A Witty Title

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Contents

1 Introduction

2 Preliminaries

2.1 Local Fields

2.2 Group Cohomology

2.3 Extensions

3 Development of Theory

Given an extension of local fields L/K, we are interested in classifying all Galois gerbes up to equivalence. Let $G = \mathrm{Gal}(L/K)$, and consider the gerbe

$$1 \longrightarrow L^\times \longrightarrow \mathcal{E} \xrightarrow{\pi} G \longrightarrow 1 \ .$$

We begin by describing lifts of elements of G.

LEMMA 1. Suppose that $\sigma \in G$ has order n, and let f be a lift of σ in $\mathcal{E}$. Then $f^n \in L^\times$.

PROOF. We have that

$$\begin{aligned} \pi(f^n) &= \pi(f)^n \\ &= \sigma^n \\ &= 1. \end{aligned}$$

Thus, $f^n \in \ker \pi = L^\times$. □

In fact, given our setup, we have the following stronger condition.

LEMMA 2. The element f^n lies in the fixed field $L^{\langle\sigma\rangle}$.

PROOF. We have from earlier that $f^n \in L^\times$. Thus,

$$\begin{aligned} \sigma(f^n) &= f \cdot f^n \cdot f^{-1} \\ &= f^n, \end{aligned}$$

and so $f^n \in L^{\langle\sigma\rangle}$. □

This will be extremely useful later, so it will help to introduce some notation.

DEFINITION 1. Let $f \in \mathcal{E}$ be a lift of $\sigma \in G$. Define $\alpha = f^n$.

Where appropriate, we will subscript α to represent lifts of different generators of G.

3.1 Cyclic Extensions

When L/K is a cyclic extension generated by σ, it is the case that $L^\times = L^{\langle\sigma\rangle}$. Thus, if we have a lift f, then we are able to specify the group law of $\mathcal{E}$.

Theorem 1. Let L/K be cyclic with $G = \langle\sigma\rangle$. Then the extension

$$1 \longrightarrow L^\times \longrightarrow \mathcal{E} \xrightarrow{\pi} G \longrightarrow 1$$

is characterized as follows:

- $\pi(f) = \sigma$
- $\mathcal{E}$ is generated by f and $L^\times$,
- for all $a \in L^\times$, $faf^{-1} = \sigma(a)$,
- $\alpha = f^n \in L^\times$.

Proof. TBD ☐

Moreover, we may identify α with a 2-cocycle in $H^2(L/K)$ as follows:

Proposition 1. Let $G = \mathrm{Gal}(L/K)$ be generated by σ and have order n. The 2-cochain defined by

$$c(\sigma^i, \sigma^j) = \begin{cases} 1 & i + j < n \\ \alpha & i + j \geq n \end{cases}$$

is a 2-cocycle.

Proof. TBD ☐

Given an extension characterized by α, we are interested in classifying it up to equivalence. This is dependent on our choice of lift. Given a cocycle c, If we take an arbitrary lift $f = (x, \sigma) \in \mathcal{E} \cong L^\times \times G$ of σ, we obtain that

$$\alpha = \mathrm{Nm}(x) \cdot \prod_{i=0}^{n-1} c(\sigma^i, \sigma),$$

where $\mathrm{Nm} : L^\times \to K$ is the norm map.

Noting this setup, we define an equivalence relation for α as follows:

Definition 2. Elements α and α' are equivalent if there exists $x \in L^\times$ such that

$$\alpha = \alpha' \cdot \mathrm{Nm}(x).$$

Because $\mathrm{Nm}(xy) = \mathrm{Nm}(x) \cdot \mathrm{Nm}(y)$, the equivalence relation is in fact well defined.

3.2 Bicyclic Extensions

We now turn to a more general case of when L/K is a bicyclic extension. In this case, we have that $G = G_1 \times G_2$, where G_1 and G_2 are cyclic. Let σ_1 and σ_2 be respective generators, with respective orders n_1 and n_2. In the cyclic case, the quantity α was sufficient to encode the necessary information about the extension $\mathcal{E}$. However, we now need to track how two lifts f_1, f_2 of σ_1, σ_2 commute with each other. To do this, we will introduce a new quantity.

Definition 3. For $i = 1, 2$, let $f_i \in \mathcal{E}$ be a lift of $\sigma_i \in G$. Define $\beta = f_1 f_2 f_1^{-1} f_2^{-1}$ to be the commutator of f_1 and f_2 in $\mathcal{E}$.

