Deep Learning for Computer Vision

Variational Auto-Encoders

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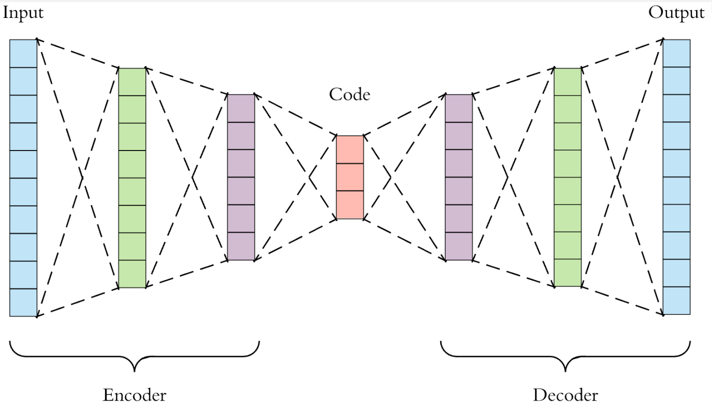
Recall: Autoencoders

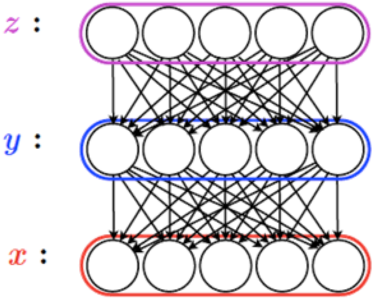


Autoencoders can reconstruct data, and can learn features to initialize a supervised model.
Can we generate images from an autoencoder?

Credit: Arden Dertat, TowardsDataScience

Variational Autoencoders

- Introduced around the same time by two groups of researchers:
  - Kingma and Welling, **Auto-Encoding Variational Bayes**, ICLR 2014
  - Rezende, Mohamed and Wiestra, **Stochastic Backpropagation and Variational Inference in Deep Latent Gaussian Models**, ICML 2014



Credit: Aaron Courville, Deep Learning Summer School, 2015

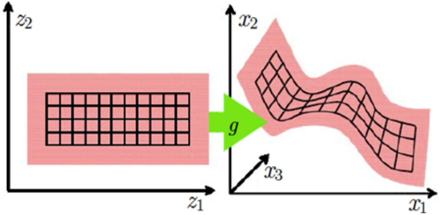
Variational Autoencoders

- **Latent Variable Model:** Learn a mapping from some latent variable z to a possibly complex distribution on x

$$p(x) = \int p(x,z)dz \quad \text{where} \quad p(x,z) = p(x|z)p(z)$$

$$p(z) = \text{something simple}; \quad p(x|z) = g(z)$$

- Can we learn to decouple the true explanatory factors (**latent variables**) underlying the data distribution (e.g. identity and expression in face images)? How?



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Variational Autoencoders

- Leverage neural networks to learn a latent variable model!

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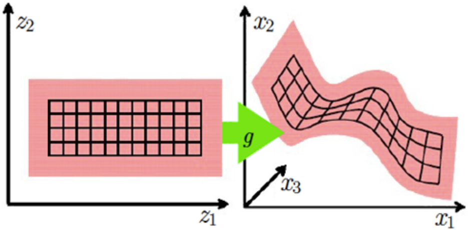

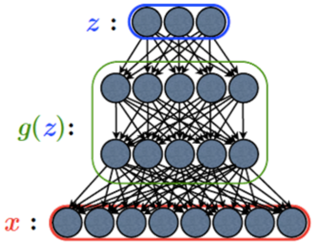

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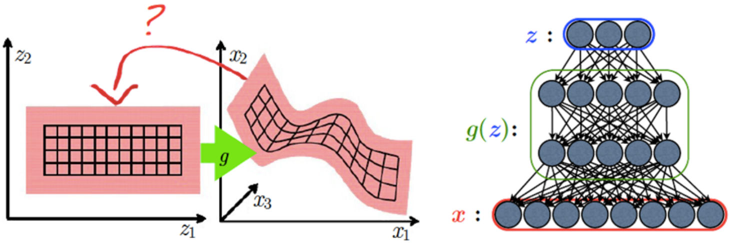
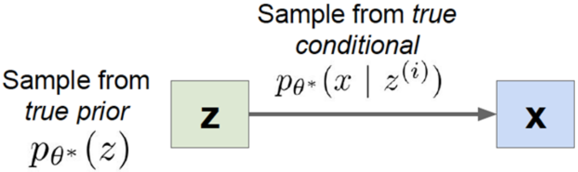
- Where does z come from? Computing the posterior $p(z|x)$ is intractable, and we need it to train the directed model



Credit: Aaron Courville, Deep Learning Summer School, 2015

A Bayesian spin on an autoencoder!

