In this note I am going to cover the main facts and techniques from section 11.2.

Suppose you have a sequence of real numbers $\{a_n\}$. In this section we will attach a precise meaning to the idea of adding together all the terms $a_n$.

In general, since there are infinitely many terms to add together, it is impossible to add them in the usual sense, since you will never be able to stop!

What we *can* do, is to try the following procedure:

1. Start with 0. Add the first term. Call this the first partial sum and denote it by $s_1$. (It's called a partial sum because we've only added some of the terms of the sequence).
2. Start with 0. Add the first two terms, and call this the second partial sum, $s_2$.
3. Start with 0. Add the first three terms, and call this the second partial sum, $s_3$.

$\vdots$

4.
5. Start with 0. Add together the first $n$ terms and call it the $n$-th partial sum, $s_n$.
6. Ask the question: is the sequence of partial sums: $\{s_1, s_2, s_3, \cdots, s_n, \cdots\}$ tending to a particular limit? If so, call this limit the sum of the sequence. If not, we say that the sum of the terms $\{a_n\}$ diverges.

We'll formalize this idea:

**Definition 1.** Suppose you have a sequence of real numbers $\{a_n\}$. We call the $n$-th partial sum the number $a_1 + a_2 + a_3 + \cdots + a_n$, and we denote this number by $s_n$, as well as by the expression $\sum_{i=1}^{n} a_i$.

**Definition 2.** Suppose you have a sequence of real numbers $\{a_n\}$. Construct the sequence of $n$-th partial sums, $\{s_n\}$. If the sequence $\{s_n\}$ has a limit, then we call this limit the **sum** of the sequence $\{a_n\}$, denoting it by $s$, as well as by the expression $\sum_{i=1}^{\infty} a_n$. An expression of the form $\sum_{i=1}^{\infty} a_n$ is also called a *series*. If the limit of the sequence of partial sums does not exist, then we say that the series *diverges*. If the limit of the sequence of partial sums does exist, then we say that the series *converges*, with the limit as its **value** or **sum**.

An important example of a series is that of a **geometric series**.

**Definition 3.** A **geometric series** is a series of the form $a + ar + ar^2 + ar^3 + \cdots + ar^n + \ldots$, where $a$ and $r$ are fixed real numbers. The number $a$ is called the **first term** of the series, and the number $r$ is called the **common ratio** of the geometric series.

**Example 4.** *An example of a geometric series is:*

$$1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \cdots + (\frac{1}{2})^n + \ldots.$$

*In this case,* $a = 1, r = \frac{1}{2}$.

**Fact 5.** *When* $|r| < 1$*, the geometric series* $\sum_{i=1}^{\infty} ar^{n-1}$ *converges and has sum* $\frac{a}{1-r}$*. When* $|r| \geq 1$*, the geometric series* $\sum_{i=1}^{\infty} ar^{n-1}$ *diverges.*

*Proof.* Denote by $s_n$ the partial sum:

$$s_n = a + ar + ar^2 + ar^3 + \cdots + ar^n$$

Multiplying both sides by $r$ gives:

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1}$$

Subtracting the expression $rs_n$ from $s_n$ gives:

$$s_n - rs_n = a - ar^{n+1}$$

Hence:

$$s_n(1-r) = a(1-r^{n+1})$$

and

$$s_n = \frac{a(1-r^{n+1})}{(1-r)}$$

Now letting $n \to \infty$, we get that:

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^{n+1})}{(1-r)} = \frac{a}{1-r}$$

since $|r| < 1$.

We'll skip the proof that the sum diverges if $|r| \geq 1$.

□

**Example 6.** *Consider the geometric series*

$$1 + \frac{1}{3} + (\frac{1}{3})^2 + (\frac{1}{3})^3 + \cdots + (\frac{1}{3})^n + \ldots.$$

*In this case, $a = 1, r = \frac{1}{3}$, and $|r| = |\frac{1}{3}| = \frac{1}{3} < 1$. By the fact above, this geometric series converges, and has sum*

$$\frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

Here's a very important:

**Fact 7.** *Suppose you have a sequence of terms $\{a_n\}$, and the series $\sum_{i=1}^{\infty} a_i$ converges. Then we must have $\lim_{n\to\infty} a_n = 0$.*

