THE MATHEMATICAL CONCEPT OF 2ND ORDER ORDINARY DIFFERENTIAL EQUATIONS

A CRISP AND CONCISE INTRODUCTION

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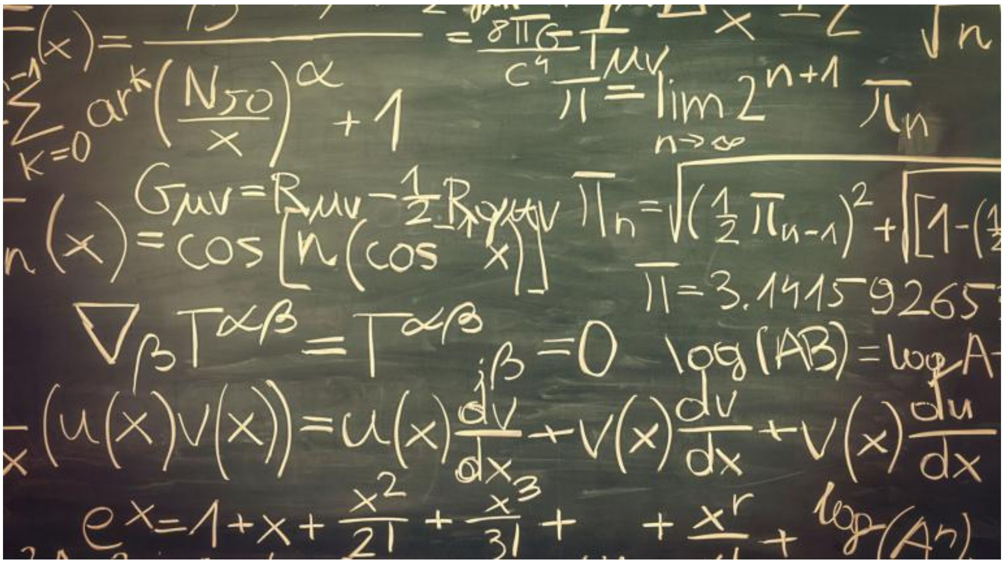

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1 Power Series, Taylor Series & Maclaurin Series

Consider the following function that is represented as a **power series.**

$\boxed{f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n}$ $f(a) = c_0$ $f^{'}(a) = c_1$

$f^{''}(a) = 2c_2 \rightarrow c_2 = \frac{1}{2}f^{''}(a)$ $f^{'''}(a) = 3 \times 2c_3 \rightarrow c_3 = \frac{1}{3!}f^{'''}(a)$ $\boxed{f^n(a) = n(n-1)(n-2)...1 \rightarrow c_n = \frac{1}{n!}f^n(a)}$.

If $f^n(x)$ exists at $x = a$, the **Taylor series** for $f(x)$ at a is:

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a) = f(a) + f^{'}(a)(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots$$

A **Maclaurin series** is a Taylor series expansion about 0. $\boxed{f(x) = f(0) + f^{'}(0)x + \frac{f^{''}(0)}{2!}x^2 + \ldots \frac{f^n(0)}{n!}x^n + \ldots}$

The series solution may or may not converge at $x = x_p$. To converge, for any ϵ, there is exists an N that satisfies $\boxed{|R_n(x_p)| = |s(x_p) - s_n(x_p)| < \epsilon \; \forall n > N}$ where $s_n(x)$ is the nth partial sum $\boxed{s_n(x) = a_0 + a_a(x-x_0) + \cdot + a_n(x-a_0)^n}$ and $R_n(x_p)$ is the remainder. $\boxed{R_n(x) = a_{n+1}(x-x_0)^{n+1} + a_{n+2}(x-x_0)^{n+2} + \ldots}$ The **convergence interval** is $|x - x_0| < R$ (radius of convergence). This means that in the case of convergence, we can approximate the sum $s(x_1)$ by $s_n(x_1)$ as accurately as we want by taking a large enough n. **$f(x)$ is called analytic at a point $x = x_0$** if it can be represented by a power series in powers of $x - x_0$ with a positive radius of convergence R.

