Real Analysis

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Abstract

Real Analysis the study of approximation on the reals.

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1 Cardinality

1.1 Brief Motivation

We want to build a metric space to measure the distance between objects.

We need

1. set X of objects.

2. need to measure closeness. func $d : X \times X \to [0, \infty)$ s/t

   (a) $d(x, y) = 0 \iff x = y$

   (b) $d(x, y) = d(y, x)$

   (c) $d(x, z) \leq d(x, y) + d(y, z)$

We call d a metric on X. (X, d) is a **metric space**.

$(\mathbb{R}, +' \circ^{\circ} (\sqrt{\ }^{=}) -)$ is a metric space. Note the BQN notation.

1.2 Function Theory

Definition 1.1: Injection. *Let A, B be non-empty sets. We say $f : A \to B$ is injective* ***iff*** $\forall a, b \in A \quad f(a) = f(b) \implies a = b$

Definition 1.2: Surjection. *$f : A \to B$ is a surjection if $\forall b \in B \quad \exists a \in A$ s/t $f(a) = b$.*

Definition 1.3: Bijective. *$f : A \to B$ is bijective* ***iff*** *its injective and surjective.*

Definition 1.4: Invertable. *$f : A \to B$ is invertable* ***iff*** *$\exists g : B \to A$ s/t $g(f(a)) = a$ and $f(g(b)) = b \quad \forall a \in A,\, b \in B$.*

We write $g = f^{-1}$ and call it "the" inverse.

Proposition 1.1. *$f : A \to B$ is invertable* ***iff*** *f is bijective.*

Proof. $(\implies)$ f is invertable. Suppose $f(a) = f(b)$. We'll show $a = b$.

$$f^{-1}f(a)) = f^{-1}f(b))$$

$$\implies a = b$$

Now we'll show $\forall b \in B\ \exists a \in A\ f(a) = b$.

$$a = f^{-1}(b) \implies \text{there is way to get from b to a, and it's } f^{-1}$$

($\Longleftarrow$) Assume $f \leftarrow$ (bijective). We'll construct f's inverse. For $b \in B$ let a_b be the unique element of A s/t $f(a_b) = b$. a_b exists b/c of surjectivity of f, and it's unique b/c of injectivity.

$$g := \{g : A \to B,\ g(b) = a_b\}$$

$$f(g(b)) = f(a_b) = b$$

$$g(f(a_b)) = g(b) = a_b$$

$$\implies g = f^{-1}$$

□

Proposition 1.2. $\exists$(*injection*) $f : A \to B \iff \exists$(*surjection*) $g : B \to A$

Proof. ($\implies$) Suppose $f : A \to B \leftarrow$ (injective). Let $b \in B$.
Case 1: $b \in f(A)$.
Let $g(b)$ be the unique element of A s/t $f(g(b)) = b$, unique b/c $f \leftarrow$ (injective)
Case 2: $b \notin f(A)$.
Fix any $z \in A$. Let $g(b) = z$.

$$\implies g(b) = \begin{cases} f^{=}(b) & b \in f(A) \\ z & b \notin f(A) \end{cases}$$

We claim g is a surjection. So we have to show $\forall a \in A, \exists b \in B$ s/t $g(b) = a$ Let $a \in A$ s/t $f(a) \in B$.

$$g(f(a)) \implies f(g(f(a))) = f(a)$$

$$\text{(injective)} \implies g(f(a)) = a$$

$$\implies g \leftarrow \text{(surjective)}$$

$\impliedby$ Suppose $(g : B \to A) \leftarrow$ (surjective). $\forall a \in A$ choose $b_a \in B$ s/t $g(b_a) = a$. $f := \{f : A \to B \quad f(a) = b_a\}$. Suppose

$$f(x) = f(y)$$

$$\implies b_x = b_y$$

$$\implies g(b_x) = g(b_y)$$

$$\implies x = y$$

$$\implies f \leftarrow \text{(injective)}$$

□

Definition 1.5: Powerset. *Let X be a set. Then $\mathcal{P}(X) := \{A : A \subseteq X\}$, called the "**powerset** of X."*

$$X = \{a, b\}$$

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Axiom 1.1: Choice. *Given $X \neq \emptyset \exists a$ choice func $f : \mathcal{P}(X) \backslash \{\emptyset\} \to X$ s/t $f(A) \in A \,\forall \neq A \subseteq A$.*

1.3 Cardinality

$$A = \{a, b\},\ B = \{c, d, e, f\}$$

$$\text{Intuitively } |A| < |B|$$

$$f \leftarrow (\text{inj}) : A \to B,\ f(a) := c,\ f(b) := d$$

$$\implies f \leftarrow (\text{inj})(A) \subset B$$

$$\implies |A| \leq |B|$$

Definition 1.6: Ordering of Cardinality on Sets. *A, B sets.*

1. $|A| \leq |B| \iff \exists f \leftarrow (\text{inj}) : A \to B$
2. $|A| = |B| \iff \exists f \leftarrow (\text{bij}) : A \to B$

$$|\mathbb{N}| \leq |\mathbb{Z}| \impliedby f \leftarrow (\text{inj}) : \mathbb{N} \to \mathbb{Z},\ f(n) := n$$

$$f \leftarrow (\text{bij}) : \mathbb{N} \to \mathbb{Z} : f(n) := \begin{cases} 2n + 2 & : n \geq 0 \\ 2(-n) - 1 & : n < 0 \end{cases} \implies |\mathbb{N}| = |\mathbb{Z}|$$

$$h \leftarrow (\text{bij}) : \mathbb{R} \to (0, 1) : h(x) := \frac{\arctan(x) + \pi/2}{\pi} \implies |\mathbb{R}| = |(0, 1)|$$

Theorem 1.1: Cantor-Schroeder-Berstien (CSB). *if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.*

Lemma 1.1: Phi has a Fixed Point. *X set. Suppose $\exists \phi : \mathcal{P}(X) \to \mathcal{P}(X)$ s/t $\phi(A) \subseteq \phi(B)$ if $A \subseteq B \subseteq X$. Then*

$$\exists F \subseteq X \ s/t \ \phi(F) = F$$

Let $F = \bigcup_{A \subseteq X : A \subseteq \phi(A)} A$.

