Finding f(x) Using Newtons Dackward Difference Table

GROUP PRESENTATION
(Group 9)

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Backward Difference

Definition

The differences $y_1 - y_0, y_2 - y_1, \cdots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \cdots, \nabla y_n$, respectively, are called first backward difference. Thus, the first backward differences are $\nabla y_r = y_r$ - y_{r-1}. This formula is useful when the value of f(x) is required near the end of the table. h is called the interval of difference and $u = \dfrac{x - x_n}{h}$, Here a_n is last term.

Newton's backward difference is a method for approximating the derivative of a function at a specific point using a finite difference approach. It is used to calculate the value of the derivative at a specific point by using the function values at the previous points. This formula is called the "backward" difference formula because it uses function values from previous points in the backward direction.

Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1 $(= x_0 + h)$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2 $(= x_0 + 2h)$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3 $(= x_0 + 3h)$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_4 $(= x_0 + 4h)$	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5 $(= x_0 + 5h)$	y_5					

Solved Problem No. 1

Question

The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

$x=$ height:	150	100	200	250	300	350	400
$y =$ distance:	13.03	10.63	15.04	16.81	18.42	19.90	21.27

Find the values of y when:
$x = 410$.

Solution

Backward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.4			
150	13.3		-0.39		
		2.01		0.15	
200	15.04		-0.24		-0.07
		1.77		0.08	
250	16.81		-0.16		-0.05
		1.61		0.03	
300	18.42		-0.13		-0.01
		1.48		0.02	
350	19.9		-0.11		
		1.37			
400	21.27				

Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$$\therefore \quad \text{Taking } x_n = 400, p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$$

Using the line of backward difference

$$y_n = 21.27, \nabla y_n = 1.37, \nabla^2 y_n = -0.11, \nabla^3 y_n = 0.02 \text{ etc.}$$

$\therefore$ Newton's backward formula gives

$$\begin{aligned}
y_{410} &= y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2!}\nabla^2 y_{400} \\
&+ \frac{p(p+1)(p+2)}{3!}\Delta^3 y_{400} + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_{400} + \cdots \\
&= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2!}(-0.11) \\
&+ \frac{0.2(1.2)(2.2)}{3!}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{4!}(-0.01) \\
&= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007 = 21.53 \text{ nautical miles}
\end{aligned}$$

Solved Problem No. 2

Question

Find the cubic polynomial which takes the following values:

x :	0	1	2	3
$f(x)$:	1	2	1	10

Hence or otherwise evaluate $f(4)$.

Solution

The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**1**			
		1		
1	2		$-\mathbf{2}$	
		-1		**12**
2	1		**10**	
		9		
3	**10**			

Using Newton's backward interpolation formula, we get

$$
\begin{aligned}
f(4) &= f(3) + p\nabla f(3) + \frac{p(p+1)}{1.2}\nabla^2 f(3) + \frac{p(p+1)(p+2)}{1.2.3}\nabla^3 f(3) \\
&= 10 + 9 + 10 + 12 = 41
\end{aligned}
$$

Practice Problems

Problem 1

Find the value of f(0.456) for the below given Newtons Backward Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.45	0.475482			
		0.009174		
0.46	0.484656		0.003622	
		0.012796		-0.011120
0.47	0.497452		-0.007498	
		0.005298		0.011118
0.48	0.502750		0.003620	
		0.008918		
0.49	0.511668			

Problem 2

Use Newton's backward interpolation formula to determine the value of $f(0.48)$ using the following table:

x	0.25	0.30	0.35	0.40	0.45	0.50
$f(x)$	2.6754	2.8765	2.9076	3.2876	3.3451	3.7139

Problem 3

The following table gives the value of x and y.

x	2	3	4	5	6
$f(x)$	5	8	12	20	37

Calculate the value of y at $x = 5.5$ by considering third degree Newton's backward interpolation polynomial. Again, find the same value by considering the fourth degree polynomial.

Problem 4

The upward velocity of a rocket is given below:

$t(\mathrm{sec})$	0	10	15	20	25	30
$v(t)(\mathrm{m/sec})$	0	126.75	350.50	510.80	650.40	920.25

Determine the value of the velocity at $t = 26\mathrm{sec}$ using Newton's backward formulae.

Problem 5

Using Newton's backward difference formula, construct an interpolating polynomial of degree 3 for the data: $f(-0.75) = -0.0718125, f(-0.5) = -0.02475, f(-0.25) = 0.3349375, f(0) = 1.10100$. Hence find $f(-1/3)$.

Solution for Problem 1

For this problem,
$x_0 = 0.45, x = 0.456, h = 0.01, u = \frac{x-x_0}{h} = \frac{0.456-0.45}{0.01} = 0.6.$ Now

$$\begin{aligned} y(0.456) =& y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 \\ =& 0.475482 + 0.6 \times 0.009174 + \frac{0.6(0.6-1)}{2} \times 0.003622 \\ &+ \frac{0.6(0.6-1)(0.6-2)}{6} \times (-0.011120) \\ =& 0.475482 + 0.0055044 - 0.0008693 - 0.0006227 \\ =& 0.4794944. \end{aligned}$$

Hence, the value of y when $x = 0.456$ is 0.479494 .

