Lecture 2: Projective Geometry

1   Homogeneous Coordinates

In Lecture 1 we derived the camera equations

$$\lambda \mathbf{x} = P\mathbf{X}, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, 1)$, $\mathbf{X} = (X_1, X_2, X_3, 1)$ and P is a 3×4 matrix. The vector $\mathbf{X}$ represents a 3D-point and $\mathbf{x}$ is its projection in the image. The 3×4 matrix P contains the parameters of the camera that captured the image. It can be decomposed into $P = K \begin{bmatrix} R & t \end{bmatrix}$ where R and t encodes orientation and position of the camera, respectively, and K contains the inner parameters.

The interpretation of this equation is that the projection (x_1, x_2) of the scene point with coordinates (X_1, X_2, X_3) can be found by first computing $v = P\mathbf{X}$ and then dividing v by its third coordinate. There are several vectors v that give the same projection. For example, $v = (3, 2, 1)$ gives the projection $(3, 2)$ and $v = (6, 4, 2)$ gives $\left(\frac{6}{2}, \frac{4}{2}\right) = (3, 2)$.

Formally, we will say that two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ are equivalent if there is a number $\lambda \neq 0$ such that $\mathbf{x} = \lambda \mathbf{y}$, and write

$$\mathbf{x} \sim \mathbf{y}. \tag{2}$$

The two vectors are said to be representing the same element of the so called two dimensional **projective space** $\mathbb{P}^2$. The space consists of all the elements that can be represented by vectors in $\mathbb{R}^3$ (with the exception of the zero vector). For example, the two vectors $(6, 9, 3)$ and $(4, 6, 2)$ are equivalent and therefore both represent the same element of $\mathbb{P}^2$. Furthermore, by dividing with the third coordinate we can interpret this element as a point in $\mathbb{R}^2$, namely $(2, 3)$. The vectors $(6, 9, 3)$ and $(4, 6, 2)$ are called **homogeneous coordinates** of $(2, 3)$.

There are elements in $\mathbb{P}^2$ that cannot be interpreted as points in $\mathbb{R}^2$. Specifically, if the third coordinate is zero we cannot divide by it. We will soon see that these points also have a simple geometric interpretation.

The projective space of n dimensions $\mathbb{P}^n$ is defined similarly as $\mathbb{P}^2$. In general the homogeneous coordinates for representing $\mathbb{P}^n$ are the vectors of $\mathbb{R}^{n+1}$, and if coordinate $n+1$ is not zero then we can interpret them as points of $\mathbb{R}^n$ by dividing with this coordinate.

2   Lines and Points in $\mathbb{P}^2$

From linear algebra we know that a line (in $\mathbb{R}^2$) can be represented by the equation

$$ax + by + c = 0, \tag{3}$$

where $(a, b, c) \neq (0, 0, 0)$. If we think of $\mathbf{x} \sim (x, y, 1)$ as a point in $\mathbb{P}^2$ then $\mathbf{x}$ belongs to the line $\mathbf{l} = (a, b, c)$ if (3) holds. Note that (3) can be seen as the scalar product of the vectors $(x, y, 1)$ and (a, b, c). If we use $(\lambda x, \lambda y, \lambda)$ to represent $\mathbf{x}$ instead of $(x, y, 1)$ we get the scalar product

$$a\lambda x + b\lambda y + c\lambda = \lambda(ax + by + c) = 0. \tag{4}$$

Therefore we see that it does not matter which representative we use for $\mathbf{x}$. They all fulfill (4). Similarly if $c \neq 0$ we can represent $\mathbf{l}$ with $\left(\frac{a}{c}, \frac{b}{c}, 1\right)$ instead of (a, b, c).

Note that lines and points behave in the same way here. They are both represented with 3-vectors. As a consequence of this we say that **points are dual to lines in** $\mathbb{P}^2$. That is, whenever we have a theorem involving lines and points in $\mathbb{P}^2$ we can always exchange points for lines and get a dual statement. For example, the statement "Two lines intersect each other in one point" is dual to "For any two points there is one line going through them both". (Note that the first statement is not true in $\mathbb{R}^2$ if the lines are parallel.)

