今まで見つけた式

神鳥奈紗

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1　定義

式中に用いられる関数や定数を定義しておく。

$$
\begin{aligned}
\varphi &\overset{\text{def.}}{=} \frac{1+\sqrt{5}}{2} \\
\gamma &\overset{\text{def.}}{=} \lim_{n\to\infty} (H_n - \log n) \\
H_n &\overset{\text{def.}}{=} \sum_{k=1}^{n} \frac{1}{k} \\
F_n &\overset{\text{def.}}{=} \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}} \\
\zeta(n) &\overset{\text{def.}}{=} \sum_{k=1}^{\infty} \frac{1}{k^n} \\
\beta(n) &\overset{\text{def.}}{=} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n} \\
\Gamma(x) &\overset{\text{def.}}{=} \int_0^{\infty} t^{x-1} e^{-t} dt \\
\mathrm{Li}_n(x) &\overset{\text{def.}}{=} \sum_{k=1}^{\infty} \frac{x^k}{k^n} \\
(n)_m &\overset{\text{def.}}{=} \prod_{k=0}^{m-1} (n+k)
\end{aligned}
$$

また、定義が複雑なので省略するが $\gamma(q)$ を q-Euler 定数とする。

2 積分

$$
\begin{aligned}
\int_0^\infty \frac{x}{2e^x-1}dx =& \frac{\pi^2}{12} - \frac{1}{2}\log^2 2 \\
\int_0^{\frac{\pi}{2}} \cos\log\tan x dx =& \frac{\pi}{2}\operatorname{sech}\frac{\pi}{2} \\
\int_0^{\frac{\pi}{2}} \frac{\sin\log\tan x}{\log\tan x}dx =& \arccos\operatorname{sech}\frac{\pi}{2} \\
\int_0^{\frac{\pi}{2}} \frac{\cos\log\tan x\log\cos x}{\sin^2 x}dx =& -\frac{\pi}{4\cosh\frac{\pi}{2}} \\
\int_0^\infty \frac{e^{\frac{1}{6}x}+e^{\frac{5}{6}x}-2e^{\frac{1}{2}x}}{x(e^x-1)}dx =& \log 2 \\
\int_0^\infty \frac{x^2e^{\frac{1}{2}x}}{e^x+1}dx =& \frac{\pi^3}{2} \\
\int_0^1 \frac{x^{19}-1}{\sqrt{x}\log x}dx =& \log 39 \\
\int_0^\infty \frac{1-\tanh x}{\sqrt{\tanh x}}dx =& \frac{\pi}{2} \\
\int_0^{\frac{\pi}{2}} \operatorname{arsinh}\sin x dx =& \beta(2) \\
\int_0^{\frac{\pi}{2}} \frac{\cos^2 x\log\sin x\log\cos x}{\sin x}dx =& -\frac{\pi^4}{576} \\
\int_0^\infty \log\left(1-e^{-x}\right)dx =& -\frac{\pi^2}{6} \\
\int_0^1 x\log\log\frac{1}{x}dx =& -\frac{\gamma+\log 2}{2} \\
\int_0^1 \frac{\log x\log(1+x)}{x(1+x)}dx =& \zeta(3) \\
\int_0^{\frac{\pi}{2}} \frac{\log\cos x}{\tan x}dx =& \frac{2}{3}\pi^2 \\
\int_0^{\frac{1}{2}} \frac{\operatorname{arsinh}^2 x}{x}dx =& \frac{1}{10}\zeta(3) \\
\int_0^1 \frac{x\operatorname{artanh}^3 x}{3+x^2}dx =& \frac{17}{5184}\pi^4 \\
\int_0^\infty \frac{x\log\tanh x}{\tanh x}dx =& -\frac{7}{16}\zeta(3) - \frac{\pi^2}{8}\log 2 \\
\int_0^\infty x\log\tanh x dx =& \frac{7}{16}\zeta(3) \\
\int_0^\infty \frac{\arctan x^2}{x^2}dx =& \frac{\pi}{\sqrt{2}} \\
\int_0^\infty \frac{\arctan^2 x}{x^2}dx =& \pi\log 2 \\
\int_0^{\frac{\pi}{2}} x\log 2\sin x dx =& \frac{7}{16}\zeta(3) \\
\int_0^{\frac{\pi}{2}} x^2\log 2\sin x dx =& \frac{3\pi}{16}\zeta(3)
\end{aligned}
$$

