PTAS For MIS On Planar Graphs

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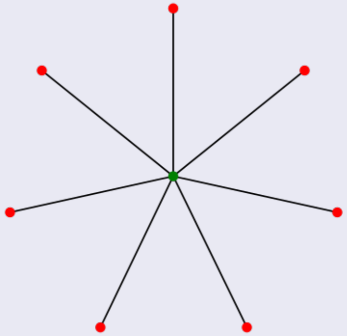
Outline

Outline

Prerequisite: Recall

Example

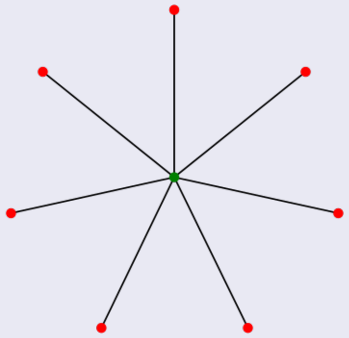
What is $|MIS|$ for S_n?



Prerequisite: Recall

MIS

What is $|MIS|$ for S_n? Ans. $n - 1$



Prerequisite: Recall

Example

What is $|MIS|$ for K_n?

Prerequisite: Recall

MIS

What is $|MIS|$ for K_n? Ans. 1

Prerequisite: Recall

Euler's formula

If a finite, connected, planar graph is drawn in the plane without any edge intersections, and v is the number of vertices, e is the number of edges and f is the number of faces (regions bounded by edges, including the outer, infinitely large region), then $v - e + f = 2$

Outline

Outerplanar Graphs

Definition

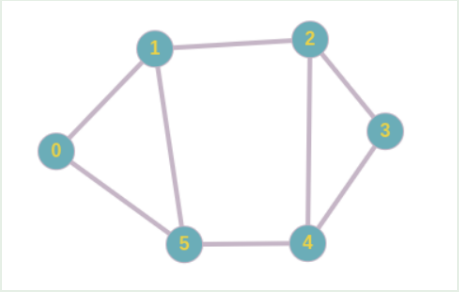
A graph is called outerplanar (1-outerplanar) if it has an embedding in the plane such that every vertex lies on the unbounded face.

In Simple Terms

$\Leftrightarrow$ has a planar drawing for which all vertices belong to the outer face of the drawing.

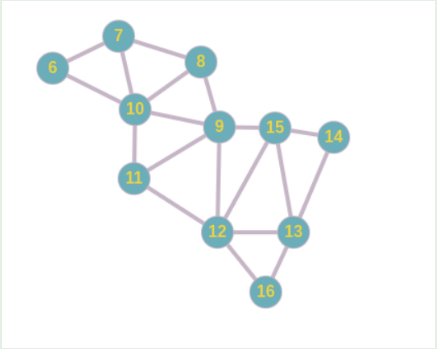
Outerplanar Graphs

Example



Outerplanar Graphs

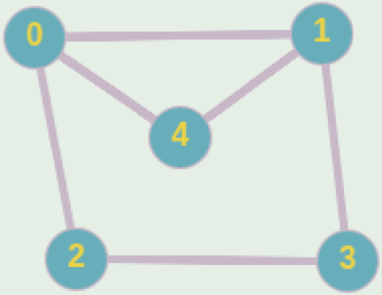
Example



Outerplanar Graphs

Example

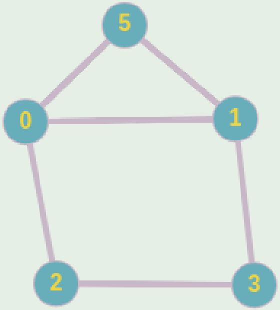
Is this outerplanar?



Outerplanar Graphs

Example

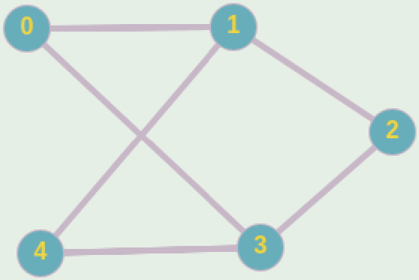
Yes, as it is isomorphic to the following graph.



Outerplanar Graphs

Example

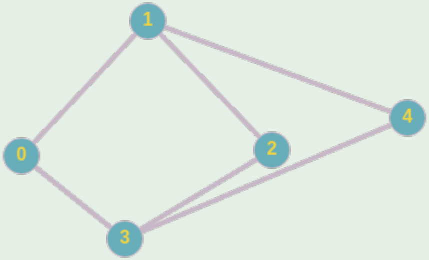
Is this outerplanar?



