THE MATHEMATICAL CONCEPT OF HIGHER ORDER DIFFERENTIAL EQUATIONS

A CRISP AND CONCISE INTRODUCTION

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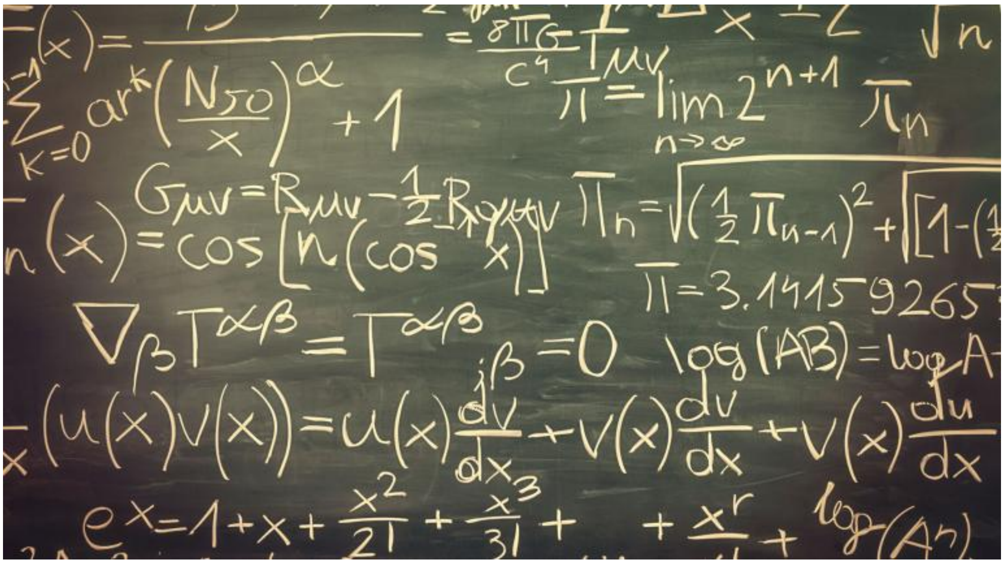

Contents

1 Higher Order ODE

The concepts of the 2nd Order ODE can be easily extended to higher order ODE which is of the form: $\boxed{y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y^{'} + p_0(x)y = r(x)}$ (standard form) For constant coefficients, we let $\boxed{y = e^{\lambda x}}$ $\rightarrow$ $\boxed{\lambda^{(n)} + a_{n-1}\lambda^{(n-1)} + \ldots + a_1\lambda + a_0 = 0}$ (characteristic equation). For n distinct roots, we have n distinct basis solutions: $\boxed{y = c_1 e^{\lambda^1 x} + c_2 e^{\lambda^2 x} + \ldots + c_n e^{\lambda^n x}}$. The **Wronkskian** is given by:

$$W(y_1, y_2, y_3, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1^{'} & y_2^{'} & \ldots & y_n^{'} \\ . & . & \ldots & . \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix} = E \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\ . & . & \ldots & . \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix} \text{where } E = e^{(\lambda_1 + \lambda_2 + \ldots + \lambda_n)x}$$

$W = 0$, only if the determinant $= 0$, **Vandermonde** or **Cauchy** determinant. $W \neq 0$, if and only if, all the n roots are different. If the roots are repeated, i.e., if λ is a real root of order m, then the corresponding basis solutions are: $\boxed{e^{\lambda x},\ xe^{\lambda x},\ x^2 e^{\lambda x},\ \ldots,\ x^{m-1}e^{\lambda x}}$. Complex roots must occur in conjugate pairs $\boxed{\lambda = \gamma \pm iw}$ since the coefficients of the ODE are real. $\boxed{y_1 = e^{\gamma x} cos\ wx,\ y_2 = e^{\gamma x} sin\ wx}$

Multple Complex roots: $\boxed{e^{\gamma x} cos\, wx, e^{\gamma x} sin\, wx}$ $\boxed{xe^{\gamma x} cos\, wx, xe^{\gamma x} sin\, wx}$

2 Higher Order Non-Homogeneous ODE

Apply **method of undetermined coefficients** for solving 2nd order ODE with a modification. If a term in the choice for $y_p(x)$ is a solution of the homogeneous equation, then multiply this term by x^k, where k is the smallest positive integer and satisfies the condition that this **term** $\times x^k$ is **NOT a solution** of the homogeneous equation. So, we try $cxe^{\lambda x}, cx^2 e^{\lambda x}, \ldots, cx^k e^{\lambda x}$ as a solution, plug into the ODE, and solve for c for the minimum k.

