In class today, we went over some examples of sequences, defined what it means for a sequence to have a limit, what it means for a sequence to tend to infinity, and we wrote out some rules for calculating limits of sums, differences, products, and quotients of sequences.

In this note, I'll cover the remaining facts we need to know, and some examples.

Question: Determine whether the sequence $a_n = 1 - \frac{1}{n}$ has a limit.

Answer: We can think of the sequence $a_n = 1 - \frac{1}{n}$ as being the difference of the sequences $\{1\}$ and $\{\frac{1}{n}\}$. Then by the rule for the limit of the difference of two sequences, we can write:

$$\begin{aligned}
&\lim_{n\to\infty} a_n \\
&\lim_{n\to\infty} (\{1\} - \frac{1}{n}) \\
&= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n} \\
&= 1 - \lim_{n\to\infty} \frac{1}{n} \\
&= 1 - 0 \\
&= 1.
\end{aligned}$$

□

Fact 1. *Suppose you have three sequences: $\{a_n\}, \{b_n\}$, and $\{c_n\}$, and the following hold:*

1. $\lim_{n\to\infty} a_n = L$
2. $\lim_{n\to\infty} c_n = L$ *(the same limit)*
3. *There is a number n_0 so that for every n occuring afterwards (that is, for every $n \geq n_0$):*

$$a_n \leq b_n \leq c_n$$

Then we must have that $\lim_{n\to\infty} b_n = L$ also.

Question: Determine the limit of the sequence $a_n = \frac{\tan^{-1}(n)}{n}$, if it exists.

Answer:

Notice that $-\frac{\pi}{2} \leq \tan^{-1}(x) \leq \frac{\pi}{2}$ for real number x.

Therefore, $-\frac{\pi}{2} \leq \tan^{-1}(n) \leq \frac{\pi}{2}$ for any natural number $n \geq 1$.

Therefore, $-\frac{\pi}{2n} \leq \frac{\tan^{-1}(n)}{n} \leq \frac{\pi}{2n}$ for any $n \geq 1$ (we are taking $n_0 = 1$ in Fact 1).

Next, we know that $\lim_{n\to\infty} -\frac{\pi}{2n} = 0$, and $\lim_{n\to\infty} \frac{\pi}{2n} = 0$.

By Fact 1, $\lim_{n\to\infty} \frac{\tan^{-1}(n)}{n} = 0$ also.

□

Fact 2. *Suppose you have a sequence $\{a_n\}$, and the following holds:*

1. $\lim_{n\to\infty} |a_n| = 0$

Then we must have that $\lim_{n\to\infty} a_n = 0$ also.

Remark 3. *Fact 2 follows from Fact 1 by considering the "squeeze" inequality:*

$$-|a_n| \leq a_n \leq |a_n|$$

which holds for every term a_n.

Fact 4. *Suppose you have a sequence $\{a_n\}$ and a function $f(x)$, and the following holds:*

1. $\lim_{n\to\infty} a_n = L$
2. *$f(x)$ is continuous at L*

Then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$.

That is, we can swap the function and the limit.

Question: Determine whether the sequence $a_n = e^{\frac{n+1}{n}}$ has a limit.

Answer:

Notice that e^x is continuous at any point $x = L$.

Next, we know that $\lim_{n\to\infty} \frac{n+1}{n} = 1$ $(= L)$.

By Fact 4,

$$\begin{aligned} &\lim_{n\to\infty} e^{\frac{n+1}{n}} \\ &= e^{\lim_{n\to\infty} \frac{n+1}{n}} \\ &= e^1 \\ &= e \end{aligned}$$

$\square$

Question: Determine whether the sequence $a_n = \sqrt{\frac{n+1}{9n+1}}$ has a limit.

Answer:

Notice that the function $z \mapsto \sqrt{z}$ is continuous at any point $z > 0$.

Next, we know that $L = \lim_{n\to\infty} \frac{n+1}{9n+1} = \frac{1}{9}$, and $L > 0$.

