

T.P. - Normal and anomalous Zeeman effect

I. INTRODUCTION

In 1896, the dutch physicist Pieter Zeeman discovered that when atom is submitted to an external magnetic field, its associated spectral lines could split in various components shifted in frequency and polarized.

In this practical work we will try to observe experimentally the "Zeeman effect" on the red emission line of Cadmium atoms in a magnetic field.

The wavelength of this red spectral line, characteristic of Cadmium is equal to $\lambda = 6438\text{\AA}$ and the very small Zeeman splitting induced by the magnetic field (≈ 1 Tesla) is typically of the order of a fraction of an Angström. While this is a very small The measurement of this splitting in wavelength is perfectly possible by using, for instance, a Fabry-Pérot interferometer.



FIG. 1. Pieter Zeeman, Nobel prize in physics in 1902..

II. THEORETICAL REMINDERS

A. Quantum numbers

The discrete spectrum of any atom can only be understood in the framework of quantum mechanics that describes the well defined energy levels on which the electrons surrounding the nucleus sit. The absorbed or emitted radiations correspond to electrons jumping from one energy level to another. In the case of the emission of a spectral line, energy is transmitted to the atoms of a vapor or a plasma by electronic collisions or by the interaction with radiation. The thus excited atoms revert back to their ground state by giving back this energy in the form of light whose frequency is linked to the energy difference between the starting level and the final level of the electron. In order to understand the quantization of

the energy levels, we must first introduce the quantum numbers n , l , s and j .

A first level of quantization concerns the more or less complete atomic layers for which a *principal number* n is assigned. Within a given layer, an electron can occupy various sub-layers or energy levels defined through the *orbital quantum number* l associated with the quantization of the angular momentum. The electron also has an intrinsic angular momentum, the spin s .

Here is a list of these quantum numbers:

- n : Principal quantum number characterizing the various electronic layers $n = 1, 2, 3, 4, \dots$ (we also use the notation K, L, M, N, ...).
- l : Orbital quantum number associated to the quantization of the angular momentum $\hat{\mathbf{L}}$ that can take any integer values between 0 and $(n - 1)$. An electronic layer n thus incorporates n sub-layers. The values $l = 0, 1, 2, 3, 4, \dots$ are referred to using the letters s, p, d, f, g, \dots
- m_l : Quantum number associated to the projection of the angular momentum $\hat{\mathbf{L}}$ on a quantization axis. $m_l = -l, -l + 1, \dots, l - 1, l$. We thus have $(2l + 1)$ possible values for m_l .
- s : Spin quantum number associated to the quantization of the intrinsic angular momentum $\hat{\mathbf{S}}$ of the electron (spin of the electron).
- m_s : Quantum number associated to the projection of the spin $\hat{\mathbf{S}}$. $m_s = -s, -s + 1, \dots, s - 1, s$. In the case of the electron, the spin s is equal to $\frac{1}{2}$ leading to two possible values of m_s : $m_s = \pm \frac{1}{2}$. In the case of an atom with many electrons, the total spin can take both integer and half-integer values.

The quantum numbers n , l and s are usually not sufficient to explain the many spectral lines and the Zeeman effect. Indeed, in general, the orbital angular momentum $\hat{\mathbf{L}}$ combines with the spin $\hat{\mathbf{S}}$ of the electron. The energy levels split in sub-levels with a very similar energies, responsible for the fine structure of the spectral lines of the atom. The resulting angular momentum leads to the so-called *spin-orbit* term and to the apparition of a new quantum number j linked to the total angular momentum $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$.

- j : Spin-orbit quantum number linked to the total angular momentum $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$. This sum can be taken in the quantum sense of the term, meaning that j can take any integer values (or half-integer if s is an half-integer) between $l + s$ and $|l - s|$.
- m_j : Quantum number associated with the projection of $\hat{\mathbf{J}}$ on a quantization axis: $m_j = -j, -j +$

$1, \dots, j-1, j$. We thus have $(2j+1)$ possible values for $m_j = m_l + m_s$. The state of the electron is then defined by 5 quantum numbers: n, l, s, j and m_j .

The Hunds rule gives the electronic distribution for increasing energies based on the levels n and sub-levels l for the atom in its ground state, i.e. in its lowest energy

$$E_{1s} < E_{2s} < E_{2p} < E_{3s} < E_{3p} < E_{4s} < E_{3d} < E_{4p} < E_{5s} < E_{4d} < E_{5p} \dots$$

with the filling of the levels such as

$$1s^2 \ 2s^2 \ 2p^6 \ 3s^2 \ 3p^6 \ 4s^2 \ 3d^{10} \ 4p^6 \ 5s^2 \ 4d^{10}.$$

The Cd atom, in its ground state, has 48 electrons distributed among 5 atomic layers: 2 electrons on the $n = 1$ layer, 8 electrons on the $n = 2$ layer, 18 electrons on the $n = 3$ layer, 18 electrons on the $n = 4$ layer and 2 electrons on the $n = 5$ layer. It is then said that the Cadmium atom (as the Mercury atom) is a *two-electron atom* because its outer shell only has 2 electrons.

