Q 10) $\sum_{n=1}^{\infty} \frac{10^n x^n}{3^n}$. Find interval and radius of convergence.

$$b_n = \frac{10^n x^n}{3^n}; \quad b_{n+1} = \frac{10^{n+1} x^{n+1}}{3^{n+1}}.$$

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{10^{n+1} \cdot x^{n+1} \cdot 3^n}{3^{n+1} \cdot 10^n x^n}\right| = \left|\frac{10 \cdot x}{3}\right|.$$

$$\lim_{n \to \infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n \to \infty} \left|\frac{10 \cdot x}{3}\right| = \frac{10}{3} \lim_{n \to \infty} |x| = \frac{10}{3} \cdot |x|.$$

(Since x does not depend on n.).

By ratio tests

$\frac{10}{3}|x| < 1 \Rightarrow AC$

$\frac{10}{3}|x| > 1 \Rightarrow D$

$\frac{10}{3}|x| = 1 \Rightarrow ?$ (Check).

$\therefore$ $|x| < 3/10 \Rightarrow AC$

$|x| > 3/10 \Rightarrow D$

$|x| = 3/10 \Rightarrow$ Need to check.

$\therefore$ Radius of convergence $= \frac{3}{10}$, with center $= 0$.

@ $x = \frac{3}{10}$, we get:

$$\sum_{n=1}^{\infty} \frac{10^n}{3^n} \cdot \left(\frac{3}{10}\right)^n = \sum_{n=1}^{\infty} \frac{\cancel{10^n}}{\cancel{3^n}} \cdot \frac{\cancel{3^n}}{\cancel{10^n}} = \sum_{n=1}^{\infty} 1 = \infty \quad D.$$

@ $x = -\frac{3}{10}$, we get:

$$\sum_{n=1}^{\infty} \frac{10^n}{3^n} \cdot \left(\frac{-3}{10}\right)^n = \sum_{n=1}^{\infty} \frac{\cancel{10^n}}{\cancel{3^n}} \cdot (-1)^n \frac{\cancel{3^n}}{\cancel{10^n}}$$

$$= \sum_{n=1}^{\infty} (-1)^n = -1+1-1+1-1+1-1+ \dots$$

D.

$\therefore$ the interval of convergence is:

$(-3/10, 3/10)$.

$\square$

12) $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$.

$b_n = \frac{x^n}{n3^n}$; $b_{n+1} = \frac{x^{n+1}}{(n+1)3^{n+1}}$

$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{x^{n+1}\cdot n\cdot 3^n}{(n+1)3^{n+1}\cdot x^n}\right| = \left|\frac{x\cdot n}{3\cdot(n+1)}\right|$.

$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{x\cdot n}{3(n+1)}\right| = \frac{|x|}{3}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x|}{3}$.

By ratio test, $\frac{|x|}{3} < 1 \Rightarrow AC$

$\frac{|x|}{3} > 1 \Rightarrow D$

$\frac{|x|}{3} = 1 \Rightarrow ?$ Check.

$|x| < 3 \Rightarrow AC$

$|x| > 3 \Rightarrow D$

$|x| = 3 \Rightarrow$ Check.

So, radius of convergence is 3, with center 0.

The interval of convergence is one of:

$[-3, 3]$

$[-3, 3)$

$(-3, 3]$

$(-3, 3)$

(p-series, $p=1$).

@ $x=3$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ D.

@ $x=-3$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ C

(alternating series test.)

So the interval of convergence is:

$[-3, 3)$.

$\square$

14) $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

$$b_n = \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} \quad ; \quad b_{n+1} = \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!}$$

$$= \frac{(-1)^{n+1} \cdot x^{2n+3}}{(2n+3)!}$$

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{(-1)^{n+1} \cdot x^{2n+3} \cdot (2n+1)!}{(2n+3)! (-1)^n \cdot x^{2n+1}}\right| = \left|\frac{(-1)\, x^2}{(2n+3)(2n+2)}\right|.$$

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)\, x^2}{(2n+3)(2n+2)}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^2}{(2n+3)(2n+2)}\right| = \cancel{\lim}$$

$$= |x^2| \lim_{n\to\infty} \left|\frac{1}{(2n+3)(2n+2)}\right|$$

$$= x^2 \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = x^2 \cdot 0 = 0.$$

By ratio test, since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 \quad \text{(for every real number } x \text{ !!!)}$$

~~then~~ we have that

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$ converges ~~absolutely~~ absolutely for every real number x.

So, the radius of convergence is ∞.

And the interval of convergence is ~~$(-\infty$~~

$(-\infty, \infty)$.

$\square$