Real Analysis

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Abstract

Real Analysis the study of approximation on the reals.

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1 Cardinality

1.1 Brief Motivation

We want to build a metric space to measure the distance between objects.

We need

1. set X of objects.
2. need to measure closeness. func $d : X \times X \to [0, \infty)$ s/t
   (a) $d(x, y) = 0 \iff x = y$
   (b) $d(x, y) = d(y, x)$
   (c) $d(x, z) \leq d(x, y) + d(y, z)$

We call d a metric on X. (X, d) is a **metric space**.

$(\mathbb{R}, +' \circ^\circ (\sqrt{}^{=}) -)$ is a metric space. Note the BQN notation.

1.2 Function Theory

Definition 1.1: Injection. *Let A, B be non-empty sets. We say $f : A \to B$ is injective **iff** $\forall a, b \in A \quad f(a) = f(b) \implies a = b$*

Definition 1.2: Surjection. *$f : A \to B$ is a surjection if $\forall b \in B \quad \exists a \in A$ s/t $f(a) = b$.*

Definition 1.3: Bijective. *$f : A \to B$ is bijective **iff** its injective and surjective.*

Definition 1.4: Invertable. *$f : A \to B$ is invertable **iff** $\exists g : B \to A$ s/t $g(f(a)) = a$ and $f(g(b)) = b \quad \forall a \in A,\, b \in B$.*

We write $g = f^{-1}$ and call it "the" inverse.

Proposition 1.1. *$f : A \to B$ is invertable **iff** f is bijective.*

Proof. ($\implies$) f is invertable. Suppose $f(a) = f(b)$. We'll show $a = b$.

$$f^{-1}f(a)) = f^{-1}f(b))$$

$$\implies a = b$$

Now we'll show $\forall b \in B\ \exists a \in A\ f(a) = b$.

$$a = f^{-1}(b) \implies \text{there is way to get from b to a, and it's } f^{-1}$$

($\Longleftarrow$) Assume $f \leftarrow$ (bijective). We'll construct f's inverse. For $b \in B$ let a_b be the unique element of A s/t $f(a_b) = b$. a_b exists b/c of surjectivity of f, and it's unique b/c of injectivity.

$$g := \{g : A \to B,\ g(b) = a_b\}$$

$$f(g(b)) = f(a_b) = b$$

$$g(f(a_b)) = g(b) = a_b$$

$$\implies g = f^{-1}$$

□

Proposition 1.2. $\exists$(*injection*) $f : A \to B \iff \exists$(*surjection*) $g : B \to A$

Proof. ($\implies$) Suppose $f : A \to B \leftarrow$ (injective). Let $b \in B$.
Case 1: $b \in f(A)$.
Let $g(b)$ be the unique element of A s/t $f(g(b)) = b$, unique b/c $f \leftarrow$ (injective)
Case 2: $b \notin f(A)$.
Fix any $z \in A$. Let $g(b) = z$.

$$\implies g(b) = \begin{cases} f^{=}(b) & b \in f(A) \\ z & b \notin f(A) \end{cases}$$

We claim g is a surjection. So we have to show $\forall a \in A, \exists b \in B$ s/t $g(b) = a$ Let $a \in A$ s/t $f(a) \in B$.

$$g(f(a)) \implies f(g(f(a))) = f(a)$$

$$\text{(injective)} \implies g(f(a)) = a$$

$$\implies g \leftarrow \text{(surjective)}$$

$\impliedby$ Suppose $(g : B \to A) \leftarrow$ (surjective). $\forall a \in A$ choose $b_a \in B$ s/t $g(b_a) = a$. $f := \{f : A \to B \quad f(a) = b_a\}$. Suppose

$$f(x) = f(y)$$

$$\implies b_x = b_y$$

$$\implies g(b_x) = g(b_y)$$

$$\implies x = y$$

$$\implies f \leftarrow \text{(injective)}$$

□

Definition 1.5: Powerset. *Let X be a set. Then $\mathcal{P}(X) := \{A : A \subseteq X\}$, called the "**powerset** of X."*

$$X = \{a, b\}$$

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Axiom 1.1: Choice. *Given $X \neq \emptyset \exists a$ choice func $f : \mathcal{P}(X) \setminus \{\emptyset\} \to X$ s/t $f(A) \in A \, \forall \neq A \subseteq A$.*

