

# 今まで見つけた式

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## 1 定義

式中に用いられる関数や定数を定義しておく。

$$\begin{aligned}\varphi &\stackrel{\text{def.}}{=} \frac{1 + \sqrt{5}}{2} \\ \gamma &\stackrel{\text{def.}}{=} \lim_{n \rightarrow \infty} (H_n - \log n) \\ H_n &\stackrel{\text{def.}}{=} \sum_{k=1}^n \frac{1}{k} \\ F_n &\stackrel{\text{def.}}{=} \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}} \\ \zeta(n) &\stackrel{\text{def.}}{=} \sum_{k=1}^{\infty} \frac{1}{k^n} \\ \beta(n) &\stackrel{\text{def.}}{=} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n} \\ \Gamma(x) &\stackrel{\text{def.}}{=} \int_0^{\infty} t^{x-1} e^{-t} dt \\ \text{Li}_n(x) &\stackrel{\text{def.}}{=} \sum_{k=1}^{\infty} \frac{x^k}{k^n} \\ (n)_m &\stackrel{\text{def.}}{=} \prod_{k=0}^{m-1} (n+k)\end{aligned}$$

また、定義が複雑なので省略するが  $\gamma(q)$  を q-Euler 定数とする。

## 2 積分

$$\begin{aligned}
\int_0^\infty \frac{x}{2e^x - 1} dx &= \frac{\pi^2}{12} - \frac{1}{2} \log^2 2 \\
\int_0^{\frac{\pi}{2}} \cos \log \tan x dx &= \frac{\pi}{2} \operatorname{sech} \frac{\pi}{2} \\
\int_0^\infty \frac{e^{\frac{1}{6}x} + e^{\frac{5}{6}x} - 2e^{\frac{1}{2}x}}{x(e^x - 1)} dx &= \log 2 \\
\int_0^\infty \frac{x^2 e^{\frac{1}{2}x}}{e^x + 1} dx &= \frac{\pi^3}{2} \\
\int_0^1 \frac{x^{19} - 1}{\sqrt{x} \log x} dx &= \log 39 \\
\int_0^\infty \frac{1 - \tanh x}{\sqrt{\tanh x}} dx &= \frac{\pi}{2} \\
\int_0^{\frac{\pi}{2}} \frac{\sin \log \tan x}{\log \tan x} dx &= \arccos \operatorname{sech} \frac{\pi}{2} \\
\int_0^{\frac{\pi}{2}} \frac{\cos \log \tan x \log \cos x}{\sin^2 x} dx &= -\frac{\pi}{4 \cosh \frac{\pi}{2}} \\
\int_0^{\frac{\pi}{2}} \operatorname{arsinh} \sin x dx &= \beta(2) \\
\int_0^{\frac{\pi}{2}} \frac{\cos^2 x \log \sin x \log \cos x}{\sin x} dx &= -\frac{\pi^4}{576} \\
\int_0^\infty \log(1 - e^{-x}) dx &= -\frac{\pi^2}{6} \\
\int_0^1 x \log \log \frac{1}{x} dx &= -\frac{\gamma + \log 2}{2} \\
\int_0^1 \frac{\log x \log(1+x)}{x(1+x)} dx &= \zeta(3) \\
\int_0^{\frac{\pi}{2}} \frac{\log \cos x}{\tan x} dx &= \frac{2}{3} \pi^2 \\
\int_0^{\frac{1}{2}} \frac{\operatorname{arsinh}^2 x}{x} dx &= \frac{1}{10} \zeta(3) \\
\int_0^1 \frac{x \operatorname{artanh}^3 x}{3 + x^2} dx &= \frac{17}{5184} \pi^4 \\
\int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos^4 x} dx &= \frac{\sqrt{1 + \sqrt{2}}}{4} \pi \\
\int_0^\infty \frac{x \log \tanh x}{\tanh x} dx &= -\frac{7}{16} \zeta(3) - \frac{\pi^2}{8} \log 2 \\
\int_0^\infty x \log \tanh x dx &= \frac{7}{16} \zeta(3) \\
\int_0^\infty \frac{\arctan x^2}{x^2} dx &= \frac{\pi}{\sqrt{2}} \\
\int_0^\infty \frac{\arctan^2 x}{x^2} dx &= \pi \log 2 \\
\int_0^{\frac{\pi}{2}} x \log 2 \sin x dx &= \frac{7}{16} \zeta(3) \\
\int_0^{\frac{\pi}{2}} x^2 \log 2 \sin x dx &= \frac{3\pi}{16} \zeta(3)
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \frac{\log^2 x}{1+x} dx = \frac{3}{2} \zeta(3) \\
& \int_0^\infty \frac{\log^2 x}{1+x^2} dx = \frac{\pi^3}{8} \\
