Section 2.4, Question 19: Sketch the graph of $f(x) = x^4 - 6x^2$.

Answer: To sketch this curve, we'll construct a chart like we did in section 2.3, of the following type:

x	.	.	.	.
f(x)	.	.	.	.
f'(x)	.	.	.	.
f''(x)	.	.	.	.

First, we find $f'(x)$, set it equal to 0 and then solve for x.

$$f(x) = x^4 - 6x^2$$

So,

$$f'(x) = 4x^3 - 12x$$

Setting this equal to 0 gives:

$$f'(x) = 4x^3 - 12x = 0$$

$$4x^3 - 12x = 0$$

$$4x(x^2 - 3) = 0$$

The quadratic in the parentheses factors into $(x^2 - 3) = (x - \sqrt{3})(x + \sqrt{3})$ using the algebraic identity $(x^2 - a^2) = (x - a)(x + a)$. In this case, we take $a = \sqrt{3}$.

So we get:

$$4x(x - \sqrt{3})(x + \sqrt{3}) = 0$$

So we get that $f'(x) = 0$ when $x = -\sqrt{3}, 0, \sqrt{3}$.

Next we find $f''(x)$, set it equal to 0 and solve for x.

We have found:

$$f'(x) = 4x^3 - 12x$$

So,

$$f''(x) = 12x^2 - 12$$

Setting this equal to 0 gives:

$$f''(x) = 12x^2 - 12 = 0$$

$$12x^2 - 12 = 0$$

$$12(x^2 - 1) = 0$$

$$12(x - 1)(x + 1) = 0$$

So we get that $f''(x) = 0$ when $x = -1, +1$.

Now we have our points that we'll use to divide up $(-\infty, \infty)$.

We write these in as a first step in making our chart:

x	.	$-\sqrt{3}$	.	-1	.	0	.	1	.	$\sqrt{3}$	.
f(x)	.	.	.	.	.	.	.	.	.	.	.
f'(x)	.	.	.	.	.	.	.	.	.	.	.
f''(x)	.	.	.	.	.	.	.	.	.	.	.

In the next step, we'll fill in the intervals in between:

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
f(x)	.	.	.	.	.	.	.	.	.	.	.
f'(x)	.	.	.	.	.	.	.	.	.	.	.
f''(x)	.	.	.	.	.	.	.	.	.	.	.

This finishes the x row.

Next, we fill in the $f(x)$ row, filling in values for the columns containing points.

x	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
f(x)	.	-9	.	-5	.	0	.	-5	.	-9	.
f'(x)	.	.	.	.	.	.	.	.	.	.	.
f"(x)	.	.	.	.	.	.	.	.	.	.	.

This finishes the $f(x)$ row.

Next we fill in the $f'(x)$ row.

Remember that we found:

$$f'(x) = 4x^3 - 12x$$

which factors as:

$$f'(x) = (4x)(x - \sqrt{3})(x + \sqrt{3})$$

So we get that $f'(x) = 0$ when $x = -\sqrt{3}, 0, \sqrt{3}$. We can fill this in immediately in the chart:

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
f(x)	.	-9	.	-5	.	0	.	-5	.	-9	.
f'(x)	.	0	.	.	.	0	.	.	.	0	.
f"(x)	.	.	.	.	.	.	.	.	.	.	.

Next, notice that in between these points, $f'(x)$ can only have one sign: either $+$ or $-$. We can see this by looking at an example. If $f'(x)$ switched from $+$ to $-$ on $(-\infty, -\sqrt{3})$ for example, then it would have to equal 0 in between the $+$ and $-$ signs. Then, there would be a point in $(-\infty, -\sqrt{3})$ where $f'(x) = 0$. But we know this is impossible, since we have already found the roots of $f'(x) = 0$, and they were $x = -\sqrt{3}, 0, \sqrt{3}$.

So on $(-\infty, -\sqrt{3})$, $f'(x)$ has only one sign.