*Proof.* Notice that since $s_n = a_1 + a_2 + \cdots + a_n$, and $s_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1}$, we have $a_n = s_{n+1} - s_n$. Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_{n+1} - s_n)$. Since we have supposed that the series converges, both $\{s_n\}$ and $\{s_{n+1}\}$ have the same limit, $s$. So,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_{n+1} - s_n = s - s = 0.$$

$\square$

This last fact has a very useful consequence:

**Fact 8. (Test for divergence):** *Suppose you have a sequence of terms $\{a_n\}$, and $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist. Then the series $\sum_{i=1}^{\infty} a_i$ diverges.*

**Example 9.** *Consider the series*

$$(1+1) + (1+\frac{1}{3}) + (1+(\frac{1}{3})^2) + (1+(\frac{1}{3})^3) + \cdots = \sum_{i=1}^{\infty}(1+(\frac{1}{3})^n).$$

*In this case, the associated sequence is $a_n = 1 + (\frac{1}{3})^n$, which does not have limit equal to $0$ (in fact its limit is $1$). By the test for divergence, this sum does not converge.*

**Example 10.** *It is important to use Fact 7 carefully. It says that if the series converges, then the terms must tend to $0$. The reverse is not true: just because the terms go to $0$ does* **NOT** *mean that the series converges.*

*Consider the series*

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{i=1}^{\infty} \frac{1}{i}.$$

*In this case, the associated sequence is $a_n = \frac{1}{n}$, which has limit equal to $0$.*

*However, this series does* **NOT** *converge!*

*To see this, notice the following pattern:*

$$s_2 = 1 + \frac{1}{2},$$

$$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 2,$$

$$s_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{3}{2},$$

*In general we have the formula:*

$$s_{2^n} > 1 + \frac{n}{2}.$$

*So, $\{s_{2^n}\} \to \infty$, and so $\{s_n\}$ does not converge to a finite number. Therefore the sum is divergent.*

**Fact 11.** *Suppose you have sequences $\{a_n\}$, $\{b_n\}$, and a real number $c$. Construct the series $\{a_n + b_n\}, \{a_n - b_n\}, \{ca_n\}$. Then:*

1. *If* $\sum_{i=1}^{\infty} a_i$ *and* $\sum_{i=1}^{\infty} b_i$ *converge, then so does* $\sum_{i=1}^{\infty}(a_i + b_i)$*, and*

$$\sum_{i=1}^{\infty}(a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i.$$

2. *If* $\sum_{i=1}^{\infty} a_i$ *and* $\sum_{i=1}^{\infty} b_i$ *converge, then so does* $\sum_{i=1}^{\infty}(a_i - b_i)$*, and*

$$\sum_{i=1}^{\infty}(a_i - b_i) = \sum_{i=1}^{\infty} a_i - \sum_{i=1}^{\infty} b_i.$$

3. *If* $\sum_{i=1}^{\infty} a_i$ *converges, then so does* $\sum_{i=1}^{\infty} ca_i$*, and*

$$\sum_{i=1}^{\infty} ca_i = c\sum_{i=1}^{\infty} a_i.$$

**Example 12.** *Here I'll present an important technique - called the method of telescoping sums - for computing the value of a series.*

*Consider the sequence* $a_n = \frac{1}{n(n+1)}$*. The associated series is:*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

*The associated partial sums are:*

$$s_1 = \frac{1}{1 \cdot 2}.$$

$$s_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3}.$$

$$s_3 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4}.$$

$$\vdots$$

$$s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}.$$

*It's hard to see whether these partial sums are tending to a limit (and if so, what the limit is).*

*On the other hand, consider the partial sum decomposition:*

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

*which you can (and* **should!***) verify yourself.*

*Then*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

*Now the partial sums look like:*

$$s_1 = \frac{1}{1} - \frac{1}{2}.$$

$$s_2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3}.$$

$$s_3 = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4}.$$

$\vdots$

$$s_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1}.$$

*Cancelling in each of the partial sums, we see that we get a* **formula** *for the terms* $\{s_n\}$*:*

$$s_n = 1 - \frac{1}{n+1}.$$

*By definition, the sum is*

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (1 - \frac{1}{n+1}) = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1 - 0 = 1.$$

*So, the series converges, with sum* $1$.

**This is a very important technique that you will use often in the course.**