Assume our data $\{x^{(i)}\}_{i=1}^{N}$ is generated like this:



Kingma and Welling, "Auto-Encoding Variational Bayes", ICLR 2014

Credit: Fei-Fei Li, Andrej Karpathy and Justin Johnson, CS231n, Stanford Univ

Variational Autoencoders

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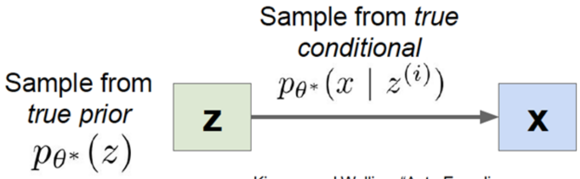

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- **Intuition:** x is an image, z gives class, orientation, attributes, etc

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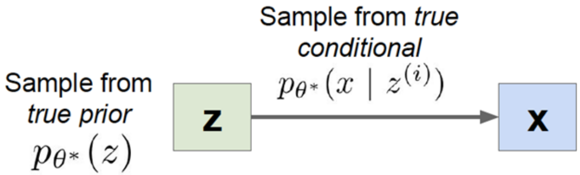
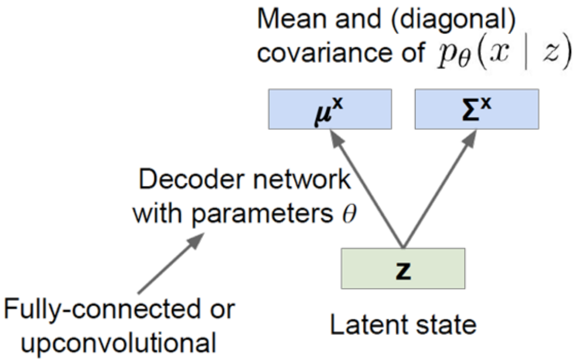


Kingma and Welling, "Auto-Encoding Variational Bayes", ICLR 2014

- **Intuition:** x is an image, z gives class, orientation, attributes, etc
- **Problem:** Estimate θ without access to latent states $z^{(i)}$

Credit: Fei-Fei Li, Andrej Karpathy and Justin Johnson, CS231n, Stanford Univ

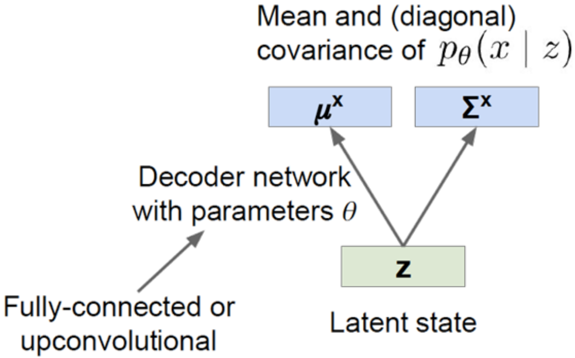
Variational Autoencoders



- **Prior:** Assume $p_\theta(z)$ is a unit Gaussian

Credit: Fei-Fei Li, Andrej Karpathy and Justin Johnson, CS231n, Stanford Univ

Variational Autoencoders



- **Prior:** Assume $p_\theta(z)$ is a unit Gaussian
- **Conditional:** Assume $p_\theta(x|z)$ is a diagonal Gaussian, predict mean and variance with neural network

Credit: Fei-Fei Li, Andrej Karpathy and Justin Johnson, CS231n, Stanford Univ

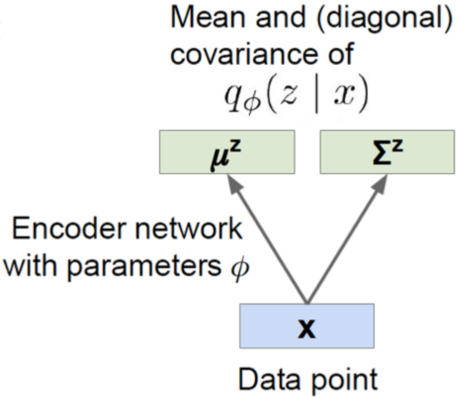
By Bayes Rule the posterior is:

$$p_\theta(z \mid x) = \frac{\boxed{p_\theta(x \mid z)}\boxed{p_\theta(z)}}{\boxed{p_\theta(x)}}$$

