THE MATHEMATICAL CONCEPT OF A FUNCTION

A CRISP AND CONCISE INTRODUCTION

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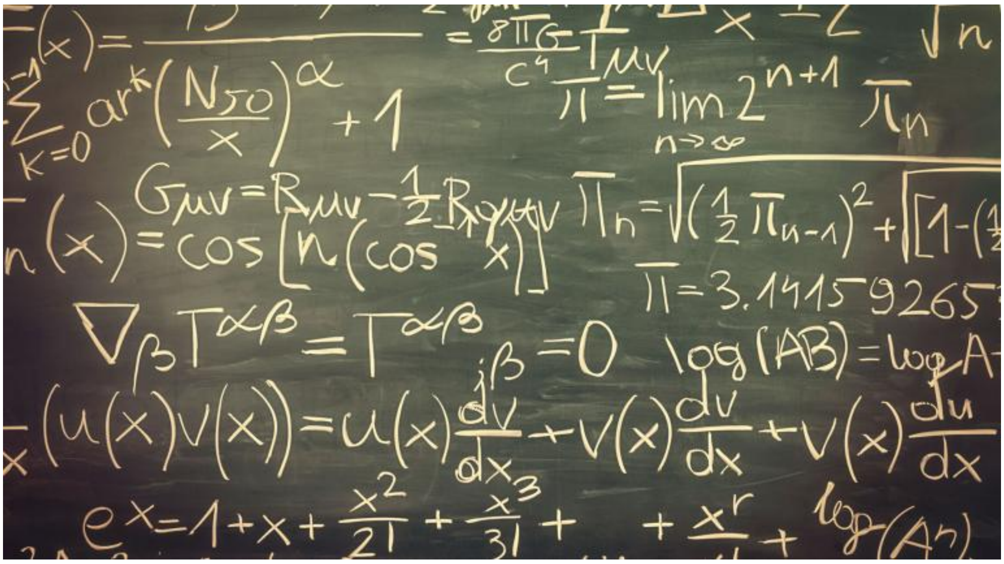
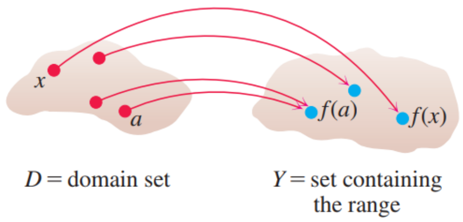


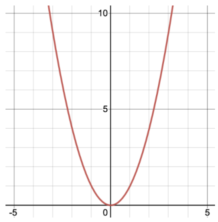
Contents

1 Function, Domain, Range

When a value of one variable x depends on another variable y, we say that *y is a function of x* and is written symbolically as $y = f(x)$ and pronounced as *"y equals f of x"*. Formally, a function *f* from a set *D*, a domain, to *Y*, a range, is a rule that assigns an *unique* value *f(x)* in *Y* to each *x* in *D*.



Example:
$y = x^2$
Domain = $[-\infty, +\infty]$
Range = $[0, +\infty]$

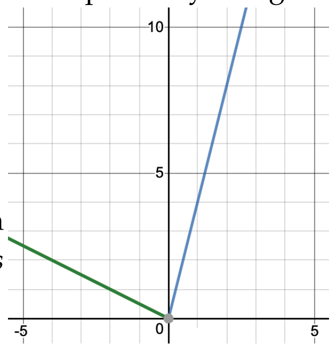


2 Piecewise Continuous & Discontinuous Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain.

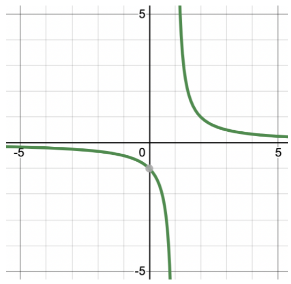
$$|y| = \begin{cases} 4x & \text{if } x \geq 0 \\ -0.5x & \text{if } x < 0 \end{cases}$$

y is *unique* for a given x. Such functions are *piecewise continous* as there are no "gaps".



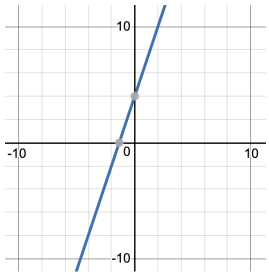
$$y = \frac{1}{x-1}$$

y does not exist for $x = 1$; The curve is not continuous at $x = 1$ and the function is *discontinous*.

