CORRECTED VERSION AS OF 2013-06-19 11.07pm

In class today, we went over the technique of writing a proper rational function whose denominator factors into linear factors, with no repetition, using a partial fraction decomposition. We called this a Type I integral.

In this note, I will show you how to compute the partial fraction decompositions for Type II, III and IV proper rational functions, with an example for each type.

1 Integrals of the form $\frac{P(x)}{Q(x)}$, where $Q(x)$ factorizes into linear factors, some (not necessarily all) with repetition; Type II

Fact 1. *Suppose that $Q(x)$ factorizes as $(a_1x+b_1)^{r_1}(a_2x+b_2)^{r_2}\cdots(a_nx+b_n)^{r_n}$, where each $r_1,\ldots,r_n$ is ≥ 1.*

Then you can find constants

$A_{11}, A_{12}, \cdots A_{1r_1},$
$A_{21}, A_{22}, \cdots A_{2r_2},$
$\cdots,$
$A_{n1}, A_{n2}, \cdots A_{nr_n}$

so that

$$\begin{aligned}\frac{P(x)}{(a_1x+b_1)^{r_1}(a_2x+b_2)^{r_2}\cdots(a_nx+b_n)^{r_n}} &= \frac{A_{11}}{(a_1x+b_1)} + \frac{A_{12}}{(a_1x+b_1)^2} + \cdots + \frac{A_{1r_1}}{(a_1x+b_1)^{r_1}} \\ &+ \frac{A_{21}}{(a_2x+b_2)} + \frac{A_{22}}{(a_2x+b_2)^2} + \cdots + \frac{A_{2r_2}}{(a_2x+b_2)^{r_2}} \\ &\vdots \\ &+ \frac{A_{n1}}{(a_nx+b_n)} + \frac{A_{n2}}{(a_nx+b_n)^2} + \cdots + \frac{A_{nr_n}}{(a_nx+b_N)^{r_n}}.\end{aligned}$$

Example:

$$\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}dx$$

Solution:

1. Fact 1 tells us that there are constants A_{11}, A_{21}, A_{22} so that

$$\frac{x^2-5x+16}{(2x+1)(x-2)^2} = \frac{A_{11}}{(2x+1)} + \frac{A_{21}}{(x-2)} + \frac{A_{22}}{(x-2)^2}.$$

2. To solve for the constants A_{11}, A_{21} and A_{22}, we write the right hand side with a common denominator equal to $(2x+1)(x-2)^2$.

$$\frac{A_{11}}{(2x+1)} + \frac{A_{21}}{(x-2)} + \frac{A_{22}}{(x-2)^2}$$

$$= \frac{A_{11}(x-2)^2}{(2x+1)(x-2)^2} + \frac{A_{21}(x-2)(2x+1)}{(2x+1)(x-2)^2} + \frac{A_{22}(2x+1)}{(2x+1)(x-2)^2}$$

$$= \frac{A_{11}(x-2)^2 + A_{21}(x-2)(2x+1) + A_{22}(2x+1)}{(2x+1)(x-2)^2}.$$

So, we have that

$$\frac{x^2-5x+16}{(2x+1)(x-2)^2} \equiv \frac{A_{11}(x-2)^2 + A_{21}(x-2)(2x+1) + A_{22}(2x+1)}{(2x+1)(x-2)^2}.$$

Since these are equal as fractions, and they have the same denominator, their numerators must be equal.

Hence,

$$x^2-5x+16 \equiv A_{11}(x-2)^2 + A_{21}(x-2)(2x+1) + A_{22}(2x+1). \tag{1}$$

3. Now we can solve for A_{11}, A_{21} and A_{22} as follows:

   When we substitute $x = 2$ into Equation (1), we get that

   $$10 = 5A_{22} \implies A_{22} = 2.$$

   When we substitute $x = -\frac{1}{2}$ into Equation (1), we get that

   $$\frac{75}{4} = \frac{25}{4}A_{11} \implies A_{11} = 3.$$

   When we substitute $x = 1$ into Equation (1), we get that

   $$12 = A_{11} - 3A_{21} + 3A_{22} = 3 - 3A_{21} + 6 = 9 - 3A_{21} \implies A_{21} = -1.$$

4. So, we get that

   $$\frac{x^2-5x+16}{(2x+1)(x-2)^2} = \frac{3}{(2x+1)} - \frac{1}{(x-2)} + \frac{2}{(x-2)^2}.$$

