Q) Find the exact area of the surface obtained by rotating

$$y = x^3 \; ; \; 0 \leq x \leq 2.$$

about the x-axis.

A) $A = \int 2\pi r \, ds.$

About x axis $\Rightarrow r = y = x^3$.

Bounds $0 \leq x \leq 2 \Rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$

$$A = \int_0^2 2\pi x^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$y = x^3$

$\frac{dy}{dx} = 3x^2$; $\left(\frac{dy}{dx}\right)^2 = 9x^4$; $\left(\frac{dy}{dx}\right)^2 + 1 = 9x^4 + 1$

$$A = \int_0^2 2\pi x^3 \sqrt{9x^4 + 1} \, dx.$$

Try: $u = 9x^4 + 1$.

$du = 36x^3 \, dx$

$\underline{dx} = \frac{du}{36x^3}$.

$$A = \int_{x=0}^{x=2} 2\pi \cancel{x^3} \sqrt{u} \cdot \frac{du}{36 \cancel{x^3}}$$

$$= \frac{\pi}{18} \int_{x=0}^{x=2} \sqrt{u}\, du$$

$$= \frac{\pi}{18} \left[\frac{2}{3} u^{3/2} \right]_{x=0}^{x=2}$$

$$= \frac{\pi}{18} \left[\frac{2}{3} \cdot (9x^4+1)^{3/2} \right]_0^2$$

$$= \frac{\pi}{27} \left[(9x^4+1)^{3/2} \right]_0^2$$

$$= \frac{\pi}{27} \left[\{(9 \cdot 16+1)^{3/2}\} - \{1\}^{3/2} \right]$$

$$= \frac{\pi}{27} \left[(145)^{3/2} - 1 \right].$$

$\square$

Q) $\{n^2 e^{-n}\}$ C(D).

$$\lim_{n\to\infty} n^2 e^{-n} = \lim_{n\to\infty} \frac{n^2}{e^n} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{2n}{e^n} = \lim_{n\to\infty} \frac{2}{e^n} = 0.$$

So, the sequence converges, with limit 0.

$\square$

Q) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ C/D.

$$\frac{3}{n(n+3)} \equiv \frac{A}{n} + \frac{B}{n+3} = \frac{A(n+3) + B(n)}{n(n+3)}.$$

$$3 \equiv A(n+3) + B(n)$$

@ $n=0$: $3 = 3A \Rightarrow A = 1$

@ $n=-3$: $3 = -3B \Rightarrow B = -1$.

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+3}.$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{7}}\right) +$$
$$\left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{8}}\right) + \left(\cancel{\frac{1}{6}} - \frac{1}{9}\right) + \left(\cancel{\frac{1}{7}} - \frac{1}{10}\right) + \text{---}$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}.$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}.$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ converges. $\square$

8) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$ C/D.

$\frac{\ln(n)}{n^3} \geq 0$ for $n \geq 1$.

Let $f(x) = \frac{\ln(x)}{x^3}$.

$$f'(x) = \frac{x^3/x - \ln(x) \cdot 3x^2}{x^6} = \frac{x^3 - 3x^3 \ln x}{x^7}$$

$$= \frac{x^3(1-3\ln x)}{x^7} = \frac{1-3\ln x}{x^4}.$$

For $x>0$, $\frac{1}{x^4} > 0$.

For $x > e^{1/3} = \sqrt[3]{e}$, $1-3\ln x < 0$.

So, $f(x)$ is decreasing on $(\sqrt[3]{e}, \infty)$.

Note: $e = 2.718\ldots$ $\Rightarrow$ $\sqrt[3]{e} \in (1,2)$.

So, $\frac{\ln(n)}{n^3}$ decreasing on $[2,\infty)$.

Integral tests

$$\sum_{n\geq 2}^{\infty} \frac{\ln(n)}{n^3} \ C \Leftrightarrow \int_2^{\infty} \frac{\ln(x)}{x^3}\,dx \ C.$$

$$\int_2^{\infty} \frac{\ln(x)}{x^3}\,dx = \frac{-1}{2x^2}\ln(x)\Big|_2^{\infty} + \int_2^{\infty} \frac{1}{2x^3}\,dx$$

$$= \frac{-1}{2x^2}\ln x\Big|_2^{\infty} + \frac{1}{2}\left(\frac{-1}{2x^2}\right)\Big]_2^{\infty}$$

$$= \left[\frac{-1}{2x^2}\ln x - \frac{1}{4x^2}\right]_2^{\infty}$$

$$= \frac{-1}{2x^2}\left(\ln x - \frac{1}{2}\right)\Big]_2^{\infty}$$

$$= \lim_{t\to\infty}\left(\frac{-1}{2t^2}\left(\ln t - \frac{1}{2}\right)\right) + \frac{1}{8}\left(\ln 2 - \frac{1}{2}\right).$$

$$= \lim_{t\to\infty} \frac{1/2 - \ln t}{2t^2} + \frac{1}{8}\left(\ln 2 - \frac{1}{2}\right)$$

$$\overset{l'H}{=} \lim_{t\to\infty} \frac{-1/t}{4t} + \frac{1}{8}\left(\ln 2 - \frac{1}{2}\right)$$

$$= \lim_{t\to\infty} \frac{-1}{4t^2} + \frac{1}{8}\left(\ln 2 - \frac{1}{2}\right)$$

$$= \frac{1}{8}\left(\ln 2 - \frac{1}{2}\right).$$

So, $\int_{2}^{\infty} \frac{\ln(x)}{x^3} dx$ C

So, $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^3}$ C.

So, $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3} = \frac{\ln(1)}{1^3} + \sum_{n=2}^{\infty} \frac{\ln(n)}{n^3}$ C.

$\square$