In class today, we went over two examples of integration of trigonometric functions. In this note, I'll show you the facts you need to know from this section, with examples for each technique.

1 Integrals of the form $\int \sin^m(x)\cos^n(x)dx$, where m is an odd natural number

Method:

1. Set aside one copy of $\sin(x)$, and place it next to the differential dx.

2. You have an even number $(m-1)$ of copies of $\sin(x)$ remaining. Since $m-1=2k$ for some number k, write:

   $$= \int \sin^{m-1}(x)\cos^n(x)\sin(x)dx$$
   $$= \int \sin^{2k}(x)\cos^n(x)\sin(x)dx$$
   $$= \int (\sin^2(x))^k\cos^n(x)\sin(x)dx$$

3. Rewrite $\sin^2(x)=1-\cos^2(x)$ to get:

   $$= \int (1-\cos^2(x))^k\cos^n(x)\sin(x)dx$$

4. Make the substitution: $u=-\cos(x)$. Then $\cos(x)=-u$, and $du=\sin(x)dx$, so we get:

   $$= \int (1-(-u)^2)^k(-u)^n du$$

5. Expand the integrand. You will get a polynomial function of u.

6. Integrate the resulting polynomial function of u, and substitute back for u.

Example: $\int \sin^5(x)\cos^{10}(x)dx$

Solution:

1. $= \int \sin^4(x)\cos^{10}(x)\sin(x)dx$

2. $= \int \sin^{2\cdot 2}(x)\cos^{10}(x)\sin(x)dx = \int (\sin^2(x))^2\cos^{10}(x)\sin(x)dx$

3. $= \int (1-\cos^2(x))^2\cos^{10}(x)\sin(x)dx$

4. $= \int (1-(-u)^2)^2(-u)^{10}du = \int (1-u^2)^2u^{10}du$

5. $= \int (1+u^4-2u^2)u^{10}du = \int (u^{10}+u^{14}-2u^{12})du$

6. $= \frac{u^{11}}{11}+\frac{u^{15}}{15}-2\frac{u^{13}}{13} = \frac{(-\cos(x))^{11}}{11}+\frac{(-\cos(x))^{15}}{15}-2\frac{(-\cos(x))^{13}}{13}+C$

□

2 Integrals of the form $\int \sin^m(x)\cos^n(x)dx$, where n is an odd natural number

Method:

1. Set aside one copy of $\cos(x)$, and place it next to the differential dx.

2. You have an even number $(n-1)$ of copies of $\cos(x)$ remaining. Since $n-1 = 2k$ for some number k, write:

   $= \int \cos^{n-1}(x)\sin^m(x)\cos(x)dx$
   $= \int \cos^{2k}(x)\sin^m(x)\cos(x)dx$
   $= \int (\cos^2(x))^k \sin^m(x)\cos(x)dx$

3. Rewrite $\cos^2(x) = 1 - \sin^2(x)$ to get:

   $= \int (1 - \sin^2(x))^k \sin^m(x)\cos(x)dx$

4. Make the substitution: $u = \sin(x)$. Then $du = \cos(x)dx$, so we get:

   $= \int (1 - (u)^2)^k (u)^m du$

5. Expand the integrand. You will get a polynomial function of u.

6. Integrate the resulting polynomial function of u, and substitute back for u.

Example: $\int \sin^4(x)\cos^5(x)dx$

Solution:

1. $= \int \sin^4(x)\cos^4(x)\cos(x)dx$

2. $= \int \cos^{2\cdot 2}(x)\sin^4(x)\cos(x)dx = \int (\cos^2(x))^2 \sin^4(x)\cos(x)dx$

3. $= \int (1 - \sin^2(x))^2 \sin^4(x)\cos(x)dx$

4. $= \int (1 - (u)^2)^2 (u)^4 du = \int (1 - u^2)^2 u^4 du$

5. $= \int (1 + u^4 - 2u^2)u^4 du = \int (u^4 + u^8 - 2u^6)du$

6. $= \frac{u^5}{5} + \frac{u^9}{9} - 2\frac{u^7}{7} = \frac{(\sin(x))^5}{5} + \frac{(\sin(x))^9}{9} - 2\frac{(\sin(x))^7}{7} + C$