In three dimensions the equation

$$ax + by + cz + d = 0 \tag{5}$$

represents a plane. Therefore, for the space $\mathbb{P}^3$ (the space consisting of all elements that can be represented by vectors in $\mathbb{R}^4$) we have duality between points and planes instead.

Exercise 1. Compute the point of intersection $\mathbf{x} \in \mathbb{P}^2$ ($\mathbf{x} \sim (x, y, z)$) of the two lines $\mathbf{l}_1 \sim (-1, 0, 1)$ and $\mathbf{l}_2 \sim (0, -1, 1)$.

Exercise 2. Compute the line $\mathbf{l} \sim (a, b, c)$ passing through the points $\mathbf{x}_1 \sim (-1, 0, 1)$ and $\mathbf{x}_2 \sim (0, -1, 1)$. (Hint: look at the previous exercise.)

3 Vanishing Points

In the space $\mathbb{P}^2$ every pair of lines have a common intersection point, even parallel ones. Consider for example the two lines $\mathbf{l}_1 = (-1, 0, 1)$ and $\mathbf{l}_2 = (1, 0, 1)$, see Figure 1.
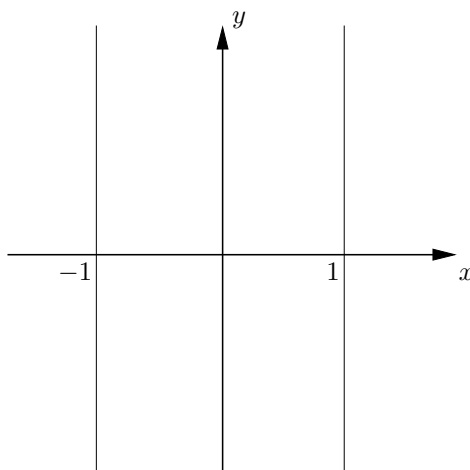

Figure 1: The two lines $(-1, 0, 1)$ and $(1, 0, 1)$.

Since $\mathbf{x}$ lies on both the lines we have to solve the system of equations

$$\begin{cases} \mathbf{l}_1^T \mathbf{x} &= 0 \\ \mathbf{l}_2^T \mathbf{x} &= 0 \end{cases} \Leftrightarrow \begin{cases} x + z &= 0 \\ -x + z &= 0 \end{cases}. \tag{6}$$

Since we have no constraints for y we get

$$\begin{cases} x + z &= 0 \\ 2z &= 0 \\ y &= t \end{cases} \Leftrightarrow \begin{cases} x &= 0 \\ y &= t \\ z &= 0 \end{cases}. \tag{7}$$

Hence, for example $(0, 1, 0)$ is a representative of our intersection point. In this case we cannot interpret the result by dividing with the third coordinate since this one is zero, which makes sense

since the lines are parallel and therefore do not intersect in $\mathbb{R}^2$. To interpret $(0, 1, 0)$ geometrically we look at $(0, 1, \epsilon)$, where ϵ is a small positive number. This point has non-zero third coordinate and is equivalent to $\left(0, \frac{1}{\epsilon}, 1\right)$, that is, it is a point with x-coordinate zero and a very large y-coordinate. Making ϵ smaller we see that $(0, 1, 0)$ can be interpreted as a point infinitely far away in the direction $(0, 1)$. We call this type of point a **vanishing point** or a point at infinity.

Note that if we instead assume that ϵ is a small negative number we get a point far away in the direction $(0, -1)$. We therefore do not differ between these points, and in addition $(0, 1, 0) \sim (0, -1, 0)$.

The line $z = 0$ is called the **vanishing line** or the line at infinity since it only contains points that has third coordinate 0.