Solution for Problem 2

The backward difference table is

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0.25	2.6754				
0.30	2.8765	0.2011			
0.35	2.9076	0.0311	-0.1700		
0.40	3.2876	0.3800	0.3489	0.5189	
0.45	3.3451	0.0575	-0.3225	-0.6714	-1.1903
0.50	3.7139	0.3688	0.3113	0.6338	1.3052

In this problem,
$x_n = 0.50, x = 0.48, h = 0.05, v = \frac{x - x_n}{h} = \frac{0.48 - 0.50}{0.05} = -0.4$. Then,

$$f(0.48) = f(x_n) + v\nabla f(x_n) + \frac{v(v+1)}{2!}\nabla^2 f(x_n) + \frac{v(v+1)(v+2)}{3!}\nabla^3 f(x_n) + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 f(x_n) + \cdots$$

$$
\begin{aligned}
=&3.7139 - 0.4 \times 0.3688 + \frac{-0.4(-0.4+1)}{2} \times 0.3113 \\
&+ \frac{-0.4(-0.4+1)(-0.4+2)}{6} \times 0.6338 \\
&+ \frac{-0.4(-0.4+1)(-0.4+2)(-0.4+3)}{24} \times 1.3052 \\
=&3.7139 - 0.14752 - 0.037356 - 0.040563 - 0.054296 \\
=&3.43416 \simeq 3.4342.
\end{aligned}
$$

Thus, $f(0.48) = 3.4342$

Solution for Problem 3

To find a third degree polynomial, we consider last four data and the corresponding backward difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
3	8			
4	12	4		
5	20	8	4	
6	37	17	9	5

In this problem, $x_n = 6, x = 5.5, h = 1, v = \frac{x - x_n}{h} = \frac{5.5-6}{1} = -0.5$.

$$
\begin{aligned}
y(5.5) =& y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n \\
=& 37 - 0.5 \times 17 + \frac{-0.5(-0.5+1)}{2} \times 9 \\
&+ \frac{-0.5(-0.5+1)(-0.5+2)}{6} \times 5 \\
=& 37 - 8.5 - 1.125 - 0.3125 = 27.0625.
\end{aligned}
$$

Now, we consider fourth degree polynomial to calculate $y(5.5)$. The difference table is shown below.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
2	5				
3	8	3			
4	12	4	1		
5	20	8	4	3	
6	37	17	9	5	2

The value of $y(5.5)$ can be determined by the following formula. Note that there is only one term we have to calculate, other terms are already determined in previous step.

$f(0.48) =$
$y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n.$
The value of the last term is

$$\frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n = \frac{-0.5(-0.5+1)(-0.5+2)(-0.5+3)}{4!} \times 2$$

=-0.078125.

Solution for Problem 4

The velocity at $t = 0$ is zero, and hence it does not give any information. Therefore, we discard this data.
Using Newton's backward formula
The backward difference table is

t	$v(t)$	∇v	$\nabla^2 v$	$\nabla^3 v$	$\nabla^4 v$
10	126.75				
15	350.50	223.75			
20	510.80	160.30	-63.45		
25	650.40	139.60	-20.70	42.45	
30	920.25	269.85	130.25	150.95	108.50

Here, $t_n = 30, t = 26, h = 5, v = \frac{t - t_n}{h} = \frac{26-30}{5} = -0.8.$

By Newton's backward formula

$$
\begin{aligned}
v(26) =& y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n \\
& + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n \\
=& 920.25 - 0.8 \times 269.85 + \frac{-0.8(-0.8+1)}{2} \times (130.25) \\
& + \frac{-0.8(-0.8+1)(-0.8+2)}{6} \times 150.95 \\
& + \frac{-0.8(-0.8+1)(-0.8+2)(-0.8+3)}{24} \times 108.50 \\
=& 687.21.
\end{aligned}
$$

Thus the velocity of the rocket at $t = 16\mathrm{sec}$ is 687.21 $\mathrm{m/s}$.

Solution for Problem 5

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-0.75	-0.0718125			
		0.0470625		
-0.50	-0.02475		0.312625	
		0.3596875		**0.09375**
-0.25	0.3349375		**0.400375**	
		0.7660625		
0	**1.10100**			

We use Newton's backward difference formula

$$y(x) = y_3 + \frac{p}{1!}\nabla y_3 + \frac{p(p+1)}{2!}\nabla^2 y_3 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_3$$

taking $\quad x_3 = 0, p = \dfrac{x-0}{h} = \dfrac{x}{0.25} = 4x \quad [\because h = 0.25]$

$$\begin{aligned} y(x) =& 1.10100 + 4x(0.7660625) + \frac{4x(4x+1)}{2}(0.400375) \\ &+ \frac{4x(4x+1)(4x+2)}{6}(0.09375) \\ &= 1.101 + 3.06425x + 3.251x^2 + 0.81275x + x^3 + 0.75x^2 + 0.125x \\ &= x^3 + 4.001x^2 + 4.002x + 1.101 \end{aligned}$$

Put $\quad x = -\frac{1}{3}$, so that

$$y\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + 4.001\left(-\frac{1}{3}\right)^2 + 4.002\left(-\frac{1}{3}\right) + 1.101 = 0.1745$$

Thank You for your Patience!!!!