rangle = \langle \delta_a \phi(y) \rangle$$

where, the Fourier transform can be identified as: - $J^\mu_a(p) = \int d^D x e^{ipx} J^\mu_a(x)$. Then, we can integrate both sides with respect to y and get ,

$$p_\mu \langle J^\mu_a(p) \phi(-p) \rangle = \int d^D y \langle \delta_a \phi(y) \rangle = \langle \delta_a \phi(p=0) \rangle$$

The term , $\langle \delta_a \phi(p=0) \rangle$ characterizes the phases and when $\langle \delta_a \phi(p=0) \rangle \neq 0$, it indicates spontaneous symmetry breaking. When the phase is broken, the correlation function must have a massless pole at zero-momentum $\langle J^\mu_a(p) \phi(-p) \rangle \sim \frac{p^\mu}{p^2}$

This indicates the presence of massless excitations and these excitations can be termed as the Goldstone Bosons which is consistent with what we had found in section(4.3.1)

6 Conclusion

A string theory, QFT and a topological route , all of these have led to the existence of Goldstone Bosons. We figure out how massless poles arise as a consequence of symmetry being either gauged/broken. A broader direction has been considered for this paper and many other approaches can also be considered as we try and investigate how global symmetries work at low energies as well as at the Planck scale. At ordinary as well as higher form symmetries, topological operators start behaving like symmetry operators and using such operators , we can perform analysis on the behavior of symmetries and also come up with a formulation on how Effective Field Theories (EFTs) behave when coupled with gravity. These are interesting and emergent areas of theories which will be discussed as we explore Global Symmetry behavior in Quantum Gravity.

7 Appendix A

One can consider a compact Gauge Group G. The corresponding representation of the matrices of $\mathcal{G}$ can be denoted T_1 and the structure constants of $\mathcal{G}$ in the basis can be written as:- f^k_{ij}.

$$[T_i, T_j] = f^k_{ij} T_k$$

This is the Yang Mills type. The field strengths are the covariant derivative of the matter fields , which are denoted by:- $f^I_{\mu,\nu}$ and $D_\mu \psi^i$ where ψ^i is the matter field, which can be fermionic or bosonic.

In the BRST formalism , the gauge paramtres are replaces by the anticommutating fields which are called the "ghost fields". BRST cohomologies capture important physical information about the system. The reason for BRST symmetries and cohomologies being important is to become a substitute for gauge redundancies , which would otherwise become obscure. [14]

8 Appendix B

In general when we are given a subgroup G, one choice that we can make is to take the manifold to be the group itself. $\mathcal{G} = \mathcal{M}$. In this case, each element , $g \in \mathcal{G}$ gives us the natural map $\mathcal{M} \to \mathcal{M}$ given by $\acute{g} \in \mathcal{M} \to g.\acute{g}$.

Another choice is to take a subgroup(coset space). This is the manifold $\mathcal{M} = \frac{G}{H}$. where $H \subset G$ is a group of G , which is exactly what we were trying to prove in section (5)

A point g in the coset $\frac{G}{H}$ is defined by the equivalence relation among the elements of G. $g \equiv g.h.$ for all $h \in H$. Again, any element of $g \in G$ gives us a natural map $\mathcal{M} \to \frac{G}{H} \to \frac{G}{H}$

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