Q) $$\int \frac{x^2-x+6}{x^3+3x}\,dx$$

$$= \int \frac{x^2-x+6}{x(x^2+3)}\,dx$$

Partial fraction decomposition is:

$$\frac{A}{x} + \frac{Bx+C}{x^2+3}.$$

$$\frac{x^2-x+6}{x(x^2+3)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$\equiv \frac{A(x^2+3)+(Bx+C)x}{x(x^2+3)}$$

$$x^2-x+6 \equiv A(x^2+3)+(Bx+C)x$$

@ $x=0$:

$+6 = 3A + 0$

$\boxed{A=2}$

@ $x=1$:

$6 = 4A + B + C$

$6 = 8 + B + C$

$\boxed{-2 = B + C}$

@ $x=-1$:

$8 = 4A + B - C$

$8 = 8 + B - C$

$\boxed{0 = B - C}$

$B + C = -2$

$B - C = 0$

$2B = -2$

$\boxed{B = -1}$

$\boxed{C = -1}$

$$\frac{x^2-x+6}{x(x^2+3)} = \frac{1}{x} + \frac{-x-1}{x^2+3}$$

$$= \frac{1}{x} - \frac{x+1}{x^2+3}.$$

$$\int \frac{x^2-x+6}{x(x^2+3)}\,dx = \int \frac{1}{x}\,dx - \int \frac{x+1}{x^2+3}\,dx$$

$$= \ln|x| - \int \frac{x+1}{x^2+3}\,dx$$

$$= \ln|x| - \int \frac{x}{x^2+3}\,dx - \int \frac{1}{x^2+3}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+3| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

$\square$

a) $\int \frac{1}{x\sqrt{4x+1}}\,dx$

$u=\sqrt{4x+1}$.

$u^2=4x+1$

$u^2-1=4x$.

$x=\frac{u^2-1}{4}$.

$du=\frac{1}{2\sqrt{4x+1}}\cdot 4=\frac{2}{\sqrt{4x+1}}\,dx$

$dx=du\,\frac{\sqrt{4x+1}}{2}$.

$$I=\int \frac{1}{\frac{u^2-1}{4}\cdot u}\cdot du\cdot\frac{\sqrt{4x+1}}{2}$$

$$=\int \frac{4}{u(u^2-1)}\,du\cdot\cancel{\sqrt{4x+1}}.$$

$$=\int \frac{4}{u^2-1}\,du$$

$$= \int \frac{4}{(u-1)(u+1)} \, du.$$

<u>Partial fractions:</u>

$$\frac{4}{(u-1)(u+1)} \equiv \frac{A}{u-1} + \frac{B}{u+1} \equiv \frac{A(u+1)+B(u-1)}{(u-1)(u+1)}.$$

$$4 \equiv A(u+1) + B(u-1)$$

@ $u=1$;

$$4 = 2A \Rightarrow A=2$$

@ $u=-1$;

$$4=-2B \Rightarrow B=-2$$

$$\frac{4}{(u-1)(u+1)} \equiv \frac{2}{u-1} - \frac{2}{u+1}.$$

$$\int \frac{4}{(u-1)(u+1)} \, du = \int \frac{2}{u-1} \, du - \int \frac{2}{u+1} \, du$$

$$= 2\ln|u-1| - 2\ln|u+1|.$$

$$= 2\ln\left|\sqrt{4x+1}-1\right| - 2\ln\left|\sqrt{4x+1}+1\right| + C$$

$\square$