Classifying Extensions of Local Fields

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Abstract

Given an extension L/K of local fields, an extension of $L^\times$ by the Galois group $\mathrm{Gal}(L/K)$ can be concretely described up to equivalence using cohomological techniques. We extend this further using data from the fields in such a way that explicit computations can be made in specific cases.

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1 Introduction

In number theory, particularly local class field theory, the Kronecker-Weber theorem gives a characterization of abelian extensions of the local field $\mathbb{Q}_p$.

THEOREM 1. (KRONECKER-WEBER). Let K be a finite abelian extension of $\mathbb{Q}_p$. Then K is contained in some cyclotomic extension of $\mathbb{Q}_p$.

Moreover, given an abelian extension L/K of local fields with Galois group $G = \mathrm{Gal}(L/K)$, we may construct a short exact sequence of the form

$$1 \longrightarrow L^\times \longrightarrow \mathcal{E} \xrightarrow{\pi} G \longrightarrow 1 \ ,$$

called a *group extension* of $L^\times$ by G. We will demonstrate a connection between these extensions, the cohomology group $H^2(L/K)$, and specific computable elements in $L^\times$. We will also view some specific examples for cyclotomic extensions of $\mathbb{Q}_p$ such that we may use the Kronecker-Weber theorem to gather useful information about abelian extensions in general.

2 Preliminaries

We first present some expository material from other sources that will be helpful in our study.

2.1 Local Fields

Let $|\cdot|$ be a discrete valuation on a field K.

DEFINITION 1. A field K is a (non-Archimedean) *local field* if it is complete with respect to a valuation $|\cdot|$ and its residue field is finite.

EXAMPLE 1. Let $|\cdot|_p$ denote the p-adic absolute value on $\mathbb{Q}$, defined on r/s by

$$\left|\frac{r}{s}\right|_p = p^{-(x-y)}$$

where x and y are the largest powers of p dividing r and s, respectively. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ generates the p-adic numbers $\mathbb{Q}_p$, a local field.

The topology of a local field K has the basis formed by balls

$$b + B_n = \{a \in K \mid |a| \leq n\}$$

for $b \in K$ and positive real n.

DEFINITION 2. Under the given topology, for a local field K, we may define the following:

- the set $\mathcal{O}_K = \{a \in K \mid |a| \leq 1\}$ is the *ring of integers* of K,
- the set $\mathcal{O}_K^\times = \{a \in K \mid |a| = 1\}$ is the units of $\mathcal{O}_K$,
- there exists a unique prime ideal $\mathfrak{m} = \{a \in K \mid |a| < 1\}$ of $\mathcal{O}_K$,
- a generator π of $\mathfrak{m}$ (called a *uniformizer* of K) and
- the residue field k of K is given by $\mathcal{O}/\mathfrak{m}$.

Using the properties of the non-Archimedean absolute value, it is easy to see that $\mathcal{O}_K$, $\mathcal{O}_K^\times$, and $\mathfrak{m}$ as defined do in fact have the correct algebraic structure.

EXAMPLE 2. For $K = \mathbb{Q}_p$, we have the ring of integers $\mathcal{O}_K = \mathbb{Z}_p$ and uniformizer $\pi = p$.

Using these definitions, there exists a description of the multiplicative group $K^\times$ of a local field K.

PROPOSITION 1. ([NeuANT], Proposition II.5.3). Let K be a local field with uniformizer π_K, prime ideal $\mathfrak{m}$, and residue field $k = \mathbb{F}_q$. Then

$$K^\times \cong \langle \pi_K \rangle \times \mu_{q-1} \times (1 + \mathfrak{m}),$$

where μ_{q-1} is the group of $q-1$-th roots of unity.

In fact, there exists a full classification of local fields.

THEOREM 2. ([MilANT], Remark 7.49). Every non-Archimedean local field K is isomorphic to one of the following:

(a) an extension of $\mathbb{Q}_p$ for some prime p,

(b) an extension of $\mathbb{F}_q((t))$ for some finite field $\mathbb{F}_q$,

For the purposes of this paper, we will be primarily interested in extensions of $\mathbb{Q}_p$, which are non-Archimedean and have characteristic zero.

Finally, given an extension L/K, we are able to describe the extension in terms of its ramification index. Given an extension of (arbitrary) fields L/K with degree n, we are able to write

$$n = \sum_{i=0}^{r} e_i f_i$$

for each prime ideal of $\mathcal{O}_L$, where e_i is the ramification index and f is the degree of the extension ℓ/k of residue fields. When L/K is an extension of local fields, there is precisely one prime ideal, so we obtain that $n = ef$.

DEFINITION 3. Let L/K be an extension of local fields with degree $n = ef$.

- if $e = 1$ and $f = n$, then L is said to be *unramified* over K,
- if $e = n$ and $f = 1$, then L is said to be *totally ramified* over K.

EXAMPLE 3. Let p be prime, and n a positive integer.

- if $p \nmid n$, then the cyclotomic extension $\mathbb{Q}_p(\zeta_n)$ is unramified,
- if $n = p^k$ for some k, then the extension $\mathbb{Q}_p(\zeta_n)$ is totally ramified.

2.2 Group Cohomology

We begin with two definitions.

DEFINITION 4. Let G be a group and $\mathbb{Z}[G]$ the integral group ring of G. A *G-module* is a $\mathbb{Z}[G]$-module. The category of G-modules is denoted G-mod.

DEFINITION 5. Let A be a G-module. Then A^G is the largest submodule of A fixed by G.

Throughout this paper, we will be primarily interested in working in the G-module $L^\times$, where G is the Galois group of a field extension L/K.

