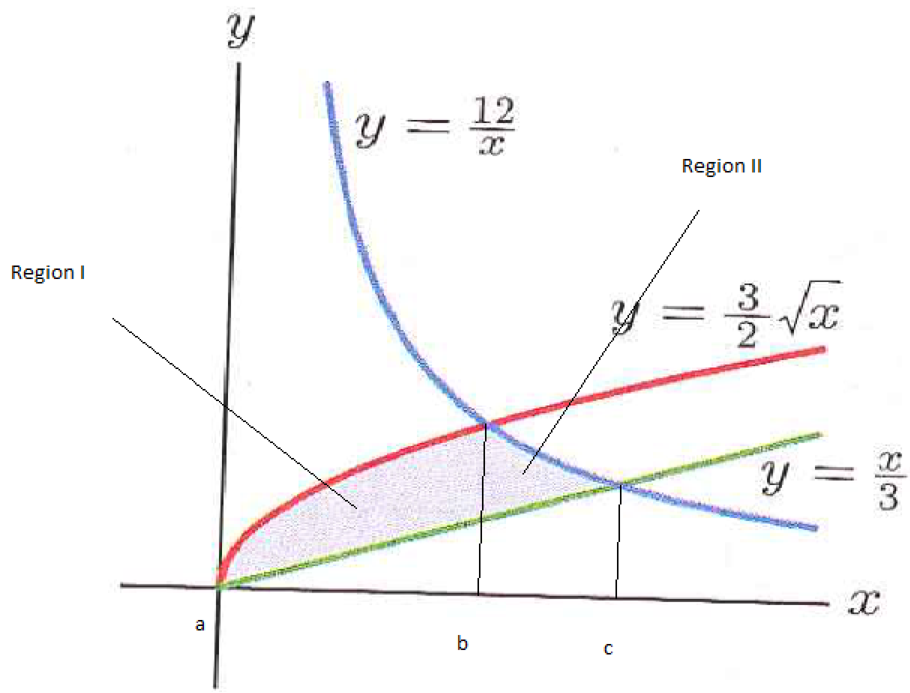
Section 6.4, Question 23: Find the area of the region bounded by $y = \frac{12}{x}$, $y = \frac{3}{2}\sqrt{x}$, and $\frac{x}{3}$.

Hints: To solve this question, we will use the following fact:

Fact 1. *Suppose that a(n upper) curve $y = f(x)$ lies above a (lower) curve $y = g(x)$ over the interval $[a, b]$. Then the area between the two curves over the interval $[a, b]$ is given by:*

$$\int_a^b [f(x) - g(x)]dx$$

First, we'll examine the region in the $x - y$ plane.



(a)

Notice that in the figure, the lower boundary of the region is the curve $y = \frac{x}{3}$, but the upper boundary of the region comes in two parts: $y = \frac{3}{2}\sqrt{x}$ over the interval $[a, b]$, and $y = \frac{12}{x}$ over the interval $[b, c]$.

Before we proceed, we'll identify the relevant intervals $[a, b]$ and $[b, c]$ in the picture.

Notice that at the point a, the two curves $y = \frac{x}{3}$ and $y = \frac{3}{2}\sqrt{x}$ intersect. So, we can find the value of a by setting the two curves equal to each other: $\frac{x}{3} = \frac{3}{2}\sqrt{x}$. This gives: $9\sqrt{x} = 2x \implies \sqrt{x}(9 - 2\sqrt{x}) = 0 \implies \sqrt{x} = 0$ or $\sqrt{x} = \frac{9}{2} \implies x = 0$ or $x = \frac{81}{4}$. Of these two solutions, it is clear from the picture that we want $x = 0$. So, we get that $a = 0$.

Similarly, at the point b, the two curves $y = \frac{3}{2}\sqrt{x}$ and $y = \frac{12}{x}$ intersect. So, we can find the value of b by setting the two curves equal to each other: $\frac{3}{2}\sqrt{x} = \frac{12}{x}$. This gives: $3x^{\frac{3}{2}} = 24 \implies x^{\frac{3}{2}} = 8 \implies x = 4$. So, we get that $b = 4$.

Lastly, at the point c, the two curves $y = \frac{x}{3}$ and $y = \frac{12}{x}$ intersect. So, we can find the value of c by setting the two curves equal to each other: $\frac{x}{3} = \frac{12}{x}$. This gives: $x^2 = 36 \implies x = 6$. So, we get that $c = 6$.

To find the area of the region in the picture, we'll use the fact to find the areas of regions I and II, and then add our answers together to get the desired area.

