

Real Analysis

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Abstract

Real Analysis the study of approximation on the reals.

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1 Cardinality

1.1 Brief Motivation

We want to build a metric space to measure the distance between objects.

We need

1. set X of objects.
2. need to measure closeness. func $d : X \times X \rightarrow [0, \infty)$ s/t

$$(a) \quad d(x, y) = 0 \iff x = y$$

$$(b) \quad d(x, y) = d(y, x)$$

$$(c) \quad d(x, z) \leq d(x, y) + d(y, z)$$

We call d a metric on X . (X, d) is a **metric space**.

$(\mathbb{R}, +', \circ^\circ (\sqrt{=}) -)$ is a metric space. Note the BQN notation.

1.2 Function Theory

Definition 1.1: Injection. Let A, B be non-empty sets. We say $f : A \rightarrow B$ is injective **iff** $\forall a, b \in A \quad f(a) = f(b) \implies a = b$

Definition 1.2: Surjection. $f : A \rightarrow B$ is a surjection if $\forall b \in B \quad \exists a \in A$ s/t $f(a) = b$.

Definition 1.3: Bijective. $f : A \rightarrow B$ is bijective **iff** its injective and surjective.

Definition 1.4: Invertible. $f : A \rightarrow B$ is invertible **iff** $\exists g : B \rightarrow A$ s/t $g(f(a)) = a$ and $f(g(b)) = b \quad \forall a \in A, b \in B$.

We write $g = f^{-1}$ and call it "the" inverse.

Proposition 1.1. $f : A \rightarrow B$ is invertible **iff** f is bijective.

Proof. (\implies) f is invertible. Suppose $f(a) = f(b)$. We'll show $a = b$.

$$f^{-1}f(a) = f^{-1}f(b)$$

$$\implies a = b$$

Now we'll show $\forall b \in B \exists a \in A f(a) = b$.

$a = f^{-1}(b) \implies$ there is way to get from b to a , and it's f^{-1}

(\Leftarrow) Assume f is (bijective). We'll construct f 's inverse. For $b \in B$ let a_b be the unique element of A s/t $f(a_b) = b$. a_b exists b/c of surjectivity of f , and it's unique b/c of injectivity.

$$g := \{g : A \rightarrow B, g(b) = a_b\}$$

$$f(g(b)) = f(a_b) = b$$

$$g(f(a_b)) = g(b) = a_b$$

$$\implies g = f^{-1}$$

□

Proposition 1.2. $\exists(\text{injection}) f : A \rightarrow B \iff \exists(\text{surjection}) g : B \rightarrow A$

Proof. (\implies) Suppose $f : A \rightarrow B$ is (injective). Let $b \in B$.

Case 1: $b \in f(A)$.

Let $g(b)$ be the unique element of A s/t $f(g(b)) = b$, unique b/c f is (injective)

Case 2: $b \notin f(A)$.

Fix any $z \in A$. Let $g(b) = z$.

$$\implies g(b) = \begin{cases} f^{-1}(b) & b \in f(A) \\ z & b \notin f(A) \end{cases}$$

We claim g is a surjection. So we have to show $\forall a \in A, \exists b \in B$ s/t $g(b) = a$. Let $a \in A$ s/t $f(a) \in B$.

$$g(f(a)) \implies f(g(f(a))) = f(a)$$

$$(\text{injective}) \implies g(f(a)) = a$$

$$\implies g \text{ is (surjective)}$$

\Leftarrow Suppose $(g : B \rightarrow A)$ is (surjective). $\forall a \in A$ choose $b_a \in B$ s/t $g(b_a) = a$.
 $f := \{f : A \rightarrow B \mid f(a) = b_a\}$. Suppose

$$f(x) = f(y)$$

$$\implies b_x = b_y$$

$$\implies g(b_x) = g(b_y)$$

$$\implies x = y$$

$$\implies f \text{ is (injective)}$$

□

Definition 1.5: Powerset. Let X be a set. Then $\mathcal{P}(X) := \{A : A \subseteq X\}$, called the "**powerset** of X ."

$$X = \{a, b\}$$

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Axiom 1.1: Choice. Given $X \neq \emptyset \exists$ a choice func $f : \mathcal{P}(X) \setminus \{\emptyset\} \rightarrow X$ s/t $f(A) \in A \forall \emptyset \neq A \subseteq X$.

1.3 Cardinality

$$A = \{a, b\}, B = \{c, d, e, f\}$$

Intuitively $|A| < |B|$

$$f_{\text{is}}(\text{inj}) : A \rightarrow B, f(a) := c, f(b) := d$$

$$\implies f_{\text{is}}(\text{inj})(A) \subset B$$

$$\implies |A| \leq |B|$$

Definition 1.6: Ordering of Cardinality on Sets. A, B sets.

1. $|A| \leq |B| \iff \exists f_{\text{is}}(\text{inj}) : A \rightarrow B$

2. $|A| = |B| \iff \exists f_{\text{is}}(\text{bij}) : A \rightarrow B$

$$|\mathbb{N}| \leq |\mathbb{Z}| \iff f_{\text{is}}(\text{inj}) : \mathbb{N} \rightarrow \mathbb{Z}, f(n) := n$$

$$f_{\text{is}}(\text{bij}) : \mathbb{N} \rightarrow \mathbb{Z} : f(n) := \begin{cases} 2n + 2 & : n \geq 0 \\ 2(-n) - 1 & : n < 0 \end{cases} \implies |\mathbb{N}| = |\mathbb{Z}|$$

$$h_{\text{is}}(\text{bij}) : \mathbb{R} \rightarrow (0, 1) : h(x) := \frac{\arctan(x) + \pi/2}{\pi} \implies |\mathbb{R}| = |(0, 1)|$$

Theorem 1.1: Cantor-Schroeder-Berstein (CSB). if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

Lemma 1.1: Phi has a Fixed Point. X set. Suppose $\exists \phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ s/t $\phi(A) \subseteq \phi(B)$ if $A \subseteq B \subseteq X$. Then

$$\exists F \subseteq X \text{ s/t } \phi(F) = F$$

Let $F = \bigcup_{A \subseteq X: A \subseteq \phi(A)} A$.

Note: $\emptyset \subseteq X$ & $\emptyset \subseteq \phi(\emptyset)$

Claim: $F = \phi(F)$. Take $A \subseteq X$ with $A \subseteq \phi(A)$. Then $A \subseteq F$.

$$\implies \phi(A) \subseteq \phi(F)$$

$$\implies A \subseteq \phi(F)$$

$$\implies \bigcup_{A \subseteq X: A \subseteq \phi(A)} A \subseteq \phi(F) \quad (\text{by properties of unions})$$

$$\implies F \subseteq \phi(F)$$

Further,

$$F \subseteq \phi(F) \implies \phi(F) \subseteq \phi(\phi(F))$$

$$\implies \phi(F) \in \{A \subseteq X : A \subseteq \phi(A)\}$$

For this step, let $\phi(F) = B$ and notice that $B \subseteq \phi(B)$. So its in the set above.

$$\implies \phi(F) \subseteq \bigcup_{A \subseteq X: A \subseteq \phi(A)} A = F$$

$$\implies F = \phi(F)$$

To motivate (CSB) [theorem 1.1](#): prove that $|N| = |N \times N|$.

$$f_{\text{is(inj)}} : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}, f(n) := (n, 1)$$

$$g_{\text{is(inj)}} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}, g((n, m)) := 2^n 3^m$$

By (CSB) $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.

