PTAS For MVC On Planar Graphs

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Outline

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   - Fixed Parameter Tractability
2. High Level Plan
3. Bakers Algorithm
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   - Main Algorithm
   - Analysis of the given Algorithm
   - Theorem 2

Outline

Outerplanar Graphs

Definition

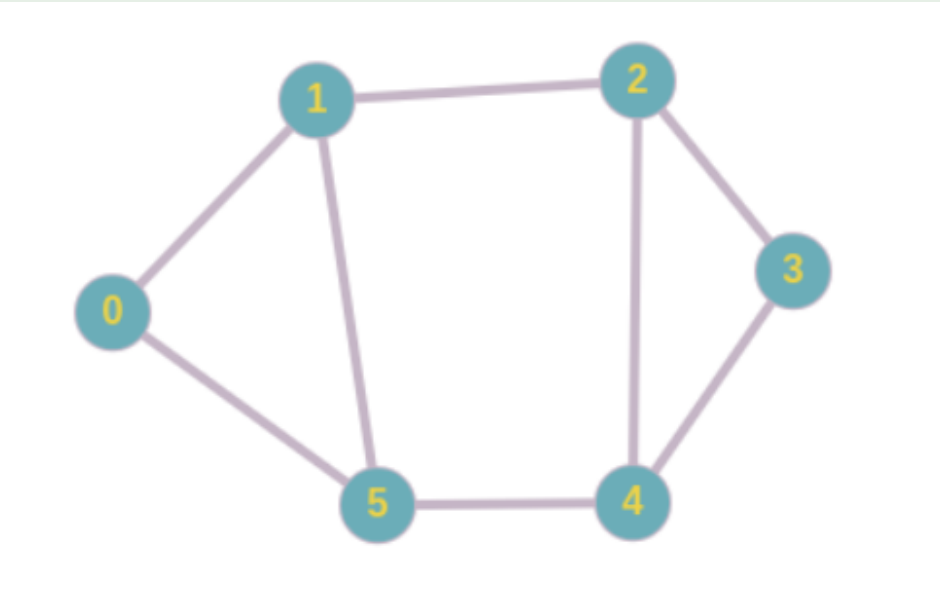
A graph is called outerplanar (1-outerplanar) if it has an embedding in the plane such that every vertex lies on the unbounded face.

In Simple Terms

$\Leftrightarrow$ has a planar drawing for which all vertices belong to the outer face of the drawing.