Outerplanar Graphs

Example

No, but it is planar. *It is 2-outerplanar



Outerplanar Graphs

Definition

Given a planar embedding E of a planar graph G, we define k-level nodes according to the faces of the embedding. Every node on the exterior face is called an exterior or level 1 node. Exterior edges are defined analogously. Call a cycle of level i nodes a *level i face* if it is an interior face in the subgraph induced by the level i nodes. For each *level i face f*, let G_f be the subgraph induced by all nodes placed inside f in this embedding. Then the nodes on the exterior face of G_f are at level $i+1$.

Outerplanar Graphs

Definition

Basically, in each step, starting at 1, remove all nodes on the exterior face. Repeat until there are no nodes left. The level of a node is the step in which it was removed. Using an appropriate data structure for the planar embedding, such as the one used by Lipton and Tarjan [Lipton and Tarjan, 1979], this can be done in linear time. A planar embedding is called k-outerplanar if there are no nodes of level $>$ k, which means the algorithm terminates after at most k steps. A planar graph is called k-outerplanar if it has a k-outerplanar embedding.

Example

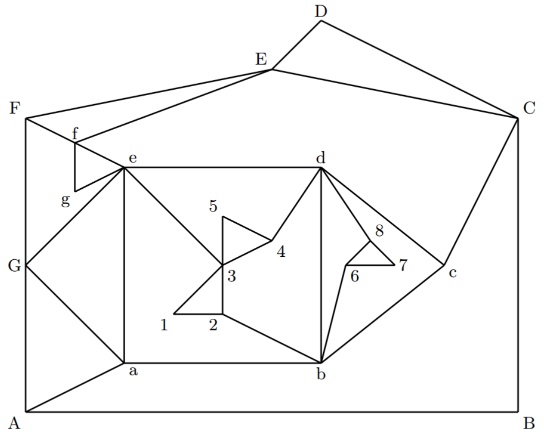


Figure 1. A planar embedding of a 3-outerplanar graph [Baker, 1994].

Outline

Definition

A problem is considered fixed parameter tractable if an algorithm exists which solves the problem in running time $f(k) * n^{O(1)}$ where k is an additional parameter depending upon input and n is the size of the input (in our case we will treat is as number of nodes) and f is an arbitrary function depending only on k.

High Level Plan

We will use this new notion of k-outerplanarity to construct a polynomial time approximation scheme for maximum independent set on planar graphs, which can be easily adapted to other NP-complete problems. The goal is an algorithm that, for a freely chosen but fixed k, solves maximum independent set with an approximation ratio of $k/(k+1)$ and a linear complexity with respect to the number of nodes n. Basically, we want to exactly solve maximum independent on disjunct subgraphs that are k-outerplanar and ignore some nodes in the graph to merge the subgraphs without violating the conditions of the maximum independent set.

PS: Approximation Ratio? For maximization problems it is at least $(1-\epsilon)$ therefore fix $k = (1/\epsilon) - 1$ (do math).

Outline

Bakers Algorithm: Theorem 1

Theorem (Baker, 1994)

Let k be a positive integer. Given a k-outerplanar embedding of a k-outerplanar graph G, an optimal solution for maximum independent set can be obtained in time $O(8^k n)$, where n is the number of nodes.

We won't have time to prove it, but we will prove it for 1-outerplanar graphs.

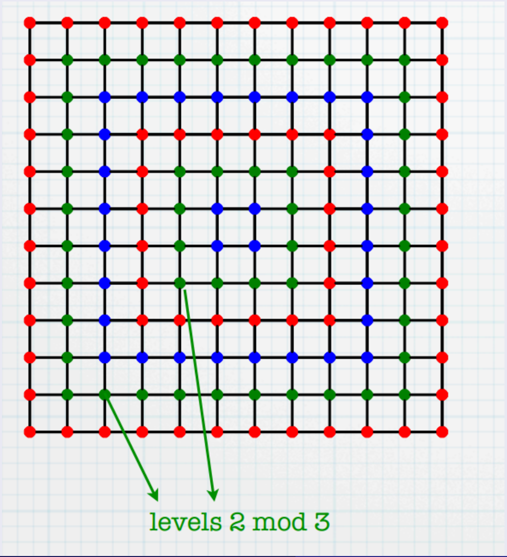
Outline

Bakers Algorithm: Main Algorithm

1. Generate a planar embedding of G using the linear time algorithm of Hopcroft and Tarjan [Hopcroft and Tarjan, 1974] and compute the level of every node.
2. For each i; $0 \leq i \leq k$, do the following:
    1. Remove all nodes where the nodes level mod (k + 1) equals i, splitting the graph into components with an (already computed) k-outerplanar embedding.
    2. Use Theorem 1 to exactly solve maximum independent set on all those components and take the union of the solutions. This is possible as the components are disjunct and no edges can exist between two components.
3. Take the best solution for all i as the solution for the entire graph.