If we fix a group G and a G-module A, then we can define cohomology groups as follows:

Definition 6. Let G be a group. For $q \geq 0$, there exist functors $H^q(G, -) : G\text{-Mod} \to \text{AbGrp}$ with the following properties:

(a) $H^0(G, -) = (-)^G$.

(b) there exist connecting homomorphisms $\delta : H^q(G, C) \to H^{q+1}(G, A)$ such that a short exact sequence $0 \to A \to B \to C \to 0$ induces a long exact sequence

$$\cdots \to H^q(G, A) \to H^q(G, B) \to H^q(G, C) \to H^{q+1}(G, A) \to \cdots ,$$

(c) $H^q(G, A) = 0$ for all $q \geq 1$ if A is co-induced.

For a G-module A, the group $H^q(G, A)$ is called the q-th cohomology group of A.

The definition above uniquely determines the cohomology groups [AtWaGC Theorem 1].

Our primary interest for this paper will be the group $H^2(G, A)$. For notational purposes, when given a Galois extension L/K with Galois group G, we will denote

$$H^2(L/K) = H^2(G, L^\times).$$

We may also explicitly construct the cohomology groups as follows: fix a group G, and consider the groups $C^q(G, A) : \text{Hom}_G(\mathbb{Z}[G^{q+1}], A)$. Define $d^{q+1} : C^q(G, A) \to C^{q+1}(G, A)$ as

$$(d^{q+1}\varphi)(g_1, ..., g_{q+1}) = g_1\varphi(g_2, ..., g_{q+1}) + \sum_{i=0}^{q}(-1)^j(g_0, ..., g_{j-1}, g_jg_{j+1}, ..., g_{q+1}) + (-1)^{q+1}\varphi(g_1, ..., g_q).$$

The G-module homomorphisms $\phi \in C^q(G, A)$ are called q-cochains. Moreover, we can define

$$Z^q(G, A) = \ker d^{q+1}$$
$$B^q(G, A) = \operatorname{im} d^q$$

called q-cocycles and q-coboundaries, respectively, so that

$$H^q(G, A) = Z^q(G, A)/B^q(G, A)$$

for all $q \geq 0$.

Because we are primarily interested in $H^2(G, A)$, it will be helpful to write down the 2-cocycle condition for a 2-cochain.

Corollary 1. Let φ be a 2-cochain. If, for all $g_1, g_2, g_3 \in G$, we have

$$g_1\varphi(g_2, g_3) \cdot \varphi(g_1g_2, g_3)^{-1} \cdot \varphi(g_1, g_2g_3) \cdot \varphi(g_1, g_2)^{-1} = 1,$$

then φ is a 2-cocycle in $Z^2(G, A)$.

Proof. Let $q = 2$ in the above construction and set $d^3\varphi = 0$. □

2.3 Local Class Field Theory

We briefly present two isomorphisms from local class field theory:

THEOREM 3. Let L/K be an abelian extension of local fields of degree n and Galois group $G = \mathrm{Gal}(L/K)$. There exists an isomorphism

$$\theta^{-1} : K^\times / \mathrm{Nm}(L^\times) \to G,$$

called the *local Artin reciprocity map*.

THEOREM 4. Let L/K be a Galois extension of local fields of degree n. There exists an isomorphism

$$\mathrm{inv} : H^2(L/K) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

called the *invariant map*.

The class $u_{L/K} \in H^2(L/K)$ such that $\mathrm{inv}\, u_{L/K} = \frac{1}{n}$ is called the *local fundamental class*.

2.4 Extensions

We begin with a definition.

DEFINITION 7. Let G be a group, and let A be a G-module. A *group extension* is a short exact sequence

$$1 \longrightarrow A \longrightarrow \mathcal{E} \xrightarrow{\pi} G \longrightarrow 1$$

such that for all $a \in A$ and $w \in \mathcal{E}$,

$$waw^{-1} = \pi(w) \cdot a.$$

Throughout this paper, we will be interested in group extensions of the multiplicative group $L^\times$ of a field extension L/K by its Galois group $\mathrm{Gal}(L/K)$.

Two extensions are *equivalent* if they are isomorphic as short exact sequences. Moreover, we are able to identify equivalence classes of extensions using cohomology.

THEOREM 5. ([ConCE]). Let G be a group and A a G-module. Equivalence classes of extensions of A by G are in bijective correspondence with equivalence classes in $H^2(G, A)$.

For an extension $1 \to A \to \mathcal{E} \to G \to 1$, the structure of $\mathcal{E}$ can be described more explicitly. By choosing a set-theoretic section $s : G \to \mathcal{E}$ that sends an element of G to a coset representative in $\mathcal{E}$, we may identify $\mathcal{E}$ as a set

$$\mathcal{E} = A \times G.$$

Because our conjugation condition in the definition of an extension has been specified in advance, we obtain that for $g, h \in G$, $s(g)s(h) \in \pi^{-1}(gh)$, and so there exists some unique $c_{g,h} \in A$ such that

$$s(g)s(h) = c_{g,h}s(gh).$$

The set of $c_{g,h}$ across all possible g, h produces a function $c : G \times G \to A$.

Proposition 2. ([ConGE]). The function $c : G \times G \to A$ defined above is a 2-cocycle.

Given this cocycle, we are able to describe the group structure of $\mathcal{E}$.

Proposition 3. ([ConGE]) Let $1 \to A \to \mathcal{E} \to G \to 1$ be a group extension with an associated cocycle c. Then the composition law on $\mathcal{E} = A \times G$ can be written as

$$(x, g) * (x', g') = (x + gx' + c(g, g'), gg').$$

3 Development of Theory

Given an extension of local fields L/K, we are interested in classifying all group extensions of $L^\times$ by $\mathrm{Gal}(L/K)$ up to equivalence. Let $G = \mathrm{Gal}(L/K)$, and consider the extension

$$1 \longrightarrow L^\times \longrightarrow \mathcal{E} \xrightarrow{\pi} G \longrightarrow 1 \ .$$

We begin by describing lifts of elements of G.