Furthermore, we will introduce some norm maps on $L^\times$ as follows:

Definition 4. For a given index i, denote the map $N_i : L^\times \to L^{\langle \sigma_i \rangle}$ as

$$N_i(x) = \prod_{\ell=0}^{n_i-1} \sigma_i^\ell(x)$$

We immediately note some properties of β.

Proposition 2. The element β satisfies the following properties:

- $\beta \in L^\times$,
- $N_1(\beta) = \alpha_1/\sigma_2(\alpha_1)$,
- $N_2(\beta^{-1}) = \alpha_2/\sigma_1(\alpha_2)$.

Proof. Let β be as defined. We have that

$$\begin{aligned}
\pi(\beta) &= \pi(f_1 f_2 f_1^{-1} f_2^{-1}) \\
&= \pi(f_1) \cdot \pi(f_2) \cdot \pi(f_1^{-1}) \cdot \pi(f_2^{-1}) \\
&= \sigma_1 \cdot \sigma_2 \cdot \sigma_1^{-1} \cdot \sigma_2^{-1} \\
&= 1,
\end{aligned}$$

and so $\beta \in \ker \pi = L^\times$.

To prove the second claim, we proceed by induction. If $n_1 = 2$, then

$$\begin{aligned}
N_1(\beta) &= \beta \cdot \sigma_1(\beta) \\
&= f_1 f_2 f_1^{-1} f_2^{-1} \cdot (f_1 \cdot f_1 f_2 f_1^{-1} f_2^{-1} \cdot f_1^{-1}) \\
&= f_1 f_2 f_1^{-1} f_2^{-1} f_1^2 f_2 f_1^{-1} f_2^{-1} f_1^{-1} \\
&= f_1 f_2 f_1^{-1} \sigma_2^{-1}(\alpha_1) f_1^{-1} f_2^{-1} f_1^{-1} \\
&= f_1 f_2 \ \sigma_1^{-1} \sigma_2^{-1}(\alpha_1) f_1^{-2} f_2^{-1} f_1^{-1} \\
&= f_1 f_2 \ \sigma_1^{-1} \sigma_2^{-1}(\alpha_1) \ \alpha_1^{-1} f_2^{-1} f_1^{-1} \\
&= \sigma_1 \sigma_2 (\sigma_1^{-1} \sigma_2^{-1}(\alpha_1) \ \alpha_1^{-1}) \\
&= \alpha_1 / \sigma_1 \sigma_2(\alpha_1) \\
&= \alpha_1 / \sigma_2(\alpha_1),
\end{aligned}$$

where we use the fact that α_1 is fixed by σ_1. Suppose the relation holds for $n = k$. Using some liberties with notation, we will use f_1^k and $f_2 f_1^k f_2^{-1}$ to indicate the result for $n = k$. Then for $n = k + 1$,

$$\begin{aligned}
N_1(\beta) &= \left(\prod_{i=0}^{k-1} \sigma_1^i(\beta) \right) \cdot \sigma^k(\beta) \\
&= f_1^k f_2 f_1^{-k} f_2^{-1} \cdot (f_1^k \cdot f_1 f_2 f_1^{-1} f_2^{-1} \cdot f_1^{-k}) \\
&= f_1^k f_2 f_1^{-k} f_2^{-1} \alpha_1 f_2 f_1^{-1} f_2^{-1} f_1^{-k} \\
&= f_1^k f_2 f_1^{-k} \ \sigma_2^{-1}(\alpha_1) \ f_1^{-1} f_2^{-1} f_1^{-k} \\
&= f_1^k f_2 f_1^{-k-1} \ \sigma_2^{-1}(\alpha_1) \ f_2^{-1} f_1^{-k} \\
&= f_1^k f_2 \ \alpha_1^{-1} \ \sigma_2^{-1}(\alpha_1) \ f_2^{-1} f_1^{-k} \\
&= f_1^k \ \sigma_2(\alpha_1^{-1}) \ \alpha_1 \ f_1^{-k} \\
&= \alpha_1 / \sigma_2(\alpha_1)
\end{aligned}$$

Thus, the second claim holds. To obtain the last claim, we simply note that $\beta^{-1} = f_2 f_1 f_2^{-1} f_1^{-1}$, such that the previous argument holds by symmetry. ☐

Remark 1. In the proof above, the element f_1^k (and indeed, all lower orders) may not necessarily be an element of $L^\times$ on which σ_2 may act. However, we may ignore this specification by working purely with f_1 and f_2, defining the "action" of σ_2 to work by its defined conjugation.

