# Real Analysis

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**Abstract**

Real Analysis the study of approximation on the reals.

# Contents

# 1 Cardinality

## 1.1 Brief Motivation

We want to build a metric space to measure the distance between objects.

We need

1. set $X$ of objects.

2. need to measure closeness. func $d : X \times X \to [0, \infty)$ s/t

   (a) $d(x, y) = 0 \iff x = y$

   (b) $d(x, y) = d(y, x)$

   (c) $d(x, z) \leq d(x, y) + d(y, z)$

We call $d$ a metric on $X$. $(X, d)$ is a **metric space**.

$(\mathbb{R}, +' \circ^{\circ} (\sqrt{=}) -)$ is a metric space. Note the BQN notation.

## 1.2 Function Theory

**Definition 1.1: Injection.** *Let $A, B$ be non-empty sets. We say $f : A \to B$ is injective **iff** $\forall a, b \in A \quad f(a) = f(b) \implies a = b$*

**Definition 1.2: Surjection.** *$f : A \to B$ is a surjection if $\forall b \in B \quad \exists a \in A$ s/t $f(a) = b$.*

**Definition 1.3: Bijective.** *$f : A \to B$ is bijective **iff** its injective and surjective.*

**Definition 1.4: Invertable.** *$f : A \to B$ is invertable **iff** $\exists g : B \to A$ s/t $g(f(a)) = a$ and $f(g(b)) = b \quad \forall a \in A, b \in B$.*

We write $g = f^{-1}$ and call it "the" inverse.

**Proposition 1.1.** *$f : A \to B$ is invertable **iff** $f$ is bijective.*

*Proof.* ($\implies$) $f$ is invertable. Suppose $f(a) = f(b)$. We'll show $a = b$.

$$f^{-1}f(a)) = f^{-1}f(b))$$

$$\implies a = b$$

Now we'll show $\forall b \in B \; \exists a \in A \; f(a) = b$.

$$a = f^{-1}(b) \implies \text{there is way to get from b to a, and it's } f^{-1}$$

($\impliedby$) Assume $f \hookleftarrow$ (bijective). We'll construct $f$'s inverse. For $b \in B$ let $a_b$ be the unique element of $A$ s/t $f(a_b) = b$. $a_b$ exists b/c of surjectivity of f, and it's unique b/c of injectivity.

$$g := \{g : A \to B, \; g(b) = a_b\}$$
$$f(g(b)) = f(a_b) = b$$
$$g(f(a_b)) = g(b) = a_b$$
$$\implies g = f^{-1}$$

$\square$

**Proposition 1.2.** $\exists (injection) \, f : A \to B \iff \exists (surjection) \, g : B \to A$

*Proof.* ( $\implies$ ) Suppose $f : A \to B \hookleftarrow$ (injective). Let $b \in B$.

Case 1: $b \in f(A)$.

Let $g(b)$ be the unique element of $A$ s/t $f(g(b)) = b$, unique b/c $f \hookleftarrow$ (injective)

Case 2: $b \notin f(A)$.

Fix any $z \in A$. Let $g(b) = z$.

$$\implies g(b) = \begin{cases} f^=(b) & b \in f(A) \\ z & b \notin f(A) \end{cases}$$

We claim $g$ is a surjection. So we have to show $\forall a \in A, \exists b \in B$ s/t $g(b) = a$ Let $a \in A$ s/t $f(a) \in B$.

$$g(f(a)) \implies f(g(f(a))) = f(a)$$

$$\text{(injective)} \implies g(f(a)) = a$$

$$\implies g \hookleftarrow \text{(surjective)}$$

$\impliedby$ Suppose $(g : B \to A) \hookleftarrow$ (surjective). $\forall a \in A$ choose $b_a \in B$ s/t $g(b_a) = a$. $f := \{f : A \to B \quad f(a) = b_a\}$. Suppose

$$f(x) = f(y)$$

$$\implies b_x = b_y$$

$$\implies g(b_x) = g(b_y)$$

$$\implies x = y$$

$$\implies f \hookleftarrow \text{(injective)}$$

□

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**Definition 1.5: Powerset.** *Let $X$ be a set. Then $\mathcal{P}(X) := \{A : A \subseteq X\}$, called the "**powerset** of $X$."*

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$$X = \{a, b\}$$

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

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**Axiom 1.1: Choice.** *Given $X \neq \emptyset \exists a$ choice func $f : \mathcal{P}(X) \backslash \{\emptyset\} \to X$ s/t $f(A) \in A \forall \neq A \subseteq A$.*