Lemma 1. Suppose that $\sigma \in G$ has order n, and let f be a lift of σ in $\mathcal{E}$. Then $f^n \in L^\times$.

Proof. We have that

$$\begin{aligned} \pi(f^n) &= \pi(f)^n \\ &= \sigma^n \\ &= 1. \end{aligned}$$

Thus, $f^n \in \ker \pi = L^\times$. □

In fact, given our setup, we have the following stronger condition.

Lemma 2. The element f^n lies in the fixed field $L^{\langle \sigma \rangle}$.

Proof. We have from earlier that $f^n \in L^\times$. Thus,

$$\begin{aligned} \sigma(f^n) &= f \cdot f^n \cdot f^{-1} \\ &= f^n, \end{aligned}$$

and so $f^n \in L^{\langle \sigma \rangle}$. □

This will be extremely useful later, so it will help to introduce some notation.

DEFINITION 8. Let $f \in \mathcal{E}$ be a lift of $\sigma \in G$. Define $\alpha = f^n$.

Where appropriate, we will subscript α to represent lifts of different generators of G (i.e. α_i corresponds to the lift f_i of σ_i).

3.1 Cyclic Extensions

When L/K is a cyclic extension generated by σ, it is the case that $L^\times = L^{\langle\sigma\rangle}$. We may identify α with a 2-cocycle in $H^2(L/K)$ as follows:

PROPOSITION 4. Let $G = \mathrm{Gal}(L/K)$ be generated by σ and have order n. The 2-cochain defined by

$$c(\sigma^i, \sigma^j) = \begin{cases} 1 & i + j < n \\ \alpha & i + j \geq n \end{cases}$$

is a 2-cocycle.

PROOF. Let $g_1 = \sigma^i$, $g_2 = \sigma^j$, and $g_3 = \sigma^k$. We show that

$$g_1 c(g_2, g_3) \cdot c(g_1 g_2, g_3)^{-1} \cdot c(g_1, g_2 g_3) \cdot c(g_1, g_2)^{-1} = 1$$

for all $g_1, g_2, g_3 \in G$ as follows: observe that $c(g_2, g_3)$ is fixed by σ, so we may disregard the action of g_1. Furthermore, $c(g_1, g_2) = \alpha$ if $i + j \geq n$ and $c(g_2, g_3) = \alpha$ if $j + k \geq n$. We analyze each case.

(a) $i + j < n$ and $j + k < n$. Then

$$c(g_1 g_2, g_3) = c(g_1, g_2 g_3) = \begin{cases} 1 & i + j + k < n \\ \alpha & i + j + k \geq n \end{cases}$$

such that the equation cancels as desired.

(b) $i + j < n$ and $j + k \geq n$. Then $c(g_2, g_3) = \alpha$ and, factoring in the carry for $g_2 g_3$,

$$c(g_1 g_2, g_3) = \begin{cases} 1 & i + j + k < n \\ \alpha & i + j + k \geq n \end{cases}$$

$$c(g_1, g_2 g_3) = \begin{cases} 1 & i + j + k < 2n \\ \alpha & i + j + k \geq 2n \end{cases}$$

Because $i + j + k \geq j + k \geq n$ and $i + j + k \leq i + 2j + k < 2n$, we have that $c(g_1 g_2, g_3) = \alpha$ and $c(g_1, g_2 g_3) = 1$, such that the equation cancels as desired.

(c) $i + j \geq n$ and $j + k < n$. Then $c(g_1, g_2) = \alpha$ and, factoring in the carry for $g_1 g_2$,

$$c(g_1 g_2, g_3) = \begin{cases} 1 & i + j + k < 2n \\ \alpha & i + j + k \geq 2n \end{cases}$$

$$c(g_1, g_2g_3) = \begin{cases} 1 & i+j+k < n \\ \alpha & i+j+k \geq n \end{cases}$$

Because $i+j+k \geq i+j \geq n$ and $i+j+k \leq i+2j+k < 2n$, we have that $c(g_1g_2, g_3) = 1$ and $c(g_1, g_2g_3) = \alpha$, such that the equation cancels as desired.

(d) $i+j \geq n$ and $j+k \geq n$. Then, factoring in both carries,

$$c(g_1g_2, g_3) = c(g_1, g_2g_3) = \begin{cases} 1 & i+j+k < 2n \\ \alpha & i+j+k \geq 2n \end{cases}$$

such that the equation cancels as desired.

Thus, we have a 2-cocycle. □

Given an extension characterized by α, we are interested in classifying it up to equivalence. This is dependent on our choice of lift. Given a cocycle c, If we take an arbitrary lift $f = (x, \sigma) \in \mathcal{E} \cong L^\times \times G$ of σ, we obtain that

$$\alpha = \mathrm{Nm}(x) \cdot \prod_{i=0}^{n-1} c(\sigma^i, \sigma),$$

where $\mathrm{Nm} : L^\times \to K$ is the norm map.

Noting this setup, we define an equivalence relation for α as follows:

Definition 9. Elements α and α' in $L^\times$ are equivalent if there exists $x \in L^\times$ such that

$$\alpha = \mathrm{Nm}(x) \cdot \alpha'.$$

Because $\mathrm{Nm}(xy) = \mathrm{Nm}(x) \cdot \mathrm{Nm}(y)$, the equivalence relation is in fact well defined.

