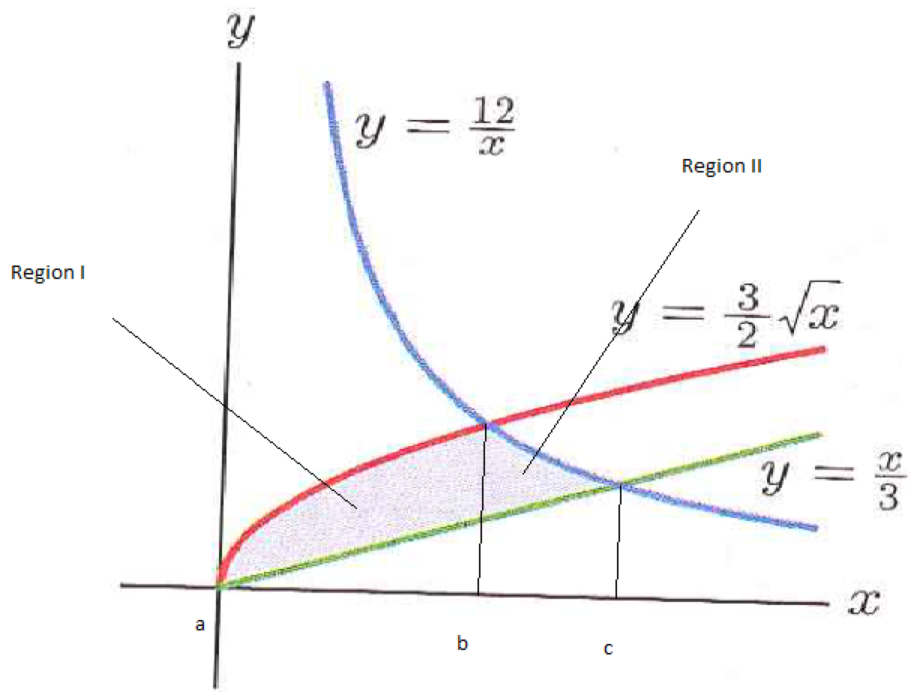
Section 6.4, Question 23: Find the area of the region bounded by $y = \frac{12}{x}$, $y = \frac{3}{2}\sqrt{x}$, and $\frac{x}{3}$.

Hints: To solve this question, we will use the following fact:

Fact 1. *Suppose that a(n upper) curve $y = f(x)$ lies above a (lower) curve $y = g(x)$ over the interval $[a, b]$. Then the area between the two curves over the interval $[a, b]$ is given by:*

$$\int_a^b [f(x) - g(x)]dx$$

First, we'll examine the region in the $x - y$ plane.



(a)

Notice that in the figure, the lower boundary of the region is the curve $y = \frac{x}{3}$, but the upper boundary of the region comes in two parts: $y = \frac{3}{2}\sqrt{x}$ over the interval $[a, b]$, and $y = \frac{12}{x}$ over the interval $[b, c]$.

Before we proceed, we'll identify the relevant intervals $[a, b]$ and $[b, c]$ in the picture.

Notice that at the point a, the two curves $y = \frac{x}{3}$ and $y = \frac{3}{2}\sqrt{x}$ intersect. So, we can find the value of a by setting the two curves equal to each other: $\frac{x}{3} = \frac{3}{2}\sqrt{x}$. This gives: $9\sqrt{x} = 2x \implies \sqrt{x}(9 - 2\sqrt{x}) = 0 \implies \sqrt{x} = 0$ or $\sqrt{x} = \frac{9}{2} \implies x = 0$ or $x = \frac{81}{4}$. Of these two solutions, it is clear from the picture that we want $x = 0$. So, we get that $a = 0$.

Similarly, at the point b, the two curves $y = \frac{3}{2}\sqrt{x}$ and $y = \frac{12}{x}$ intersect. So, we can find the value of b by setting the two curves equal to each other: $\frac{3}{2}\sqrt{x} = \frac{12}{x}$. This gives: $3x^{\frac{3}{2}} = 24 \implies x^{\frac{3}{2}} = 8 \implies x = 4$. So, we get that $b = 4$.