Variation of Parameters Extending the concept that we used for 2nd order ODE:

$$\boxed{y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)} r(x)dx = y_1(x) \int \frac{W_1(x)}{W(x)} r(x)dx + y_2(x) \int \frac{W_2(x)}{W(x)} r(x)dx + \ldots + y_n(x) \int \frac{W_n(x)}{W(x)} r(x)dx}$$

3 Series Solutions of ODEs

Higher order linear ODEs with constant coefficients can be solved by algebraic methods as their solutions are often elementary functions which are known from calculus. For ODEs with variable coefficients the situation is complicated and their solutions are nonelementary **special functions**, e.g., Legendre and Bessel functions. **Power Series Method** $\boxed{y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3^3 + \ldots}$ Compute $y^{'}, y^{''}, \ldots, y^{(n)}$, substitute in the ODE and compute the coefficients of the powers of $x, x^2, x^3, \ldots, x^n$. Equate each of the coefficients to 0 to determine $a_0, a_1, a_2, \ldots, a_n$.

4 Existence of Power Series Solutions

Consider the ODE $\boxed{y'' + p(x)y' + q(x)y = r(x)}$ If p, q, r have Taylor series representations (analytic) then every solution of the ODE can be represented by a power series in powers of $x - x_0$ with a positive radius of convergence R. Operations on Power Series: A power series can be differentiated term by term. Two power series can be added term by term. Two power series can be multiplied term by term. **Identity Theorem** If a power series has a positive radius of convergence and a sum that is identically zero throughout its interval of convergence, then each coefficient of the series must be zero.

5 Some Classical Differential Equations

$$\begin{aligned}
&Legendre: && \boxed{(1-x^2)y^{''} - 2xy^{'} + k(k+1)y = 0}\\
&Chebyshev: && \boxed{(1-x^2)y'' - xy' + k^2y = 0}\\
&Herimite: && \boxed{y'' - 2xy' + 2ky = 0}\\
&Laguerre: && \boxed{xy'' + (1-x)y' + ky = 0}
\end{aligned}$$

where k is a constant.

6 Legendre's Equation & Polynomials

$\boxed{(1-x^2)y^{''} - 2xy^{'} + k(k+1)y = 0}$ k is a constant Let $\boxed{y = a_n \sum_{n=0}^{\infty} x^n}$, compute $y, y^{'}, y^{''}$ and substitute.

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} na_n x^{n-1} + k(k+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n} - 2\sum_{n=1}^{\infty} na_n x^{n} + k(k+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

Set $m = n-2, \implies n = m+2$

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} ma_m x^m$$
$$+k(k+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

$$m = 0 \to 2a_2 + k(k+1)a_0 = 0 \implies a_2 = -\frac{-k(k+1)}{2!}a_0$$
$$m = 1 \to 6a_3 + [-2 + k(k+1)]a_1 = 0 \implies a_3 = -\frac{(k-1)(k+2)}{3!}a_1$$
$$m > 1 \to (m+2)(m+1)a_{m+2} + [-m(m-1) - 2m + k(k+1)]a_m = 0$$

$$\to \boxed{a_{n+2} = -\frac{(k-n)(k+n+1)}{(n+2)(n+1)}a_m} \quad n = 0, 1, \ldots \text{ (recurrence relation)}$$

$$\begin{aligned}
a_2 &= -\frac{k(k+1)}{2!}a_0\\
a_3 &= -\frac{(k-1)(k+2)}{3!}a_1\\
a_4 &= -\frac{(k-2)(k+3)}{4\times 3}a_2 = \frac{(k-2)k(k+1)(k+3)}{4!}a_0\\
a_5 &= -\frac{(k-3)(k+4)}{5\times 4}a_3 = \frac{(k-3)(k-1)(k+2)(k+4)}{5!}a_1
\end{aligned}$$

$$\begin{aligned}
y =& \quad a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots + a_nx^n + \ldots\\
y =& \quad a_0 + a_1x - \frac{k(k+1)}{2!}a_0x^2 - \frac{(k-1)(k+2)}{3!}a_1x^3 +\\
& \frac{(k-2)k(k+1)(k+3)}{4!}a_0x^4 + \frac{(k-3)(k-1)(k+2)(k+4)}{5!}a_1x^5 + \ldots
\end{aligned}$$

$$\begin{aligned}
y_1 =& \quad \left(1 - \frac{k(k+1)}{2!}x^2 + \frac{(k-2)k(k+1)(k+3)}{4!}x^4 + \ldots\right)a_0\\
y_2 =& \quad \left(x - \frac{(k-1)(k+2)}{3!}x^3 + \frac{(k-3)(k-1)(k+2)(k+4)}{5!}x^5 + \ldots\right)a_1\\
y =& \quad a_0y_1(x) + a_1y_2(x) \ (y_1 \text{ is the even series } \& \ y_2 \text{ is the odd series})
\end{aligned}$$

$y_1/y_2 \neq$ constant Hence, they are independent. The general solution is $\boxed{y = a_0y_1(x) + a_1y_2(x)}$.

