THE MATHEMATICAL CONCEPT OF A LIMIT

A CRISP AND CONCISE INTRODUCTION

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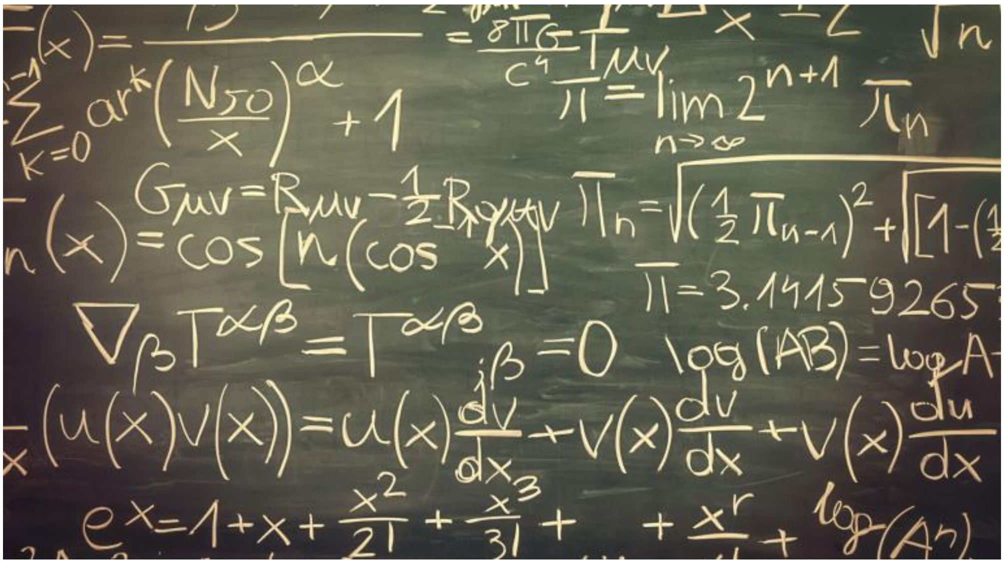

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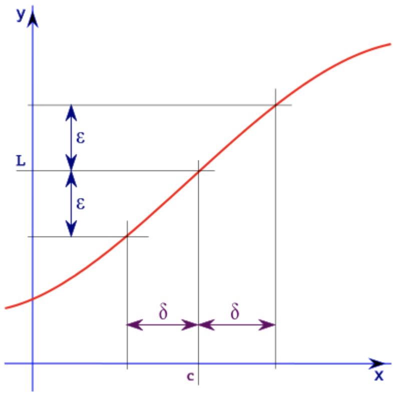
1 Definition of Limit

Consider a function $f(x)$ that is defined in a domain D which includes c. The function may or may not be defined at c. If, for all x that is *close* to c except for c, f(x) is arbitrarily close to a number L (as close to L as we like), then it is said that f approaches the limit L as x approaches c and is written as $\lim_{x \to c} f(x) = L$. If the function can be evaluated at c, the limit L is simply $f(c)$. But, what if the function is not evaluable at c? For example, the following function cannot be evalauted at $x = 1$.

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{x - 1} \text{ So, how do we compute: } \lim_{x \to 1} \frac{x^2 - 2x + 1}{x - 1} \text{ ?}$$

But this function can be simplified to: $f(x) = \frac{x^2 - 2x + 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$. Hence, $\lim_{x \to 1} f(x) = 2$.

2 Formal Definition of Limit



Let $f(x)$ be a function that is defined on an interval that contains x = c, except possibly at c. Then, $\lim_{x \to c} f(x) = L$ if for every number $\epsilon > 0$, there is some number $\delta > 0$ such that, when $0 < |x - a| < \delta, \quad |f(x) - L| < \epsilon$.

For any number $\epsilon > 0$ that we pick, one can go to the graph and sketch two horizontal lines at $L + \epsilon$ and $L - \epsilon$. Then there must be another number $\delta > 0$ that can be determined to enable us to add in two vertical lines in the graph $a + \delta$ and $a - \delta$.

