In class today, we went over the technique of writing a proper rational function whose denominator factors into linear factors, with no repetition, using a partial fraction decomposition. We called this a Type I integral.

In this note, I will show you how to compute the partial fraction decompositions for Type II, III and IV proper rational functions, with an example for each type.

1 Integrals of the form $\frac{P(x)}{Q(x)}$, where Q(x) factorizes into linear factors, some (not necessarily all) with repetition; Type II

Fact 1. Suppose that Q(x) factorizes as $(a_1x + b_1)^{r_1}(a_2x + b_2)^{r_2} \cdots (a_nx + b_n)^{r_n}$, where each r_1, \ldots, r_n is ≥ 1 .

Then you can find constants

$$A_{11}, A_{12}, \cdots A_{1r_1}, \\ A_{21}, A_{22}, \cdots A_{2r_2}, \\ \cdots, \\ A_{n1}, A_{n2}, \cdots A_{nr_n}$$

so that

$$\begin{split} \frac{P(x)}{(a_1x+b_1)^{r_1}(a_2x+b_2)^{r_2}\cdots(a_nx+b_n)^{r_n}} &= \frac{A_{11}}{(a_1x+b_1)} + \frac{A_{12}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1r_1}}{(a_1x+b_1)^{r_1}} \\ &\quad + \frac{A_{21}}{(a_2x+b_2)} + \frac{A_{22}}{(a_2x+b_2)^2} + \cdots + \frac{A_{2r_2}}{(a_2x+b_2)^{r_2}} \\ &\quad \vdots \\ &\quad + \frac{A_{n1}}{(a_nx+b_n)} + \frac{A_{n2}}{(a_nx+b_n)^2} + \cdots + \frac{A_{nr_n}}{(a_nx+b_n)^{r_n}}. \end{split}$$

Example:

$$\int \frac{x^2 - 5x + 16}{(2x+1)(x-2)^2} dx$$

Solution:

1. Fact 1 tells us that there are constants A_{11}, A_{21}, A_{22} so that

$$\frac{x^2 - 5x + 16}{(2x+1)(x-2)^2} = \frac{A_{11}}{(2x+1)} + \frac{A_{21}}{(x-2)} + \frac{A_{22}}{(x-2)^2}.$$

2. To solve for the constants A_{11} , A_{21} and A_{22} , we write the right hand side with a common denominator equal to $(2x+1)(x-2)^2$.

$$\frac{A_{11}}{(2x+1)} + \frac{A_{21}}{(x-2)} + \frac{A_{22}}{(x-2)^2}$$

$$= \frac{A_{11}(x-2)^2}{(2x+1)(x-2)^2} + \frac{A_{21}(x-2)(2x+1)}{(2x+1)(x-2)^2} + \frac{A_{22}(2x+1)}{(2x+1)(x-2)^2}$$

$$=\frac{A_{11}(x-2)^2+A_{21}(x-2)(2x+1)+A_{22}(2x+1)}{(2x+1)(x-2)^2}.$$

So, we have that

$$\frac{x^2 - 5x + 16}{(2x+1)(x-2)^2} \equiv \frac{A_{11}(x-2)^2 + A_{21}(x-2)(2x+1) + A_{22}(2x+1)}{(2x+1)(x-2)^2}.$$

Since these are equal as fractions, and they have the same denominator, their numerators must be equal.

Hence,

$$x^{2} - 5x + 16 \equiv A_{11}(x - 2)^{2} + A_{21}(x - 2)(2x + 1) + A_{22}(2x + 1). \tag{1}$$

3. Now we can solve for A_{11} , A_{21} and A_{22} as follows:

When we substitute x = 2 into Equation (1), we get that

$$10 = 5A_{22} \implies A_{22} = 2.$$

When we substitute $x = -\frac{1}{2}$ into Equation (1), we get that

$$\frac{75}{4} = \frac{25}{4} A_{11} \implies A_{11} = 3.$$

When we substitute x = 1 into Equation (1), we get that

$$12 = A_{11} - 3A_{21} + 3A_{22} = 3 - 3A_{21} + 6 = 9 - 3A_{21} \implies A_{21} = -1.$$

