Bifurcations, chaos and applications of the Lorenz Equations

Sourabh Choudhary; Reg. No. 201918213

GitHub: https://github.com/sourabh-ch/ph456-computational/tree/master/complex-sys

1. Introduction

The Lorenz equations are a set 3 coupled ordinary differential equations (ODE) in 3 dimensions. They form a deterministic dynamical system, these equations were first proposed by Edward Lorenz in the 1960s as a simplified model of atmospheric convection (E. Lorenz, 1960). The parameters of these equations have no inherent random aspect. However, their behavior over time is challenging to predict. As per the equations shown below, with a given set of parameters $\sigma,\ \rho,\ \beta$ combined with the initial conditions for the state space (x, y, z) the behavior of the system can be studied.

$$\frac{dx}{dt} = \sigma\,(\,y - x\,)$$

$$\frac{dx}{dt} = x\,(\,\rho - x\,)$$

$$\frac{dx}{dt} = xy - \ \beta$$

The system exhibits periodic behavior for some parameters, while for other parameters it behaves like an attractor bounded around a certain region but exhibiting non periodic and non-convergent behavior. There are also other parameters for which the system converges to one point or goes towards infinity (C. Moler, 2004). In this essay we will look first look at a python model of the above equations with a popular set of parameters with miniscule variation in the initial state vector. Followed by this we will look at some applications of these equations in engineering and economics.

2. Python model

The python code used in for this assignment can be found in the GitHub link given below the title of this essay. It is available in the form of a Jupyter notebook.

A python model of the parameters $\sigma = 10, \rho = 28$, and, $\beta = 8/3$ which is a popular version of parameters to showing a state vector resembling wings of a butterfly is shown in the Figure 1. Plots shown in the figure have been made with two sets of initial position, the first diagram is for the initial state vector of x, y, z = (1.0, 1.0, 1.0) and the second one with a difference of +0.001 in the z co-ordinate with starting position x, y, z = (1.0, 1.0, 1.001). This set of parameters exhibit a bounded, chaotic behavior. The plot was made by integrating the three ODEs at time steps of 0.01 s over a total of 50 s.

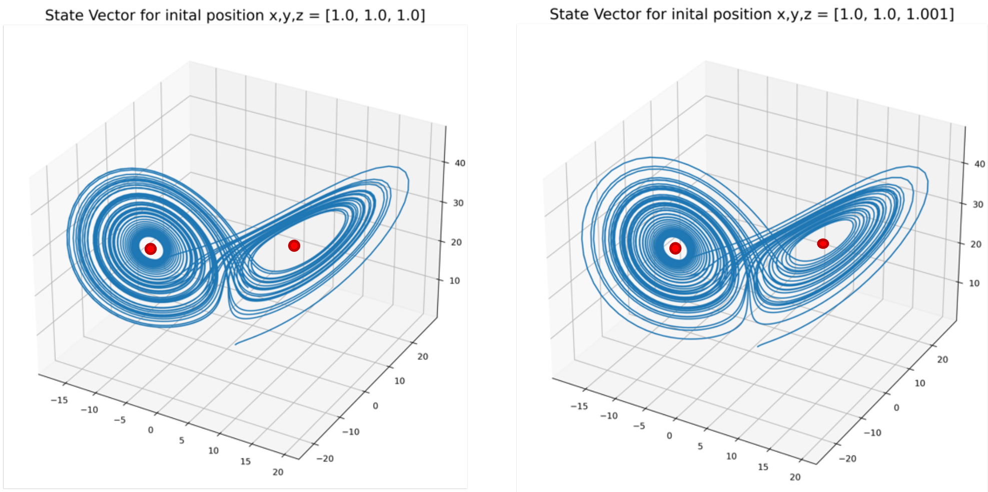

Figure 1: Left: State vector with initial condition x, y, z = (1.0, 1.0, 1.0) for 50 s at a resolution of 0.01 s.
Right: State vector with initial condition x, y, z = (1.0, 1.0, 1.001) for 50 s at a resolution of 0.01 s.