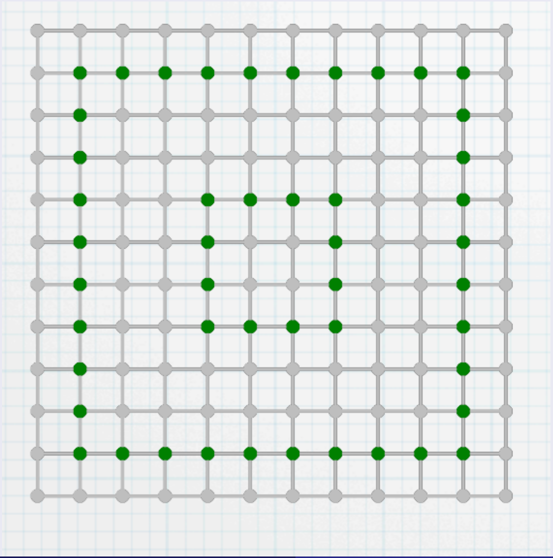
Bakers Algorithm: Main Algorithm

Illustration



Bakers Algorithm: Main Algorithm

Illustration



Outline

Bakers Algorithm: Analysis of the given Algorithm

To see that this gives a solution with approximation ratio $k/(k+1)$, consider the optimal solution S_{OPT} . For some i as defined above, at most $1/(k+1)$ of the nodes in S_{OPT} are at a level congruent to i
[As suppose this was not the case $\rightarrow S_{OPT} > (k+1) * (1/(k+1) * S_{OPT})$ i.e. $S_{OPT} > S_{OPT}$ which is absurd].
As the other components are solved exactly, the union of their solutions has at least the size of $S_{OPT} - 1/(k+1) * S_{OPT}$ i.e. $k/(k+1) * S_{OPT}$ giving our desired approximation factor.
[As $Rem \leq S_{OPT}/(k+1) \rightarrow -Rem \geq -S_{OPT}/(k+1) \rightarrow S_{OPT} - Rem \geq k/(k+1) * S_{OPT}$ and *our solution* $\geq S_{OPT} - Rem$]
This leads to the second theorem:

Outline

Bakers Algorithm: Theorem 2

Theorem (Baker, 1994)

For fixed k, there is an $O(8^k kn)$-time algorithm for maximum independent set that achieves a solution of size at least $k/k+1$ optimal for general planar graphs.

We have proved this [assuming theorem 1]. So, what remains is to prove theorem 1.

Outline

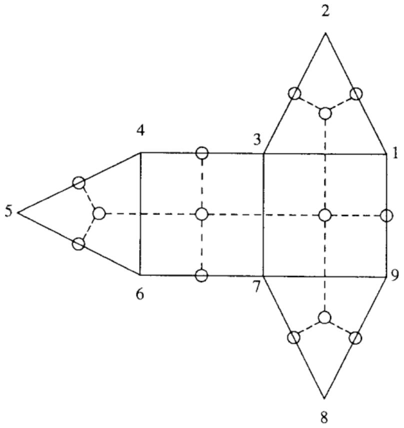
Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

Given a connected outerplanar graph, do the following

- Replace each bridge (x, y) by two edges between x and y. This will allow us to treat a bridge as a face rather than as a special case. Call the resulting graph G. [Will the $|MIS|$ change?]
- The maximum independent set for G will be computed by recursively processing an ordered, rooted tree $\bar{G}$ that represents the structure of G. Each leaf of $\bar{G}$ will represent an exterior edge of G and every other vertex will represent a face of G and will be called a *face* vertex. $\bar{G}$ is constructed as follows:
  1. Construct a graph vertex for every interior face and every exterior edge.
  2. Draw edges between every edge vertex and the vertex of the face the edge is contained in, and between every two vertices of faces that share an edge.
  3. Need to handle cut points (presented later)

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

FIG. 3. An outerplanar graph (with no cutpoints) and a tree (drawn with dashed lines).



Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

Question

Why are we getting a Tree?