2 2nd Order Linear ODE

The **standard form** of a **2nd Order ODE** is $\boxed{y^{''} + p(x)y^{'} + q(x)y = r(x)}$ It is linear in y, y' and y''. If $r(x) = 0$, the ODE is homogeneous, else it is non-homogeneous.

3 2nd Order Linear ODE

Choose $e^{\lambda x}$ as a solution and substitute in the the homogeneous ODE. $\boxed{(\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \rightarrow \lambda^2 + a\lambda + b = 0}$

$\boxed{\lambda = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right) \rightarrow y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$ This is the **superposition principle** or the **linearity principle** and is true **only for linear homogeneous ODE**. The arbitrary constants c_1 and c_2 are determined from the **initial conditions**, $y(x_0) = k_0$ and $y^{'}(x_0) = k_1$. y_1 and y_2 are **linearly independent** and are called **basis of solutions**. and $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ is a **general solution**. A **particular solution** is obtained if we assign specific values to c_1 and c_2.

4 Lagrange's Method of Reduction of Order

Consider a linear homogeneous 1st Order ODE in its **standard form** $y^{''} + p(x)y^{'} + q(x)y = 0$ If y_1 is a **basis solution**, we can find y_2 as follows. Let, $\boxed{y = y_2 = uy_1}$ $\boxed{y_2^{'} = u^{'}y_1 + uy_1^{'}}$ $\boxed{y_2^{''} = u^{''}y_1 + 2u^{'}y_1^{'} + uy_1^{''}}$

$\boxed{(u^{''}y_1 + 2u^{'}y_1^{'} + uy_1^{''}) + p(u^{'}y_1 + uy_1^{'}) + q(uy_1) = 0}$ $\boxed{y_1 u'' + (2y_1^{'} + py_1)u' + (y_1^{''} + py_1^{'} + qy_1)u = 0}$

Since y_1 is a solution, $\boxed{u^{''} + u^{'}\frac{2y_1^{'} + py_1}{y_1} = 0}$ $\boxed{\text{Let } u^{'} = U}$ $\boxed{U^{'} + U\left(\frac{2y_1^{'}}{y_1} + p\right) = 0}$ $\boxed{\frac{U'}{U} = -\left(\frac{2y_1^{'}}{y_1} + p\right)}$

$\boxed{\ln|U| = -2\ln|y_1| - \int p dx}$ $\boxed{\ln|Uy_1^2| = -\int p dx}$ $\boxed{U = \frac{1}{y_1^2}e^{\int -p dx}}$ $\boxed{u = \int U dx}$ $\boxed{y_2 = y_1 \int U dx = y_1 \int \frac{1}{y_1^2}e^{\int -p dx} dx}$

5 Homogeneous Linear ODE with Constant Coefficients

$\boxed{y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}, \lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)}$

Case 1: 2 Real Roots when $\boxed{a^2 - 4b > 0}$ General solution is $\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$

Case 2: $\boxed{\lambda = -a/2,\ y_1 = e^{-ax/2}}$. Determine y_2 using reduction method as stated above.

$\boxed{y_1^2 = e^{-ax}}$ $\boxed{U = e^{ax}e^{-ax} = 1}$ $\boxed{u = x}$ $\boxed{y_2 = xe^{-ax/2}}$ $\boxed{y = c_1e^{-ax/2} + c_2xe^{-ax/2} = (c_1 + c_2x)e^{-ax/2}}$

Case 3 $\boxed{\lambda = \frac{1}{2}a \pm iw, w = \sqrt{|a^2 - 4b|}}$ $\boxed{z = x + iy, e^z = e^{x+iy} = e^x e^{iy}}$ Using Maclaurin series:

$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots}$ $\boxed{e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \ldots = (1 - \frac{y^2}{2!} + \ldots) + i\,(y - \frac{y^3}{3!} + \ldots)}$

$\boxed{e^{iw} = \cos w + i\sin w}$ (Euler's formula or Euler's Identity) $\boxed{e^{iwx} = \cos wx + i\sin wx}$ (DeMoivre's theorem)

$\boxed{y = e^{-ax/2}(c_1\cos wx + c_2\sin wx)}$ (general solution) where $\boxed{c_1 \& c_2}$ are arbitrary constants.