Use decoder network ☺

Gaussian ☺

Intractible integral ☹



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Variational Autoencoders

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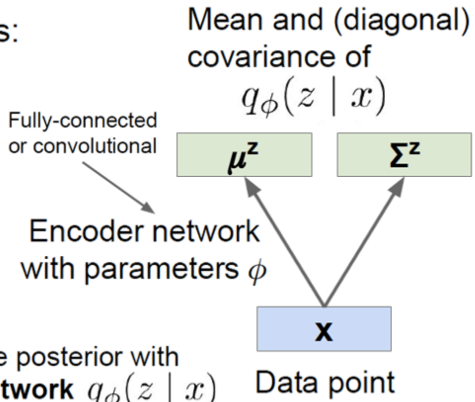
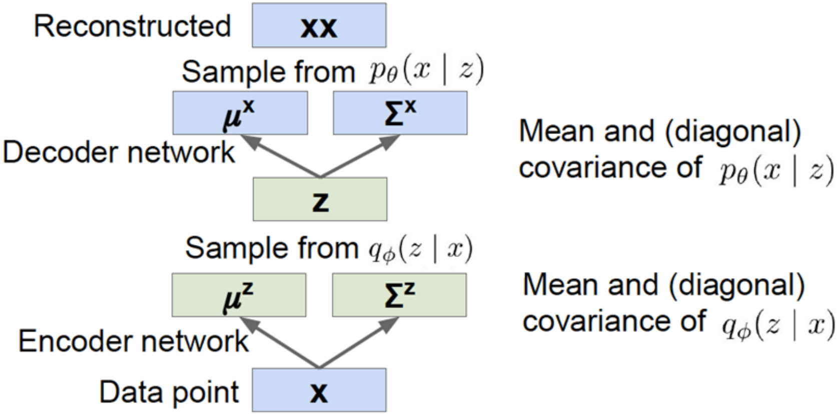


Approximate posterior with **encoder network** $q_\phi(z \mid x)$

Credit: Fei-Fei Li, Andrej Karpathy and Justin Johnson, CS231n, Stanford Univ

Variational Autoencoders



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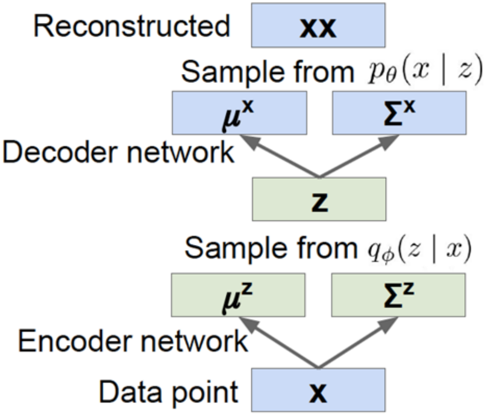
Variational Autoencoders



Training like a normal autoencoder: **reconstruction loss** at the end, **regularization toward prior** in middle

Mean and (diagonal) covariance of $p_\theta(x \mid z)$ **(should be close to data x)**

Mean and (diagonal) covariance of $q_\phi(z \mid x)$ **(should be close to prior $p_\theta(z)$)**

Credit: Fei-Fei Li, Andrej Karpathy and Justin Johnson, CS231n, Stanford Univ

Variational Autoencoder: The Math

$$\theta^* = \arg\max_\theta \prod_{i=1}^{N} p_\theta(x^{(i)})$$

Maximize likelihood of dataset $\{x^{(i)}\}_{i=1}^{N}$

Credit: Fei-Fei Li, Andrej Karpathy and Justin Johnson, CS231n, Stanford Univ

Variational Autoencoder: The Math

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$$= \arg\max_\theta \sum_{i=1}^{N} \log p_\theta(x^{(i)})$$

Maximize log-likelihood instead because sums are nicer

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$$p_\theta(x^{(i)}) = \int p_\theta(x^{(i)}, z)dz$$

Marginalize joint distribution

Credit: Fei-Fei Li, Andrej Karpathy and Justin Johnson, CS231n, Stanford Univ

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$$= \int p_\theta(x^{(i)} \mid z)p_\theta(z)dz$$

Intractible integral ☹

Credit: Fei-Fei Li, Andrej Karpathy and Justin Johnson, CS231n, Stanford Univ

Variational Autoencoder: The Math

$$\log p_\theta(x^{(i)}) = \mathbf{E}_{z\sim q_\phi(z|x^{(i)})}\left[\log p_\theta(x^{(i)})\right] \qquad (p_\theta(x^{(i)}) \text{ Does not depend on } z)$$

$$= \mathbf{E}_z\left[\log \frac{p_\theta(x^{(i)} \mid z)p_\theta(z)}{p_\theta(z \mid x^{(i)})}\right] \qquad (\text{Bayes' Rule})$$

$$= \mathbf{E}_z\left[\log \frac{p_\theta(x^{(i)} \mid z)p_\theta(z)}{p_\theta(z \mid x^{(i)})} \frac{q_\phi(z \mid x^{(i)})}{q_\phi(z \mid x^{(i)})}\right] \qquad (\text{Multiply by constant})$$