Proof of (CSB) [theorem 1.1](#). Let $f, g : A \rightarrow B$. For $X \subseteq Y \subseteq A$

$$f(X) \subseteq f(Y)$$

$$\implies B \setminus f(Y) \subseteq B \setminus f(X)$$

$$\implies g(B \setminus f(Y)) \subseteq g(B \setminus f(X))$$

$$\implies A \setminus g(B \setminus f(X)) \subseteq A \setminus g(B \setminus f(Y))$$

This letting $\phi : \mathcal{P}(A) \rightarrow \mathcal{P}(A) : \phi(x) := A \setminus g(B \setminus f(x))$ insures it preserves \subseteq . So by the [lemma 1.1](#), $\exists F \subseteq A$ s.t $F = \phi(F) = A \setminus g(B \setminus f(F))$. In particular, $A \setminus F = g(B \setminus f(F)) \implies g : B \setminus f(F) \rightarrow A \setminus F$ is(bij).

Note. It's a surjection b/c everyone in $A \setminus F$ gets mapped to b/c it's the image if $g(B \setminus f(F))$.

Moreover, $g^{-1} : A \setminus F \rightarrow B \setminus f(F)$ is a bijection, and $f : F \rightarrow f(F)$ is a bijection (for the same reason as above; restriction of domain of an injective function is injective, and a function that maps to its image is automatically a surjection). Hence

$$h : A \rightarrow B : h(x) := \begin{cases} g^{-1}(x) & : x \in A \setminus F \\ f(x) & : x \in F \end{cases}$$

□

Show $|\mathbb{Q}| = |\mathbb{N}|$.

Proof.

$$f : \mathbb{N} \rightarrow \mathbb{Q} : f(x) := x \implies |\mathbb{N}| \leq |\mathbb{Q}|$$

$q \in \mathbb{Q}$ can be written in the form $q = \frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

$$g : \mathbb{Q} \rightarrow \mathbb{Z} \times \mathbb{N} : g(q) := (m, n) : q = \frac{m}{n}$$

$$\implies |\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{N}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

So by (CSB) [theorem 1.1](#), $|\mathbb{Q}| = |\mathbb{N}|$.

□

Definition 1.7: Finite, Countably Infinite, Countable.

1. a set A is ***finite*** iff $|A| = |\{1, 2, \dots, n\}|$ for some $n \in \mathbb{N}$. In this case, $|A| = n$.
2. $|\emptyset| := 0$
3. A is ***countably infinite*** iff $|A| = |\mathbb{N}| := \aleph_0$.
4. A is countable iff A is finite or ctbly infinite.

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{Q}| = \aleph_0$$

Proposition 1.3: Aleph Null is the Smallest Infinity. If A is infinite, then $|\mathbb{N}| \leq |A|$.

Proof. By (Choice) [axiom 1.1](#), $\exists f : \mathcal{P}(A) \setminus \{\emptyset\} \rightarrow A$ s/t $f(X) \in X$, $\forall \emptyset \neq X \subseteq A$.

Let $a_1 = f(A) \in A$

$a_2 = f(A \setminus \{a_1\}) \in A \setminus \{a_1\}$

\vdots

$\implies \aleph_0 = |\{a_1, \dots\}| \leq |A|$.

□

Proposition 1.4: The Reals are Uncountable. \mathbb{R} is uncountable. So $\nexists f \text{ is(bij)} : \mathbb{N} \rightarrow \mathbb{R}$.

Proof. Cantor's Diagonal Element Proof. Since $|\mathbb{R}| = |(0, 1)|$, we'll show that $(0, 1)$ is uncountable.

For contradiction, assume that there exists a bijection $f : \mathbb{N} \rightarrow (0, 1)$.

So let's say

$$f(1) = 0.a_{11}a_{12}a_{13} \cdots$$

$$f(2) = 0.a_{21}a_{22}a_{23} \cdots$$

$$f(3) = 0.a_{31}a_{32}a_{33} \cdots$$

$$\vdots = \vdots$$

Where we avoid repeated nines.

Choose $b_i \in \{1, \dots, 8\}$ s/t $b_i \neq a_{ii}$.

$$\implies \nexists n \in \mathbb{N}, f(n) = 0.b_1b_2b_3 \cdots$$

Thats a contradiction. □

Definition 1.8: Continuum. We write $|\mathbb{R}| = c$, where c stands for **continuum**.

So we have 3 cardinals: n, \aleph_0, c .

Axiom 1.2: Continuum Hypothesis. If A is a set with $\aleph_0 \leq |A| \leq c$, then $\aleph_0 = |A|$ or $|A| = c$.

1.4 Cardinality of Power Sets

Proposition 1.5. If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.

Proof.

$$|\mathcal{P}(A)| = \sum_{k=1}^n \binom{n}{k} = (1+1)^n = 2^n$$

□

Definition 1.9: Cartesian Product. Let I be a set. $\forall i \in I$ Let $A_i \text{ is (set) } \implies \prod_{i \in I} A_i := \{f \mid f : I \rightarrow \bigcup A_i, f(i) \in A_i\}$

$$f(i) \in A_i$$

$$I = \mathbb{N} \implies f : \mathbb{N} \rightarrow \bigcup A_i : f(i) \in A_i \equiv (f(1), f(2), \dots)$$

Definition 1.10: Set Power. $A, B \text{ is (set) } \implies A^B = \{f : B \rightarrow A\}$

$$|A|^{|B|} := |A^B| = |\{f : B \rightarrow A\}|$$

Proposition 1.6: Cardinality of a Power Set. if $X \text{ is (set) }, \mathcal{P}(X) = 2^{|X|} = |\{f : X \rightarrow \{0, 1\}\}|$.

Proof.

$$\phi : \mathcal{P}(X) \rightarrow \{f : X \rightarrow \{0, 1\}\} : \phi(A) := \chi_A$$

$$\chi_A : X \rightarrow \{0, 1\} : \chi_A(x) := \begin{cases} 1 & : x \in A \\ 0 & : x \notin A \end{cases}$$

Show ϕ is (bij). First show it's injective.

$$\phi(A) = \phi(B)$$

$$\implies \chi_A = \chi_B$$

$$\implies A = B \implies \phi \text{ is (inj)}$$

Now show it's surjective. $\forall f \in \{f : X \rightarrow \{0, 1\}\} \exists P \in \mathcal{P}(X)$ s/t $\phi(P) = f$.

$$\text{Let } f^{-1}(\{0, 1\}) = F^{-1}$$

$$\implies \chi_{F^{-1}} : F^{-1} \rightarrow \{0, 1\}$$

$$\implies \chi_{F^{-1}} = \phi(F^{-1}) \text{ w/ } \phi(F^{-1}) = f$$

$$F^{-1} \subseteq X \implies F^{-1} \in \mathcal{P}(X) \implies \phi \text{ is (surj)}$$

□

Proposition 1.7: The Powerset is Larger than the Set. If X is (set), then $|X| < |\mathcal{P}(X)|$.