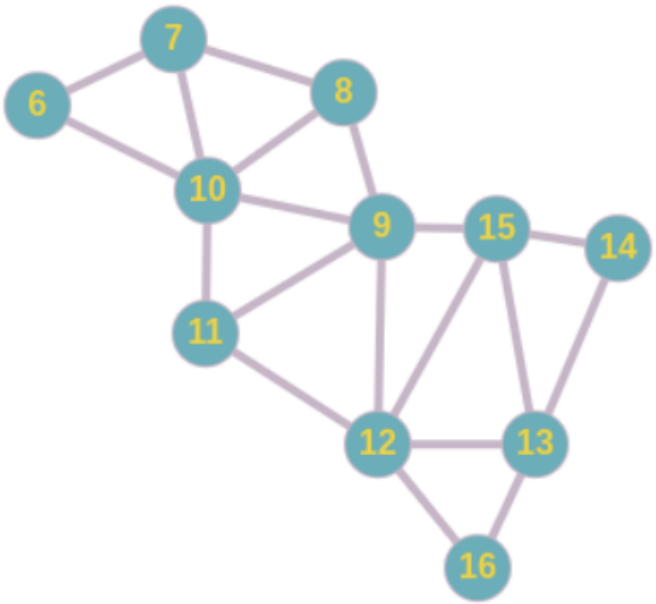
Outerplanar Graphs

Example



Outerplanar Graphs

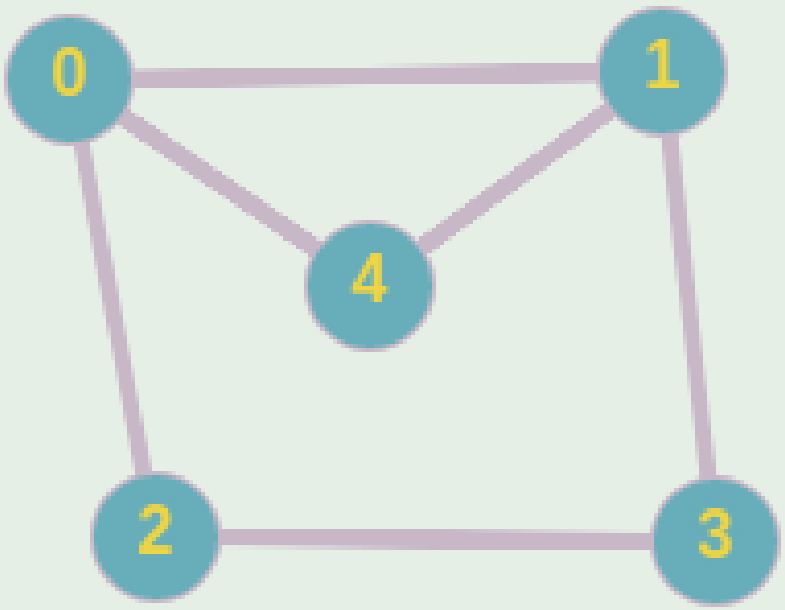
Example



Outerplanar Graphs

Example

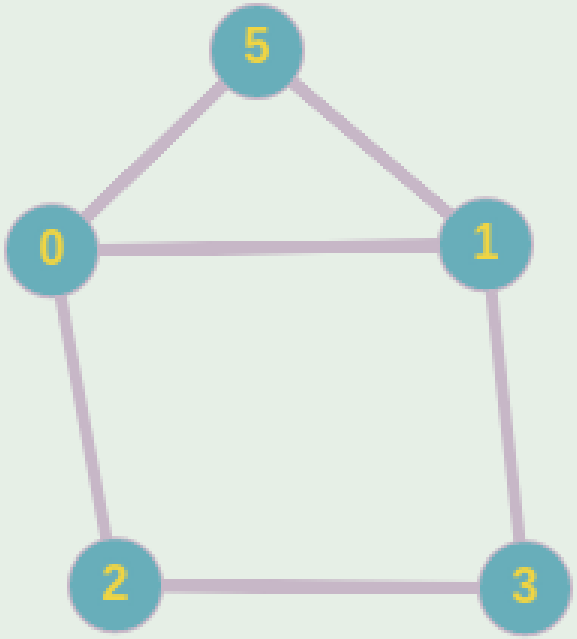
Is this outerplanar?



Outerplanar Graphs

Example

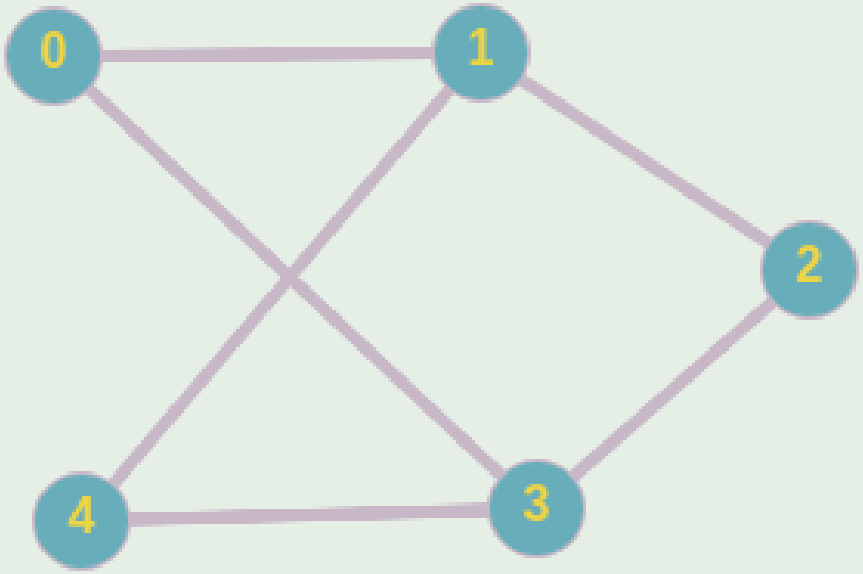
Yes, as it is isomorphic to the following graph.