## 1.3 Cardinality

$$A = \{a, b\}, \ B = \{c, d, e, f\}$$

$$\text{Intuitively } |A| < |B|$$

$$f \leftharpoonup (\text{inj}) : A \to B, \ f(a) := c, \ f(b) := d$$

$$\implies f \leftharpoonup (\text{inj})(A) \subset B$$

$$\implies |A| \leq |B|$$

**Definition 1.6: Ordering of Cardinality on Sets.** *$A, B$ sets.*

*1. $|A| \leq |B| \iff \exists f \leftharpoonup (\text{inj}) : A \to B$*

*2. $|A| = |B| \iff \exists f \leftharpoonup (\text{bij}) : A \to B$*

$$|\mathbb{N}| \leq |\mathbb{Z}| \impliedby f \leftharpoonup (\text{inj}) : \mathbb{N} \to \mathbb{Z}, \ f(n) := n$$

$$f \leftharpoonup (\text{bij}) : \mathbb{N} \to \mathbb{Z} : f(n) := \begin{cases} 2n + 2 & : n \geq 0 \\ 2(-n) - 1 & : n < 0 \end{cases} \implies |\mathbb{N}| = |\mathbb{Z}|$$

$$h \leftharpoonup (\text{bij}) : \mathbb{R} \to (0, 1) : h(x) := \frac{\arctan(x) + \pi/2}{\pi} \implies |\mathbb{R}| = |(0, 1)|$$

**Theorem 1.1: Cantor-Schroeder-Berstien (CSB).** *if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.*

**Lemma 1.1: Phi has a Fixed Point.** *$X$ set. Suppose $\exists \phi : \mathcal{P}(X) \to \mathcal{P}(X)$ s/t $\phi(A) \subseteq \phi(B)$ if $A \subseteq B \subseteq X$. Then*

$$\exists F \subseteq X \ s/t \ \phi(F) = F$$

Let $F = \bigcup_{A \subseteq X : A \subseteq \phi(A)} A$.

**Note:** $\emptyset \subseteq X$ & $\emptyset \subseteq \phi(\emptyset)$

**Claim:** $F = \phi(F)$. Take $A \subseteq X$ with $A \subseteq \phi(A)$. Then $A \subseteq F$.

$$\implies \phi(A) \subseteq \phi(F)$$

$$\implies A \subseteq \phi(F)$$

$$\implies \bigcup_{A \subseteq X : A \subseteq \phi(A)} A \subseteq \phi(F) \quad \text{(by properties of unions)}$$

$$\implies F \subseteq \phi(F)$$

Further,

$$F \subseteq \phi(F) \implies \phi(F) \subseteq \phi(\phi(F))$$

$$\implies \phi(F) \in \{A \subseteq X : A \subseteq \phi(A)\}$$

For this step, let $\phi(F) = B$ and notice that $B \subseteq \phi(B)$. So its in the set above.

$$\implies \phi(F) \subseteq \bigcup_{A \subseteq X : A \subseteq \phi(A)} A = F$$

$$\implies F = \phi(F)$$

**To motivate (CSB) theorem 1.1:** prove that $|N| = |N \times N|$.

$$f \leftharpoonup (\text{inj}) : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, \ f(n) := (n, 1)$$

$$g \leftharpoonup (\text{inj}) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \ g((n, m)) := 2^n 3^m$$

By (CSB) $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.

*Proof of (CSB) theorem 1.1.* Let $f, g \hookleftarrow (\text{inj}) : A : B$. For $X \subseteq Y \subseteq A$

$$f(X) \subseteq f(Y)$$

$$\implies B\backslash f(Y) \subseteq B\backslash f(X)$$

$$\implies g(B\backslash f(Y)) \subseteq g(B\backslash f(X))$$

$$\implies A\backslash g(B\backslash f(X)) \subseteq A\backslash g(B\backslash f(Y))$$

This letting $\phi : \mathcal{P}(A) \to \mathcal{P}(A) : \phi(x) := A\backslash g(B\backslash f(x))$ insures it preserves $\subseteq$. So by the lemma 1.1, $\exists F \subseteq A$ s/t $F = \phi(F) = A\backslash g(B\backslash f(F))$. In particular, $A\backslash F = g(B\backslash f(F)) \implies g : B\backslash f(F) \to A\backslash F :\hookleftarrow (\text{bij})$.

> **Note.** It's a surjection b/c everyone in $A\backslash F$ gets mapped to b/c it's the image if $g(B\backslash f(F))$.

Moreover, $g^{-1} : A\backslash F \to B\backslash f(F)$ is a bijection, and $f : F \to f(F)$ is a bijection (for the same reason as above; restriction of domain of an injective function is injective, and a function that maps to its image is automatically a surjection). Hence

$$h : A \to B : h(x) := \begin{cases} g^{-1}(x) & : x \in A\backslash F \\ f(x) & : x \in F \end{cases}$$

$\square$

Show $|\mathbb{Q}| = |\mathbb{N}|$.