3 Laws of Limit

Given L, M, c, k are real numbers and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then,

Sum Rule	$\lim_{x \to c}(f(x) + g(x)) = L + M$
Difference Rule	$\lim_{x \to c}(f(x) - g(x)) = L - M$
Constant Rule	$\lim_{x \to c}(kf(x)) = k.L$
Product Rule	$\lim_{x \to c}(f(x)g(x)) = LM$
Quotient Rule	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$
Power Rule	$\lim_{x \to c}[f(x)]^n = L^n \ (n > 0)$
Root Rule	$\lim_{x \to c} \sqrt[n]{(f(x)} = \sqrt[n]{(L)} \ (n > 0)$

4 Example 1

A simple example: $\lim_{x \to 3} \sqrt{(2x^3 + 10} = 8$

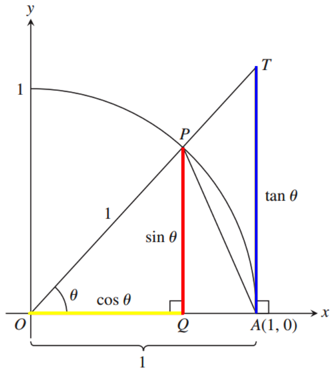
$$\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$

This function is not evaluable at $x = 0$. The standard trick is to multiply both numerator and denominator by the conjugate radical expression.

$$\frac{\sqrt{x^2+9} - 3}{x^2} = \frac{\sqrt{x^2+9} - 3}{x^2} . \frac{\sqrt{x^2+9} + 3}{\sqrt{x^2+9} + 3} = \frac{1}{\sqrt{x^2+9} + 3} = \lim_{x \to 0} \frac{1}{\sqrt{x^2+9} + 3} = \frac{1}{6}$$

5 Example 2

$$\lim_{\theta \to 0} \frac{Sin\ \theta}{\theta} = 1 \quad \text{where } \theta \text{ is in radians}$$



Consider the circle with a unit radius.

Area △ OAP < area sector OAP < area △ OAT

$\frac{1}{2} sin\ \theta \leq \pi 1^2 \left(\frac{\theta}{2\pi}\right) \leq \frac{1}{2} \tan\ \theta$ (θ is in radians)

$1 \leq \frac{\theta}{sin\ \theta} \leq \frac{1}{cos\ \theta}$ or $1 \geq \frac{sin\ \theta}{\theta} \geq cos\ \theta$

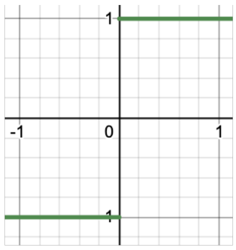
Hence, $\lim_{\theta \to 0} \frac{Sin\ \theta}{\theta} = 1$ where θ is in radians

Now consider the function $\frac{1}{sin\ \theta}$. Does it have a limit as $t \to \theta$ from either side? As θ approaches 0, its reciprocal, $1/x$, grows without bound and the values of function cycle repeatedly from -1 to 1. But is no single number L that the function's values stay increasingly close to as θ approaches 0. The function has neither a right-hand limit nor a lefthand limit at $\theta = 0$.

6 One Sided Limits

$$\lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0^-} f(x) = -1$$



7 Continuous Function

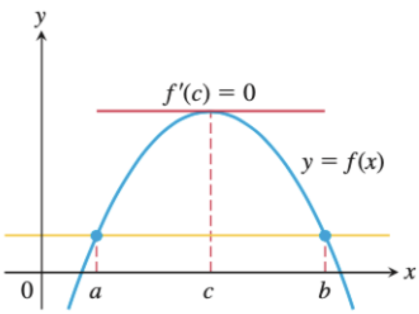
Function is right-continous at c (continuous from right) if $\lim_{x \to c^+} f(x) = f(c)$
Function is left-continous at c (continuous from left) if $\lim_{x \to c^-} f(x) = f(c)$
A function is continous at c if $\lim_{x \to c} f(x) = f(c)$ If a function is discontinuous at one or more points of its domain, it is called a discontinuous function.

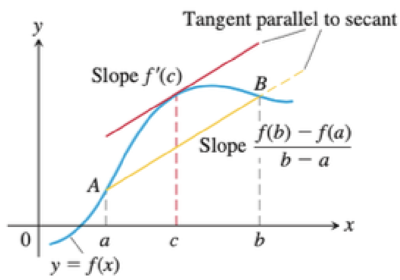
8 Infinite Limits

$\lim_{x \to 0^+} \frac{1}{x} = \infty$, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

Note that this does not mean that the limit exists as there is no real number such as ∞. It is simply a concise way of saying that the limit does not exist.

9 Theorems





Rolle's Theorem: If f is a continuous function on a closed interval $[a, b]$ and If $f(a) = f(b)$, then there is at least one point c in (a, b) where $f'(c) = 0$.

Mean Value Theorem: There is at least one number c in the interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$