In fact, using our formula for the cocycle c corresponding to α, we are able to prescribe a structure on our equivalence classes of $[\alpha]$.

Theorem 6. Let the equivalence classes $[\alpha]$ and $[\alpha']$ correspond to $[c], [c'] \in H^2(L/K)$, respectively. Then the correspondence of $[\alpha\alpha']$ to the class of cocycles equivalent to

$$(cc')(\sigma^i, \sigma^j) = \begin{cases} 1 & i+j < n \\ \alpha\alpha' & i+j \geq n \end{cases}$$

endows a group structure on the classes of α.

Proof. Because α lies in the multiplicative group $L^\times$, we only need to note that

$$(cc')(\sigma^i, \sigma^j) = 1$$

is the identity element of $H^2(L/K)$. □

3.2 Bicyclic Extensions

We now turn to a more general case of when L/K is a bicyclic extension. In this case, we have that $G = G_1 \times G_2$, where G_1 and G_2 are cyclic. Let σ_1 and σ_2 be respective generators, with respective orders n_1 and n_2. In the cyclic case, the quantity α was sufficient to encode the necessary information about the extension $\mathcal{E}$. However, we now need to track how two lifts f_1, f_2 of σ_1, σ_2 commute with each other. To do this, we will introduce a new quantity.

Definition 10. For $i = 1, 2$, let $f_i \in \mathcal{E}$ be a lift of $\sigma_i \in G$, respectively. Define $\beta = f_1 f_2 f_1^{-1} f_2^{-1}$ to be the commutator of f_1 and f_2 in $\mathcal{E}$.

Furthermore, we will introduce some norm maps on $L^\times$ as follows:

Definition 11. For a given index i, denote the map $N_i : L^\times \to L^{\langle \sigma_i \rangle}$ as

$$N_i(x) = \prod_{\ell=0}^{n_i-1} \sigma_i^\ell(x)$$

We immediately note some properties of β.

Proposition 5. The element β satisfies the following properties:

- $\beta \in L^\times$,
- $N_1(\beta) = \alpha_1 / \sigma_2(\alpha_1)$,
- $N_2(\beta^{-1}) = \alpha_2 / \sigma_1(\alpha_2)$.

Proof. Let β be as defined. We have that

$$\begin{aligned}
\pi(\beta) &= \pi(f_1 f_2 f_1^{-1} f_2^{-1}) \\
&= \pi(f_1) \cdot \pi(f_2) \cdot \pi(f_1^{-1}) \cdot \pi(f_2^{-1}) \\
&= \sigma_1 \cdot \sigma_2 \cdot \sigma_1^{-1} \cdot \sigma_2^{-1} \\
&= 1,
\end{aligned}$$

and so $\beta \in \ker \pi = L^\times$.

To prove the second claim, we proceed by induction. If $n_1 = 2$, then

$$\begin{aligned}
N_1(\beta) &= \beta \cdot \sigma_1(\beta) \\
&= f_1 f_2 f_1^{-1} f_2^{-1} \cdot (f_1 \cdot f_1 f_2 f_1^{-1} f_2^{-1} \cdot f_1^{-1}) \\
&= f_1 f_2 f_1^{-1} f_2^{-1} f_1^2 f_2 f_1^{-1} f_2^{-1} f_1^{-1} \\
&= f_1 f_2 f_1^{-1} \sigma_2^{-1}(\alpha_1) f_1^{-1} f_2^{-1} f_1^{-1} \\
&= f_1 f_2 \; \sigma_1^{-1} \sigma_2^{-1}(\alpha_1) f_1^{-2} f_2^{-1} f_1^{-1} \\
&= f_1 f_2 \; \sigma_1^{-1} \sigma_2^{-1}(\alpha_1) \; \alpha_1^{-1} f_2^{-1} f_1^{-1} \\
&= \sigma_1 \sigma_2 (\sigma_1^{-1} \sigma_2^{-1}(\alpha_1) \; \alpha_1^{-1}) \\
&= \alpha_1 / \sigma_1 \sigma_2(\alpha_1) \\
&= \alpha_1 / \sigma_2(\alpha_1),
\end{aligned}$$

where we use the fact that α_1 is fixed by σ_1. Suppose the relation holds for $n = k$. Using some liberties with notation, we will use f_1^k and $f_2 f_1^k f_2^{-1}$ to indicate the result for $n = k$. Then for $n = k + 1$,

$$\begin{aligned}
N_1(\beta) &= \left(\prod_{i=0}^{k-1} \sigma_1^i(\beta) \right) \cdot \sigma^k(\beta) \\
&= f_1^k f_2 f_1^{-k} f_2^{-1} \cdot (f_1^k \cdot f_1 f_2 f_1^{-1} f_2^{-1} \cdot f_1^{-k}) \\
&= f_1^k f_2 f_1^{-k} f_2^{-1} \alpha_1 f_2 f_1^{-1} f_2^{-1} f_1^{-k} \\
&= f_1^k f_2 f_1^{-k} \; \sigma_2^{-1}(\alpha_1) \; f_1^{-1} f_2^{-1} f_1^{-k} \\
&= f_1^k f_2 f_1^{-k-1} \; \sigma_2^{-1}(\alpha_1) \; f_2^{-1} f_1^{-k} \\
&= f_1^k f_2 \; \alpha_1^{-1} \; \sigma_2^{-1}(\alpha_1) \; f_2^{-1} f_1^{-k} \\
&= f_1^k \; \sigma_2(\alpha_1^{-1}) \; \alpha_1 \; f_1^{-k} \\
&= \alpha_1 / \sigma_2(\alpha_1)
\end{aligned}$$

Thus, the second claim holds. To obtain the last claim, we simply note that $\beta^{-1} = f_2 f_1 f_2^{-1} f_1^{-1}$, such that the previous argument holds by symmetry. ☐

Remark 1. In the proof above, the element f_1^k (and indeed, all lower orders) may not necessarily be an element of $L^\times$ on which σ_2 may act. However, we may ignore this specification by working purely with f_1 and f_2, defining the "action" of σ_2 to work by its defined conjugation.

