Section 4.5, Question 27: Find the coordinates of the maximum point on the graph of $f(x) = \frac{\ln(x)}{\sqrt{x}}$.

Answer: Remember from class that we are able to do this problem using the following steps:

1. Find the derivative $f'(x)$.

$$f'(x) = \frac{d}{dx}[\frac{\ln(x)}{\sqrt{x}}]$$

We'll use the quotient rule:

$$f'(x) = \frac{\sqrt{x}\frac{d}{dx}[\ln(x)] - \ln(x)\frac{d}{dx}[\sqrt{x}]}{(\sqrt{x})^2}$$

$$f'(x) = \frac{\sqrt{x}(\frac{1}{x}) - \ln(x)(\frac{1}{2}x^{-\frac{1}{2}})}{(\sqrt{x})^2}$$

$$f'(x) = \frac{\sqrt{x}(\frac{1}{x}) - \ln(x)(\frac{1}{2}x^{-\frac{1}{2}})}{x}$$

Using the rules for exponents of the same base:

$$f'(x) = \frac{x^{-\frac{1}{2}} - \ln(x)(\frac{1}{2}x^{-\frac{1}{2}})}{x}$$

2. Set $f'(x) = 0$.

$$f'(x) = \frac{x^{-\frac{1}{2}} - \ln(x)(\frac{1}{2}x^{-\frac{1}{2}})}{x} = 0$$

$$\frac{x^{-\frac{1}{2}} - \ln(x)(\frac{1}{2}x^{-\frac{1}{2}})}{x} = 0$$

3. Solve for x and y.

$$\frac{x^{-\frac{1}{2}} - \ln(x)(\frac{1}{2}x^{-\frac{1}{2}})}{x} = 0$$

Cross multiplying:

$$x^{-\frac{1}{2}} - \ln(x)(\frac{1}{2}x^{-\frac{1}{2}}) = 0$$

$$x^{-\frac{1}{2}}[1 - \ln(x)(\frac{1}{2})] = 0$$

Note that this is where we left off in class today. Next, we'll finish solving for x:

$$x^{-\frac{1}{2}}[1 - \ln(x)(\frac{1}{2})] = 0$$

$$\frac{1 - \ln(x)(\frac{1}{2})}{x^{\frac{1}{2}}} = 0$$

Cross multiplying again:

$$1 - \ln(x)(\frac{1}{2}) = 0$$

$$1 = \ln(x)(\frac{1}{2})$$

$$2 = \ln(x)$$

$$\ln(x) = 2$$

Next, we can take exponentials of both sides:

$$x = e^2$$

This gives us a solution for x.

Then we can solve for y by plugging in our solution to $f(x)$:

$$y = f(e^2) = \frac{\ln(e^2)}{\sqrt{e^2}}$$

$$y = f(e^2) = \frac{2}{e}$$

So we get:

$$y = \frac{2}{e}$$

4. Check the answers using the second derivative.

   We need to check that the point $(e^2, \frac{2}{e})$ is a maximum.

   To do this, we'll compute the second derivative:

$$f''(x) = \frac{d}{dx}[f'(x)]$$

$$f''(x) = \frac{d}{dx}[\frac{x^{-\frac{1}{2}} - \ln(x)(\frac{1}{2}x^{-\frac{1}{2}})}{x}]$$

   First, we'll simplify $f'(x)$ a bit to make it easier to work with:

$$f''(x) = \frac{d}{dx}[\frac{x^{-\frac{1}{2}}[1 - (\frac{1}{2})\ln(x)]}{x}]$$

$$f''(x) = \frac{d}{dx}[\frac{[1 - (\frac{1}{2})\ln(x)]}{x^{\frac{1}{2}} * x}]$$

$$f''(x) = \frac{d}{dx}[\frac{[1 - (\frac{1}{2})\ln(x)]}{x^{\frac{3}{2}}}]$$

   Now we'll use the quotient rule:

$$f''(x) = \frac{x^{\frac{3}{2}}\frac{d}{dx}[1 - (\frac{1}{2})\ln(x)] - (1 - (\frac{1}{2})\ln(x))\frac{d}{dx}[x^{\frac{3}{2}}]}{(x^{\frac{3}{2}})^2}$$

$$f''(x) = \frac{x^{\frac{3}{2}}(-\frac{1}{2} * \frac{1}{x}) - (1 - (\frac{1}{2})\ln(x))\frac{3}{2}x^{\frac{1}{2}}}{(x^{\frac{3}{2}})^2}$$

$$f''(x) = \frac{x^{\frac{3}{2}}(-\frac{1}{2} * \frac{1}{x}) - (1 - (\frac{1}{2})\ln(x))\frac{3}{2}x^{\frac{1}{2}}}{x^3}$$

$$f''(x) = \frac{x^{\frac{3}{2}}(-\frac{1}{2x}) - (1 - \frac{\ln(x)}{2})\frac{3}{2}x^{\frac{1}{2}}}{x^3}$$

We can evaluate $f''(x)$ at $x = e^2$ to check the concavity of $f(x)$ at $x = e^2$:

$$f''(e^2) = \frac{(e^2)^{\frac{3}{2}}(-\frac{1}{2(e^2)}) - (1 - \frac{\ln(e^2)}{2})\frac{3}{2}(e^2)^{\frac{1}{2}}}{(e^2)^3}$$

Clearly, this is an absolutely horrendous expression to evaluate!

We're going to make life easier by working this in steps:

Firstly, the denominator is equal to e^6, which is positive.

Therefore, the fraction is positive if the numerator is positive, and the fraction is negative if the numerator is negative.

So now we'll check the sign of the numerator. We get lucky here because the second term in the numerator simplifies a lot (remember that $\ln(e^2) = 2$):

$$(e^2)^{\frac{3}{2}}(-\frac{1}{2(e^2)}) - (1 - \frac{2}{2})\frac{3}{2}(e^2)^{\frac{1}{2}}$$

$$(e^2)^{\frac{3}{2}}(-\frac{1}{2(e^2)}) - (1 - 1)\frac{3}{2}(e^2)^{\frac{1}{2}}$$

$$(e^2)^{\frac{3}{2}}(-\frac{1}{2(e^2)}) - (0)\frac{3}{2}(e^2)^{\frac{1}{2}}$$

$$(e^2)^{\frac{3}{2}}(-\frac{1}{2(e^2)}) - 0$$

$$(e^2)^{\frac{3}{2}}(-\frac{1}{2(e^2)})$$

$$(e^3)(-\frac{1}{2e^2})$$

Now, $\frac{1}{2e^2} > 0$, so $-\frac{1}{2e^2} < 0$. Also, $e^3 > 0$. So, we get that the numerator:

$$(e^3)(1 - \frac{1}{2e^2}) < 0$$

Since the numerator is negative, the fraction is negative.

So, we get that $f''(x) < 0$ when $x = e^2$. This means that the function $f(x)$ is concave down at the point $(e^2, \frac{2}{e})$, and so this is a maximum.

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