Lastly, at the point c, the two curves $y = \frac{x}{3}$ and $y = \frac{12}{x}$ intersect. So, we can find the value of c by setting the two curves equal to each other: $\frac{x}{3} = \frac{12}{x}$. This gives: $x^2 = 36 \implies x = 6$. So, we get that $c = 6$.

To find the area of the region in the picture, we'll use the fact to find the areas of regions I and II, and then add our answers together to get the desired area.

To find the area of region I, we use the fact with upper curve $f(x) = \frac{3}{2}\sqrt{x}$ and lower curve $g(x) = \frac{x}{3}$. By the fact, the area of region I is:

$$\int_0^4 \frac{3}{2}\sqrt{x} - \frac{x}{3} dx$$

To solve this, we'll use the following steps:

1. Find the bounds of integration. In this case, we have $a = 0, b = 4$.

2. Find an antiderivative. In this case, the family of antiderivatives is:

$$\int \frac{3}{2}\sqrt{x} - \frac{x}{3} dx$$

$$\int \frac{3}{2}\sqrt{x} dx - \int \frac{x}{3} dx$$

$$\frac{3}{2}\int \sqrt{x} dx - \frac{1}{3}\int x dx$$

$$\frac{3}{2}\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{3}\frac{x^2}{2} + C$$

$$x^{\frac{3}{2}} - \frac{x^2}{6} + C$$

   So we can set $C = 0$ and take $F(x) = x^{\frac{3}{2}} - \frac{x^2}{6}$ for our antiderivative.

3. Compute $F(b) - F(a)$.

   We get:

$$F(b) - F(a) = F(4) - F(0) = [(4)^{\frac{3}{2}} - \frac{(4)^2}{6}] - [(0)^{\frac{3}{2}} - \frac{(0)^2}{6}] = 8 - \frac{8}{3} = \frac{16}{3}$$

So, the area of region I is $\frac{16}{3}$ $units^2$.

We find the area of region II similarly:

We use the fact with upper curve $f(x) = \frac{12}{x}$ and lower curve $g(x) = \frac{x}{3}$. By the fact, the area of region II is:

$$\int_4^6 \frac{12}{x} - \frac{x}{3} dx$$

To solve this, we'll use the following steps:

1. Find the bounds of integration. In this case, we have $a = 4, b = 6$.

2. Find an antiderivative. In this case, the family of antiderivatives is:

$$\int \frac{12}{x} - \frac{x}{3} dx$$

$$= \int \frac{12}{x} dx - \int \frac{x}{3} dx$$

$$= 12 \int \frac{1}{x} dx - \frac{1}{3} \int x dx$$

$$= 12 \ln(x) - \frac{x^2}{6} + C$$

   So we can set $C = 0$ and take $F(x) = 12 \ln(x) - \frac{x^2}{6}$ for our antiderivative.

3. Compute $F(b) - F(a)$.

   We get:

$$F(b) - F(a) = F(6) - F(4)$$

$$= [12 \ln(6) - \frac{(6)^2}{6}] - [12 \ln(4) - \frac{(4)^2}{6}]$$

$$= [12 \ln(6) - 6] - [12 \ln(4) - \frac{8}{3}]$$

$$= 12\ln(\frac{3}{2}) - \frac{10}{3}$$

So, the area of region II is $12\ln(\frac{3}{2}) - \frac{10}{3}$ $units^2$.

Finally, adding the areas of regions I and II together, we get that the area of the region bounded by $y = \frac{12}{x}$, $y = \frac{3}{x}$, and $y = \frac{3}{2}\sqrt{x}$ is:

$$\frac{16}{3} + 12\ln(\frac{3}{2}) - \frac{10}{3}$$

$$= 2 + 12\ln(\frac{3}{2})\ units^2$$

$\square$