> *Proof.*
> $$f : \mathbb{N} \to \mathbb{Q} : f(x) := x \implies |\mathbb{N}| \leq |\mathbb{Q}|$$
>
> $q \in \mathbb{Q}$ can be written in the form $q = \frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$.
>
> $$g : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N} : g(q) := (m, n) \; : \; q = \frac{m}{n}$$
>
> $$\implies |\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{N}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$
>
> So by (CSB) theorem 1.1, $|\mathbb{Q}| = |\mathbb{N}|$. $\square$

**Definition 1.7: Finite, Countably Infinite, Countable.**

1. *a set $A$ is **finite** <u>iff</u> $|A| = |\{1, 2, \ldots, n\}|$ for some $n \in \mathbb{N}$. In this case, $|A| = n$.*

2. $|\emptyset| := 0$

3. *$A$ is **countably infinite** <u>iff</u> $|A| = |\mathbb{N}| := \aleph_0$.*

4. *$A$ is countable <u>iff</u> $A$ is finite or ctbly infinite.*

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{Q}| = \aleph_0$$

**Proposition 1.3: Aleph Null is the Smallest Infinity.** *If $A$ is infinite, then $|\mathbb{N}| \leq |A|$.*

*Proof.* By (Choice) axiom 1.1, $\exists f : \mathcal{P}(A) \backslash \{\emptyset\} \to A$ s/t $f(X) \in X$, $\forall \emptyset \neq X \subseteq A$.

$$\text{Let } a_1 = f(A) \in A$$
$$a_2 = f(A \backslash \{a_1\}) \in A \backslash \{a_1\}$$
$$\vdots$$

$\implies \aleph_0 = |\{a_1, \ldots\}| \leq |A|$. $\qquad\qquad\square$

**Proposition 1.4: The Reals are Uncountable.** $\mathbb{R}$ *is uncountable. So* $\nexists f \hookleftarrow$ (bij) $: \mathbb{N} \to \mathbb{R}$.

> *Proof.* **Cantor's Diagonal Element Proof.** Since $|\mathbb{R}| = |(0,1)|$, we'll show that $(0,1)$ is uncountable.
>
> For contradiction, assume that there exists a bijection $f : \mathbb{N} \to (0,1)$.
> So let's say
>
> $$f(1) = 0.a_{11}a_{12}a_{13}\cdots$$
> $$f(2) = 0.a_{21}a_{22}a_{23}\cdots$$
> $$f(3) = 0.a_{31}a_{32}a_{33}\cdots$$
> $$\vdots = \vdots$$
>
> Where we avoid repeated nines.
> Choose $b_i \in \{1, \ldots, 8\}$ s/t $b_i \neq a_{ii}$.
>
> $$\implies \nexists n \in \mathbb{N}, \ f(n) = 0.b_1 b_2 b_3 \cdots$$
>
> Thats a contradiction. $\square$

**Definition 1.8: Continuum.** *We write* $|\mathbb{R}| = c$, *where c stands for* ***continuum***.

So we have 3 cardinals: $n, \aleph_0, c$.

**Axiom 1.2: Continuum Hypothesis.** *If A is a set with* $\aleph_0 \leq |A| \leq c$, *then* $\aleph_0 = |A|$ *or* $|A| = c$.

## 1.4   Cardinality of Power Sets

**Proposition 1.5.** *If* $|A| = n$, *then* $|\mathcal{P}(A)| = 2^n$.

> *Proof.*
> $$|\mathcal{P}(A)| = \sum_{k=1}^{n} \binom{n}{k} = (1+1)^n = 2^n$$
> $\square$

**Definition 1.9: Cartesian Product.** *Let $I$ be a set.* $\forall i \in I$ *Let* $A_i \hookleftarrow$ (set) $\implies$
$\prod_{i \in I} A_i := \{f | f : I \to \bigcup A_i, \ f(i) \in A_i\}$

$$f(i) \in A_i$$

$$I = \mathbb{N} \implies f : \mathbb{N} \to \bigcup A_i : f(i) \in A_i \equiv (f(1), f(2), \dots)$$

**Definition 1.10: Set Power.** $A, B \hookleftarrow$ (set) $\implies A^B = \{f : B \to A\}$

$$|A|^{|B|} := \left|A^B\right| = |\{f : B \to A\}|$$

**Proposition 1.6: Cardinality of a Power Set.** *if* $X \hookleftarrow$ (set)*,* $\mathcal{P}(X) = 2^{|X|} = |\{f : X \to \{0,1\}\}|$.

*Proof.*

$$\phi : \mathcal{P}(X) \to \{f : X \to \{0,1\}\} : \phi(A) := \chi_A$$

$$\chi_A : X \to \{0,1\} : \chi_A(x) := \begin{cases} 1 & : x \in A \\ 0 & : x \notin A \end{cases}$$

Show $\phi \hookleftarrow$ (bij). First show it's injective.

$$\phi(A) = \phi(B)$$

$$\implies \chi_A = \chi_B$$

$$\implies A = B \implies \phi \hookleftarrow \text{(inj)}$$

Now show it's surjective. $\forall f \in \{f : X \to \{0,1\}\} \ \exists P \in \mathcal{P}(X) \text{ s/t } \phi(P) = f$.

$$\text{Let } f^{-1}(\{0,1\}) = F^{-1}$$

$$\implies \chi_{F^{-1}} : F^{-1} \to \{0,1\}$$

$$\implies \chi_{F^{-1}} = \phi(F^{-1}) \text{ w/ } \phi(F^{-1}) = f$$

$$F^{-1} \subseteq X \implies F^{-1} \in \mathcal{P}(X) \implies \phi \hookleftarrow \text{(surj)}$$

$\square$

**Proposition 1.7: The Powerset is Larger than the Set.** *If $X \leftarrow$ (set), then $|X| < |\mathcal{P}(X)|$.*

*Proof.* Show $|X| \leq |\mathcal{P}(X)|$. $f(x) = \{x\} \leftarrow$ (inj) $\implies |X| \leq |\mathcal{P}(X)|$.