3.3 Some Remarks on Abelian Extensions

Much of the commentary on cyclic and bicyclic extensions extends to a finite abelian extension L/K. Suppose that $G = \mathrm{Gal}(L/K) = G_1 \times \cdots \times G_r$ is a decomposition into cyclic groups generated by $\sigma_1, ..., \sigma_r$. In the bicyclic case, we represented the commutator between lifts of σ_1, σ_2 using β; to extend this, we will introduce a β term for each pair of indices analogously:

$$\beta_{ij} = f_i f_j f_i^{-1} f_j^{-1}.$$

We then have the following relation.

Proposition 3. Suppose $i < j < k$. Then

$$\frac{\beta_{ik}}{\sigma_j(\beta_{ik})} = \frac{\beta_{ij}}{\sigma_k(\beta_{ij})} \cdot \frac{\beta_{jk}}{\sigma_i(\beta_{jk})}$$

4 Cyclotomic Extensions of $\mathbb{Q}_p$

Let p be an odd prime.

5 Cyclotomic Extensions of $\mathbb{Q}_2$

We now consider the special case of $p = 2$. We observed that when p is an odd prime, a cyclotomic extension of $\mathbb{Q}_p$ is bicyclic. However, in the case of $\mathbb{Q}_2$, our diagram of fields splits as

$$\begin{tikzcd}
 & & & \mathbb{Q}_2(\zeta_n) \arrow[dl, dash] \arrow[ddr, dash] & \\
 & & \mathbb{Q}_2(\zeta_{2^k}) \arrow[dl, dash] \arrow[d, dash] & & \\
 & \mathbb{Q}_2(i) \arrow[dr, dash] & \mathbb{Q}_2(\zeta_{2^k} + \zeta_{2^k}^{-1}) \arrow[d, dash] & & \mathbb{Q}_2(\zeta_m) \arrow[dll, dash] \\
 & & \mathbb{Q}_2 & &
\end{tikzcd}$$

where the extension $\mathbb{Q}_2(\zeta_{2^k})/\mathbb{Q}_2$ is bicyclic over $\mathbb{Q}_2$.

5.1 Some Explicit Results

Consider the specific case of $K = \mathbb{Q}_2(\zeta_8) = \mathbb{Q}_2(i, \sqrt{2})$, with the following diagram of fields:

$$\begin{tikzcd}
 & \mathbb{Q}_2(\zeta_8) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}_2(i) \arrow[dr, dash] & & \mathbb{Q}_2(\sqrt{2}) \arrow[dl, dash] \\
 & \mathbb{Q}_2 &
\end{tikzcd}$$

This bicyclic extension is totally ramified and of degree 4. We seek to fully classify its equivalence classes of triples as described in Section 3.

In this case, the Galois group $\mathrm{Gal}(K/\mathbb{Q}_2)$ is generated by the automorphisms

$$\sigma_1 : \sqrt{2} \mapsto -\sqrt{2}$$
$$\sigma_2 : i \mapsto -i$$

such that $K_1 = \mathbb{Q}_2(i)$ and $K_2 = \mathbb{Q}_2(\sqrt{2})$. As observed earlier, a triple $(\alpha_1, \alpha_2, \beta)$ that has order 4 cannot have $\alpha_i = N_i(x)$ for an element $x \in K^\times$. Thus, it is useful to explicitly compute the norm groups of the subfields of this extension.

Observe that $[\mathbb{Q}_2(\zeta_8) : \mathbb{Q}_2(i)] = [\mathbb{Q}_2(\zeta_8) : \mathbb{Q}_2(\sqrt{2})] = 2$. Because K is an abelian extension, the norm groups $N_1(K^\times)$ and $N_2(K^\times)$ have index 2 over their respective fields. Thus, $N_i(K^\times)$ induces an order 2 subgroup of $K_i^\times/(K_i^\times)^2$. We'll consider each subfield separately.