& \int_0^\infty e^{-x} \log^2 x dx = \gamma^2 + \zeta(2) \\
& \int_0^\infty \frac{\sin x}{1+e^{\pi x}} dx = \frac{e^2 - 2e - 1}{2(e+1)(e-1)} \\
& \int_0^1 \frac{\gamma(x)}{x} dx = \frac{\pi^4}{36} - \frac{\pi^2}{6} \\
& \int_0^{\frac{\pi}{2}} \frac{\arctan \frac{\tan x}{\cos x}}{\tan x} dx = \frac{4}{3} \beta(2) \\
& \int_0^\infty \frac{\sin^3 x}{x(1+\sin^4 x)} dx = \frac{\pi}{4} \sqrt{\sqrt{2}-1} \\
& \int_0^\infty \frac{\sin x \log^2 \sin x}{\cos^3 x} dx = \frac{\pi^2}{24} \\
& \int_0^{\frac{\pi}{2}} \frac{\sin x \log^3 \sin x}{\cos^3 x dx} = -\frac{3}{8} \zeta(3) \\
& \int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{3+\cos 4x} dx = \frac{\pi}{4\sqrt{2}} \\
& \int_{-\infty}^\infty \frac{\operatorname{arccoth} \sqrt{x^2+\pi^2}}{\sqrt{x^2+\pi^2}} dx = \pi \arcsin \frac{1}{\pi} \\
& \lim_{t \rightarrow \infty} t \int_t^\infty e^{-(x+t)(x-t)} dx = \frac{1}{2} \\
& \int_0^{\frac{\pi}{2}} \log \left( \frac{9}{16} + \cos^2 x \right) dx = 0 \\
& \int_0^{\frac{\pi}{2}} \frac{\operatorname{arccoth} \sqrt{1+2\tan^2 x}}{\sqrt{1+2\tan^2 x}} dx = \frac{\pi^2}{8} \\
& \int_0^1 \left( \frac{x}{\log x} + \frac{1}{1-x} \right) dx = \gamma + \log 2 \\
& \int_0^\infty \frac{1}{x+1} \log \left( \frac{x^2+2x+1}{x^2+x+1} \right) dx = \frac{\pi^2}{18} \\
& \int_0^\pi \log \left( \frac{5}{4} + \cos x \right) dx = 0 \\
& \int_0^1 \frac{\operatorname{artanh} \sqrt{1-x^2}}{1-x} dx = \frac{3}{8} \pi^2 \\
& \int_0^{\frac{1}{\sqrt{3}}} \frac{x \arctan x}{(1-x^2)\sqrt{1-2x^2}} dx = \frac{\pi^2}{96} \\
& \int_0^{\frac{\pi}{2}} \frac{x}{\sin^8 x + \cos^8 x} dx = \frac{\sqrt{10-\sqrt{2}}}{8} \pi^2 \\
& \int_0^{\frac{\pi}{2}} x^2 \log \tan x dx = \frac{7}{16} \pi \zeta(3) \\
& \int_0^{\frac{\pi}{2}} \log^4 \tan x dx = \frac{5\pi^5}{64} \\
& \int_0^\infty \frac{x \log \sinh x}{\sinh x \cosh x} dx = -\frac{7}{16} \zeta(3) \\
& \int_0^{\sqrt{\frac{2}{3}}} \frac{\arctan \sqrt{1-x^2}}{2+x^2} dx = \frac{11\pi^2}{288\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\Re \int_0^\infty H_{\frac{i}{x}} dx &= \frac{\pi^3}{12} \\
\int_0^{\frac{\pi}{4}} x \operatorname{artanh} \tan x dx &= \frac{7}{32} \zeta(3) \\
\int_0^{\frac{\pi}{2}} \frac{\arctan \frac{\cos x}{2}}{\sqrt{\sin x - \sin^2 x}} dx &= \frac{\pi}{\sqrt{2}} \operatorname{artanh} \sqrt{\frac{5 - \sqrt{5}}{8}} \\
\int_1^{\sqrt{2}} \frac{\operatorname{artanh} \sqrt{2 - x^2}}{1 + x} dx &= \frac{\pi^2}{48} \\
\int_0^1 \frac{\operatorname{artanh}^2 x}{x^2 + 3} dx &= \frac{\pi^3}{81\sqrt{3}} \\
\int_0^\pi \frac{\sinh x - \sin x}{\cosh x - \cos x} dx &= \frac{\pi}{2} \\
\int_0^{\frac{\pi}{2}} \frac{\log \cos x}{\sin x} \log \left( \frac{1 + \sin x}{1 - \sin x} \right) dx &= -2\pi\beta(2) \\
\int_0^1 \frac{\operatorname{artanh}^2 x}{x^2} dx &= \frac{\pi^2}{6} \\
\int_0^1 \frac{x^2}{\operatorname{artanh}^2 x} dx &= 124 \frac{\zeta(5)}{\pi^4} - \frac{14}{3} \frac{\zeta(3)}{\pi^2} \\
\int_0^\pi \arctan \frac{\log \sin x}{x} dx &= -\pi \arctan \frac{2 \log 2}{\pi}
\end{aligned}$$

