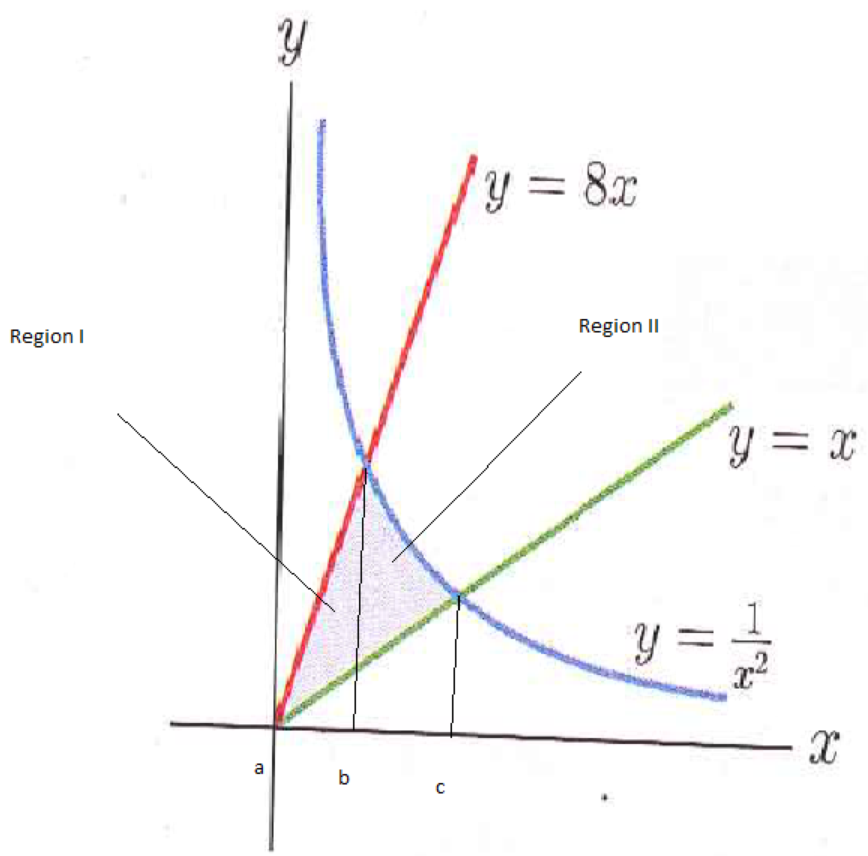
Section 6.4, Question 21: Find the area of the region bounded by $y = \frac{1}{x^2}$, $y = x$, and $y = 8x$, for $x \geq 0$.

Answer: To solve this question, we will use the following fact:

Fact 1. *Suppose that a(n upper) curve $y = f(x)$ lies above a (lower) curve $y = g(x)$ over the interval $[a, b]$. Then the area between the two curves over the interval $[a, b]$ is given by:*

$$\int_a^b [f(x) - g(x)]dx$$

First, we'll examine the region in the $x - y$ plane.



Notice that in the figure, the lower boundary of the region is the curve $y = x$, but the upper boundary of the region comes in two parts: $y = 8x$ over the interval $[a, b]$, and $y = \frac{1}{x^2}$ over the interval $[b, c]$.

Before we proceed, we'll identify the relevant intervals $[a, b]$ and $[b, c]$ in the picture.

Notice that at the point a, the two curves $y = 8x$ and $y = x$ intersect. So, we can find the value of a by setting the two curves equal to each other: $8x = x$. This gives: $7x = 0 \implies x = 0$. So, we get that $a = 0$.

Similarly, at the point b, the two curves $y = 8x$ and $y = \frac{1}{x^2}$ intersect. So, we can find the value of b by setting the two curves equal to each other: $8x = \frac{1}{x^2}$. This gives: $8x^3 = 1 \implies x^3 = \frac{1}{8} \implies x = \frac{1}{2}$. So, we get that $b = \frac{1}{2}$.

Lastly, at the point c, the two curves $y = x$ and $y = \frac{1}{x^2}$ intersect. So, we can find the value of c by setting the two curves equal to each other: $x = \frac{1}{x^2}$. This gives: $x^3 = 1 \implies x^3 = 1 \implies$

$x = 1$. So, we get that $c = 1$.

To find the area of the region in the picture, we'll use the fact to find the areas of regions I and II, and then add our answers together to get the desired area.

To find the area of region I, we use the fact with upper curve $f(x) = 8x$ and lower curve $g(x) = x$. By the fact, the area of region I is:

$$\int_0^{\frac{1}{2}} 8x - xdx$$

$$\int_0^{\frac{1}{2}} 7xdx$$

To solve this, we'll use the following steps:

1. Find the bounds of integration. In this case, we have $a = 0, b = \frac{1}{2}$.

2. Find an antiderivative. In this case, the family of antiderivatives is:

$$\int 7xdx = 7\int xdx = 7\frac{x^2}{2} + C$$

   So we can set $C = 0$ and take $F(x) = 7\frac{x^2}{2}$ for our antiderivative.

3. Compute $F(b) - F(a)$.

   We get:

$$F(b) - F(a) = F(\frac{1}{2}) - F(0) = 7\frac{(\frac{1}{2})^2}{2} - 7\frac{(0)^2}{2} = \frac{7}{8} - 0 = \frac{7}{8}$$

So, the area of region I is $\frac{7}{8}$ $units^2$.

We find the area of region II similarly:

We use the fact with upper curve $f(x) = \frac{1}{x^2}$ and lower curve $g(x) = x$. By the fact, the area of region II is:

$$\int_{\frac{1}{2}}^{1} \frac{1}{x^2} - xdx$$

To solve this, we'll use the following steps:

1. Find the bounds of integration. In this case, we have $a = \frac{1}{2}, b = 1$.

2. Find an antiderivative. In this case, the family of antiderivatives is:

$$\int \frac{1}{x^2} - x dx = \int \frac{1}{x^2} dx - \int x dx = \int x^{-2} dx - \int x dx = \frac{x^{-1}}{-1} - \frac{x^2}{2} + C = -\frac{1}{x} - \frac{x^2}{2} + C$$

So we can set $C = 0$ and take $F(x) = -\frac{1}{x} - \frac{x^2}{2}$ for our antiderivative.

3. Compute $F(b) - F(a)$.

We get:

$$F(b) - F(a) = F(1) - F(\frac{1}{2})$$

$$= [-\frac{1}{(1)} - \frac{(1)^2}{2}] - [-\frac{1}{(\frac{1}{2})} - \frac{(\frac{1}{2})^2}{2}]$$

$$= [-1 - \frac{1}{2}] - [-2 - \frac{1}{8}]$$

$$= [-\frac{3}{2}] - [-\frac{17}{8}]$$

$$= \frac{17}{8} - \frac{3}{2}$$

$$= \frac{17}{8} - \frac{12}{8}$$

$$= \frac{5}{8}$$

So, the area of region II is $\frac{5}{8}$ $units^2$.

Finally, adding the areas of regions I and II together, we get that the area of the region bounded by $y = \frac{1}{x^2}$, $y = x$, and $y = 8x$, for $x \geq 0$ is:

$$\frac{7}{8} + \frac{5}{8}$$

$$= \frac{12}{8}$$

$$= \frac{3}{2} \; units^2$$

$\square$