4. So, we get that

$$\frac{x^2 - 5x + 16}{(2x+1)(x-2)^2} = \frac{3}{(2x+1)} - \frac{1}{(x-2)} + \frac{2}{(x-2)^2}.$$

5. Therefore,

$$\int \frac{x^2 - 5x + 16}{(2x+1)(x-2)^2} dx = \int \frac{3}{(2x+1)} - \frac{1}{(x-2)} + \frac{2}{(x-2)^2} dx$$
$$= \frac{3}{2} \ln|2x+1| - \ln|x-2| - \frac{2}{(x-2)} + C$$

2 Integrals of the form $\frac{P(x)}{Q(x)}$, where the factorization of Q(x) includes irreducible quadratic factors with no repetition; Type III

Fact 2. Suppose that Q(x) factorizes as

 $(d_1x + e_1)^{r_1}(d_2x + e_2)^{r_2} \cdots (d_mx + e_m)^{r_m}(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2) \cdots (a_nx^2 + b_nx + c_n),$ where the quadratics are irreducible and each r_1, \ldots, r_m is ≥ 1 .

Then you can find constants

$$A_{11}, A_{12}, \cdots A_{1r_1}, \\ A_{21}, A_{22}, \cdots A_{2r_2}, \\ \cdots, \\ A_{m1}, A_{m2}, \cdots A_{mr_m}, \\ B_1, B_2, \cdots, B_n, \\ C_1, C_2, \cdots, C_n$$

so that

$$\frac{P(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\cdots(a_nx^2+b_nx+c_n)} = \frac{A_{11}}{(d_1x+e_1)} + \frac{A_{12}}{(d_1x+e_1)^2} + \cdots + \frac{A_{1r_1}}{(d_1x+e_1)^{r_1}} + \frac{A_{21}}{(d_2x+e_2)} + \frac{A_{22}}{(d_2x+e_2)^2} + \cdots + \frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} = \frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} + \frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} + \frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} + \cdots + \frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} + \frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} + \cdots + \frac{A_{2r_2}}{(d_2x+e_$$

Example:

$$\int \frac{x^2 - 2x - 1}{(x - 1)^2 (x^2 + 1)} dx$$

Solution:

1. Fact 2 tells us that there are constants A_{11}, A_{12}, B_1, C_1 so that

$$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{A_{11}}{(x-1)} + \frac{A_{12}}{(x-1)^2} + \frac{B_1x+C_1}{x^2+1}.$$

2. To solve for the constants A_{11} , A_{12} , B_1 , C_1 , we write the right hand side with a common denominator equal to $(x-1)^2(x^2+1)$.

$$\frac{A_{11}}{(x-1)} + \frac{A_{12}}{(x-1)^2} + \frac{B_1x + C_1}{x^2 + 1}$$

$$= \frac{A_{11}(x-1)(x^2+1)}{(x-1)^2(x^2+1)} + \frac{A_{12}(x^2+1)}{(x-1)^2(x^2+1)} + \frac{(B_1x + C_1)(x-1)^2}{(x-1)^2(x^2+1)}$$

$$= \frac{A_{11}(x-1)(x^2+1) + A_{12}(x^2+1) + (B_1x + C_1)(x-1)^2}{(x-1)^2(x^2+1)}$$

So, we have that

$$\frac{x^2 - 2x - 1}{(x - 1)^2(x^2 + 1)} \equiv \frac{A_{11}(x - 1)(x^2 + 1) + A_{12}(x^2 + 1) + (B_1x + C_1)(x - 1)^2}{(x - 1)^2(x^2 + 1)}.$$

Since these are equal as fractions, and they have the same denominator, their numerators must be equal.

Hence,

$$x^{2} - 2x - 1 \equiv A_{11}(x - 1)(x^{2} + 1) + A_{12}(x^{2} + 1) + (B_{1}x + C_{1})(x - 1)^{2}.$$
 (2)