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

Question

Why are we getting a Tree? Will we always get a tree for a planar embedding?

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

Question

Why are we getting a Tree? Will we always get a tree for a planar embedding - Ans. No, consider planar embedding of K_4. So, this property is actually the result of outerplanarity condition.

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

Labelling our tree

1. The planar embedding induces a cyclic ordering on the edges of each vertex in the tree [for our discussion, we will follow counterclockwise orientation]. Choosing a face vertex v as the root and choosing which child of v is to be its leftmost child determine the parent and ordering of children for every other vertex of G [We will see this in diagram].
2. Label the vertices of $\bar{G}$ recursively, as follows: Label each leaf of the tree with the oriented exterior edge it represents. Label each face vertex with the first and last nodes in its childrens labels.
3. If a face vertex is labeled (x, y), the leaves of its subtree represent a directed walk of exterior edges in a counterclockwise direction from x to y. For the root, $x = y$ and the directed walk covers all the exterior edges. For any other face vertex v, $x \neq y$, and (x, y) is an interior edge shared by the face represented by v and the face represented by its parent in the tree.

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

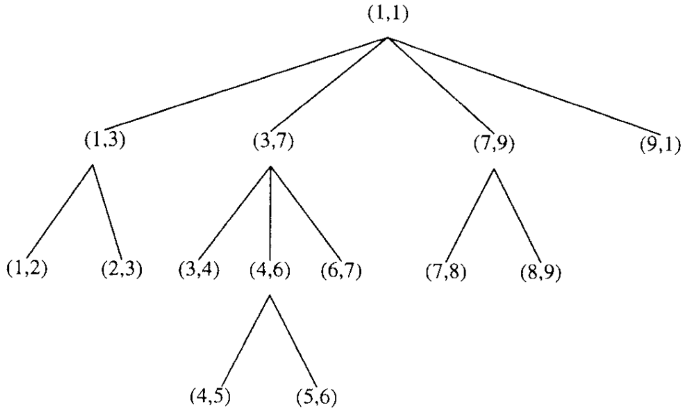


FIG. 4. A rooted, ordered, labeled tree $\overline{G}$ for the graph G of Figure 3.

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

Handling Cutpoints

Cutpoints are nodes, whose removal disconnects the graph. As we removed bridges, a cutpoint is a node where at least two faces meet without sharing an edge, therefore disconnecting the tree for G. Draw an edge between two vertices of faces that both contain the cutpoint and are part of different components until G is connected.

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

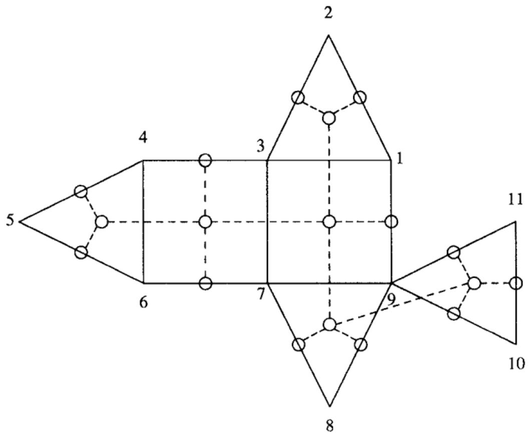


FIG. 5. An outerplanar graph with a cutpoint (node 9) and a corresponding tree (shown by dashed lines).

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

Labelling our tree

Labelling is done same as before. Note that now it is possible for a face vertex labelled (x, y) and x = y in which case, the label represents a cutpoint shared by two faces, as for the vertex labeled (9,9) in below figure.

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

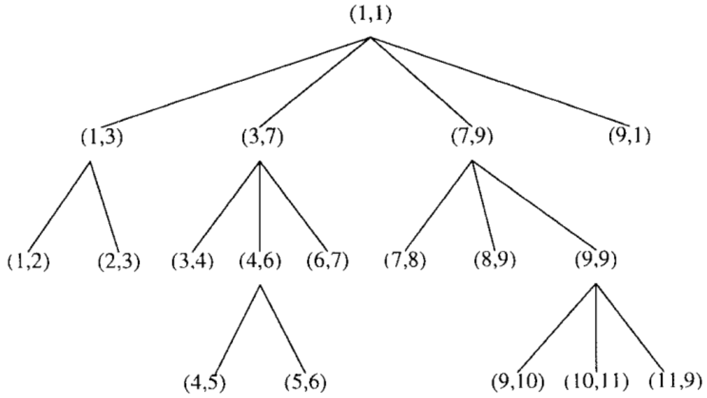


FIG. 6. A rooted ordered, labeled tree $\overline{G}$ for the graph G of Figure 5.