1.3 Cardinality

$$A = \{a, b\},\ B = \{c, d, e, f\}$$

Intuitively $|A| < |B|$

$$f \leftarrow (\text{inj}) : A \to B,\ f(a) := c,\ f(b) := d$$

$$\implies f \leftarrow (\text{inj})(A) \subset B$$

$$\implies |A| \leq |B|$$

Definition 1.6: Ordering of Cardinality on Sets. *A, B sets.*

1. $|A| \leq |B| \iff \exists f \leftarrow (\text{inj}) : A \to B$
2. $|A| = |B| \iff \exists f \leftarrow (\text{bij}) : A \to B$

$$|\mathbb{N}| \leq |\mathbb{Z}| \impliedby f \leftarrow (\text{inj}) : \mathbb{N} \to \mathbb{Z},\ f(n) := n$$

$$f \leftarrow (\text{bij}) : \mathbb{N} \to \mathbb{Z} : f(n) := \begin{cases} 2n+2 & : n \geq 0 \\ 2(-n) - 1 & : n < 0 \end{cases} \implies |\mathbb{N}| = |\mathbb{Z}|$$

$$h \leftarrow (\text{bij}) : \mathbb{R} \to (0,1) : h(x) := \frac{\arctan(x) + \pi/2}{\pi} \implies |\mathbb{R}| = |(0,1)|$$

Theorem 1.1: Cantor-Schroeder-Berstien (CSB). *if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.*

Lemma 1.1: Phi has a Fixed Point. *X set. Suppose $\exists \phi : \mathcal{P}(X) \to \mathcal{P}(X)$ s/t $\phi(A) \subseteq \phi(B)$ if $A \subseteq B \subseteq X$. Then*

$$\exists F \subseteq X \ s/t \ \phi(F) = F$$

Let $F = \bigcup_{A \subseteq X : A \subseteq \phi(A)} A$.

Note: $\emptyset \subseteq X$ & $\emptyset \subseteq \phi(\emptyset)$

Claim: $F = \phi(F)$. Take $A \subseteq X$ with $A \subseteq \phi(A)$. Then $A \subseteq F$.

$$\implies \phi(A) \subseteq \phi(F)$$

$$\implies A \subseteq \phi(F)$$

$$\implies \bigcup_{A \subseteq X : A \subseteq \phi(A)} A \subseteq \phi(F) \quad \text{(by properties of unions)}$$

$$\implies F \subseteq \phi(F)$$

Further,

$$F \subseteq \phi(F) \implies \phi(F) \subseteq \phi(\phi(F))$$

$$\implies \phi(F) \in \{A \subseteq X : A \subseteq \phi(A)\}$$

For this step, let $\phi(F) = B$ and notice that $B \subseteq \phi(B)$. So its in the set above.

$$\implies \phi(F) \subseteq \bigcup_{A \subseteq X : A \subseteq \phi(A)} A = F$$

$$\implies F = \phi(F)$$

To motivate (CSB) theorem 1.1: prove that $|N| = |N \times N|$.

$$f \leftarrow \text{(inj)} : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, \;\; f(n) := (n, 1)$$

$$g \leftarrow \text{(inj)} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \;\; g((n, m)) := 2^n 3^m$$

By (CSB) $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.

Proof of (CSB) theorem 1.1. Let $f, g \leftarrow \text{(inj)} : A : B$. For $X \subseteq Y \subseteq A$

$$f(X) \subseteq f(Y)$$

$$\implies B\backslash f(Y) \subseteq B\backslash f(X)$$

$$\implies g(B\backslash f(Y)) \subseteq g(B\backslash f(X))$$

$$\implies A\backslash g(B\backslash f(X)) \subseteq A\backslash g(B\backslash f(Y))$$

This letting $\phi : \mathcal{P}(A) \to \mathcal{P}(A) : \phi(x) := A\backslash g(B\backslash f(x))$ insures it preserves $\subseteq$. So by the lemma 1.1, $\exists F \subseteq A$ s/t $F = \phi(F) = A\backslash g(B\backslash f(F))$. In particular, $A\backslash F = g(B\backslash f(F)) \implies g : B\backslash f(F) \to A\backslash F :\leftarrow \text{(bij)}$.

Note. It's a surjection b/c everyone in $A\backslash F$ gets mapped to b/c it's the image if $g(B\backslash f(F))$.

Moreover, $g^{-1} : A\backslash F \to B\backslash f(F)$ is a bijection, and $f : F \to f(F)$ is a bijection (for the same reason as above; restriction of domain of an injective function is injective, and a function that maps to its image is automatically a surjection). Hence

$$h : A \to B : h(x) := \begin{cases} g^{-1}(x) & : x \in A\backslash F \\ f(x) & : x \in F \end{cases}$$

□

Show $|\mathbb{Q}| = |\mathbb{N}|$.