3. Now we can solve for A_{11}, A_{12}, B_1, C_1 as follows:

When we substitute x = 1 into Equation (2), we get that

$$-2 = 2A_{12} \implies A_{12} = -1.$$

When we substitute x = 0 into Equation (2), we get that

$$-1 = -A_{11} - 1 + C_1 \implies C_1 = A_{11}.$$

When we substitute x = -1 into Equation (2), we get that

$$2 = -4A_{11} - 2 + (A_{11} - B_1)4 \implies 2 = -2 - 4B_1 \implies B_1 = -1.$$

When we substitute x = 2 into Equation (2), we get that

$$-1 = 5A_{11} - 5 + (2B_1 + C_1) \implies 5A_{11} + 5A_{12} + C_1 - 2 = -1 \implies 5A_{11} + C_1 = 6$$

$$\implies 6A_{11} = 6 \implies A_{11} = 1 \implies C_1 = 1.$$

4. So, we get that

$$\frac{x^2 - 2x - 1}{(x - 1)^2(x^2 + 1)} = \frac{1}{(x - 1)} - \frac{1}{(x - 1)^2} + \frac{1 - x}{(x^2 + 1)}.$$

5. Therefore,

$$\int \frac{x^2 - 2x - 1}{(x - 1)^2 (x^2 + 1)} dx = \int \frac{1}{(x - 1)} - \frac{1}{(x - 1)^2} + \frac{1 - x}{(x^2 + 1)} dx$$

$$= \int \frac{1}{(x - 1)} dx - \int \frac{1}{(x - 1)^2} dx + \int \frac{1 - x}{(x^2 + 1)} dx$$

$$= \int \frac{1}{(x - 1)} dx - \int \frac{1}{(x - 1)^2} dx + \int \frac{1}{(x^2 + 1)} dx - \int \frac{x}{(x^2 + 1)} dx$$

$$= \ln|x - 1| + \frac{1}{(x - 1)} + \tan^{-1}(x) - \frac{1}{2} \ln|x^2 + 1| + C$$

3 Integrals of the form $\frac{P(x)}{Q(x)}$, where the factorization of Q(x) includes irreducible quadratic factors with some repetition; Type IV

Fact 3. Suppose that Q(x) factorizes as

$$(d_1x+e_1)^{r_1}(d_2x+e_2)^{r_2}\cdots(d_mx+e_m)^{r_m}(a_1x^2+b_1x+c_1)^{s_1}(a_2x^2+b_2x+c_2)^{s_2}\cdots(a_nx^2+b_nx+c_n)^{s_n},$$

where the quadratics are irreducible and each $r_1,\ldots,r_m,\ s_1,\ldots,s_n\ is\geq 1.$

Then you can find constants

$$A_{11}, A_{12}, \cdots A_{1r_1}, \\ A_{21}, A_{22}, \cdots A_{2r_2}, \\ \cdots, \\ A_{m1}, A_{m2}, \cdots A_{mr_m}, \\ B_{11}, B_{12}, \cdots, B_{1s_1}, \\ B_{21}, B_{22}, \cdots, B_{2s_2}, \\ \cdots, \\ B_{n1}, B_{n2}, \cdots, C_{1s_1}, \\ C_{11}, C_{12}, \cdots, C_{1s_1}, \\ C_{21}, C_{22}, \cdots, C_{2s_2}, \\ \cdots, \\ C_{n1}, C_{n2}, \cdots, C_{ns_n}$$

so that

$$\frac{P(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\cdots(a_nx^2+b_nx+c_n)} = \frac{A_{11}}{(d_1x+e_1)} + \frac{A_{12}}{(d_1x+e_1)^2} + \cdots + \frac{A_{1r_1}}{(d_1x+e_1)^{r_1}} + \frac{A_{21}}{(d_2x+e_2)} + \frac{A_{22}}{(d_2x+e_2)^2} + \cdots + \frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} = \frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} + \frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} + \cdots + \frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} + \frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} + \cdots + \frac{A_{2r_2}}{(d_2x^2+e_2)^{r_2}} + \cdots + \frac{A_{2r_2}}{(d_2x^2+e_2)^{r_2}} + \cdots + \frac{B_{1s_1}x+C_{1s_1}}{(a_1x^2+b_1x+c_1)^{s_1}} + \frac{B_{2s_1}x+C_{1s_1}}{(a_1x^2+b_1x+c_1)^2} + \cdots + \frac{B_{2s_2}x+C_{2s_2}}{(a_2x^2+b_2x+c_2)^{s_2}} + \cdots + \frac{B_{2s_2}x+C_{2s_2}}{(a_2x^2+b_2x+c_2)^{s_2}} + \cdots + \frac{B_{ns_n}x+C_{ns_n}}{(a_nx^2+b_nx+c_n)^{s_n}} + \frac{B_{ns_n}x+C_{ns_n}$$