Proof. Show $|X| \leq |\mathcal{P}(X)|$. $f(x) = \{x\}$ is (inj) $\implies |X| \leq |\mathcal{P}(X)|$.

For the sake on contradiction, assume there is a surjection $g : X \rightarrow \mathcal{P}(X)$. Consider $B := \{x \in X : x \notin g(x)\}$. Hence there must be (by surjectivity of g) $z \in X$ s/t $g(z) = B$. Someone has to map to it.

$$z \in B \implies z \notin g(z) = B$$

$$z \notin B \implies z \in g(z) = B$$

This is a contradiction. So $|X| < |\mathcal{P}(X)|$. □

Infinite Infinities. $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$

Proposition 1.8: The Natural Powerset Cardinal is the Continuum Cardinal. $|\mathcal{P}(\mathbb{N})| = c$ ($\equiv 2^{\aleph_0} = c \equiv |\{0, 1\}^{\mathbb{N}}| = |\mathbb{R}|$)

Proof.

We'll use the continuum hyphthesis, however there's an alternative proof in the course notes.

Consider $X = \{f : \mathbb{N} \rightarrow \{0, 1\}\}$.

$$\phi : X \rightarrow \mathbb{R} : \phi(f) := 0.f(1)f(2)f(3)\dots$$

We can see that ϕ is injective. So

$$2^{\aleph_0} = |X| \leq |\mathbb{R}| = c$$

Also, $\aleph_0 < 2^{\aleph_0} \leq c$. So by (CH), we know $2^{\aleph_0} = c$. □

Proof (without (CH)). ... □

1.5 Cardinal Arithmetic

Definition 1.11. A, B is (sets)

1. $A \cap B = \emptyset \implies |A| + |B| := |A \cup B|$
2. $|A| \cdot |B| := |A \times B|$
3. $|A|^{|B|} := |\{f : B \rightarrow A\}|$

Example. $\aleph_0 + \aleph_0 = \aleph_0$. Let $A = \{a_1, \dots\}$, $B = \{b_1, \dots\}$, so that $|A| = |B| = \aleph_0$, and $A \cap B = \emptyset$.

Then $\phi : A \cup B \rightarrow \mathbb{N} : \phi(a_i) := 2i, \phi(b_i) := 2i - 1$. This is a bijection. Hence $|A \cup B| = \aleph_0$.

Example. $\aleph_0 + c = c$.

$\aleph_0 = |\mathbb{N}|, |(0, 1)| = c$.

$$(0, 1) \subseteq \mathbb{N} \cup (0, 1) \subseteq \mathbb{R}$$

$$\implies c \leq \aleph_0 + c \leq c$$

$$\implies \aleph_0 + c = c$$

Proposition 1.9: Cardinal Exponent Laws. A, B, C is (sets).

1. $(|A|^{|B|})^{|C|} = |A|^{|B| \cdot |C|}$
2. $(|A|^{|B|})(|A|^{|C|}) = |A|^{|B| + |C|}$

Example. Show that $c \cdot c = c$.

$$c \cdot c = (2^{\aleph_0})(2^{\aleph_0}) = 2^{\aleph_0 + \aleph_0} = 2^{\aleph_0} = c$$

Proof of 2. We must show

$$|\{f|f : B \cup C \rightarrow A\}| = |\{f|f : B \cup A\} \times \{f|f : B \rightarrow A\}|$$

$$\text{Let } X := \{f|f : B \rightarrow A\}$$

$$\text{Let } Y := \{f|f : C \rightarrow A\}$$

$$\text{Let } Z := \{f|f : B \cup C \rightarrow A\}$$

So, equivcally we need to show $|Z| = |X \times Y|$.

$$\text{Consider } \varphi(f, g)(x) = \begin{cases} f(x) & x \in B \\ g(x) & x \in C \end{cases}.$$

$$\varphi(f_1, g_1) = \varphi(f_2, g_2)$$

$$\implies \forall x \in B \cup C, \varphi(f_1, g_1)(x) = \varphi(f_2, g_2)(x)$$

$$\implies \forall x \in B, f_1(x) = f_2(x) \implies f_1 = f_2$$

$$\implies \forall x \in C, g_1(x) = g_2(x) \implies g_1 = g_2$$

Consider $h : B \cup C \rightarrow A$. Let $f = h|_B$, $g = h|_C$. Then $\varphi(f, g) = h$.

So φ is bijective, so proposition 2 holds. □

Example: $c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$.

2 Topology

2.1 Metric Spaces

Definition 2.1: Metric Space. $X \text{ is (set) } X$. A metric on X is a function $d : X \times X \rightarrow [0, \infty)$ s/t

1. $d(x, y) = 0 \iff x = y$
2. **Abelian:** $d(x, y) = d(y, x)$
3. **Triangle:** $d(x, y) \leq d(x, z) + d(z, y)$

Definition 2.2: Normed Vector Space (NVS). Let V is (Vector Space) over \mathbb{R} . A norm on V is a fn $\| \cdot \| : V \rightarrow [0, \infty)$ s/t

$$1. \|v\| = 0 \iff v = \vec{0}$$

$$2. \|\alpha v\| = |\alpha| \cdot \|v\|$$

$$3. \|v + u\| \leq \|v\| + \|u\|$$

BQN: $\| \times \| = \| \circ l \cdot \| \circ r \quad | + \leq \leq + \square$

Proposition 2.1: NVS have trivial Metrics. Let $V, \| \cdot \|$ is (NVS). $d(v, w) = \|v - w\|$ is a metric on V .

2.2 Examples of Metric Spaces

Example 2.1: Discrete Metric. X is (set).

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Example 2.2: Absolute Value Norm. $(\mathbb{R}, | \cdot |)$ is (NVS)

Example 2.3: Euclidean Norm. $(\mathbb{R}^n, \| \cdot \|_2)$ is (NVS) where $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$.

Example 2.4: P-Norm. $p \geq 1$, $(\mathbb{R}^n, \| \cdot \|_p)$ is (NVS) where

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Note: see posted notes for the proof that this is a norm. OPTIONAL.

Example 2.5: Infinity Norm. $p = \infty$, $(\mathbb{R}^n, \| \cdot \|_\infty)$ is (NVS) where

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$$

Example 2.6: P-Norm on Sequences of Reals. $\mathbb{R}^{\mathbb{N}} := \{f | f : \mathbb{N} \rightarrow \mathbb{R}\} = \{(a_n)_{n=1}^{\infty} : a_n \in \mathbb{R}\}$. For $p \geq 1$,

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \quad (1)$$

$l^p := \{x \in \mathbb{R} : \|x\|_p < \infty\} \implies (l^p, \|\cdot\|_p)$ is (NVS). This is the p -norm on sequences of reals. Notice how this solve the divergence issue (by ignoring it lol).

