THE MATHEMATICAL CONCEPT OF A DERIVATIVE

A CRISP AND CONCISE INTRODUCTION

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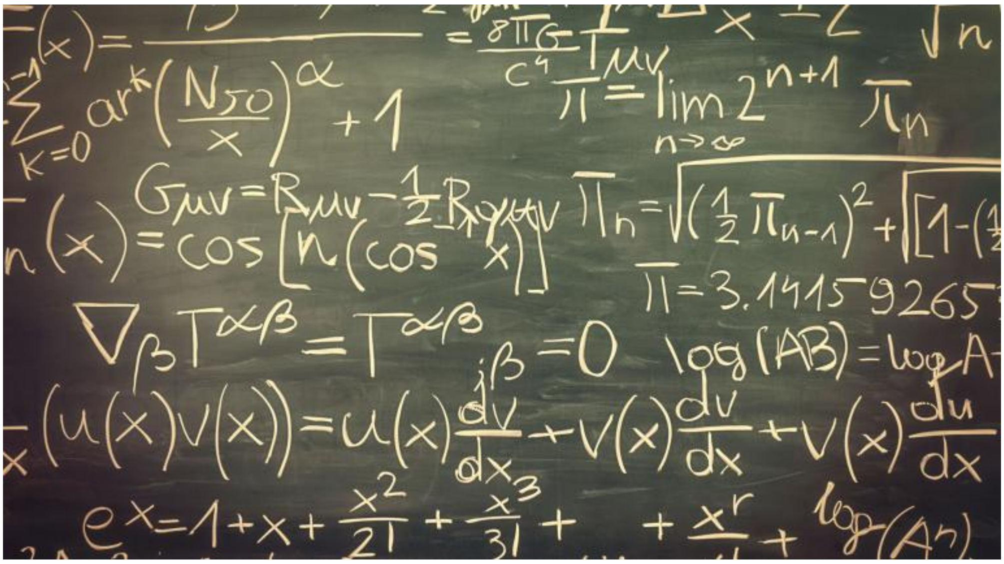

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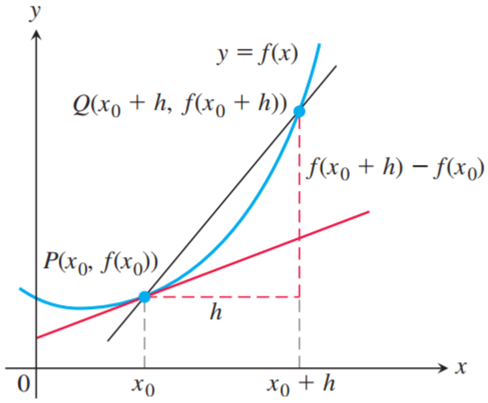
1 Definition of a Derivative



Consider the limit: $\lim_{h \to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$

This limit is called the *derivative* and is shown as: $\dfrac{df}{dx} = \dfrac{dy}{dx} = f'(x)$

Its value at a is represented as: $f'(a) = \dfrac{dy}{dx}\Big|_{x=a}$

A derivative is rate of change, it is the *tangent* at the point.

A function $f(x)$ is differentiable at $x = a$ if $f'(a)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval. If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$. $\boxed{\dfrac{d}{dx}}$ is known as the **Differential Operator**.

2 Derivative of a Polynomial Term

$$f'(a) = \lim_{x \to a} \tfrac{f(x)-f(a)}{x-a} = \lim_{x \to a} \tfrac{x^n - a^n}{x-a}$$

$$\text{Now, } x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})$$

$$\implies f'(a) = \lim_{x \to a}(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots a^{n-2}x + a^{n-1}) = na^{n-1}$$

Hence, $\boxed{\dfrac{dx^n}{dx} = nx^{n-1}}$ and obviously, $\frac{d}{dx}(constant) = 0$

3 Derivatives of a Trigonometric Function

$$\frac{d}{dx}(sin(x)) = \lim_{h \to 0} \frac{sin(x+h) - sin(x)}{h} = \lim_{h \to 0} \frac{sin(x)\ cos(h) + cos(x)\ sin(h) - sin(x)}{h}$$

$$= sin(x) \lim_{h \to 0} \frac{cos(h) - 1}{h} + cos(x) = \lim_{h \to 0} \frac{sin(h)}{h} = sinx(x).0 + cos(x).1 = cos(x)$$

4 Important Derivatives

$$\boxed{\frac{d}{dx} ln\ x = \frac{1}{x}}$$

$$\lim_{h \to 0} \frac{ln(x+h) - ln(x)}{h} = \lim_{h \to 0} \frac{ln\frac{(x+h)}{x}}{h} = \lim_{h \to 0} \frac{1}{h} ln\left(1 + \frac{h}{x}\right) = \lim_{h \to 0} ln\left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$$