Proof.

$$f : \mathbb{N} \to \mathbb{Q} : f(x) := x \implies |\mathbb{N}| \leq |\mathbb{Q}|$$

$q \in \mathbb{Q}$ can be written in the form $q = \frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

$$g : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N} : g(q) := (m, n) \ : \ q = \frac{m}{n}$$

$$\implies |\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{N}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

So by (CSB) theorem 1.1, $|\mathbb{Q}| = |\mathbb{N}|$. □

Definition 1.7: Finite, Countably Infinite, Countable.

1. *a set A is **finite** iff $|A| = |\{1, 2, \ldots, n\}|$ for some $n \in \mathbb{N}$. In this case, $|A| = n$.*
2. *$|\emptyset| := 0$*
3. *A is **countably infinite** iff $|A| = |\mathbb{N}| := \aleph_0$.*
4. *A is countable iff A is finite or ctbly infinite.*

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{Q}| = \aleph_0$$

Proposition 1.3: Aleph Null is the Smallest Infinity. *If A is infinite, then $|\mathbb{N}| \leq |A|$.*

Proof. By (Choice) axiom 1.1, $\exists f : \mathcal{P}(A) \backslash \{\emptyset\} \to A$ s/t $f(X) \in X$, $\forall \emptyset \neq X \subseteq A$.

$$\begin{aligned} \text{Let } a_1 &= f(A) \in A \\ a_2 &= f(A \backslash \{a_1\}) \in A \backslash \{a_1\} \\ &\vdots \end{aligned}$$

$\implies \aleph_0 = |\{a_1, \ldots\}| \leq |A|$. $\square$

Proposition 1.4: The Reals are Uncountable. *$\mathbb{R}$ is uncountable. So $\nexists f \leftarrow$ (bij) $: \mathbb{N} \to \mathbb{R}$.*

Proof. **Cantor's Diagonal Element Proof.** Since $|\mathbb{R}| = |(0,1)|$, we'll show that $(0,1)$ is uncountable.
For contradiction, assume that there exists a bijection $f : \mathbb{N} \to (0,1)$.
So let's say

$$\begin{aligned} f(1) &= 0.a_{11}a_{12}a_{13}\cdots \\ f(2) &= 0.a_{21}a_{22}a_{23}\cdots \\ f(3) &= 0.a_{31}a_{32}a_{33}\cdots \\ \vdots &= \vdots \end{aligned}$$

Where we avoid repeated nines.
Choose $b_i \in \{1, \ldots, 8\}$ s/t $b_i \neq a_{ii}$.

$$\implies \nexists n \in \mathbb{N},\ f(n) = 0.b_1b_2b_3\cdots$$

Thats a contradiction. □

Definition 1.8: Continuum. *We write $|\mathbb{R}| = c$, where c stands for* ***continuum****.*

So we have 3 cardinals: $n, \aleph_0, c$.

Axiom 1.2: Continuum Hypothesis. *If A is a set with $\aleph_0 \leq |A| \leq c$, then $\aleph_0 = |A|$ or $|A| = c$.*

1.4 Cardinality of Power Sets

Proposition 1.5. *If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.*

Proof.

$$|\mathcal{P}(A)| = \sum_{k=1}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

□

Definition 1.9: Cartesian Product. *Let I be a set. $\forall i \in I$ Let $A_i \leftarrow$* (set) $\implies \prod_{i\in I} A_i := \{f | f : I \to \bigcup A_i,\ f(i) \in A_i\}$

$$f(i) \in A_i$$

$$I = \mathbb{N} \implies f : \mathbb{N} \to \bigcup A_i : f(i) \in A_i \equiv (f(1), f(2), \dots)$$

Definition 1.10: Set Power. $A, B \leftarrow$ (set) $\implies A^B = \{f : B \to A\}$

$$|A|^{|B|} := \left|A^B\right| = |\{f : B \to A\}|$$

Proposition 1.6: Cardinality of a Power Set. *if $X \leftarrow$* (set)*, $\mathcal{P}(X) = 2^{|X|} = |\{f : X \to \{0,1\}\}|$.*

Proof.

$$\phi : \mathcal{P}(X) \to \{f : X \to \{0,1\}\} : \phi(A) := \chi_A$$

$$\chi_A : X \to \{0,1\} : \chi_A(x) := \begin{cases} 1 & : x \in A \\ 0 & : x \notin A \end{cases}$$

Show $\phi \leftarrow$ (bij). First show it's injective.