In Figure 1 we can observer that the evolution of the state vector is chaotic, one simple way of inferring this is by counting the bands at the top right corner of the wings and observing the gaps in between these bands. In the 3D plot, the trajectory of the state vector never overlaps, however for the time period under observation here we can see that its behavior bounded and loops around the two fixed red points in space.

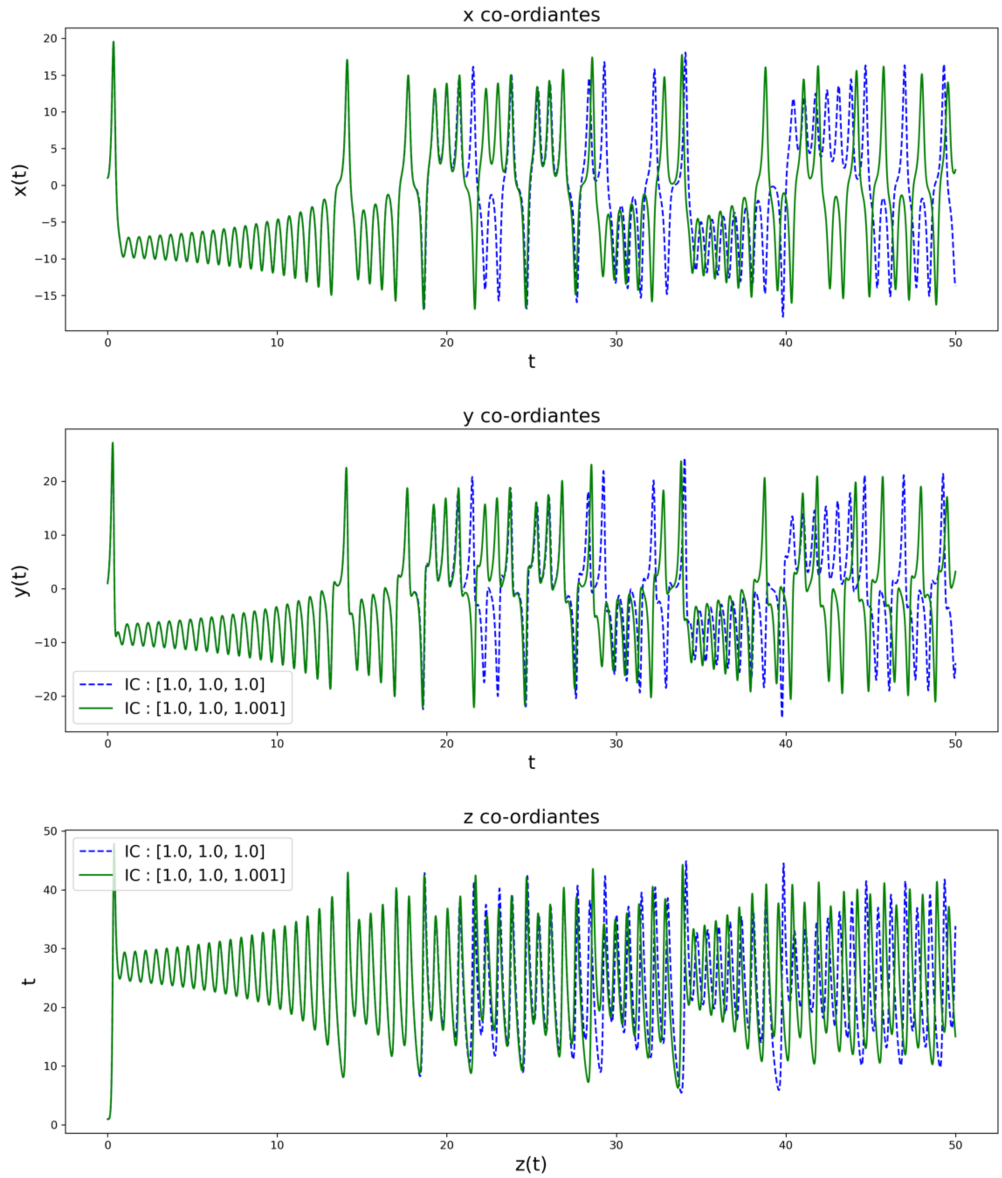

Figure 2: Evolution of x, y, z co-ordinates over time for both initial conditions
Top: Evolution of the x-coordinate of the state vector over 50 s time at a resolution of 0.01 s
Middle: Evolution of the y-coordinate of the state vector over 50 s time at a resolution of 0.01 s
Bottom: Evolution of the z-coordinate of the state vector over 50 s time at a resolution of 0.01 s

In Figure 2 we can further visualize the differences in the evolution of the trajectory by such a minuscule change in one initial variable by looking at the x, y, z components of both the state vectors