THE MATHEMATICAL CONCEPT OF ORDINARY DIFFERENTIAL EQUATIONS

A CRISP AND CONCISE INTRODUCTION

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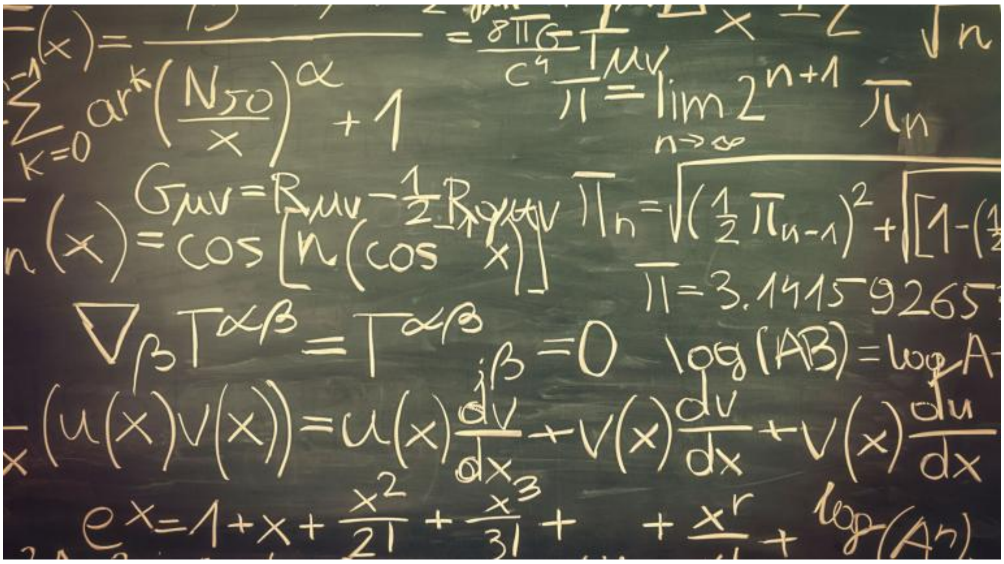
Last updated on 02-Nov-2022



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1 Differential Equation

A differential equation (DE) is an equation involving an **unknown function and its derivatives**. A DE is an **ordinary differential equation (ODE)** if the unknown function depends on only one variable. If the unknown function depends on **2 or more independent variables**, the DE is a **partial differential equation**. A DE along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable, constitutes an **initial-value problem**. The subsidiary conditions are initial conditions. If the subsidiary conditions are given at more than one value of the independent variable, the problem is a **boundary-value problem** and the conditions are boundary conditions.

2 Standard & Differential forms of an ODE

Standard form for 1st order DE is $\boxed{\dfrac{dy}{dx} = f(x,y)}$ and the **differential form** is $\boxed{M(x,y)dx + N(x,y)dy = 0}$

3 Order & Degree of a Differential Equation (DE)

Order of a DE is the **order of the highest derivative** which is also known as the differential coefficient. E.g., $\boxed{\dfrac{d^3x}{dx} + 3x\dfrac{dy}{dx} = e^y}$, the order of DE is 3. A 1st order DE is of the form: $\boxed{\dfrac{dy}{dx} + Py = Q}$ where P & Q are constants or functions of independent variables. E.g., $\boxed{\dfrac{dy}{dx} + (x^2+5)y = \dfrac{x}{5}}$. The degree of the DE is represented by the **power of the highest order derivative** in the given differential equation. $\boxed{\left[\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2\right]^4 = k^2\left(\dfrac{d^3y}{dx^3}\right)^2}$, degree of DE is 2. For the eq. $\boxed{tan\left(\dfrac{dy}{dx}\right) = x + y}$, the degree is undefined.

4 Solving ODE- Separation of Variables - Method 1

Through algebraic manipulations, some ODEs can be reduced to $\boxed{g(y)\dfrac{dy}{dx} = f(x)}$. Simply integrate both sides: $\boxed{\int g(y)dy = \int f(x)dx + c}$. Example: $\boxed{\dfrac{dy}{dx} = 1 + y^2 \rightarrow \dfrac{dy}{1+y^2} = dx}$ Let $y = tan\,\theta$. We have,

$$\boxed{\frac{dy}{d\theta} = sec^2\,\theta \rightarrow \frac{sec^2\,\theta}{1+tan^2\,\theta}d\theta = dx \rightarrow d\theta = dx \rightarrow \theta = x + c \rightarrow arctan\; y = x + c}$$

5 Solving ODE - Reduction to Separable Form - Method 2

Consider the ODE of the form: $\boxed{\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)}$

$$\boxed{\text{Let } y = ux \rightarrow \frac{dy}{dx} = x\frac{du}{dx} + u \rightarrow f(u) = x\frac{du}{dx} + u \rightarrow \frac{du}{f(u)-u} = \frac{dx}{x} \rightarrow \int \frac{du}{f(u)-u} = \int \frac{dx}{x} + c}$$

6 Solving ODE - Exact ODE & Integrating Factor - Method 3

If an ODE has an **implicit solution** $\boxed{u(x,y) = c = constant}$, then: $\boxed{du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy = 0}$ $\boxed{M(x,y)dx + N(x,y)dy = 0}$ where $\boxed{M = \dfrac{\partial u}{\partial x}}$, $\boxed{N = \dfrac{\partial u}{\partial y}}$. A 1st order ODE is an **exact DE** if

$\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = \dfrac{\partial^2 u}{\partial x \partial y} \rightarrow u = \int M dx + k(y) = \int N dx + l(x)}$. E.g.,: $\boxed{cos(x+y)dx + (3y^2 + 2y + cos(x+y))dy = 0}$

$$\boxed{M = \frac{\partial u}{\partial x} = cos(x+y) \rightarrow u = sin(x+y) + k(y) \rightarrow \frac{\partial u}{\partial y} = cos(x+y) + \frac{dk}{dy} = 3y^2 + 2y + cos(x+y)}$$