Corollary 2. Let β be as given in Definition 11. Then $\mathrm{Nm}(\beta) = 1$.

Proof. We observe that

$$\begin{aligned}
\mathrm{Nm}(\beta) &= N_2 N_1(\beta) \\
&= N_2(\alpha_1 \cdot \sigma_2(\alpha_1^{-1})) \\
&= N_2(\alpha_1) \cdot N_2(\sigma_2(\alpha_1^{-1})) \\
&= N_2(\alpha_1) \cdot N_2(\alpha_1^{-1}) \\
&= 1.
\end{aligned}$$

Thus, the result is proven. ☐

Because the information given by the α's and β is useful, we will henceforth, when given a bicyclic extension, refer to the element $(\alpha_1, \alpha_2, \beta)$ as a "triple".

We continue to be able to associate cocycles in $H^2(L/K)$ to extensions via triples. Because we have introduced the element β, our cocycle formula becomes slightly more complex.

Proposition 6. Let $G = \mathrm{Gal}(L/K)$ be generated by σ_1, σ_2 with respective orders n_1, n_2. The 2-cochain defined by

$$c(\sigma_1^{r_1}\sigma_2^{r_2}, \sigma_1^{s_1}\sigma_2^{s_2}) = \left(\sigma_1^{r_1}\prod_{k=0}^{s_1-1}\prod_{\ell=0}^{r_2-1}\sigma_1^k\sigma_2^\ell(\beta^{-1})\right)\left(\alpha_1^{\chi_1}\cdot\sigma_1^{r_1+s_1}(\alpha_2^{\chi_2})\right)$$

where χ_i is an indicator function given by

$$\chi_i = \begin{cases} 0 & r_i + s_i < n \\ 1 & r_i + s_i \geq n \end{cases}$$

is a 2-cocycle.

Proof. Let $g_1 = \sigma_1^{x_1}\sigma_2^{x_2}$, $g_2 = \sigma_1^{y_1}\sigma_2^{y_2}$, and $g_3 = \sigma_1^{z_1}\sigma_2^{z_2}$. We will first analyze the α component and then the β component of the formula

$$g_1c(g_2, g_3)\cdot c(g_1, g_2g_3)\cdot c(g_1g_2, g_3)^{-1}\cdot c(g_1, g_2)^{-1},$$

which verifies that we have a 2-cocycle. We observe that the α component is given by the proof in the cyclic case. For the β component, we compute

$$\begin{aligned}
&\sigma_1^{x_1}\sigma_2^{x_2}\left(\sigma_1^{y_1}\prod_{k=0}^{z_1-1}\prod_{\ell=0}^{y_2-1}\sigma_1^k\sigma_2^\ell(\beta^{-1})\right)\left(\sigma_1^{x_1}\prod_{k=0}^{y_1+z_1-1}\prod_{\ell=0}^{x_2-1}\sigma_1^k\sigma_2^\ell(\beta^{-1})\right)\\
&\qquad\left(\sigma_1^{x_1+y_1}\prod_{k=0}^{z_1-1}\prod_{\ell=0}^{x_2+y_2-1}\sigma_1^k\sigma_2^\ell(\beta^{-1})\right)^{-1}\left(\sigma_1^{x_1}\prod_{k=0}^{y_1-1}\prod_{\ell=0}^{x_2-1}\sigma_1^k\sigma_2^\ell(\beta^{-1})\right)^{-1}\\
&= \sigma_1^{x_1}\left[\left(\prod_{k=y_1}^{y_1+z_1-1}\prod_{\ell=x_2}^{x_2+y_2-1}\sigma_1^k\sigma_2^\ell(\beta^{-1})\cdot\prod_{k=0}^{y_1+z_1-1}\prod_{\ell=0}^{x_2-1}\sigma_1^k\sigma_2^\ell(\beta^{-1})\right)\right.\\
&\qquad\left.\left(\prod_{k=y_1}^{y_1+z_1-1}\prod_{\ell=0}^{x_2+y_2-1}\sigma_1^k\sigma_2^\ell(\beta^{-1})\cdot\prod_{k=0}^{y_1-1}\prod_{\ell=0}^{x_2-1}\sigma_1^k\sigma_2^\ell(\beta^{-1})\right)^{-1}\right]\\
&= \sigma_1^{x_1}\left[\left(\prod_{k=y_1}^{y_1+z_1-1}\prod_{\ell=0}^{x_2-1}\sigma_1^k\sigma_2^\ell(\beta^{-1})\right)^{-1}\cdot\left(\prod_{k=y_1}^{y_1+z_1-1}\prod_{\ell=0}^{x_2-1}\sigma_1^k\sigma_2^\ell(\beta^{-1})\right)\right]\\
&= \sigma_1^{x_1}(1)\\
&= 1.
\end{aligned}$$

Thus, we have a 2-cocycle.

Remark 2. If, for either $i = 1, 2$, $r_i = s_i = 0$, the formula reduces to exactly that of the corresponding cyclic subextensions.