over time. The figure shows the evolutions of each dimension over time with the green line showing the evolution of the variables with initial conditions of (1.0, 1.0, 1.0) and the dotted blue line shows the evolution of initial condition (1.0,1.0,1.001), observing the counts and amplitude of the aperiodic peaks and trough helps understand the characteristic chaotic behavior of the system and its sensitivity to initial conditions. More exotic examples of Lorenz equations exhibiting a periodic and stable system can be found in a comprehensive study (Colin Sparrow, 1982) that explores the characteristics of these systems at a deeper level.

From this example we can understand how uncertainty in measurements which, no matter how sophisticated, always have a certain limit for precision. These uncertainties will affect the system modelled after these measurements and it would only an approximation of reality.

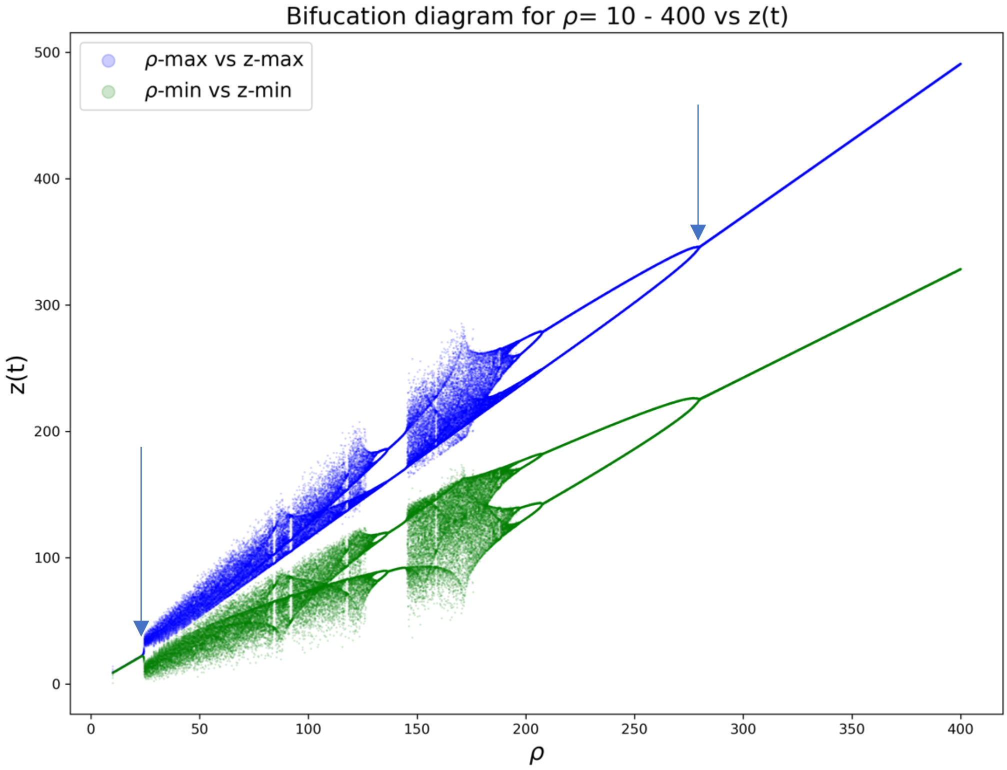

Figure 3: Shows a bifurcation diagram for a Lorenz system with a varying value for ρ

