卐 Short Revision Notes 卐

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Contents

Think twice code once!

1 Maths

1.1 Game Theory

Games like chess or checkers are partizan type.

1.1.1 What is a Combinatorial Game?

1. There are 2 players.
2. There is a set of possible positions of Game
3. If both players have same options of moving from each position, the game is called impartial; otherwise partizan
4. The players move alternating.
5. The game ends when a postion is reached from which no moves are possible for the player whose turn it is to move. Under **normal play rule**, the last player to move wins. Under **misere play rule** the last player to move loses.
6. The game ends in a finite number of moves no matter how it is played.

P - Previous Player, **N** - Next Player

1. Label every terminal position as P - postion
2. Position which can move to a P position is N position
3. Position whose all moves are to N positoin is P position.

Note: Every Position is either a P or N. For games using misere play all is same except that step 1 is replaced by the condition that all terminal positions are **N** postions.

Directed graph G = (X, F), where X is positions (vertices) and F is a function that gives for each $x \in X$ a subset of X, i.e. *followers of* x. If $F(x)$ is empty, x is called a terminal position.

$g(x) = min\{n \geq 0 : n \neq g(y) \text{ for } y \in F(x)\}$

Positions x for which $g(x)$ is 0 are P postions and all others are N positions. **Note:** $g(x)$ is 0 if x is a terminal position

4.1 The Sum of n Graph Games. Suppose we are given n progressively bounded graphs, $G_1 = (X_1, F_1), G_2 = (X_2, F_2), \ldots, G_n = (X_n, F_n)$. One can combine them into a new graph, $G = (X, F)$, called the **sum** of $G_1, G_2, \ldots, G_n$ and denoted by $G = G_1 + \cdots + G_n$ as follows. The set X of vertices is the Cartesian product, $X = X_1 \times \cdots \times X_n$. This is the set of all n-tuples $(x_1, \ldots, x_n)$ such that $x_i \in X_i$ for all i. For a vertex $x = (x_1, \ldots, x_n) \in X$, the set of followers of x is defined as

$$\begin{aligned} F(x) = F(x_1, \ldots, x_n) = &F_1(x_1) \times \{x_2\} \times \cdots \times \{x_n\} \\ &\cup \{x_1\} \times F_2(x_2) \times \cdots \times \{x_n\} \\ &\cup \cdots \\ &\cup \{x_1\} \times \{x_2\} \times \cdots \times F_n(x_n). \end{aligned}$$

Theorem 2. *If g_i is the Sprague-Grundy function of G_i, $i = 1, \ldots, n$, then $G = G_1 + \cdots + G_n$ has Sprague-Grundy function $g(x_1, \ldots, x_n) = g_1(x_1) \oplus \cdots \oplus g_n(x_n)$.*

Thus, if a position is a **N** position, we can cleverly see which position should we go to (what move of a component game to take) such that we reach **P** position.

1.2 Mobius

Just read this and this.

Prob, Sol: $\sum_{g=1}^{i} h(g) * cnt[g]$ where $cnt[g]$ = no. of arrays with $gcd(a_1, a_2, a_3, ..., a_n) = g$ and where each $a_k \leq i$. Now $h(g)$ = Dirichlet identity function. Thus it is summation of mobius function. Ans thus we get $\sum_{d=1}^{i} \mu(d) * f(d)$ where $f(d)$ is the number of arrays with elements in range $[1, i]$ such that $gcd(a_1, \ldots, a_n)$ is divisible by j. Obviously $f(j) = (\lfloor i/j \rfloor)^n$.

1.3 Bell, Burnside, etc

Just read this

1.4 Modulo

$(a + b)modm = (amodm + bmodm)modm$

.... − ....

.... ∗ ....

```
const int m1 = (int) 1e9 + 7;
template <typename T>
inline T add(T a, T b) {
    a += b;
    if (a >= m1) a -= m1;
    return a;
}
template <typename T>
inline T sub(T a, T b) {
    a -= b;
    if (a < 0) a += m1;
    return a;
}

template <typename T>
inline T mul(T a, T b) {
    return (T) (((long long) a * b) % m1);
}

template <typename T>
inline T power(T a, T b) {
    int res = 1;
    while (b > 0) {
        if (b & 1) {
            res = mul<T>(res, a);
        }
        a = mul<T>(a, a);
        b >>= 1;
    }
    return res;
}

template <typename T>
inline T inv(T a) {
    return power<T>(a, m1 - 2);
}
```

1.5 Prob and Comb

- $E[X] = \sum E(X|A_i)P(A_i)$
- k, p_a, p_b prob, Sol, if $n + m \geq k \rightarrow p_b(i + j) + p_a * p_b * (i + j + 1) + p_a^2 * p_b * (i + j + 2) \cdots = (i + j) + \frac{p_a}{p_b}$ Also

$$\begin{aligned} dp[0][0] &= p_a * dp[1][0] + p_b * dp[0][0] && (1)\\ &= p_a * dp[1][0]/(1 - p_b) && (2)\\ &= dp[1][0] && (3) \end{aligned}$$

- **Dearrangement of n objects**
  $n! * \sum_{k=0}^{n} (-1)^k / k! = \,!n$
  $!n = (n-1) * [!(n-1) + !(n-2)]$ for n ≥ 2
- **Gambler ruin's problem:** Probability that first player (p for each step) wins. $(1 - (q/p)^{n_1})/(1 - (q/p)^{n_1+n_2})$. $n_1 = \lceil ev_1/d \rceil$, $n_2 = \lceil ev_2/d \rceil$. In case $p = q = 1/2$, formula is $n_1/(n_1 + n_2)$.
- UVA 10491, $ans = (N_{cows}/(N_{cows} + N_{cars})) * (N_{cars}/(N_{cows} + N_{cars} - N_{shows} - 1)) + (N_{cars}/(N_{cows} + N_{cars})) * ((N_{cars} - 1)/(N_{cows} + N_{cars} - N_{shows} - 1))$
- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$
- $\binom{n}{r} = n/r * \binom{n-1}{r-1}$
- $\sum_{r=0}^{n} \binom{n}{r} = 2^n$
- $\sum_{m=0}^{n} \binom{m}{r} = \binom{n+1}{r+1}$
- $\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m-1}{m}$
- $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$

1.6 jk

1.7 Euler's Totient Function

Also known as phi-function $\phi(n)$, counts the number of integers between 1 and n inclusive, which are coprime to n.

