Section 2.1, Question 19: Refer to the graph given.

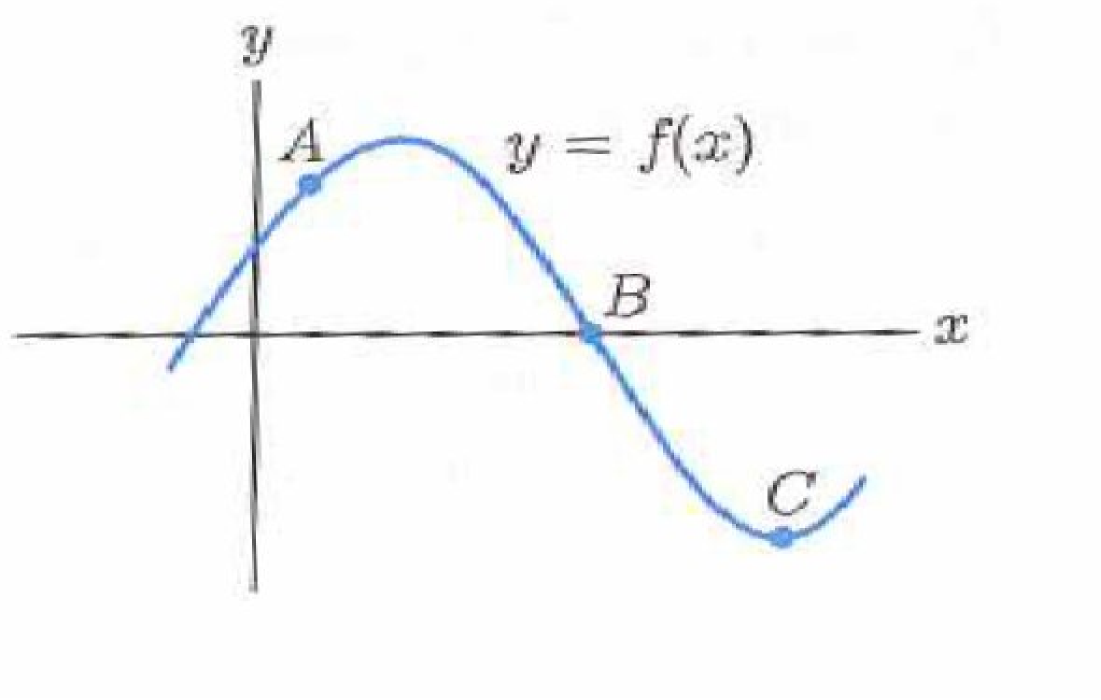

Figure 1:

Fill in each entry of the grid below with POS, NEG or 0.

	f(x)	f'(x)	f"(x)
A			
B			
C			

Answer: To answer this question, we will need to use the following two facts from class today:

- **First derivative rule:** If $f'(x) > 0$ at $x = a$, then $f(x)$ is increasing at $x = a$. If $f'(x) < 0$ at $x = a$, then $f(x)$ is decreasing at $x = a$.

- **Second derivative rule:** If $f''(x) > 0$ at $x = a$, then $f(x)$ is concave up at $x = a$. If $f''(x) < 0$ at $x = a$, then $f(x)$ is concave down at $x = a$.

We will answer this question by filling in each entry in the grid. We will use the coordinates (i, j) to refer to the entry in row i and column j. For example, the entry in the third row and first column has coordinates $(3, 1)$.

To fill in the $(1,1)$th entry, we need to find out whether $f(x)$ is positive (POS), negative (NEG) or zero (0) at the point A.

Looking at the graph, we see that $f(x)$ is positive at the point A (it is above the x-axis). So we can fill in this entry by writing in POS:

	f(x)	f'(x)	f"(x)
A	POS		
B			
C			

To fill in the $(2,1)$th entry, we need to find out whether $f(x)$ is positive, negative or zero at B.

Looking at the graph, we see that $f(x) = 0$ at B. (It crosses the x-axis at the point B). So we can fill in this entry by writing in 0:

	f(x)	f'(x)	f"(x)
A	POS		
B	0		
C			

Similarly, to fill in the $(3,1)$th entry, we need to find out whether $f(x)$ is positive, negative or zero at C.

Looking at the graph, we see that $f(x)$ is negative at C. (It lies below the x-axis at the point C). So we can fill in this entry by writing in NEG:

	f(x)	f'(x)	f"(x)
A	POS		
B	0		
C	NEG		

Next, let's fill in the second column.

To fill in the $(1, 2)$th entry, we need to find out whether $f'(x)$ is positive, negative or zero at the point A.

