Approximation Algorithms
Assignment 1

Instructions

- Please refer to the roll list to know your serial number in this course.
- You need to submit written answer only for the question with question number same as your roll no. For your beneift, solve all the questions given.
- You may discuss the answer with anyone. But you should write down the proof on your own and should be ready to explain every detail of the proof.
- In your proofs, you are supposed to write all necessary arguments with proper justifications.
- If you are using a fact which is fully proven in class, still you need to write the statement of the fact correctly and mention that it is proven in class.
- **Submit by : 12 noon, Tuesday, 5th February, 2019.**

Questions

1. Given a directed graph $G = (V, E)$, pick a maximum cardinality set of edges from E so that the resulting subgraph is acyclic.
   `Hint:`Arbitrarily number the vertices and pick the bigger of the two sets, the forward-going edges and the backward-going edges. What scheme are you using for upper bounding OPT?
2. Consider the following factor 2 approximation algorithm for the cardinality vertex cover problem. Find a depth first search tree in the given graph, G, and output the set, say S, of all the non-leaf vertices of this tree. Show that S is indeed a vertex cover for G and $|S| \leq 2\ OPT$.
   `Hint:` Show that G has a matching in which all non-leaf vertices (and may be some leaf vertices) of the dfs tree are matched. What is the least size of such a matching?
3. Perhaps the first strategy one tries when designing an algorithm for an optimization problem is the greedy strategy. For the vertex cover problem, this would involve iteratively picking a maximum degree vertex and removing it, together with edges incident at it, until there are no edges

left. Show that this algorithm achieves an approximation guarantee of $O(\log n)$. Give a tight example for this algorithm.
`Hint:` Analysis similar to that of greedy set cover algorithm.

4. Give a greedy algorithm for the following problem, achieving an approximation guarantee of factor 1/4.
Maximum directed cut : Given a directed graph $G = (V, E)$ with nonnegative edge costs, find a subset $S \subseteq V$ so as to maximize the total cost of edges out of S, i.e., $cost(\{uv : u \in S \text{ and } v \in \overline{S}\})$.
`Hint:` Algorithm similar to that of undirected version. Choose the appropriate side of the partition and analyse.

5. Use Algorithm 2.13 and the fact that the vertex cover problem is polynomial time solvable for bipartite graphs to give a factor $\lceil \log_2 \Delta \rceil$ algorithm for vertex cover, where Δ is the degree of the vertex having highest degree.
`Hint:` Let H denote the subgraph consisting of edges in the maximum cut found by Algorithm 2.13. Clearly, H is bipartite, and for any vertex v, $deg_H(v) \geq (1/2)deg_G(v)$ (why?). After finding vertex cover of H, delete edges of H and recurse to cover remaining edges. Complete the analysis.

6. Let 2SC denote the restriction of set cover to instances having $f = 2$, where f is the frequency of the most frequent element. Show that 2SC is equivalent to the vertex cover problem, with arbitrary costs, under approximation factor preserving reductions.

7. Give a tigher analysis of the greedy set cover algorithm and show that it achieves an approximation factor of H_k, where k is the cardinality of the largest specified subset of U.
`Hint:` Consider an optimum setcover $F^* = \{S_1, S_2, \ldots, S_t\}$. For each S_i, upper bound price of covering elements of S_i using $H_k\ cost(S_i)$. This can be done by comparing cost of covering each element e_j of S_i using the greedy cover with the price of covering the same element using S_i itself. Order elements of S_i according to the order in which they were covered by the greedy algorithm.

8. A tournament is a directed graph $G = (V, E)$, such that for each pair of vertices, $u, v \in V$, exactly one of (u, v) and (v, u) is in E. A feedback vertex set for G is a subset of the vertices of G whose removal leaves an acyclic graph. Give a factor 3 algorithm for the problem of finding a minimum feedback vertex set in a directed graph.
`Hint:` Show that if there are no length 3 directed cycles, there are no directed cycles at all. Then, it is sufficient to kill all length 3 cycles. Reduce to a set cover instance and use the factor f set cover algorithm.

9. Let $G = (V, E)$ be a graph with nonnegative edge costs. S, the senders and R, the receivers, are disjoint subsets of V. The problem is to find a minimum cost subgraph of G that has a path connecting each receiver to a sender (any sender suffices). Partition the instances into two cases: $S \cup R = V$ and $S \cup R \neq V$. Show that these two cases are in P and NP-hard, respectively. For the second case, give a factor 2 approximation algorithm.
`Hint:` Add a new vertex which is connected to each sender by a zero

cost edge. Consider the new vertex and all receivers as required and the remaining vertices as Steiner, and find a minimum cost Steiner tree.

10. Asymmetric TSP : We are given a directed graph G on vertex set V, with a nonnegative cost specified for edge $(u \to v)$, for each pair $u, v \in V$. The edge costs satisfy the directed triangle inequality, i.e., for any three vertices $u, v, andw$, $cost(u \to v) \leq cost(u \to w) + cost(w \to v)$. The problem is to find a minimum cost cycle visiting every vertex exactly once.
    `Hint:` Use the fact that a minimum cost cycle cover (i.e., disjoint cycles covering all the vertices) can be found in polynomial time. Shrink the cycles and recurse.

11. Let $G = (V, E)$ be a graph with even number of vertices, edge costs satisfying the triangle inequality, and $V' \subseteq V$ be a set of even cardinality. Prove or disprove: The cost of a minimum cost perfect matching on V' is bounded above by the cost of a minimum cost perfect matching on V.

Reference

Approximation Algorithms, Vijay Vazirani, Springer, Corrected Second Printing 2003.