Credit: Fei-Fei Li, Andrej Karpathy and Justin Johnson, CS231n, Stanford Univ

Variational Autoencoder: The Math

$$
\begin{aligned}
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&= \mathbf{E}_z\left[\log p_\theta(x^{(i)} \mid z)\right] - \mathbf{E}_z\left[\log \frac{q_\phi(z \mid x^{(i)})}{p_\theta(z)}\right] + \mathbf{E}_z\left[\log \frac{q_\phi(z \mid x^{(i)})}{p_\theta(z \mid x^{(i)})}\right] \quad \text{(Logarithms)}\\
&= \underbrace{\mathbf{E}_z\left[\log p_\theta(x^{(i)} \mid z)\right] - D_{KL}(q_\phi(z \mid x^{(i)}) \,||\, p_\theta(z))}_{\mathcal{L}(x^{(i)},\theta,\phi)\ \text{“Elbow”}} + \underbrace{D_{KL}(q_\phi(z \mid x^{(i)}) \,||\, p_\theta(z \mid x^{(i)}))}_{\geq 0}
\end{aligned}
$$

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$$\log p_\theta(x^{(i)}) \geq \mathcal{L}(x^{(i)},\theta,\phi)$$

Variational lower bound (elbow)

$$\theta^*, \phi^* = \arg\max_{\theta,\phi} \sum_{i=1}^{N} \mathcal{L}(x^{(i)},\theta,\phi)$$

Training: Maximize lower bound

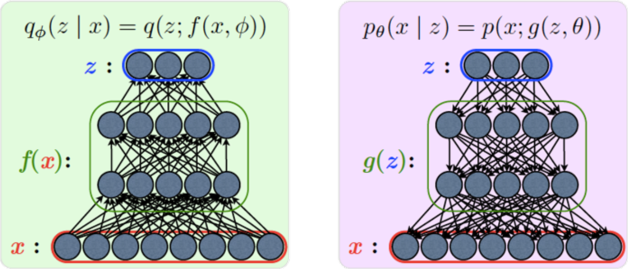
Credit: Fei-Fei Li, Andrej Karpathy and Justin Johnson, CS231n, Stanford Univ

Variational Autoencoder: Inference

- Introduce an inference model $q_\phi(z|x)$ that learns to approximates the intractable posterior $p_\theta(z|x)$ by optimizing the variational lower bound:

$$\mathcal{L}(\theta, \phi, x) = -D_{KL}(q_\phi(z|x) || p_\theta(z)) + \mathbb{E}_{q_\phi(z|x)}[\log p_\theta(x|z)]$$

- We parametrize $q_\phi(z|x)$ with another neural network:



Credit: Aaron Courville, Deep Learning Summer School, 2015

Variational Autoencoder: How to train?

$$\begin{aligned}\mathcal{L}_{VAE} &= \mathbb{E}_{q_\phi(z|x)}\left[\log \frac{p_\theta(z,x)}{q_\phi(z|x)}\right] \\ &= -D_{KL}(q_\phi(z|x)||p_\theta(z)) + \mathbb{E}_{q_\phi(z|x)}[\log p_\theta(x|z)]\end{aligned}$$

- $z \sim q_\phi(z|x)$: need to differentiate through the sampling process; how to update ϕ? (encoder is probabilistic)

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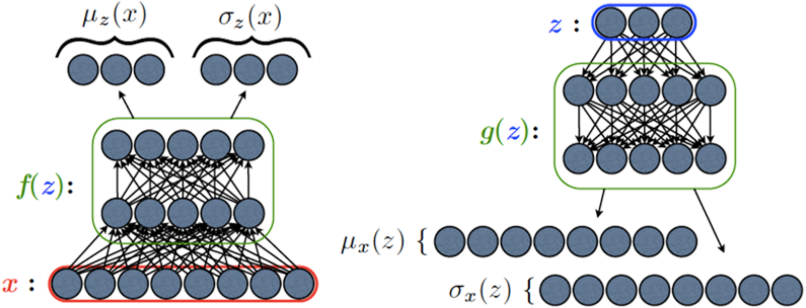
- $z \sim q_\phi(z|x)$: need to differentiate through the sampling process; how to update ϕ? (encoder is probabilistic)
- **Solution**: Make the randomness independent of encoder output, thus making the encoder deterministic; how?

