Real Analysis

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Abstract

Real Analysis the study of approximation on the reals.

Contents

1 Cardinality

1.1 Brief Motivation

We want to build a metric space to measure the distance between objects.

We need

1. set X of objects.
2. need to measure closeness. func $d : X \times X \to [0, \infty)$ s/t
   (a) $d(x, y) = 0 \iff x = y$
   (b) $d(x, y) = d(y, x)$
   (c) $d(x, z) \leq d(x, y) + d(y, z)$

We call d a metric on X. (X, d) is a **metric space**.

$(\mathbb{R}, +' \circ^{\circ} (\sqrt{\ }^{=})-)$ is a metric space. Note the BQN notation.

1.2 Function Theory

Definition 1.1: Injection. *Let A, B be non-empty sets. We say $f : A \to B$ is injective **iff** $\forall a, b \in A \quad f(a) = f(b) \implies a = b$*

Definition 1.2: Surjection. *$f : A \to B$ is a surjection if $\forall b \in B \quad \exists a \in A$ s/t $f(a) = b$.*

Definition 1.3: Bijective. *$f : A \to B$ is bijective **iff** its injective and surjective.*

Definition 1.4: Invertable. *$f : A \to B$ is invertable **iff** $\exists g : B \to A$ s/t $g(f(a)) = a$ and $f(g(b)) = b \quad \forall a \in A,\ b \in B$.*

We write $g = f^{-1}$ and call it "the" inverse.

Proposition 1.1. *$f : A \to B$ is invertable **iff** f is bijective.*

Proof. $(\implies)$ f is invertable. Suppose $f(a) = f(b)$. We'll show $a = b$.

$$f^{-1}f(a)) = f^{-1}f(b))$$

$$\implies a = b$$

Now we'll show $\forall b \in B\ \exists a \in A\ f(a) = b$.

$$a = f^{-1}(b) \implies \text{there is way to get from b to a, and it's } f^{-1}$$

($\Longleftarrow$) Assume $f$is(bijective). We'll construct f's inverse. For $b \in B$ let a_b be the unique element of A s/t $f(a_b) = b$. a_b exists b/c of surjectivity of f, and it's unique b/c of injectivity.

$$g := \{g : A \to B,\ g(b) = a_b\}$$

$$f(g(b)) = f(a_b) = b$$

$$g(f(a_b)) = g(b) = a_b$$

$$\implies g = f^{-1}$$

□

Proposition 1.2. $\exists(\textit{injection})\, f : A \to B \iff \exists(\textit{surjection})\, g : B \to A$

Proof. ($\implies$) Suppose $f : A \to B$is(injective). Let $b \in B$.
Case 1: $b \in f(A)$.
Let $g(b)$ be the unique element of A s/t $f(g(b)) = b$, unique b/c $f$is(injective)
Case 2: $b \notin f(A)$.
Fix any $z \in A$. Let $g(b) = z$.

$$\implies g(b) = \begin{cases} f^{=}(b) & b \in f(A) \\ z & b \notin f(A) \end{cases}$$

We claim g is a surjection. So we have to show $\forall a \in A$, $\exists b \in B$ s/t $g(b) = a$ Let $a \in A$ s/t $f(a) \in B$.

$$g(f(a)) \implies f(g(f(a))) = f(a)$$

$$(\text{injective}) \implies g(f(a)) = a$$

$$\implies g\text{is(surjective)}$$

$\impliedby$ Suppose $(g : B \to A)$is(surjective). $\forall a \in A$ choose $b_a \in B$ s/t $g(b_a) = a$. $f := \{f : A \to B \quad f(a) = b_a\}$. Suppose

$$f(x) = f(y)$$

$$\implies b_x = b_y$$

$$\implies g(b_x) = g(b_y)$$

$$\implies x = y$$

$$\implies f\text{is(injective)}$$

$\square$

Definition 1.5: Powerset. *Let X be a set. Then $\mathcal{P}(X) := \{A : A \subseteq X\}$, called the "**powerset** of X."*

$$X = \{a, b\}$$

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Axiom 1.1: Choice. *Given $X \neq \emptyset \exists a$ choice func $f : \mathcal{P}(X) \backslash \{\emptyset\} \to X$ s/t $f(A) \in A \,\forall \neq A \subseteq A$.*

1.3 Cardinality

$$A = \{a, b\},\ B = \{c, d, e, f\}$$

$$\text{Intuitively } |A| < |B|$$

$$f \text{is(inj)} : A \to B,\ f(a) := c,\ f(b) := d$$

$$\implies f\text{is(inj)}(A) \subset B$$

$$\implies |A| \leq |B|$$

Definition 1.6: Ordering of Cardinality on Sets. *A, B sets.*

1. $|A| \leq |B| \iff \exists f\text{is(inj)} : A \to B$
2. $|A| = |B| \iff \exists f\text{is(bij)} : A \to B$

$$|\mathbb{N}| \leq |\mathbb{Z}| \impliedby f\text{is(inj)} : \mathbb{N} \to \mathbb{Z},\ f(n) := n$$

$$f\text{is(bij)} : \mathbb{N} \to \mathbb{Z} : f(n) := \begin{cases} 2n + 2 & : n \geq 0 \\ 2(-n) - 1 & : n < 0 \end{cases} \implies |\mathbb{N}| = |\mathbb{Z}|$$

$$h\text{is(bij)} : \mathbb{R} \to (0, 1) : h(x) := \frac{\arctan(x) + \pi/2}{\pi} \implies |\mathbb{R}| = |(0, 1)|$$

Theorem 1.1: Cantor-Schroeder-Berstien (CSB). *if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.*

Lemma 1.1: Phi has a Fixed Point. *X set. Suppose $\exists \phi : \mathcal{P}(X) \to \mathcal{P}(X)$ s/t $\phi(A) \subseteq \phi(B)$ if $A \subseteq B \subseteq X$. Then*

$$\exists F \subseteq X \text{ s/t } \phi(F) = F$$

Let $F = \bigcup_{A \subseteq X : A \subseteq \phi(A)} A$.

Note: $\emptyset \subseteq X$ & $\emptyset \subseteq \phi(\emptyset)$

Claim: $F = \phi(F)$. Take $A \subseteq X$ with $A \subseteq \phi(A)$. Then $A \subseteq F$.

$$\implies \phi(A) \subseteq \phi(F)$$

$$\implies A \subseteq \phi(F)$$

$$\implies \bigcup_{A \subseteq X : A \subseteq \phi(A)} A \subseteq \phi(F) \quad \text{(by properties of unions)}$$

$$\implies F \subseteq \phi(F)$$

Further,

$$F \subseteq \phi(F) \implies \phi(F) \subseteq \phi(\phi(F))$$

$$\implies \phi(F) \in \{A \subseteq X : A \subseteq \phi(A)\}$$

For this step, let $\phi(F) = B$ and notice that $B \subseteq \phi(B)$. So its in the set above.

$$\implies \phi(F) \subseteq \bigcup_{A \subseteq X : A \subseteq \phi(A)} A = F$$

$$\implies F = \phi(F)$$

To motivate (CSB) theorem 1.1: prove that $|N| = |N \times N|$.

$$f \text{is(inj)} : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, \;\; f(n) := (n, 1)$$

$$g \text{is(inj)} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \;\; g((n, m)) := 2^n 3^m$$

By (CSB) $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.

Proof of (CSB) theorem 1.1. Let $f, g \text{is(inj)} : A : B$. For $X \subseteq Y \subseteq A$

$$f(X) \subseteq f(Y)$$

$$\implies B \backslash f(Y) \subseteq B \backslash f(X)$$

$$\implies g(B \backslash f(Y)) \subseteq g(B \backslash f(X))$$

$$\implies A \backslash g(B \backslash f(X)) \subseteq A \backslash g(B \backslash f(Y))$$

This letting $\phi : \mathcal{P}(A) \to \mathcal{P}(A) : \phi(x) := A \backslash g(B \backslash f(x))$ insures it preserves $\subseteq$. So by the lemma 1.1, $\exists F \subseteq A$ s/t $F = \phi(F) = A \backslash g(B \backslash f(F))$. In particular, $A \backslash F = g(B \backslash f(F)) \implies g : B \backslash f(F) \to A \backslash F : \text{is(bij)}$.

Note. It's a surjection b/c everyone in $A \backslash F$ gets mapped to b/c it's the image if $g(B \backslash f(F))$.

Moreover, $g^{-1} : A \backslash F \to B \backslash f(F)$ is a bijection, and $f : F \to f(F)$ is a bijection (for the same reason as above; restriction of domain of an injective function is injective, and a function that maps to its image is automatically a surjection). Hence

$$h : A \to B : h(x) := \begin{cases} g^{-1}(x) & : x \in A \backslash F \\ f(x) & : x \in F \end{cases}$$

□

Show $|\mathbb{Q}| = |\mathbb{N}|$.