B. Energy levels of an atom

Within this classification, we must also take into account the various spin-orbit couplings between the electrons of an atom. To note the energy state of an atom in a given electronic configuration, we use the usual atomic spectroscopic notation: $^{2S+1}L_J$ with

- L : Orbital quantum number of the electrons of the atom. The values $L = 0, 1, 2, 3, 4, \dots$ are noted, as for electronic levels, by a letter: $S, P, D, F \dots$
- S : Total spin quantum number electrons of the atom. The multiplicity of the state is $(2S+1)$.
- J : Total angular momentum quantum number (spin-orbit coupling) of the atom $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$.
- M_J : Quantum number associated with $\hat{\mathbf{J}}$ that can take the following values: $M_J = -J, -J+1, \dots, J-1, J$. We thus have $(2J+1)$ possible values.

Example:

1D_2 is a level for which $L = 2, S = 0$ and $J = 2$. For a given atomic state, there is one and only one corresponding energy value, whether or not an external field is applied, but several states can have the same energy: they are said to be degenerate. For an "isolated" atom, its energy only depends on the quantum numbers L, S and J and does not depend on M_J . An energy level characterized by a given value of J is thus $(2J+1)$ times

configuration, the energy E_{nl} of an orbital increases with $(n+l)$ and, for a given value of $(n+l)$, with n . The number of electrons in a given sub-layer (n, l) is, at the maximum $2(2l+1)$.

Example:

The Cadmium atom (Cd) has 48 electrons. The energy levels in increasing order of energy are

degenerate. For instance, the 1D_2 level will split into 5 energy levels under the influence of a magnetic field B . The distance between two consecutive energy levels is proportional to B .

C. Transition between two energy levels of Cadmium

The frequency of an emission line is linked to the energy difference between two levels. Let us choose the Oz axis aligned with the magnetic field \mathbf{B} . It is conceivable that a coherent description would demand to take into account all the magnetic effects simultaneously. For this reason the Hamiltonian \hat{H}_0 of the atom must include the spin-orbit coupling, responsible for the fine structure of the atom. That being said, \hat{H}_0 can be written as

$$\hat{H}_0 = \hat{H}_{\text{electrostatic}} + a(r)\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}. \quad (1)$$

It is not necessary here to specify the function $a(r)$. In the presence of a magnetic field, and by neglecting the diamagnetic term, the Hamiltonian must be completed by \hat{V} (Zeeman Hamiltonian)

$$\hat{V} = -\frac{e}{2m} \left(\hat{L}_z + g_e \hat{S}_z \right) B, (g_e = 2.0023 \dots \simeq 2) \quad (2)$$

such that the Hamiltonian in the presence of magnetic field is

$$\hat{H} = \hat{H}_{\text{electrostatic}} + a(r)\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} - \frac{e}{2m} \left(\hat{L}_z + g_e \hat{S}_z \right) B. \quad (3)$$

Obviously, for a very weak field, the coupling \hat{V} is much smaller than the spin-orbit term, even if it is itself smaller than the first two terms of \hat{H}_0 . It is this situation that we will be interested in here which is referred to as, using the orthodox terminology, the Zeeman effect. Once we have made that choice, the eigenstates in a first order approximation are those of \hat{H}_0 and are written $|n, J, M_J, L, S\rangle$. They are common to $\{\hat{H}_0, \hat{\mathbf{J}}^2, \hat{J}_z, \hat{\mathbf{L}}^2, \hat{\mathbf{S}}^2\}$.

Except in the case of an accidental degeneracy, each level is thus $(2J+1)$ times degenerate ($M_J = -J, -J+1, \dots, J$).

$1, \dots, J-1, J$) and we must then use the stationary perturbation theory for a degenerate level: we only consider the matrix associated to \hat{V} in the subspace that we are interested in, forget the rest and diagonalize. In order to find the \hat{V} matrix in this subspace we can use two methods:

- \hat{V} can be written in terms of \hat{L}_z and \hat{S}_z , and as such its matrix in the $\{|n, L, M_L, M_S\rangle\}$ basis is easily found. One must then use the relations between this basis and the $\{|n, J, M_J, L, s\rangle\}$ basis - making use of the Clebsch-Gordan coefficients - in order to deduce the matrix of \hat{V} in the eigen basis of \hat{H}_0 .

- We can also use the Wigner-Eckart theorem, according to which, within a subspace $\{|n, L, M_L, M_S\rangle\}$, $(\hat{\mathbf{L}} + g_e \hat{\mathbf{S}})$ is a vectorial operator proportional to $\hat{\mathbf{J}}$ with a well defined proportionality factor $g_{L,S,J}$:

$$(\hat{\mathbf{L}} + g_e \hat{\mathbf{S}}) = \frac{\langle (\hat{\mathbf{L}} + g_e \hat{\mathbf{S}}) \cdot \hat{\mathbf{J}} \rangle}{J(J+1)\hbar^2} \hat{\mathbf{J}} \equiv g_{L,S,J} \hat{\mathbf{J}}. \quad (4)$$

