**Section 6.5, Question 13:** Find the consumer surplus for the demand curve $p = \frac{500}{x+10} - 3$ at the sales level $x = 40$.

**Hints:** We'll use the following fact to solve this question:

**Fact 1.** *The consumers' surplus in a market with demand function $p = f(x)$ and sales level $A$ is:*

$$CS = \int_0^A [f(x) - B]dx$$

*where the market price is $B = f(A)$.*

We'll use this fact with:

1. Demand function $p = \frac{500}{x+10} - 3$
2. Sales level $A = 40$
3. Market price $B = f(40) = \frac{500}{(40)+10} - 3 = \frac{500}{50} - 3 = 10 - 3 = 7$

By the fact, the consumers' surplus is:

$$CS = \int_0^{40} [\frac{500}{x+10} - 3 - 7]dx$$

$$CS = \int_0^{40} [\frac{500}{x+10} - 10]dx$$

We'll solve this definite integral using the following three steps:

1. Find the bounds of integration. In this question, we have $a = 0, b = 40$.
2. Find an antiderivative. In this question, we'll need the following additional fact:

   **Fact 2.** *If $m$ and $c$ are any constants, then:*

   $$\int \frac{1}{mx+c}\, dx = \frac{\ln(mx+c)}{m} + C$$

   *where $C$ is an arbitrary constant.*

So, we can find the family of antiderivatives:

$$\int \frac{500}{x+10} - 10dx$$

$$\int \frac{500}{x+10}dx - \int 10dx$$

$$500\int \frac{1}{x+10}dx - 10\int 1dx$$

$$500\ln(x+10) - 10x + C$$

So, we can set $C = 0$ and take $F(x) = 500\ln(x+10) - 10x$ as our antiderivative.

3. Compute $F(b) - F(a)$.

   We get:

$$F(b) - F(a)$$

$$F(40) - F(0)$$

$$[500\ln((40)+10) - 10(40)] - [500\ln((0)+10) - 10(0)]$$

$$[500\ln(50) - 400] - [500\ln(10)]$$

$$500\ln(50) - 400 - 500\ln(10)$$

$$500(\ln(50) - \ln(10)) - 400$$

$$500(\ln(5)) - 400 \cong 404.71896$$

So, the consumers' surplus is \$404.72.

$\square$

**Section 6.5, Question 19:** Find the point of intersection $(A, B)$ and the consumer surplus and producer surplus for the demand curve $p = 12 - \frac{x}{50}$ and the supply curve $p = \frac{x}{20} + 5$.

**Hints:** To find the point of intersection of the demand and supply curves, we'll set their functions equal to each other:

$$12 - \frac{x}{50} = \frac{x}{20} + 5$$

$$12 - 5 = \frac{x}{20} + \frac{x}{50}$$

$$7 = \frac{5x}{100} + \frac{2x}{100}$$

$$7 = \frac{7x}{100}$$

$$700 = 7x$$

$$x = 100$$

So we get $A = 100$.

We can solve for $B$ by substituting $A$ into either the demand or supply curve:

$$B = 12 - \frac{100}{50} = 12 - 2 = 10$$

To find the consumer's surplus, we'll use the following fact:

**Fact 3.** *The consumers' surplus in a market with demand function $p = f(x)$ and sales level $A$ is:*

$$CS = \int_0^A [f(x) - B]dx$$

*where the market price is $B = f(A)$.*

We'll use this fact with:

1. Demand function $p = 12 - \frac{x}{50}$

2. Sales level $A = 100$

3. Market price $B = 10$

By the fact, the consumers' surplus is:

$$CS = \int_0^{100} [12 - \frac{x}{50} - 10]dx$$

$$CS = \int_0^{100} 2 - \frac{x}{50}dx$$

We'll solve this definite integral using the following three steps:

1. Find the bounds of integration. In this question, we have $a = 0, b = 100$.

2. Find an antiderivative. We can find the family of antiderivatives:

$$\int 2 - \frac{x}{50}dx$$

$$\int 2dx - \int \frac{x}{50}dx$$

$$2\int 1dx - \frac{1}{50}\int xdx$$

$$2x - \frac{x^2}{100} + C$$

   So, we can set $C = 0$ and take $F(x) = 2x - \frac{x^2}{100}$ as our antiderivative.

3. Compute $F(b) - F(a)$.

   We get:

$$F(b) - F(a)$$

$$F(100) - F(0)$$

$$[2(100) - \frac{(100)^2}{100}] - [2(0) - \frac{(0)^2}{100}]$$

$$[200 - 100] - [0]$$

$$100$$

So, the consumers' surplus is $100.

To find the producers' surplus, we'll use the following fact:

**Fact 4.** *The producers' surplus in a market with supply function $p = g(x)$ and sales level $A$ is:*

$$PS = \int_0^A [B - g(x)]dx$$

*where the market price is $B = g(A)$.*

*Remark* 5. The derivation of this formula is in the exercises to this section, immediately before Question 15.

We'll use this fact with:

1. Supply function $p = \frac{x}{20} + 5$
2. Sales level $A = 100$
3. Market price $B = 10$

By the fact, the producers' surplus is:

$$PS = \int_0^{100} 10 - (\frac{x}{20} + 5)dx$$

$$PS = \int_0^{100} 10 - \frac{x}{20} - 5dx$$

$$PS = \int_0^{100} 5 - \frac{x}{20}dx$$

We'll solve this definite integral using the following three steps:

1. Find the bounds of integration. In this question, we have $a = 0, b = 100$.

2. Find an antiderivative. We can find the family of antiderivatives:

$$\int 5 - \frac{x}{20} dx$$

$$\int 5dx - \int \frac{x}{20} dx$$

$$5 \int 1dx - \frac{1}{20} \int xdx$$

$$5x - \frac{x^2}{40} + C$$

   So, we can set $C = 0$ and take $F(x) = 5x - \frac{x^2}{40}$ as our antiderivative.

3. Compute $F(b) - F(a)$.

   We get:

$$F(b) - F(a)$$

$$F(100) - F(0)$$

$$[5(100) - \frac{(100)^2}{40}] - [5(0) - \frac{(0)^2}{40}]$$

$$[500 - \frac{10000}{40}]$$

$$[500 - 250]$$

$$250$$

So, the producers' surplus is \$250.

$\square$