$$\phi(A) = \phi(B)$$

$$\implies \chi_A = \chi_B$$

$$\implies A = B \implies \phi \leftarrow \text{(inj)}$$

Now show it's surjective. $\forall f \in \{f : X \to \{0,1\}\}\ \exists P \in \mathcal{P}(X)$ s/t $\phi(P) = f$.

$$\text{Let } f^{-1}(\{0,1\}) = F^{-1}$$

$$\implies \chi_{F^{-1}} : F^{-1} \to \{0,1\}$$

$$\implies \chi_{F^{-1}} = \phi(F^{-1}) \text{ w/ } \phi(F^{-1}) = f$$

$$F^{-1} \subseteq X \implies F^{-1} \in \mathcal{P}(X) \implies \phi \leftarrow \text{(surj)}$$

□

Proposition 1.7: The Powerset is Larger than the Set. *If* $X \leftarrow$ (set), *then* $|X| < |\mathcal{P}(X)|$.

Proof. Show $|X| \leq |\mathcal{P}(X)|$. $f(x) = \{x\} \leftarrow$ (inj) $\implies |X| \leq |\mathcal{P}(X)|$.
For the sake on contradiction, assume there is a surjection $g : X \to \mathcal{P}(X)$. Consider $B := \{x \in X : x \notin g(x)\}$. Hence there must be (by surjectivity of g) $z \in X$ s/t $g(z) = B$. Someone has to map to it.

$$z \in B \implies z \notin g(z) = B$$

$$z \notin B \implies z \in g(z) = B$$

This is a contradiction. So $|X| < |\mathcal{P}(X)|$. □

Infinite Infinities. $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$

Proposition 1.8: The Natural Powerset Cardinal is the Continuum Cardinal. $|\mathcal{P}(\mathbb{N})| = c$ $(\equiv 2^{\aleph_0} = c \equiv |\{0,1\}|^{|\mathbb{N}|} = |\mathbb{R}|)$

Proof.

> We'll use the continuum hypthesis, however there's an alternative proof in the course notes.

Consider $X = \{f : \mathbb{N} \to \{0,1\}\}$.

$$\phi : X \to \mathbb{R} : \phi(f) := 0.f(1)f(2)f(3)\dots$$

We can see that ϕ is injective. So

$$2^{\aleph_0} = |X| \leq |\mathbb{R}| = c$$

Also, $\aleph_0 < 2^{\aleph_0} \leq c$. So by (CH), we know $2^{\aleph_0} = c$. □

Proof (without (CH)). ... □

1.5 Cardinal Arithmetic

Definition 1.11. $A, B \leftarrow$ (sets)

1. $A \cap B = \emptyset \implies |A| + |B| := |A \cup B|$
2. $|A| \cdot |B| := |A \times B|$
3. $|A|^{|B|} := |\{f : B \to A\}|$

Example. $\aleph_0 + \aleph_0 = \aleph_0$. Let $A = \{a_1, \dots\}$, $B = \{b_1, \dots\}$, so that $|A| = |B| = \aleph_0$, and $A \cap B = \emptyset$.
Then $\phi : A \cup B \to \mathbb{N} : \phi(a_i) := 2i,\ \phi(b_i) := 2i - 1$. This is a bijection. Hence $|A \cup B| = \aleph_0$.

Example. $\aleph_0 + c = c$.
$\aleph_0 = |\mathbb{N}|,\ |(0,1)| = c$.

$$(0,1) \subseteq \mathbb{N} \cup (0,1) \subseteq \mathbb{R}$$

$$\implies c \leq \aleph_0 + c \leq c$$

$$\implies \aleph_0 + c = c$$

Proposition 1.9: Cardinal Exponent Laws. $A, B, C \leftarrow$ (sets).

1. $(|A|^{|B|})^{|C|} = |A|^{|B| \cdot |C|}$
2. $(|A|^{|B|})(|A|^{|C|}) = |A|^{|B|+|C|}$

Example. Show that $c \cdot c = c$.

$$c \cdot c = (2^{\aleph_0})(2^{\aleph_0}) = 2^{\aleph_0 + \aleph_0} = 2^{\aleph_0} = c$$

Proof of 2. We must show

$$|\{f|f : B \cup C \to A\}| = |\{f|f : B \cup A\} \times \{f|f : B \to A\}|$$

$$\text{Let } X := \{f|f : B \to A\}$$

$$\text{Let } Y := \{f|f : C \to A\}$$

$$\text{Let } Z := \{f|f : B \cup C \to A\}$$

So, equivically we need to show $|Z| = |X \times Y|$.