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

So what have we achieved so far?

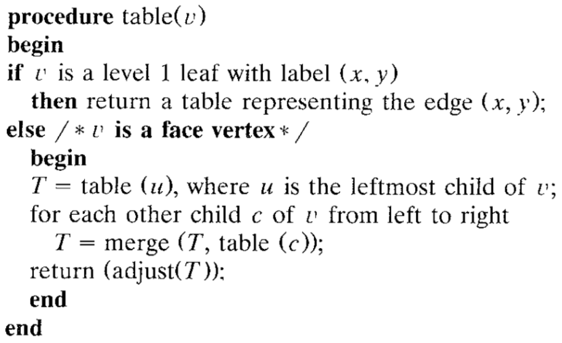
Our aim is to calculate MIS for G with the help of DP, for that we have converted G to a representable tree. Now DP on trees and on DAG is almost always possible.
So, now we will calculate MIS for G using $\bar{G}$ in an unquestionable manner so that it is clear as to why this tree structure represents G.
Also recall that in $\bar{G}$ each vertex other than root represents an edge of the original graph (provided $x \neq y$) and if we construct MIS solution considering all edges of G then that solution will necessarily be correct.

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

MIS Calculation

Computation of the maximum independent set proceeds as follows: The result of processing a vertex *v* labeled (x, y) is a table that gives the sizes of maximum independent sets for the subgraph represented by the subtree rooted at *v*, according to whether the nodes x and y are in the set. That is, for each of the four possible bit pairs representing whether each of x and y is in the set, the table contains the size of the maximum independent set for the subgraph.

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs



```
procedure table($v$)
begin
if $v$ is a level 1 leaf with label $(x, y)$
   then return a table representing the edge $(x, y)$;
else /* $v$ is a face vertex */
   begin
   $T$ = table $(u)$, where $u$ is the leftmost child of $v$;
   for each other child $c$ of $v$ from left to right
      $T$ = merge ($T$, table $(c)$);
   return (adjust($T$));
   end
end
```

FIG. 7. Procedure for computing a table for vertex v.

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

Details of psuedo code

The details not given in above figure are as follows: The table for a leaf u representing an edge (x, y) specifies that the size of a maximum independent set for the subgraph is 1 if exactly one endpoint of (x, y) is in the set, 0 if neither is in it, and undefined (illegal) if both are in it. The procedure merge works as follows: For some z, the current table T has a value for each bit pair representing x and z, the child c has label (z, w) for some w, and table(c) has a value for each bit pair representing z and w. The updated table T has a value for each bit pair representing x and w. This updated value is found for bit pair (b_1, b_3) (representing x and w) by taking the maximum over 0-1 values of bit b_2 of V(T) + V(table(c)) - b_2, where V(T) is the value of T for (b_1, b_2) (representing x and z) and V(table(c)) is the value of table(c) for (b_2, b_3) (representing z and w). Subtracting b_2 prevents counting z twice if z is in the maximum independent set.

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

Details of psuedo code

When all of the children's tables have been processed, T has a value for each bit pair representing x and y. Finally, the label (x, y) is incorporated into T by the procedure adjust. In particular, if $x = y$, the table entry is set to "undefined" for each bit pair requiring exactly one of x and y to be in the maximum independent set, and the value representing the inclusion of both x and y in the set is decremented by one to avoid counting $x = y$ twice. If $x \neq y$, the table entry for the bit pair specifying that both x and y are in the set is set to "undefined" because (x, y) represents an edge. When table is called on the root (r, r) of the tree, it results in a table with entries for four bit pairs. Thanks to the procedure adjust, two of the values are undefined because they represent inconsistency as to whether r is in the maximum independent set. The larger of the two remaining values is the size of the maximum independent set for the graph.

Bakers Algorithm: Linear Time Algorithm for Outerplanar Graphs

Analysing Time Complexity

Clearly in constructing $\bar{G}$, number of nodes are for each graph edge, for cutpoints and our special vertex root. So it is linear in n (original number of nodes). And our DP solution is more or less like a tree traversal and hence that is also linear in n. So the time complexity is $O(n)$.

For Further Reading I

BRENDA S. BAKER
Approximation Algorithms for NP-Complete Problems on Planar Graphs
Journal of the ACM, Volume 41 Issue 1, Jan. 1994, Pages 153-180.