In Figure 3 a bifurcation plot of the chaotic system with initial position x, y, z = (1.0, 1.0, 1.0) and static parameters $\sigma = 10,\ \beta = 8/3$ and a variable parameter $\rho = [10, 400]$ is shown. For a time period of 10 s with a time step size of 0.001 s and starting with a initial value of $\rho = 10$ the maximum and the minimum value of z (t) and ρ was stored. Incrementing ρ by 0.1 for each iteration and plotting the corresponding $\rho - max$, $z(t) - max$ and $\rho - min$ and $z(t) - min$ are plotted in blue and green respectively. Marked with blue arrows in the figure we can see the system bifurcating and entering a chaotic state and then at the 2nd arrow entering again to a bounded state for $z(t)$ vs $\rho(t)$.

These models and their features related to uncertainties have given rise to a new field dedicated to studying chaotic systems with a probabilistic approach. This new approach helps extract more reliable information of the dynamics system being studied (Nicolis and Gaspard, 1994).

3. Applications

There are numerous applications of Lorenz equations. They have been used to study the weather and atmospheric convections, fluid dynamics, etc. Here we will look at 2 applications of these equations for use in communication and in economic analysis.

a. Communication Systems

A secure communication system implemented in non-linear electronic circuits by synchronizing the sender and receiver signals using different methods to adapt Lorenz equations (Chen, Liau and Hou, 2013). These implementations can be used to send as a practical encrypted communication link.

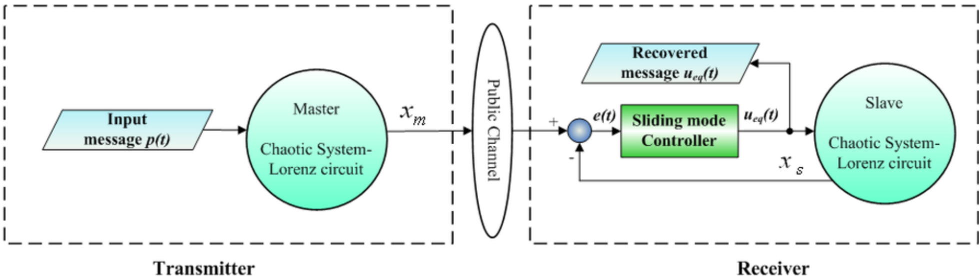

Figure 4: Block diagram of a sender-receiver communication system utilizing a public channel and a slide mode controller to decrypt the receiving message (Chen, Liau and Hou, 2013)

Figure 4 illustrators such a communication system, the main component being the decrypting element, in this case the Slide Mode Controller, which is synchronized via simultaneous transmission of the message and the lorenz state of the master system.

b. Economic Analysis

A model based on the Lorenz equations describing the properties of an urban system situated in a metropolitan was demonstrated by researchers at University of Stuttgart, Germany in 1996. The urban system assumed to be sufficiently isolated from the area such that there are no effects of the short-term changes in the metropolitan area on the subset urban system. The model was implemented to study the changing population, output of the urban system and housing costs (Haag, Hagel and Sigg, 1997).

4. Conclusion

In this essay we first described what the 3 Lorenz equations are and how they emerged. We then looked at some graphical implantations of these and studied the characteristics of the model for different parameters while exploring general ideas that can be derived from them. Finally, we briefly looked at some areas where applications of these models have had an innovative impact.

References

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