$$\int_0^1 \frac{\log^2 x}{1+x} dx = \frac{3}{2}\zeta(3)$$
$$\int_0^\infty \frac{\log^2 x}{1+x^2} dx = \frac{\pi^3}{8}$$
$$\int_0^\infty e^{-x} \log^2 x dx = \gamma^2 + \zeta(2)$$
$$\int_0^\infty \frac{\sin x}{1+e^{\pi x}} dx = \frac{e^2-2e-1}{2(e+1)(e-1)}$$
$$\int_0^{\frac{\pi}{2}} \frac{\arctan \frac{\tan x}{\cos x}}{\tan x} dx = \frac{4}{3}\beta(2)$$
$$\int_0^\infty \frac{\sin^3 x}{x(1+\sin^4 x)} dx = \frac{\pi}{4}\sqrt{\sqrt{2}-1}$$
$$\int_0^\infty \frac{\sin x \log^2 \sin x}{\cos^3 x} dx = \frac{\pi^2}{24}$$
$$\int_0^{\frac{\pi}{2}} \frac{\sin x \log^3 \sin x}{\cos^3 x dx} = -\frac{3}{8}\zeta(3)$$
$$\int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{3+\cos 4x} dx = \frac{\pi}{4\sqrt{2}}$$
$$\int_{-\infty}^\infty \frac{\operatorname{arcoth}\sqrt{x^2+\pi^2}}{\sqrt{x^2+\pi^2}} dx = \pi \arcsin \frac{1}{\pi}$$
$$\int_0^{\frac{\pi}{2}} \frac{\operatorname{arcoth}\sqrt{1+2\tan^2 x}}{\sqrt{1+2\tan^2 x}} dx = \frac{\pi^2}{8}$$
$$\int_0^1 \left(\frac{x}{\log x} + \frac{1}{1-x}\right) dx = \gamma + \log 2$$
$$\int_0^\infty \frac{1}{x+1} \log\left(\frac{x^2+2x+1}{x^2+x+1}\right) dx = \frac{\pi^2}{18}$$
$$\int_0^\pi \log\left(\frac{5}{4} + \cos x\right) dx = 0$$
$$\int_0^{\frac{\pi}{2}} \log\left(\frac{9}{16} + \cos^2 x\right) dx = 0$$
$$\int_0^1 \frac{\operatorname{artanh}\sqrt{1-x^2}}{1-x} dx = \frac{3}{8}\pi^2$$
$$\int_0^{\frac{1}{\sqrt{3}}} \frac{x \arctan x}{(1-x^2)\sqrt{1-2x^2}} d = \frac{\pi^2}{96}$$
$$\int_0^{\frac{\pi}{2}} \frac{x}{\sin^8 x + \cos^8 x} dx = \frac{\sqrt{10-\sqrt{2}}}{8}\pi^2$$
$$\int_0^{\frac{\pi}{2}} x^2 \log \tan x dx = \frac{7}{16}\pi\zeta(3)$$
$$\int_0^{\frac{\pi}{2}} \log^4 \tan x dx = \frac{5\pi^5}{64}$$
$$\int_0^\infty \frac{x \log \sinh x}{\sinh x \cosh x} dx = -\frac{7}{16}\zeta(3)$$
$$\int_0^{\sqrt{\frac{2}{3}}} \frac{\arctan\sqrt{1-x^2}}{2+x^2} dx = \frac{11\pi^2}{288\sqrt{2}}$$
$$\int_1^{\sqrt{2}} \frac{\operatorname{artanh}\sqrt{2-x^2}}{1+x} dx = \frac{\pi^2}{48}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos^4 x} dx = \frac{\sqrt{1+\sqrt{2}}}{4}\pi$$

$$\int_0^{\frac{\pi}{4}} x \operatorname{artanh} \tan x dx = \frac{7}{32}\zeta(3)$$

$$\int_0^{\frac{\pi}{2}} \frac{\arctan \frac{\cos x}{2}}{\sqrt{\sin x - \sin^2 x}} dx = \frac{\pi}{\sqrt{2}} \operatorname{artanh} \sqrt{\frac{5-\sqrt{5}}{8}}$$