If p is prime $\phi(p) = p - 1$.

If p is a prime number and $k \geq 1$, then there are exactly p^k/p numbers between 1 and p^k that are divisible by p. Which gives us:$\phi(p^k) = p^k - p^{k-1}$.

If a and b are relatively prime, then: $\phi(ab) = \phi(a) \cdot \phi(b)$. This relation is not trivial to see. It follows from the Chinese remainder theorem.

In general, for not coprime a and b, the equation

$$\phi(ab) = \phi(a) \cdot \phi(b) \cdot \frac{d}{\phi(d)}$$

with $d = \gcd(a, b)$ holds.

$\phi(n) = \phi({p_1}^{a_1}) \cdot \phi({p_2}^{a_2}) \cdots \phi({p_k}^{a_k})$

$= \left({p_1}^{a_1} - {p_1}^{a_1-1}\right) \cdot \left({p_2}^{a_2} - {p_2}^{a_2-1}\right) \cdots \left({p_k}^{a_k} - {p_k}^{a_k-1}\right)$

$= p_1^{a_1} \cdot \left(1 - \frac{1}{p_1}\right) \cdot p_2^{a_2} \cdot \left(1 - \frac{1}{p_2}\right) \cdots p_k^{a_k} \cdot \left(1 - \frac{1}{p_k}\right)$

$= n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$

Eulers Theorem:

$$a^{\phi(m)} \equiv 1 \pmod m$$

if a and m are relatively prime.

In the particular case when m is prime, Euler's theorem turns into Fermat's little theorem:

$$a^{m-1} \equiv 1 \pmod m$$

1.8 Catalan

$Cat(n) = \binom{2n}{n}/(n + 1)$

$Cat(m) = (2m * (2m - 1)/(m * (m + 1))) * Cat(m - 1)$

$Cat(n) =$

1. the number of ways a convex polygon with n+2 sides can be cut into n triangles
2. the number of ways to use n rectangles to tile a stairstep shape (1, 2, ..., n1, n).
3. No. of expressions containing n pairs of parentheses which are correctly matched.

4. the number of planar binary trees with n+1 leaves
5. No. of distinct binary trees with n vertices
6. No. of different ways in which n + 1 factors can be completely parenthesized. Like for {a, b, c, d}, one parenthing will be ((ab)c)d.
7. the number of monotonic paths of length 2n through an n-by-n grid that do not rise above the main diagonal
8. n pair of people on circle can do non cross hand shakes. i.e. no of ways to connect the 2n points on a circle to form n disjoint chords
9. no. of permutations of length n that an be stack sorted
10. no. of non crossing partitions of a set of n elements

Note: Its better to use bigint for catalan computations. Also no. of binary trees with n labelled nodes = $cat[n] * fact[n]$

1.9 Floyd Cycle Finding

```
// mu = start of the cycle
// lam = its length
// O (mu + lam) time complexity
// O (1) space complexity
ii floydCycleFinding(int x0) {
    // 1st part: finding k * lam
    int tortoise = f(x0), hare = f (f (x0));
    // hare moves at twice speed
    while (tortoise != hare) {
        tortoise = f (tortoise); hare = f(f(hare));
    }
    // thus tor = x_i; hare = x_2i
    // i.e. x_2i = x_{i + k * lam}
    // i.e. k * lam = i.
    // Now if hare is set to beginning
    // i.e. hare = x_0, tor = x_i
    // thus if both now move same no. of steps and in between
    they become equal, i.e.
    // x_l = x_{i + l}
    // i.e. x_l = x_{l + k * lam}
    // Thus l must be the minimum index and therefore l = mu
    int mu = 0;
    hare = x0;
    while (tortoise != hare) {
        tortoise = f (tortoise); hare = f(hare); mu++
    }
    // finding lam
    int lam = 1; hare = f (tortoise);
    while (tortoise != hare) {
        hare = f (hare); lambda++;
    }
    return ii (mu, lambda);
}
```

1.10 Base Conversion

```
// decimal no. to some base
stack<int> S;
while (q) {
    s.push (q % b);
    q /= b;
}
while (!s.empty ()) {
    cout << process (s.top ()) << " ";
    s.pop ();
}
// base to decimal no.
ll baseToDec () {
    ll ret = 0;
    for (auto &c : num) {
        ret = (ret * base + (c - 48)); // can take mod if final
        answer is required in mod
    }
    return ret;
}
```

1.11 Extended Euclid

$ax + by = c$ this is called diophantine eqn and is solvable only when $d = gcd(a, b)$ divides c. so first solve $ax + by = d$ then multiply x, y with c/d. Also once we have found a particular soln to this eqn then their exist infinite solns of the form $(x0 + (b/d) * n, y0 - (a/d) * n)$ where n is any integer, note that these infinite solutions are as well the solution to original diophantine eqn. Assume we found the coefs (x1, y1) for (b, a mod b) $\rightarrow b * x1 + (a \bmod b)y1 = g$
$\rightarrow b * x1 + (a - \lfloor (a/b) \rfloor * b) * y1 = g$
$\rightarrow a * y1 + b * (x1 - \lfloor (a/b) \rfloor * y1) = g$
$\rightarrow x = y1 \;\&\; y = x1 - \lfloor (a/b) \rfloor * y1$

```
void extendedEuclid(int a, int b) {
   if (b == 0) { x = 1; y = 0; d = a; return; } // base case
   extendedEuclid(b, a % b); // similar as the original gcd
   int x1 = y;
   int y1 = x - (a / b) * y;
   x = x1;
   y = y1;
}
```