We are interested in classifying triples up to equivalence. Fix a cocycle c, and suppose we have lifts $f_1 = (x_1, \sigma_1)$ and $f_2 = (x_2, \sigma_2)$. From our remark above, the formula for α_i is the same as discussed in the cyclic case. Moreover, our β may be written as

$$\beta = \frac{x_1}{x_2} \cdot \frac{\sigma_1(x_2)}{\sigma_2(x_1)} \cdot \frac{c(\sigma_1, \sigma_2)}{c(\sigma_2, \sigma_1)}$$

Thus, we have equivalence as follows.

Definition 12. Triples $(\alpha_1, \alpha_2, \beta)$ and $(\alpha_1', \alpha_2', \beta')$ are equivalent if there exist $x_1, x_2 \in L^\times$ such that

$$\begin{aligned} \alpha_i &= N_i(x_i) \cdot \alpha_i', \\ \beta &= \frac{x_1}{x_2} \cdot \frac{\sigma_1(x_2)}{\sigma_2(x_1)} \cdot \beta'. \end{aligned}$$

From here, we note the relationship between equivalence classes of triples and $H^2(L/K)$, as previously done in the cyclic case.

Theorem 7. Let the equivalence classes of triples $[(\alpha_1, \alpha_2, \beta)]$ and $[(\alpha_1', \alpha_2', \beta')]$ correspond to $[c], [c'] \in H^2(L/K)$, respectively. Then the correspondence of $[(\alpha_1\alpha_1', \alpha_2\alpha_2', \beta\beta')]$ to the class of cocycles equivalent to cc' endows a group structure on the classes of triples.

Proof. It is clear from the multiplicative structure of the defining relations that $(\alpha_1\alpha_1', \alpha_2\alpha_2', \beta\beta')$ is also a triple. We observe that taking $(cc')(g,h) = c(g,h) \cdot c'(g,h)$ produces a group structure. $\square$

For a given bicyclic extension, it is sometimes helpful for computational purposes to understand each cyclic subextension, which are each parametrized by a singular α term.

Theorem 8. Let L/K be a bicyclic extension with Galois group $G = G_1 \times G_2$, and let K_i/K be the subextension such that $G_i = \mathrm{Gal}(L/K_i)$ for $i = 1, 2$. Suppose that G_1 and G_2 have orders n_1 and n_2, respectively. Then a triple $(\alpha_1, \alpha_2, \beta)$ has order $n_1 n_2$ if α_1 has order n_1 in K/K_1 and α_2 has order n_2 in K/K_2.

Proof. The projection map $\pi : (\alpha_1, \alpha_2, \beta) \mapsto \alpha_1$ corresponds to a restriction map

$$H^2(L/K) \xrightarrow{\mathrm{Res}} H^2(L/K_1)$$

in cohomology via the diagram

$$\begin{tikzcd} \{(\alpha_1, \alpha_2, \beta)\} \arrow[r, "\pi"] \arrow[d, leftrightarrow] & \{\alpha_1\} \arrow[d, leftrightarrow] \\ H^2(L/K) \arrow[r, "\mathrm{Res}"] & H^2(L/K_1) \end{tikzcd}$$

Thus, if a cocycle c corresponding to $(\alpha_1, \alpha_2, \beta)$ has order $n_1 n_2$ in $H^2(L/K)$, then $\mathrm{Res}\, c$ will have order n_1 in $H^2(L/K_1)$ and correspond to α_1. By symmetry, this is also the case for α_2. $\square$

3.3 Some Remarks on Finite Abelian Extensions

Much of the commentary on cyclic and bicyclic extensions extends to a finite abelian extension L/K. Suppose that $G = \mathrm{Gal}(L/K) = G_1 \times \cdots \times G_r$ is a decomposition into cyclic groups generated by $\sigma_1, ..., \sigma_r$. In the bicyclic case, we represented the commutator between lifts of σ_1, σ_2 using β; to extend this, we will introduce a β term for each pair of indices analogously: for respective lifts f_i, f_j of σ_i, σ_j, define

$$\beta_{ij} = f_i f_j f_i^{-1} f_j^{-1}.$$

We then have the following relation.

Proposition 7. Suppose $i < j < k$. Then

$$\frac{\beta_{ik}}{\sigma_j(\beta_{ik})} = \frac{\beta_{ij}}{\sigma_k(\beta_{ij})} \cdot \frac{\beta_{jk}}{\sigma_i(\beta_{jk})}$$

Proof. We will prove the equivalent statement that

$$\sigma_i(\beta_{jk})\,\beta_{ik}\,\sigma_k(\beta_{ij}) = \beta_{ij}\,\sigma_j(\beta_{ik})\,\beta_{jk}.$$

By computation,

$$\begin{aligned}
\sigma_i(\beta_{jk})\,\beta_{ik}\,\sigma_k(\beta_{jk}) &= (f_i f_j f_k f_j^{-1} f_k^{-1} f_i^{-1})(f_i f_k f_i^{-1} f_k^{-1})(f_k f_i f_j f_i^{-1} f_j^{-1} f_k^{-1}) \\
&= f_i f_j f_k f_i^{-1} f_j^{-1} f_k^{-1} \\
&= f_i f_j (f_i^{-1} f_j^{-1} f_j f_i) f_k f_i^{-1} (f_k^{-1} f_j^{-1} f_j f_k) f_j^{-1} f_k^{-1} \\
&= \beta_{ij}\,\sigma_j(\beta_{ik})\,\beta_{jk}.
\end{aligned}$$

Thus, the relation holds. □

By taking a set-theoretic lift as described in [ConGE], we may explicitly construct a cocycle in terms of (α_i, β_{ij}) and the generators σ_i of G. However, because it is not relevant to the remainder of the paper, we have elected not to do so.