Proof.

$$f : \mathbb{N} \to \mathbb{Q} : f(x) := x \implies |\mathbb{N}| \leq |\mathbb{Q}|$$

$q \in \mathbb{Q}$ can be written in the form $q = \frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

$$g : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N} : g(q) := (m, n) \ : \ q = \frac{m}{n}$$

$$\implies |\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{N}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

So by (CSB) theorem 1.1, $|\mathbb{Q}| = |\mathbb{N}|$. □

Definition 1.7: Finite, Countably Infinite, Countable.

1. *a set A is* ***finite*** *iff* $|A| = |\{1, 2, \ldots, n\}|$ *for some* $n \in \mathbb{N}$. *In this case,* $|A| = n$.
2. $|\emptyset| := 0$
3. *A is* ***countably infinite*** *iff* $|A| = |\mathbb{N}| := \aleph_0$.
4. *A is countable iff A is finite or ctbly infinite.*

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{Q}| = \aleph_0$$

Proposition 1.3: Aleph Null is the Smallest Infinity. *If A is infinite, then* $|\mathbb{N}| \leq |A|$.

Proof. By (Choice) axiom 1.1, $\exists f : \mathcal{P}(A) \backslash \{\emptyset\} \to A$ s/t $f(X) \in X$, $\forall \emptyset \neq X \subseteq A$.

$$\begin{aligned} \text{Let } a_1 &= f(A) \in A \\ a_2 &= f(A \backslash \{a_1\}) \in A \backslash \{a_1\} \\ &\vdots \end{aligned}$$

$\implies \aleph_0 = |\{a_1, \ldots\}| \leq |A|$. $\square$

Proposition 1.4: The Reals are Uncountable. *$\mathbb{R}$ is uncountable. So $\nexists f \text{is(bij)} : \mathbb{N} \to \mathbb{R}$.*

Proof. **Cantor's Diagonal Element Proof.** Since $|\mathbb{R}| = |(0,1)|$, we'll show that $(0,1)$ is uncountable.
For contradiction, assume that there exists a bijection $f : \mathbb{N} \to (0,1)$.
So let's say

$$\begin{aligned} f(1) &= 0.a_{11}a_{12}a_{13}\cdots \\ f(2) &= 0.a_{21}a_{22}a_{23}\cdots \\ f(3) &= 0.a_{31}a_{32}a_{33}\cdots \\ \vdots &= \vdots \end{aligned}$$

Where we avoid repeated nines.
Choose $b_i \in \{1, \dots, 8\}$ s/t $b_i \neq a_{ii}$.

$$\implies \nexists n \in \mathbb{N},\ f(n) = 0.b_1b_2b_3\cdots$$

Thats a contradiction. ∎

Definition 1.8: Continuum. *We write $|\mathbb{R}| = c$, where c stands for **continuum**.*

So we have 3 cardinals: $n, \aleph_0, c$.

Axiom 1.2: Continuum Hypothesis. *If A is a set with $\aleph_0 \leq |A| \leq c$, then $\aleph_0 = |A|$ or $|A| = c$.*

1.4 Cardinality of Power Sets

Proposition 1.5. *If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.*

Proof.

$$|\mathcal{P}(A)| = \sum_{k=1}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

∎

Definition 1.9: Cartesian Product. *Let I be a set.* $\forall i \in I$ *Let* $A_i \text{is(set)} \implies \prod_{i\in I} A_i := \{f | f : I \to \bigcup A_i,\ f(i) \in A_i\}$

$$f(i) \in A_i$$

$$I = \mathbb{N} \implies f : \mathbb{N} \to \bigcup A_i : f(i) \in A_i \equiv (f(1), f(2), \dots)$$

Definition 1.10: Set Power. $A, B \text{is(set)} \implies A^B = \{f : B \to A\}$

$$|A|^{|B|} := \left|A^B\right| = |\{f : B \to A\}|$$

Proposition 1.6: Cardinality of a Power Set. *if* $X \text{is(set)},\ \mathcal{P}(X) = 2^{|X|} = |\{f : X \to \{0,1\}\}|$.

Proof.

$$\phi : \mathcal{P}(X) \to \{f : X \to \{0,1\}\} : \phi(A) := \chi_A$$

$$\chi_A : X \to \{0,1\} : \chi_A(x) := \begin{cases} 1 & : x \in A \\ 0 & : x \notin A \end{cases}$$

Show $\phi \text{is(bij)}$. First show it's injective.

$$\phi(A) = \phi(B)$$

$$\implies \chi_A = \chi_B$$

$$\implies A = B \implies \phi \text{is(inj)}$$

Now show it's surjective. $\forall f \in \{f : X \to \{0,1\}\}\ \exists P \in \mathcal{P}(X)$ s/t $\phi(P) = f$.

$$\text{Let } f^{-1}(\{0,1\}) = F^{-1}$$

$$\implies \chi_{F^{-1}} : F^{-1} \to \{0,1\}$$

$$\implies \chi_{F^{-1}} = \phi(F^{-1}) \text{ w/ } \phi(F^{-1}) = f$$

$$F^{-1} \subseteq X \implies F^{-1} \in \mathcal{P}(X) \implies \phi \text{is(surj)}$$

$\square$

Proposition 1.7: The Powerset is Larger than the Set. *If* $X$is(set)*, then* $|X| < |\mathcal{P}(X)|$.

Proof. Show $|X| \leq |\mathcal{P}(X)|$. $f(x) = \{x\}$is(inj) $\implies |X| \leq |\mathcal{P}(X)|$.
For the sake on contradiction, assume there is a surjection $g : X \to \mathcal{P}(X)$. Consider $B := \{x \in X : x \notin g(x)\}$. Hence there must be (by surjectivity of g) $z \in X$ s/t $g(z) = B$. Someone has to map to it.

$$z \in B \implies z \notin g(z) = B$$

$$z \notin B \implies z \in g(z) = B$$

This is a contradiction. So $|X| < |\mathcal{P}(X)|$. □

Infinite Infinities. $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$

Proposition 1.8: The Natural Powerset Cardinal is the Continuum Cardinal. $|\mathcal{P}(\mathbb{N})| = c$ $(\equiv 2^{\aleph_0} = c \equiv |\{0,1\}|^{|\mathbb{N}|} = |\mathbb{R}|)$

Proof.

We'll use the continuum hypthesis, however there's an alternative proof in the course notes.

Consider $X = \{f : \mathbb{N} \to \{0,1\}\}$.

$$\phi : X \to \mathbb{R} : \phi(f) := 0.f(1)f(2)f(3)\dots$$

We can see that ϕ is injective. So

$$2^{\aleph_0} = |X| \leq |\mathbb{R}| = c$$

Also, $\aleph_0 < 2^{\aleph_0} \leq c$. So by (CH), we know $2^{\aleph_0} = c$. □

Proof (without (CH)). ... □

1.5 Cardinal Arithmetic

Definition 1.11. $A, B$is(sets)

1. $A \cap B = \emptyset \implies |A| + |B| := |A \cup B|$
2. $|A| \cdot |B| := |A \times B|$
3. $|A|^{|B|} := |\{f : B \to A\}|$

Example. $\aleph_0 + \aleph_0 = \aleph_0$. Let $A = \{a_1, \dots\}$, $B = \{b_1, \dots\}$, so that $|A| = |B| = \aleph_0$, and $A \cap B = \emptyset$.
Then $\phi : A \cup B \to \mathbb{N} : \phi(a_i) := 2i,\ \phi(b_i) := 2i - 1$. This is a bijection. Hence $|A \cup B| = \aleph_0$.

Example. $\aleph_0 + c = c$.
$\aleph_0 = |\mathbb{N}|,\ |(0,1)| = c$.

$$(0,1) \subseteq \mathbb{N} \cup (0,1) \subseteq \mathbb{R}$$

$$\implies c \leq \aleph_0 + c \leq c$$

$$\implies \aleph_0 + c = c$$

Proposition 1.9: Cardinal Exponent Laws. $A, B, C$is(sets).

1. $(|A|^{|B|})^{|C|} = |A|^{|B| \cdot |C|}$
2. $(|A|^{|B|})(|A|^{|C|}) = |A|^{|B| + |C|}$

Example. Show that $c \cdot c = c$.

$$c \cdot c = (2^{\aleph_0})(2^{\aleph_0}) = 2^{\aleph_0 + \aleph_0} = 2^{\aleph_0} = c$$

Proof of 2. We must show

$$|\{f|f : B \cup C \to A\}| = |\{f|f : B \cup A\} \times \{f|f : B \to A\}|$$

$$\text{Let } X := \{f|f : B \to A\}$$

$$\text{Let } Y := \{f|f : C \to A\}$$

$$\text{Let } Z := \{f|f : B \cup C \to A\}$$

So, equivically we need to show $|Z| = |X \times Y|$.