For the sake on contradiction, assume there is a surjection $g : X \to \mathcal{P}(X)$. Consider $B := \{x \in X : x \notin g(x)\}$. Hence there must be (by surjectivity of $g$) $z \in X$ s/t $g(z) = B$. Someone has to map to it.

$$z \in B \implies z \notin g(z) = B$$

$$z \notin B \implies z \in g(z) = B$$

This is a contradiction. So $|X| < |\mathcal{P}(X)|$. $\qquad\square$

**Infinite Infinities.** $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$

**Proposition 1.8: The Natural Powerset Cardinal is the Continuum Cardinal.** $|\mathcal{P}(\mathbb{N})| = c$ ($\equiv 2^{\aleph_0} = c \equiv |\{0,1\}|^{|\mathbb{N}|} = |\mathbb{R}|$)

*Proof.*

> We'll use the continuum hypthesis, however there's an alternative proof in the course notes.

Consider $X = \{f : \mathbb{N} \to \{0,1\}\}$.

$$\phi : X \to \mathbb{R} : \phi(f) := 0.f(1)f(2)f(3)\dots$$

We can see that $\phi$ is injective. So

$$2^{\aleph_0} = |X| \leq |\mathbb{R}| = c$$

Also, $\aleph_0 < 2^{\aleph_0} \leq c$. So by (CH), we know $2^{\aleph_0} = c$. $\qquad\square$

*Proof (without (CH)).* $\dots$ $\qquad\square$

## 1.5 Cardinal Arithmetic

**Definition 1.11.** $A, B \hookleftarrow$ (sets)

1. $A \cap B = \emptyset \implies |A| + |B| := |A \cup B|$

2. $|A| \cdot |B| := |A \times B|$

3. $|A|^{|B|} := |\{f : B \to A\}|$

**Example.** $\aleph_0 + \aleph_0 = \aleph_0$. Let $A = \{a_1, \dots\}$, $B = \{b_1, \dots\}$, so that $|A| = |B| = \aleph_0$, and $A \cap B = \emptyset$.
Then $\phi : A \cup B \to \mathbb{N} : \phi(a_i) := 2i, \ \phi(b_i) := 2i - 1$. This is a bijection. Hence $|A \cup B| = \aleph_0$.

**Example.** $\aleph_0 + c = c$.
$\aleph_0 = |\mathbb{N}|, \ |(0,1)| = c$.
$$(0,1) \subseteq \mathbb{N} \cup (0,1) \subseteq \mathbb{R}$$
$$\implies c \leq \aleph_0 + c \leq c$$
$$\implies \aleph_0 + c = c$$

**Proposition 1.9: Cardinal Exponent Laws.** $A, B, C \hookleftarrow$ (sets).

1. $(|A|^{|B|})^{|C|} = |A|^{|B| \cdot |C|}$

2. $(|A|^{|B|})(|A|^{|C|}) = |A|^{|B| + |C|}$

**Example.** Show that $c \cdot c = c$.
$$c \cdot c = (2^{\aleph_0})(2^{\aleph_0}) = 2^{\aleph_0 + \aleph_0} = 2^{\aleph_0} = c$$

*Proof of 2.* We must show

$$|\{f|f : B \cup C \to A\}| = |\{f|f : B \cup A\} \times \{f|f : B \to A\}|$$

$$\text{Let } X := \{f|f : B \to A\}$$

$$\text{Let } Y := \{f|f : C \to A\}$$

$$\text{Let } Z := \{f|f : B \cup C \to A\}$$

So, equivically we need to show $|Z| = |X \times Y|$.

Consider $\varphi(f,g)(x) = \begin{cases} f(x) & x \in B \\ g(x) & y \in C \end{cases}$.

$$\varphi(f_1, g_1) = \varphi(f_2, g_2)$$

$$\implies \forall x \in B \cup C, \ \varphi(f_1, g_1)(x) = \varphi(f_2, g_2)(x)$$

$$\implies \forall x \in B, \ f_1(x) = f_2(x) \implies f_1 = f_2$$

$$\implies \forall x \in C, \ g_1(x) = g_2(x) \implies g_1 = g_2$$

Consider $h : B \cup C \to A$. Let $f = h|_B$, $g = h|_C$. Then $\varphi(f,g) = h$.
So $\varphi$ is bijective, so proposition 2 holds. $\qquad\square$

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**Example:** $c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$.

