Section 2.1, Question 19: Refer to the graph given.

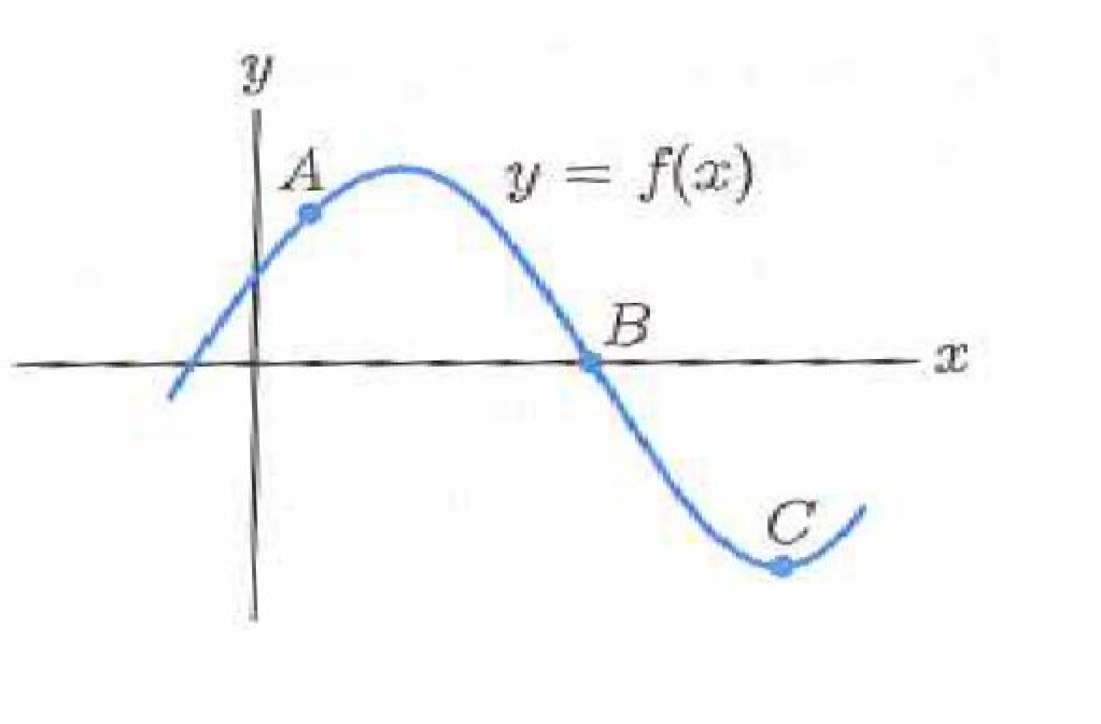

Figure 1:

Fill in each entry of the grid below with POS, NEG or 0.

	f(x)	f'(x)	f''(x)
A			
B			
C			

Answer: To answer this question, we will need to use the following two facts from class today:

- **First derivative rule:** If $f'(x) > 0$ at $x = a$, then $f(x)$ is increasing at $x = a$. If $f'(x) < 0$ at $x = a$, then $f(x)$ is decreasing at $x = a$.

- **Second derivative rule:** If $f''(x) > 0$ at $x = a$, then $f(x)$ is concave up at $x = a$. If $f''(x) < 0$ at $x = a$, then $f(x)$ is concave down at $x = a$.

We will answer this question by filling in each entry in the grid. We will use the coordinates (i, j) to refer to the entry in row i and column j. For example, the entry in the third row and first column has coordinates $(3, 1)$.

To fill in the $(1,1)$th entry, we need to find out whether $f(x)$ is positive (POS), negative (NEG) or zero (0) at the point A.

Looking at the graph, we see that $f(x)$ is positive at the point A (it is above the x-axis). So we can fill in this entry by writing in POS:

	f(x)	f'(x)	f"(x)
A	POS		
B			
C			

To fill in the $(2,1)$th entry, we need to find out whether $f(x)$ is positive, negative or zero at B.

Looking at the graph, we see that $f(x) = 0$ at B. (It crosses the x-axis at the point B). So we can fill in this entry by writing in 0:

	f(x)	f'(x)	f"(x)
A	POS		
B	0		
C			

Similarly, to fill in the $(3,1)$th entry, we need to find out whether $f(x)$ is positive, negative or zero at C.

Looking at the graph, we see that $f(x)$ is negative at C. (It lies below the x-axis at the point C). So we can fill in this entry by writing in NEG:

	f(x)	f'(x)	f"(x)
A	POS		
B	0		
C	NEG		

Next, let's fill in the second column.

To fill in the $(1, 2)$th entry, we need to find out whether $f'(x)$ is positive, negative or zero at the point A.