Outerplanar Graphs

Example

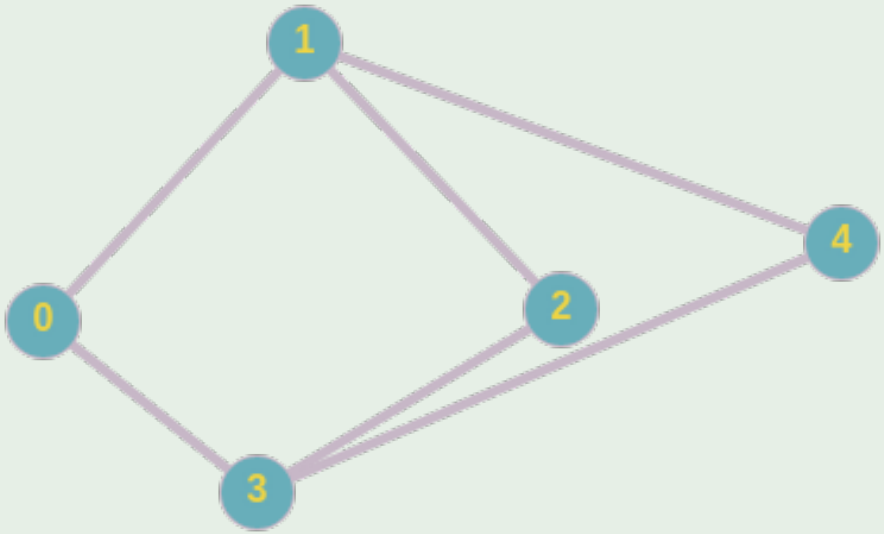
Is this outerplanar?



Outerplanar Graphs

Example

No, but it is planar.



Outerplanar Graphs

Definition

Given a planar embedding E of a planar graph G, we define k-level nodes according to the faces of the embedding. Every node on the exterior face is called an exterior or level 1 node. Exterior edges are defined analogously. In each step, starting at 1, remove all nodes on the exterior face. Repeat until there are no nodes left. The level of a node is the step in which it was removed. Using an appropriate data structure for the planar embedding, such as the one used by Lipton and Tarjan [Lipton and Tarjan, 1979], this can be done in linear time. A planar embedding is called k-outerplanar if there are no nodes of level $>$ k, which means the algorithm terminates after at most k steps. A planar graph is called k-outerplanar if it has a k-outerplanar embedding.