Example: $l^1 = \{x \in \mathbb{R} : \sum |x_i| < \infty\} \implies l^1$ is the set of absolutely convergent sequences.

Example 2.7: Suprema Norm (Infinity Norm on Sequences of Reals).

$$\|x\|_{\infty} = \sup\{|x_n| : n \in \mathbb{N}\}.$$

if we let $l^{\infty} := \{x \in \mathbb{R}^{\mathbb{N}} : \|x\|_{\infty} < \infty\}$, noting that l^{∞} is the set of all bounded sequences, then $(l^{\infty}, \|\cdot\|_{\infty})$ is (NVS).

Example 2.8: P-Norm on Function. $C([a, b]) := \{f : [a, b] \rightarrow \mathbb{R} | f \text{ is (cts)}\}$.

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}, \quad p \geq 1$$

Example 2.9: Infinity Norm on Functions. $\|f\|_{\infty} = \sup\{|f(x)| : x \in [a, b]\}$

Example 2.10: Bounded Functions and the Infinity Norm are a NVS.

$$\mathbb{B}([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} | f \text{ is (bd)}\}, \quad (\mathbb{B}([a, b]), \|\cdot\|_{\infty}) \text{ is (NVS)}.$$

Example 2.11: Sequence Metric. $X = \mathbb{R}^{\mathbb{N}} = \{f \mid f : \mathbb{N} \rightarrow \mathbb{R}\}.$

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

Prove that d isn't induced by a metric. If $d(x, y) = \|x - y\|$ for some norm, then $\|\alpha x - \alpha y\| = |\alpha| \|x - y\|.$

$$d(ax, ay) = \sum_{i=1}^{\infty} \frac{|ax_i - ay_i|}{2^i(1 + |ax_i - ay_i|)}$$

$$d(ax, ay) = \sum_{i=1}^{\infty} \frac{|a||x_i - y_i|}{2^i(1 + |a||x_i - y_i|)}$$

$$|a|d(x, y) = |a| \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

$$|a|d(x, y) = \sum_{i=1}^{\infty} \frac{|a||x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

$$\text{b/c } |a||x_i - y_i| \neq |x_i - y_i|$$

\implies not induced by a norm

□

Example 2.12: Cantor Space. $X = 2^{\mathbb{N}} := \{f \mid f : \mathbb{N} \rightarrow \{0, 1\}\}.$

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

Example 2.13: Hamming Distance. X is finite. $A, B \in \mathcal{P}(X).$

$$d(A, B) := |A \Delta B| = |(A \cup B) \setminus (A \cap B)|$$

Example 2.14: Hausdorff Metric. $\mathcal{H} = \{K \subseteq \mathbb{R}^n : K \text{ compact}\}$. Let $a \in A, b \in B, A, B \in \mathcal{H}$.

$$d(a, B) = \min\{\|a - b\| : b \in B\}$$

$$d(b, A) = \min\{\|a - b\| : a \in A\}$$

$$d(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}$$

Note 2.1. $\sup_{a \in A} d(a, B)$ represents the biggest shortest path between A and B .

Note 2.2. Metrics give a sense of convergence on a space.

Example 2.15: P-adic Metric. Let p be prime, $X = \mathbb{Q}$.

$$\text{Let } 0 \neq q \in X = \mathbb{Q} \quad q = p^a \frac{n}{m}$$

$$\text{where } \gcd(n, m) = \gcd(p, n) = \gcd(p, m) = 1$$

$$|q|_p = \frac{1}{p^a}, \quad |0|_p = 0$$

$$d(q_1, q_2) := |q_1 - q_2|_p$$

Note 2.3. This notion of distance implies the more factors of p , the closer. This gives a sense of optimizing for a certain adjective. These numbers aren't close using $\|\cdot\|_2$, but are using the p -adic norm.

Definition 2.3: Subspace of a Metric Space. $(X, d), Y \subseteq X \implies (Y, d)$. (Y, d) is called a **subspace** of (X, d) .

Definition 2.4. $(X, d_1), (Y, d_2)$. Consider $(X \times Y, d)$ with

$$d((x_1, y_1), (x_2, y_2)) := d_1(x_1, x_2) + d_2(y_1, y_2) \text{ (1-norm)}$$

$$\text{or } d((x_1, y_1), (x_2, y_2)) := \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \text{ (\infty-norm)}$$

Example 2.16: Product Metric. $(X_i, d_i) \ i \in \mathbb{N}, X := \prod_{n=1}^{\infty} X_i$. Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$.

$$d(x, y) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i(1 + d_i(x_i, y_i))}$$

2.3 Convergence

Definition 2.5: Convergence of a Sequence. $(X, d), (x_n) \subseteq X$, and $x \in X$.

Notation 2.1. $(x_n) \subseteq X$ means (x_n) is a sequence in X .

(x_n) conv to $x, x_n \rightarrow x$ **iff**

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s/t } \forall n \geq N, d(x_n, x) < \epsilon$$

Definition 2.6: Divergence. (x_n) diverges of $\nexists x \in X$ s/t $x_n \rightarrow x$.

Note 2.4: Convergence is Distance going to Zero. $(X, d), (x_n) \subseteq X, x \in X$.
Then $x_n \rightarrow x$ **iff** $d(x_n, x) \rightarrow 0$.

Definition 2.7: Cauchy. $(X, d), (x_n) \subseteq X$ is a *cauchy seq* **iff**

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s/t } \forall n, m \geq N, d(x_n, x_m) < \epsilon$$

Proposition 2.2: Convergence implies Cauchyness. (X, d) . If $(x_n) \subseteq X$ converges, then (x_n) is *cauchy*.

Epsilon/2. Suppose $(x_n) \rightarrow x$. Let $\epsilon > 0$. So $\exists N$ s/t $\forall n \geq N$,

$$d(x_n, x) < \gamma$$

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x) + d(x, x_m) \text{ by } \triangle \\ &< \gamma + \gamma = 2\gamma \\ &:= 2\frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

□

Example 2.17: Cauchy doesn't imply Convergence. $X = (0, 1]$ with the std metric.

$$\frac{1}{n} \rightarrow 0 \implies \left(\frac{1}{n}\right) \subseteq X \text{ is cauchy}$$

Note 2.5. I think this is trying to say that $1/n$ is cauchy in X , but $1/n \rightarrow 0$, which is not in X , so it diverges (in X).

Definition 2.8: Bounded.

1. $A \subseteq X$ is bd iff $\sup\{d(x, y) : x, y \in A\} < \infty$
2. $(x_n) \subseteq X$ is (bd) $\iff \{x_1, x_2, \dots\}$ is (bd) $\iff \sup\{n, m \in \mathbb{N} : d(x_n, x_m)\} < \infty$

Definition 2.9: Open and Closed Balls.

