Q) Find the exact length of the curve:

$$y = \ln(\cos(x)) \; ; \quad 0 \le x \le \frac{\pi}{3}.$$

A.) Recall:

$$S = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

$$y = \ln(\cos(x))$$

$$\frac{dy}{dx} = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$$

$$\left(\frac{dy}{dx}\right)^2 = \tan^2 x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sec x.$$

So, $$S = \int_0^{\pi/3} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\pi/3} \sec x \, dx$$

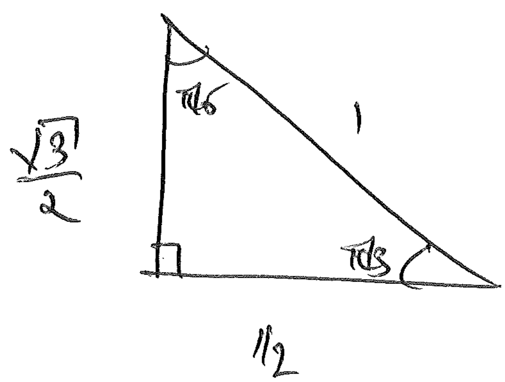
$$= \Big[\ln|\tan x + \sec x|\Big]_0^{\pi/3}$$

$$= \left\{ \ln\left|\tan\left(\frac{\pi}{3}\right) + \sec\left(\frac{\pi}{3}\right)\right| \right\} - \left\{ \ln\left|\tan 0 + \sec 0\right| \right\} \qquad (*)$$

$\tan 0 = 0$

$\sec 0 = 1$

$\ln|\tan 0 + \sec 0| = \ln|1| = 0.$



$\tan \frac{\pi}{3} = \sqrt{3}$

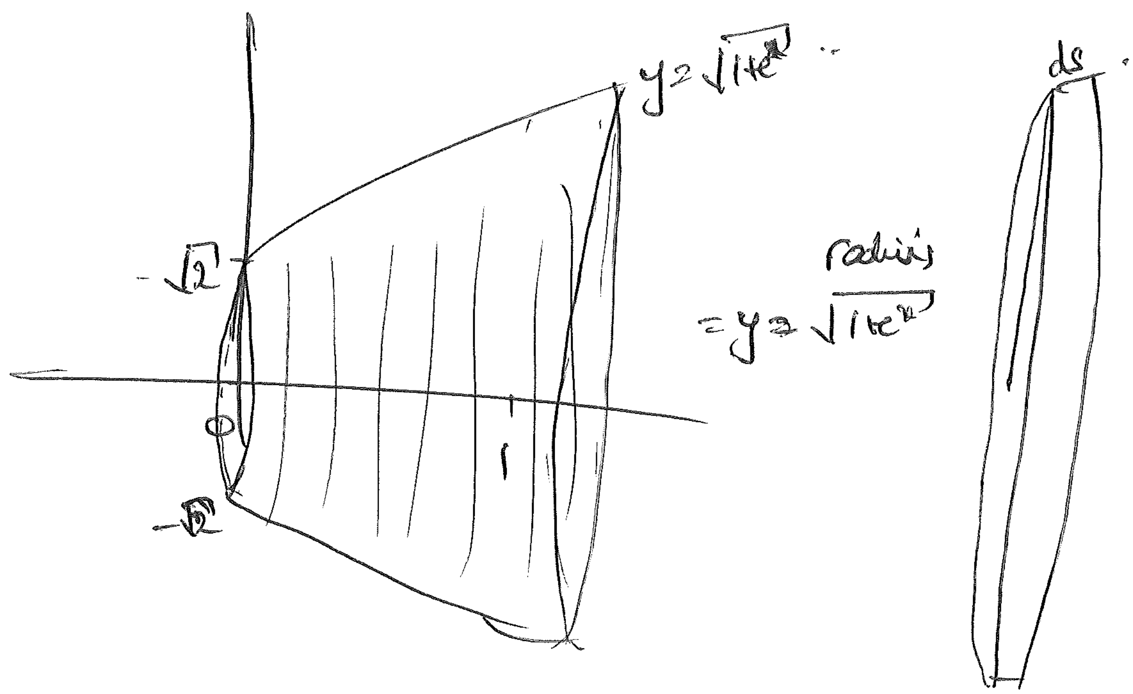
$\sec \frac{\pi}{3} = 2.$

$$(*) = \ln\left|\sqrt{3} + 2\right|.$$

$\square$

Q) Find the area of the surface obtained by rotating $y = \sqrt{1+e^x}$ between $x=0$ and $x=1$ about the x-axis.