Consider $\varphi(f,g)(x) = \begin{cases} f(x) & x \in B \\ g(x) & y \in C \end{cases}$.

$$\varphi(f_1, g_1) = \varphi(f_2, g_2)$$

$$\implies \forall x \in B \cup C,\ \varphi(f_1, g_1)(x) = \varphi(f_2, g_2)(x)$$

$$\implies \forall x \in B,\ f_1(x) = f_2(x) \implies f_1 = f_2$$

$$\implies \forall x \in C,\ g_1(x) = g_2(x) \implies g_1 = g_2$$

Consider $h : B \cup C \to A$. Let $f = h|_B,\ g = h|_C$. Then $\varphi(f,g) = h$.

So φ is bijective, so proposition 2 holds. □

Example: $c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$.

2 Topology

2.1 Metric Spaces

Definition 2.1: Metric Space. $X$is(set). *A metric on X is a function $d : X \times X \to [0, \infty)$ s/t*

1. $d(x,y) = 0 \iff x = y$
2. ***Abelian:*** $d(x,y) = d(y,x)$
3. ***Triangle:*** $d(x,y) \leq d(x,z) + d(z,y)$

Definition 2.2: Normed Vector Space (NVS). *Let* $V$is(Vector Space) *over* $\mathbb{R}$. *A norm on* V *is a fn* $\|\cdot\| : V \to [0, \infty)$ *s/t*

1. $\|v\| = 0 \iff v = \vec{0}$
2. $\|\alpha v\| = |\alpha| \cdot \|v\|$
3. $\|v + u\| \leq \|v\| + \|u\|$

BQN: $\| \times == | \circ l \cdot \| \circ r \quad | + \leq\leq + \Box|$

Proposition 2.1: NVS have trivial Metrics. *Let* $V, \|\cdot\|$is(NVS). $d(v, w) = \|v - w\|$ *is a metric on* V.

2.2 Examples of Metric Spaces

Example 2.1: Discrete Metric. $X$is(set).

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Example 2.2: Absolute Value Norm. $(\mathbb{R}, |\cdot|)$is(NVS)

Example 2.3: Euclidean Norm. $(\mathbb{R}^n, \|\cdot\|_2)$is(NVS) *where* $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$.

Example 2.4: P-Norm. $p \geq 1$, $(\mathbb{R}^n, \|\cdot\|_p)$is(NVS) *where*

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Note: *see posted notes for the proof that this is a norm. OPTIONAL.*

Example 2.5: Infinity Norm. $p = \infty$, $(\mathbb{R}^n, \|\cdot\|_\infty)$is(NVS) *where*

$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$$

Example 2.6: P-Norm on Sequences of Reals. $\mathbb{R}^{\mathbb{N}} := \{f | f : \mathbb{N} \to \mathbb{R}\} = \{(a_n)_{n=1}^{\infty} : a_n \in \mathbb{R}\}$. *For* $p \geq 1$,

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \tag{1}$$

$l^p := \{x \in \mathbb{R} : \|x\|_p < \infty\} \implies (l^p, \|\cdot\|_p)\text{is(NVS)}$. *This is the p-norm on sequences of reals. Notice how this solve the divergence issue (by ignoring it lol).*

Example: $l^1 = \{x \in \mathbb{R} : \sum |x_i| < \infty\} \implies l^p$ is the set of absolutly convergent sequences.

Example 2.7: Suprema Norm (Infinity Norm on Sequences of Reals). $\|x\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}$.
if we let $l^\infty := \{x \in \mathbb{R}^{\mathbb{N}} : \|x\|_\infty < \infty\}$, *noting that* l^∞ *is the set of all bounded sequences, then* $(l^\infty, \|\cdot\|_\infty)\text{is(NVS)}$.

Example 2.8: P-Norm on Function. $C([a, b]) := \{f : [a, b] \to \mathbb{R} | f\text{is(cts)}\}$.

$$\|f\|_p = \left(\int_a^b |f(x)| \,\mathrm{d}x\right)^{\frac{1}{p}},\ p \geq 1$$

Example 2.9: Infinity Norm on Functions. $\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\}$

Example 2.10: Bounded Functions and the Infinity Norm are a NVS. $\mathbb{B}([a, b]) = \{f : [a, b] \to \mathbb{R} | f\text{is(bd)}\}$, $(\mathbb{B}([a, b]), \|\cdot\|_\infty)\text{is(NVS)}$.

Example 2.11: Sequence Metric. $X = \mathbb{R}^{\mathbb{N}} = \{f | f : \mathbb{N} \to \mathbb{R}\}$.

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

Prove that d isn't induced by a metric. If $d(x,y) = \|x - y\|$ for some norm, then $\|\alpha x - \alpha y\| = |\alpha| \|x - y\|$.

$$d(ax, ay) = \sum_{i=1}^{\infty} \frac{|ax_i - ay_i|}{2^i(1 + |ax_i - ay_i|)}$$

$$d(ax, ay) = \sum_{i=1}^{\infty} \frac{|a||x_i - y_i|}{2^i(1 + |a||x_i - y_i|)}$$

$$|a| d(x,y) = |a| \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

$$|a| d(x,y) = \sum_{i=1}^{\infty} \frac{|a||x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

b/c $|a||x_i - y_i| \neq |x_i - y_i|$

$\implies$ not induced by a norm

$\square$

Example 2.12: Cantor Space. $X = 2^{\mathbb{N}} := \{f | f : \mathbb{N} \to \{0, 1\}\}$.

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

Example 2.13: Hamming Distance. $X \text{is(finite)}$. $A, B \in \mathcal{P}(X)$.

$$d(A, B) := |A \triangle B| = |(A \cup B) \backslash (A \cap B)|$$

Example 2.14: Hausdorff Metric. $\mathcal{H} = \{K \subseteq \mathbb{R}^n : K \text{ compact}\}$. *Let* $a \in A,\ b \in B,\ A, B \in \mathcal{H}$.

$$d(a, B) = \min\{\|a - b\| : b \in B\}$$

$$d(b, A) = \min\{\|a - b\| : a \in A\}$$

$$d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$$

Note 2.1. $\sup_{a\in A} d(a, B)$ *represents the biggest shortest path between* A *and* B.

Note 2.2. *Metrics give a sense of convergence on a space.*

Example 2.15: P-adic Metric. *Let* p *be prime,* $X = \mathbb{Q}$.

$$\text{Let } 0 \neq q \in X = \mathbb{Q} \quad q = p^a \frac{n}{m}$$

$$\text{where } \gcd(n, m) = \gcd(p, n) = \gcd(p, m) = 1$$

$$|q|_p = \frac{1}{p^a},\ |0|_p = 0$$

$$d(q_1, q_2 := |q_1 - q_2|_p$$

Note 2.3. *This notion of distance implies the more factors of p, the closer.*
This gives a sense of optimizing for a certain adjective.
These numbers aren't close using $\|\cdot\|_2$, *but are using the p-adic norm.*

Definition 2.3: Subspace of a Metric Space. $(X, d),\ Y \subseteq X \implies (Y, d)$. (Y, d) *is called a* ***subspace*** *of* (X, d).

Definition 2.4. $(X, d_1),\ (Y, d_2)$. *Consider* $(X \times Y, d)$ *with*

$$d((x_1, y_1), (x_2, y_2)) := d_1(x_1, x_2) + d_2(y_1, y_2) \text{ (1-norm)}$$
$$\text{or } d((x_1, y_1), (x_2, y_2)) := \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \text{ (∞-norm)}$$

Example 2.16: Product Metric. (X_i, d_i) $i \in \mathbb{N}$, $X := \prod_{n=1}^{\infty} X_i$. *Let* $x = (x_1, x_2, \dots)$ *and* $y = (y_1, y_2, \dots)$.

$$d(x,y) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i(1 + d_i(x_i, y_i))}$$

2.3 Convergence

Definition 2.5: Convergence of a Sequence. (X, d), $(x_n) \subseteq X$, *and* $x \in X$.

Notation 2.1. $(x_n) \subseteq X$ *means* (x_n) *is a sequence in* X.

$$(x_n) \text{ conv to } x,\ x_n \to x \textbf{ iff}$$

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s/t } \forall n \geq N,\ d(x_n, x) < \epsilon$$

Definition 2.6: Divergence. (x_n) *diverges of* $\nexists x \in X$ *s/t* $x_n \to x$.

Note 2.4: Convergence is Distance going to Zero. (X, d), $(x_n) \in X$, $x \in x$. *Then* $x_n \to x$ ***iff*** $d(x_n, x) \to 0$.