Looking at the graph, we see that $f(x)$ has positive slope at the point A (the tangent line at A is upwards sloping). This means that $f'(x)$ is positive at A. We can fill in this entry by writing in POS:

	f(x)	f'(x)	f"(x)
A	POS	POS	
B	0		
C	NEG		

To fill in the $(2, 2)$th entry, we need to find out whether $f'(x)$ is positive, negative or zero at the point B.

Looking at the graph, we see that $f(x)$ has negative slope at the point B (the tangent line at B is downwards sloping). This means that $f'(x)$ is negative at B. We can fill in this entry by writing in NEG:

	f(x)	f'(x)	f"(x)
A	POS	POS	
B	0	NEG	
C	NEG		

To fill in the $(3, 2)$th entry, we need to find out whether $f'(x)$ is positive, negative or zero at the point C.

Looking at the graph, we see that $f(x)$ has slope 0 at the point C (the tangent line is horizontal at C). This means that $f'(x)$ is 0 at C. We can fill in this entry by writing in 0:

	f(x)	f'(x)	f"(x)
A	POS	POS	
B	0	NEG	
C	NEG	0	

Lastly, let's fill in the third column.

To fill in the $(1, 3)$th entry, we need to find out whether $f''(x)$ is positive, negative or zero at the point A.

Looking at the graph, we see that $f(x)$ is concave down at the point A (near A, the tangent lines lie above the curve).

Firstly, this means that $f''(x)$ cannot be positive at A, because then the second derivative rule would tell us that f(x) was concave *up* at A!

So, $f''(x)$ must be either negative or zero at A.

If we work a little harder, we can see that $f''(x)$ must be negative at A. Let's look at how the tangent lines to the curve change near A as we move from left to right. If you try to physically roll a straight-edge ruler on the curve near A, you will see that the slopes of the tangent lines decrease from left to right. So, $f'(x)$ is a decreasing function near A. This means that the derivative of $f'(x)$ is negative. So, $f''(x)$ is negative.

Remark: In the above paragraph, we did a lot of work to show that $f''(x)$ was negative! The reason we had to do this is that we cannot just conclude that since $f(x)$ was concave down, $f''(x)$ was negative. It is perfectly possible for a curve to be concave down and for $f''(x) = 0$. Here's an example: take $g(x) = -x^4$ and look at $x = 0$. The graph of $g(x) = -x^4$ is shown below. At $x = 0$, the curve is concave down. However, $g''(x) = -12x^2$. So at $x = 0$, $g''(x) = 0$!

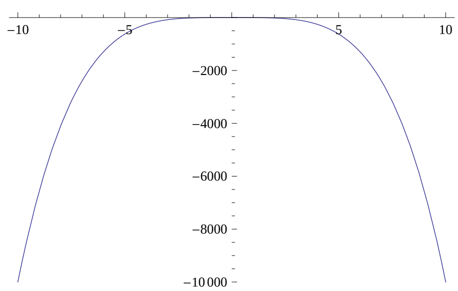

Figure 2: $y = -x^4$

We can fill in this entry by writing in NEG:

	f(x)	f'(x)	f''(x)
A	POS	POS	NEG
B	0	NEG	
C	NEG	0	

To fill in the $(2,3)$th entry, we need to find out whether $f''(x)$ is positive, negative or zero at the point B.

Looking at the graph, we see that $f(x)$ is neither concave down nor concave up at the point B (at B, the tangent line *crosses* the curve - it does not lie on one side or the other of the curve). The second derivative rule tells us that $f''(x)$ is neither negative nor positive at B (otherwise it would be either concave down or concave up at B!). So, $f''(x)$ must be 0 at B. We can fill in this entry by writing in 0:

	f(x)	f'(x)	f"(x)
A	POS	POS	NEG
B	0	NEG	0
C	NEG	0	

To fill in the $(3,3)$th entry, we need to find out whether $f''(x)$ is positive, negative or zero at the point C.

Looking at the graph, we see that $f(x)$ is concave up at the point C (near C, the tangent lines lie below the curve).

Firstly, this means that $f''(x)$ cannot be negative at C, because then the second derivative rule would tell us that f(x) was concave *down* at C!

So, $f''(x)$ must be either positive or zero at C.

If we work a little harder, we can see that $f''(x)$ must be positive at C. Let's look at how the tangent lines to the curve change near C as we move from left to right. If you try to physically roll a straight-edge ruler on the curve near C, you will see that the slopes of the tangent lines increase from left to right. So, $f'(x)$ is an increasing function near A. This means that the derivative of $f'(x)$ is positive. So, $f''(x)$ is positive.

We can fill in this entry by writing in POS:

	f(x)	f'(x)	f"(x)
A	POS	POS	NEG
B	0	NEG	0
C	NEG	0	POS

$\square$