SGU 106 Equation, Sol
Prob: To find the soln with minimum value of $x + y$ and obviously there has to be range of x, y. Sol: Now $x + y = x_0 + y_0 + n * (b/d - a/d)$. If $a < b$, select smallest possible value of n. If $a > b$ select the largest. And if $a = b$, all solutions have same sum of $x + y$

1.12 Linear Congruence Equation

This equation is of the form:

$$a \cdot x = b \pmod n,$$

where a, b and n are given integers and x is an unknown integer.

It is required to find the value x from the interval $[0, n-1]$ (clearly, on the entire number line there can be infinitely many solutions that will differ from each other in $n \cdot k$, where k is any integer). If the solution is not unique, then we will consider how to get all the solutions.

1.12.1 Solution by finding the inverse element

Let us first consider a simpler case where a and n are coprime ($\gcd(a, n) = 1$). Then one can find the inverse of a, and multiplying both sides of the equation with the inverse, and we can get a unique solution.

$$x = b \cdot a^{-1} \pmod n$$

Now consider the case where a and n are not coprime ($\gcd(a, n) \neq 1$). Then the solution will not always exist (for example $2 \cdot x = 1 \pmod 4$ has no solution).

Let $g = \gcd(a, n)$, i.e. the greatest common divisor of a and n (which in this case is greater than one).

Then, if b is not divisible by g, there is no solution. In fact, for any x the left side of the equation $a \cdot x \pmod n$, is always divisible by g, while the right-hand side is not divisible by it, hence it follows that there are no solutions.

If g divides b, then by dividing both sides of the equation by g (i.e. dividing a, b and n by g), we receive a new equation:

$$a' \cdot x = b' \pmod{n'}$$

in which a' and n' are already relatively prime, and we have already learned how to handle such an equation. We get x' as solution for x.

It is clear that this x' will also be a solution of the original equation. However it will not be the only solution. It can be shown that the original equation has exactly g solutions, and they will look like this:

$$x_i = (x' + i \cdot n') \pmod n \quad \text{for } i = 0 \ldots g-1$$

Summarizing, we can say that the number of solutions of the linear congruence equation is equal to either $g = \gcd(a, n)$ or to zero.

1.12.2 Solution with the Advanced Euclidean Algorithm

We can rewrite the linear congruence to the following Diophantine equation:

$$a \cdot x + n \cdot k = b,$$

where x and k are unknown integers.

The method of solving this equation is described in the corresponding article Linear Diophantine equations and it consists of applying the Extended Euclidean Algorithm.

It also describes the method of obtaining all solutions of this equation from one found solution, and incidentally this method, when carefully considered, is absolutely equivalent to the method described in the previous section.

1.13 Sieve

```
 ll _sieve_size; // ll is defined as: typedef long long ll;
bitset<10000010> bs; // 10^7 should be enough for most cases
vi primes; // compact list of primes in form of vector<int>
void sieve(ll upperbound) { // create list of primes in
[0..upperbound]
   _sieve_size = upperbound + 1; // add 1 to include upperbound
   bs.set(); // set all bits to 1
   bs[0] = bs[1] = 0; // except index 0 and 1
   for (ll i = 2; i <= _sieve_size; i++) if (bs[i]) {
// cross out multiples of i starting from i * i!
           for (ll j = i * i; j <= _sieve_size; j += i) bs[j] =
           0;
           primes.push_back((int)i); // add this prime to the
           list of primes
       } } // call this method in main method
bool isPrime(ll N) { // a good enough deterministic prime
tester
```

```
    // O(#primes < sqrt(N))
    // O(sqrt(N)/ln(sqrt(N)))
   if (N <= _sieve_size) return bs[N]; // O(1) for small primes
   for (int i = 0; i < (int)primes.size(); i++)
       if (N % primes[i] == 0) return false;
   return true; // it takes longer time if N is a large prime!
} // note: only work for N <= (last prime in vi "primes")^2

vi primeFactors(ll N) { // remember: vi is vector<int>, ll is
long long
   vi factors;
   ll PF_idx = 0, PF = primes[PF_idx]; // primes has been
   populated by sieve
   while (PF * PF <= N) { // stop at sqrt(N); N can get smaller
       while (N % PF == 0) { N /= PF; factors.push_back(PF); }
       // remove PF
       PF = primes[++PF_idx]; // only consider primes!
   }
   if (N != 1) factors.push_back(N); // special case if N is a
   prime
   return factors; // if N does not fit in 32-bit integer and
   is a prime
} // then 'factors' will have to be changed to vector<ll>


memset(numDiffPF, 0, sizeof numDiffPF);
//Modified Sieve.
void pre() {
   for (int i = 2; i < MAX_N; i++)
       if (numDiffPF[i] == 0) // i is a prime number
           for (int j = i; j < MAX_N; j += i)
               numDiffPF[j]++; // increase the values of
               multiples of i
}
// Bottom up euler totient function
for (int i = 0; i <= limit; i++) eu[i] = i;
for (int i = 2; i <= limit; i++) {
    if (eu[i] == i) {
        for (int j = i; j <= limit; j += i) {
            eu[j] -= eu[j] / i;
        }
    }
}
```