$$\int_0^1 \frac{\operatorname{artanh}^2 x}{x^2+3} dx = \frac{\pi^3}{81\sqrt{3}}$$

$$\int_0^{\pi} \frac{\sinh x - \sin x}{\cosh x - \cos x} dx = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \frac{\log \cos x}{\sin x} \log\left(\frac{1+\sin x}{1-\sin x}\right) dx = -2\pi\beta(2)$$

$$\int_0^1 \frac{\operatorname{artanh}^2 x}{x^2} dx = \frac{\pi^2}{6}$$

$$\int_0^1 \frac{x^2}{\operatorname{artanh}^2 x} dx = 124\frac{\zeta(5)}{\pi^4} - \frac{14}{3}\frac{\zeta(3)}{\pi^2}$$

$$\int_0^{\pi} \arctan \frac{\log \sin x}{x} dx = -\pi \arctan \frac{2\log 2}{\pi}$$

$$\int_{0<x_1<x_2<\frac{\pi}{4}} \cot\left(x_1 + \frac{\pi}{4}\right) \cot x_2 dx_1 dx_2 = \frac{5}{96}\pi^2$$

$$\int_0^1 \frac{\gamma(x)}{x} dx = \frac{\pi^4}{36} - \frac{\pi^2}{6}$$

$$\Re \int_0^{\infty} H_{\frac{i}{x}} dx = \frac{\pi^3}{12}$$

$$\lim_{t\to\infty} t \int_t^{\infty} e^{-(x+t)(x-t)} dx = \frac{1}{2}$$

3 級数

$$\begin{aligned}
\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} &= \frac{3}{4}\pi \\
\sum_{n=1}^{\infty} \arctan \frac{1}{2n^2} &= \frac{1}{4}\pi \\
\sum_{n=0}^{\infty} \frac{1}{(4n+1)^3} &= \frac{7}{16}\zeta(3) + \frac{\pi^3}{64} \\
\sum_{n=0}^{\infty} \frac{1}{(4n+3)^3} &= \frac{7}{16}\zeta(3) - \frac{\pi^3}{64} \\
\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} &= \frac{\pi^3}{32} \\
\sum_{n=1}^{\infty} \frac{(4n)!}{(n-1)!(n+2)!(2n)!2^{6n}} &= \frac{8\sqrt{2}}{35\pi} \\
\sum_{n=0}^{\infty} \frac{(-1)^n}{1+\pi^2 n^2} &= \frac{e^2+2e-1}{2(e+1)(e-1)} \\
\sum_{n=0}^{\infty} \frac{(-1)^n(1+e^2\pi^2 n^2)}{(\pi^2 n^2+1)^2} &= \frac{1}{2} \\
\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} &= \frac{1}{4} \\
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n}{(2n+1)2^{4n}} &= 2\log\varphi \\
\sum_{0<n_1,n_2,n_3} \frac{1}{n_1 n_2 n_3 (n_1+n_2+n_3)} &= \frac{\pi^4}{15} \\
\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} &= \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3) \\
\sum_{0\le n_1<n_2} \frac{1}{(2n_1+1)^2(2n_2+1)^2} &= \frac{\pi^4}{384} \\
\sum_{n=0}^{\infty} \mathrm{Li}_3\left(-e^{-(2n+1)\pi}\right) &= -\frac{\pi^3}{720} \\
\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n^{2n}} &= \frac{1}{2} \\
\sum_{n=0}^{\infty} \frac{3n^2-1}{(2n+1)\binom{2n}{n}} &= 0 \\
\sum_{n=3}^{\infty} \frac{(-1)^{n-1}F_{2n}}{F_n^4-1} &= \frac{1}{3} \\
\sum_{n=1}^{\infty} (-1)^n \log\left(1+\frac{1}{n}\right) &= \log\frac{2}{\pi} \\
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{F_n}{2^{3n}} &= \sqrt{\frac{2}{5}}
\end{aligned}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{H_n}{(n+1)2^{4n}} = 4-\frac{16\log 2}{\pi}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2H_{2n}-H_n}{(2n+1)2^{2n}} = \pi\log 2$$