If $k=1$	$a_3 = a_5 = \ldots = 0$	y_1 does not terminate	$y_2 = a_1x$
If $k=2$	$a_4 = a_6 = \ldots = 0$	$y_1 = (1-3x^2)a_0$	y_2 does not terminate
If $k=3$	$a_4 = a_6 = \ldots = 0$	y_1 does not terminate	$y_2 = (x - \frac{5}{3}x^3)a_1$

a_0, a_1 can be arbitrarily chosen. Choose a_n (coefficient of x^n) as $\frac{(2n)!}{2^n(n!)^2}$ and calculate the other coefficients using $a_n = -\frac{(n+2)(n+1)}{(k-n)(k+n+1)}a_{n+2}$

$$\begin{aligned}
\text{with } n &= k-2\\
a_{n-2} &= -\frac{k(k-1)}{2(2k-1)}a_n = -\frac{k(k-1)}{2(2k-1)} \times \frac{(2k)!}{2^k(k!)^2}\\
&= -\frac{k(k-1)}{2(2k-1)} \times \frac{2k(2k-1)(2k-2)!}{2^k k(k-1)!k(k-1)(k-2)!} = \frac{(2k-2)!}{2^k(k-1)!(k-2)!}\\
a_{n-2m} &= (-1)^m\frac{(2k-2m)!}{2^n m!(k-m)!(k-2m)!}\\
P_n(x) &= \sum_{m=0}^{M}(-1)^m\frac{(2k-2m)!}{2^k m!(k-m)!(k-2m)!}
\end{aligned}$$

$P_n(x)$ is the $\boxed{\text{Legendre polynomial}}$ of degree n. The reduction of power series to polynomials is a great advantage because then we have solutions for all x, without convergence restrictions.

Several important 2nd order ODEs have coefficients that are not analytic. Yet these ODEs can be solved through an extension of the power serties method that is credited to Frobenius. Consider the ODE $\boxed{y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0 \; (b(x), c(x) \text{ are analytic at } x = 0)}$ This ODE has at least one solution of the form $\boxed{y(x) = x^r \sum_{m=0}^{\infty} a_m x^m \text{ where r is real or complex and } a_0 \neq 0}$ Multiply the ODE by x^2

$$x^2y'' + xb(x)y' + c(x)y = 0$$

and expand b(x) and c(x) in Taylor series

$$b(x) = b_0 + b_1x + b_2x^2 + \ldots$$

$$c(x) = c_0 + c_1x + c_2x^2 + \ldots$$

7 Frobenius Method

$$\begin{aligned}
y(x) &= x^r \sum_{m=0}^{\infty} a_m x^m = x^r(a_0 + a_1x + a_2x^2 + \ldots) \\
y'(x) &= \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} = x^{r-1}[ra_0 + (r+1)a_1x + (r+2)a_2x^2 + \ldots] \\
y''(x) &= \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2} \\
&= x^{r-2}[r(r-1)a_0 + (r+1)ra_1x + (r+2)(r+1)a_2x^2 + \ldots]
\end{aligned}$$

Substituting in the ODE

$$\begin{aligned}
&x^r[r(r-1)a_0 + (r+1)ra_1x + (r+1)(r+2)a_2x^2 + \ldots] + \\
&(b_0 + b_1x + c_bx^2 + \ldots)x^r[ra_0 + (r+1)a_1x + (r+2)a_2x^2 + \ldots] + \\
&(c_0 + c_1x + c_2x^2 + \ldots)x^r(a_0 + a_1x + a_2x^2 + \ldots) = 0
\end{aligned}$$

From coefficient of x^r $\boxed{[r(r-1) + b_0r + c_0] = 0}$ (quadratic equation)

Distinct roots not differing by an integer.
$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \ldots)$
$y_2(x) = x^{r_2}(A_0 + A_1x + A_2x^2 + \ldots)$

Double root $r_1 = r_2 = r = \frac{1}{2}(1 - b_0)$
$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \ldots)$
$y_2(x) = y_1(x)ln\, x + x^{r_1}(A_0 + A_1x + A_2x^2 + \ldots)$

Roots differing by an integer
$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \ldots)$
$y_2(x) = ky_1(x)ln\, x + x^{r_2}(A_0 + A_1x + A_2x^2 + \ldots)$ $r_1 > r_2, k$ can be 0

For cases 2 and 3, the second independent solution can be obtained by $\boxed{\text{reduction of order}}$.