Example:

$$\int \frac{1}{x(x^2+4)^2} dx$$

Solution:

1. Fact 3 tells us that there are constants $A, B_{11}, B_{12}, C_{11}, C_{12}$ so that

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{B_{11}x + C_{11}}{(x^2+4)} + \frac{B_{12}x + C_{12}}{(x^2+4)^2}.$$

2. To solve for the constants $A, B_{11}, B_{12}, C_{11}, C_{12}$, we write the right hand side with a common denominator equal to $x(x^2 + 4)^2$.

$$\frac{A}{x} + \frac{B_{11}x + C_{11}}{(x^2 + 4)} + \frac{B_{12}x + C_{12}}{(x^2 + 4)^2}$$

$$= \frac{A(x^2 + 4)^2}{x(x^2 + 4)^2} + \frac{(B_{11}x + C_{11})x(x^2 + 4)}{x(x^2 + 4)^2} + \frac{(B_{12}x + C_{12})x}{x(x^2 + 4)^2}$$

$$= \frac{A(x^2 + 4)^2 + (B_{11}x + C_{11})x(x^2 + 4) + (B_{12}x + C_{12})x}{x(x^2 + 4)^2}$$

So, we have that

$$\frac{1}{x(x^2+4)^2} \equiv \frac{A(x^2+4)^2 + (B_{11}x + C_{11})x(x^2+4) + (B_{12}x + C_{12})x}{x(x^2+4)^2}.$$

Since these are equal as fractions, and they have the same denominator, their numerators must be equal.

Hence,

$$1 \equiv A(x^2 + 4)^2 + (B_{11}x + C_{11})x(x^2 + 4) + (B_{12}x + C_{12})x.$$
 (3)

3. Now we can solve for $A, B_{11}, B_{12}, C_{11}, C_{12}$ as follows:

When we substitute x = 0 into Equation (3), we get that:

$$1 = 16A \implies A = \frac{1}{16}.\tag{4}$$

When we substitute x = 1 into Equation (3), we get that :

$$1 = \frac{25}{16} + 5(B_{11} + C_{11}) + (B_{12} + C_{12}). \tag{5}$$

When we substitute x = -1 into Equation (3), we get that:

$$1 = \frac{25}{16} + 5(B_{11} - C_{11}) + B_{12} - C_{12}. (6)$$

When we substitute x = 2 into Equation (3), we get that :

$$1 = 4 + 32B_{11} + 16C_{11} + 4B_{12} + 2C_{12}. (7)$$

When we substitute x = -2 into Equation (3), we get that:

$$1 = 4 + 32B_{11} - 16C_{11} + 4B_{12} - 2C_{12}. (8)$$

Adding Equation 5 to Equation 6, gives that $5B_{11} + B_{12} = -\frac{9}{16}$.

Subtracting Equation 6 from Equation 5, gives that $5C_{11} + B_{12} = 0$.

Adding Equation 7 to Equation 8, gives that $32B_{11} + 4B_{12} = -3$.

Subtracting Equation 8 from Equation 7, gives that $8C_{11} + C_{12} = 0$.

These equations imply that $C_{11} = C_{12} = 0$, and $B_{11} = -\frac{1}{16}$, $B_{12} = -\frac{1}{4}$.

4. So, we get that

$$\frac{1}{x(x^2+4)^2} \equiv \frac{1}{16x} - \frac{x}{16(x^2+4)} - \frac{x}{4(x^2+4)^2}.$$

5. Therefore,

$$\int \frac{1}{x(x^2+4)^2} dx = \int \frac{1}{16x} - \frac{x}{16(x^2+4)} - \frac{x}{4(x^2+4)^2} dx$$
$$= \int \frac{1}{16x} dx - \int \frac{x}{16(x^2+4)} dx - \int \frac{x}{4(x^2+4)^2} dx$$
$$= \frac{1}{16} \ln|x| - \frac{1}{32} \ln|x^2+4| + \frac{1}{8(x^2+4)} + C$$