4 Extensions of $\mathbb{Q}_p$

Let p be an odd prime. To demonstrate our theory, consider a cyclotomic extension $\mathbb{Q}_p(\zeta_n)$, and write $n = p^k m$, where $p \nmid m$. We obtain a diagram of fields

$$\begin{tikzcd}
& \mathbb{Q}_p(\zeta_n) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}_p(\zeta_{p^k}) \arrow[dr, dash] & & \mathbb{Q}_p(\zeta_m) \arrow[dl, dash] \\
& \mathbb{Q}_p &
\end{tikzcd}$$

where $\mathbb{Q}_p(\zeta_{p^k}) \cap \mathbb{Q}_p(\zeta_m) = \mathbb{Q}_p$. We have that both of these subfields are cyclic extensions, so we may write

$$\mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p(\zeta_{p^k})) \times \mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p(\zeta_m))$$

such that our bicyclic theory may be applied.

We now present an explicit example. Let $K = \mathbb{Q}_p(\zeta_m)$ be an unramified extension of $\mathbb{Q}_p$. The Galois group $\mathrm{Gal}(K/\mathbb{Q}_p)$ is generated by a Frobenius element σ. Moreover, it is known ([MilCFT, Section III.1) that the local fundamental class $u_{K/\mathbb{Q}_p} \in H^2(K/\mathbb{Q}_p)$ is given by

$$u_{K/\mathbb{Q}_p}(\sigma^i, \sigma^j) = \begin{cases} 1 & i + j < n \\ \pi_K & i + j \geq n \end{cases}$$

Thus, in the case, if we identify α with π_K, our classification is as follows:

Theorem 9. Let $K/\mathbb{Q}_p$ be an unramified extension of $\mathbb{Q}_p$ of degreen n with uniformizer π_K. Then, up to equivalence, all extensions of $K^\times$ by $\mathrm{Gal}(K/\mathbb{Q}_p)$ are classified by powers of π_K modulo n.

Proof. Identifying α with π_K, we have that the cocycle that corresponds to α is the local fundamental class. Because $\mathrm{inv}\, u_{K/\mathbb{Q}_p} = \frac{1}{n}$, we have from the structure on the set of α that the cocycle corresponding to π_K^m maps to $\frac{m}{n}$ under the invariant map for all $m < n$. Because the invariant map as defined in Theorem 4 is an isomorphism, we obtain all classes in $H^2(L/K)$, and thus all group extensions by Theorem 5. $\square$

5 Extensions of $\mathbb{Q}_2$

We now consider the special case of $p = 2$. We observed that when p is an odd prime, a cyclotomic extension of $\mathbb{Q}_p$ is bicyclic. However, in the case of $\mathbb{Q}_2$, because the Galois group

$$\mathrm{Gal}(\mathbb{Q}_2(\zeta_{2^k})) \cong \mathbb{Z}/2^{k-1}\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{k-2}\mathbb{Z}$$

is bicyclic, our diagram of fields splits as

$$\begin{tikzcd}
 & & & \mathbb{Q}_2(\zeta_n) \arrow[dll, dash] \arrow[dd, dash] \\
 & \mathbb{Q}_2(\zeta_{2^k}) \arrow[dl, dash] \arrow[d, dash] & & \\
\mathbb{Q}_2(i) \arrow[dr, dash] & \mathbb{Q}_2(\zeta_{2^k} + \zeta_{2^k}^{-1}) \arrow[d, dash] & & \mathbb{Q}_2(\zeta_m) \arrow[dll, dash] \\
 & \mathbb{Q}_2 & &
\end{tikzcd}$$

where $\mathbb{Q}_2(i)$ and $\mathbb{Q}_2(\zeta_{2^k} + \zeta_{2^k}^{-1})$ are the corresponding fixed fields.

As with $\mathbb{Q}_p$, we now present an explicit example. Consider the specific case of $K = \mathbb{Q}_2(\zeta_8) = \mathbb{Q}_2(i, \sqrt{2})$, with the following diagram of fields:

$$\begin{tikzcd}
 & \mathbb{Q}_2(\zeta_8) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}_2(i) \arrow[dr, dash] & & \mathbb{Q}_2(\sqrt{2}) \arrow[dl, dash] \\
 & \mathbb{Q}_2 &
\end{tikzcd}$$

This bicyclic extension is totally ramified and of degree 4. We seek to classify its equivalence classes of triples as described in Section 3. To do this, we will characterize a triple $(\alpha_1, \alpha_2, \beta)$ with order 4 so that all extensions are generated by $(\alpha_1, \alpha_2, \beta)$.

THEOREM 10. Let $K/\mathbb{Q}_2$ be the bicyclic extension described above. Then the triple $(\alpha_1, \alpha_2, \beta)$ such that $\alpha_1 = i \cdot N_1(x_1)$ and $\alpha_2 = \sqrt{2} \cdot N_2(x_2)$, where $x_1, x_2 \in K$, is of order 4.

PROOF. In this case, the Galois group $\mathrm{Gal}(K/\mathbb{Q}_2)$ is generated by the automorphisms

$$\begin{aligned} \sigma_1 &: \sqrt{2} \mapsto -\sqrt{2} \\ \sigma_2 &: i \mapsto -i \end{aligned}$$

such that $K_1 = \mathbb{Q}_2(i)$ and $K_2 = \mathbb{Q}_2(\sqrt{2})$. As observed earlier, a triple $(\alpha_1, \alpha_2, \beta)$ that has order 4 cannot have $\alpha_i = N_i(x)$ for an element $x \in K^\times$. Thus, it is useful to explicitly compute the norm groups of the subfields of this extension.

To compute the norm groups, we will make use of the following result:

LEMMA 3. Let L/K be an abelian extension of local fields with degree n. Then the norm group $\mathrm{Nm}(L^\times)$ is of index n in $K^\times$.