1.14 Frac lib and Eqn solving

To be revised.

```
class Frac {
public:
   long long a, b;
   Frac() {
       a = 0, b = 1;
   }
   Frac(int x, int y) {
       a = x, b = y;
       reduce(); ///So we are always reducing out fractions...
   }
   Frac operator+(const Frac &y) {
       long long ta, tb;
       tb = this->b/gcd(this->b, y.b)*y.b;
       ta = this->a*(tb/this->b) + y.a*(tb/y.b);
       Frac z(ta, tb);
       return z;
   }
   Frac operator-(const Frac &y) {
       long long ta, tb;
       tb = this->b/gcd(this->b, y.b)*y.b;
       ta = this->a*(tb/this->b) - y.a*(tb/y.b);
       Frac z(ta, tb);
       return z;
   }
   Frac operator*(const Frac &y) {
       long long tx, ty, tz, tw, g;
       tx = this->a, ty = y.b;
       g = gcd(tx, ty), tx /= g, ty /= g;
       tz = this->b, tw = y.a;
       g = gcd(tz, tw), tz /= g, tw /= g;
       Frac z(tx*tw, ty*tz);
       return z;
   }
   Frac operator/(const Frac &y) {
       long long tx, ty, tz, tw, g;
       tx = this->a, ty = y.a;
       g = gcd(tx, ty), tx /= g, ty /= g;
       tz = this->b, tw = y.b;
       g = gcd(tz, tw), tz /= g, tw /= g;
       Frac z(tx*tw, ty*tz);
       return z;
   }
   bool operator == (const frac &other) const {
        return a == other.a and b == other.b;
    }
    bool operator < (const frac &other) const {
        if (a != other.a) return a < other.a;
        else return b > other.b;
    }
private:
   static long long gcd(long long x, long long y) {
       return b == 0 ? a : gcd (b, a % b);
   }
   void reduce() {
       if (a == 0) {  // to handle case when b == 0 (not
       required in this problem) a = inf (so as to have same
       ground)
           b = 1;
           return;
       } else {
            long long g = gcd(abs(a), abs(b));
            a /= g, b /= g;
            if(b < 0)   a *= -1, b *= -1;
       }
   }

};
ostream& operator<<(ostream& out, const Frac&x) {
   out << x.a;
   if(x.b != 1)
       out << '/' << x.b;
   return out;
}
int main() {//UVA 10109
   int n, m, i, j, k, N;
   char NUM[100], first = 0;
   long long X, Y;
   Frac matrix[100][100];
   while(scanf("%d", &N) == 1 && N) {
       scanf("%d %d", &n, &m);
       for(i = 0; i < m; i++) {
           for(j = 0; j <= n; j++) {
               scanf("%s", NUM);
               if(sscanf(NUM, "%lld/%lld", &X, &Y) == 2) {
                   matrix[i][j].a = X;
                   matrix[i][j].b = Y;
               } else
                   sscanf(NUM, "%lld", &matrix[i][j].a),
                   matrix[i][j].b = 1;
           }
       }
       Frac tmp, one(1,1);
       int idx = 0, rank = 0;
       for(i = 0; i < m; i++) {
           while(idx < n) {
               int ch = -1;
               for(j = i; j < m; j++)
                   if(matrix[j][idx].a) {///This means that idx
                   must be incremented to check
                       ///the pivot at correct row...
                       ch = j;
                       break;
                   }
               if(ch == -1) {
                   idx++;
                   continue;
               }///this if condition executes if all the
               elements in desired column
               ///and below the i-1 th row are zero so we need
               to go to the next column...
               if(i != ch)///So if the desired pivot element is
               zero we swap that row with
                   ///the closest row that has non zero
                   pivot...
                   for(j = idx; j <= n; j++)
                       swap(matrix[ch][j], matrix[i][j]);
               break;
```

```
        }
        if(idx >= n) break;
        tmp = one/matrix[i][idx];
        rank++;
        for(j = idx; j <= n; j++)
            matrix[i][j] = matrix[i][j]*tmp;///So here we
            are making pivot element 1.
        for(j = 0; j < m; j++) {
            if(i == j)  continue;///This condition executes
            means that we are ignoring the
            ///pivot row...
            tmp = matrix[j][i]/matrix[i][idx];
            for(k = idx; k <= n; k++) {
                matrix[j][k] = matrix[j][k] -
                tmp*matrix[i][k];///Thus now we are making
                ///all the elements below pivot as zero..
            }
        }
        idx++;
    }
    if(first)   puts("");
    first = 1;
    printf("Solution for Matrix System # %d\n", N);
    int sol = 0;
    for(i = 0; i < m; i++) {
        for(j = 0; j < n; j++) {
            if(matrix[i][j].a)
                break;
        }
        if(j == n) {
            if(matrix[i][n].a == 0 && sol != 1)
                sol = 0; // INFINITELY
            else
                sol = 1; // No Solution.
        }
    }
    if(rank == n && sol == 0) {
        for(i = 0; i < n; i++) {
            printf("x[%d] = ", i+1);
            cout << matrix[i][n] << endl;
        }
        continue;
    }
    if(sol == 1)
        puts("No Solution.");
    else
        printf("Infinitely many solutions containing %d
        arbitrary constants.\n", n-rank);
  }
  return 0;
}
```

1.15 Balanced Ternary

It is a ternary (base 3) no. system in which digits have values $-1, 0, 1$. Balanced ternary can represent all integers without using a seperate minus sign. To convert unbalanced ternary (0, 1, 2) to balanced ternary (which may have digits after a radix point) start from the right most digit and proceed towards left, if the digit is 0, 1, then ignore, but if it is 2, replace it with z and add 1 to the digit towards left.
Ex: $200120_{unbal} = 2002z0 = 201zz0 = 1z01zz0_{bal}$
Ex: $022_{unbal} = 0003z = 0010z_{bal}$
Just see both the power of 3 problems

1.16 15 Puzzle Game: Existence Of The Solution

This game is played on a 4×4 board. On this board there are 15 playing tiles numbered from 1 to 15. One cell is left empty (denoted by 0). You need to get the board to the position presented below by repeatedly moving one of the tiles to the free space:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	0

1.16.1 Existence Of The Solution

j

Let's consider this problem: given position on the board, determine whether a sequence of moves which leads to a solution exists.

