Signal & System HW1

Problem 1

(a) $y(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau$

① $y(t)$ consider previous signal → not memoryless

② $x(t) \longrightarrow \boxed{h} \longrightarrow \int_{-\infty}^{2t} x(\tau)\,d\tau = y(t)$ → not time-invariant

let $x'(t) = x(t-\alpha) \longrightarrow \boxed{h} \longrightarrow \int_{-\infty}^{2t} x'(\tau)\,d\tau = \int_{-\infty}^{2t} x(\tau-\alpha)\,d\tau \neq y(t-\alpha)$

③ $x_1(t) \Rightarrow y_1(t) = \int_{-\infty}^{2t} x_1(\tau)\,d\tau$ let $x(t) = \alpha x_1(t) + \beta x_2(t)$

$x_2(t) \Rightarrow y_2(t) = \int_{-\infty}^{2t} x_2(\tau)\,d\tau$ $\Rightarrow y(t) = \int_{-\infty}^{2t} x(\tau)\,d\tau = \int_{-\infty}^{2t} \alpha x_1(\tau) + \beta x_2(\tau)\,d\tau$

$= \alpha y_1(t) + \beta y_2(t)$ → linear

④ Consider future signal → not causal

⑤ let $x(t) = u(t)$ (bounded input)

$y(t) = \int_{-\infty}^{2t} u(\tau)\,d\tau = \infty$ when $t \to \infty$ → not stable

(b) $y(t) = \cos^2(2t)\,x(t)$

① don't consider previous signal → memoryless

② let $x'(t) = x(t-\alpha)$. $y'(t) = \cos^2(2t)x'(t) = \cos^2(2t)x(t-\alpha) \neq y(t-\alpha)$ → not time-invariant

③ let $y_1(t) = \cos^2(2t)x_1(t)$, $y_2(t) = \cos^2(2t)x_2(t)$, $x(t) = \alpha x_1(t) + \beta x_2(t)$.

$y(t) = \cos^2(2t)x(t) = \cos^2(2t)[\alpha x_1(t) + \beta x_2(t)] = \alpha\cos^2(2t)x_1(t) + \beta\cos^2(2t)x_2(t)$

$= \alpha y_1(t) + \beta y_2(t)$ → linear

④ $y(t)$ doesn't depend on future $x(t)$ → causal

⑤ let $x(t) < \infty\ \forall t$, $y(t) = \cos^2(2t)x(t) < \infty$ $(0 \le \cos^2(2t) \le 1)$ → stable

(c) $y[n] = x[n-2] - 2x[n-6]$

① consider past → not memoryless

② let $x'[n] = x[n-\alpha]$ $y'[n] = x'[n-2] - 2x'[n-6] = x[n-\alpha-2] - 2x[n-\alpha-6] = y[n-\alpha]$ → time invariant

③ let $y_1[n] = x_1[n-2] - 2x_1[n-6]$, $y_2[n] = x_2[n-2] - 2x_2[n-6]$. $x[n] = \alpha x_1[n] + \beta x_2[n]$

$y[n] = x[n-2] - 2x[n-6] = \alpha x_1[n-2] + \beta x_2[n-2] - 2(\alpha x_1[n-6] + \beta x_2[n-6])$

$= \alpha(x_1[n-2] - 2x_1[n-6]) + \beta(x_2[n-2] - 2x_2[n-6]) = \alpha y_1[n] + \beta y_2[n]$ → linear

④ $y[n]$ doesn't depend on future → causal

⑤ LTI ⇒ $h[n] = \delta[n-2] - 2\delta[n-6]$ $\sum_{n=-\infty}^{\infty} |h[n]| = 3 < \infty$ → stable

(d) $y[n] = \begin{cases} x[n-1] & n \geq 1 \\ 0, & n=0 \\ x[n] & n \leq -1 \end{cases}$

① Considers past when $n \geq 1$ → not memoryless

④ doesn't consider future → causal

② let $x'[n] = x[n-\alpha]$ $y'[n] = \begin{cases} x'[n-1] = x[n-\alpha-1], & n \geq 1 \\ 0, & n=0 \\ x'[n] = x[n-\alpha], & n \leq -1. \end{cases} \neq y[n-\alpha] = \begin{cases} x[n-\alpha-1], & n-\alpha \geq 1 \\ 0, & n-\alpha = 0 \\ x[n-\alpha], & n-\alpha \leq -1 \end{cases}$

→ not time-invariant.