We can find it by taking a test point, say $x = -3$ and plugging it into:

$$f'(x) = (4x)(x - \sqrt{3})(x + \sqrt{3})$$

When $x = -3$, the three factors are negative, negative and negative respectively. So, $f'(-3) < 0$ overall. So we can write in a "$-$" in all the cells where $x < -\sqrt{3}$:

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
f(x)	.	-9	.	-5	.	0	.	-5	.	-9	.
f'(x)	$-$	0	.	.	.	0	.	.	.	0	.
f"(x)	.	.	.	.	.	.	.	.	.	.	.

On $(-\sqrt{3}, 0)$, $f'(x)$ has only one sign.

We can find it by taking a test point, say $x = -1.5$ and plugging it into:

$$f'(x) = (4x)(x - \sqrt{3})(x + \sqrt{3})$$

When $x = -1.5$, the three factors are negative, negative and positive respectively. So, $f'(-1.5) > 0$ overall. So we can write in a "+" in all the cells where $-\sqrt{3} < x < 0$:

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
f(x)	.	-9	.	-5	.	0	.	-5	.	-9	.
f'(x)	$-$	0	$+$	$+$	$+$	0	.	.	.	0	.
f"(x)	.	.	.	.	.	.	.	.	.	.	.

On $(0, \sqrt{3})$, $f'(x)$ has only one sign.

We can find it by taking a test point, say $x = 1.5$ and plugging it into:

$$f'(x) = (4x)(x - \sqrt{3})(x + \sqrt{3})$$

When $x = 1.5$, the three factors are positive, negative and positive respectively. So, $f'(1.5) < 0$ overall. So we can write in a "$-$" in all the cells where $0 < x < \sqrt{3}$:

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
f(x)	.	-9	.	-5	.	0	.	-5	.	-9	.
f'(x)	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	.
f"(x)	.	.	.	.	.	.	.	.	.	.	.

On $(\sqrt{3}, \infty)$, $f'(x)$ has only one sign.

We can find it by taking a test point, say $x = 3$ and plugging it into:

$$f'(x) = (4x)(x - \sqrt{3})(x + \sqrt{3})$$

When $x = 3$, the three factors are positive, positive and positive respectively. So, $f'(3) > 0$ overall. So we can write in a "+" in all the cells where $0 < x < \sqrt{3}$:

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
f(x)	.	-9	.	-5	.	0	.	-5	.	-9	.
f'(x)	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
f"(x)	.	.	.	.	.	.	.	.	.	.	.

This finishes the $f'(x)$ row.

Next we'll do the $f''(x)$ row.

Remember that we found:

$$f''(x) = 12x^2 - 12$$

which factorizes as:

$$12(x-1)(x+1)$$

So we get that $f''(x) = 0$ when $x = -1, +1$. We can fill this in immediately in the chart:

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
f(x)	.	-9	.	-5	.	0	.	-5	.	-9	.
f'(x)	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
f''(x)	.	.	.	0	.	.	.	0	.	.	.

Next, notice that in between these points, $f''(x)$ can only have one sign: either $+$ or $-$. We can see this by looking at an example. If $f''(x)$ switched from $+$ to $-$ on $(-\infty, -1)$ for example, then it would have to equal 0 in between the $+$ and $-$ signs. Then, there would be a point in $(-\infty, -1)$ where $f''(x) = 0$. But we know this is impossible, since we have already found the roots of $f''(x) = 0$, and they were $x = -1, +1$.

So on $(-\infty, -1)$, $f''(x)$ has only one sign .

We can find it by taking a test point, say $x = -2$ and plugging it into:

$$f''(x) = 12(x-1)(x+1)$$

When $x = -2$, the two factors are negative and negative respectively. So, $f''(-2) > 0$ overall. So we can write in a "+" in all the cells where $x < -1$:

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
f(x)	.	-9	.	-5	.	0	.	-5	.	-9	.
f'(x)	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
f''(x)	$+$	$+$	$+$	0	.	.	.	0	.	.	.

On $(-1, 1)$, $f''(x)$ has only one sign.

We can find it by taking a test point, say $x = 0$ and plugging it into:

$$f''(x) = 12(x-1)(x+1)$$

When $x = 0$, the two factors are negative and positive respectively. So, $f''(0) < 0$ overall. So we can write in a "$-$" in all the cells where $-1 < x < 1$:

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
f(x)	.	-9	.	-5	.	0	.	-5	.	-9	.
f'(x)	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
f''(x)	$+$	$+$	$+$	0	$-$	$-$	$-$	0	.	.	.