1. **Open:** $B_r(a) := \{x \in X : d(x, a) < r\}$
2. **Clsd:** $B_r[a] := \{x \in X : d(x, a) \leq r\}$

Proposition 2.3: Boundedness iff subset of a Closed Ball. $(X, d), A \subseteq X$.
Then A is bd iff $\exists r > 0, \exists x \in X$ s/t $A \subseteq B_r[x]$

Proof. Suppose $\sup\{d(x, y) : x, y \in A\} = r < \infty$. Assume $A \neq \emptyset$, taking $a \in A$. For $b \in A$, $d(a, b) \leq r \implies A \subseteq B_r[a]$.

Assume $A \subseteq B_r[a] \implies \forall a, b \in A, d(a, b) \leq d(a, x) + d(x, b) \leq 2r$

□

Proposition 2.4: Cauchy implies Bounded. $(x_n)_{\text{is(cauchy)}} \implies (x_n)_{\text{is(bd)}}$

Example 2.18: Counter Example. $(0, 1, 0, 1, 0, \dots)$ is bounded but not cauchy.

Note 2.6. $CONVERGENCE \implies CAUCHY \implies BOUNDED$

Proof. Suppose (x_n) is cauchy. Let $\epsilon = 1$.

$$\exists N \in \mathbb{N} \text{ s/t } \forall n, m \geq N \ d(x_n, x_m) < 1$$

Let $r := \max\{d(x_1, x_N), \dots, d(x_{N-1}, x_N)\}$

Note 2.7. We did this because the web's edges are no longer than 1. So, we can look at the finite part left. Finite sets are bounded, and the "web" is bounded.

$$(x_n) \subseteq B_r[x_N]$$

□

2.4 Convergence Examples

Example 2.19: 2-Adic Norm. Consider $(\mathbb{Q}, |\cdot|_2)$. Let $x_n := \frac{1+2^n}{3}$.

Note 2.8: P-Adic Convergence Claims. Looking at $x_n = \frac{1}{3} + \frac{2^n}{3} = \frac{1}{3} + \overset{0}{\cancel{\frac{2^n}{3}}} = \frac{1}{3}$.

We claim $x_n \rightarrow \frac{1}{3}$.

Proof.

$$\begin{aligned} \left| x_n - \frac{1}{3} \right|_2 &= \left| \frac{2^n}{3} \right|_2 \\ &= \frac{1}{2^n} \\ &\rightarrow 0 \end{aligned}$$

So $x_n \rightarrow \frac{1}{3}$ under $|\cdot|_2$.

□

Example 2.20: Bounded Sequences and the Infinity Norm. $(l^\infty, \|\cdot\|_\infty)$.

Let $x_n := (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, 0, \dots)$ and $x := (1, \frac{1}{2}, \frac{1}{3}, \dots)$

We claim that $x_n \rightarrow x$.

Proof.

$$\begin{aligned} & \|x_n - x\|_\infty \\ &= \sup \left(0, 0, \dots, 0, \frac{1}{n+1}, \frac{1}{n+2}, \dots \right) \\ &= \frac{1}{n+1} \rightarrow 0 \end{aligned}$$

□

Example 2.21: Zero Tailed Sequences aren't Cauchy under Sup Norm.

$$y_n := (\underbrace{1, 1, \dots, 1}_n, 0, 0, 0, \dots)$$

$$y := (1, 1, 1, 1, \dots)$$

$$n \neq m, \|y_n - y_m\|_\infty = 1$$

2.5 Completeness

Definition 2.10: Complete, Complete Metric Space, Banach Space.

(X, d) , $A \subseteq X$. Then

1. A is **complete** iff every cauchy seq in A converges to some $a \in A$.
2. if X is complete, we call it a **Complete Metric Space**.
3. A complete normed vector space is called a **Banach Space**

Example 2.22.

$$X = (0, 1], \frac{1}{n} \rightarrow 0 \notin X \implies (\text{div}) \implies X \text{ is not comp}$$

Example 2.23.

$$A = [1/2, 1] \subseteq X \text{ is comp}$$

Example 2.24. $(X, d := (\text{discrete}))$ Let $(x_n) \in \mathbb{R}^{\mathbb{N}}$ be *cauchy*.

$$\exists N \in \mathbb{N} \text{ s.t. } \forall n, m \geq N \implies d(x_n, x_m) < 1$$

$$\implies d(x_n, x_m) = 0$$

$$\implies x_n = x_m$$

$$\implies x_n = (x_1, x_2, \dots, x_N, x_N, x_N, \dots) \rightarrow x_N$$

$$\implies X \text{ is (complete)}$$

Note 2.9. *So nice sets or nice metrics can cause completeness.*

Example 2.25. Show $(l^\infty, \|\cdot\|_\infty)$ is a Banach Space.

Let $(x_n) \subseteq l^\infty$, [example 2.7](#). We know already that $(l^\infty, \|\cdot\|_\infty)$ is a (NVS), so we have to show it's complete. Let $\epsilon > 0$ be given. Then,

$$\exists N \in \mathbb{N} \text{ s/t } n, m \geq N \implies \|x_n - x_m\|_\infty < \epsilon$$

$$x_k = (x_k[1], x_k[2], \dots)$$

$$\text{for } n, m \geq N, |x_n[i] - x_m[i]|$$

$$\leq \sup\{|x_n[i] - x_m[i]| : i \in \mathbb{N}\}$$

$$= \|x_n - x_m\|_\infty < \epsilon$$

$$\implies \forall i \in \mathbb{N}, \text{ the seq } (x_n[i])_{n=1}^\infty \text{ is (cauchy in } \mathbb{R})$$

$$\mathbb{R} \text{ is (comp)} \implies x_n[i] \xrightarrow{n} b_i$$

$$\text{Claim: } x_n \rightarrow b := (b_1, b_2, \dots)$$

$$\forall n, m \geq N, |x_n[i] - x_m[i]| < \epsilon$$

$$\implies \lim_{m \rightarrow \infty} |x_n[i] - x_m[i]| \leq \epsilon$$

$$\implies \forall n \geq N, |x_n[i] - b_i| \leq \epsilon$$

$$\text{Consider } \|x_n - b\|_\infty$$

$$= \sup\{|x_n[i] - b_i| : i \in \mathbb{N}\}$$

$$\leq \epsilon < 2\epsilon$$

Hence $x_n \rightarrow b$.

Note: we have that $x_N - b \in l^\infty$, and $x_N \in l^\infty$. However l^∞ is a (VS), so $b \in l^\infty$.

Proposition 2.5: Set of bounded Sequences on the P-Norm is Banach.

$(\ell^p, \|\cdot\|_p)$ is (banach).

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

$$\ell^p := \{x \in \mathbb{R} : \|x\|_p < \infty\} \implies (\ell^p, \|\cdot\|_p) \text{ is (NVS)}$$

Proof. Let $(a_k) \subseteq \ell^p$ be cauchy.