Reparametrization Trick

- Let's consider z to be real and $q_\phi(z|x) = \mathcal{N}(z; u_z(x), \sigma_z(x))$
- Parametrize z as $z = \mu_z(z) + \sigma_z(x)\epsilon_z$ where $\epsilon_z = \mathcal{N}(0, 1)$

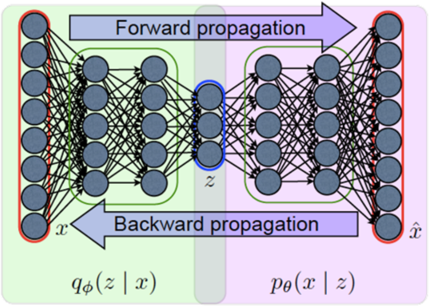


Credit: Aaron Courville, Deep Learning Summer School, 2015

Training with Backpropagation

With the **reparametrization trick**, we can simultaneously train both the **generative model** $p_\theta(x|z)$ and the **inference model** $q_\phi(z|x)$ using backpropagation

Objective function: $\mathcal{L}(\theta, \phi, x) = -D_{KL}(q_\phi(z|x)||p_\theta(z)) + \mathbb{E}_{q_\phi(z|x)}[\log p_\theta(x|z)]$



Credit: Aaron Courville, Deep Learning Summer School, 2015

VAE: Summary

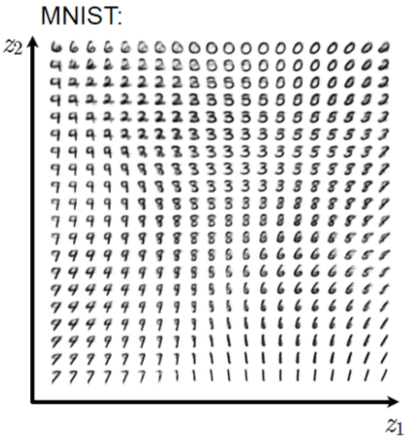
Traditional Autoencoders

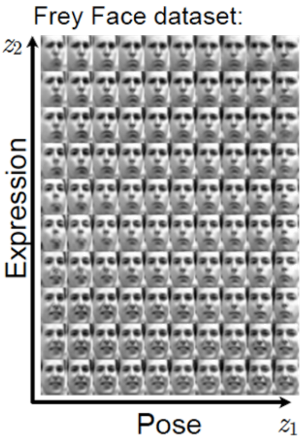
- Learned by reconstructing input
- Used to learn features, initialize supervised models (not much anymore though)

Variational Autoencoders

- Bayesian learning meets deep learning
- Sample from model to generate images

VAE: What can they do?





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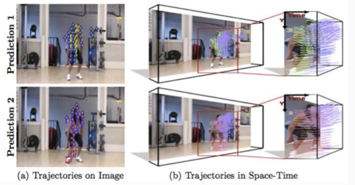
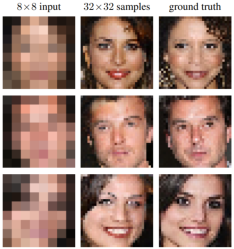
Applications of VAEs

- Image and video generation
- Superresolution
- Forecasting from static images
- Image inpainting
- many more...





Credit: Dahl et al, Pixel Recursive Super Resolution, ICCV 2017

A Few Variants and Extensions

- Semi-Supervised VAEs
  - Kingma et al, Semi-Supervised Learning with Deep Generative Models, NeurIPS 2014
- Conditional VAE
  - Sohn et al, Learning Structured Output Representation using Deep Conditional Generative Models, NeurIPS 2015
- Importance-Weighted VAE
  - Burda et al, Importance Weighted Autoencoders, ICLR 2016
- Denoising VAE
  - Jiwoong et al, Denoising Criterion for Variational Auto-encoding Framework, AAAI 2017
- Inverse Graphics Network
  - Kulkarni et al, Deep Convolutional Inverse Graphics Network, NeurIPS 2015
- Adversarial Autoencoders
  - Makhzani et al, Adversarial Autoencoders, ICLR 2016

Homework

Readings

- Carl Doersch, Tutorial on Variational Autoencoder, arXiv 2016
- VAE example in PyTorch
- Kingma and Welling, Auto-Encoding Variational Bayes, ICLR 2014

Question

- Why does the encoder of a VAE map to a vector of means and a vector of standard deviations? Why does it not instead map to a vector of means and a covariance matrix?
- What about the decoder? If we assume a Mean Squared Error for the reconstruction loss, what is the covariance of the $p(x|z)$ Gaussian distribution?