In class we went over improper integrals of Type I, and we also saw that:

Theorem 1.
$$\int_1^\infty \frac{1}{x^p} dx$$
converges when $p > 1$ but diverges when $0 \le p \le 1$.

In this note, I will discuss Type II improper integrals, and the Comparison Theorem.

1 Type II Improper Integrals

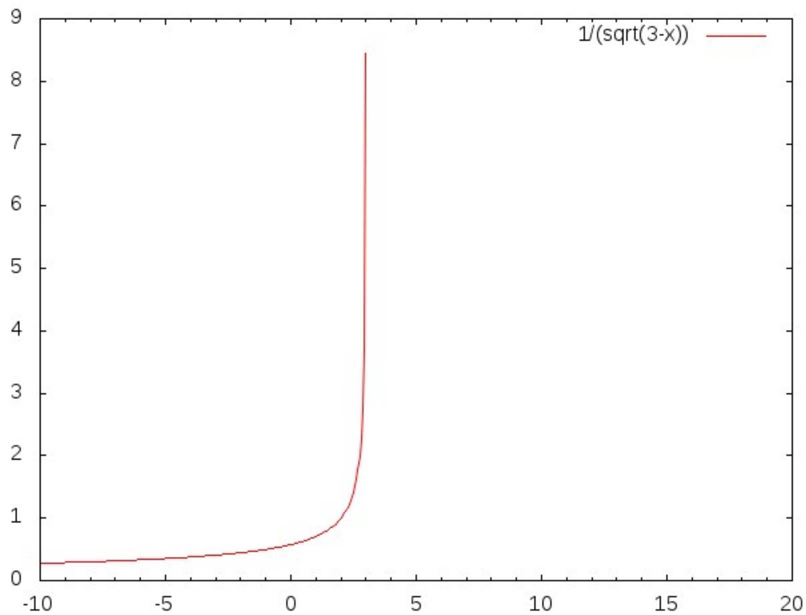
1. Suppose $f(x)$ is a continuous function on $(a, b]$, but has a discontinuity or a vertical asymptote at $x = a$. Then we define
$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx.$$
When the limit exists, we say the integral is convergent, and when it does not exist (or is infinite) we say that the integral diverges.

2. Suppose $f(x)$ is a continuous function on $[a, b)$, but has a discontinuity or a vertical asymptote at $x = b$. Then we define
$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx.$$
When the limit exists, we say the integral is convergent, and when it does not exist (or is infinite) we say that the integral diverges.

3. Suppose $f(x)$ is a continuous function on (a, b), but has a discontinuity or a vertical asymptote at $x = c$, where $a < c < b$, and $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ both converge. Then we define
$$\int_a^c f(x)dx + \int_c^b f(x)dx.$$
When the limit exists, we say the integral is convergent, and when it does not exist (or is infinite) we say that the integral diverges.

Example: Compute
$$\int_2^3 \frac{1}{\sqrt{3-x}} dx.$$

Solution:



Notice that $\frac{1}{\sqrt{3-x}}$ has a vertical asymptote at $x = 3$. Using the definition, we have that

$$\begin{aligned}
\int_2^3 \frac{1}{\sqrt{3-x}}dx &= \lim_{t\to 3^-} \int_2^t \frac{1}{\sqrt{3-x}}dx \\
&= \lim_{t\to 3^-} \left[-2\sqrt{3-x}\right]_2^t \\
&= \lim_{t\to 3^-} \left[-2\sqrt{3-t} + 2\sqrt{3-2}\right] \\
&= \lim_{t\to 3^-} \left[-2\sqrt{3-t} + 2\right] \\
&= \lim_{t\to 3^-} \left[2 - 2\sqrt{3-t}\right] \\
&= \lim_{t\to 3^-} \left[2 - 2(0)\right] \\
&= \lim_{t\to 3^-} \left[2\right] \\
&= 2.
\end{aligned}$$

□

2 Comparison Theorem

Suppose that f and g are continuous functions, such that whenever $x \geq a$, we have

$$f(x) \geq g(x) \geq 0.$$

Then:

1. If $\int_a^\infty f(x)dx$ converges, then so does $\int_a^\infty g(x)dx$.
2. If $\int_a^\infty g(x)dx$ diverges, then so does $\int_a^\infty f(x)dx$.

The idea here is that since $f(x) \geq g(x)$ on the interval $[a, \infty)$, the area under the curve $y = g(x)$ lying above $[a, \infty)$ is less than or equal to the area under the curve $y = f(x)$.

So, if the area under the curve $y = f(x)$ lying above $[a, \infty)$ is finite, then so is the area under the curve $y = g(x)$. On the other hand, if the area under the curve $y = g(x)$ lying above $[a, \infty)$ is infinite, then so is the area under the curve $y = f(x)$.

Example: Determine whether $\int_1^\infty \frac{2+e^{-x}}{x} dx$ is convergent.

Solution:

The trick to these problems is that given the function $f(x)$ in the problem, we need to find a function $g(x)$ so that either $f(x) \leq g(x)$, or $g(x) \leq f(x)$, and so that $\int_1^\infty g(x)dx$ is known to either converge or diverge. Then we can try to use the Comparison Theorem,

In this example, we can find a suitable $g(x)$ by simplifying the given function in the following way: notice that $0 \leq e^{-x}$. Therefore $2 \leq 2 + e^{-x}$, and

$$\frac{2}{x} \leq \frac{2+e^{-x}}{x}$$

for $x \geq 1$.

By the Comparison Theorem, $\int_1^\infty \frac{2+e^{-x}}{x} dx$ diverges if $\int_1^\infty \frac{2}{x} dx$ does. But

$$\int_1^\infty \frac{2}{x} dx$$

$$= 2 \int_1^\infty \frac{1}{x} dx$$

which diverges by Theorem 1 from the beginning of this note.

Therefore $\int_1^\infty \frac{2+e^{-x}}{x} dx$ is divergent.

$\square$