Real Analysis

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Abstract

Real Analysis the study of approximation on the reals.

Contents

Chapter 1

Cardinality

1.1 Brief Motivation

We want to build a metric space to measure the distance between objects.

We need

1. set X of objects.
2. need to measure closeness. func $d : X \times X \to [0, \infty)$ s/t
   (a) $d(x, y) = 0 \iff x = y$
   (b) $d(x, y) = d(y, x)$
   (c) $d(x, z) \leq d(x, y) + d(y, z)$

We call d a metric on X. (X, d) is a **metric space**.

$(\mathbb{R}, +' \circ^{\circ} (\surd^{=})-)$ is a metric space. Note the BQN notation.

1.2 Function Theory

Definition 1.2.1: Injection. *Let A, B be non-empty sets. We say $f : A \to B$ is injective* ***iff*** $\forall a, b \in A \quad f(a) = f(b) \implies a = b$

Definition 1.2.2: Surjection. *$f : A \to B$ is a surjection if $\forall b \in B \quad \exists a \in A$ s/t $f(a) = b$.*

Definition 1.2.3: Bijective. *$f : A \to B$ is bijective* ***iff*** *its injective and surjective.*

Definition 1.2.4: Invertable. *$f : A \to B$ is invertable **iff** $\exists g : B \to A$ s/t $g(f(a)) = a$ and $f(g(b)) = b \quad \forall a \in A, b \in B$.*

We write $g = f^{-1}$ and call it "the" inverse.

Proposition 1.2.1. *$f : A \to B$ is invertable **iff** f is bijective.*

Proof. ($\implies$) f is invertable. Suppose $f(a) = f(b)$. We'll show $a = b$.

$$f^{-1}f(a)) = f^{-1}f(b))$$

$$\implies a = b$$

Now we'll show $\forall b \in B \; \exists a \in A \; f(a) = b$.

$$a = f^{-1}(b) \implies \text{there is way to get from b to a, and it's } f^{-1}$$

($\impliedby$) Assume $f \leftarrow$ (bijective). We'll construct f's inverse. For $b \in B$ let a_b be the unique element of A s/t $f(a_b) = b$. a_b exists b/c of surjectivity of f, and it's unique b/c of injectivity.

$$g := \{g : A \to B, \, g(b) = a_b\}$$

$$f(g(b)) = f(a_b) = b$$

$$g(f(a_b)) = g(b) = a_b$$

$$\implies g = f^{-1}$$

□

Proposition 1.2.2. *$\exists$(injection) $f : A \to B \iff \exists$(surjection) $g : B \to A$*

Proof. ($\implies$) Suppose $f : A \to B \leftarrow$ (injective). Let $b \in B$.

Case 1: $b \in f(A)$.

Let $g(b)$ be the unique element of A s/t $f(g(b)) = b$, unique b/c $f \leftarrow$ (injective)

Case 2: $b \notin f(A)$.

Fix any $z \in A$. Let $g(b) = z$.

$$\implies g(b) = \begin{cases} f^{=}(b) & b \in f(A) \\ z & b \notin f(A) \end{cases} \tag{1.1}$$

We claim g is a surjection. So we have to show $\forall a \in A, \exists b \in B$ s/t $g(b) = a$ Let

$a \in A$ s/t $f(a) \in B$.

$$g(f(a)) \implies f(g(f(a))) = f(a)$$

$$\text{(injective)} \implies g(f(a)) = a$$

$$\implies g \leftarrow \text{(surjective)}$$

$\impliedby$ Suppose $(g : B \to A) \leftarrow$ (surjective). $\forall a \in A$ choose $b_a \in B$ s/t $g(b_a) = a$. $f := \{f : A \to B \quad f(a) = b_a\}$. Suppose

$$f(x) = f(y)$$

$$\implies b_x = b_y$$

$$\implies g(b_x) = g(b_y)$$

$$\implies x = y$$

$$\implies f \leftarrow \text{(injective)}$$

$\square$