Let $n = \dfrac{h}{x}$ or $h = nx$ or $\dfrac{1}{h} = \dfrac{1}{n}.\dfrac{1}{x} \to \lim_{h \to 0} ln(1+n)^{\frac{1}{n}\frac{1}{x}} = \lim_{h \to 0} ln\left((1+n)^{\frac{1}{n}}\right)^{\frac{1}{x}} = \dfrac{1}{x} ln\left(\lim_{h \to 0}(1+n)^{\frac{1}{n}}\right) = \dfrac{1}{x}$

$$\boxed{\frac{d}{dx} a^x = ln(a)a^x}$$

Let $a = e^{ln(a)}$

$$\frac{d}{dx} a^x = \frac{d}{dx}(e^{(ln(a))^x}) = \tfrac{d}{dx}(e^{ln(a)x}) = ln(a)e^{ln(a)x} = ln(a)a^x$$

5 Chain Rule

$$\tfrac{d}{dx}(v(u(x))) = \tfrac{dv}{dx} = \tfrac{dv}{du} \times \tfrac{du}{dx}$$

because $\frac{dv}{dx} = \lim_{x \to 0} \frac{\Delta v}{\Delta x} = \lim_{x \to 0} \left(\frac{\Delta v}{\Delta u} \times \frac{\Delta u}{\Delta x}\right) = \lim_{x \to 0} \left(\frac{\Delta v}{\Delta u}\right) \times \lim_{x \to 0} \left(\frac{\Delta u}{\Delta x}\right) = \frac{dv}{du} \times \frac{du}{dx}$

6 Implicit Differentiation

In implicit differentiation, we differentiate each side of an equation with two variables (usually x and y) by treating one of the variables as a function of the other. This calls for using the chain rule. Example:

$$x^2+y^2=1 \rightarrow \tfrac{d}{dx}(x^2+y^2)=\tfrac{d}{dx}x^2+\tfrac{d}{dx}y^2=2x+2y\tfrac{dy}{dx}=0 \rightarrow \tfrac{dy}{dx}=-\tfrac{x}{y}$$

7 Product Rule

$$\begin{aligned}(fg)' &= fg'+gf' = \lim_{h\to 0}\tfrac{f(x+h)g(x+h)-f(x)g(x)}{h} = \lim_{h\to 0}\tfrac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\ &= \lim_{h\to 0} f(x+h)\tfrac{(g(x+h)-g(x))}{h}+\lim_{h\to 0} g(x)\tfrac{f(x+h)-f(x)}{h} = \boxed{f(x)g'(x)+g(x)f'(x)}\end{aligned}$$

8 Quotient Rule

$$\begin{aligned}\left(\tfrac{f}{g}\right)' &= \lim_{h\to 0}\tfrac{f'g-fg'}{g^2} = \lim_{h\to 0}\tfrac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\tfrac{1}{h}\tfrac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\\ &= \lim_{h\to 0}\tfrac{1}{g(x+h)g(x)}\tfrac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{h} = \lim_{h\to 0}\left(g(x)\tfrac{f(x+h)-f(x)}{h}-f(x)\tfrac{g(x+h)-g(x)}{h}\right) = \boxed{\frac{f'g-fg'}{g^2}}\end{aligned}$$

9 L'Hôpital's rule

First, need to do mathematical manipulations to get the limit into a l'Hôpital form, i.e., $0/0$ or ∞/∞ form. Let $f(x)$ and $g(x)$ be continuous functions on an interval containing $x=a$, with $f(a)=g(a)=0$. Suppose that f and g are differentiable, and that f' and g' are continuous. and, suppose that $g'(a)\neq 0$. Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}=\lim_{x\to a}\frac{(f(x)-f(a))/(x-a)}{(g(x)-f(a))/(x-a)}=\frac{\lim_{x\to a}(f(x)-f(a))/(x-a)}{\lim_{x\to a}(g(x)-f(a))/(x-a)}=\frac{f'(x)}{g'(x)}$$

10 Concave Up (Convex) & Concave Down

Let $y=f(x)$ be twice-differentiable on an interval I. If f" > 0 on I, the graph of f over I is concave up (also called convex).If f" < 0 on I, the graph of f over I is concave down.

11 Euler's number - e

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{n\to 0}(1+n)^{\frac{1}{n}}=\sum_{n=0}^{\infty}\frac{1}{n!}\text{ such that, }\lim_{h\to 0}\frac{e^h-1}{h}=1,\ e=2.718281\ldots\boxed{\frac{d}{dx}e^x=e^x}$$

12 Hyperbolic Functions

$$sinh\ x=\tfrac{e^x-e^{-x}}{2}\qquad cosh\ x=\tfrac{e^x+e^{-x}}{2}\qquad tanh\ x=\tfrac{e^x-e^{-x}}{e^x+e^{-x}}\qquad cosh^2x-sinh^2x=1$$

13 Partial Derivatives

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\qquad f_y(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

$$f_x=f_x(x,y)=\frac{\partial}{\partial x}f(x,y)\qquad f_y=f_x(x,y)=\frac{\partial}{\partial y}f(x,y)$$

Example, $f(x,y)=x^2y-10y^2z^3+43x-7tan(4y)$

$$\tfrac{\partial}{\partial x}f(x,y,z)=2xy+43\qquad \tfrac{\partial}{\partial y}f(x,y,z)=x^2-20yz^3-28sec^2(4y)\qquad \tfrac{\partial}{\partial z}f(x,y,z)=-30y^2z^2$$