PROOF. Denote $G = \mathrm{Gal}(L/K)$. Because L/K is Galois, $|G| = n$. The local Artin map gives an isomorphism

$$K^\times / \mathrm{Nm}(L^\times) \cong G,$$

and so $\mathrm{Nm}(L^\times)$ has index n. □

Because $K^\times$ is infinite and may be particularly complicated, we will first track through the n-th powers of $K^\times$. Since $K^\times$ is fixed by $\mathrm{Gal}(L/K)$, we have that for $x \in K^\times$, $\mathrm{Nm}(x) = x^n$, and so the n-th powers $(K^\times)^n$ form a subgroup of $\mathrm{Nm}(L^\times)$.

Observe that $[\mathbb{Q}_2(\zeta_8) : \mathbb{Q}_2(i)] = [\mathbb{Q}_2(\zeta_8) : \mathbb{Q}_2(\sqrt{2})] = 2$. Because K is an abelian extension, the norm groups $N_1(K^\times)$ and $N_2(K^\times)$ have index 2 over their respective fields. Thus, $N_i(K^\times)$ induces an order 2 subgroup of $K_i^\times/(K_i^\times)^2$. We'll consider each subfield separately.

Consider $K_1 = \mathbb{Q}_2(i)$. Observe that K_1 has uniformizer $\pi_{K_1} = i - 1$, so the unit group of K_1 has the structure

$$\begin{aligned} K_1^\times &\cong \langle \pi_{K_1} \rangle \times (1 + \mathfrak{m}) \\ &= \langle i - 1 \rangle \times (1 + (i-1)\mathbb{Z}_2[i]). \end{aligned}$$

By a basic computation,

$$\begin{aligned} (K_1^\times)^2 &\cong \langle -2i \rangle \times (1 + 2(i-1)\mathbb{Z}_2[i] - 2i\mathbb{Z}_2[i]) \\ &= \langle -2i \rangle \times (1 + 2\mathbb{Z}_2[i]) \\ &= \langle \pi_{K_1}^2 \rangle \times (1 + \mathfrak{m}^2) \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} K_1^\times/(K_1^\times)^2 &= \langle \pi_{K_1} \rangle / \langle \pi_{K_1}^2 \rangle \times (1+\mathfrak{m})/(1+\mathfrak{m}^2) \\ &= \langle i - 1 \rangle \times \langle i \rangle \\ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

The norm group $N_1(K) \subseteq K_1$ has index 2, so it is generated by an order 2 subgroup of $K_1^\times/(K_1^\times)^2$. We observe that these subgroups are precisely those generated by i, $i-1$, and $-1-i$. Furthermore, the uniformizer $\pi_K = \zeta_8 - 1$ must map to a uniformizer of K_1 under the norm map, so because

$$N_1(\pi_K) = (\zeta_8 - 1)(-\zeta_8 - 1) = -(i-1)$$

and $-1 = i^2$ is a norm in K_1, we conclude that $N_1(K)$ is generated by $(K_1^\times)^2$ and $i-1$. Moreover, we determine the quotient group to be

$$K_1^\times/N_1(K) = \langle i \rangle.$$

Thus, up to norm, we have that $\alpha_1 \equiv i$.

Now consider $K_2 = \mathbb{Q}_2(\sqrt{2})$. Repeating the process, observe that K_2 has uniformizer $\pi_{K_2} = \sqrt{2}$, so the unit group of K_2 has the structure

$$\begin{aligned} K_2^\times &\cong \langle \pi_{K_2} \rangle \times (1 + \mathfrak{m}) \\ &= \langle \sqrt{2} \rangle \times (1 + \sqrt{2}\,\mathbb{Z}_2[\sqrt{2}]). \end{aligned}$$

Again by computation,

$$\begin{aligned} (K_2^\times)^2 &\cong \langle 2 \rangle \times (1 + 2\sqrt{2}\,\mathbb{Z}_2[\sqrt{2}] + 2\mathbb{Z}_2[\sqrt{2}]) \\ &= \langle \pi_{K_2}^\times \rangle \times (1 + \mathfrak{m}^2). \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} K_2^\times/(K_2^\times)^2 &= \langle \pi_{K_2} \rangle / \langle \pi_{K_2}^2 \rangle \times (1+\mathfrak{m})/(1+\mathfrak{m}^2) \\ &= \langle \sqrt{2} \rangle \times \langle 1 + \sqrt{2} \rangle \\ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

As with before, the norm group $N_2(K) \subseteq K_2$ has index 2, and the subgroups of $K_2^\times/(K_2^\times)^2$ are generated by $\sqrt{2}$, $1+\sqrt{2}$, and $2+\sqrt{2}$. Given the uniformizer π_K, we have that

$$\begin{aligned} N_2(\pi_K) &= (\zeta_8 - 1)(\zeta_8^{-1} - 1) \\ &= 2 - (\zeta_8 + \zeta_8^{-1}) \\ &= 2 - \sqrt{2} \\ &= \frac{2}{2+\sqrt{2}}. \end{aligned}$$

We note that since $2+\sqrt{2}$ is a generator of one of our subgroups, so is $(2+\sqrt{2})^{-1}$, and because 2 is a norm, we have that the norm group $N_2(K)$ is generated by $(K_2^\times)^2$ and $2+\sqrt{2}$ and that the quotient group is computed to be

$$K_2^\times/N_2(K) = \langle \sqrt{2} \rangle = \langle 1 + \sqrt{2} \rangle$$

Thus, up to norm, $\alpha_2 \equiv \sqrt{2}$. □

6 References

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