$\square$

3 Integrals of the form $\int \sec^m(x)\tan^n(x)dx$, where n is an odd natural number

Method:

1. Set aside one copy of $\sec(x)\tan(x)$, and place it next to the differential dx.

2. You have an even number $(n-1)$ of copies of $\tan(x)$ remaining. Since $n-1=2k$ for some number k, write:

   $= \int \tan^{n-1}(x)\sec^{m-1}(x)\text{sec}(x)\tan(x)dx$
   $= \int \tan^{2k}(x)\sec^{m-1}(x)\text{sec}(x)\tan(x)dx$
   $= \int (\tan^2(x))^k\sec^{m-1}(x)\text{sec}(x)\tan(x)dx$

3. Rewrite $\tan^2(x)=\sec^2(x)-1$ to get:

   $= \int (\sec^2(x)-1)^k\sec^{m-1}(x)\text{sec}(x)\tan(x)dx$

4. Make the substitution: $u=\sec(x)$. Then $du=\sec(x)\tan(x)dx$, so we get:

   $= \int (u^2-1)^k u^{m-1}du$

5. Expand the integrand. You will get a polynomial function of u.

6. Integrate the resulting polynomial function of u, and substitute back for u.

Example: $\int \sec^4(x)\tan^5(x)dx$

Solution:

1. $= \int \sec^3(x)\tan^4(x)\text{sec}(x)\tan(x)dx$

2. $= \int \sec^3(x)\tan^{2\cdot 2}(x)\text{sec}(x)\tan(x)dx = \int (\tan^2(x))^2\sec^3(x)\text{sec}(x)\tan(x)dx$

3. $= \int (\sec^2(x)-1)^2\sec^3(x)\text{sec}(x)\tan(x)dx$

4. $= \int ((u)^2-1)^2(u)^3du = \int (u^2-1)^2u^3du$

5. $= \int (u^4+1-2u^2)u^3du = \int (u^7+u^3-2u^5)du$

6. $= \frac{u^8}{8}+\frac{u^4}{4}-2\frac{u^6}{6} = \frac{(\sec(x))^8}{8}+\frac{(\sec(x))^4}{4}-2\frac{(\sec(x))^6}{6}+C$

□

4 Integrals of the form $\int \sec^m(x)\tan^n(x)dx$, where m is an even natural number

Method:

1. Set aside two copies of $\sec(x)$ (that is, set aside $\sec^2(x)$), and place them next to the differential dx.

2. You have an even number $(m-2)$ of copies of $\sec(x)$ remaining. Since $m-2=2k$ for some number k, write:

   $= \int \tan^n(x)\sec^{m-2}(x)\text{sec}^2(x)dx$
   $= \int \tan^n(x)\sec^{2k}(x)\text{sec}^2(x)dx$
   $= \int (\sec^2(x))^k \tan^n(x)\text{sec}^2(x)dx$

3. Rewrite $\sec^2(x) = \tan^2(x)+1$ to get:

   $= \int (\tan^2(x)+1)^k \tan^n(x)\text{sec}^2(x)dx$

4. Make the substitution: $u=\tan(x)$. Then $du = \sec^2(x)dx$, so we get:

   $= \int (u^2+1)^k u^n du$

5. Expand the integrand. You will get a polynomial function of u.

6. Integrate the resulting polynomial function of u, and substitute back for u.

Example: $\int \sec^4(x)\tan^5(x)dx$

Solution:

1. $= \int \sec^2(x)\tan^5(x)\text{sec}^2(x)dx$

2. $= \int \sec^2(x)\tan^5(x)\text{sec}^2(x)dx = \int (\sec^2(x))^1\tan^5(x)\text{sec}^2(x)dx$

3. $= \int (\tan^2(x)+1)^1\tan^5(x)\text{sec}^2(x)dx$

4. $= \int (u^2+1)u^5du = \int u^7+u^5du$

5. $= \frac{u^8}{8}+\frac{u^6}{6} = \frac{(\tan(x))^8}{8}+\frac{(\tan(x))^6}{6}+C$

□

5 Integrals of the form $\int \sin^m(x)\cos^n(x)dx$, where both m and n are even natural numbers

This time, we aren't able to use a substitution immediately. Instead, we'll use the double angle formulae to transform the integrand.

