Section 6.3, Question 7: Compute $\int_{-1}^{1} x\ dx$.

Answer:

To answer this question, we'll use our three step method for computing the definite integral $\int_a^b f(x)dx$:

1. Find a and b.
2. Find an antiderivative $F(x)$ for $f(x)$.
3. Compute $F(b) - F(a)$.

Implementing these steps:

1. We can read off from the question that $a = -1$ and $b = 1$.
2. We'll find the family of antiderivatives $\int x\ dx$:

$$\int x\ dx$$

Using the power rule, we get:

$$\frac{x^2}{2} + C$$

where C is an arbitrary constant.

Since we only need one antiderivative, we can choose the constant C as we please. A good candidate is to set $C = 0$, since then we have one fewer term to deal with.

So, we get an antiderivative $F(x) = \frac{x^2}{2}$.

3. Lastly, we compute:

$$F(b) - F(a)$$

$$F(x)|_b - F(x)|_a$$

$$(\frac{x^2}{2})|_b - (\frac{x^2}{2})|_a$$

$$(\frac{x^2}{2})|_1 - (\frac{x^2}{2})|_{-1}$$

$$(\frac{(1)^2}{2}) - (\frac{(-1)^2}{2})$$

$$(\frac{1}{2}) - (\frac{1}{2})$$

$$= 0.$$

Notice that the answer is 0!

This may seem strange, since the area between the graph of $y = x$ and the x-axis over the interval $[-1, 1]$ is clearly not zero. The actual area can be found by adding two triangles: the first triangle is the triangle between the line $y = x$ and the x-axis over the interval $[-1, 0]$, and the second triangle is the triangle between the line $y = x$ and the x-axis over the interval $[0, 1]$. These two triangles are actually equal, and each has area $\frac{1}{2} * base * height = \frac{1}{2} * (1) * (1) = \frac{1}{2}$. So, the total area should be $\frac{1}{2} + \frac{1}{2} = 1$.

This example illustrates the fact that the definite integral is an *oriented* measure of area: since the first triangle lies below the x-axis, it is counted as a *negative* area, equal to $-\frac{1}{2}$. So, the definite integral adds up the areas as $-\frac{1}{2} + \frac{1}{2} = 0$ and the triangles "cancel out".

$\square$