5. Therefore,

   $$\begin{aligned}\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}dx &= \int \frac{3}{(2x+1)} - \frac{1}{(x-2)} + \frac{2}{(x-2)^2}dx \\ &= \frac{3}{2}\ln|2x+1| - \ln|x-2| - \frac{2}{(x-2)} + C\end{aligned}$$

$\square$

2 Integrals of the form $\frac{P(x)}{Q(x)}$, where the factorization of $Q(x)$ includes irreducible quadratic factors with no repetition; Type III

Fact 2. *Suppose that $Q(x)$ factorizes as*

$$(d_1x+e_1)^{r_1}(d_2x+e_2)^{r_2}\cdots(d_mx+e_m)^{r_m}(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\cdots(a_nx^2+b_nx+c_n),$$

where the quadratics are irreducible and each $r_1,\ldots,r_m$ is ≥ 1.

Then you can find constants

$A_{11}, A_{12}, \cdots A_{1r_1}$,
$A_{21}, A_{22}, \cdots A_{2r_2}$,
$\cdots$,
$A_{m1}, A_{m2}, \cdots A_{mr_m}$,
$B_1, B_2, \cdots, B_n$,
$C_1, C_2, \cdots, C_n$

so that

$$\begin{aligned}
\frac{P(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\cdots(a_nx^2+b_nx+c_n)} = \\
\frac{A_{11}}{(d_1x+e_1)}+\frac{A_{12}}{(d_1x+e_1)^2}+\cdots+\frac{A_{1r_1}}{(d_1x+e_1)^{r_1}} \\
+\frac{A_{21}}{(d_2x+e_2)}+\frac{A_{22}}{(d_2x+e_2)^2}+\cdots+\frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} \\
\vdots \\
+\frac{A_{m1}}{(d_mx+e_m)}+\frac{A_{m2}}{(d_mx+e_m)^2}+\cdots+\frac{A_{mr_m}}{(d_mx+e_m)^{r_m}} \\
+\frac{B_1x+C_1}{(a_1x^2+b_1x+c_1)}+\frac{B_2x+C_2}{(a_2x^2+b_2x+c_2)}+\cdots+\frac{B_nx+C_n}{(a_nx^2+b_nx+c_n)}.
\end{aligned}$$

Example:

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx$$

Solution:

1. Fact 2 tells us that there are constants A_{11}, A_{12}, B_1, C_1 so that
$$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{A_{11}}{(x-1)}+\frac{A_{12}}{(x-1)^2}+\frac{B_1x+C_1}{x^2+1}.$$

2. To solve for the constants A_{11}, A_{12}, B_1, C_1, we write the right hand side with a common denominator equal to $(x-1)^2(x^2+1)$.
$$\frac{A_{11}}{(x-1)}+\frac{A_{12}}{(x-1)^2}+\frac{B_1x+C_1}{x^2+1}$$
$$= \frac{A_{11}(x-1)(x^2+1)}{(x-1)^2(x^2+1)}+\frac{A_{12}(x^2+1)}{(x-1)^2(x^2+1)}+\frac{(B_1x+C_1)(x-1)^2}{(x-1)^2(x^2+1)}$$
$$= \frac{A_{11}(x-1)(x^2+1)+A_{12}(x^2+1)+(B_1x+C_1)(x-1)^2}{(x-1)^2(x^2+1)}$$

So, we have that

$$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} \equiv \frac{A_{11}(x-1)(x^2+1)+A_{12}(x^2+1)+(B_1x+C_1)(x-1)^2}{(x-1)^2(x^2+1)}.$$

Since these are equal as fractions, and they have the same denominator, their numerators must be equal.

