Section 2.7, Question 11: Until recently, hamburgers at the city sports arena cost \$4 each. The food concessionaire sold an average of 10,000 hamburgers on a game night. When the price was raised to \$4.40, hamburger sales dropped off to an average of 8000 per night.

a. Assuming a linear demand curve, find the price of a hamburger that will maximize the nightly hamburger revenue.

b. If the concessionaire has fixed costs of \$1000 per night and the variable cost is \$.60 per hamburger, find the price of a hamburger that will maximize the nightly hamburger profit.

Answer:

a. Our first step is to state the objective of the problem. The objective is to maximize revenue:

Obj.: Maximize revenue

To do this, we need to find an expression for revenue. We'll do this by first constructing a demand/sales function.

The question asks us to assume a linear demand curve: that is, a linear relationship between sales x and price $p(x)$. So, we are looking for a function $p(x)$ whose equation is that of a straight line.

To find the equation of a straight line, we need the slope of the line, and a point on the line.

The questions gives us the information that:

1. when the price $p(x)$ is \$4, sales x are 10000
2. when the price $p(x)$ is \$4.40, sales x are 8000

So we have 2 data points:

1. $(x_1, y_1) = (10000, 4)$
2. $(x_2, y_2) = (8000, 4.4)$

So the slope of the function $p(x)$ is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.4 - 4}{8000 - 10,000} = -\frac{0.4}{2000} = -\frac{1}{5000}$$

We can find the equation of the line by using the point slope form along with the point $(x_1, y_1) = (10,000, 4)$:

$$p - 4 = -\frac{1}{5000}(x - 10,000)$$

$$p = 4 - \frac{1}{5000}(x - 10,000)$$

$$p = 6 - \frac{1}{5000}x$$

So now we have a sales function that gives us the price in terms of the sales level.

Remember that $revenue = price * sales$.

So,

$$revenue = p(x) * x = (6 - \frac{1}{5000}x) * x = 6x - \frac{1}{5000}x^2$$

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So our objective is:

Max. $R(x) = 6x - \frac{1}{5000}x^2$

where $R(x)$ stands for revenue.

There are no constraints in this problem. Since the objective function is already a function of only one variable, we do not need to simplify the objective function any further.

To maximize $R(x)$, we'll take the first derivative:

$R'(x) = 6 - (2)\frac{1}{5000}(x)$

and set it equal to 0:

$R'(x) = 6 - (2)\frac{1}{5000}(x) = 0 \implies x = 15,000.$

So $x = 15,000$ is a possible maximum for the revenue.

To check this, we'll take the second derivative:

$R''(x) = -(2)\frac{1}{5000}.$

Note that at $x = 15,000$, the second derivative is $R''(15,000) = -(2)\frac{1}{5000} < 0$. $R(x)$ is concave down at that point, and so $x = 15,000$ is a maximum.

So, revenue is maximized when $x = 15,000$ and price is $p = 6 - \frac{1}{5000}(15,000) = 6 - 3 = \$3$.

The maximum revenue is $R(15,000) = 6(15,000) - \frac{1}{5000}(15,000)^2 = \$45,000$.

b. In this part, our objective is to maximize profit:

Obj.: Max. Profit

We now need to find a function that represents profit.

Remember that $profit = revenue - cost$.

We found the revenue function in the previous part: it was:

$$R(x) = 6x - \frac{1}{5000}x^2$$

The question says that the concessionaire has a *fixed cost* of \$1000. This part of the cost does not change with the amount sold, and is an upfront expense. Further, the variable cost is

\$.60 per hamburger sold. So if x hamburgers are sold, then the concessionaire's costs are:

$$C(x) = 1000 + 0.6(x)$$

So the profit function is:

$$\Pi(x) = R(x) - C(x) = 6x - \frac{1}{5000}x^2 - (1000 + 0.6(x)) = -\frac{1}{5000}x^2 + 5.4x - 1000$$

So, our objective is:

Max. $\Pi(x) = -\frac{1}{5000}x^2 + 5.4x - 1000$.

There are no other constraints in this problem. As the objective function is already a function of one variable, we do not need to simplify it further.

To find the maximum, we take the first derivative:

$$\Pi'(x) = -(2)\frac{1}{5000}x + 5.4$$

set it equal to 0, and solve for x:

$$\Pi'(x) = -(2)\frac{1}{5000}x + 5.4 = 0 \implies x = 13500$$

So $x = 13500$ is a possible maximum for profit. To verify this, we'll take the second derivative:

$$\Pi''(x) = -(2)\frac{1}{5000}$$

and note that when $x = 13500$, $\Pi''(x) = -(2)\frac{1}{5000} < 0$. So, $\Pi(x)$ is concave down at $x = 13500$, so it is a maximum.

This means that the concessionaire can maximize profit when $x = 13500$ and $p = 6 - \frac{1}{5000}(13500) = \$3.3$.

The maximum profit is $\Pi(13500) = \$35,450$.

$\square$

Section 2.7, Question 13: An artist is planning to sell signed prints of her latest work. If 50 prints are offered for sale, she can charge \$400 each. However, if she makes more than 50 prints, she must lower the price of all the prints by \$5 for each print in excess of the 50. How many prints should the artist make to maximize her revenue?

Answer:

Our first step is to define the objective for this problem.