A) Sketch of curve:



$$A = \int_0^1 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \sqrt{1+e^x}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+e^x}} \cdot e^x$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{e^{2x}}{4(1+e^x)}$$

Check this step!

$$\left(\frac{dy}{dx}\right)^2 + 1 = \frac{e^{2x}}{4(1+e^x)} + 1 = \frac{e^{2x}+4+4e^x}{4(1+e^x)} = \frac{(e^x+2)^2}{4(e^x+1)}$$

$$\sqrt{\left(\frac{dy}{dx}\right)^2+1} = \frac{e^x+2}{2\sqrt{e^x+1}}$$

$$A = \int_0^1 2\pi \cancel{\sqrt{e^x+1}} \cdot \frac{e^x+2}{2\cancel{\sqrt{e^x+1}}}\, dx$$

$$= \int_0^1 \frac{\cancel{2}\pi}{\cancel{2}}(e^x+2) = \pi \int_0^1 e^x+2 \; dx$$

$$= \pi \cdot \left[e^{x} + 2x \right]_{0}^{1}$$

$$= \pi \cdot \left[\left\{ e^{1} + 2 \right\} - \left\{ e^{0} + 0 \right\} \right]$$

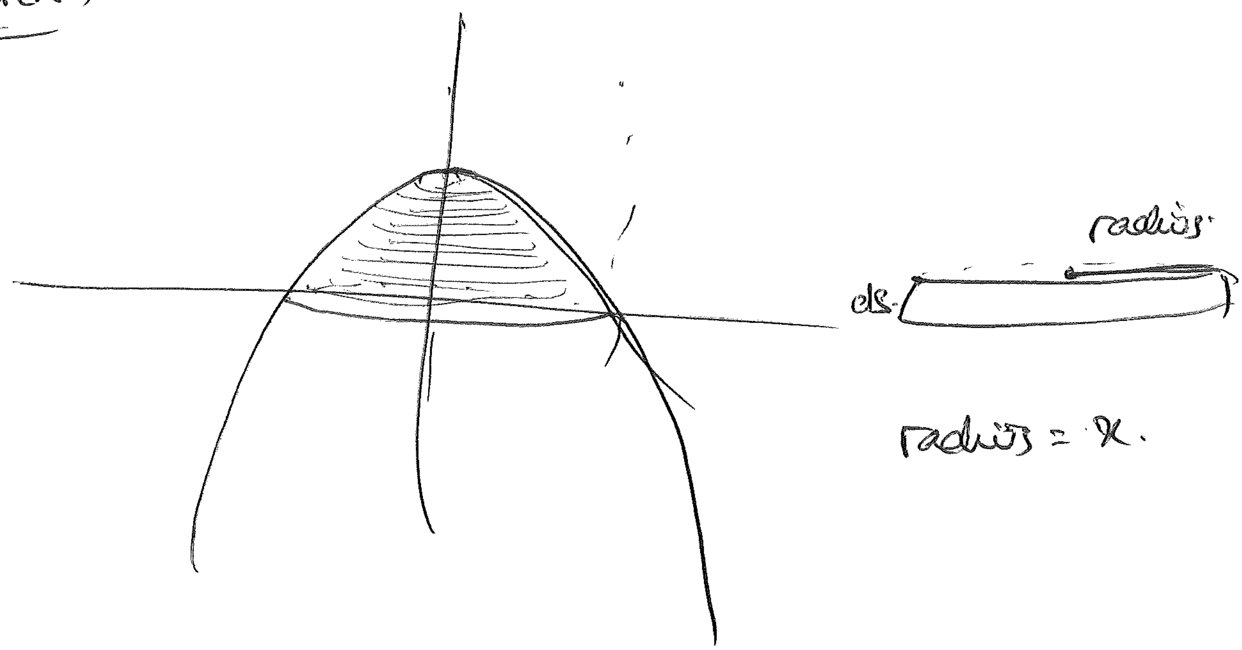
$$= \pi \cdot \left[e + 2 - 1 \right]$$

$$= \pi \left[e + 1 \right] .$$

$\square$

Q) Find the area of the surface obtained by rotating $y = 1 - x^2$ between $x = 0$, and $x = 1$ about the y-axis.

A) Sketch:



$$A = \int_0^1 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$y = 1 - x^2$

$\frac{dy}{dx} = -2x$

$\left(\frac{dy}{dx}\right)^2 = 4x^2$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 4x^2$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+4x^2}.$$

$$A = \int_0^1 2\pi x\sqrt{1+4x^2}\,dx$$

$$= 2\pi \cdot \int_0^1 x\sqrt{1+4x^2}\,dx$$

$u = 1+4x^2$

$du = 8x\,dx$

$dx = \frac{du}{8x}.$

$$= 2\pi \int_{x=0}^{x=1} x\sqrt{u}\,\frac{du}{8x}$$

$$= \frac{\pi}{4} \cdot \int_{x=0}^{x=1} \sqrt{u}\,du$$

$$= \frac{\pi}{4} \cdot \left[\frac{2}{3}u^{3/2}\right]_{x=0}^{x=1}$$

$$= \frac{\pi}{4}\left[\frac{2}{3}(1+4x^2)^{3/2}\right]_{x=0}^{x=1}$$

$$= \frac{\pi}{6}\cdot\left[(1+4x^2)^{3/2}\right]_{x=0}^{x=1}$$

$$= \frac{\pi}{6}\left[\left\{(1+4)^{3/2}\right\} - \left\{(1)^{3/2}\right\}\right]$$

$$= \frac{\pi}{6}\cdot 6^{3/2}$$

$$= \frac{\sqrt{125}\cdot\pi}{6}.$$

$\rbrack.$