Looking at the graph, we see that $f(x)$ has positive slope at the point A (the tangent line at A is upwards sloping). This means that $f'(x)$ is positive at A. We can fill in this entry by writing in POS:

	f(x)	f'(x)	f"(x)
A	POS	POS	
B	0		
C	NEG		

To fill in the $(2, 2)$th entry, we need to find out whether $f'(x)$ is positive, negative or zero at the point B.

Looking at the graph, we see that $f(x)$ has negative slope at the point B (the tangent line at B is downwards sloping). This means that $f'(x)$ is negative at B. We can fill in this entry by writing in NEG:

	f(x)	f'(x)	f"(x)
A	POS	POS	
B	0	NEG	
C	NEG		

To fill in the $(3, 2)$th entry, we need to find out whether $f'(x)$ is positive, negative or zero at the point C.

Looking at the graph, we see that $f(x)$ has slope 0 at the point C (the tangent line is horizontal at C). This means that $f'(x)$ is 0 at C. We can fill in this entry by writing in 0:

	f(x)	f'(x)	f"(x)
A	POS	POS	
B	0	NEG	
C	NEG	0	

Lastly, let's fill in the third column.

To fill in the $(1,3)$th entry, we need to find out whether $f''(x)$ is positive, negative or zero at the point A.

Looking at the graph, we see that $f(x)$ is concave down at the point A (near A, the tangent lines lie above the curve).

Firstly, this means that $f''(x)$ cannot be positive at A, because then the second derivative rule would tell us that f(x) was concave *up* at A!

So, $f''(x)$ must be either negative or zero at A.

If we work a little harder, we can see that $f''(x)$ must be negative at A. Let's look at how the tangent lines to the curve change near A as we move from left to right. If you try to physically roll a straight-edge ruler on the curve near A, you will see that the slopes of the tangent lines decrease from left to right. So, $f'(x)$ is a decreasing function near A. This means that the derivative of $f'(x)$ is negative. So, $f''(x)$ is negative.

Remark: In the above paragraph, we did a lot of work to show that $f''(x)$ was negative! The reason we had to do this is that we cannot just conclude that since $f(x)$ was concave down, $f''(x)$ was negative. It is perfectly possible for a curve to be concave down and for $f''(x) = 0$. Here's an example: take $g(x) = -x^4$ and look at $x = 0$. The graph of $g(x) = -x^4$ is shown below. At $x = 0$, the curve is concave down. However, $g''(x) = -12x^2$. So at $x = 0$, $g''(x) = 0$!

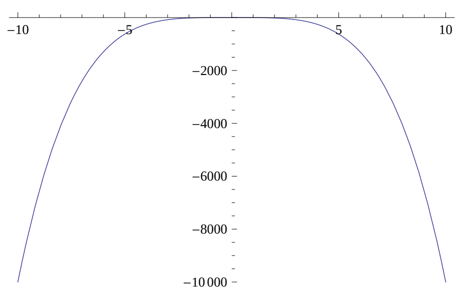

Figure 2: $y = -x^4$

We can fill in this entry by writing in NEG:

	f(x)	f'(x)	f''(x)
A	POS	POS	NEG
B	0	NEG	
C	NEG	0	

To fill in the $(2,3)$th entry, we need to find out whether $f''(x)$ is positive, negative or zero at the point B.

Looking at the graph, we can see that $f(x)$ is neither concave down nor concave up at the point B:

At the point B, the tangent line *crosses over* the curve: to the left of B, the tangent line lies above the curve, and to the right of B, the tangent line lies below the curve. So, the tangent line does not lie on one side or the other of the curve.

The second derivative rule tells us that $f''(x)$ is neither negative nor positive at B (otherwise it would be either concave down or concave up at B and the tangent line would lie to one side of the curve!). So, $f''(x)$ must be 0 at B. We can fill in this entry by writing in 0:

	f(x)	f'(x)	f''(x)
A	POS	POS	NEG
B	0	NEG	0
C	NEG	0	

To fill in the $(3, 3)$th entry, we need to find out whether $f''(x)$ is positive, negative or zero at the point C.

Looking at the graph, we see that $f(x)$ is concave up at the point C (near C, the tangent lines lie below the curve).

Firstly, this means that $f''(x)$ cannot be negative at C, because then the second derivative rule would tell us that f(x) was concave *down* at C!

So, $f''(x)$ must be either positive or zero at C.

If we work a little harder, we can see that $f''(x)$ must be positive at C. Let's look at how the tangent lines to the curve change near C as we move from left to right. If you try to physically roll a straight-edge ruler on the curve near C, you will see that the slopes of the tangent lines increase from left to right. So, $f'(x)$ is an increasing function near A. This means that the derivative of $f'(x)$ is positive. So, $f''(x)$ is positive.

We can fill in this entry by writing in POS:

	f(x)	f'(x)	f''(x)
A	POS	POS	NEG
B	0	NEG	0
C	NEG	0	POS

□