Suppose we have some position on the board:

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{matrix}$$

where one of the elements equals zero and indicates an empty cell $a_z = 0$

Let's consider the permutation:

$$a_1 a_2 ... a_{z-1} a_{z+1} ... a_{15} a_{16}$$

(i.e. the permutation of numbers corresponding to the position on the board without a zero element)

Let N be the number of inversions in this permutation (i.e. the number of such elements a_i and a_j that $i > j$, but $a_i < a_j$).

Suppose K is an index of a row where the empty element is located (i.e. in our indications $K = (z-1)$ div $4 + 1$).

Then, the solution exists iff $N + K$ is even.

1.17 Stern Broco Tree

At zero iteration, we have 2 fractions (0/1 and 1/0) further at each subsequent iteration, this list of fractions is taken and between each 2 adjacent fractions a/b and c/d their median is inserted $((a+c)/(b+d))$. It is stated that it is possible to obtain the set of all non negative fractions. Moreover all recieved fractions will be different and irreducible. Any tow adjacent fractions a/b, c/d with $(a/b < c/d)$ satisfies $bc - ad = 1$, this statement aswell holds for farey sequence of any order. Converse is aswell true, so if a/b and c/d are first two terms and next term is p/q then $c/d = (p+a)/(b+q)$ Note that we cannot say $c = p + a, d = b + q$. So, $kc = a + p, kd = b + q$ i.e. $p(k) = kc - a, q(k) = kd - b$, and k happens to be $\lfloor (n+b)/d \rfloor$. Thus for constructing farey sequence of order n, we can start with a/b = 0/1 and c/d = 1/n.
Algo for constructing this tree:

```
void build (int a = 0, int b = 1, int c = 1, int d = 0, int
level = 1) {
    int x = a + c, y = b + d;
    build (a, b, x, y, level + 1);
    build (x, y, c, d, level + 1);
}
// Search algo for fractions, algo will not terminate
irrational nos
string find (int x, int y, int a = 0, int b = 1, int c = 1, int
d = 0) {
    int m = a + c, n = b + d;
    if (x == m and y == n) return "";
    if (x * n < y * m) return "L" + find (x, y, a, b, m, n);
    else return "R" + find (x, y, m, n, c, d);
}
```

Farey sequence of order n is the sequence of completely reduced fractions between 0 and 1 which when in irreducible form have denominators less than or equal to n, arranged in order of increasing size.
$F_5 = \{0/1, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 1/1\}$
The order (n) farey sequence contains all the elements of farey sequence with order $n-1$ and also all irreducible fractions with denominators equal to n but this no. is $\phi(n)$. Thus $L_n = L_{n-1} + \phi(n) = 1 + \sum_{k=1}^{n} \phi(k)$

1.18 Finding Power Of Factorial Divisor

You are given two numbers n and k. Find the largest power of k (say x) such that $n!$ is divisible by k^x.

1.18.1 Prime k

Let's first consider the case of prime k. The explicit expression for factorial

$$n! = 1 \cdot 2 \cdot 3 \ldots (n-1) \cdot n$$

Note that every k-th element of the product is divisible by k, i.e. adds $+1$ to the answer; the number of such elements is $\left\lfloor \frac{n}{k} \right\rfloor$.

Next, every k^2-th element is divisible by k^2, i.e. adds another $+1$ to the answer (the first power of k has already been counted in the previous paragraph). The number of such elements is $\left\lfloor \frac{n}{k^2} \right\rfloor$.

And so on, for every i each k^i-th element adds another $+1$ to the answer, and there are $\left\lfloor \frac{n}{k^i} \right\rfloor$ such elements.

The final answer is

$$\left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{n}{k^2} \right\rfloor + \ldots + \left\lfloor \frac{n}{k^i} \right\rfloor + \ldots$$

The sum is of course finite, since only approximately the first $\log_k n$ elements are not zeros. Thus, the runtime of this algorithm is $O(\log_k n)$.
Implementation:

```
int fact_pow (int n, int k) {
  int res = 0;
  while (n) {
    n /= k;
    res += n;
  }
  return res;
}
```

1.18.2 Composite k

The same idea can't be applied directly. Instead we can factor k, representing it as $k = k_1^{p_1} \cdot \ldots \cdot k_m^{p_m}$. For each k_i, we find the number of times it is present in $n!$ using the algorithm described above - let's call this value a_i. The answer for composite k will be

$$\min_{i=1\ldots m} \frac{a_i}{p_i}$$

1.19 GCD, LCM

```
// time complexity O(log(min(a, b) / gcd(a, b)))
int gcd (int a, int b) { return b == 0 ? a : gcd (b, a %
b); }
int lcm (int a, int b) { return a * (b / gcd (a, b)); }
```