$g_{L,S,J}$ is a number (called *Landé factor*), that can be written using quantum numbers L , S and J . Writing $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = \frac{1}{2} (\hat{\mathbf{J}}^2 - \hat{\mathbf{L}}^2 - \hat{\mathbf{S}}^2)$ we find

$$(\hat{\mathbf{L}} + g_e \hat{\mathbf{S}}) \cdot \hat{\mathbf{J}} = (\hat{\mathbf{L}} + g_e \hat{\mathbf{S}}) (\hat{\mathbf{L}} + \hat{\mathbf{S}}) = \frac{1}{2} [(g_e + 1)\hat{\mathbf{J}}^2 - (g_e - 1)\hat{\mathbf{L}}^2 + (g_e - 1)\hat{\mathbf{S}}^2],$$

and thus

$$\langle (\hat{\mathbf{L}} + g_e \hat{\mathbf{S}}) \cdot \hat{\mathbf{J}} \rangle \hat{\mathbf{J}} = \frac{1}{2} [(g_e + 1)J(J+1)\hbar^2 - (g_e - 1)L(L+1)\hbar^2 + (g_e - 1)S(S+1)\hbar^2] \hat{\mathbf{J}}$$

If we now chose $g_e = 2$, the Landé factor $g_{L,S,J}$ is then a simple rational number

$$g_{L,S,J} \simeq 1 + \frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)} \quad (5)$$

The direct consequence of equations 2 and 4 is that \hat{V}_{ap} , the projection of the matrix of \hat{V} , is already diagonal. The first order corrections for the energy can be read directly from the diagonal and are thus

$$\varepsilon_{n,J}^{(1)} = +g_{L,S,J} \frac{eB}{2m} M_J \hbar \equiv -g_{L,S,J} M_J \mu_B B. \quad (6)$$

The magnetic field completely lifts the degeneracy and splits each atomic level into $(2J+1)$ equidistant levels. The energy difference between two consecutive levels is proportional to the intensity of the external field B . This degeneracy lifting results from the fact that the magnetic field, unlike the electric field, breaks the time reversal invariance: two states with M_J of opposite sign have opposite energies in relation with the zero-field energy.

$$\Delta E = h\Delta\nu = \left(\frac{e\hbar}{2m}\right) B(g_{J'}M_{J'} - g_J M_J) = |\mu_B| B(g_{J'}M_{J'} - g_J M_J)$$

where g_J is the Landé factor that can be written using L , S and J using relation 5

$$g_J = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}$$

and where the quantity $\mu_B = -\frac{e\hbar}{2m}$ ($\mu_B = -9.27408 \cdot 10^{-24} \text{J.T}^{-1}$) is called the *Bohr magneton* (For more details on the calculation of Landé factors, refer to the appendix at the end of this document).

The frequency of an emission line is linked to the energy difference between two levels of an atom

$$h\nu = |E_{L',S',J'} - E_{L,S,J}|. \quad (7)$$

This transition, called radiative, is only possible if some conditions on the quantum numbers are met. These conditions are called *selection rules*. They can be established by using the Wigner-Eckart theorem:

$$\Delta L = \pm 1 \quad \Delta J = 0, \pm 1 \quad \Delta M_J = 0, \pm 1.$$

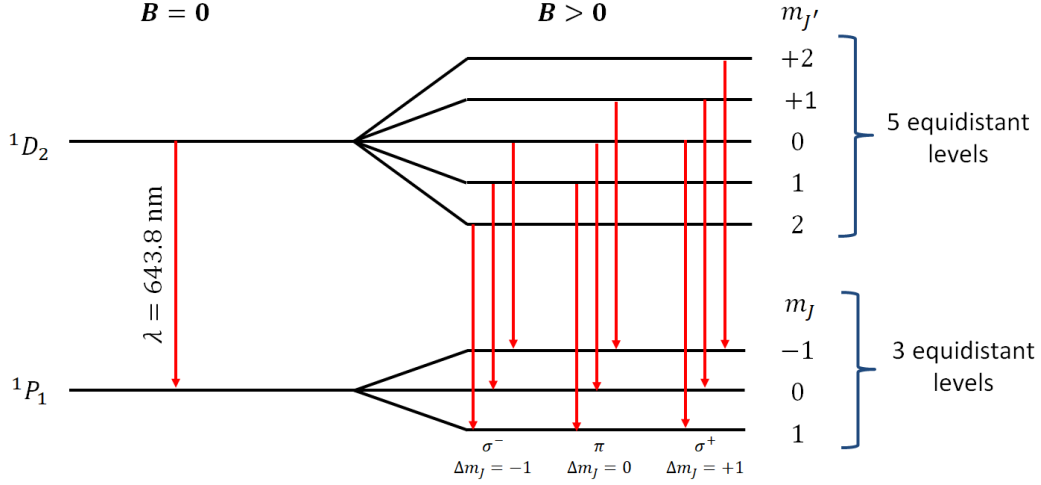


FIG. 2. Zeeman effect on the $^1D_2 \rightarrow ^1P_1$ transition (red line of Cadmium at 643.847 nm).

The red line of Cadmium is at 643.847 nm and corresponds to a transition from the 1D_2 ($L = 2$, $S = 0$, $J = 2$) level to the 1P_1 ($L = 1$, $S = 0$, $J = 1$) level. The Landé factor of these levels is equal to

$$^1D_2 \rightarrow g_{J'} = 1 \quad ^1P_1 \rightarrow g_J = 1.$$

When a radiating Cadmium atom is placed in an external magnetic field, there is then a degeneracy lifting leading to a splitting of the lines between various components.

There are thus in total 9 allowed transitions, but only three different frequencies and as such three different lines. These transitions can be interpreted without using the spin as in this case $S = 0$ and thus $J = L$. This is called the "normal" Zeeman effect (see fig. 2).