$$\sum_{n=0}^{\infty}\frac{H_n-H_{2n}+\log 2}{2n+1} = \frac{\pi^2}{12}$$

$$\sum_{\substack{0<n_1,n_2\\ n_1\neq n_2}}\frac{1}{n_1(n_1^2-n_2^2)} = \frac{3}{4}\zeta(3)$$

$$\sum_{n=1}^{\infty}\frac{(3n+1)2^{4n}}{n^2(2n+1)^2\binom{2n}{n}^2} = \pi$$

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{\left(\sqrt{5}-2\right)^n}{n^2} = \frac{2}{15}\pi^2-3\log^2\varphi$$

$$\sum_{n=1}^{\infty}\frac{H_n^2}{\varphi^{2n-1}} = \frac{\pi^2}{15}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\varphi^{n-1}}H_n^2 = \frac{\pi^2}{15}-\frac{3}{2}\log^2\varphi$$

$$\sum_{0<n_1<n_2}\binom{2n_2}{n_2}\frac{1}{(2n_1+1)^2n_2 2^{2n_2}} = 2\pi\beta(2)-\frac{7}{2}\zeta(3)$$

$$\sum_{n=1}^{\infty}\frac{n^3\pi^3\cosh n\pi}{\sinh^3 n\pi} = \frac{\Gamma\left(\frac{1}{4}\right)^8}{1024\pi^7}-\frac{1}{16\pi^3}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\varphi^{6n+3}} = \frac{\pi^2}{24}-\frac{3}{4}\log^2\varphi$$

$$\sum_{n=0}^{\infty}\frac{(3n)!}{n!^3}\frac{H_n}{54^n} = \frac{\Gamma\left(\frac{1}{3}\right)^4}{8\sqrt[3]{2}\pi^2}\left(2\pi\sqrt{3}-9\log 3\right)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\cosh n\pi}{\sinh^2 n\pi} = -\frac{1}{12}$$

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{H_{2n}}{n2^{2n}} = \frac{5}{12}\pi^2$$

$$\sum_{0\leq n_1<n_2}\binom{2n_2}{n_2}\frac{1}{(2n_1+1)n_2^2 2^{2n_2}} = \frac{7}{2}\zeta(3)-\frac{\pi^2}{2}\log 2$$

4　一般化

$$\int_{-\infty}^{\infty} \operatorname{sech}^n x dx = \frac{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$$

$$\int_0^{\infty} \frac{\cos ax \sin bx}{x} dx = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \tan^n x dx = \frac{\pi}{2} \csc \frac{n}{2}\pi$$

$$\int_0^{\frac{\pi}{2}} \frac{1-\cos^{2z} x}{\sin x \cos^z x} dx = \frac{\pi}{2} \tan \frac{z}{2}\pi$$

$$\int_0^{\infty} \frac{x^{-(1-z)-x^{-z}}}{x+1} dx = 0$$

$$\int_0^{\infty} \frac{x^{z-1}}{x+1} dx = \frac{\pi}{\sin \pi z}$$

$$\int_0^{1} \frac{\arctan x^n}{x} dx = \frac{1}{n}\beta(2)$$

$$\int_0^{1} x^{z-1} \log\log \frac{1}{x} dx = -\frac{\gamma + \log z}{z}$$

$$\int_0^{\infty} \frac{e^{-kx}\sin x}{1+x^2} dx = \int_0^{\infty} \frac{\sin x}{1+(x+k)^2} dx$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{a+\cos^2 x} dx = \frac{\pi}{2\sqrt{a(1+a)}}$$

$$\int_0^{\infty} \sin x^a dx = \Gamma\left(1+\frac{1}{n}\right) \sin \frac{\pi}{2n}$$

$$\int_0^{\infty} x^n \log\left(1-\frac{e^{-x}}{2}\right) dx = -n!\operatorname{Li}_{n+2}\left(\frac{1}{2}\right)$$

$$\int_0^{\frac{\pi}{2}} \log\left(a^2+\cos^2 x\right) dx = \pi \operatorname{arsinh} a - \pi \log 2$$