- For a series of numbers if you want next no. to have gcd 1 with all previous no. then $GCD(a_j, LCM(a_1, a_2, \ldots, a_{j-1})) = 1$.
- if $p|N\&p|M$ then $p|gcd(N, M)$ as $N = pk, M = pl \rightarrow$ N, M have p common so gcd will also have p.
- $N|P\&M|P \rightarrow lcm(N, M)|P$.
- $N = gcd(N, m) \Leftrightarrow N|M$
- $M = lcdm(N, M) \Leftrightarrow N|M$
- $gcd(P * N, P * M) = P * gcd(N, M)$
- $lcm(P * N, P * M) = P * lcm(N, M)$
- If $gcd(N_1, N_2) = 1$ then $gcd(N_1 * N_2, M) = gcd(N_1, M) * gcd(N_2, M)$ and $lcm(N_1 * N_2, M) = lcm(N_1, M) * lcm(N_2, M)/M$
- gcd(gcd(N, M), P) = gcd(N, gcd(M, P))
- lcm(lcm(N, M), P) = lcm(N, lcm(M, P))
- gcd(M, N, P) = gcd(gcd(M, N), P) = gcd(M, gcd(N, P))
- lcm(M, N, P) = lcm(lcm(M, N), P) = lcm(M, lcm(N, P))
- for integers $N_1, \ldots, N_k$ and $k \geq 2$,

$$lcm(gcd(N_1, M), gcd(N_2, M), \ldots, gcd(N_k, M)) = gcd(lcm(N_1, \ldots, N_k), M)$$

$$gcd(lcm(N_1, M), lcm(N_2, M), \ldots, lcm(N_k, M)) = lcm(gcd(N_1, \ldots, N_k), M)$$

1.20 Some properties of Fibonacci numbers

- $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$
- Cassini's identity: $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$
- The "addition" rule: $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$
- Applying the previous identity to the case $k = n$, we get: $F_{2n} = F_n(F_{n+1} + F_{n-1})$
- From this we can prove by induction that for any positive integer k, F_{nk} is multiple of F_n. The inverse is also true: if F_m is multiple of F_n, then m is multiple of n.
- GCD identity: $GCD(F_m, F_n) = F_{GCD(m,n)}$
- Every positive integer can be expressed uniquely as a sum of fibonacci numbers such that no two numbers are equal or consecutive fibonacci numbers. This can be done greedily by taking the highest fibonacci no. at each point.
- Fibonacci nos are periodic under modulo. The period of the fibonacci sequence modula a positive integer j is the smallest positive integer m such that such that $F_m \equiv 0(modj)\&F_{m+1} \equiv 1(modj)$
- $$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^p = \begin{bmatrix} fib(p+1) & fib(p) \\ fib(p) & fib(p-1) \end{bmatrix}$$

  Thus higher fibs can be computed in $O(\log p)$
- $$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$
- You can immediately notice that the second term's absolute value is always less than 1, and it also decreases very rapidly (exponentially). Hence the value of the first term alone is "almost" F_n. This can be written strictly as:

$$F_n = \left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}} \right]$$

1.21 Wilson Theorem

States that for a prime no. p, $(p-1)! \mod p = p - 1$.
Note that $n! \mod p$ is 0 if $n \geq p$. Suppose p is prime and is close to input number n. For example 25! mod 29. From wilson theorem, we know that $28! \mod 29 = -1 \equiv 28$, so we basically need to find $(28 * inverse(28, 29) * inverse(27, 29) * inverse(26, 29)) \mod 29$
Time complexity $O((p-n) * \log n)$

1.22 Factorial modulo p in $O(p \log n)$

In some cases it is necessary to consider complex formulas modulo p, containing factorials in both numerator and denominator. We consider the case when p is relatively small. This problem makes sense only when factorials are included in both numerator and denominator of fractions. Otherwise $p!$ and subsequent terms will reduce to zero, but in fractions all multipliers containing p can be reduced, and the resulting expression will be non-zero modulo p.

Thus, formally the task is: You want to calculate $n! \bmod p$, without taking all the multiple factors of p into account that appear in the factorial. Imaging you write down the prime factorization of $n!$, remove all factors p, and compute the product modulo p. We will denote this modified factorial with $n!_{\mod p}$.

Learning how to effectively calculate this modified factorial allows us to quickly calculate the value of the various combinatorial formulae (for example, Binomial coefficients).

1.22.1 Algorithm

Let's write this modified factorial explicitly.

$$\begin{aligned} n!_{\mod p} &= 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-2) \cdot (p-1) \cdot \underbrace{1}_{p} \cdot (p+1) \cdot (p+2) \cdot \ldots \cdot (2p - \\ &\quad \cdot (2p+1) \cdot \ldots \cdot (p^2-1) \cdot \underbrace{1}_{p^2} \cdot (p^2+1) \cdot \ldots \cdot n \pmod p \\ &= 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-2) \cdot (p-1) \cdot \underbrace{1}_{p} \cdot 2 \cdot \ldots \cdot (p-1) \cdot \underbrace{2}_{2p} \cdot 1 \cdot 2 \\ &\quad \cdot \ldots \cdot (p-1) \cdot \underbrace{1}_{p^2} \cdot 1 \cdot 2 \cdot \ldots \cdot (n \bmod p) \pmod p \end{aligned}$$

It can be clearly seen that factorial is divided into several blocks of same length expect for the last one.

$$\begin{aligned} n!_{\mod p} &= \underbrace{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-2) \cdot (p-1) \cdot 1}_{1st} \cdot \underbrace{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-2) \cdot (p-1) \cdot}_{2nd} \\ &\quad \underbrace{\cdot 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (p-2) \cdot (p-1) \cdot 1}_{pth} \cdot \ldots \cdot \underbrace{1 \cdot 2 \cdot \ldots \cdot (n \bmod p)}_{tail} \end{aligned}$$

The general part of the blocks it is easy to count — it's just $(p-1)! \bmod p$ that you can calculate programmatically or via Wilson theorem, according to which $(p-1)! \bmod p = p - 1$. To multiply these common parts of all blocks, we can raise the value to the higher power modulo p, which can be done in $O(\log n)$ operations using Binary Exponentiation; however, you may notice that the result will always be either 1 or $p-1$, depending on the parity of the index. The value of the last partial block can be calculated separately in $O(p)$. Leaving only the last elements of the blocks, we can examine that:

$$n!_{\mod p} = \underbrace{\ldots \cdot 1}\cdot\underbrace{\ldots \cdot 2}\cdot \ldots \cdot \underbrace{\ldots \cdot (p-1)}\cdot\underbrace{\ldots \cdot 1}\cdot\underbrace{\ldots \cdot 1}\cdot\underbrace{\ldots \cdot 2}\cdots$$

And again, by removing the blocks that we already computed, we receive a "modified" factorial but with smaller dimension ($\lfloor n/p \rfloor$ blocks remain). Thus, in the calculation of "modified" the factorial $n! \mod p$ we did $O(p)$ operations and are left with the calculation of $(n/p)! \mod p$. Revealing this recursive dependence, we obtain that the recursion depth is $O(\log_p n)$, the total asymptotic behavior of the algorithm is thus $O(p \log_p n)$.