### 3 級数

$$\begin{aligned}
& \sum_{n=1}^{\infty} \arctan \frac{2}{n^2} = \frac{3}{4}\pi \\
& \sum_{n=1}^{\infty} \arctan \frac{1}{2n^2} = \frac{1}{4}\pi \\
& \sum_{n=0}^{\infty} \frac{1}{(4n+1)^3} = \frac{7}{16}\zeta(3) + \frac{\pi^3}{64} \\
& \sum_{n=0}^{\infty} \frac{1}{(4n+3)^3} = \frac{7}{16}\zeta(3) - \frac{\pi^3}{64} \\
& \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32} \\
& \sum_{n=1}^{\infty} \frac{(4n)!}{(n-1)!(n+2)!(2n)!2^{6n}} = \frac{8\sqrt{2}}{35\pi} \\
& \sum_{n=0}^{\infty} \frac{(-1)^n}{1+\pi^2 n^2} = \frac{e^2+2e-1}{2(e+1)(e-1)} \\
& \sum_{n=0}^{\infty} \frac{(-1)^n(1+e^2\pi^2 n^2)}{(\pi^2 n^2+1)^2} = \frac{1}{2} \\
& \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} = \frac{1}{4} \\
& \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n}{(2n+1)2^{4n}} = 2\log \varphi \\
& \sum_{0 < n_1, n_2, n_3} \frac{1}{n_1 n_2 n_3 (n_1 + n_2 + n_3)} = \frac{\pi^4}{15} \\
& \sum_{n=1}^{\infty} \frac{H_n^2}{n^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3) \\
& \sum_{0 \leq n_1 < n_2} \frac{1}{(2n_1+1)^2(2n_2+1)^2} = \frac{\pi^4}{384} \\
& \sum_{n=0}^{\infty} \text{Li}_3 \left( -e^{-(2n+1)\pi} \right) = -\frac{\pi^3}{720} \\
& \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n^{2n}} = \frac{1}{2} \\
& \sum_{n=0}^{\infty} \frac{3n^2-1}{(2n+1)\binom{2n}{n}} = 0 \\
& \sum_{n=3}^{\infty} \frac{(-1)^{n-1}F_{2n}}{F_n^4-1} = \frac{1}{3} \\
& \sum_{n=1}^{\infty} (-1)^n \log \left( 1 + \frac{1}{n} \right) = \log \frac{2}{\pi} \\
& \sum_{n=0}^{\infty} \binom{2n}{n} \frac{F_n}{2^{3n}} = \sqrt{\frac{2}{5}}
\end{aligned}$$