- The $M_J = 0$ transitions have the same wavelength as the red line of Cadmium without magnetic field. They are rectilinearly polarized (π transition), parallel to the applied magnetic field. They thus do not appear when observing along the direction of the field B (see figure 3).
- The $\Delta M_J = \pm 1$ transitions have a wavelength shift $\Delta\lambda$. They are called σ transitions (σ^+ and σ^- respectively). Their polarization is elliptical, they appear circular when observing along the direction of the field B and rectilinear when observing in a direction perpendicular to the direction of B (see figure 3).

In practice, the wavelength variations $\Delta\lambda$ that we will measure are of the order of 0.05 Å. The Fabry-Pérot interferometer is the ideal instrument with which to observe such variations.

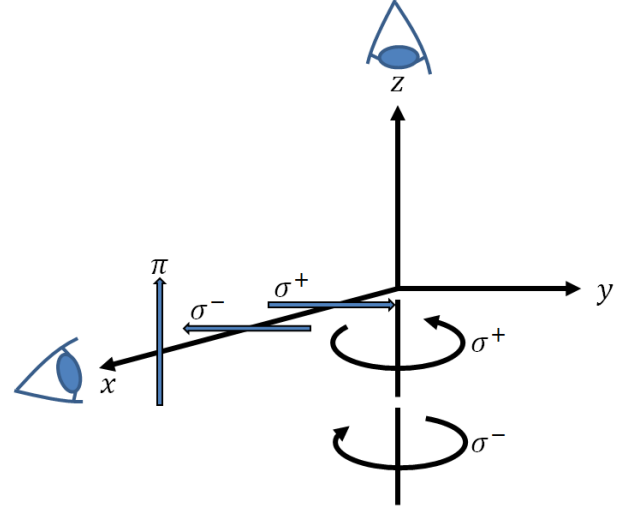


FIG. 3. Definition of the polarizations

III. EXPERIMENT AND MANIPULATIONS

A. Reminders on the Fabry-Pérot interferometer

The Fabry-Pérot interferometer consists of two plane mirrors with reflectivity R close to 1. Interferences happen between the waves reflecting on both faces. They are multiple waves interferences localized at infinity. As such we will observe them at the focus of a lens or by using a camera with a lens.

For an incidence angle i , the difference in distance traveled of the two light rays successively reflected can be written as

$$\delta = 2nd \cos i$$

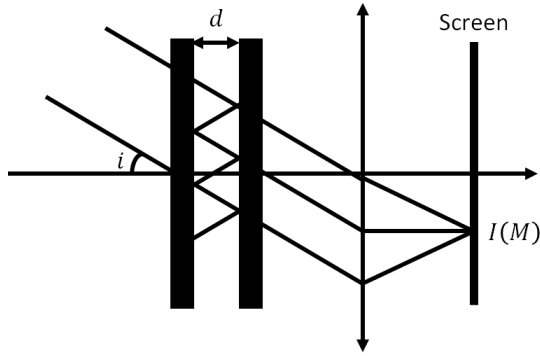


FIG. 4. Fabry-Pérot in a parallel air plates configuration ($d = 12.02$ mm for a tunable FP, $d = 3$ mm for a commercial FP)

where d is the distance between the mirrors and n the refractive index of the medium separating the two mirrors (in this experiment it is air for which $n = 1$).

B. Radius of the Zeeman effect rings

In a Fabry-Pérot interferometer, the measurements are made on the bright rings, thus those with a delay equal to $\delta = k\lambda = 2d \cos i_k$ (with k an integer) obtained for a specific value of the incidence angle $i = i_k$ (same inclination rings). [1]

We will note k_0 the interference order of the first bright ring observed starting from the center and we will note that, from the formula, when the angle i_k increases (or the radius of the rings increases) the order of k diminishes (see fig. 7). The interference order of the center, at $i_k = 0$, does not have to be integer or half-integer, but can take a value slightly larger than k_0 , without nevertheless be higher than the next order at $k_0 + 1$. We will then write this particular order at the center $k_0 + \varepsilon$ with $0 \leq \varepsilon < 1$



FIG. 5. Experimental setup.

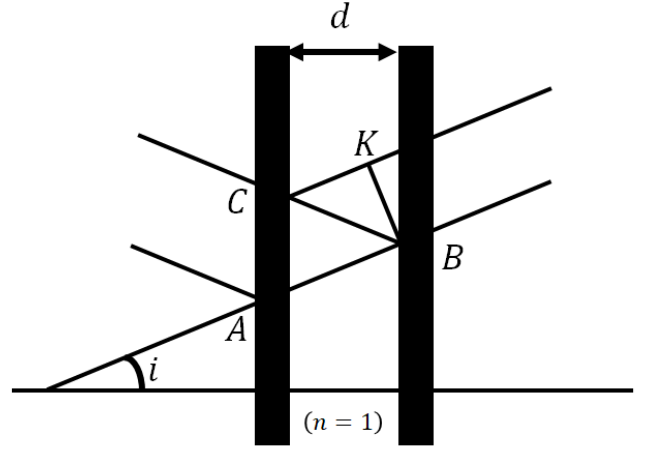


FIG. 6. Calculation of the delay.