1.22.2 Implementation

We don't need recursion because this is a case of tail recursion and thus can be easily implemented using iteration.

```
int factmod(int n, int p) {
  int res = 1;
  while (n > 1) {
    res = (res * ((n/p) % 2 ?  p-1 : 1)) % p;
    for (int i = 2; i <= n%p; ++i)
      res = (res * i) % p;
    n /= p;
  }
  return res % p;
}
```

This implementation works in $O(p \log_p n)$.

1.23 Modular Inverse

A modular multiplicative inverse of an integer a is an integer x such that $a \cdot x$ is congruent to 1 modular some modulus m. $a \cdot x \equiv 1 \mod m$. We will also denote x with a^{-1}.

It can be proven that the modular inverse exists if and only if a and m are relatively prime (i.e. $\gcd(a, m) = 1$).

For an arbitrary (but coprime) modulus m: $a^{\phi(m)-1} \equiv a^{-1} \mod m$
For a prime modulus m: $a^{m-2} \equiv a^{-1} \mod m$ From these results, we can easily find the modular inverse using the binary exponentiation algorithm,

which works in O(logm) time.
Or Consider the following equation (with unknown x and y):

$$a \cdot x + m \cdot y = 1$$

This is a Linear Diophantine equation in two variables. As shown in the linked article, when $\gcd(a, m) = 1$, the equation has a solution which can be found using the extended Euclidean algorithm. Note that $\gcd(a, m) = 1$ is also the condition for the modular inverse to exist.

Now, if we take modulo m of both sides, we can get rid of $m \cdot y$, and the equation becomes:

$$a \cdot x \equiv 1 \mod m$$

Thus, the modular inverse of a is x.

The implementation is as follows:

```
int x, y;
int g = extended_euclidean(a, m, x, y);
if (g != 1) {
    cout << "No solution!";
}
else {
    x = (x % m + m) % m;
    cout << x << endl;
}
```

Notice that we way we modify x. The resulting x from the extended Euclidean algorithm may be negative, so x

The problem is the following: we want to compute the modular inverse for every number in the range $[1, m-1]$.

Applying the algorithms described in the previous sections, we can obtain a solution with complexity $O(m \log m)$.

Here we present a better algorithm with complexity $O(m)$. However for this specific algorithm we require that the modulus m is prime.

We denote by $\text{inv}[i]$ the modular inverse of i. Then for $i > 1$ the following equation is valid:

$$\text{inv}[i] = -\left\lfloor \frac{m}{i} \right\rfloor \cdot \text{inv}[m \bmod i] \bmod m$$

Thus the implementation is very simple:

```
inv[1] = 1;
for(int i = 2; i < m; ++i)
    inv[i] = (m - (m/i) * inv[m%i] % m) % m;
```

Proof

We have:

$$m \bmod i = m - \left\lfloor \frac{m}{i} \right\rfloor \cdot i$$

Taking both sides modulo m yields:

$$m \bmod i \equiv -\left\lfloor \frac{m}{i} \right\rfloor \cdot i \mod m$$

Multiply both sides by $i^{-1} \cdot (m \bmod i)^{-1}$ yields

$$(m \bmod i) \cdot i^{-1} \cdot (m \bmod i)^{-1} \equiv -\left\lfloor \frac{m}{i} \right\rfloor \cdot i \cdot i^{-1} \cdot (m \bmod i)^{-1} \mod m,$$

which simplifies to:

$$i^{-1} \equiv -\left\lfloor \frac{m}{i} \right\rfloor \cdot (m \bmod i)^{-1} \mod m,$$

Binomial coeff modulo large prime no

```
fact[0] = 1;
for (int i = 1; i <= maxn; i++) {
    fact[i] = (fact[i - 1] * (i % m)) % m;
}
// afterwards we can compute binomial coeff in O(log m)
ll getC(int n, int k) {
    ll res = fact[n];
    ll div = fact[k] * fact[n - k] % m;
    div = pow (div, m - 2, m);
    return (res * div) % m;
}
```

Binomial Coeff modulo prime power Here we want to compute the binomial coefficient modulo some prime power, i.e. $m = p^b$ for some prime p. If $p > \max(k, n-k)$, then we can use the same method as described in the previous section. But if $p \le \max(k, n-k)$, then at least one of $k!$ and $(n-k)!$ are not coprime with m, and therefore we cannot compute the inverses - they don't exist. Nevertheless we can compute the binomial coefficient.

The idea is the following: We compute for each $x!$ the biggest exponent c such that p^c divides $x!$, i.e. $p^c | x!$. Let $c(x)$ be that number. And let $g(x) := \frac{x!}{p^{c(x)}}$. Then we can write the binomial coefficient as:

$$\binom{n}{k} = \frac{g(n)p^{c(n)}}{g(k)p^{c(k)}g(n-k)p^{c(n-k)}} = \frac{g(n)}{g(k)g(n-k)}p^{c(n)-c(k)-c(n-k)}$$

The interesting thing is, that $g(x)$ is now free from the prime divisor p. Therefore $g(x)$ is coprime to m, and we can compute the modular inverses of $g(k)$ and $g(n-k)$.