Note: $\emptyset \subseteq X$ & $\emptyset \subseteq \phi(\emptyset)$

Claim: $F = \phi(F)$. Take $A \subseteq X$ with $A \subseteq \phi(A)$. Then $A \subseteq F$.

$$\implies \phi(A) \subseteq \phi(F)$$

$$\implies A \subseteq \phi(F)$$

$$\implies \bigcup_{A \subseteq X : A \subseteq \phi(A)} A \subseteq \phi(F) \quad \text{(by properties of unions)}$$

$$\implies F \subseteq \phi(F)$$

Further,

$$F \subseteq \phi(F) \implies \phi(F) \subseteq \phi(\phi(F))$$

$$\implies \phi(F) \in \{A \subseteq X : A \subseteq \phi(A)\}$$

For this step, let $\phi(F) = B$ and notice that $B \subseteq \phi(B)$. So its in the set above.

$$\implies \phi(F) \subseteq \bigcup_{A \subseteq X : A \subseteq \phi(A)} A = F$$

$$\implies F = \phi(F)$$

To motivate (CSB) theorem 1.1: prove that $|N| = |N \times N|$.

$$f \leftarrow (\text{inj}) : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, \;\; f(n) := (n, 1)$$

$$g \leftarrow (\text{inj}) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \;\; g((n, m)) := 2^n 3^m$$

By (CSB) $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.

Proof. of (CSB) theorem 1.1. Let $f, g \leftarrow$ (inj) : $A : B$. For $X \subseteq Y \subseteq A$

$$f(X) \subseteq f(Y)$$

$$\implies B \backslash f(Y) \subseteq B \backslash f(X)$$

$$\implies g(B \backslash f(Y)) \subseteq g(B \backslash f(X))$$

$$\implies A \backslash g(B \backslash f(X)) \subseteq A \backslash g(B \backslash f(Y))$$

This letting $\phi : \mathcal{P}(A) \to \mathcal{P}(A) : \phi(x) := A \backslash g(B \backslash f(x))$ insures it preserves $\subseteq$. So by the lemma 1.1, $\exists F \subseteq A$ s/t $F = \phi(F) = A \backslash g(B \backslash f(F))$. In particular, $A \backslash F = g(B \backslash f(F)) \implies g : B \backslash f(F) \to A \backslash F :\leftarrow$ (bij).

Note. It's a surjection b/c everyone in $A \backslash F$ gets mapped to b/c it's the image if $g(B \backslash f(F))$.

Moreover, $g^{-1} : A \backslash F \to B \backslash f(F)$ is a bijection, and $f : F \to f(F)$ is a bijection (for the same reason as above; restriction of domain of an injective function is injective, and a function that maps to its image is automatically a surjection). Hence

$$h : A \to B : h(x) := \begin{cases} g^{-1}(x) & : x \in A \backslash F \\ f(x) & : x \in F \end{cases}$$

□

Show $|\mathbb{Q}| = |\mathbb{N}|$.

Proof.

$$f : \mathbb{N} \to \mathbb{Q} : f(x) := x \implies |\mathbb{N}| \leq |\mathbb{Q}|$$

$q \in \mathbb{Q}$ can be written in the form $q = \frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

$$g : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N} : g(q) := (m, n) \ : \ q = \frac{m}{n}$$

$$\implies |\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{N}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

So by (CSB) theorem 1.1, $|\mathbb{Q}| = |\mathbb{N}|$. □

Definition 1.7: Finite, Countably Infinite, Countable.

1. *a set A is* ***finite*** *<u>iff</u> $|A| = |\{1, 2, \ldots, n\}|$ for some $n \in \mathbb{N}$. In this case, $|A| = n$.*

2. $|\emptyset| := 0$

3. *A is* ***countably infinite*** *<u>iff</u> $|A| = |\mathbb{N}| := \aleph_0$.*

4. *A is countable <u>iff</u> A is finite or ctbly infinite.*

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{Q}| = \aleph_0$$

Proposition 1.3: Aleph Null is the Smallest Infinity. *If A is infinite, then $|\mathbb{N}| \leq |A|$.*

Proof. By (Choice) axiom 1.1, $\exists f : \mathcal{P}(A)\backslash\{\emptyset\} \to A$ s/t $f(X) \in X$, $\forall \emptyset \neq X \subseteq A$.

$$\begin{aligned} \text{Let } a_1 &= f(A) \in A \\ a_2 &= f(A\backslash\{a_1\}) \in A\backslash\{a_1\} \\ &\vdots \end{aligned}$$

$\implies \aleph_0 = |\{a_1, \ldots\}| \leq |A|$. □

$\mathbb{R}$ is uncountable. So $\nexists f \leftarrow$ (bij) : $\mathbb{N} \to \mathbb{R}$.

Proof. Since $|\mathbb{R}| = |(0, 1)|$, we'll show that $(0, 1)$ is uncountable. □