Definition 2.7: Cauchy. (X, d). $(x_n) \subseteq X$ *is a cauchy seq* ***iff***

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s/t } \forall n, m \geq N,\ d(x_n, x_m) < \epsilon$$

Proposition 2.2: Convergence implies Cauchyness. (X, d). *If* $(x_n) \subseteq X$ *converges, then* (x_n) *is cauchy.*

Epsilon/2. Suppose $(x_n) \to x$. Let $\epsilon > 0$. So $\exists N$ s/t $\forall n \geq N$,

$$d(x_n, x) < \gamma$$

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x, x_m) \text{ by } \triangle \\ &< \gamma + \gamma = 2\gamma \\ &:= 2\frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

□

Example 2.17: Cauchy doesn't imply Convergence. *$X = (0, 1]$ with the std metric.*

$$\frac{1}{n} \to 0 \implies \left(\frac{1}{n}\right) \subseteq X \textit{ is cauchy}$$

Note 2.5. *I think this is trying to say that $1/n$ is cauchy in X, but $1/n \to 0$, which is not in X, so it diverges (in X).*

Definition 2.8: Bounded.

1. *$A \subseteq X$ is bd <u>iff</u>* $\sup\{d(x, y) : x, y \in A\} < \infty$
2. $(x_n) \subseteq X \text{is(bd)} \iff \{x_1, x_2, \ldots\}\text{is(bd)} \iff \sup\{n, m \in \mathbb{N} : d(x_n, x_m)\} < \infty$

Definition 2.9: Open and Closed Balls.

1. ***Open:*** $B_r(a) := \{x \in X : d(x, a) < r\}$
2. ***Clsd:*** $B_r[a] := \{x \in X : d(x, a) \leq r\}$

Proposition 2.3: Boundedness iff subset of a Closed Ball. *(X, d), $A \subseteq X$. Then A is bd <u>iff</u> $\exists r > 0$, $\exists x \in X$ s/t $A \subseteq B_r[x]$*

Proof. Suppose $\sup\{d(x, y) : x, y \in A\} = r < \infty$. Assume $A \neq \emptyset$, taking $a \in A$. For $b \in A$, $d(a, b) \leq r \implies A \subseteq B_r[a]$.
Assume $A \subseteq B_r[a] \implies \forall a, b \in A,\ d(a, b) \leq d(a, x) + d(x, b) \leq 2r$ □

Proposition 2.4: Cauchy implies Bounded. $(x_n)\text{is}(\text{cauchy}) \implies (x_n)\text{is}(\text{bd})$

Example 2.18: Counter Example. $(0, 1, 0, 1, 0, \dots)$ *is bounded but not cauchy.*

Note 2.6. *CONVERGENCE* $\implies$ *CAUCHY* $\implies$ *BOUNDED*

Proof. Suppose (x_n) is cauchy. Let $\epsilon = 1$.

$$\exists N \in \mathbb{N} \text{ s/t } \forall n, m \geq N \; d(x_n, x_m) < 1$$

Let $r := \max\{d(x_1, x_N), \dots, d(x_{N-1}, x_N)\}$

Note 2.7. *We did this because the web's edges are no longer than 1. So, we can look at the* finite *part left. Finite sets are bounded, and the "web" is bounded.*

$$(x_n) \subseteq B_r[x_N]$$

□

2.4 Convergence Examples

Example 2.19: 2-Adic Norm. *Consider* $(\mathbb{Q}, |\cdot|_2)$*. Let* $x_n := \frac{1+2^n}{3}$.

Note 2.8: P-Adic Convergence Claims. *Looking at* $x_n = \frac{1}{3} + \frac{2^n}{3} = \frac{1}{3} + \cancelto{0}{\frac{2^n}{3}} = \frac{1}{3}$.

We claim $x_n \to \frac{1}{3}$.

Proof.

$$\begin{aligned} \left| x_n - \frac{1}{3} \right|_2 &= \left| \frac{2^n}{3} \right|_2 \\ &= \frac{1}{2^n} \\ &\to 0 \end{aligned}$$

So $x_n \to \frac{1}{3}$ under $|\cdot|_2$. □

Example 2.20: Bounded Sequences and the Infinity Norm. $(l^\infty, \|\cdot\|_\infty)$.
Let $x_n := \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0, \ldots\right)$ *and* $x := \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$
We claim that $x_n \to x$.

Proof.

$$\begin{aligned} &\|x_n - x\|_\infty \\ &= \sup\left(0, 0, \ldots, 0, \frac{1}{n+1}, \frac{1}{n+2}, \ldots\right) \\ &= \frac{1}{n+1} \to 0 \end{aligned}$$

□

Example 2.21: Zero Tailed Sequences aren't Cauchy under Sup Norm.

$$y_n := (\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, 0, \ldots)$$

$$y := (1, 1, 1, 1, \ldots)$$

$$n \neq m, \ \|y_n - y_m\|_\infty = 1$$

2.5 Completeness

Definition 2.10: Complete, Complete Metric Space, Banach Space. (X, d), $A \subseteq X$. *Then*

1. *A is **complete** iff every cauchy seq in A converges to some $a \in A$.*
2. *if X is complete, we call it a **Complete Metric Space**.*
3. *A complete normed vector space is called a **Banach Space***

Example 2.22.

$$X = (0, 1], \ \frac{1}{n} \to 0 \notin X \implies (\text{div}) \implies X \not\text{is}(\text{comp})$$

Example 2.23.

$$A = [1/2, 1] \subseteq X \text{is}(\text{comp})$$

Example 2.24. $(X, d := \text{(discrete)})$ *Let* $(x_n) \in \mathbb{R}^{\mathbb{N}}$ *be cauchy.*

$$\exists N \in \mathbb{N} \ s/t \ \forall n, m \geq N \implies d(x_n, x_m) < 1$$

$$\implies d(x_n, x_m) = 0$$

$$\implies x_n = x_m$$

$$\implies x_n = (x_1, x_2, \ldots, x_N, x_N, x_N, \ldots) \to x_N$$

$$\implies X \text{is(complete)}$$

Note 2.9. *So nice sets or nice metrics can cause completeness.*

Example 2.25. *Show* $(l^\infty, \|\cdot\|_\infty)$ *is a Banach Space.*
Let $(x_n) \subseteq l^\infty$, *example 2.7. We know already that* $(l^\infty, \|\cdot\|_\infty)$ *is a (NVS), so we have to show it's complete. Let* $\epsilon > 0$ *be given. Then,*

$$\exists N \in \mathbb{N}\ s/t\ n, m \geq N \implies \|x_n - x_m\|_\infty < \epsilon$$

$$x_k = (x_k[1], x_k[2], \dots)$$

$$for\ n, m \geq N,\ |x_n[i] - x_m[i]|$$

$$\leq \sup\{|x_n[i] - x_m[i]| : i \in \mathbb{N}\}$$

$$= \|x_n - x_m\|_\infty < \epsilon$$

$$\implies \forall i \in \mathbb{N},\ the\ seq\ (x_n[i])_{n=1}^{\infty}\text{is(cauchy in }\mathbb{R})$$

$$\mathbb{R}\text{is(comp)} \implies x_n[i] \underset{n}{\to} b_i$$

$$Claim\text{: } x_n \to b := (b_1, b_2, \dots)$$

$$\forall n, m \geq N,\ |x_n[i] - x_m[i]| < \epsilon$$

$$\implies \lim_{m\to\infty} |x_n[i] - x_m[i]| \leq \epsilon$$

$$\implies \forall n \geq N,\ |x_n[i] - b_i| \leq \epsilon$$

$$Consider\ \|x_n - b\|_\infty$$

$$= \sup\{|x_n[i] - b_i| : i \in \mathbb{N}\}$$

$$\leq \epsilon < 2\epsilon$$

Hence $x_n \to b$.
Note: *we have that* $x_N - b \in l^\infty$, *and* $x_N \in l^\infty$. *However* l^∞ *is a (VS), so* $b \in l^\infty$.

Proposition 2.5: Set of bounded Sequences on the P-Norm is Banach.
$(\ell^p, \|\cdot\|_p)$is(banach).