4 Conics

The conics are all second order curves of the form

$$\mathbf{x}^T C \mathbf{x} = 0 \tag{8}$$

where C is a symmetric matrix. For example the circle of radius 1 can be written

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = x^2 + y^2 - 1 = 0. \tag{9}$$

The line $\mathbf{l} = C\mathbf{x}$ is a tangent line to to the conic at the point $\mathbf{x}$. The dual conic is the set of all lines that tangent to the conic. If the matrix C is invertible then the expression for the dual conic can be computed by noting that

$$0 = \mathbf{x}^T C \mathbf{x} = \mathbf{x}^T C C^{-1} C \mathbf{x} = (C\mathbf{x})^T C^{-1} C \mathbf{x} = \mathbf{l}^T C^{-1} \mathbf{l}. \tag{10}$$

Therefore any line that is tangent to the conic has to fulfill $\mathbf{l}^T C^{-1} \mathbf{l} = 0$ which is a conic in the space of all lines. If C is not invertible one can use the pseudo inverse instead.

5 Projective Transformations

A **projective transformation** is an invertible mapping $\mathbb{P}^n \mapsto \mathbb{P}^n$ defined by

$$\mathbf{x} \sim H \mathbf{y} \tag{11}$$

where $\mathbf{x} \in \mathbb{R}^{n+1}$ and $\mathbf{y} \in \mathbb{R}^{n+1}$ are homogeneous coordinates representing elements of $\mathbb{P}^n$ and H is an invertible $(n + 1) \times (n + 1)$ matrix. Projective transformations are also often called **homographies**.

Exercise 3. Show that it does not matter what representative we choose, the result will be the same. (Hint: $\mathbf{y}$ and $\lambda \mathbf{y}$ are two representatives of the same point.)

Projective mappings occur often when working with images. In Lecture 1 we saw one example of such a mapping, namely the K-matrix. In the camera equation

$$\mathbf{x} \sim K \begin{bmatrix} R & t \end{bmatrix} \mathbf{X} \tag{12}$$

the matrix K transforms the point $\begin{bmatrix} R & t \end{bmatrix} \mathbf{X}$ in the image plane to the real image coordinate system (with the unit pixels).

Another example is point transfer via a plane. Figure 2 shows two images of a desk with a roughly planar surface. With this setup the there is a homography that transforms the points of one image to the other given by (11), where H is a 3×3 matrix. To find the transformation we need to determine the elements of H. There are 9 elements but since thee scale does not matter (H and λH represents
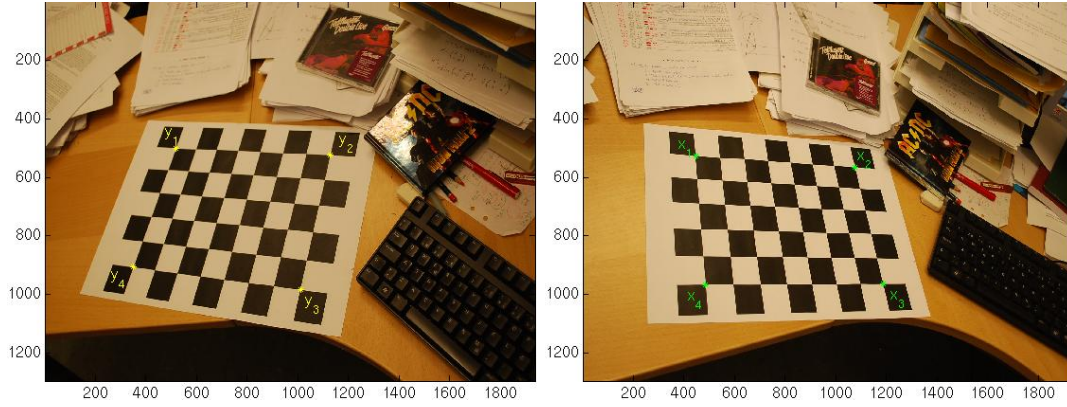
Figure 2: Two images of a roughly planar surface with 4 detected point correspondences.