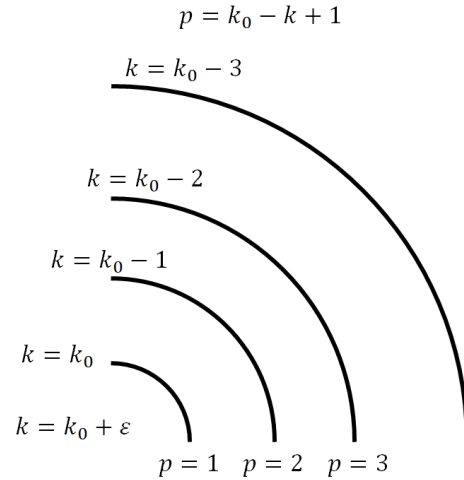


FIG. 7. Ring numbering.

as the fractional excess. We then have the relation

$$k_0 + \varepsilon = \frac{2d}{\lambda}. \quad (8)$$

During the course of the experimental measurements on the radii, it is often preferable, in order to simplify the numbering of the rings, to introduce a new numbering p such as presented in fig. 7.

If we write f the focus of the camera's lens and G the magnification between the CCD sensor of the camera and any support (screen, printer...) we have the exact relation

$$\cos i = \frac{1}{\sqrt{1 + \left(\frac{R}{b}\right)^2}}, \quad (9)$$

where R is the radius of the rings directly measured on the chosen support (in practice it is obviously preferable

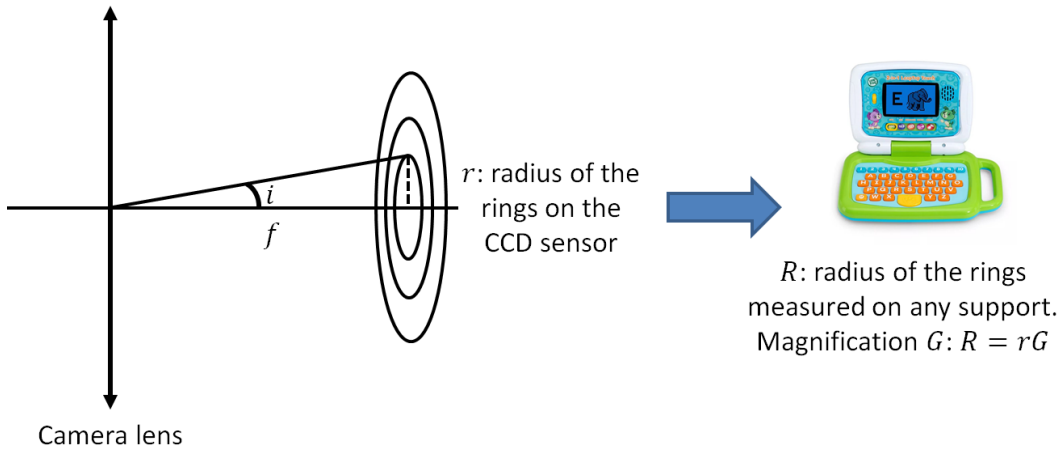


FIG. 8. Observation of rings on the camera.

to measure the diameter of the rings). We have used $b = fG$.

We then obtain, using the relation $\delta = k\lambda = 2d \cos i$, the exact expression for the radius

$$R^2 = b^2 \left[\left(\frac{2d}{k\lambda} \right)^2 - 1 \right]. \quad (10)$$

In order to eliminate the constant term b , always hard to estimate and often with little precision, it is wise to determine the wavelength variation $\Delta\lambda$ from a graphical analysis of the various radii R_{λ_1} , R_λ and R_{λ_2} . Relation 10 allows us to compute the spectral gap $\Delta\lambda = \lambda - \lambda_2 = \lambda_1 - \lambda$ due to the Zeeman effect

$$\frac{R_\lambda^2}{R_{\lambda_2}^2 - R_{\lambda_1}^2} = -\frac{k^2\lambda^3}{(4d)^2\Delta\lambda} + \frac{\lambda}{4\Delta\lambda}. \quad (11)$$

Or, by only considering R_{λ_1} and R_{λ_2}

$$\frac{R_{\lambda_1}^2 + R_{\lambda_2}^2}{2(R_{\lambda_1}^2 - R_{\lambda_2}^2)} = -\frac{k^2\lambda^3}{(4d)^2\Delta\lambda} + \frac{\lambda}{4\Delta\lambda}. \quad (12)$$

It is often more convenient to work with the p order given by the relation

$$p = k_0 - k + 1 = \frac{2d}{\lambda} - \varepsilon - k + 1.$$

We then have

$$k = \frac{2d}{\lambda} - \varepsilon - p + 1 = \frac{2d}{\lambda} \left[1 - \frac{\lambda}{2d}(p - 1 + \varepsilon) \right] = \frac{2d}{\lambda}(1 - a)$$

with $a = \frac{\lambda}{2d}(p - 1 + \varepsilon)$.

Using a limited expansion on k^2 we can then write

$$\frac{R_\lambda^2}{R_{\lambda_2}^2 - R_{\lambda_1}^2} = \frac{R_{\lambda_1}^2 + R_{\lambda_2}^2}{2(R_{\lambda_1}^2 - R_{\lambda_2}^2)} \simeq \frac{\lambda^2}{4d\Delta\lambda}(p - 1 + \varepsilon). \quad (13)$$

Because the fractional excess ε is hard to estimate, you will draw a diagram representing $\frac{R_\lambda^2}{R_{\lambda_2}^2 - R_{\lambda_1}^2}$ or $\frac{R_{\lambda_1}^2 + R_{\lambda_2}^2}{2(R_{\lambda_1}^2 - R_{\lambda_2}^2)}$ as a function of p . The slope of the resulting straight line then allows to compute the spectral gap $\Delta\lambda$ that we are looking for.