# 2 Topology

## 2.1 Metric Spaces

**Definition 2.1: Metric Space.** $X \leftarrow$ (set). *A metric on $X$ is a function* $d : X \times X \to [0, \infty)$ *s/t*

1. $d(x,y) = 0 \iff x = y$

2. **Abelian:** $d(x,y) = d(y,x)$

3. **Triangle:** $d(x,y) \leq d(x,z) + d(z,y)$

**Definition 2.2: Normed Vector Space (NVS).** *Let $V \hookleftarrow$ (Vector Space) over* $\mathbb{R}$. *A norm on $V$ is a fn* $\|\cdot\| : V \to [0, \infty)$ *s/t*

1. $\|v\| = 0 \iff v = \vec{0}$

2. $\|\alpha v\| = |\alpha| \cdot \|v\|$

3. $\|v + u\| \leq \|v\| + \|u\|$

**BQN:** $\|\times == |\circ l \cdot \| \circ r \quad |+ \leq \leq +\square|$

**Proposition 2.1: NVS have trivial Metrics.** *Let* $V, \|\cdot\| \hookleftarrow$ (NVS). $d(v, w) = \|v - w\|$ *is a metric on $V$.*

## 2.2 Examples of Metric Spaces

**Example 2.1: Discrete Metric.** $X \hookleftarrow$ (set).

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

**Example 2.2: Absolute Value Norm.** $(\mathbb{R}, |\cdot|) \hookleftarrow$ (NVS)

**Example 2.3: Euclidean Norm.** $(\mathbb{R}^n, \|\cdot\|_2) \hookleftarrow$ (NVS) *where* $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$.

**Example 2.4: P-Norm.** $p \geq 1$, $(\mathbb{R}^n, \|\cdot\|_p) \hookleftarrow$ (NVS) *where*

$$\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

*Note: see posted notes for the proof that this is a norm. OPTIONAL.*

**Example 2.5: Infinity Norm.** $p = \infty$, $(\mathbb{R}^n, \|\cdot\|_\infty) \hookleftarrow$ (NVS) *where*

$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$$

**Example 2.6: P-Norm on Sequences of Reals.** $\mathbb{R}^{\mathbb{N}} := \{f | f : \mathbb{N} \to \mathbb{R}\} = \{(a_n)_{n=1}^{\infty} : a_n \in \mathbb{R}\}$. *For $p \geq 1$,*

$$\|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \tag{1}$$

$l^p := \{x \in \mathbb{R} : \|x\|_p < \infty\} \implies (l^p, \|\cdot\|_p) \hookleftarrow (\text{NVS})$. *This is the p-norm on sequences of reals. Notice how this solve the divergence issue (by ignoring it lol).*

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**Example:** $l^1 = \{x \in \mathbb{R} : \sum |x_i| < \infty\} \implies l^p$ is the set of absolutly convergent sequences.

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**Example 2.7: Suprema Norm (Infinity Norm on Sequences of Reals).**
$\|x\|_{\infty} = \sup\{|x_n| : n \in \mathbb{N}\}$.
*if we let $l^{\infty} := \{x \in \mathbb{R}^{\mathbb{N}} : \|x\|_{\infty} < \infty\}$, noting that $l^{\infty}$ is the set of all bounded sequences, then $(l^{\infty}, \|\cdot\|_{\infty}) \hookleftarrow (\text{NVS})$.*

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**Example 2.8: P-Norm on Function.** $C([a,b]) := \{f : [a,b] \to \mathbb{R} | f \hookleftarrow (\text{cts})\}$.

$$\|f\|_p = \left( \int_a^b |f(x)| \, \mathrm{d}x \right)^{\frac{1}{p}}, \; p \geq 1$$

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**Example 2.9: Infinity Norm on Functions.** $\|f\|_{\infty} = \sup\{|f(x)| : x \in [a,b]\}$

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**Example 2.10: Bounded Functions and the Infinity Norm are a NVS.**
$\mathbb{B}([a,b]) = \{f : [a,b] \to \mathbb{R} | f \hookleftarrow (\text{bd})\}$, $(\mathbb{B}([a,b]), \|\cdot\|_{\infty}) \hookleftarrow (\text{NVS})$.

**Example 2.11: Sequence Metric.** $X = \mathbb{R}^{\mathbb{N}} = \{f | f : \mathbb{N} \to \mathbb{R}\}$.

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

*Prove that $d$ isn't induced by a metric.* If $d(x,y) = \|x - y\|$ for some norm, then $\|\alpha x - \alpha y\| = |\alpha|\|x - y\|$.