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$$

$$\ell^p := \{x \in \mathbb{R} : \|x\|_p < \infty\} \implies (\ell^p, \|\cdot\|_p)\text{is(NVS)}$$

Proof. Let $(a_k) \subseteq \ell^p$ be cauchy.

$$\text{Say } a_k = (a_k[1], a_k[2], \dots)$$

$$\text{Let } \epsilon > 0$$

$$\exists N \in \mathbb{N} \text{ s/t } \|a_k - a_m\| < \epsilon,\ \forall k, m \geq N$$

$$\text{Fixing } i \in \mathbb{N},\ \text{Since } |a_k[i] - a_m[i]| \leq \|a_k - a_m\|_p < \epsilon$$

$$\text{We see that } (a_k[i])_{k=1}^{\infty} \text{is(cauchy in } \mathbb{R})$$

$$\mathbb{R}\text{is(comp)} \implies a_k[i] \to b_i \text{ for some } b_i \in \mathbb{R}$$

$$\text{Claim: } a_k \to b = (b_1, b_2, \dots)$$

$$\forall k, m \geq N,\ \text{ we see that}$$

$$\sum_{i=1}^{M} |a_k[i] - a_m[i]|^p$$

$$\leq \sum_{i=1}^{\infty} |a_k[i] - a_m[i]|^p$$

$$= \|a_k - a_m\|_p^p < \epsilon^p$$

$$\sum_{i=1}^{M} |a_k[i] - b_i|^p \leq \epsilon^p,\ \forall M \in \mathbb{N}$$

$$M \to \infty : \sum_{i=1}^{\infty} |a_k[i] = b_i|^p \leq \epsilon^p$$

$$\implies \|a_k - b\|_p \leq \epsilon,\ \forall k \geq N$$

Noting that $a_N, a_N - b \in \ell^p \implies b \in \ell^p$. $\square$

Example 2.26.

$$C_{00} = \{(x_n) \in \ell^\infty : \exists N \in \mathbb{N} \ s/t \ \forall n \geq N, \ x_n = 0\}$$

$$\text{is(NVS)}, \ via \ \|\cdot\|_\infty$$

Consider

$$x_n = (1, 1/2, \ldots, 1/n, 0, 0, 0, \ldots)$$

$$x_n \text{is(cauchy, divergence)} \ b/c$$

$$x_n \to (1, 1/2, 1/3, \ldots) \notin C_{00} \implies C_{00}\text{is}(\neg\text{comp})$$

2.6 Topological Spaces

Definition 2.11: Topology. *X set. A* ***topology*** *on X is a set $\mathcal{T} \subseteq \mathcal{P}(X)$ s/t*

1. $\emptyset, X \in \mathcal{T}$
2. $U, V \in \mathcal{T} \implies U \cap V \in \mathcal{T}$
3. $U_i \in \mathcal{T}, \ (i \in I) \implies \bigcup_{i \in I} U_i \in \mathcal{T}$

Example 2.27: Discrete Topology. *X set.* $\mathcal{T} := \mathcal{P}(X)$

Example 2.28: Indiscrete Topology. $\mathcal{T} := \{\emptyset, X\}$

Example 2.29. $X = \{a, b, c\}$

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{b, c\}\}$$

$$\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$$

Notation 2.2: Topological Space. *If $\mathcal{T}$ is a topology on X, then $(X, \mathcal{T})$ is called a* ***topological space****.*

2.7 Metric Topology

Definition 2.12: Open. *$(X, \mathcal{T})$. $U \subseteq X$ is open iff $\forall x \in U$, $\exists r > 0$ s/t $B_r(x) \subseteq U$.*

Proposition 2.6: The Set of Open sets form a Topology. *$\mathcal{T} = \{U \subseteq X : U$ open$\}$ is a topology.*

Proof. $\emptyset, X \subseteq \mathcal{T}$ trivially.
Let $U, V \in \mathcal{T}$. Since U & V are open

$$\exists r_1, r_2 > 0 \text{ s/t} \quad B_{r_1}(x) \subseteq U \quad B_{r_2}(x) \subseteq V$$

$$r := \min\{r_1, r_2\} \quad B_r(x) \subseteq U \cap V \implies U \cap V \in \mathcal{T}$$

Let $U_i \in \mathcal{T}$ for all $i \in I$. Let $x \in \bigcup_{i\in I} U_i$.

$$\text{So } \exists i \in I \text{ s/t } x \in U_i$$

$$\text{So } \exists r > 0 \text{ s/t } B_r(x) \subseteq U_i \subseteq \bigcup U_i$$

□

Example 2.30: Counterexample for Infinte Intersections are Open.

$$(\mathbb{R}, \mathcal{T}_{\text{open}}) \quad U_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \quad \bigcap U_n = \{0\} \notin \mathcal{T}$$

Proposition 2.7: Metric Spaces are Haussdorff. *(X, d) $\forall x \neq y \in X$, $\exists U, V \subseteq X$ open s/t $x \in U$, $y \in V$, $U \cap V = \emptyset$*

Proof. Let $r = d(x, y) > 0$. Let $U = B_{r/2}(x)$ $V = B_{r/2}(y)$.

Open Balls are Open Proof. Let $x \in B_r(a)$ for some $a \in X$, $r > 0$. Then

$$d(x, a) < r \text{ by open ball def}$$

$$y :\in B_{r-d(x,a)}(x)$$

$$\implies d(x, y) < r - d(x, a)$$

$$d(y, a) \overset{\text{(TI)}}{<} d(x, y) + d(x, a) < r$$

$$\implies y \in B_r(a) \implies B_{r-d(x,a)}(x) \subseteq B_r(a) \implies B_r(a) \text{ is open}$$

□

Assume for the sake of contradiction $\exists z \in B_{r/2}(x) \cap B_{r/2}(y)$.

$$\implies d(z, x) < r/2 \quad d(z, y) < r/2$$

$$r > d(x, z) + d(z, y) > d(x, y) = r$$

This is a contradiction. □

Example 2.31. $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, X, \{a, b\}\}$. *Since a cannot be "seperated" from b, then there is no possible metric on X, so $\mathcal{T}$ isn't a metric topology.*
All metrics make a topology, not all topologies make a metric.

2.8 Closed Sets

Definition 2.13: Closed Sets. $(X, \mathcal{T})$. *$C \subseteq X$ is closed iff $X \backslash C \in \mathcal{T}$*

Proposition 2.8: Properties of Closed Sets. $(X, \mathcal{T})$.

1. *$\emptyset, X$ closed.*
2. *C, D closed then $C \cup D$ closed.*
3. *C_i, $i \in I$, $\bigcap_{i \in I} C_i$ is closed.*

Proof by "Boolean Nonsense". $X \setminus C,\ X \setminus D \in \mathcal{T}$.

$$X \setminus (C \cup D) \in \mathcal{T}$$

$$X \setminus C \cap X \setminus D \in \mathcal{T}$$

$\square$

Definition 2.14: Limit Point. $(X, \mathcal{T})$, $A \subseteq X$. *We say $x \in X$ is a **limit point** of A iff*

$$\forall U \in \mathcal{T} \text{ with } x \in U \quad A \cap U \neq \emptyset$$

Note 2.10. $x \in A \implies x \text{is(limit point)}$

Proposition 2.9: Limits Points in a Metric Topology are the Limit of a Sequence. $A \in X$. *Then $x \in X$ is a limit point of A iff $\exists (a_n) \in A$ s/t $a_n \to x$.*

$(\implies)$ *Proof.* Assume x is a limit a point of A. Then $\forall U \in \mathcal{T},\ x \in U,\ A \cap U \neq \emptyset$. Then $\forall n \in \mathbb{N},\ \exists a_n \in B_{\frac{1}{n}}(x) \cap A \quad (\neq \emptyset)$. Then $d(x, a_n) < \frac{1}{n} \to 0 \implies a_n \to x$ $\square$

$(\impliedby)$ *Proof.* Assume $\exists a_n \to x$. Then $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ s/t $d(a_n, x) < \epsilon$. Let $U \subseteq X$ be open with $x \in U$.

$$\implies \exists r > 0 \text{ s/t } B_r(x) \subseteq U$$

$$\text{Then } \exists N \in \mathbb{N} \text{ s/t } d(a_N, x) < r$$

$$\text{So } a_N \in U \implies A \cap U \ni a_N$$

$\square$

Proposition 2.10: Closed Sets Attain Limits. *C is closed iff C attains its limits points.*

$(\implies)$ *Proof.* Suppose C is clsd. Let $x \in X$ be a limit point of C. Since $X \setminus C$ is open and $(X \setminus C) \cap C = \emptyset$, $x \notin X \setminus C \implies x \in C$. $\square$

($\Longleftarrow$) *Proof.* Suppose C attains its limit points. Show $X \setminus C \in \mathcal{T}$.

$$\text{Let} x \in X \setminus C$$

$$\text{So } x \text{ isn't a limit point of } C$$

$$\text{then} \exists U_x \in \mathcal{T},\ x \in U_x \text{ s/t } U_x \cap C = \emptyset$$

$$\text{then} U_x \subseteq U_x \cap C \quad (?)$$

$$\text{then} X \setminus C = \bigcup_{x \in X \setminus C} U_x \in \mathcal{T}$$

$\square$

Corollary 2.1: Closed Sets Attain Sequence Limits. $(X, d),\ C \subseteq X$. *Then C is closed* <u>*iff*</u>

$$\forall (c_n) \subseteq C,\ c_n \to x \in X \text{then} x \in C$$

Definition 2.15: Subspace Topology. $(X, \mathcal{T})$, $Y \subseteq X$.
Note *that* $\varphi, X \in \mathcal{T}$, $\mathcal{T}$ *closed under* $(\cup, \cap)$
The ***subspace topology*** *is*

$$\mathcal{T}' = \{Y \cap U : U \in \mathcal{T}\}$$

Prove subspace topologies are Topologies.

