Section 1.6, Question 53: The tangent line to the curve $y = \frac{1}{3}x^3 - 4x^2 + 18x + 22$ is parallel to the line $6x - 2y = 1$ at two points on the curve. Find the two points.

Answer: The question is essentially asking us to find the two points on the curve $y = \frac{1}{3}x^3 - 4x^2 + 18x + 22$ where the slope of the curve is the same as the slope of the straight line $6x - 2y = 1$. (Notice that the question does *not* require that the straight line be tangent to the curve.)

We can solve this question if we can find the slope of the straight line, and find out where the slope of the curve equals the slope of the straight line.

The straight line has equation

$$6x - 2y = 1,$$

which we can rewrite as:

$$2y = 6x - 1$$

or

$$y = 3x - \frac{1}{2}.$$

So the slope of the straight line is 3.

Let's now find the derivative of the function $y = \frac{1}{3}x^3 - 4x^2 + 18x + 22$. We'll differentiate on both sides to get:

$$\frac{dy}{dx} = \frac{d}{dx}[\frac{1}{3}x^3 - 4x^2 + 18x + 22]$$

$$\frac{dy}{dx} = \frac{d}{dx}[\frac{1}{3}x^3] - \frac{d}{dx}[4x^2] + \frac{d}{dx}[18x] + \frac{d}{dx}[22]$$

(sum rule)

$$\frac{dy}{dx} = \frac{1}{3}\frac{d}{dx}[x^3] - 4\frac{d}{dx}[x^2] + \frac{d}{dx}[18x] + \frac{d}{dx}[22]$$

(constant multiple rule)

$$\frac{dy}{dx} = \frac{1}{3} * 3x^2 - 4 * 2x + 18 + 0$$

(simple power rule, derivative of a linear function, derivative of a constant function)

$$\frac{dy}{dx} = x^2 - 8x + 18$$

So the slope function for the curve $y = \frac{1}{3}x^3 - 4x^2 + 18x + 22$ is $\frac{dy}{dx} = x^2 - 8x + 18$.

We can find our two points by setting the slope function equal to 3 and solving for x (and then y).

So, $x^2 - 8x + 18 = 3 \implies x^2 - 8x + 15 = 0 \implies (x-5)(x-3) = 0 \implies x = 3$ or $x = 5$.

When $x = 3$, $y = \frac{1}{3} * 3^3 - 4 * 3^2 + 18 * 3 + 22 = 9 - 36 + 54 + 22 = 49$.

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When $x = 5$, $y = \frac{1}{3} * 5^3 - 4 * 5^2 + 18 * 5 + 22 = \frac{125}{3} - 100 + 90 + 22 = \frac{161}{3}$.

So the two points are

$$(3, 49), (5, \frac{161}{3}).$$

$\square$