the same transformation) there are only 8 degrees of freedom. Now suppose that we have n points $\mathbf{y}_i$, $i = 1, ..., n$ that we know are transformed to n corresponding points $\mathbf{x}_i$, $i = 1, ..., n$ in the second image. Each point pair gives us 3 equations

$$\lambda_i \mathbf{x}_i = H \mathbf{y}_i \tag{13}$$

(recall that $\mathbf{y}_i$ and $H\mathbf{x}_i$ are vectors of size 3) but one new unknown λ_i is introduced. We now have $3n$ equations and $8 + n$ degrees of freedom. To be able to find H we therefore need

$$3n \geq 8 + n \Leftrightarrow 2n \geq 8 \Leftrightarrow n \geq 4 \tag{14}$$

point correspondences. Figure 2 shows 4 point correspondences that can be used to compute H. (An approach for doing this, the so called DLT method, will be presented in Lecture 3.)
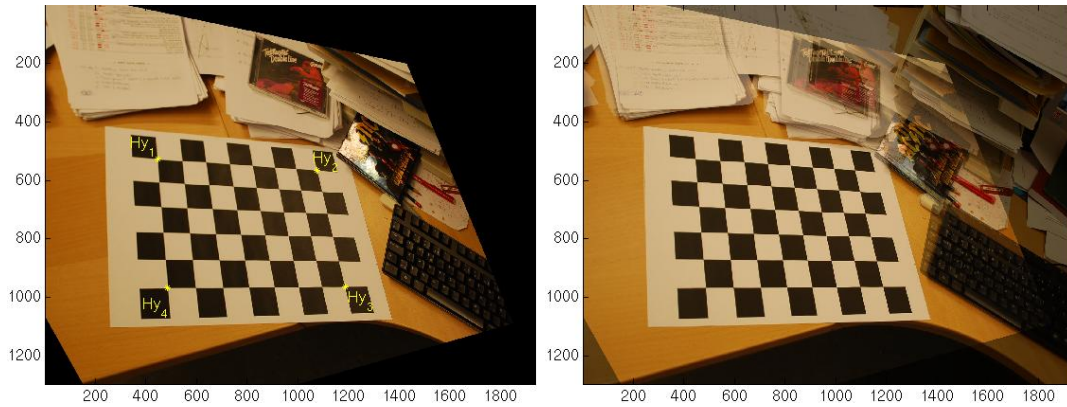

Figure 3: Left: the result of appying H to the left image in Figure 2. Right: The transformed image overlaid on the right image of Figure 2.

When H has been computed we can transform the other points in the image. Figure 3 shows the transformation of the left image and the transformed image overlaid on the right image. It can be seen that the images agree well where the scene is roughly planar.

When looking carefully at the checkerboard pattern it is evident that this transformation preserves lines, that is, points on a line in the original image is mapped to another line in the new figure. This is always the case when we have projective transformations.

Exercise 4. Assume that $\mathbf{y}$ lies on the line $\mathbf{l}$, that is, $\mathbf{l}^T \mathbf{y} = 0$, and that $\mathbf{x} \sim H\mathbf{y}$. Show that $\mathbf{x}$ lies on the line $\hat{\mathbf{l}} = (H^{-1})^T \mathbf{l}$.

5.1   Special Cases of Transformations

A special case of projective transformation $\mathbb{P}^n \mapsto \mathbb{P}^n$ is the **affine transformation**. For this type of mapping the matrix H has the special shape

$$H = \left[\begin{array}{cc} A & t \\ 0 & 1 \end{array} \right], \tag{15}$$

where A is an invertible $n \times n$ matrix, t is an $n \times 1$ vector and 0 is a $1 \times n$ vector of all zeros. Besides being projective, the affine transformation has the special property that parallel lines are mapped to parallel lines. Furthermore, it preserves the line at infinity, that is, vanishing points are mapped to vanishing points and regular points to regular points. If we only consider points in $\mathbb{R}^2$ (with regular Cartesian coordinates) then the transformation can be written $x = Ay + t$. An example of an affine transformation is shown in Figure 4.

Figure 4: The affine transformation preserves parallel lines.

The **similarity transformation** has the form

$$H = \left[\begin{array}{cc} sR & t \\ 0 & 1 \end{array} \right], \tag{16}$$

where R is an $n \times n$ rotation and s is a positive number. This mapping also preserves angles between lines. An example of a similarity transformation is shown in Figure 5.

Figure 5: The similarity transformation preserves angles between lines.

If $s = 1$ in (16) then the transformation is called **Euclidean**. This mapping also preserves distances between points. An example of an affine transformation is shown in Figure 6.

Figure 6: The Euclidean transformation preserves distances between points.