Consider $\varphi(f, g)(x) = \begin{cases} f(x) & x \in B \\ g(x) & y \in C \end{cases}$.

$$\varphi(f_1, g_1) = \varphi(f_2, g_2)$$

$$\implies \forall x \in B \cup C,\ \varphi(f_1, g_1)(x) = \varphi(f_2, g_2)(x)$$

$$\implies \forall x \in B,\ f_1(x) = f_2(x) \implies f_1 = f_2$$

$$\implies \forall x \in C,\ g_1(x) = g_2(x) \implies g_1 = g_2$$

Consider $h : B \cup C \to A$. Let $f = h|_B,\ g = h|_C$. Then $\varphi(f, g) = h$.
So φ is bijective, so proposition 2 holds. □

Example: $c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$.

2 Topology

2.1 Metric Spaces

Definition 2.1: Metric Space. *$X \leftarrow$ (set). A metric on X is a function $d : X \times X \to [0, \infty)$ s/t*

1. $d(x, y) = 0 \iff x = y$
2. ***Abelian:*** $d(x, y) = d(y, x)$
3. ***Triangle:*** $d(x, y) \leq d(x, z) + d(z, y)$

Definition 2.2: Normed Vector Space (NVS). *Let* $V \leftarrow$ (Vector Space) *over* $\mathbb{R}$. *A norm on* V *is a fn* $\|\cdot\| : V \to [0, \infty)$ *s/t*

1. $\|v\| = 0 \iff v = \vec{0}$
2. $\|\alpha v\| = |\alpha| \cdot \|v\|$
3. $\|v + u\| \leq \|v\| + \|u\|$

BQN: $\| \times == | \circ l \cdot \| \circ r \quad |+ \leq\leq + \square|$

Proposition 2.1: NVS have trivial Metrics. *Let* $V, \|\cdot\| \leftarrow$ (NVS). $d(v, w) = \|v - w\|$ *is a metric on* V.

2.2 Examples of Metric Spaces

Example 2.1: Discrete Metric. $X \leftarrow$ (set).

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Example 2.2: Absolute Value Norm. $(\mathbb{R}, |\cdot|) \leftarrow$ (NVS)

Example 2.3: Euclidean Norm. $(\mathbb{R}^n, \|\cdot\|_2) \leftarrow$ (NVS) *where* $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$.

Example 2.4: P-Norm. $p \geq 1$, $(\mathbb{R}^n, \|\cdot\|_p) \leftarrow$ (NVS) *where*

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Note: *see posted notes for the proof that this is a norm. OPTIONAL.*

Example 2.5: Infinity Norm. $p = \infty$, $(\mathbb{R}^n, \|\cdot\|_\infty) \leftarrow$ (NVS) *where*

$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$$

Example 2.6: P-Norm on Sequences of Reals. $\mathbb{R}^{\mathbb{N}} := \{f | f : \mathbb{N} \to \mathbb{R}\} = \{(a_n)_{n=1}^{\infty} : a_n \in \mathbb{R}\}$. *For* $p \geq 1$,

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \tag{1}$$

$l^p := \{x \in \mathbb{R} : \|x\|_p < \infty\} \implies (l^p, \|\cdot\|_p) \leftarrow (\text{NVS})$. *This is the p-norm on sequences of reals. Notice how this solve the divergence issue (by ignoring it lol).*

Example: $l^1 = \{x \in \mathbb{R} : \sum |x_i| < \infty\} \implies l^p$ is the set of absolutly convergent sequences.

Example 2.7: Suprema Norm (Infinity Norm on Sequences of Reals). $\|x\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}$.
if we let $l^\infty := \{x \in \mathbb{R}^{\mathbb{N}} : \|x\|_\infty < \infty\}$, *noting that* l^∞ *is the set of all bounded sequences, then* $(l^\infty, \|\cdot\|_\infty) \leftarrow (\text{NVS})$.

Example 2.8: P-Norm on Function. $C([a, b]) := \{f : [a, b] \to \mathbb{R} | f \leftarrow (\text{cts})\}$.

$$\|f\|_p = \left(\int_a^b |f(x)| \, \mathrm{d}x\right)^{\frac{1}{p}}, \; p \geq 1$$

Example 2.9: Infinity Norm on Functions. $\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\}$

Example 2.10: Bounded Functions and the Infinity Norm are a NVS. $\mathbb{B}([a, b]) = \{f : [a, b] \to \mathbb{R} | f \leftarrow (\text{bd})\}$, $(\mathbb{B}([a, b]), \|\cdot\|_\infty) \leftarrow (\text{NVS})$.