Hence,

$$x^2-2x-1 \equiv A_{11}(x-1)(x^2+1)+A_{12}(x^2+1)+(B_1x+C_1)(x-1)^2. \tag{2}$$

3. Now we can solve for A_{11}, A_{12}, B_1, C_1 as follows:

   When we substitute $x=1$ into Equation (2), we get that

   $$-2 = 2A_{12} \implies A_{12} = -1.$$

   When we substitute $x=0$ into Equation (2), we get that

   $$-1 = -A_{11} - 1 + C_1 \implies C_1 = A_{11}.$$

   When we substitute $x=-1$ into Equation (2), we get that

   $$2 = -4A_{11} - 2 + (A_{11} - B_1)4 \implies 2 = -2 - 4B_1 \implies B_1 = -1.$$

   When we substitute $x=2$ into Equation (2), we get that

   $$-1 = 5A_{11} - 5 + (2B_1 + C_1) \implies 5A_{11} + 5A_{12} + C_1 - 2 = -1 \implies 5A_{11} + C_1 = 6$$
   $$\implies 6A_{11} = 6 \implies A_{11} = 1 \implies C_1 = 1.$$

4. So, we get that

   $$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{1}{(x-1)} - \frac{1}{(x-1)^2} + \frac{1-x}{(x^2+1)}.$$

5. Therefore,

$$\begin{aligned}
\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}dx &= \int \frac{1}{(x-1)} - \frac{1}{(x-1)^2} + \frac{1-x}{(x^2+1)}dx \\
&= \int \frac{1}{(x-1)}dx - \int \frac{1}{(x-1)^2}dx + \int \frac{1-x}{(x^2+1)}dx \\
= \int \frac{1}{(x-1)}dx - \int \frac{1}{(x-1)^2}dx &+ \int \frac{1}{(x^2+1)}dx - \int \frac{x}{(x^2+1)}dx \\
&= \ln|x-1| + \frac{1}{(x-1)} + \tan^{-1}(x) - \frac{1}{2}\ln|x^2+1| + C
\end{aligned}$$

□

3 Integrals of the form $\frac{P(x)}{Q(x)}$, where the factorization of $Q(x)$ includes irreducible quadratic factors with some repetition; Type IV

Fact 3. *Suppose that $Q(x)$ factorizes as*

$$(d_1x+e_1)^{r_1}(d_2x+e_2)^{r_2}\cdots(d_mx+e_m)^{r_m}(a_1x^2+b_1x+c_1)^{s_1}(a_2x^2+b_2x+c_2)^{s_2}\cdots(a_nx^2+b_nx+c_n)^{s_n},$$

where the quadratics are irreducible and each $r_1,\ldots,r_m$, $s_1,\ldots,s_n$ is ≥ 1.

Then you can find constants

$A_{11}, A_{12}, \cdots A_{1r_1}$,
$A_{21}, A_{22}, \cdots A_{2r_2}$,
$\cdots$,
$A_{m1}, A_{m2}, \cdots A_{mr_m}$,
$B_{11}, B_{12}, \cdots, B_{1s_1}$,
$B_{21}, B_{22}, \cdots, B_{2s_2}$,
$\cdots$,
$B_{n1}, B_{n2}, \cdots, B_{ns_n}$,
$C_{11}, C_{12}, \cdots, C_{1s_1}$,
$C_{21}, C_{22}, \cdots, C_{2s_2}$,
$\cdots$,
$C_{n1}, C_{n2}, \cdots, C_{ns_n}$

so that

$$\begin{aligned}
\frac{P(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\cdots(a_nx^2+b_nx+c_n)} = \\
\frac{A_{11}}{(d_1x+e_1)}+\frac{A_{12}}{(d_1x+e_1)^2}+\cdots+\frac{A_{1r_1}}{(d_1x+e_1)^{r_1}} \\
+\frac{A_{21}}{(d_2x+e_2)}+\frac{A_{22}}{(d_2x+e_2)^2}+\cdots+\frac{A_{2r_2}}{(d_2x+e_2)^{r_2}} \\
\vdots \\
+\frac{A_{m1}}{(d_mx+e_m)}+\frac{A_{m2}}{(d_mx+e_m)^2}+\cdots+\frac{A_{mr_m}}{(d_mx+e_m)^{r_m}} \\
+\frac{B_{11}x+C_{11}}{(a_1x^2+b_1x+c_1)}+\frac{B_{12}x+C_{12}}{(a_1x^2+b_1x+c_1)^2}+\cdots+\frac{B_{1s_1}x+C_{1s_1}}{(a_1x^2+b_1x+c_1)^{s_1}} \\
+\frac{B_{21}x+C_{21}}{(a_2x^2+b_2x+c_2)}+\frac{B_{22}x+C_{@2}}{(a_2x^2+b_2x+c_2)^2}+\cdots+\frac{B_{2s_2}x+C_{2s_2}}{(a_2x^2+b_2x+c_2)^{s_2}} \\
\vdots \\
+\frac{B_{n1}x+C_{n1}}{(a_nx^2+b_nx+c_n)}+\frac{B_{n2}x+C_{n2}}{(a_nx^2+b_nx+c_n)^2}+\cdots+\frac{B_{ns_n}x+C_{ns_n}}{(a_nx^2+b_nx+c_n)^{s_n}} \\
.
\end{aligned}$$