Say $a_k = (a_k[1], a_k[2], \dots)$

Let $\epsilon > 0$

$\exists N \in \mathbb{N}$ s/t $\|a_k - a_m\| < \epsilon, \forall k, m \geq N$

Fixing $i \in \mathbb{N}$, Since $|a_k[i] - a_m[i]| \leq \|a_k - a_m\|_p < \epsilon$

We see that $(a_k[i])_{k=1}^\infty$ is (cauchy in \mathbb{R})

\mathbb{R} is (comp) $\implies a_k[i] \rightarrow b_i$ for some $b_i \in \mathbb{R}$

Claim: $a_k \rightarrow b = (b_1, b_2, \dots)$

$\forall k, m \geq N$, we see that

$$\begin{aligned} & \sum_{i=1}^M |a_k[i] - a_m[i]|^p \\ & \leq \sum_{i=1}^{\infty} |a_k[i] - a_m[i]|^p \\ & = \|a_k - a_m\|_p^p < \epsilon^p \end{aligned}$$

$$\sum_{i=1}^M |a_k[i] - b_i|^p \leq \epsilon^p, \forall M \in \mathbb{N}$$

$$M \rightarrow \infty : \sum_{i=1}^{\infty} |a_k[i] - b_i|^p \leq \epsilon^p$$

$$\implies \|a_k - b\|_p \leq \epsilon, \forall k \geq N$$

Noting that $a_N, a_N - b \in \ell^p \implies b \in \ell^p$. □

Example 2.26.

$$C_{00} = \{(x_n) \in \ell^\infty : \exists N \in \mathbb{N} \text{ s/t } \forall n \geq N, x_n = 0\}$$

is(NVS), via $\|\cdot\|_\infty$

Consider

$$x_n = (1, 1/2, \dots, 1/n, 0, 0, 0, \dots)$$

x_n is (cauchy, divergence) b/c

$$x_n \rightarrow (1, 1/2, 1/3, \dots) \notin C_{00} \implies C_{00} \text{ is } (\neg\text{-comp})$$

2.6 Topological Spaces

Definition 2.11: Topology. X set. A **topology** on X is a set $\mathcal{T} \subseteq \mathcal{P}(X)$ s/t

1. $\emptyset, X \in \mathcal{T}$
2. $U, V \in \mathcal{T} \implies U \cap V \in \mathcal{T}$
3. $U_i \in \mathcal{T}, (i \in I) \implies \bigcup_{i \in I} U_i \in \mathcal{T}$

Example 2.27: Discrete Topology. X set. $\mathcal{T} := \mathcal{P}(X)$

Example 2.28: Indiscrete Topology. $\mathcal{T} := \{\emptyset, X\}$

Example 2.29. $X = \{a, b, c\}$

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{b, c\}\}$$

$$\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$$

Notation 2.2: Topological Space. If \mathcal{T} is a topology on X , then (X, \mathcal{T}) is called a **topological space**.

2.7 Metric Topology

Definition 2.12: Open. (X, \mathcal{T}) . $U \subseteq X$ is open iff $\forall x \in U, \exists r > 0$ s/t $B_r(x) \subseteq U$.

Proposition 2.6: The Set of Open sets form a Topology. $\mathcal{T} = \{U \subseteq X : U \text{ open}\}$ is a topology.

Proof. $\emptyset, X \subseteq \mathcal{T}$ trivially.

Let $U, V \in \mathcal{T}$. Since U & V are open

$$\exists r_1, r_2 > 0 \text{ s/t } B_{r_1}(x) \subseteq U \quad B_{r_2}(x) \subseteq V$$

$$r := \min\{r_1, r_2\} \quad B_r(x) \subseteq U \cap V \implies U \cap V \in \mathcal{T}$$

Let $U_i \in \mathcal{T}$ for all $i \in I$. Let $x \in \bigcup_{i \in I} U_i$.

$$\text{So } \exists i \in I \text{ s/t } x \in U_i$$

$$\text{So } \exists r > 0 \text{ s/t } B_r(x) \subseteq U_i \subseteq \bigcup U_i$$

□

Example 2.30: Counterexample for Infinte Intersections are Open.

$$(\mathbb{R}, \mathcal{T}_{\text{open}}) \quad U_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \quad \bigcap U_n = \{0\} \notin \mathcal{T}$$

Proposition 2.7: Metric Spaces are Hausdorff. $(X, d) \quad \forall x \neq y \in X, \exists U, V \subseteq X \text{ open s/t } x \in U, y \in V, U \cap V = \emptyset$

Proof. Let $r = d(x, y) > 0$. Let $U = B_{r/2}(x)$ $V = B_{r/2}(y)$.

Open Balls are Open Proof. Let $x \in B_r(a)$ for some $a \in X$, $r > 0$. Then

$$d(x, a) < r \text{ by open ball def}$$

$$y \in B_{r-d(x,a)}(x)$$

$$\implies d(x, y) < r - d(x, a)$$

$$d(y, a) \stackrel{(\text{TI})}{<} d(x, y) + d(x, a) < r$$

$$\implies y \in B_r(a) \implies B_{r-d(x,a)}(x) \subseteq B_r(a) \implies B_r(a) \text{ is open}$$

□

Assume for the sake of contradiction $\exists z \in B_{r/2}(x) \cap B_{r/2}(y)$.

$$\implies d(z, x) < r/2 \quad d(z, y) < r/2$$

$$r > d(x, z) + d(z, y) > d(x, y) = r$$

This is a contradiction.

□

Example 2.31. $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, X, \{a, b\}\}$. Since a cannot be "separated" from b , then there is no possible metric on X , so \mathcal{T} isn't a metric topology. All metrics make a topology, not all topologies make a metric.

2.8 Closed Sets

Definition 2.13: Closed Sets. (X, \mathcal{T}) . $C \subseteq X$ is closed iff $X \setminus C \in \mathcal{T}$

Proposition 2.8: Properties of Closed Sets. (X, \mathcal{T}) .

1. \emptyset, X closed.
2. C, D closed then $C \cup D$ closed.
3. $C_i, i \in I, \bigcap_{i \in I} C_i$ is closed.

Proof by "Boolean Nonsense". $X \setminus C, X \setminus D \in \mathcal{T}$.

$$X \setminus (C \cup D) \in \mathcal{T}$$

$$X \setminus C \cap X \setminus D \in \mathcal{T}$$

□

Definition 2.14: Limit Point. $(X, \mathcal{T}), A \subseteq X$. We say $x \in X$ is a **limit point** of A iff

$$\forall U \in \mathcal{T} \text{ with } x \in U \quad A \cap U \neq \emptyset$$

Note 2.10. $x \in A \implies x \text{ is (limit point)}$

Proposition 2.9: Limits Points in a Metric Topology are the Limit of a Sequence. $A \subseteq X$. Then $x \in X$ is a limit point of A iff $\exists (a_n) \in A$ s/t $a_n \rightarrow x$.