Our objective is to maximize revenue:

Obj.: Max. Revenue

Now, we need a function defining revenue.

We'll let the variable x denote the number of prints sold.

We'll first construct a sales/demand function for the business.

The question says that if upto 50 prints are sold, then the price is \$400 per print. If *more than* 50 prints are sold, then the price per print is reduced by \$5 for each print sold in excess of 50.

For examples, if 51 prints are sold, then the price per print will be \$400 − \$5 = \$395. If 52 prints are sold, then the price per print will be \$400 − \$10 = \$390.

We can summarize this information in the following demand function:

$$p(x) = \left\{ \begin{array}{ll} 400 & x \leq 50 \\ 400 - 5(x-50) & x > 50 \end{array} \right\}.$$

Now that we have the demand function, which gives us a formula for price in terms of x, we can construct the revenue function.

Remember that revenue = price * sales. So, we get that revenue is $p(x) * x$. This gives a formula for revenue:

$$R(x) = \left\{ \begin{array}{ll} 400x & x \leq 50 \\ (400 - 5(x-50))x & x > 50 \end{array} \right\}.$$

or,

$$R(x) = \left\{ \begin{array}{ll} 400x & x \leq 50 \\ 400x - 5x^2 + 250x & x > 50 \end{array} \right\}.$$

or,

$$R(x) = \left\{ \begin{array}{ll} 400x & x \leq 50 \\ 650x - 5x^2 & x > 50 \end{array} \right\}.$$

So now we have a revenue function.

There are no relevant constraints in this problem. Since the objective function is already a function of one variable, we do not need to simplify it any further.

Our objective is:

Max.

$$R(x) = \left\{ \begin{array}{ll} 400x & x \leq 50 \\ 650x - 5x^2 & x > 50 \end{array} \right\}.$$

To do this, we'll take the first derivative:

$$R'(x) = \left\{ \begin{array}{ll} 400 & x \leq 50 \\ 650 - 10x & x > 50 \end{array} \right\}.$$

and set $R'(x) = 0$.

Now, $400 \neq 0$, so there cannot be any maxima when $x \leq 50$.

We look at the second part of the derivative:

$650 - 10x = 0 \implies x = 65$.

So there is a possible maximum where $x = 65$. To verify this, we'll take the second derivative:

$$R''(x) = \left\{ \begin{array}{ll} 0 & x \leq 50 \\ -10 & x > 50 \end{array} \right\}.$$

and note that at $x = 65$, $R''(65) = -10 < 0$. So $R(x)$ is concave down when $x = 65$, which means that $x = 65$ is a max.

So, we get that the artist can maximize revenue by selling 65 prints.

$\square$

Section 2.7, Question 15: In the planning of a sidewalk cafe, it is estimated that for 12 tables the daily profit will be \$10 per table. Because of overcrowding, for each additional table the profit per table will be reduced by \$.50. How many tables should be provided to maximize the profit from the cafe?

Answer: Our first step is to define the objective for this problem.

Our objective is to maximize profit:

Obj.: Max. Profit

Now, we need a function defining profit.

We'll let the variable x denote the number of tables provided.

The question says that if upto 12 tables are provided, then the profit is \$10 per table. If *more than* 12 tables are provided, then the profit per table is reduced by \$.50 for each table provided in excess of 12.

For examples, if 11 tables are provided, then the profit per table will be $\$12 - \$.50 = \$11.50$. If 12 tables are provided, then the profit per table will be $\$12 - \$1 = \$11$.

We can summarize this information in the following profit function:

$$\Pi(x) = \left\{ \begin{array}{ll} 10x & x \leq 12 \\ (10 - 0.5(x - 12))x & x > 12 \end{array} \right\}.$$

or,

$$\Pi(x) = \left\{ \begin{array}{ll} 10x & x \leq 12 \\ 10x - 0.5x^2 + 6x & x > 12 \end{array} \right\}.$$

or,

$$\Pi(x) = \left\{ \begin{array}{ll} 10x & x \leq 12 \\ 16x - 0.5x^2 & x > 12 \end{array} \right\}.$$

There are no relevant constraints in this problem. Since the objective function is already a function of one variable, we do not need to simplify it any further.

Our objective is:

Max.

$$\Pi(x) = \left\{ \begin{array}{ll} 10x & x \leq 12 \\ 16x - 0.5x^2 & x > 12 \end{array} \right\}.$$

To do this, we'll take the first derivative:

$$\Pi'(x) = \left\{ \begin{array}{ll} 10 & x \leq 12 \\ 16 - x & x > 12 \end{array} \right\}.$$

and set $\Pi'(x) = 0$.

Now, $10 \neq 0$, so there cannot be any maxima when $x \leq 12$.

We look at the second part of the derivative:

$16 - x = 0 \implies x = 16$.

So there is a possible maximum where $x = 16$. To verify this, we'll take the second derivative:

$$\Pi''(x) = \left\{ \begin{array}{ll} 0 & x \leq 12 \\ -1 & x > 12 \end{array} \right\}.$$

and note that at $x = 16$, $\Pi''(16) = -1 < 0$. So $\Pi(x)$ is concave down when $x = 16$, which means that $x = 16$ is a max.

So, we get that the profit can be maximized by providing 16 tables.

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