Section 2.5, Question 9: Find the positive values of x and y that minimize $S = x + y$ if $xy = 36$, and find this mininum value.

Answer: We'll do this question along the lines of Quiz 4.

1. Write down the objective function.

In this question we are being asked to minimize the quantity $S = x + y$.

We can summarize this by saying that the objective is:

$$minimize : S = x + y.$$

2. Write down the constraint.

The constraint given is that:

$$xy = 36$$

.

Note also that we must have $x > 0$ and $y > 0$. This means that if we get a solution with $x \leq 0$ or $y \leq 0$, we must discard it.

3. Use the constraint to rewrite/simplify the objective function.

We can solve for y to incorporate the constraint and simplify the objective.

$$xy = 36$$

.

$$\implies y = \frac{36}{x}$$

.

So we can rewrite the objective as:

$$minimize : S = x + y = x + \frac{36}{x}.$$

4. Use the second derivative test to find the minimum value of Q as well as the coordinates (x, y) where the minimum occurs.

To implement the second derivative test, we first take the first derivative:

$$\frac{dS}{dx} = 1 + (-1)\frac{36}{x^2}$$

and set it equal to zero:

$$\frac{dS}{dx} = 1 + (-1)\frac{36}{x^2} = 0$$

$$\implies 1 + (-1)\frac{36}{x^2} = 0$$

$$\implies 1 = \frac{36}{x^2}$$

$$\implies x^2 = 36$$

$$\implies x = \pm 6.$$

We can discard the negative solution since we are only looking for positive values of x and y.

So we get that:

$$x = 6.$$

To verify that this is a minimum, we take the second derivative:

$$\frac{d^2S}{dx^2} = (2)\frac{36}{x^3}.$$

When $x = 6$, we get that:

$$\frac{d^2S}{dx^2}|_{x=6} = (2)\frac{36}{6^3} > 0$$

so S is concave up when $x = 6$, which means that $x = 6$ must be a minimum.

We conclude that $S = x + y$ is minimized when:

1. $x = 6$
2. $y = \frac{36}{6} = 6$
3. $S = 6 + 6 = 12.$

$\square$