On $(1, \infty)$, $f''(x)$ has only one sign.

We can find it by taking a test point, say $x = 2$ and plugging it into:

$$f''(x) = 12(x - 1)(x + 1)$$

When $x = 2$, the two factors are positive and positive respectively. So, $f''(2) > 0$ overall. So we can write in a "$+$" in all the cells where $x > 1$:

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
f(x)	.	-9	.	-5	.	0	.	-5	.	-9	.
f'(x)	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
f''(x)	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$

This finishes the $f''(x)$ row, and the chart.

Lastly, we'll use this chart to sketch the graph.

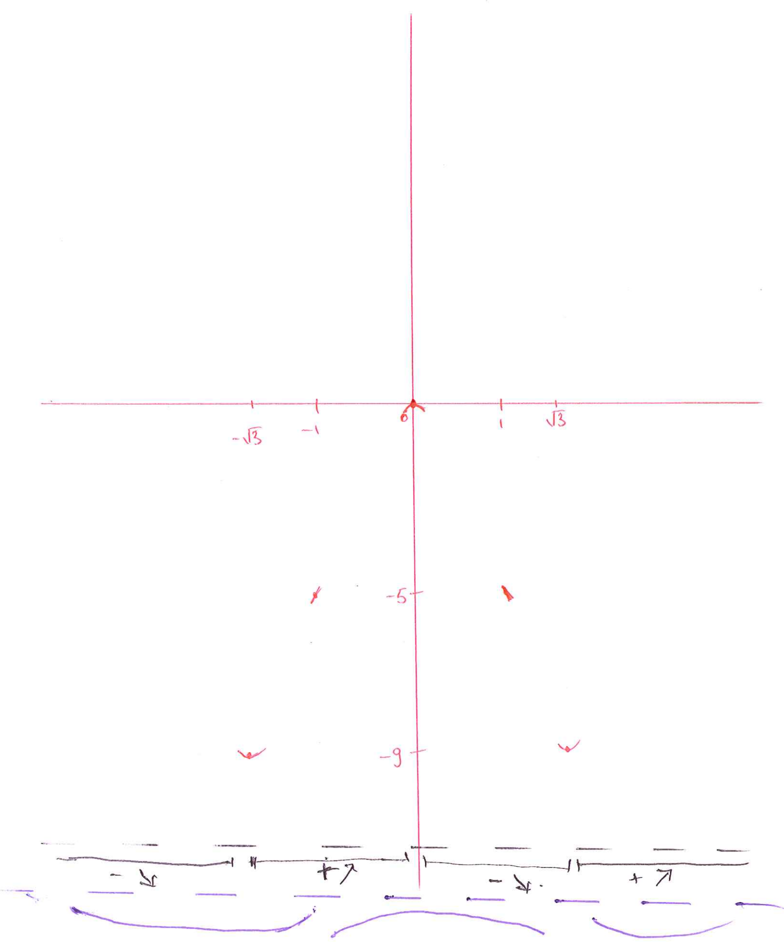
Let's look at the chart again:

x	$(-\infty,-\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3},-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},\infty)$
f(x)	.	-9	.	-5	.	0	.	-5	.	-9	.
f'(x)	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
f''(x)	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$

It tells us that:

1. $f(x)$ has a local maximum at $(0,0)$
2. $f(x)$ has local minima at $(-\sqrt{3},-9)$ and $(\sqrt{3},-9)$
3. $f(x)$ has inflection points at $(-1,-5)$ and $(1,-5)$
4. $f(x)$ is increasing on $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$
5. $f(x)$ is decreasing on $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$
6. $f(x)$ is concave up on $(-\infty,-1)$ and $(1,\infty)$
7. $f(x)$ is concave down on $(-1,1)$

We can sketch these details into a graph as follows:



and then fill in the rest of graph as smoothly as possible, making sure that the concavity and direction are correct everywhere.

$\square$