6 Euler-Cauchy Equations

The **Euler-Cauchy** equations are of the form $\boxed{x^2y'' + axy' + by = 0}$ a, b are constants.

$\boxed{\text{Let } y = x^m \rightarrow y' = mx^{m-1} \rightarrow y'' = m(m-1)x^{m-2}}$ $\rightarrow$ $\boxed{x^2m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0}$

$\boxed{m^2 + (a-1)m + b = 0}$ $\boxed{m = \frac{1}{2}(1-a) \pm \sqrt{\frac{1}{4}(a-1)^2 - b}}$

Case 1: Roots are distinct $\boxed{y_1(x) = x^{m_1},\ y_2(x) = x^{m_2}}$ (basis solutions) $\boxed{y = c_1x^{m_1} + c_2x^{m_2}}$ (general solution)

Case 2: Double roots $\boxed{m = \frac{1}{2}(1-a), y_1 = x^{\frac{1}{2}(1-a)}, y_2 = uy_1, U = \frac{1}{y_1^2}e^{\int -pdx}, y_2 = uy_1 = y_1\int Udx \text{ where } p = \frac{a}{x}}$

7 Existence and Uniqueness of Solutions, Wronskian

Two solutions y_1 and y_2 are **linearly dependent** if their **Wronkskian** $\boxed{W(y_1, y_2) = y_1y_2' - y_2y_1' = 0}$ This is quite obvious, because if the solutions are dependent: $\boxed{y_1 = ky_2, \text{ or } y_2 = ly_1 \text{ where } k, l \text{ are constants}}$

$\boxed{W(y_1, y_2) = y_1y_2' - y_2y_1' = ky_2y_2' - y_2ky_2' = 0}$

The Wronksian is expressed as a **Wronski Determinant** $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

8 Non-homogeneous ODE

Consider $\boxed{y'' + p(x)y' + q(x)y = r(x)}$ The total solution is sum of homogeneous and particular solutions $\boxed{y(x) = y_h(x) + y_p(x)}$ $\boxed{y_h = c_1y_1 + c_2y_2}$ (general solution) y_p is a solution of the non-homogeneous equation without any constants. A particular solution is obtained by assigning specific values to the constants. The **Method of Undetermined Coeffcients** is an approach to finding a particular solution to nonhomogeneous ODEs. If the term in $r(x)$ contains the following term, the choice for $y_p(x)$ is given by:

$ke^{\gamma x} \rightarrow Ce^{\gamma x}$, $\quad Kx^n(n = 0, 1, \ldots) \rightarrow K_nx^n + K_{n-1}x^{n-1} + \ldots + K_1x + K_0$

$k\cos wx$ or $k\sin wx \rightarrow K\cos wx + M\sin wx$, $\quad ke^{\alpha x}\cos wx$ or $ke^{\alpha x}\sin wx \rightarrow e^{\alpha x}(K\cos wx + M\sin wx)$

9 Solution Variation of Parameters (Lagrange)

Particular solution for **standard form** ODEs $\boxed{y'' + p(x)y' + q(x)y = r(x)}$

Find a pair of functions $u_1(x)$ and $u_2(x)$ such that $\boxed{y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)}$

$\boxed{y_p'(x) = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'}$ and set constraint $\boxed{u_1'y_1 + u_2'y_2 = 0}$ $\boxed{y_p''(x) = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''}$

$\boxed{(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) = r}$

$\boxed{(y_1'' + py_1' + qy_1)u_1 + (y_2'' + py_2' + qy_2)u_2 + (u_1'y_1' + u_2'y_2') = r}$ $\rightarrow$ $\boxed{u_1'y_1' + u_2'y_2' = r}$

$\boxed{u_1' = -\frac{y_2r}{y_1y_2' - y_2y_1'} = -\frac{y_2r}{W}}$ $\boxed{u_2' = -\frac{y_1r}{y_1y_2' - y_2y_1'} = -\frac{y_1r}{W}}$ $\rightarrow$ $\boxed{y_p(x) = -y_1\int\frac{y_2r}{W}dx + y_2\int\frac{y_1r}{W}dx}$