1. Show
$$\emptyset, Y \in \mathcal{T}'$$
So,
$$\emptyset \in \mathcal{T},\ Y \cap \emptyset = \emptyset \implies \emptyset \in \mathcal{T}'$$
$$X \in \mathcal{T},\ Y \cap X = Y \implies Y \in \mathcal{T}'$$

2. Show
$$U, V \in \mathcal{T}' \implies U \cap V \in \mathcal{T}'$$
So let $U, V \in \mathcal{T}'$.
$$\implies \exists \text{open } U_X \in X \text{ s/t } U = U_X \cap Y$$
$$\implies \exists \text{open } V_X \in X \text{ s/t } V = V_X \cap Y$$
$$\implies U \cap V = (U_X \cap Y) \cap (V_X \cap Y) = (U_X \cap V_X) \cap Y$$
$$U_X \cap V_X \in \mathcal{T} \implies U \cap V \in \mathcal{T}'$$

3. Show
$$U_i \in \mathcal{T}',\ (i \in I) \implies \bigcup_{i \in I} U_i \in \mathcal{T}'$$
So
$$U_i \in \mathcal{T}' \implies U_i = Y \cap U_i^X \text{ for some open } U_i^X \in \mathcal{T}$$
$$\bigcup U_i = Y \cap \bigcup U_i^X \implies U_i \in \mathcal{T}'$$

$\square$

Note 2.11. $(X, \mathcal{T})$, $Y \subseteq X$, $\mathcal{T}'$ *as above.* $C \subseteq Y$ *clsd.*

$$\implies Y \setminus C \in \mathcal{T}'$$

$$\implies Y \setminus C = Y \cap U,\ U \in \mathcal{T}$$

$$\implies C = Y \cap \underbrace{(X \setminus U)}_{clsd}$$

Note 2.12. (X, d), $Y \subseteq X$. *Define* (Y, d) *as a subspace metric space. Suppose* $U \subseteq Y$ *is open wrt* Y.

$$\implies \forall x \in U\ \exists r_x > 0\ s/t\ \underbrace{B_{r_x}(x)}_{in\ Y} \subseteq U$$

$$\implies \forall x \in U\ \exists r_x > 0\ s/t\ \underbrace{B_{r_x}(x)}_{in\ X} \cap Y \subseteq U$$

$$U = \bigcup_{x \in U} (Y \cap B_{r_x}(x)) = Y \cap \underbrace{\left(\bigcup_x B_{r_x}(x) \right)}_{open\ in\ X}$$

2.9 Closure and Interior

Definition 2.16: Closure and Interior.

1. *the* ***Closure*** *of* A *is defined as follows:*

$$\overline{A} = \bigcup_{\substack{C \supseteq A, \\ C\ clsd}} C$$

2. *the* ***Interior*** *of* A *is defined as follows:*

$$\mathrm{Int}(A) = \bigcup_{\substack{U \subseteq A \\ U \in \mathcal{T}}} U$$

Note 2.13. *1.* $\overline{A}$is*clsd* $\text{Int}(A) \in \mathcal{T}$

2. $\text{Int}(A) \subseteq A \subseteq \overline{A}$

3.

$$A \textit{ closed} \iff A = \overline{A}$$

$$A \textit{ open} \iff A = \text{Int}(A)$$

Note 2.14. $(X, \mathcal{T})$, $Y \subseteq X$, $A \subseteq Y$. $(Y, \mathcal{T}')$.

$$(\textit{wrt } Y) \longrightarrow \overline{A} = \bigcap \{C : A \subseteq C,\ C \text{ clsd in } Y\}$$

$$= \bigcap \{Y \cap C : A \subseteq C,\ C \textit{ clsd in } X\}$$

$$= Y \cap \left(\bigcap \{C : A \subseteq C,\ C \textit{ clsd in } X\} \right)$$

$$= Y \cap \overline{A} \longleftarrow (\textit{wrt } X)$$

Similiarly, $\text{Int}(A)$ *wrt* $(Y) = Y \cap \text{Int}(A)$ *wrt* (X).

Proposition 2.11: The Closure is the Set of Limit Points. $A \subseteq X$.

$$\overline{A} = \{x \in X : x \textit{ is a limit point of } A\}$$

Proof. Let $L := \{x \in X : x \text{ is a limit point of } A\}$. Let $x \in \overline{A}$. Let $U \in \mathcal{T}$ s/t $x \in U$. Suppose $A \cap U = \emptyset$

$$\implies A \subseteq \underbrace{X \setminus U}_{\textit{clsd}} \implies x \in X \setminus U$$

Contradiction, $x \in U$ and $x \in X \setminus U$.
Let $x \in L$, and let C be clsd, with $A \subseteq C$. Suppose

$$x \notin C \implies x \in X \setminus C := U$$

$$\implies (X \setminus C) \cap A \neq \emptyset$$

Contradiction of $A \subseteq C$. $\square$

Note 2.15: Norms.

1. *P-Norm, $x \in \mathbb{R}^n$.*
$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$
2. *Inf-Norm, $x \in \mathbb{R}^n$.*
$$\max\{|x_i| : 0 \leq i \leq n\}$$
3. *P-Norm, $x \in \mathbb{R}^{\mathbb{N}}$*
$$\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$$
4. *Inf-Norm, $x \in \mathbb{R}^{\mathbb{N}}$.*
$$\sup\{|x_i| : i \in \mathbb{N}\}$$
5. *P-Norm, $x \in \mathbb{R}^{\mathbb{R}}$.*
$$\left(\int_{-\infty}^{\infty} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$
6. *Inf-Norm, $x \in \mathbb{R}^{\mathbb{R}}$.*
$$\sup\{|f(x)| : x \in \mathbb{R}\}$$
7. $\ell^p = \{x \in \mathbb{R}^{\mathbb{N}} : \|x\|_p < \infty\}$

Corollary 2.2: Closure is the Set of Reachable Points. $\overline{A} = \{x \in X : \exists (a_n) \subseteq A,\ a_n \to x\}$

Definition 2.17: Interior Points. $(X, \mathcal{T}),\ A \subseteq X$. *Then $x \in A$ is an **interior point** iff*
$$\exists U \in \mathcal{T}\ s/t\ x \in U \subseteq A$$

Note 2.16. *Notice that this is similiar to how we define openness in a metric space.*

Note 2.17. $(X, d),\ A \subseteq X$. *Then $x \in A$ is an interior point of A iff*
$$\exists r > 0\ s/t\ x \in B_r(x) \subseteq A$$

Proposition 2.12: Interior Points of A build the Interior of A.

$$\text{Int}(A) = \{x \in A : x \textit{is} \textit{an interior point of } A\}$$

Proof. Let $I := \{x : x \text{isint pt}\}$, $x :\in \text{Int}(A)$.

$$\iff \exists U \in \mathcal{T},\ x \in U \subseteq A \iff x \in I$$

$\square$

Example 2.32: Closures and Closed "Versions" of Sets are Inequal. $(\mathbb{N}, |\cdot|)$

$$B_1(1) = \{1\} \implies \overline{B_1(1)} = \{1\} = B_1(1)$$

$$B_1[1] = \{1, 2\}$$

Proof. **Prove** $\overline{B_r(x)} = B_r[x]$, under a (NVS).
NVS: vectors, w/ buildin norm. Basically this is a ordinary norm, rather than a metric. So we have norm TI, norm zero uniqueness, AND a scalar property.
$(\subseteq)$. Let $y \in \overline{B_r(x)}$. So

$$B_r[x] = \{y \in \mathbb{R} : \|x - y\| \leq r\}$$

$$\text{Show } \|x - y\| \leq r$$

$$y \in \overline{B_r(x)} \implies \exists y_n \in \mathbb{R}^{\mathbb{N}} \text{ s/t } y_n \to y$$

$$\implies \exists N \in \mathbb{N} \text{ s/t } \forall n > 0,\ \|x - y_n\| < r$$

$$\implies \|x - y\| \leq r$$

$(\supseteq)$ Let $y \in B_r[x]$.

$$\|y - x\| \leq r$$

$$\text{Show } \exists (a_n) \in \mathbb{R}^{\mathbb{N}} \text{ s/t } a_n \to y$$

$$\text{Let} a_i = y + \frac{x - y}{i}$$

$$\|x - a_i\| = \left\| x - y - \frac{x - y}{i} \right\| = \|x - x/i - y + y/i\|$$

$$\|x - x/i + (y/i - y)\| \leq \|x - x/i\| + \|y - y/i\| < 2r$$

$\square$

Example 2.33. *(X,d), $A \subseteq X$ complete. Prove A is closed.*

$$(a_n) \subseteq A,\ a_n \to x \in X$$

$$(a_n) \text{ cauchy},\ (a_n) \text{ conv to a pt in } A.$$

$$\implies x \in A \implies \text{closed}$$

Definition 2.18: Subsequence.

$$(X,d),\ (X_n) \subseteq X$$

A subsequence of (X_n) is a sequence

$$(X_{n_k})_{k=1}^{\infty},\ \text{where } n_1 < n_2 < n_3 < \cdots$$

Note 2.18.

$$\mathbb{N} \to \mathbb{N} \to X : x(n(k)),\ n \text{ increasing}$$

$$K \geq N,\ n_K \geq K \geq N$$

$$K \geq N,\ n(K) \geq K \geq N$$

Example 2.34.

$$(X,d),\ (X_n) \subseteq X. \text{ Prove } x_n \to x \implies x_{n_k} \to x$$

Proof.

$$K \geq N,\ n_K \geq K \geq N$$

$$\implies d(x_{n_k}, x) < \epsilon$$

□

Example 2.35.

$$\text{Let}(X,d)\ (x_n) \subseteq X\ \textit{cauchy}$$

$$\textit{Show } x_n \textit{ conv} \iff x_n \textit{ has a conv subseq}$$

Proof. The forward direction is trivial.