C. Study of the normal Zeeman effect

In the first part of this study, we will focus on the red line of Cadmium that corresponds to the transition between the 1D_2 and 1P_1 levels.

Q1: Before starting the study of the Zeeman effect, you must first make a precise calibration of the magnetic field created by the Helmholtz coils. To do so, gently remove the Cadmium lamp from its holder and measure the magnetic field B using the gaussmeter as a function of the current I that is injected in the coils (WARNING, do not exceed $I_{\max} = 10 \text{ A}$). Give the relation $B(I)$.

Q2: Install and adjust the various elements on the optical bench as described by fig. 9. The diaphragm that is illuminated by the Cd lamp behaves as a point source. The L_1 lens of 50 mm focus is optional. The additional lens L_2 is included in the Fabry-Pérot holder. To observe the 643.8 nm line, a red filter must be added in the space provided. A lens L_3 inserted in front of the camera allows to focus and observe the rings. In the first stage, do not include the analyzer. Using the image analysis software, mark the position (diameter) of the various interference orders discussed in the introduction.

Q3: Place yourself in a geometry where the magnetic field is perpendicular to the light beam. Study the effect of the magnetic field on the rings and use the analyzer to identify the various lines.

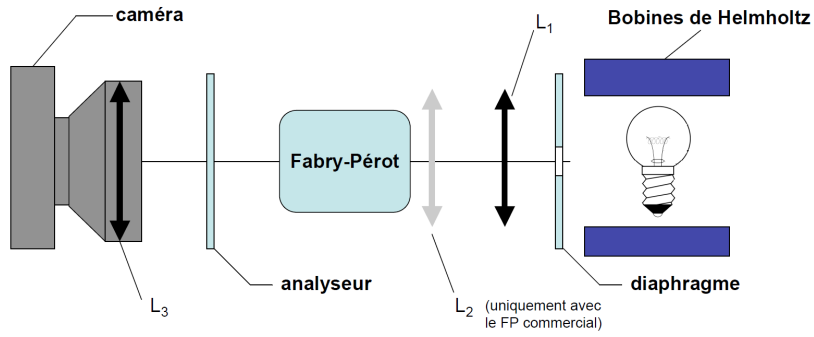


FIG. 9. Experimental device.

Q4: Place yourself in a geometry where the magnetic field is parallel to the light beam. Study the effect of the magnetic field on the rings and use the analyzer and the quarter-wave plate to identify the various lines.

Q5: Measure the splitting of the Zeeman levels as a function of the magnetic field (WARNING, do not exceed $I_{\max} = 10$ A). Deduce the value of the Bohr magneton μ_B . Discuss the various sources of uncertainties.

D. Study of the anomalous Zeeman effect

In the second part of this study, we will focus on the green line of Cadmium that corresponds to the transition between the 3S_1 and 3P_2 levels, $\lambda = 508.588$ nm.

Q6: What are the values of L , S and J for those two levels? Why do we speak of an anomalous Zeeman effect? Deduce the values of the Landé factors.

Q7: Determine the Zeeman structure of these two levels, the possible transitions as well as their energies and polarization.

Q8: Using the equipment at your disposal, study the anomalous Zeeman effect. In particular, you will show that it is possible to identify all the lines by playing with the orientation of the magnetic field, the polarization analyzer as well as the quarter wave plate.

Appendix A: Landé factors of the isolated atoms

NB: Sign convention used for the Landé factors:

The nuclear Landé factor is considered to be positive when the nuclear angular momentum and the angular momentum are pointing in the same direction. It is negative otherwise. This convention is opposite to that used for electronic momenta for which the Landé factors g_J is considered to be positive while the magnetic momenta (orbitals or spins) point in the direction opposite to the corresponding angular momenta.

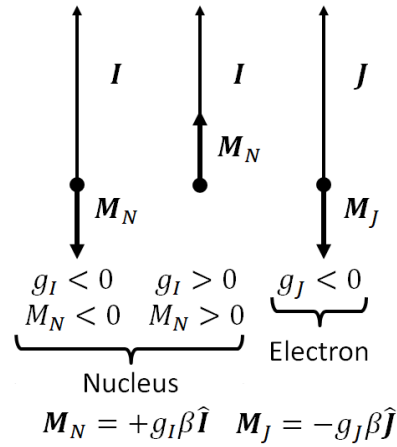


FIG. 10. Landé factors

The sign conventions vary from one author to the other and the tables used. We recommend to pay attention to this point. In the same way, the role of \hbar is a conventional one, it is possible to rigorously define the electronic Bohr magneton as $\mu_B = \frac{-e\hbar}{2m}$ ($\mu_B = -9.27408 \cdot 10^{-24}$ J.T⁻¹) and in this case $\hat{M}_J = g_J \frac{\mu_B}{\hbar} \hat{J}$.