Example:

$$\int \frac{1}{x(x^2+4)^2}dx$$

Solution:

1. Fact 3 tells us that there are constants $A, B_{11}, B_{12}, C_{11}, C_{12}$ so that

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{B_{11}x + C_{11}}{(x^2+4)} + \frac{B_{12}x + C_{12}}{(x^2+4)^2}.$$

2. To solve for the constants $A, B_{11}, B_{12}, C_{11}, C_{12}$, we write the right hand side with a common denominator equal to $x(x^2+4)^2$.

$$\frac{A}{x} + \frac{B_{11}x + C_{11}}{(x^2+4)} + \frac{B_{12}x + C_{12}}{(x^2+4)^2}$$

$$= \frac{A(x^2+4)^2}{x(x^2+4)^2} + \frac{(B_{11}x + C_{11})x(x^2+4)}{x(x^2+4)^2} + \frac{(B_{12}x + C_{12})x}{x(x^2+4)^2}$$

$$= \frac{A(x^2+4)^2 + (B_{11}x + C_{11})x(x^2+4) + (B_{12}x + C_{12})x}{x(x^2+4)^2}$$

So, we have that

$$\frac{1}{x(x^2+4)^2} \equiv \frac{A(x^2+4)^2 + (B_{11}x + C_{11})x(x^2+4) + (B_{12}x + C_{12})x}{x(x^2+4)^2}.$$

Since these are equal as fractions, and they have the same denominator, their numerators must be equal.

Hence,

$$1 \equiv A(x^2+4)^2 + (B_{11}x + C_{11})x(x^2+4) + (B_{12}x + C_{12})x. \tag{3}$$

3. Now we can solve for $A, B_{11}, B_{12}, C_{11}, C_{12}$ as follows:

When we substitute $x = 0$ into Equation (3), we get that:

$$1 = 16A \implies A = \frac{1}{16}. \tag{4}$$

When we substitute $x = 1$ into Equation (3), we get that :

$$1 = \frac{25}{16} + 5(B_{11} + C_{11}) + (B_{12} + C_{12}). \tag{5}$$

When we substitute $x = -1$ into Equation (3), we get that:

$$1 = \frac{25}{16} + 5(B_{11} - C_{11}) + B_{12} - C_{12}. \tag{6}$$

When we substitute $x = 2$ into Equation (3), we get that :

$$1 = 4 + 32B_{11} + 16C_{11} + 4B_{12} + 2C_{12}. \tag{7}$$

When we substitute $x = -2$ into Equation (3), we get that:

$$1 = 4 + 32B_{11} - 16C_{11} + 4B_{12} - 2C_{12}. \tag{8}$$

Adding Equation 5 to Equation 6, gives that $5B_{11} + B_{12} = -\frac{9}{16}$.

Subtracting Equation 6 from Equation 5, gives that $5C_{11} + B_{12} = 0$.

Adding Equation 7 to Equation 8, gives that $32B_{11} + 4B_{12} = -3$.

Subtracting Equation 8 from Equation 7, gives that $8C_{11} + C_{12} = 0$.

These equations imply that $C_{11} = C_{12} = 0$, and $B_{11} = -\frac{1}{16}, B_{12} = -\frac{1}{4}$.

4. So, we get that

$$\frac{1}{x(x^2+4)^2} \equiv \frac{1}{16x} - \frac{x}{16(x^2+4)} - \frac{x}{4(x^2+4)^2}.$$

5. Therefore,

$$\begin{aligned}
\int \frac{1}{x(x^2+4)^2} dx &= \int \frac{1}{16x} - \frac{x}{16(x^2+4)} - \frac{x}{4(x^2+4)^2} dx \\
&= \int \frac{1}{16x} dx - \int \frac{x}{16(x^2+4)} dx - \int \frac{x}{4(x^2+4)^2} dx \\
&= \frac{1}{16} \ln|x| - \frac{1}{32} \ln|x^2+4| + \frac{1}{8(x^2+4)} + C
\end{aligned}$$

$\square$