Section 6.3, Question 41: Suppose that the marginal cost function of a handbag manufacturer is $C'(x) = \frac{3}{32}x^2 - x + 200$ dollars per unit at production level x, where x is measured in units of 100 handbags).

(a) Find the total cost of producing 6 additional units if 2 units are currently being produced.

(b) Describe the answer to part (a) as an area. (Give a written description rather than a sketch).

Answer:

(a) To solve this problem, we will need to use the fundamental theorem of calculus, which says:

If $f(x)$ is continuous on $[a, b]$, and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

The question is asking for the total cost of producing 6 additional units if the current production level is 2.

If we let $C(x)$ denote the cost function for the handbag manufacturer, then the question is asking us to find $C(8) - C(2)$.

We can use the fundamental theorem of calculus ($\int_a^b f(x)dx = F(b) - F(a)$) to compute $C(8) - C(2)$ by taking $a = 2, b = 8$ and $F(x) = C(x)$ in the right hand side. This gives us that:

$$\int_2^8 C'(x)dx = C(8) - C(2)$$

(Remember that since the derivative of $C(x)$ is $C'(x)$, an antiderivative of $C'(x)$ is $C(x)$!)

So, to find $C(8) - C(2)$, we need to compute:

$$\int_2^8 C'(x)dx = \int_2^8 \frac{3}{32}x^2 - x + 200\, dx$$

To solve this, we'll use our three step rule for computing definite integrals:

1. Find a and b.
2. Find an antiderivative for $\frac{3}{32}x^2 - x + 200$. (We'll call this antiderivative $H(x)$, since we've previously used $F(x)$.
3. Compute $H(b) - H(a)$.

Implementing these steps:

1. We see from the problem that $a = 2$, $b = 8$.

2. We need an antiderivative for $\frac{3}{32}x^2 - x + 200$. That is, we need to find a member of the family:

$$\int \frac{3}{32}x^2 - x + 200 \, dx$$

We can use the sum rule for antiderivatives:

$$\int \frac{3}{32}x^2 \, dx + \int -x \, dx + \int 200 \, dx$$

and the constant multiple rule for antiderivatives:

$$\frac{3}{32} \int x^2 \, dx - \int x \, dx + \int 200 \, dx$$

and finally the power rule for antiderivatives:

$$\frac{3}{32}\frac{x^3}{3} - \frac{x^2}{2} + 200x + D$$

$$\frac{x^3}{32} - \frac{x^2}{2} + 200x + D$$

where D is an arbitrary constant.

Since we only need one antiderivative, we can set D equal to a constant of our choice. A good candidate is $D = 0$, since this reduces the number of terms we need to deal with.

This gives us an antiderivative $H(x) = \frac{x^3}{32} - \frac{x^2}{2} + 200x$.

3. Lastly, we need to compute:

$$H(b) - H(a)$$

$$H(x)|_b - H(x)|_a$$

$$(\frac{x^3}{32} - \frac{x^2}{2} + 200x)|_b - (\frac{x^3}{32} - \frac{x^2}{2} + 200x)|_a$$

$$\left(\frac{x^3}{32} - \frac{x^2}{2} + 200x\right)|_8 - \left(\frac{x^3}{32} - \frac{x^2}{2} + 200x\right)|_2$$

$$\left(\frac{8^3}{32} - \frac{8^2}{2} + 200(8)\right) - \left(\frac{2^3}{32} - \frac{2^2}{2} + 200(2)\right)$$

$$= \$1185.75$$

So, we get that the total cost of producing 6 additional units if 2 units are currently being produced is $1185.75.

(b) Remember that the definite integral $\int_a^b f(x)dx$ measures the (oriented) area under the graph of $y = f(x)$ and above the x-axis, over the interval $[a, b]$.

So,

$$\int_2^8 C'(x)dx = \int_2^8 \frac{3}{32}x^2 - x + 200 \; dx$$

measures the area under the graph of the curve $y = C'(x)$ and above the x-axis over the interval $[2, 8]$.

So,

$$C(8) - C(2) = \int_2^8 C'(x)dx = \int_2^8 \frac{3}{32}x^2 - x + 200 \; dx$$

can be described as the area under the graph of the curve $y = \frac{3}{32}x^2 - x + 200$ and above the x-axis over the interval $[2, 8]$.

$\square$