By Fact 4,

$$\begin{aligned} &\lim_{n\to\infty} \sqrt{\frac{n+1}{9n+1}} \\ &= \sqrt{\lim_{n\to\infty} \frac{n+1}{9n+1}} \\ &= \sqrt{\frac{1}{9}} \\ &= \frac{1}{3} \end{aligned}$$

$\square$

Fact 5. *Suppose you have a real number r, and you construct the sequence $\{a_n\}$ where $a_n = r^n$. Let's denote this sequence by $\{r^n\}$. Then:*

1. *If $-1 < r < 1$, then $\lim_{n\to\infty} r^n = 0$.*
2. *If $r = 1$, then $\lim_{n\to\infty} r^n = 1$.*
3. *If $r > 1$ or $r \le -1$, then $\lim_{n\to\infty} r^n$ does not exist.*

Remark 6. *The main point of this Fact is that if you take higher and higher powers of any number that is in between -1 and 1, the result gets closer and closer to 0.*

Question: Determine whether the sequence $a_n = 1 - (0.1)^n$ has a limit.

Answer: We can think of the sequence $a_n = 1 - (0.1)^n$ as being the difference of the sequences $\{1\}$ and $(0.1)^n$. Then by the rule for the limit of the difference of two sequences, we can write:

$$\begin{aligned} &\lim_{n\to\infty} a_n \\ &\lim_{n\to\infty} (\{1\} - (0.1)^n) \\ &= \lim_{n\to\infty} \{1\} - \lim_{n\to\infty} (0.1)^n \\ &= 1 - \lim_{n\to\infty} (0.1)^n \end{aligned}$$

Since we have $r = 0.1$ and $-1 < 0.1 < 1$, we get:

$$\begin{aligned} &= 1 - 0 \\ &= 1. \end{aligned}$$

□

Definition 7. Suppose you have a sequence $\{a_n\}$. Then:

1. If for every term a_n, we have $a_n < a_{n+1}$, then we call this sequence **increasing**.
2. If for every term a_n, we have $a_n > a_{n+1}$, then we call this sequence **decreasing**.
3. If $\{a_n\}$ is either **increasing or decreasing**, we call this sequence **monotonic**.

Definition 8. Suppose you have a sequence $\{a_n\}$. Then:

1. If there is a real number M so that for every term a_n, we have $a_n \le M$, then we call this sequence **bounded above** (by M).
2. If there is a real number M so that for every term a_n, we have $a_n \ge M$, then we call this sequence **bounded below** (by M).
3. If $\{a_n\}$ is **both bounded above and bounded below**, we call this sequence **bounded**.

Fact 9. *Suppose you have a sequence $\{a_n\}$. Then:*

1. *If $\{a_n\}$ is bounded above and increasing, then $\{a_n\}$ has a limit.*
2. *If $\{a_n\}$ is bounded below and decreasing, then $\{a_n\}$ has a limit.*

3. *If $\{a_n\}$ is bounded and monotonic, then $\{a_n\}$ has a limit.*

Question: Determine whether the sequence $a_n = \frac{1}{2n+3}$ has a limit. Is the sequence monotonic? bounded?

Answer:

Notice that $\frac{1}{2n+3} > 0$ for every number $n \geq 1$, so the sequence is **bounded below** (by 0).

Notice that $\frac{1}{2x+3}$ is a strictly decreasing function when $x > 0$. Therefore $\{\frac{1}{2n+3}\}$ is a **decreasing sequence**. (Therefore it is also **monotonic**).

By Fact 8 (Part 2), $\{\frac{1}{2n+3}\}$ **has a limit**.

Lastly, notice that $\frac{1}{2n+3} \leq \frac{1}{5}$ for every number $n \geq 1$, so the sequence is also **bounded above** (by $\frac{1}{5}$).

Hence the sequence is **bounded**.

$\square$