8 Bessel's Equation

$\boxed{x^2y'' + xy' + (x^2 - \nu^2)y = 0}$ (ν is a real number ≥ 0)

Applying $\boxed{\text{Frobenius}}$ technique, the solution is of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y'(x) = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} = x^{r-1}[ra_0 + (r+1)a_1x + (r+2)a_2x^2 + \ldots]$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2} = x^{r-2}[r(r-1)a_0 + (r+1)ra_1x + (r+2)(r+1)a_2x^2 + \ldots]$$

substituting in the ODE

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0$ (m = 0)
$(r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0$ (m = 1)
$(m+r)(m+r-1)a_m + (m+r)a_m + a_{m-2} - \nu^2 a_m = 0$ (m = 2, 3, . . .)

$(r+\nu)(r-\nu) = 0, \implies \boxed{r = \pm\nu}$
$((\nu+1)^2 - \nu^2)a_1 = 0 \implies (2\nu+1)a_1 = 0 \implies \boxed{a_1 = 0}$
$(m+\nu)[(m+\nu) - 1 + 1 - \nu^2]a_m + a_{m-2} = 0 \implies \boxed{m(m+2\nu)a_m + a_{m-2} = 0}$

since $a_1 = 0 \implies \boxed{a_3 = a_5 = \ldots = 0}$
$2m(2m+2\nu)a_{2m} + a_{2m-2} = 0$ (even numbers only)
$a_{2m} = -\frac{1}{2^2 m(m+\nu)} a_{2m-2}$ (m = 0, 1, 2, . . .)
$a_2 = -\frac{a_0}{2^2(\nu+1)}$
$a_4 = -\frac{a_2}{2^2 2(\nu+2)} = \frac{a_0}{2^4 2!(\nu+1)(\nu+2)}$

$$\boxed{a_{2m} = -\frac{(-1)^m a_0}{2^{2m} m!(\nu+1)(\nu+2)\ldots(\nu+m)}} \text{ (n = 0, 1, 2, . . .)}$$

When ν is an integer, denote it as by n

$$\boxed{a_{2m} = -\frac{(-1)^n a_0}{2^{2m} m!(n+1)(n+2)\ldots(n+m)}} \text{ (n = 0, 1, 2, . . .)}$$

choose $a_0 = \frac{1}{2^n n!}$

$$a_{2m} = -\frac{(-1)^n a_0}{2^{2m} m!(n+1)(n+2)\ldots(n+m)} \times \frac{1}{2^n n!} = \frac{(-1)^m}{2^{2m+n} m!(n+m)!}$$

The solution is the Bessel function of the first kind of order n, convergent $\forall x$

$$\boxed{J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}}$$
$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} + \ldots$$
$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m} m!(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} + \ldots$$

Bessel functions for real number

Choose $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$,Gamma function defined as $\Gamma(\nu+1) = \int_0^\infty e^t t^\nu dt, \nu \geq 0$

$$\Gamma(\nu+1) = -e^{-t}t^\nu \Big|_0^\infty + \nu \int_0^\infty e^{-t}t^{\nu-1}dt = 0 + \nu\Gamma(\nu)$$
$$\boxed{\Gamma(\nu+1) = \Gamma(\nu)}\ \Gamma(1) = 1,\ \Gamma(2) = 1 \times \Gamma(1) = 1!,\ \Gamma(3) = 2 \times \Gamma(2) = 2!,\ \ldots$$
$$\boxed{\Gamma(n+1) = n!}$$
$a_{2m} = -\frac{(-1)^m a_0}{2^{2m} m!(\nu+1)(\nu+2)\ldots(\nu+m)2^\nu\Gamma(\nu+1)}$ (n = 0, 1, 2, . . .)
$a_{2m} = -\frac{(-1)^m a_0}{2^{2m+\nu} m!\Gamma(\nu+m+1)}$
$$\boxed{J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\Gamma(\nu+m+1)!}}$$

General Solution

If ν is not an integer, the general solution is

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

$[x^{\nu}J_{\nu}(x)]' = x^{\nu}j_{\nu-1}$	$[x^{-\nu}J_{\nu}(x)]' = -x^{-\nu}j_{\nu+1}$
$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}j_{\nu}(x)$	$J_{\nu-1}(x) - J_{\nu+1}(x) = \frac{2\nu}{x}j'_{\nu}(x)$
$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}sinx$	$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}cosx$

If ν is an integer, the second independent solution is

$$y_2(x) = J_0(x)ln\,x + \sum_{m=1}^{\infty}$$

A standard second solution known as the **Bessel function of the 2nd kind** of order ν or **Neumann's function** of order ν is defined as

$$Y_{\nu}(x) = \frac{1}{sin\,\nu\pi}[J_{\nu}(x)cos\,\nu\pi - J_{-\nu}(x)]$$
$$Y_n(x) = \lim_{\nu\to n} Y_{\nu}(x)$$
$$y(x) = C_1 J_{\nu}(x) + C_2 Y_{\nu}(x)$$