$$d(ax, ay) = \sum_{i=1}^{\infty} \frac{|ax_i - ay_i|}{2^i(1 + |ax_i - ay_i|)}$$

$$d(ax, ay) = \sum_{i=1}^{\infty} \frac{|a||x_i - y_i|}{2^i(1 + |a||x_i - y_i|)}$$

$$|a|d(x, y) = |a| \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

$$|a|d(x, y) = \sum_{i=1}^{\infty} \frac{|a||x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

b/c $|a||x_i - y_i| \neq |x_i - y_i|$

$$\implies \text{not induced by a norm}$$

$\square$

**Example 2.12: Cantor Space.** $X = 2^{\mathbb{N}} := \{f | f : \mathbb{N} \to \{0, 1\}\}$.

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

**Example 2.13: Hamming Distance.** $X \leftarrow$ (finite). $A, B \in \mathcal{P}(X)$.

$$d(A, B) := |A \triangle B| = |(A \cup B) \backslash (A \cap B)|$$

**Example 2.14: Hausdorff Metric.** $\mathcal{H} = \{K \subseteq \mathbb{R}^n : K \text{ compact}\}$. Let $a \in A$, $b \in B$, $A, B \in \mathcal{H}$.

$$d(a, B) = \min\{\|a - b\| : b \in B\}$$

$$d(b, A) = \min\{\|a - b\| : a \in A\}$$

$$d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

**Note 2.1.** $\sup_{a \in A} d(a, B)$ *represents the biggest shortest path between $A$ and $B$.*

**Note 2.2.** *Metrics give a sense of convergence on a space.*

**Example 2.15: P-adic Metric.** *Let $p$ be prime, $X = \mathbb{Q}$.*

$$\text{Let } 0 \neq q \in X = \mathbb{Q} \quad q = p^a \frac{n}{m}$$

$$\text{where } \gcd(n, m) = \gcd(p, n) = \gcd(p, m) = 1$$

$$|q|_p = \frac{1}{p^a}, \quad |0|_p = 0$$

$$d(q_1, q_2 := |q_1 - q_2|_p$$

**Note 2.3.** *This notion of distance implies the more factors of $p$, the closer. This gives a sense of optimizing for a certain adjective. These numbers aren't close using $\|\cdot\|_2$, but are using the p-adic norm.*

**Definition 2.3: Subspace of a Metric Space.** $(X, d)$, $Y \subseteq X \implies (Y, d)$. $(Y, d)$ *is called a **subspace** of $(X, d)$.*

**Definition 2.4.** $(X, d_1)$, $(Y, d_2)$. *Consider $(X \times Y, d)$ with*

$$d((x_1, y_1), (x_2, y_2)) := d_1(x_1, x_2) + d_2(y_1, y_2) \ (1\text{-norm})$$

$$\text{or } d((x_1, y_1), (x_2, y_2)) := \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \ (\infty\text{-norm})$$

**Example 2.16: Product Metric.** $(X_i, d_i)$ $i \in \mathbb{N}$, $X := \prod_{n=1}^{\infty} X_i$. *Let* $x = (x_1, x_2, \dots)$ *and* $y = (y_1, y_2, \dots)$.

$$d(x, y) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i(1 + d_i(x_i, y_i))}$$

## 2.3   Convergence

**Definition 2.5: Convergence of a Sequence.** $(X, d)$, $(x_n) \subseteq X$, *and* $x \in X$.

> **Notation 2.1.** $(x_n) \subseteq X$ *means* $(x_n)$ *is a sequence in* $X$.

$$(x_n) \text{ conv to } x, \ x_n \to x \ \textbf{iff}$$

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} \ s/t \ \forall n \geq N, \ d(x_n, x) < \epsilon$$

**Definition 2.6: Divergence.** $(x_n)$ *diverges of* $\nexists x \in X$ *s/t* $x_n \to x$.

**Note 2.4: Convergence is Distance going to Zero.** $(X, d)$, $(x_n) \in X$, $x \in x$. *Then* $x_n \to x$ *iff* $d(x_n, x) \to 0$.

**Definition 2.7: Cauchy.** $(X, d)$. $(x_n) \subseteq X$ *is a cauchy seq* **iff**

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} \ s/t \ \forall n, m \geq N, \ d(x_n, x_m) < \epsilon$$

**Proposition 2.2: Convergence implies Cauchyness.** $(X, d)$. *If* $(x_n) \subseteq X$ *converges, then* $(x_n)$ *is cauchy.*

*Epsilon/2.* Suppose $(x_n) \to x$. Let $\epsilon > 0$. So $\exists N$ s/t $\forall n \geq N$,

$$d(x_n, x) < \gamma$$

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) \text{ by } \triangle$$
$$< \gamma + \gamma = 2\gamma$$
$$:= 2\frac{\epsilon}{2}$$
$$= \epsilon$$

$\square$

**Example 2.17: Cauchy doesn't imply Convergence.** $X = (0, 1]$ *with the std metric.*

$$\frac{1}{n} \to 0 \implies \left(\frac{1}{n}\right) \subseteq X \text{ is cauchy}$$

**Note 2.5.** *I think this is trying to say that $1/n$ is cauchy in X, but $1/n \to 0$, which is not in X, so it diverges (in X).*