Method 1:

Remember that

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x).$$

Rearranging this, we get:

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

and

$$\sin^2(x) = \frac{1-\cos(2x)}{2}.$$

Example: $\int \cos^2(x)\sin^2(x)dx$.

Solution:

$= \int (\frac{1+\cos(2x)}{2})(\frac{1-\cos(2x)}{2})dx$

$= \frac{1}{4}\int (1+\cos(2x))(1-\cos(2x))dx$

$= \frac{1}{4}\int 1-\cos^2(2x)dx$

$= \frac{1}{4}\int 1dx - \frac{1}{4}\int \cos^2(2x)dx$

$= \frac{1}{4}x - \frac{1}{4}\int \cos^2(2x)dx$

At this step, we can apply the formula above again, to get:

$= \frac{1}{4}x - \frac{1}{4}\int \frac{1+\cos(4x)}{2}dx$

$= \frac{1}{4}x - \frac{1}{8}\int 1dx - \frac{1}{8}\int \cos(4x)dx$

$= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8}\int \cos(4x)dx$

Finally, we can use a substitution $u = 4x$ to get:

$= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$

$\square$

Method 2:

Remember that

$$\sin(2x) = 2\sin(x)\cos(x).$$

Rearranging this, we get:

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x).$$

Example: $\int \cos^2(x)\sin^2(x)dx$.

Solution:

$= \int (\cos(x)\sin(x))^2 dx$

$= \int (\frac{1}{2}\sin(2x))^2 dx$

$= \frac{1}{4}\int \sin^2(2x)dx$

Using the formulas from Method 1, we get:

$= \frac{1}{4}\int \frac{1-\cos(4x)}{2}dx$

$= \frac{1}{8}\int 1 - \cos(4x)dx$

$= \frac{1}{8}\int 1dx - \frac{1}{8}\int \cos(4x)dx$

$= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$

$\square$

6 The integral $\int \tan(x)dx$

Fact: $\int \tan(x)dx = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$

Proof:

$\int \tan(x)dx$

$= \int \frac{\sin(x)}{\cos(x)}dx$

Using the substitution $u = \cos(x)$, we get that $du = -\sin(x)dx$, so $dx = \frac{-1}{\sin(x)}du$, and

$= \int \frac{\sin(x)}{u}\frac{-1}{\sin(x)}du$

$= -\int \frac{1}{u}du$

$= -\ln|u| + C$

$= -\ln|\cos(x)| + C$

giving the first result. Next,

$= \ln|(\cos(x))^{-1}| + C$

$= \ln|\frac{1}{(\cos(x))}| + C$

$= \ln|\sec(x)| + C$

giving the second result.

$\square$

7 The integral $\int \sec(x)dx$

Fact: $\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$

Proof:

This proof is clearer if we go backwards:

$\frac{d}{dx}[\ln|\sec(x) + \tan(x)|]$

$= \frac{1}{\sec(x)+\tan(x)} \frac{d}{dx}[\sec(x) + \tan(x)]$

$= \frac{1}{\sec(x)+\tan(x)}[\sec(x)\tan(x) + \sec^2(x)]$

$= \frac{1}{\sec(x)+\tan(x)}[\tan(x) + \sec(x)]\sec(x)$

$= \sec(x)$

Therefore, integrating both sides of the equation:

$$\sec(x) = \frac{d}{dx}[\ln|\sec(x) + \tan(x)|],$$

we get:

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$$

which is the result.

$\square$