To find the area of region I, we use the fact with upper curve $f(x) = \frac{3}{2}\sqrt{x}$ and lower curve $g(x) = \frac{x}{3}$. By the fact, the area of region I is:

$$\int_0^4 \frac{3}{2}\sqrt{x} - \frac{x}{3} dx$$

To solve this, we'll use the following steps:

1. Find the bounds of integration. In this case, we have $a = 0, b = 4$.

2. Find an antiderivative. In this case, the family of antiderivatives is:

$$\int \frac{3}{2}\sqrt{x} - \frac{x}{3} dx$$

$$\int \frac{3}{2}\sqrt{x} dx - \int \frac{x}{3} dx$$

$$\frac{3}{2}\int \sqrt{x} dx - \frac{1}{3}\int x dx$$

$$\frac{3}{2}\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{3}\frac{x^2}{2} + C$$

$$x^{\frac{3}{2}} - \frac{x^2}{6} + C$$

So we can set $C = 0$ and take $F(x) = x^{\frac{3}{2}} - \frac{x^2}{6}$ for our antiderivative.

3. Compute $F(b) - F(a)$.

We get:

$$F(b) - F(a) = F(4) - F(0) = [(4)^{\frac{3}{2}} - \frac{(4)^2}{6}] - [(0)^{\frac{3}{2}} - \frac{(0)^2}{6}] = 8 - \frac{8}{3} = \frac{16}{3}$$

So, the area of region I is $\frac{16}{3}$ $units^2$.

We find the area of region II similarly:

We use the fact with upper curve $f(x) = \frac{12}{x}$ and lower curve $g(x) = \frac{x}{3}$. By the fact, the area of region II is:

$$\int_4^6 \frac{12}{x} - \frac{x}{3} dx$$

To solve this, we'll use the following steps:

1. Find the bounds of integration. In this case, we have $a = 4, b = 6$.

2. Find an antiderivative. In this case, the family of antiderivatives is:

$$\int \frac{12}{x} - \frac{x}{3} dx$$

$$= \int \frac{12}{x} dx - \int \frac{x}{3} dx$$

$$= 12 \int \frac{1}{x} dx - \frac{1}{3} \int x dx$$

$$= 12 \ln(x) - \frac{x^2}{6} + C$$

   So we can set $C = 0$ and take $F(x) = 12\ln(x) - \frac{x^2}{6}$ for our antiderivative.

3. Compute $F(b) - F(a)$.

   We get:

$$F(b) - F(a) = F(6) - F(4)$$

$$= [12\ln(6) - \frac{(6)^2}{6}] - [12\ln(4) - \frac{(4)^2}{6}]$$

$$= [12\ln(6) - 6] - [12\ln(4) - \frac{8}{3}]$$

$$= 12\ln(\frac{3}{2}) - \frac{10}{3}$$

So, the area of region II is $12\ln(\frac{3}{2}) - \frac{10}{3}\ units^2$.

Finally, adding the areas of regions I and II together, we get that the area of the region bounded by $y = \frac{12}{x}$, $y = \frac{3}{x}$, and $y = \frac{3}{2}\sqrt{x}$ is:

$$\frac{16}{3} + 12\ln(\frac{3}{2}) - \frac{10}{3}$$

$$= 2 + 12\ln(\frac{3}{2})\ units^2$$

$\square$

Section 6.5, Question 13: Find the consumer surplus for the demand curve $p = \frac{500}{x+10} - 3$ at the sales level $x = 40$.

Hints: We'll use the following fact to solve this question:

Fact 2. *The consumers' surplus in a market with demand function $p = f(x)$ and sales level A is:*

$$CS = \int_0^A [f(x) - B]dx$$

where the market price is $B = f(A)$.

We'll use this fact with:

1. Demand function $p = \frac{500}{x+10} - 3$
2. Sales level $A = 40$
3. Market price $B = f(40) = \frac{500}{(40)+10} - 3 = \frac{500}{50} - 3 = 10 - 3 = 7$

By the fact, the consumers' surplus is:

$$CS = \int_0^{40} [\frac{500}{x+10} - 3 - 7]dx$$

$$CS = \int_0^{40} [\frac{500}{x+10} - 10]dx$$

We'll solve this definite integral using the following three steps:

1. Find the bounds of integration. In this question, we have $a = 0, b = 40$.
2. Find an antiderivative. In this question, we'll need the following additional fact:

   Fact 3. *If m and c are any constants, then:*

   $$\int \frac{1}{mx+c}\, dx = \frac{\ln(mx+c)}{m} + C$$

   where C is an arbitrary constant.

So, we can find the family of antiderivatives:

$$\int \frac{500}{x+10} - 10dx$$

$$\int \frac{500}{x+10}dx - \int 10dx$$

$$500\int \frac{1}{x+10}dx - 10\int 1dx$$

$$500\ln(x+10) - 10x + C$$

So, we can set $C = 0$ and take $F(x) = 500\ln(x+10) - 10x$ as our antiderivative.