Backwards: Let (x_n) be cauchy, and let some subsequence $x_{n_k} \to x$.

$$\text{Let } \epsilon > 0, \text{let } N \in \mathbb{N} \text{ s/t } \forall n, m \geq N \implies d(x_n, d_m) < \frac{\epsilon}{2}$$

$$\text{Let } k \in \mathbb{N}\ k \geq K \implies d(x_{n_k}, x) < \frac{\epsilon}{2}$$

$$\text{Assume } K \geq N \implies n_K \geq N$$

$$\begin{aligned} \forall n \geq N,\ d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$\square$

Example 2.36.

$$\text{Let } (X,d),\ x_n \to x$$

$$\text{Prove } C = \{x_n : n \in \mathbb{N}\} \cup \{x\} \text{ is closed}$$

Proof.

$$\text{if } \exists N \text{ inf many } y_n = x_N$$

$$\text{then } (x_N)_n \text{is a subseq of } y_n$$

$$y_n \to y,\ (x_N)_n \to x_N \implies y = x_N \in C\ \blacksquare$$

$$\text{if } \forall n,\ \text{only finitly many } y_K = x_n$$

$$\text{then } \exists (y_{n_k})$$

which is also a subseq of (x_n). Then $y = x$. $\blacksquare$ $\square$

Example 2.37. *Let V be a (NVS), $U \subseteq V$, $U \in \mathcal{T}$, $x \in V$. Show*

$$x + U = \{x + u \colon u \in U\} \in \mathcal{T}$$

Proof.

$$\text{Let } x + u \in x + U,\ u \in U \in \mathcal{T}$$

$$u \in U \in \mathcal{T} \implies \exists r > 0 \text{ s/t } B_r(u) \subseteq U$$

$$B_r(x+u) = x + B_r(u) \subseteq x + U \implies x + U \in \mathcal{T}$$

□

Example 2.38. V is NVS, $\mathcal{T} \ni U \subseteq V$, $A \subseteq V$.

$$\text{Show } A + U = \{a + u : a \in A,\ u \in U\} \in \mathcal{T}$$

Proof.

$$A + U = \bigcup_{a \in A} a + U$$

$$a + U \in \mathcal{T} \implies A + U \in \mathcal{T}$$

□

Example 2.39.

$$V \text{ (NVS)},\ C \subseteq V \text{ clsd},\ x \in V$$

prove $x + C$ is closed.

Proof.

$$V \setminus C \in \mathcal{T} \implies x + V \setminus C \in \mathcal{T} \text{ from above}$$

$$V \setminus (x + V \setminus C) \text{ is closed by defn}$$

$$= x + C$$

$$-C \in \mathcal{T}.\ x + (-C).\ -(x + (-C)) = x + C$$

□

Example 2.40. *Find $C, D \subseteq \mathbb{R}$ closed s/t $C + D$ isn't closed.*

Proof. $C := \mathbb{N},\ D := \{-n + 1/n : n \geq 2\}$.

$$\frac{1}{n} \in C + D,\ n \geq 2$$

$$0 \notin C + D \implies C + D \neq \overline{C + D}$$

□

Example 2.41.

$$(X, d),\ A \subseteq X. \text{ Show } X \setminus \mathrm{Int}(A) = \overline{X \setminus A}$$

($\subseteq$) *Proof.* Let $x \in X \setminus \mathrm{Int}(A)$. Let $(x_n) \subseteq X \setminus A$ s/t $x_n \in B_{\frac{1}{n}}(x)$.

$$x_n \to x \implies X \setminus \mathrm{Int}(A) \subseteq \overline{X \setminus A}$$

□

($\supseteq$) *Proof.* Let $x \in \overline{X \setminus A}$.
Then $\exists x_n \subseteq X \setminus A$ s/t $x_n \to x$.
Then $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ s/t $x_N \in B_\epsilon(x)$.
Note that since $x_N \notin A$ then $B_\epsilon(x) \nsubseteq A$.
Since $y \notin \mathrm{Int}(A) \iff \forall \epsilon > 0,\ B_\epsilon(x) \nsubseteq A$, then $x \notin \mathrm{Int}(A)$. □

Example 2.42. *Prove $X \setminus \overline{A} = \mathrm{Int} X \setminus A$.*

Proof. We know from above that $X \setminus \mathrm{Int}(X \setminus A) = \overline{X \setminus (X \setminus A)}$.
So $X \setminus \mathrm{Int}(X \setminus A) = \overline{A}$.
Then $X \setminus \overline{A} = \mathrm{Int}(X \setminus A)$. □

Definition 2.19: Boundary. $(X, d),\ A \subseteq X$. *The* ***boundary*** *of A is*

$$\partial A = \overline{A} \setminus \mathrm{Int}(A)$$

Example 2.43. *Prove ∂A is closed.*

Proof.

$$\partial A = \overline{A} \setminus \mathrm{Int}(A)$$
$$= \overline{A} \cap (X \setminus \mathrm{Int}(A))$$
$$= \overline{A} \cap \overline{X \setminus A}$$

So ∂A is closed, because intersections of closed sets are closed. □

Example 2.44. *Prove A is closed $\underline{\textit{iff}}$ $\partial A \subseteq A$.*

($\implies$) *Proof.*

$$A \text{ is closed} \implies A = \overline{A}$$
$$\partial A = \overline{A} - A^0 = A - A^0 \subseteq A$$

□

($\impliedby$) *Proof.*

$$\partial A \subseteq A \implies \overline{A} \setminus A^0 \subseteq A$$
$$\implies \overline{A} \subseteq A \cup A^0 = A$$

□

Definition 2.20: Hausdorf. $(X, \mathcal{T})$ *is Hausdorf* $\underline{\textit{iff}}$

$$\forall x \neq y \in X$$

$$\exists \text{ disjoint } U, V \in \mathcal{T} \ \ s/t \ x \in U, \ \ y \in V.$$

Example 2.45. *Let $(X, \mathcal{T})$ be Hausdorf. Show $\{x\}$ is closed.*

Proof.

$$\forall y \neq x, \ U_y, V_y \in \mathcal{T}, \ U_y \cap V_y = \emptyset, \ y \in U_y, \ x \in V_y$$
$$X \setminus \{x\} = \bigcup_{y \neq x} U_y \in \mathcal{T}$$

□

Example 2.46. $(X, \mathcal{P}(X))$. *Prove $\mathcal{T} = \mathcal{P}(X)$ is induced by a metric.*

Proof. Let
$$d(x,y) = \delta(x,y) := \begin{cases} 1 & x = y \\ 0 & o/w \end{cases}$$
We need to show $\forall A \in \mathcal{T},\ A$ is open. Let $A \in \mathcal{T}$.
$$A = \bigcup_{a \in A} \{a\} = \bigcup_{a \in A} B_1(a)$$
Since $B_1(a)$ is open, and arbitrary unions of open sets wrt δ are open, then A is open. □

3 Continuity

3.1 Continuity

Definition 3.1: Topologic Continuity. *Let $(X, \mathcal{T})$, $(Y, \mathcal{T}')$ be topological spaces.*
$f : X \to Y$ is cts iff
$$f^{-1}(U) \in \mathcal{T},\ \forall U \in \mathcal{T}'$$
Noting that $f^{-1}(U) = \{x \in X : f(x) \in U\}$.