$$\begin{aligned}
& \sum_{n=0}^{\infty} \binom{2n}{n}^2 \frac{H_n}{(n+1)2^{4n}} = 4 - \frac{16 \log 2}{\pi} \\
& \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2H_{2n} - H_n}{(2n+1)2^{2n}} = \pi \log 2 \\
& \sum_{n=0}^{\infty} \frac{H_n - H_{2n} + \log 2}{2n+1} = \frac{\pi^2}{12} \\
& \sum_{\substack{0 < n_1, n_2 \\ n_1 \neq n_2}} \frac{1}{n_1(n_1^2 - n_2^2)} = \frac{3}{4} \zeta(3) \\
& \sum_{n=1}^{\infty} \binom{2n}{n} \frac{(\sqrt{5} - 2)^n}{n^2} = \frac{2}{15} \pi^2 - 3 \log^2 \varphi \\
& \sum_{n=1}^{\infty} \frac{(3n+1)2^{4n}}{n^2(2n+1)^2 \binom{2n}{n}^2} = \pi \\
& \sum_{n=1}^{\infty} \frac{H_n^2}{\varphi^{2n-1}} = \frac{\pi^2}{15} \\
& \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\varphi^{n-1}} H_n^2 = \frac{\pi^2}{15} - \frac{3}{2} \log^2 \varphi \\
& \sum_{0 < n_1 < n_2} \binom{2n_2}{n_2} \frac{1}{(2n_1+1)^2 n_2 2^{2n_2}} = 2\pi \beta(2) - \frac{7}{2} \zeta(3) \\
& \sum_{n=1}^{\infty} \frac{n^3 \pi^3 \cosh n\pi}{\sinh^3 n\pi} = \frac{\Gamma\left(\frac{1}{4}\right)^8}{1024\pi^7} - \frac{1}{16\pi^3} \\
& \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \varphi^{6n+3}} = \frac{\pi^2}{24} - \frac{3}{4} \log^2 \varphi \\
& \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \frac{H_n}{54^n} = \frac{\Gamma\left(\frac{1}{3}\right)^4}{8\sqrt[3]{2}\pi^2} \left(2\pi\sqrt{3} - 9 \log 3\right) \\
& \sum_{n=1}^{\infty} \frac{(-1)^n \cosh n\pi}{\sinh^2 n\pi} = -\frac{1}{12} \\
& \sum_{n=1}^{\infty} \binom{2n}{n} \frac{H_{2n}}{n2^{2n}} = \frac{5}{12} \pi^2 \\
& \sum_{0 \leq n_1 < n_2} \binom{2n_2}{n_2} \frac{1}{(2n_1+1)n_2^2 2^{2n_2}} = \frac{7}{2} \zeta(3) - \frac{\pi^2}{2} \log 2
\end{aligned}$$

## 4 一般化

$$\begin{aligned}
\int_{-\infty}^{\infty} \operatorname{sech}^n x dx &= \frac{\sqrt{\pi} \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \\
\int_0^{\infty} \frac{\cos ax \sin bx}{x} dx &= \frac{\pi}{2} \\
\int_0^{\frac{\pi}{2}} \tan^n x dx &= \frac{\pi}{2} \csc \frac{n}{2} \pi \\
\int_0^{\frac{\pi}{2}} \frac{1 - \cos^{2z} x}{\sin x \cos^z x} dx &= \frac{\pi}{2} \tan \frac{z}{2} \pi \\
\int_0^{\infty} \frac{x^{-(1-z)-x^{-z}}}{x+1} dx &= 0 \\
\int_0^{\infty} \frac{x^{z-1}}{x+1} dx &= \frac{\pi}{\sin \pi z} \\
\int_0^1 \frac{\arctan x^n}{x} dx &= \frac{1}{n} \beta(2) \\
\sum_{k=1}^n \zeta(2k) \zeta(4n+2-2k) &= \frac{4n+3}{4} \zeta(4n+2) \\
\int_0^1 x^{z-1} \log \log \frac{1}{x} dx &= -\frac{\gamma + \log z}{z} \\
\int_0^{\infty} \frac{e^{-kx} \sin x}{1+x^2} dx &= \int_0^{\infty} \frac{\sin x}{1+(x+k)^2} dx \\
\int_0^{\frac{\pi}{2}} \frac{1}{a + \cos^2 x} dx &= \frac{\pi}{2\sqrt{a(1+a)}} \\
\int_0^{\infty} \sin x^a dx &= \Gamma\left(1 + \frac{1}{n}\right) \sin \frac{\pi}{2n} \\
\sum_{n=1}^{\infty} \frac{1}{(n)_{m+1}} &= \frac{1}{mm!} \\
\int_0^{\infty} x^n \log \left(1 - \frac{e^{-x}}{2}\right) dx &= -n! \operatorname{Li}_{n+2} \left(\frac{1}{2}\right) \\
\sum_{k=1}^n \binom{2n-1}{2k-1} \frac{5^{k-1}}{4^{n-1}} &= F_{2n-1} \\
\int_0^{\frac{\pi}{2}} \log(a^2 + \cos^2 x) dx &= \pi \operatorname{arsinh} a - \pi \log 2 \\
\int_0^{\frac{\pi}{2}} \frac{\operatorname{arcoth} \sqrt{1+a \sec^2 x}}{\sqrt{1+a \sec^2 x}} dx &= \frac{\pi}{2\sqrt{a}} \arcsin \sqrt{\frac{a}{a+1}} \\
\int_0^{\infty} \frac{1}{1+x^n} dx &= \frac{\pi}{n} \csc \frac{\pi}{n} \\
\sum_{k=1}^n \sin^2 \left(\frac{a-k}{n} \pi\right) &= \frac{n}{2} \\
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{H_{2n}}{2^{2n}} x^n &= \frac{1}{\sqrt{1-x}} \log \left(\frac{1+\sqrt{1-x}}{2(1-x)}\right)
\end{aligned}$$