③ let $x[n] = \alpha x_1[n] + \beta x_2[n]$.

if $n \geq 1$, $y[n] = x[n-1] = \alpha x_1[n-1] + \beta x_2[n-1] = \alpha y_1[n] + \beta y_2[n]$

if $n=0$. $y[n] = 0 = \alpha \cdot 0 + \beta \cdot 0 = \alpha y_1[n] + \beta y_2[n]$. → linear.

if $n \leq -1$, $y[n] = x[n] = \alpha x_1[n] + \beta x_2[n] = \alpha y_1[n] + \beta y_2[n]$

⑤ if x is bounded, i.e. $x[n] < \infty\ \forall n \Rightarrow y[n] < \infty\ \forall n$ → stable

Problem 2

(a) $x(t) = \cos(W_n t)\cos(\sin(W_m t))$

fundamental period for $\cos(w_n t)$: $T_1 = \frac{2\pi}{W_n}$

for $\cos(\sin(W_m t))$: $T_2 = \frac{2\pi}{W_m} \cdot \frac{1}{2} = \frac{\pi}{W_m}$

$x(t)$ is periodic if there exist T which is common multiple of $\frac{2\pi}{W_n}$ & $\frac{\pi}{W_m}$

$\therefore T = \frac{2\pi}{W_n} \cdot n = \frac{\pi}{W_m} \cdot m$. $n, m \in \mathbb{Z}$ $\frac{2W_m}{W_n} = \frac{m}{n} \in \mathbb{Q}$

$\therefore$ if $\frac{2W_m}{W_n} \in \mathbb{Q}$, $x(t)$ is periodic and fundamental period $= \frac{2\pi}{\gcd(W_n, 2W_m)}$

otherwise, $x(t)$ is not periodic

(b) $x[n] = \cos(\pi n) + \cos(2\pi\sqrt{3}\, n)$.

if $x[n]$ is periodic and has fundamental period T

$2\pi\sqrt{3} \cdot T = 2\pi \cdot m$. $T, m \in \mathbb{N}$, $\sqrt{3} = \frac{m}{T} \in \mathbb{Q}$ → ×

$\therefore$ $x[n]$ is not periodic

Problem 3

(a) $y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau)\, d\tau$

$$\frac{d}{dt} y(t) = \frac{d}{dt}\int_{-\infty}^{t} e^{-(t-\tau)} x(\tau)\, d\tau = e^{-(t-t)} x(t) + \int_{-\infty}^{t} \frac{d}{dt}\left(e^{-(t-\tau)} x(\tau)\right) d\tau$$

$$= x(t) - \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau)\, d\tau = x(t) - y(t).$$

$$\Rightarrow x(t) = y(t) + \frac{d}{dt} y(t).$$

$\therefore$ invertible, inverse system: $y(t) = x(t) + \frac{d}{dt} x(t)$

(b) $y[n] = x[n]x[n-2]$

Consider two signal $x_1[n] = \delta[n]$, $x_2[n] = \delta[n-1]$.

$y_1[n] = x_1[n]x_1[n-2] = \delta[n]\delta[n-2] = 0$. $\quad \because y_1[n] = y_2[n]$ for different $x_1[n]$, $x_2[n]$.

$y_2[n] = x_2[n]x_2[n-2] = \delta[n-1]\delta[n-3] = 0$. $\quad \rightarrow$ not invertible.

Problem 4

(a) $H^{-1}: y(t) = x(t+1)$.

$$x(t) - \boxed{H} \overset{y(t)}{-} \boxed{H^{-1}} - z(t).$$

$y(t) = x(t-1)$, $z(t) = y(t+1) = x(t)$

(b) $G^{-1}: y(t) = x(\frac{t}{2})$.

$$x(t) - \boxed{G} \overset{y(t)}{-} \boxed{G^{-1}} - z(t).$$

$y(t) = x(2t)$, $z(t) = y(\frac{t}{2}) = x(t)$.

(c).

$$x(t) \longrightarrow \boxed{H} \xrightarrow{y(t)} \boxed{G} \longrightarrow w(t).$$

$w(t) = y(2t) = x(2t-1)$.

(d)

$$x(t) \rightarrow \boxed{G^{-1}} \xrightarrow{y(t)} \boxed{H^{-1}} \rightarrow z(t)$$

$z(t) = y(t+1) = x(\frac{t+1}{2})$