$$\int_0^{\frac{\pi}{2}} \frac{\operatorname{arcoth}\sqrt{1+a\sec^2 x}}{\sqrt{1+a\sec^2 x}} dx = \frac{\pi}{2\sqrt{a}} \arcsin\sqrt{\frac{a}{a+1}}$$

$$\int_0^{\infty} \frac{1}{1+x^n} dx = \frac{\pi}{n} \csc \frac{\pi}{n}$$

$$\int_0^{\frac{\pi}{2}} \cos^s x \cos sx dx = \frac{\pi}{2^{1+s}}$$

$$\sum_{n=1}^{\infty}\frac{1}{(n)_{m+1}}=\frac{1}{mm!}$$

$$\sum_{0\le n_1,n_2}\frac{1}{(n_1+n_2+1)^{k+1}}=\zeta(k)$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\zeta(4n+2)}{2^{2n-1}\pi^{4n+2}}x^{4n+1}=\frac{\sinh x-\sin x}{\cosh x-\cos x}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{H_{2n}}{2^{2n}}x^n=\frac{1}{\sqrt{1-x}}\log\left(\frac{1+\sqrt{1-x}}{2(1-x)}\right)$$

$$\sum_{0\le n_1<n_2}\binom{2n_2}{n_2}\frac{1}{(2n_1+1)2^{2n_2}}x^{n_2}=\frac{1}{\sqrt{1-x}}\log\frac{1}{\sqrt{1-x}}$$

$$\sum_{0\le n_1\le n_2}\frac{x^{n_2}}{n_1!}=\frac{e^x}{1-x}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{n^2}{(n)_a2^{4n}}=\frac{\Gamma(a-2)}{4\Gamma\left(a-\frac{1}{2}\right)^2}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{n^3}{(n)_a2^{4n}}=\frac{(4a-3)\Gamma(a-3)}{16\Gamma\left(a-\frac{1}{2}\right)^3}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{2n}}{3^{3n}}\sin^{2n}x=\frac{\cos\frac{x}{3}}{\cos x}$$

$$\sum_{k=1}^{n}\zeta(2k)\zeta(4n+2-2k)=\frac{4n+3}{4}\zeta(4n+2)$$

$$\sum_{k=1}^{n}\sin^2\left(\frac{a-k}{n}\pi\right)=\frac{n}{2}$$

$$\sum_{k=0}^{n}\frac{k}{n^k(n-k)!}=\frac{1}{(n-1)!}$$

$$\sum_{k=1}^{n}\binom{2n-1}{2k-1}\frac{5^{k-1}}{4^{n-1}}=F_{2n-1}$$

$$\sum_{\mathrm{lcm}(i,j)=n}1=\sum_{i|n^2}1$$

$$\sum_{m=0}^{n-1}\binom{2m}{m}\frac{1}{(n+m)2^{2m}}=\frac{2^{2n}}{2n\binom{2n}{n}}$$

$$\sum_{k=0}^{n}\frac{\binom{n}{k}^2}{\binom{2n}{2k}}=\frac{2^{2n}}{\binom{2n}{n}}$$

$$\sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{(n-k)2^{2k}}=\binom{2n}{n}\frac{2H_{2n}-H_n}{2^{2n}}$$

$$\sum_{0<a_1,a_2,\dots,a_m}\frac{n!}{\prod_{k=1}^{m}a_k\left(\sum_{k=1}^{m}a_k\right)^n}=\sum_{0<a_1,a_2,\dots,a_n}\frac{m!}{\prod_{k=1}^{n}a_k\left(\sum_{k=1}^{n}a_k\right)^m}$$

$$\sum_{0\le n_0\le n_1\le\ldots\le n_k}\binom{2n_0}{n_0}^2\frac{1}{2^{4n_0}}\prod_{i=1}^{k}\frac{1}{(2n_i+1)^2}=\frac{4}{\pi}\left(1-\frac{1}{2^{2k+1}}\right)\zeta(2k+1)$$

$$\sum_{0\le n_0\le n_1\le\ldots\le n_k}\frac{\binom{2n_0}{n_0}^2}{\binom{2n_k}{n_k}}\frac{1}{2^{4n_0-2n_k}}\prod_{i=1}^{k}\frac{1}{(2n_i+1)^2}=2\left(1-\frac{1}{2^{2k}}\right)\zeta(2k)$$