$$\sum_{k=0}^n \frac{\binom{n}{k}^2}{\binom{2n}{2k}} = \frac{2^{2n}}{\binom{2n}{n}}$$

$$\sum_{k=0}^{n-1} \binom{2k}{k} \frac{1}{(n-k)2^{2k}} = \binom{2n}{n} \frac{2H_{2n} - H_n}{2^{2n}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \zeta(4n+2)}{2^{2n-1} \pi^{4n+2}} x^{4n+1} = \frac{\sinh x - \sin x}{\cosh x - \cos x}$$

$$\sum_{0 \leq n_1 < n_2} \binom{2n_2}{n_2} \frac{1}{(2n_1+1)2^{2n_2}} x^{n_2} = \frac{1}{\sqrt{1-x}} \log \frac{1}{\sqrt{1-x}}$$

$$\sum_{0 \leq n_1 \leq n_2} \frac{x^{n_2}}{n_1!} = \frac{e^x}{1-x}$$

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{2n}}{3^{3n}} \sin^{2n} x = \frac{\cos \frac{x}{3}}{\cos x}$$

$$\sum_{k=0}^n \frac{k}{n^k (n-k)!} = \frac{1}{(n-1)!}$$

$$\int_0^{\frac{\pi}{2}} \cos^s x \cos s x dx = \frac{\pi}{2^{1+s}}$$

$$\sum_{m=0}^{n-1} \binom{2m}{m} \frac{1}{(n+m)2^{2m}} = \frac{2^{2n}}{2n \binom{2n}{n}}$$

$$\sum_{0 < a_1, a_2, \dots, a_m} \frac{n!}{\prod_{k=1}^m a_k (\sum_{k=1}^m a_k)^n} = \sum_{0 < a_1, a_2, \dots, a_n} \frac{m!}{\prod_{k=1}^n a_k (\sum_{k=1}^n a_k)^m}$$

$$\sum_{0 \leq n_0 \leq n_1 \leq \dots \leq n_k} \binom{2n_0}{n_0}^2 \frac{1}{2^{4n_0}} \prod_{i=1}^k \frac{1}{(2n_i+1)^2} = \frac{4}{\pi} \left(1 - \frac{1}{2^{2k+1}}\right) \zeta(2k+1)$$

$$\sum_{0 \leq n_0 \leq n_1 \leq \dots \leq n_k} \frac{\binom{2n_0}{n_0}^2}{\binom{2n_k}{n_k}} \frac{1}{2^{4n_0-2n_k}} \prod_{i=1}^k \frac{1}{(2n_i+1)^2} = 2 \left(1 - \frac{1}{2^{2k}}\right) \zeta(2k)$$