Consider $K_1 = \mathbb{Q}_2(i)$. Observe that K_1 has uniformizer $\pi_{K_1} = i - 1$, so the unit group of K_1 has the structure

$$\begin{aligned} K_1^\times &\cong \langle \pi_{K_1} \rangle \times (1 + \mathfrak{m}) \\ &= \langle i - 1 \rangle \times (1 + (i-1)\mathbb{Z}_2[i]). \end{aligned}$$

By a basic computation,

$$\begin{aligned} (K_1^\times)^2 &\cong \langle -2i \rangle \times (1 + 2(i-1)\mathbb{Z}_2[i] - 2i\mathbb{Z}_2[i]) \\ &= \langle -2i \rangle \times (1 + 2\mathbb{Z}_2[i]) \\ &= \langle \pi_{K_1}^2 \rangle \times (1 + \mathfrak{m}^2) \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} K_1^\times/(K_1^\times)^2 &= \langle \pi_{K_1} \rangle / \langle \pi_{K_1}^2 \rangle \times (1 + \mathfrak{m})/(1 + \mathfrak{m}^2) \\ &= \langle i - 1 \rangle \times \langle i \rangle \\ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

The norm group $N_1(K) \subseteq K_1$ has index 2, so it is generated by an order 2 subgroup of $K_1^\times/(K_1^\times)^2$. We observe that these subgroups are precisely those generated by i, $i - 1$, and $-1 - i$. Furthermore, the uniformizer $\pi_K = \zeta_8 - 1$ must map to a uniformizer of K_1 under the norm map, so because

$$N_1(\pi_K) = (\zeta_8 - 1)(-\zeta_8 - 1) = -(i - 1)$$

and $-1 = i^2$ is a norm in K_1, we conclude that

$$N_1(K) = (K_1^\times)^2 \times \langle i - 1 \rangle.$$

Moreover, we determine the quotient group to be

$$K_1^\times/N_1(K) = \langle i \rangle.$$

Thus, up to norm, we have that $\alpha_1 \equiv i$.

Now consider $K_2 = \mathbb{Q}_2(\sqrt{2})$. Repeating the process, observe that K_2 has uniformizer $\pi_{K_2} = \sqrt{2}$, so the unit group of K_2 has the structure

$$\begin{aligned} K_2^\times &\cong \langle \pi_{K_2} \rangle \times (1 + \mathfrak{m}) \\ &= \langle \sqrt{2} \rangle \times (1 + \sqrt{2}\,\mathbb{Z}_2[\sqrt{2}]). \end{aligned}$$

Again by computation,

$$\begin{aligned} (K_2^\times)^2 &\cong \langle 2 \rangle \times (1 + 2\sqrt{2}\,\mathbb{Z}_2[\sqrt{2}] + 2\mathbb{Z}_2[\sqrt{2}]) \\ &= \langle \pi_{K_2}^\times \rangle \times (1 + \mathfrak{m}^2). \end{aligned}$$

Thus, we obtain that

$$\begin{aligned}K_2^\times/(K_2^\times)^2 &= \langle \pi_{K_2}\rangle/\langle \pi_{K_2}^2\rangle \times (1+\mathfrak{m})/(1+\mathfrak{m}^2)\\ &= \langle \sqrt{2}\rangle \times \langle 1+\sqrt{2}\rangle\\ &\cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

As with before, the norm group $N_2(K) \subseteq K_2$ has index 2, and the subgroups of $K_2^\times/(K_2^\times)^2$ are generated by $\sqrt{2}$, $1+\sqrt{2}$, and $2+\sqrt{2}$. Given the uniformizer π_K, we have that

$$\begin{aligned}N_2(\pi_K) &= (\zeta_8 - 1)(\zeta_8^{-1}-1)\\ &= 2 - (\zeta_8+\zeta_8^{-1})\\ &= 2-\sqrt{2}\\ &= \frac{2}{2+\sqrt{2}}.\end{aligned}$$

We note that since $2+\sqrt{2}$ is a generator of one of our subgroups, so is $(2+\sqrt{2})^{-1}$, and because 2 is a norm, we have that the norm group is

$$N_2(K) = (K_2)^\times \times \langle 2+\sqrt{2}\rangle$$

and that the quotient group is computed to be

$$K_2^\times/N_2(K) = \langle\sqrt{2}\rangle = \langle 1+\sqrt{2}\rangle$$

Thus, up to norm, $\alpha_2 \equiv \sqrt{2}$.