Some write using this definition of μ_B : $\hat{M}_J = g_J \mu_B \hat{J}$. In this case the \hbar originates from the Bohr magneton μ_B and the values of the angular momentum \hat{J} can be written without any \hbar . Everything is possible if you know what you are doing! You must thus choose a convention, report it and keep it. The main point is obviously that the final calculated angular momentum should have the correct sign and the physical dimension of... an angular momentum.

1. Calculation of the electronic Landé factor g_e

This quantity is sometimes written g_J .

$$\hat{\mu}_L = \gamma \hat{L} = g_L \left(\frac{-e}{2m_e} \right) \hat{L} = g_L \frac{\mu_B}{\hbar} \hat{L} \Rightarrow g_L = 1$$

$$\hat{\mu}_Q = \gamma \hat{\mathbf{L}} = g_S \left(\frac{-e}{2m_e} \right) \hat{\mathbf{Q}} = g_Q \frac{\mu_B}{\hbar} \hat{\mathbf{Q}} \Rightarrow g_S = 2$$

where $\langle \hat{\mathbf{L}} \cdot \hat{\mathbf{J}} \rangle_{E_0, L, S, J}$ and $\langle \hat{\mathbf{S}} \cdot \hat{\mathbf{J}} \rangle_{E_0, L, S, J}$ respectively refer to the mean values the operators $\hat{\mathbf{L}} \cdot \hat{\mathbf{J}}$ and $\hat{\mathbf{S}} \cdot \hat{\mathbf{J}}$ for the states of the atom belonging to (E_0, L, S, J) . However, we can write

$$\hat{\mu}_e = \hat{\mu}_L + \hat{\mu}_S = g_e \frac{\mu_B}{\hbar} (\hat{\mathbf{L}} + \hat{\mathbf{S}}) = g_e \frac{\mu_B}{\hbar} \hat{\mathbf{J}} = \frac{\mu_B}{\hbar} (g_L \hat{\mathbf{L}} + g_S \hat{\mathbf{S}}) \quad \hat{\mathbf{L}} \cdot \hat{\mathbf{J}} = \hat{\mathbf{L}} \cdot (\hat{\mathbf{L}} + \hat{\mathbf{S}}) = \hat{\mathbf{L}}^2 + \frac{1}{2} (\hat{J}^2 - \hat{L}^2 - \hat{S}^2),$$

We can then write $g_e \hat{\mathbf{J}} = g_L \hat{\mathbf{L}} + g_S \hat{\mathbf{S}}$.

as well as

Within the subspace (E_0, L, S, J) , eigenspace of $\{\hat{J}^2, \hat{J}_z\}$, we have from the projection theorem

$$\hat{\mathbf{S}} \cdot \hat{\mathbf{J}} = \hat{\mathbf{S}} \cdot (\hat{\mathbf{L}} + \hat{\mathbf{S}}) = \hat{\mathbf{S}}^2 + \frac{1}{2} (\hat{J}^2 - \hat{L}^2 - \hat{S}^2),$$

$$\hat{\mathbf{L}} = \frac{\langle \hat{\mathbf{L}} \cdot \hat{\mathbf{J}} \rangle_{E_0, L, S, J}}{J(J+1)\hbar^2} \hat{\mathbf{J}}, \text{ and } \hat{\mathbf{S}} = \frac{\langle \hat{\mathbf{S}} \cdot \hat{\mathbf{J}} \rangle_{E_0, L, S, J}}{J(J+1)\hbar^2} \hat{\mathbf{J}}$$

We then have

$$\langle \hat{\mathbf{L}} \cdot \hat{\mathbf{J}} \rangle_{E_0, L, S, J} = L(L+1)\hbar^2 + \frac{\hbar^2}{2} [J(J+1) - L(L+1) - S(S+1)]$$

and

$$\langle \hat{\mathbf{S}} \cdot \hat{\mathbf{J}} \rangle_{E_0, L, S, J} = L(L+1)\hbar^2 + \frac{\hbar^2}{2} [J(J+1) - L(L+1) - S(S+1)]$$

from which

$$\begin{aligned} \hat{\mathbf{L}} &= \frac{\langle \hat{\mathbf{L}} \cdot \hat{\mathbf{J}} \rangle}{J(J+1)\hbar^2} \hat{\mathbf{J}} = \frac{L(L+1)\hbar^2 + \frac{\hbar^2}{2} [J(J+1) - L(L+1) - S(S+1)]}{J(J+1)\hbar^2} \hat{\mathbf{J}} \\ &= \frac{J(J+1) - S(S+1) + L(L+1)}{2J(J+1)} \hat{\mathbf{J}} \end{aligned}$$

$$\hat{\mathbf{S}} = \frac{\langle \hat{\mathbf{S}} \cdot \hat{\mathbf{J}} \rangle}{J(J+1)\hbar^2} \hat{\mathbf{J}} = \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \hat{\mathbf{J}}.$$

By reporting the values of $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ as a function of $\hat{\mathbf{J}}$ in the equality $g_e \hat{\mathbf{J}} = g_L \hat{\mathbf{L}} + g_S \hat{\mathbf{S}}$ we get

$$g_e = g_L \frac{J(J+1) - S(S+1) + L(L+1)}{2J(J+1)} + g_S \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}.$$