(\implies) *Proof.* Assume x is a limit point of A . Then $\forall U \in \mathcal{T}, x \in U, A \cap U \neq \emptyset$. Then $\forall n \in \mathbb{N}, \exists a_n \in B_{\frac{1}{n}}(x) \cap A \quad (\neq \emptyset)$. Then $d(x, a_n) < \frac{1}{n} \rightarrow 0 \implies a_n \rightarrow x$ □

(\impliedby) *Proof.* Assume $\exists a_n \rightarrow x$. Then $\forall \epsilon > 0 \exists N \in \mathbb{N}$ s/t $d(a_n, x) < \epsilon$. Let $U \subseteq X$ be open with $x \in U$.

$$\implies \exists r > 0 \text{ s/t } B_r(x) \subseteq U$$

$$\text{Then } \exists N \in \mathbb{N} \text{ s/t } d(a_N, x) < r$$

$$\text{So } a_N \in U \implies A \cap U \ni a_N$$

□

Proposition 2.10: Closed Sets Attain Limits. C is closed iff C attains its limit points.

(\implies) *Proof.* Suppose C is clsd. Let $x \in X$ be a limit point of C . Since $X \setminus C$ is open and $(X \setminus C) \cap C = \emptyset, x \notin X \setminus C \implies x \in C$. □

(\Leftarrow) *Proof.* Suppose C attains its limit points. Show $X \setminus C \in \mathcal{T}$.

Let $x \in X \setminus C$

So x isn't a limit point of C

then $\exists U_x \in \mathcal{T}, x \in U_x$ s.t. $U_x \cap C = \emptyset$

then $U_x \subseteq U_x \cap C$ (?)

then $X \setminus C = \bigcup_{x \in X \setminus C} U_x \in \mathcal{T}$

□

Corollary 2.1: Closed Sets Attain Sequence Limits. $(X, d), C \subseteq X$. Then C is closed iff

$\forall (c_n) \subseteq C, c_n \rightarrow x \in X$ then $x \in C$

Definition 2.15: Subspace Topology. $(X, \mathcal{T}), Y \subseteq X$.

Note that $\varphi, X \in \mathcal{T}, \mathcal{T}$ closed under (\cup, \cap)

The **subspace topology** is

$$\mathcal{T}' = \{Y \cap U : U \in \mathcal{T}\}$$

Prove subspace topologies are Topologies.

1. Show

$$\emptyset, Y \in \mathcal{T}'$$

So,

$$\emptyset \in \mathcal{T}, Y \cap \emptyset = \emptyset \implies \emptyset \in \mathcal{T}'$$

$$X \in \mathcal{T}, Y \cap X = Y \implies Y \in \mathcal{T}'$$

2. Show

$$U, V \in \mathcal{T}' \implies U \cap V \in \mathcal{T}'$$

So let $U, V \in \mathcal{T}'$.

$$\implies \exists \text{ open } U_X \in X \text{ s/t } U = U_X \cap Y$$

$$\implies \exists \text{ open } V_X \in X \text{ s/t } V = V_X \cap Y$$

$$\implies U \cap V = (U_X \cap Y) \cap (V_X \cap Y) = (U_X \cap V_X) \cap Y$$

$$U_X \cap V_X \in \mathcal{T} \implies U \cap V \in \mathcal{T}'$$

3. Show

$$U_i \in \mathcal{T}', (i \in I) \implies \bigcup_{i \in I} U_i \in \mathcal{T}'$$

So

$$U_i \in \mathcal{T}' \implies U_i = Y \cap U_i^X \text{ for some open } U_i^X \in \mathcal{T}$$

$$\bigcup U_i = Y \cap \bigcup U_i^X \implies \bigcup U_i \in \mathcal{T}'$$

□

Note 2.11. (X, \mathcal{T}) , $Y \subseteq X$, \mathcal{T}' as above. $C \subseteq Y$ clsd.

$$\implies Y \setminus C \in \mathcal{T}'$$

$$\implies Y \setminus C = Y \cap U, U \in \mathcal{T}$$

$$\implies C = Y \cap \underbrace{(X \setminus U)}_{\text{clsd}}$$

Note 2.12. (X, d) , $Y \subseteq X$. Define (Y, d) as a subspace metric space. Suppose $U \subseteq Y$ is open wrt Y .

$$\implies \forall x \in U \exists r_x > 0 \text{ s/t } \underbrace{B_{r_x}(x)}_{\text{in } Y} \subseteq U$$

$$\implies \forall x \in U \exists r_x > 0 \text{ s/t } \underbrace{B_{r_x}(x)}_{\text{in } X} \cap Y \subseteq U$$

$$U = \bigcup_{x \in U} (Y \cap B_{r_x}(x)) = Y \cap \underbrace{\left(\bigcup_x B_{r_x}(x) \right)}_{\text{open in } X}$$

2.9 Closure and Interior

Definition 2.16: Closure and Interior.

1. the **Closure** of A is defined as follows:

$$\overline{A} = \bigcup_{\substack{C \supseteq A, \\ C \text{ clsd}}} C$$

2. the **Interior** of A is defined as follows:

$$\text{Int}(A) = \bigcup_{\substack{U \subseteq A \\ U \in \mathcal{T}}} U$$

Note 2.13. 1. $\overline{A} \text{ is clsd } \text{Int}(A) \in \mathcal{T}$

$$2. \text{Int}(A) \subseteq A \subseteq \overline{A}$$

3.

$$A \text{ closed} \iff A = \overline{A}$$

$$A \text{ open} \iff A = \text{Int}(A)$$

Note 2.14. $(X, \mathcal{T}), Y \subseteq X, A \subseteq Y. (Y, \mathcal{T}')$.

$$(\text{wrt } Y) \longrightarrow \overline{A} = \bigcap \{C : A \subseteq C, C \text{ clsd in } Y\}$$

$$= \bigcap \{Y \cap C : A \subseteq C, C \text{ clsd in } X\}$$

$$= Y \cap \left(\bigcap \{C : A \subseteq C, C \text{ clsd in } X\} \right)$$

$$= Y \cap \overline{A} \longleftarrow (\text{wrt } X)$$

Similarly, $\text{Int}(A) \text{ wrt } (Y) = Y \cap \text{Int}(A) \text{ wrt } (X)$.

Proposition 2.11: The Closure is the Set of Limit Points. $A \subseteq X$.

$$\overline{A} = \{x \in X : x \text{ is a limit point of } A\}$$

Proof. Let $L := \{x \in X : x \text{ is a limit point of } A\}$. Let $x \in \overline{A}$. Let $U \in \mathcal{T}$ s/t $x \in U$. Suppose $A \cap U = \emptyset$

$$\implies A \subseteq \underbrace{X \setminus U}_{\text{clsd}} \implies x \in X \setminus U$$

Contradiction, $x \in U$ and $x \in X \setminus U$.

Let $x \in L$, and let C be clsd, with $A \subseteq C$. Suppose

$$x \notin C \implies x \in X \setminus C := U$$

$$\implies (X \setminus C) \cap A \neq \emptyset$$

Contradiction of $A \subseteq C$. □

Note 2.15: Norms.