Example

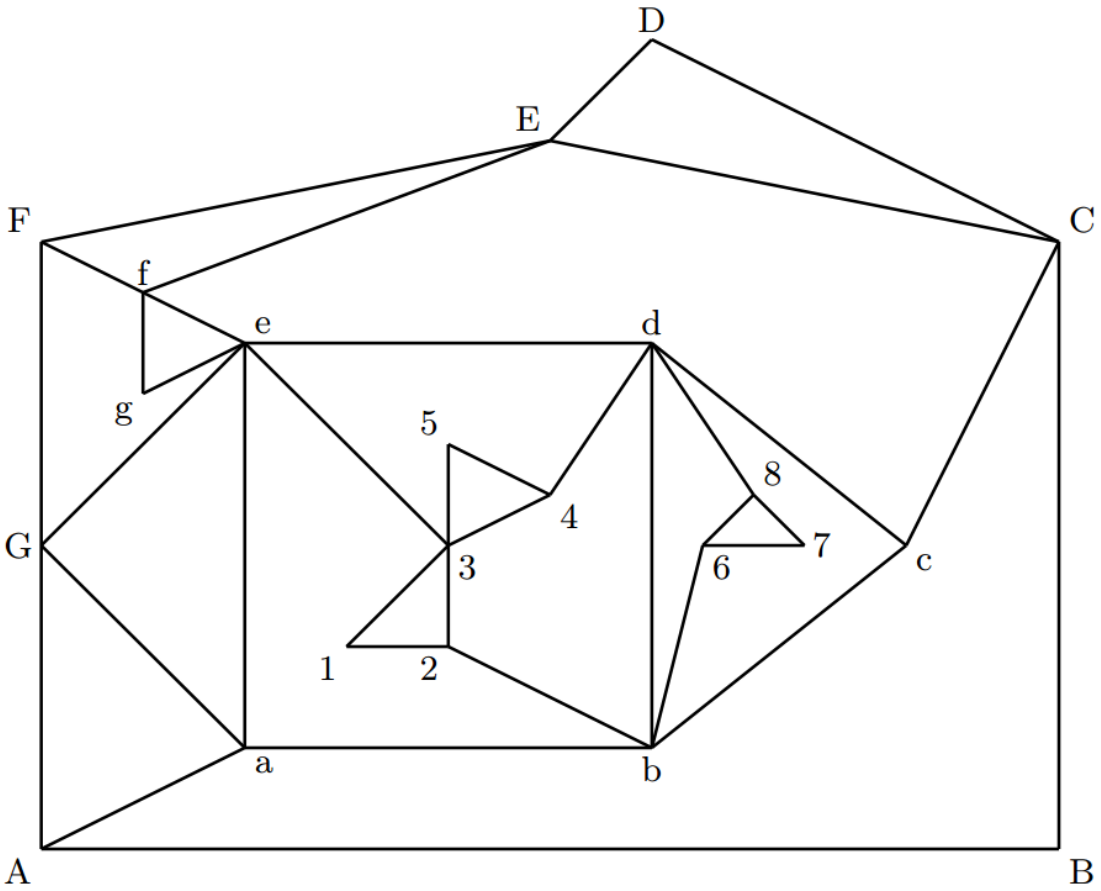


Figure 1. A planar embedding of a 3-outerplanar graph [Baker, 1994].

Outline

Definition

A problem is considered fixed parameter tractable if an algorithm exists which solves the problem in running time $f(k) * n^{O(1)}$ where k is an additional parameter depending upon input and n is the size of the input (in our case we will treat is as number of nodes) and f is an arbitrary function depending only on k.

High Level Plan

We will use this new notion of k-outerplanarity to construct a polynomial time approximation scheme for maximum independent set on planar graphs, which can be easily adapted to other NP-complete problems. The goal is an algorithm that, for a freely chosen but fixed k, solves maximum independent set with an approximation ratio of $k/(k+1)$ and a linear complexity with respect to the number of nodes n. Basically, we want to exactly solve maximum independent on disjunct subgraphs that are k-outerplanar and ignore some nodes in the graph to merge the subgraphs without violating the conditions of the maximum independent set.

PS: Approximation Ratio? For maximization problems it is atmost $(1-\epsilon)$ therefore fix $k = (1/\epsilon) - 1$ (do math).

Outline

Bakers Algorithm: Theorem 1

Theorem (Baker, 1994)

Let k be a positive integer. Given a k-outerplanar embedding of a k-outerplanar graph G, an optimal solution for maximum independent set can be obtained in time $O(8^k n)$, where n is the number of nodes.

Its proof will be done later.

Outline

Bakers Algorithm: Main Algorithm

1. Generate a planar embedding of G using the linear time algorithm of Hopcroft and Tarjan [Hopcroft and Tarjan, 1974] and compute the level of every node.
2. For each i; $0 \leq i \leq k$, do the following:
   1. Remove all nodes where the nodes level mod (k + 1) equals i, splitting the graph into components with an (already computed) k-outerplanar embedding.
   2. Use Theorem 1 to exactly solve maximum independent set on all those components and take the union of the solutions. This is possible as the components are disjunct and no edges can exist between two components.
3. Take the best solution for all i as the solution for the entire graph.

Outline

Bakers Algorithm: Analysis of the given Algorithm

To see that this gives a solution with approximation ratio $k/(k+1)$, consider the optimal solution S_{OPT} . For some i as defined above, at most $1/(k+1)$ of the nodes in S_{OPT} are at a level congruent to i
[As suppose this was not the case $\rightarrow S_{OPT} > (k+1) * (1/(k+1) * S_{OPT})$ i.e. $S_{OPT} > S_{OPT}$ which is absurd].
As the other components are solved exactly, the union of their solutions has at least the size of $S_{OPT} - 1/(k+1) * S_{OPT}$ i.e. $k/(k+1) * S_{OPT}$ giving our desired approximation factor.
[As $Rem \leq S_{OPT}/(k+1) \rightarrow -Rem \geq -S_{OPT}/(k+1) \rightarrow S_{OPT} - Rem \geq k/(k+1) * S_{OPT}$]
This leads to the second theorem:

Outline

Bakers Algorithm: Theorem 2

Theorem (Baker, 1994)

For fixed k, there is an $O(8^k kn)$-time algorithm for maximum independent set that achieves a solution of size at least $k/k+1$ optimal for general planar graphs.

We have proved this [assuming theorem 1]. So, what remains is to prove theorem 1