**Definition 2.8: Bounded.**

1. $A \subseteq X$ is bd *iff* $\sup\{d(x, y) : x, y \in A\} < \infty$

2. $(x_n) \subseteq X \leftharpoonup (\text{bd}) \iff \{x_1, x_2, \dots\} \leftharpoonup (\text{bd}) \iff \sup\{n, m \in \mathbb{N} : d(x_n, x_m)\} < \infty$

**Definition 2.9: Open and Closed Balls.**

1. ***Open:*** $B_r(a) := \{x \in X : d(x, a) < r\}$

2. ***Clsd:*** $B_r[a] := \{x \in X : d(x, a) \leq r\}$

**Proposition 2.3: Boundedness iff subset of a Closed Ball.** $(X, d)$, $A \subseteq X$. *Then $A$ is bd iff $\exists r > 0$, $\exists x \in X$ s/t $A \subseteq B_r[x]$*

*Proof.* Suppose $\sup\{d(x,y) : x, y \in A\} = r < \infty$. Assume $A \neq \emptyset$, taking $a \in A$.

For $b \in A$, $d(a,b) \leq r \implies A \subseteq B_r[a]$.

Assume $A \subseteq B_r[a] \implies \forall a, b \in A, \ d(a,b) \leq d(a,x) + d(x,b) \leq 2r$ $\qquad \square$

**Proposition 2.4: Cauchy implies Bounded.** $(x_n) \leftarrow (\text{cauchy}) \implies (x_n) \leftarrow$ (bd)

**Example 2.18: Counter Example.** $(0, 1, 0, 1, 0, \dots)$ *is bounded but not cauchy.*

**Note 2.6.** $CONVERGENCE \implies CAUCHY \implies BOUNDED$

*Proof.* Suppose $(x_n)$ is cauchy. Let $\epsilon = 1$.

$$\exists N \in \mathbb{N} \text{ s/t } \forall n, m \geq N \ d(x_n, x_m) < 1$$

Let $r := \max\{d(x_1, x_N), \dots, d(x_{N-1}, x_N)\}$

> **Note 2.7.** *We did this because the web's edges are no longer than 1. So, we can look at the finite part left. Finite sets are bounded, and the "web" is bounded.*

$$(x_n) \subseteq B_r[x_N]$$

$\qquad \square$

## 2.4 Examples

**Example 2.19: 2-Adic Norm.** *Consider $(\mathbb{Q}, |\cdot|_2)$. Let $x_n := \frac{1+2^n}{3}$.*

> **Note 2.8: P-Adic Convergence Claims.** *Looking at $x_n = \frac{1}{3} + \frac{2^n}{3} = \frac{1}{3} + \frac{\cancel{2^n}^{\,0}}{\cancel{3}} = \frac{1}{3}$.*

*We claim $x_n \to \frac{1}{3}$.*

> *Proof.*
>
> $$\left| x_n - \frac{1}{3} \right|_2 = \left| \frac{2^n}{3} \right|_2$$
> $$= \frac{1}{2^n}$$
> $$\to 0$$
>
> So $x_n \to \frac{1}{3}$ under $|\cdot|_2$. $\qquad\square$

**Example 2.20: Bounded Sequences and the Infinity Norm.** $(l^\infty, \|\cdot\|_\infty)$.
*Let $x_n := \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0, \ldots\right)$ and $x := \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$*
*We claim that $x_n \to x$.*

> *Proof.*
>
> $$\|x_n - x\|_\infty$$
> $$= \sup\left(0, 0, \ldots, 0, \frac{1}{n+1}, \frac{1}{n+2}, \ldots\right)$$
> $$= \frac{1}{n+1} \to 0$$
>
> $\qquad\square$

**Example 2.21: Zero Tailed Sequences aren't Cauchy under Sup Norm.**

$$y_n := (\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, 0, \ldots)$$

$$y := (1, 1, 1, 1, \ldots)$$

$$n \neq m, \ \|y_n - y_m\|_\infty = 1$$

## 2.5   Completeness

> **Definition 2.10:   Complete, Complete Metric Space, Banach Space.**
> $(X, d)$, $A \subseteq X$. *Then*
>
> 1. *$A$ is **complete** <u>iff</u> every cauchy seq in $A$ converges to some $a \in A$.*
>
> 2. *if $X$ is complete, we call it a **Complete Metric Space**.*
>
> 3. *A complete normed vector space is called a **Banach Space***

> **Example 2.22.**
> $$X = (0, 1], \ \frac{1}{n} \to 0 \notin X \implies (\text{div}) \implies X \not\vdash (\text{comp})$$

> **Example 2.23.**
> $$A = [1/2, 1] \subseteq X \hookleftarrow (\text{comp})$$

> **Example 2.24.**   $(X, d := (\text{discrete}))$ *Let $(x_n) \in \mathbb{R}^{\mathbb{N}}$ be cauchy.*
>
> $$\exists N \in \mathbb{N} \ s/t \ \forall n, m \geq N \implies d(x_n, x_m) < 1$$
>
> $$\implies d(x_n, x_m) = 0$$
>
> $$\implies x_n = x_m$$
>
> $$\implies x_n = (x_1, x_2, \ldots, x_N, x_N, x_N, \ldots) \to x_N$$
>
> $$\implies X \hookleftarrow (\text{complete})$$