Because $g_L = 1$ and $g_S \simeq 2$, that can be written as

$$g_e \simeq 1 + \frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)}.$$

2. Calculation of the atomic Landé factor g_F

If we now turn our attention to the nuclear magnetic momentum

$$\hat{\mu}_N = \gamma \hat{\mathbf{I}} = g_N \left(\frac{-e}{2m_p} \right) \hat{\mathbf{I}} = g_N \frac{\mu_N}{\hbar} \hat{\mathbf{I}},$$

where μ_N is now the nuclear Bohr magneton. We note that

$$\hat{\mu}_N = g_N \left(\frac{-e}{2m_p} \right) \frac{m_e}{m_p} \hat{\mathbf{I}} = g_N \left(\frac{-e}{2m_e} \right) \frac{m_e}{m_p} \hat{\mathbf{I}} = g_N \frac{\mu_B}{\hbar} \frac{m_e}{m_p} \hat{\mathbf{I}} \simeq \frac{g_N}{1836} \frac{\mu_B}{\hbar} \hat{\mathbf{I}}$$

$$\hat{\mu}_N = g_I \frac{\mu_B}{\hbar} \hat{\mathbf{I}}, \text{ with } g_I = \frac{g_N}{1836} \Rightarrow |\hat{\mu}|_N \ll |\hat{\mu}_B|,$$

keeping in mind that this time g_I can be **either positive or negative** (unlike the electronic angular momentum where g_e is always positive).

Taking into account the nuclear momentum, the total magnetic momentum of the atom can now be written as

$$\hat{\mu} = \hat{\mu}_e + \hat{\mu}_N = g_F \frac{\mu_B}{\hbar} (\hat{\mathbf{J}} + \hat{\mathbf{I}}) = g_F \frac{\mu_B}{\hbar} \hat{\mathbf{F}} = \frac{\mu_B}{\hbar} (g_e \hat{\mathbf{J}} + g_I \hat{\mathbf{I}})$$

and thus

$$g_F \hat{\mathbf{F}} = g_e \hat{\mathbf{J}} + g_I \hat{\mathbf{I}}.$$

As we did previously, we now use the projection theorem on the sub-space associated to $\{\hat{F}^2, \hat{F}_z\}$ to compute $\hat{\mathbf{J}}$ and $\hat{\mathbf{I}}$ as a function of $\hat{\mathbf{F}}$

$$\begin{aligned} \hat{\mathbf{J}} &= \frac{\langle \hat{\mathbf{J}} \cdot \hat{\mathbf{F}} \rangle}{F(F+1)\hbar^2} \hat{\mathbf{F}} = \frac{F(F+1) - I(I+1) + J(J+1)}{2F(F+1)} \hat{\mathbf{F}} \\ \hat{\mathbf{I}} &= \frac{\langle \hat{\mathbf{I}} \cdot \hat{\mathbf{F}} \rangle}{F(F+1)\hbar^2} \hat{\mathbf{F}} = \frac{F(F+1) - I(I+1) - J(J+1)}{2F(F+1)} \hat{\mathbf{F}}. \end{aligned}$$

By inputting those mathematical expressions in $g_F \hat{\mathbf{F}} = g_e \hat{\mathbf{J}} + g_I \hat{\mathbf{I}}$ we obtain

$$g_F = g_e = \frac{F(F+1) - I(I+1) + J(J+1)}{2F(F+1)} + g_I \frac{F(F+1) + I(I+1) - J(J+1)}{2F(F+1)}$$

that can be approximated by

$$g_F \simeq g_e = \frac{F(F+1) - I(I+1) + J(J+1)}{2F(F+1)},$$

the simplification coming from the fact that $g_I \simeq g_e \frac{m_e}{m_p} \ll 1$.

3. Remarks on nuclear magnetism

The situation is not as simple when talking about nuclear magnetism compared to electronic magnetism. Indeed, in an atomic nucleus the interaction forces are not central ones and we thus cannot separate neutron magnetism from proton magnetism (both the neutron and the proton have spin $\frac{1}{2}$). Nuclear magnetism thus comes from an experimental observation

- All isotopes with an even mass number A and an even atomic number Z (even-even nuclei) have a zero nuclear spin and a zero nuclear angular momentum. Examples: ${}^4\text{He}$, ${}^{16}\text{O}$, ${}^{20}\text{Ne}$, etc.
- All isotopes with an even mass number A and odd atomic number Z (even-odd nuclei) have an integer nuclear spin. Examples: ${}^2\text{D}$ ($I = 1$), ${}^6\text{Li}$ ($I = 1$), ${}^{10}\text{B}$ ($I = 3$), etc.
- All isotopes with an odd mass number A have a half integer spin. Examples: ${}^{39}\text{K}$ ($I = \frac{3}{2}$, $g_I = 1.42 \cdot 10^{-4}$, $\mu_N = 2.3 \cdot 10^{-4} |\mu_B|$), ${}^{85}\text{Rb}$ ($I = \frac{5}{2}$), etc.

[1] The delay between the wavefronts of two adjacent rays (for instance AB and CD in figure 6) is

$$\delta = BC + CK$$

where BK is perpendicular to CK . Therefore

$$CK = BC \cos 2i, \text{ and } BC \cos i = d.$$

and we have $\delta = BCK = BC(1 + \cos 2i) = 2BC \cos^2 i = 2d \cos i$. In order to have constructive interferences (bright rings), this delay must be equal to an integer number of wavelengths, i.e. $\Delta = k\lambda = 2d \cos i$ (with k an integer).