3. Compute $F(b) - F(a)$.

   We get:

$$F(b) - F(a)$$

$$F(40) - F(0)$$

$$[500\ln((40)+10) - 10(40)] - [500\ln((0)+10) - 10(0)]$$

$$[500\ln(50) - 400] - [500\ln(10)]$$

$$500\ln(50) - 400 - 500\ln(10)$$

$$500(\ln(50) - \ln(10)) - 400$$

$$500(\ln(5)) - 400 \cong 404.71896$$

So, the consumers' surplus is \$404.72.

$\square$

Section 6.5, Question 19: Find the point of intersection (A, B) and the consumer surplus and producer surplus for the demand curve $p = 12 - \frac{x}{50}$ and the supply curve $p = \frac{x}{20} + 5$.

Hints: To find the point of intersection of the demand and supply curves, we'll set their functions equal to each other:

$$12 - \frac{x}{50} = \frac{x}{20} + 5$$

$$12 - 5 = \frac{x}{20} + \frac{x}{50}$$

$$7 = \frac{5x}{100} + \frac{2x}{100}$$

$$7 = \frac{7x}{100}$$

$$700 = 7x$$

$$x = 100$$

So we get $A = 100$.

We can solve for B by substituting A into either the demand or supply curve:

$$B = 12 - \frac{100}{50} = 12 - 2 = 10$$

To find the consumer's surplus, we'll use the following fact:

Fact 4. *The consumers' surplus in a market with demand function $p = f(x)$ and sales level A is:*

$$CS = \int_0^A [f(x) - B]dx$$

where the market price is $B = f(A)$.

We'll use this fact with:

1. Demand function $p = 12 - \frac{x}{50}$

2. Sales level $A = 100$

3. Market price $B = 10$

By the fact, the consumers' surplus is:

$$CS = \int_0^{100} [12 - \frac{x}{50} - 10]dx$$

$$CS = \int_0^{100} 2 - \frac{x}{50}dx$$

We'll solve this definite integral using the following three steps:

1. Find the bounds of integration. In this question, we have $a = 0, b = 100$.

2. Find an antiderivative. We can find the family of antiderivatives:

$$\int 2 - \frac{x}{50}dx$$

$$\int 2dx - \int \frac{x}{50}dx$$

$$2\int 1dx - \frac{1}{50}\int xdx$$

$$2x - \frac{x^2}{100} + C$$

   So, we can set $C = 0$ and take $F(x) = 2x - \frac{x^2}{100}$ as our antiderivative.

3. Compute $F(b) - F(a)$.

   We get:

$$F(b) - F(a)$$

$$F(100) - F(0)$$

$$[2(100) - \frac{(100)^2}{100}] - [2(0) - \frac{(0)^2}{100}]$$

$$[200 - 100] - [0]$$

$$100$$

So, the consumers' surplus is \$100.

To find the producers' surplus, we'll use the following fact:

Fact 5. *The producers' surplus in a market with supply function $p = g(x)$ and sales level A is:*

$$PS = \int_0^A [B - g(x)]dx$$

where the market price is $B = g(A)$.

Remark 6. The derivation of this formula is in the exercises to this section, immediately before Question 15.

We'll use this fact with:

1. Supply function $p = \frac{x}{20} + 5$

2. Sales level $A = 100$

3. Market price $B = 10$

By the fact, the producers' surplus is:

$$PS = \int_0^{100} 10 - (\frac{x}{20} + 5)dx$$

$$PS = \int_0^{100} 10 - \frac{x}{20} - 5dx$$

$$PS = \int_0^{100} 5 - \frac{x}{20}dx$$

We'll solve this definite integral using the following three steps:

1. Find the bounds of integration. In this question, we have $a = 0, b = 100$.

2. Find an antiderivative. We can find the family of antiderivatives:

$$\int 5 - \frac{x}{20} dx$$

$$\int 5dx - \int \frac{x}{20} dx$$

$$5 \int 1dx - \frac{1}{20} \int xdx$$

$$5x - \frac{x^2}{40} + C$$

   So, we can set $C = 0$ and take $F(x) = 5x - \frac{x^2}{40}$ as our antiderivative.

3. Compute $F(b) - F(a)$.

   We get:

$$F(b) - F(a)$$

$$F(100) - F(0)$$

$$[5(100) - \frac{(100)^2}{40}] - [5(0) - \frac{(0)^2}{40}]$$

$$[500 - \frac{10000}{40}]$$

$$[500 - 250]$$

$$250$$

So, the producers' surplus is \$250.

$\square$