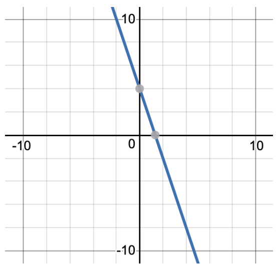


3 Increasing & Decreasing Functions

Increasing function
$f(x_2) > f(x_1)$ when $x_2 > x_1$
Example: $y = 3x + 4$

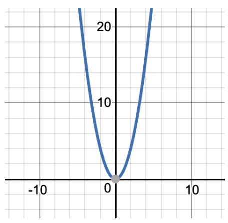


Decreasing function
$f(x_2) < f(x_1)$ when $x_2 > x_1$
Example: $y = -3x + 4$

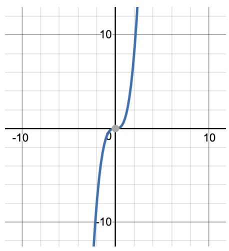


4 Even & Odd Functions

Even function $f(-x) = f(x)$
Example: $y = x^2$



Odd function $f(-x) = -f(x)$
Example: $y = x^3$



5 Types of Functons

- Linear Functions $f(x) = mx + b$
- Polynomial Functions $f(x) = a_0 + a_1x + a_2x^2 + \cdots a_{n-1}x^{n-1} + a_nx^n$; $n = 2 \rightarrow$ Quadratic, $n = 3 \rightarrow$ Cubic.
- Rational Functions $f(x) = p(x)/q(x)$
- Algebraic Functions - constructed from polynomials using algebraic operations ($+$, $-$, $\times$, $\div$, and roots)
- Trigonometric functions, e.g., $f(x) = sin(x)$
- Exponential Functions, e.g., $y = 2^x$, Logarithmic Functions $y = log_5^x$
- Transcendental Functions - functions that are not expressible as a finite combination of algebraic operations of addition, subtraction, multiplication, division, raising to a power, and extracting a root. E.g., $log\, x$, $sin\, x$, e^x and any functions containing them. Such functions are expressible in algebraic terms only as infinite series. In general, the term transcendental means nonalgebraic.

6 Sums, Differences, Products & Quotients of Functions

Much like numbers, functions can be added, subtracted, multiplied, and divided. By defnition:

$$\begin{aligned}
(f+g)(x) &= f(x)+g(x)\\
(f-g)(x) &= f(x)-g(x)\\
(fg)(x) &= f(x)g(x)\\
\frac{f}{g}(x) &= \frac{f(x)}{g(x)} \text{ where } g(x)\neq 0
\end{aligned}$$

7 Function Composition

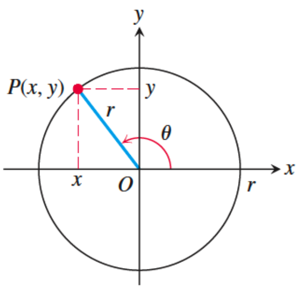
The output from is one function is the input to the second function.

$$(f\circ g)(x) = f(g(x))$$

8 Vertical & Horizontal Scaling, Reflecting a Function

$$\begin{aligned}
y = &\; cf(x) && \text{for c > 1, stretch vertically}\\
y = &\; \tfrac{1}{c}f(x) && \text{for c > 1, compress vertically}\\
y = &\; f(cx) && \text{for c > 1, stretch horizontally}\\
y = &\; f\left(\tfrac{x}{c}\right) && \text{for c > 1, compress horizontally}\\
y = &\; -f(x) && \text{for c = -1, reflect across x axis}\\
y = &\; f(-x) && \text{for c = -1, reflect across y axis}
\end{aligned}$$

9 Basic Trigonometric Function Definitions



b = base
p = perpendicular
r = h (hypoetenuse)
$b^2+p^2=r^2$ (Pythagoras)

$$\begin{aligned}
sine\ \theta &= \frac{p}{h}\\
cosine\ \theta &= \frac{b}{h}\\
tangent\ \theta &= \frac{p}{b} = \frac{sin\ \theta}{cos\ \theta}
\end{aligned}$$

abbreviated as: sin, cos and tan

$$\begin{aligned}
cosecant\quad \theta &= \frac{1}{sin\ \theta}\\
secant\quad \theta &= \frac{1}{cos\ \theta}\\
cotangent\quad \theta &= \frac{1}{tan\ \theta}
\end{aligned}$$

abbreviated as: csc, sec and cot

10 Basic Trigonometric Identities (can be easily proven from the above)

$$\boxed{\begin{gathered} sin^2\ \theta + \cos^2\ \theta = 1 sec^2\ \theta = 1 + tan^2\ \theta \\ csc^2\ \theta = 1 + cot^2\ \theta \end{gathered}}$$

$$\boxed{\begin{gathered} sin\ (\theta_1+\theta_2) = sin\ \theta_1\ cos\ \theta_2 + cos\ \theta_1\ sin\ \theta_2 \\ cos\ (\theta_1+\theta_2) = cos\ \theta_1\ cos\ \theta_2 - sin\ \theta_1\ sin\ \theta_2 \end{gathered}}$$

$$\boxed{\begin{gathered} c^2 = a^2 + b^2 - 2ab\ cos\ \theta \ \text{ (Law of Cosines)} \\ \left(\frac{sin\ A}{a}\right) = \left(\frac{sin\ B}{b}\right) = \left(\frac{sin\ C}{c}\right) \\ \text{(Law of Sines, a,b,c are angles, A,B,C are lengths)} \end{gathered}}$$