Remarks on the second derivative test:

Recall that the second derivative test says the following:

1. If $f'(a) = 0$ and $f''(a) > 0$, then $f(x)$ has a local minimum at $x = a$.
2. If $f'(a) = 0$ and $f''(a) < 0$, then $f(x)$ has a local maximum at $x = a$.

Note that the second derivative test does not cover the case when $f'(a) = 0$ and $f''(a) = 0$.

This is because $f'(a) = 0$ and $f''(a) = 0$ does not give us enough information about $f(x)$ for us to be able to conclude anything.

Here are three examples of functions $f(x), g(x)$, and $h(x)$; in each case, we will take $a = 0$.

Each function $f(x), g(x), h(x)$ will satisfy $f'(0) = 0, g'(0) = 0, h'(0) = 0$ and also $f''(0) = 0, g''(0) = 0, h''(0) = 0$.

However, $f(x)$ will have a local minimum at $x = 0$, $g(x)$ will have a local maximum at $x = 0$, and $h(x)$ will have neither a local maximum nor a local minimum at $x = 0$.

Example 1. Let $f(x) = x^4$.

We have $f'(x) = 4x^3$, so $f'(0) = 4 * 0^3 = 0$.

Also, $f''(x) = 12x^2$, so $f''(0) = 12 * 0^2 = 0$.

The graph of this function given below shows that $f(x)$ has a local minimum at $x = 0$.

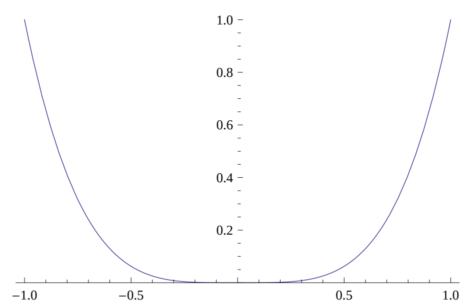

Figure 1:

Example 2. Let $f(x) = -x^4$.

We have $f'(x) = -4x^3$, so $f'(0) = -4 * 0^3 = 0$.

Also, $f''(x) = -12x^2$, so $f''(0) = -12 * 0^2 = 0$.

The graph of this function given below shows that $f(x)$ has a local maximum at $x = 0$.

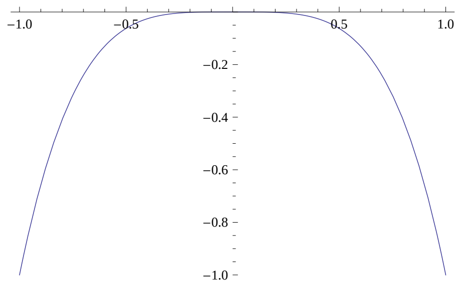

Figure 2:

Example 3. Let $f(x) = 1$.

We have $f'(x) = 0$, so $f'(0) = 0$.

Also, $f''(x) = 0$, so $f''(0) = 0$.

The graph of this function given below shows that $f(x)$ has neither a local minimum nor a local maximum at $x = 0$.

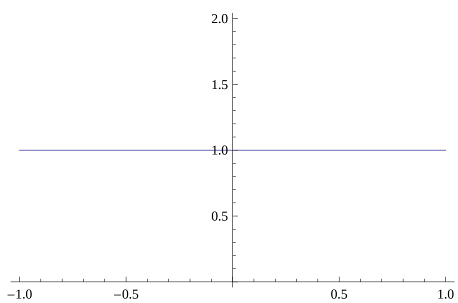

Figure 3:

These examples show that we cannot reliably use the second derivative test if $f''(a) = 0$. In such cases, it is better to use the first derivative test and construct a variation chart.