Proposition 3.1: Closedness and Continuity. $(X, \mathcal{T})$, $(Y, \mathcal{T}')$, $f : X \to Y$. *TFAE:*

1. f cts
2. $\forall A \subseteq X,\ f(\overline{A}) \subseteq \overline{f(A)}$
3. $\forall$closed $C \subseteq Y,\ f^{-1}(C)$ is closed in X

(1) $\implies$ (2) *Proof.* Assume f is cts.
Let $y \in f(\overline{A})$.
Show $\forall U \in \mathcal{T}'$ s/t $y \in U,\ U \cap f(A) \neq \emptyset$.
Let $y = f(x),$ for some $x \in \overline{A}$.
So $\forall V \in \mathcal{T}$ s/t $x \in V,\ V \cap A \neq \emptyset$.
Let $U \in \mathcal{T}'$ s/t $y \in U$.
$\implies f(x) \in U$.
$\implies x \in f^{-1}(U)$, and $f^{-1}(U) \in \mathcal{T}$ by continuity.
Since $f^{-1}(U) \in \mathcal{T},\ x \in f^{-1}(U),\ x \in \overline{A}$, then by the defn of the closure, $f^{-1}(U) \cap A \neq \emptyset$.
Let $a \in f^{-1}(U) \cap A \neq \emptyset$
$\implies f(a) \in U \cap f(A)$
$\implies y \in \overline{f(A)}$ $\square$

(2) $\implies$ (3) *Proof.* Let : $C \subseteq Y$ be closed and $A = f^{-1}(C)$.
Show A is closed. Show $\overline{A} \subseteq A$.
For $x \in \overline{A},\ f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{C} = C$
$\implies x \in f^{-1}(C) = A$
$\implies A = \overline{A} \implies A$ is closed $\square$

(3) $\implies$ (1) *Proof.* For $U \subseteq Y$ open, $Y \setminus U$ clsd
$\implies f^{-1}(Y \setminus U)$ clsd by hypothesis
$= X \setminus f^{-1}(U)$
$\implies f^{-1}(U)$ open $\square$

Proposition 3.2: Continuity preserves Convergence in Metric Spaces.
$(X,d),\ (Y,d'),\ f : X \to Y$.
Then f is cts *<u>iff</u>*
$f(x_n) \to f(x)$ whenever $(x_n) \subseteq X,\ x_n \to x \in X$.

$(\implies)$ *Proof.* Assume f is cts.
Let : $(x_n) \subseteq X, \ x_n \to x \in X$.
Let : $\epsilon > 0$.
Consider : $U = B_\epsilon(f(x))$.
$\implies x \in f^{-1}(U)$ is open by cts.
$\implies \exists r > 0, \ B_r(x) \subseteq f^{-1}(U)$ by openess
Since $x_n \to x, \ \exists N \in \mathbb{N}$ s/t $n \geq N \implies f(x_n, x) < r$
Hence, $n \geq N \implies x_n \in f^{-1}(U)$
$\implies$ if $n \geq N$ then $d'(f(x_n), f(x)) < \epsilon$
$\implies f(x_n) \to f(x)$. □

$(\Longleftarrow)$ *Proof.*

$$\text{Assume : } f(z_n) \to f(z) \text{ if } z_n \to z$$

$$\text{Let : } A \subseteq X \text{ be open}$$

$$\text{Show : } f(\overline{A}) \subseteq \overline{f(A)}$$

$$\text{Let : } y \in f(\overline{A}) \text{ s/t } y = f(x)$$

$$\text{Let : } (a_n) \subseteq A \text{ s/t } a_n \to x$$

$$\implies f(a_n) \to f(x) = y$$

$$\implies y \in \overline{f(A)}$$

□

3.2 Bounded Linear Maps

Definition 3.2: Operator Norm, Bounded Linear Map. V, W, NVS, $T : V \to W$ is linear.
T is bd $\iff$

$$\|T\|_{\text{op}} := \sup\{\|T(x)\| : \|x\| = 1\} < \infty$$

Proposition 3.3. $B(V, W) := \{T : V \to W \mid T \text{ linear and bd}\}$ *is a vector space. Prove* $\|\cdot\|_{\text{op}}$ *is a norm on* $B(V, W)$.

Proof. Show

1. $\|sT\|_{\text{op}} = |s|\|T\|_{\text{op}}$
2. $\|T\|_{\text{op}} = 0 \iff T = 0$
3. $\|T + S\|_{\text{op}} \le \|T\|_{\text{op}} + \|S\|_{\text{op}}$

1: $\|sT\|_{\text{op}} = \sup\{\|sT(x)\| : \|x\| = 1\}$
$= \sup\{|s|\|T(x)\| : \|x\| = 1\}$
$= |s|\|T\|_{\text{op}}$
2 ($\implies$): $\|T\|_{\text{op}} = \sup\{\|T(x)\| : \|x\| = 1\} = 0$
$\implies \|T(x)\| = 0, \text{ if } \|x\| = 1$
$\implies T(x) = 0, \text{ if } \|x\| = 1$
$\implies T = 0_{\text{op}}$
2 ($\impliedby$): $T = 0_{\text{op}}$
$\implies T(x) = \vec{0} \implies \|T(x)\| = 0$
$\implies \sup\{\|T(x)\| : \|x\| = 1\} = 0$
3: $\|T + S\|_{\text{op}} = \sup\{\|(T + S)(x)\| : \|x\| = 1\}$
$= \sup\{\|T(x) + S(x)\| : \|x\| = 1\}$
$\le \sup\{\|T(x)\| + \|S(x)\| : \|x\| = 1\}$
$= \|T\|_{\text{op}} + \|S\|_{\text{op}}$ $\square$

Note 3.1. $T \in B(V, W)$.
if $\vec{0} \neq x \in V$ then $\left\|T\left(\frac{x}{\|x\|}\right)\right\| \le \|T\|_{\text{op}}$
$\implies \frac{1}{\|x\|}\|T(x)\| \le \|T\|_{\text{op}}$
$\implies \|T(x)\| \le \|x\| \cdot \|T\|_{\text{op}}$

Proposition 3.4: Continuous Linear iff Bounded Linear. V, W NVS, $T : V \to W$ linear.
then T is cts $\iff$ T is bd

($\neg \impliedby \neg$) *Proof.* Assume T isn't bd. So $\forall \|x_n\| = 1,\ \|T(x_n)\| \ge n$.
Consider : $\|\frac{1}{n}x_n\| = \frac{1}{n} \to 0$
$\|T(\frac{1}{n}x_n)\| = \frac{1}{n}\|T(x_n)\| \ge \frac{1}{n}n \ge 1$
So T doesn't preserve convergence, so T isn't cts. $\square$

($\Longleftarrow$) *Proof.* Assume T is bd. Show T is cts.

$$\|T\|_{\text{op}} = \sup\{\|T(x)\| : \|x\| = 1\} < \infty$$

Let $(x_n) \subseteq V$ s/t $x_n \to x \in V$.
$\|T(x_n) - T(x)\| = \|T(x_n - x)\| \leq \|x_n - x\|\|T\|_{\text{op}} < \frac{\epsilon}{\|T\|_{\text{op}}}\|T\|_{\text{op}} = \epsilon$ □

3.3 More Continuity

Definition 3.3: Uniform Continuity. (X, d), (Y, d'), $f : X \to Y$.
*f is **uniform continuous** iff*
$\forall \epsilon > 0\ \exists \delta > 0$ *s/t* $d'(f(a), f(b)) < \epsilon$ if $a, b \in X$ w/ $d(a, b) < \delta$

Note 3.2. *f is unif cts iff*
$\forall \epsilon > 0\ \exists \delta > 0$ *which works to establish continuity at every* $b \in X$.

Definition 3.4: Lipschitz. (X, d) (Y, d'), $f : X \to Y$.
*We say f is **Lipschitz** iff*
$\exists M > 0,\ d'(f(x), f(x)) \leq Md(x, y),\ \forall x, y \in X$.

Kind of like uniform uniform cts.

Proposition 3.5: Lipschitz implies Uniform Continuous. $f : X \to Y$ *Lipschitz then f is unif cts.*

Proof. Let : $\epsilon > 0$, $M > 0$ be the Lipschitz constant for f.
Let : $\delta = \frac{\epsilon}{M}$
Assume : $d(a, b) < \delta$, $a, b \in X$.
$\implies d'(f(a), f(b)) \leq Md(a, b) < \epsilon$ □

Example 3.1. $f : [0,1] \to \mathbb{R},\ f(x) = \sqrt{x}.$

Claim : f is unif cts

Note 3.3.

$$\begin{aligned} |\sqrt{x} - \sqrt{y}|^2 &= |\sqrt{x} - \sqrt{y}||\sqrt{x} - \sqrt{y}| \\ &\leq |\sqrt{x} + \sqrt{y}||\sqrt{x} - \sqrt{y}| \\ &= |x - y| \end{aligned}$$

Let : $\epsilon > 0,\ \delta = \epsilon^2.$

if $a, b \in [0,1]$ w/ $|a - b| < \delta = \epsilon^2$ then $|\sqrt{a} - \sqrt{b}| < \epsilon$

Example 3.2. Claim : f is not Lipschitz

Suppose : f is Lipschitz

WLOG, assume : $M > 1$

$\frac{1}{M^4} \in [0,1]$

$\left|\frac{1}{\sqrt{M^4}} - 0\right| \leq M \left|\frac{1}{M^4} - 0\right|$

$\implies \frac{1}{M^2} \leq \frac{1}{M^3} \implies M^3 \leq M^2$

That's a contradiction.