> **Note 2.9.**   *So nice sets or nice metrics can cause completeness.*

**Example 2.25.** *Show $(l^\infty, \|\cdot\|_\infty)$ is a Banach Space.*

*Let $(x_n) \subseteq l^\infty$, example 2.7. We know already that $(l^\infty, \|\cdot\|_\infty)$ is a (NVS), so we have to show it's complete. Let $\epsilon > 0$ be given. Then,*

$$\exists N \in \mathbb{N} \ s/t \ n, m \geq N \implies \|x_n - x_m\|_\infty < \epsilon$$

$$x_k = (x_k[1], x_k[2], \dots)$$

$$for \ n, m \geq N, \ |x_n[i] - x_m[i]|$$

$$\leq \sup\{|x_n[i] - x_m[i]| : i \in \mathbb{N}\}$$

$$= \|x_n - x_m\|_\infty < \epsilon$$

$$\implies \forall i \in \mathbb{N}, \ the \ seq \ (x_n[i])_{n=1}^\infty \hookleftarrow (\text{cauchy in } \mathbb{R})$$

$$\mathbb{R} \hookleftarrow (\text{comp}) \implies x_n[i] \underset{n}{\to} b_i$$

$$Claim: \ x_n \to b := (b_1, b_2, \dots)$$

$$\forall n, m \geq N, \ |x_n[i] - x_m[i]| < \epsilon$$

$$\implies \lim_{m \to \infty} |x_n[i] - x_m[i]| \leq \epsilon$$

$$\implies \forall n \geq N, \ |x_n[i] - b_i| \leq \epsilon$$

$$Consider \ \|x_n - b\|_\infty$$

$$= \sup\{|x_n[i] - b_i| : i \in \mathbb{N}\}$$

$$\leq \epsilon < 2\epsilon$$

*Hence $x_n \to b$.*

**Note:** *we have that $x_N - b \in l^\infty$, and $x_N \in l^\infty$. However $l^\infty$ is a (VS), so $b \in l^\infty$.*

---

**Proposition 2.5: Set of bounded Sequences on the P-Norm is Banach.**

$(\ell^p, \|\cdot\|_p) \hookleftarrow (\text{banach})$.

$$\|x\|_p = \left(\sum_{n=1}^\infty |x_n|^p\right)^{\frac{1}{p}}$$

$$\ell^p := \{x \in \mathbb{R} : \|x\|_p < \infty\} \implies (\ell^p, \|\cdot\|_p) \hookleftarrow (\text{NVS})$$

*Proof.* Let $(a_k) \subseteq \ell^p$ be cauchy.

$$\text{Say } a_k = (a_k[1], a_k[2], \dots)$$

$$\text{Let } \epsilon > 0$$

$$\exists N \in \mathbb{N} \text{ s/t } \|a_k - a_m\| < \epsilon, \ \forall k, m \geq N$$

$$\text{Fixing } i \in \mathbb{N}, \text{ Since } |a_k[i] - a_m[i]| \leq \|a_k - a_m\|_p < \epsilon$$

$$\text{We see that } (a_k[i])_{k=1}^{\infty} \hookleftarrow (\text{cauchy in } \mathbb{R})$$

$$\mathbb{R} \hookleftarrow (\text{comp}) \implies a_k[i] \to b_i \text{ for some } b_i \in \mathbb{R}$$

$$\text{Claim: } a_k \to b = (b_1, b_2, \dots)$$

$$\forall k, m \geq N, \text{ we see that}$$

$$\sum_{i=1}^{M} |a_k[i] - a_m[i]|^p$$

$$\leq \sum_{i=1}^{\infty} |a_k[i] - a_m[i]|^p$$

$$= \|a_k - a_m\|_p^p < \epsilon^p$$

$$\sum_{i=1}^{M} |a_k[i] - b_i|^p \leq \epsilon^p, \ \forall M \in \mathbb{N}$$

$$M \to \infty : \sum_{i=1}^{\infty} |a_k[i] = b_i|^p \leq \epsilon^p$$

$$\implies \|a_k - b\|_p \leq \epsilon, \ \forall k \geq N$$

Noting that $a_N, a_N - b \in \ell^p \implies b \in \ell^p$. $\qquad\square$

**Example 2.26.**

$$C_{00} = \{(x_n) \in \ell^\infty : \exists N \in \mathbb{N} \ s/t \ \forall n \geq N, \ x_n = 0\}$$

$$\leftarrow \text{(NVS)}, \ via \ \|\cdot\|_\infty$$

*Consider*

$$x_n = (1, 1/2, \ldots, 1/n, 0, 0, 0, \ldots)$$

$$x_n \leftarrow \text{(cauchy, divergence)} \ b/c$$

$$x_n \to (1, 1/2, 1/3, \ldots) \notin C_{00} \implies C_{00} \leftarrow (\neg\text{comp})$$