$\boxed{k = y^3 + y^2 + c^* \rightarrow}$ $\boxed{u = sin(x+y) + y^3 + y^2 + c}$

7 Inexact ODE

Consider the ODE $\boxed{-ydx + xdy = 0}$. Here the above approach will not work:

$\boxed{M = \frac{\partial u}{\partial x} = -y}$ $\boxed{N = \frac{\partial u}{\partial y} = x}$ $\boxed{\frac{\partial M}{\partial y} = \frac{\partial^2 M}{\partial x \partial y} = -1}$ $\boxed{\frac{\partial N}{\partial x} = \frac{\partial^2 N}{\partial x \partial y} = 1}$ $\boxed{\frac{\partial^2 M}{\partial x \partial y} \neq \frac{\partial^2 N}{\partial x \partial y} \text{ (inexact)}}$

From above: $\boxed{u = -y \int dx + k(y) = -xy + k(y)}$ $\boxed{\frac{\partial u}{\partial y} = -x + \frac{dk}{dy}}$ But, $\boxed{\frac{\partial u}{\partial y} = x}$. Contradictory!

8 Integrating Factor to transform to an exact ODE

Multiply the equation by a factor $F(x, y)$ to make it exact. $\boxed{FMdx + FNdy = 0}$ and impose the conditions

$\boxed{\frac{\partial}{\partial y}(FM) = \frac{\partial}{\partial x}(FN) \rightarrow F_y M + FM_y = F_x N + FN_x}$ Let F depend only on x, $\boxed{FM_y = F' N + FN_x}$

$\boxed{\frac{M_y}{N} = \frac{F'}{F} + \frac{N_x}{N}} \rightarrow \boxed{\int \frac{df}{F} = \int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx = \int Rdx} \rightarrow \boxed{ln(F) = \int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx = \int Rdx}$

Integrating Factor $\boxed{F(x) = e^{\int R(x)dx}}$. Similarly, $\boxed{F(y) = e^{\int R(y)dy}}$ E.g., Solve $\boxed{(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0}$

$\boxed{M = \frac{\partial u}{\partial x} = e^{x+y} + ye^y}$ $\boxed{N = \frac{\partial u}{\partial y} = xe^y - 1}$ $\boxed{\frac{\partial M}{\partial y} = e^{x+y} + ye^y + e^y}$ $\boxed{\frac{\partial N}{\partial x} = e^y}$ $\boxed{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = e^{x+y} + ye^y}$

$\boxed{R = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{xe^y - 1} \left(e^{x+y} + ye^y \right) \text{ doesn't work as } R = R(x, y)}$

$\boxed{R^* = \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{-1}{e^{x+y} + ye^y} \left(e^{x+y} + ye^y \right) = -1, \int e^{R^* dy} = e^{-y}}$

Multiplying the ODE by $e^{R^*} = e^{-y}$, we have, $\boxed{(e^x + y)dx + (x - e^{-y})dy = 0}$ $\boxed{\text{Note: } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = 1}$

$\boxed{M = \frac{\partial u}{\partial x} = e^x + y}$ $\boxed{u = e^x + xy + k(y)}$ $\boxed{\frac{\partial u}{\partial y} = x + \frac{dk}{dy} = x - e^{-y}}$ $\boxed{k = e^y + c^*}$ $\boxed{u = e^x + xy + e^y + c}$

9 1st Order Linear ODE - Homogeneous

A first order ODE is $\boxed{\textbf{linear}}$ if it is of the following form: $\boxed{\frac{dy}{dx} + p(x)y = r(x)}$ and is $\boxed{\textbf{non-linear}}$ if it cannot be brought to the above form. The above ODE is linear in both y and y' where p and q are any function of x. When $r(x) = 0$, the ODE is called $\boxed{\textbf{homogeneous}}$ and is $\boxed{\frac{dy}{dx} + p(x)y = 0}$ By separation of variables as stated in Method 1, $\boxed{\int \frac{dy}{y} = -\int p(x)dx \ln|y| = -\int p(x)dx + c^* y = ce^{-\int p(x)dx}}$ (homogeneous solution y_h)

10 1st Order ODE - Non Homogeneous

When $r(x) \neq 0$, the ODE is called $\boxed{\textbf{non homogeneous}}$. We multiply the ODE by a function $F(x)$.

$\boxed{Fy' + p(x)Fy = r(x)F \text{ and let } p(x)Fy = F'y \rightarrow \frac{F'}{F} = p(x) \rightarrow \ln|F| = \int p(x)dx \text{ Let } h = \int p(x)dx \rightarrow F = e^h}$

$\boxed{(e^h y)' = re^h \rightarrow e^h y = \int e^h r dx + c}$ $\boxed{y_p = e^{-h} \int e^h r dx + c}$ $\boxed{y = y_h + y_p = ce^{-h} + e^{-h} \int e^h r dx + c}$

11 Reduction to Linear Form - Bernoulli Equation

The $\boxed{\textbf{Bernoulli equation, a non-linear ODE}}$ is given by $\boxed{y' + p(x)y = r(x)y^n}$ where n is any real number.

Let $\boxed{u = y^{1-n}} \rightarrow u' = (1-n)y^{-n}y' \rightarrow u' = (1-n)y^{-n}(ry^n - py) \rightarrow u' = (1-n)(r - py^{1-n})$

$u' = (1-n)(r - pu) \rightarrow \boxed{u' + (1-n)pu = (1-n)r}$ (Linear ODE)