1. *P-Norm*, $x \in \mathbb{R}^n$.

$$\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

2. *Inf-Norm*, $x \in \mathbb{R}^n$.

$$\max\{|x_i| : 0 \leq i \leq n\}$$

3. *P-Norm*, $x \in \mathbb{R}^{\mathbb{N}}$

$$\left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}$$

4. *Inf-Norm*, $x \in \mathbb{R}^{\mathbb{N}}$.

$$\sup\{|x_i| : i \in \mathbb{N}\}$$

5. *P-Norm*, $x \in \mathbb{R}^{\mathbb{R}}$.

$$\left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{\frac{1}{p}}$$

6. *Inf-Norm*, $x \in \mathbb{R}^{\mathbb{R}}$.

$$\sup\{|f(x)| : x \in \mathbb{R}\}$$

7. $\ell^p = \{x \in \mathbb{R}^{\mathbb{N}} : \|x\|_p < \infty\}$

Corollary 2.2: Closure is the Set of Reachable Points. $\overline{A} = \{x \in X : \exists(a_n) \subseteq A, a_n \rightarrow x\}$

Definition 2.17: Interior Points. (X, \mathcal{T}) , $A \subseteq X$. Then $x \in A$ is an *interior point* iff

$$\exists U \in \mathcal{T} \text{ s/t } x \in U \subseteq A$$

Note 2.16. Notice that this is similiar to how we define openness in a metric space.

Note 2.17. (X, d) , $A \subseteq X$. Then $x \in A$ is an interior point of A iff

$$\exists r > 0 \text{ s/t } x \in B_r(x) \subseteq A$$

Proposition 2.12: Interior Points of A build the Interior of A.

$$\text{Int}(A) = \{x \in A : x \text{ is an interior point of } A\}$$

Proof. Let $I := \{x : x \text{ is an int pt}\}$, $x \in \text{Int}(A)$.

$$\iff \exists U \in \mathcal{T}, x \in U \subseteq A \iff x \in I$$

□

Example 2.32: Closures and Closed “Versions” of Sets are Inequal. $(\mathbb{N}, |\cdot|)$

$$B_1(1) = \{1\} \implies \overline{B_1(1)} = \{1\} = B_1(1)$$

$$B_1[1] = \{1, 2\}$$

Proof. Prove $\overline{B_r(x)} = B_r[x]$, under a (NVS).

NVS: vectors, w/ buildin norm. Basically this is a ordinary norm, rather than a metric. So we have norm TI, norm zero uniqueness, AND a scalar property.

(\subseteq) . Let $y \in \overline{B_r(x)}$. So

$$B_r[x] = \{y \in \mathbb{R} : \|x - y\| \leq r\}$$

$$\text{Show } \|x - y\| \leq r$$

$$y \in \overline{B_r(x)} \implies \exists y_n \in \mathbb{R}^{\mathbb{N}} \text{ s/t } y_n \rightarrow y$$

$$\implies \exists N \in \mathbb{N} \text{ s/t } \forall n > 0, \|x - y_n\| < r$$

$$\implies \|x - y\| \leq r$$

(\supseteq) Let $y \in B_r[x]$.

$$\|y - x\| \leq r$$

$$\text{Show } \exists (a_n) \in \mathbb{R}^{\mathbb{N}} \text{ s/t } a_n \rightarrow y$$

$$\text{Let } a_i = y + \frac{x - y}{i}$$

$$\|x - a_i\| = \left\| x - y - \frac{x - y}{i} \right\| = \|x - x/i - y + y/i\|$$

$$\|x - x/i + (y/i - y)\| \leq \|x - x/i\| + \|y - y/i\| < 2r$$

□

Example 2.33. (X, d) , $A \subseteq X$ complete. Prove A is closed.

$$(a_n) \subseteq A, a_n \rightarrow x \in X$$

(a_n) *cauchy*, (a_n) *conv to a pt in A*.

$$\implies x \in A \implies \text{closed}$$

Definition 2.18: Subsequence.

$$(X, d), (X_n) \subseteq X$$

A *subsequence of (X_n) is a sequence*

$$(X_{n_k})_{k=1}^{\infty}, \text{ where } n_1 < n_2 < n_3 < \dots$$

Note 2.18.

$$\mathbb{N} \rightarrow \mathbb{N} \rightarrow X : x(n(k)), n \text{ increasing}$$

$$K \geq N, n_K \geq K \geq N$$

$$K \geq N, n(K) \geq K \geq N$$

Example 2.34.

$$(X, d), (X_n) \subseteq X. \text{ Prove } x_n \rightarrow x \implies x_{n_k} \rightarrow x$$

Proof.

$$K \geq N, n_K \geq K \geq N$$

$$\implies d(x_{n_k}, x) < \epsilon$$

□

Example 2.35.

Let (X, d) $(x_n) \subseteq X$ *cauchy*

Show x_n *conv* $\iff x_n$ *has a conv subseq*

Proof. The forward direction is trivial.

Backwards: Let (x_n) be cauchy, and let some subsequence $x_{n_k} \rightarrow x$.

$$\text{Let } \epsilon > 0, \text{ let } N \in \mathbb{N} \text{ s/t } \forall n, m \geq N \implies d(x_n, x_m) < \frac{\epsilon}{2}$$

$$\text{Let } k \in \mathbb{N} \text{ } k \geq K \implies d(x_{n_k}, x) < \frac{\epsilon}{2}$$

$$\text{Assume } K \geq N \implies n_K \geq N$$

$$\begin{aligned} \forall n \geq N, \quad d(x_n, x) &\leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

□

Example 2.36.

Let (X, d) , $x_n \rightarrow x$

Prove $C = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is closed

Proof.

if $\exists N$ inf many $y_n = x_N$

then $(x_N)_n$ is a subseq of y_n

$$y_n \rightarrow y, \quad (x_N)_n \rightarrow x_N \implies y = x_N \in C \blacksquare$$

if $\forall n$, only finitly many $y_K = x_n$

then $\exists (y_{n_k})$

which is also a subseq of (x_n) . Then $y = x$. \blacksquare

□

Example 2.37. Let V be a (NVS), $U \subseteq V$, $U \in \mathcal{T}$, $x \in V$. Show

$$x + U = \{x + u : u \in U\} \in \mathcal{T}$$

Proof.

$$\text{Let } x + u \in x + U, u \in U \in \mathcal{T}$$

$$u \in U \in \mathcal{T} \implies \exists r > 0 \text{ s.t. } B_r(u) \subseteq U$$

$$B_r(x + u) = x + B_r(u) \subseteq x + U \implies x + U \in \mathcal{T}$$

□

Example 2.38. V is NVS, $\mathcal{T} \ni U \subseteq V$, $A \subseteq V$.

$$\text{Show } A + U = \{a + u : a \in A, u \in U\} \in \mathcal{T}$$

Proof.

$$A + U = \bigcup_{a \in A} a + U$$

$$a + U \in \mathcal{T} \implies A + U \in \mathcal{T}$$

□