Section 2.4, Question 7: Show that the function $f(x) = \frac{1}{3}x^3 - 2x^2 + 5x$ has no relative extreme points.

Answer: Remember that a relative extreme point is a relative max or a relative min.

So, we can detect all the relative maxima by constructing a chart of the following type:

x	.	.	.	.
f(x)	.	.	.	.
f'(x)	.	.	.	.
f''(x)	.	.	.	.

and finding all points where we see the pattern: $+, 0, -$ in the $f'(x)$ row.

Similarly, we can detect all the relative minima by looking at the $f'(x)$ row and looking for the pattern: $-, 0, +$.

Since we only need to check the $f'(x)$ row, we can probably get away with only constructing part of the chart. We'll stick to constructing the following smaller chart:

x	.	.	.	.
f'(x)	.	.	.	.

To do this, we'll need to figure out how many columns there will be.

The way we've done this is to find $f'(x)$, set it equal to 0, and solve for x. After that, we divided up the interval $(-\infty, \infty)$ using these points and the intervals in between.

So let's find $f'(x)$.

$$f(x) = \frac{1}{3}x^3 - 2x^2 + 5x$$

So,

$$f'(x) = x^2 - 4x + 5$$

Setting this equal to 0 gives:

$$f'(x) = x^2 - 4x + 5 = 0$$

$$x^2 - 4x + 5 = 0$$

Now we need to solve for the values of x. We'll use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our case, we have $a = 1, b = -4, c = 5$. So, we get that the values of x are:

$$x = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2}$$

$$x = \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$x = \frac{4 \pm \sqrt{-4}}{2}$$

$$x = \frac{4 \pm 2i}{2}$$

$$x = 2 \pm i$$

where i is the imaginary (complex) number $i = \sqrt{-1}$.

But neither of these values are real numbers. So, $f'(x)$ is never equal to 0 on the interval $(-\infty, \infty)$.

This means that we do not need to divide up the interval $(-\infty, \infty)$, and that the only column in our chart will be for the interval $(-\infty, \infty)$. So, our chart looks like this:

x	$(-\infty, \infty)$
f'(x)	.

To fill in this chart, we can take a test point in $(-\infty, \infty)$, say 0 and compute $f'(x)$ at that point.

$$f'(0) = (0)^2 - 4(0) + 5 = 5 > 0$$

So we'll fill in a + sign in the cell:

x	$(-\infty, \infty)$
f'(x)	+

In particular, there is no pattern $-, 0, +$ or $+, 0, -$ (in fact, there can't possibly be, since there is only one column)!

So this function does not have any relative maxima or minima.

Remark 1. We can actually draw a stronger conclusion in this question. By looking at out chart, we see that $f(x)$ is always increasing on $(-\infty, \infty)$. This means that $f(x)$ is always increasing on $(-\infty, \infty)$, which is also another way of seeing that it doesn't have any relative extrema.

$\square$

Section 2.4, Question 8: Show that the function $f(x) = -x^3 + 2x^2 - 6x + 3$ is always decreasing.

Answer: We can solve this question in the same way as we did Question 7.

We can detect where $f(x)$ is decreasing by constructing a chart of the following type:

x	.	.	.	.
f(x)	.	.	.	.
f'(x)	.	.	.	.
f"(x)	.	.	.	.

and finding all the intervals and points where we see a $-$ in the $f'(x)$ row.

Since we only need to check the $f'(x)$ row, we can get away with only constructing part of the chart. We'll stick to constructing the following smaller chart:

x	.	.	.	.
f'(x)	.	.	.	.

Now we need to figure out how many columns there will be.

The way we'll do this is to find $f'(x)$, set it equal to 0, and solve for x. After that, we'll divide up the interval $(-\infty, \infty)$ using these points and the intervals in between.

So let's find $f'(x)$.

$$f(x) = -x^3 + 2x^2 - 6x + 3$$

So,

$$f'(x) = -3x^2 + 4x - 6$$

Setting this equal to 0 gives:

$$f'(x) = -3x^2 + 4x - 6 = 0$$

$$-3x^2 + 4x - 6 = 0$$

Now we need to solve for the values of x. We'll use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our case, we have $a = -3, b = 4, c = -6$. So, we get that the values of x are:

$$x = \frac{-4 \pm \sqrt{16 - 4(-3)(-6)}}{2}$$

$$x = \frac{-4 \pm \sqrt{16 - 72}}{2}$$

$$x = \frac{-4 \pm \sqrt{-56}}{2}$$

$$x = \frac{4 \pm i\sqrt{56}}{2}$$

$$x = 2 \pm i\frac{\sqrt{56}}{2}$$

where i is the imaginary (complex) number $i = \sqrt{-1}$.

But neither of these values are real numbers. So, $f'(x)$ is never equal to 0 on the interval $(-\infty, \infty)$.

This means that we do not need to divide up the interval $(-\infty, \infty)$, and that the only column in our chart will be for the interval $(-\infty, \infty)$. So, our chart looks like this:

x	$(-\infty, \infty)$
f'(x)	.

To fill in this chart, we can take a test point in $(-\infty, \infty)$, say 0 and compute $f'(x)$ at that point.

$$f'(0) = -3(0)^2 + 4(0) - 6 = -6 < 0$$

So we'll fill in a $-$ sign in the cell:

x	$(-\infty, \infty)$
f'(x)	$-$

But this means that the function is decreasing on the entire interval $(-\infty, \infty)$, which is the same as saying that it is always decreasing.

$\square$