After precomputing all values for g and c, which can be done efficiently using dynamic programming in $\mathcal{O}(n)$, we can compute the binomial coefficient in $O(\log m)$ time. Or precompute all inverses and all powers of p, and then compute the binomial coefficient in $O(1)$.

Notice, if $c(n) - c(k) - c(n-k) \ge b$, than $p^b | p^{c(n)-c(k)-c(n-k)}$, and the binomial coefficient is 0.

Binomial coefficient modulo an arbitrary number Now we compute the binomial coefficient modulo some arbitrary modulus m.

Let the prime factorization of m be $m = p_1^{e_1} p_2^{e_2} \cdots p_h^{e_h}$. We can compute the binomial coefficient modulo $p_i^{e_i}$ for every i. This gives us h different congruences. Since all moduli $p_i^{e_i}$ are coprime, we can apply the Chinese Remainder Theorem to compute the binomial coefficient modulo the product of the moduli, which is the desired binomial coefficient modulo m.

Lucas's theorem For non-negative integers m and n and a prime p, the following congruence relation holds:
$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod p$
where
$m = m_k p^k + m_{k-1} p^{k-1} + \cdots + m_1 p + m_0$
and
$n = n_k p^k + n_{k-1} p^{k-1} + \cdots + n_1 p + n_0$
are the base p expansions of m and n respectively. This uses the convention that $\binom{m}{n} = 0$ if m ¡ n.

Binomial coefficient for large n and small modulo When n is too large, the $\mathcal{O}(n)$ algorithms discussed above become impractical. However, if the modulo m is small there are still ways to calculate $\binom{n}{k} \bmod m$.

When the modulo m is prime, there are 2 options:

Lucas's theorem can be applied which breaks the problem of computing $\binom{n}{k} \bmod m$ into $\log_m n$ problems of the form $\binom{x_i}{y_i} \bmod m$ where $x_i, y_i < m$. If each reduced coefficient is calculated using precomputed factorials and inverse factorials, the complexity is $\mathcal{O}(m + \log_m n)$.
The method of computing factorial modulo P can be used to get the required g and c values and use them as described in the section of modulo prime power. This takes $\mathcal{O}(m \log_m n)$.

When m is not prime but square-free, the prime factors of m can be obtained and the coefficient modulo each prime factor can be calculated using either of the above methods, and the overall answer can be obtained by the Chinese Remainder Theorem.

When m is not square-free, a generalization of Lucas's theorem for prime powers can be applied instead of Lucas's theorem. I don't think that I need to consider that.

1.24 Gray Code

Gray code is a binary numeral system where two successive values differ in only one bit.

For example, the sequence of Gray codes for 3-bit numbers is: 000, 001, 011, 010, 110, 111, 101, 100, so $G(4) = 6$.

1.24.1 Finding Gray Code

Let's look at the bits of number n and the bits of number $G(n)$. Notice that i-th bit of $G(n)$ equals 1 only when i-th bit of n equals 1 and $i+1$-th bit equals 0 or the other way around (i-th bit equals 0 and $i+1$-th bit equals 1). Thus, $G(n) = n \oplus (n >> 1)$:

```
int g (int n) {
    return n ^ (n >> 1);
}
```

1.24.2 Finding inverse gray code

Given Gray code g, restore the original number n.

We will move from the most significant bits to the least significant ones (the least significant bit has index 1 and the most significant bit has index k). The relation between the bits n_i of number n and the bits g_i of number g:

$$\begin{aligned}
n_k &= g_k, && (10)\\
n_{k-1} &= g_{k-1} \oplus n_k = g_k \oplus g_{k-1}, && (11)\\
n_{k-2} &= g_{k-2} \oplus n_{k-1} = g_k \oplus g_{k-1} \oplus g_{k-2}, && (12)\\
n_{k-3} &= g_{k-3} \oplus n_{k-2} = g_k \oplus g_{k-1} \oplus g_{k-2} \oplus g_{k-3}, && (13)\\
&\vdots && (14)
\end{aligned}$$

The easiest way to write it in code is:

```
int rev_g (int g) {
  int n = 0;
  for (; g; g >>= 1)
    n ^= g;
  return n;
}
```

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1.25 Discrete Logarithm

The discrete logarithm is an integer x solving the equation

$a^x \equiv b \pmod m$

where a and m are relatively prime. (Note: if they are not relatively prime, then the algorithm described below is incorrect, though it can be modified so that it can work).

In this article, we describe the Baby step - giant step algorithm, proposed by Shanks in 1971, which has complexity $O(\sqrt{m}\log m)$. This algorithm is also known as meet-in-the-middle, because of it uses technique separation of tasks in half.

1.25.1 Algorithm

Consider the equation:

$a^x \equiv b \pmod m$

where a and m are relatively prime.

Let $x = np - q$, where n is some pre-selected constant (we will describe how to select n later). p is known as giant step, since increasing it by one increases x by n. Similarly, q is known as baby step.

Obviously, any value of x in the interval $[0; m)$ can be represented in this form, where $p \in [1; \lceil \frac{m}{n} \rceil]$ and $q \in [0; n]$.

Then, the equation becomes:

$a^{np-q} \equiv b \pmod m$.

Using the fact that a and m are relatively prime, we obtain:

$a^{np} \equiv ba^q \pmod m$

This new equation can be rewritten in a simplified form:

$f_1(p) = f_2(q)$.

This problem can be solved using the method meet-in-the-middle as follows:

We calculate f_1 for all possible values of p. Sort these values. For each value of q, calculate f_2, and look for the corresponding value of p using the sorted array of f_1 using binary search.

1.25.2 Complexity

For each value of p, we can calculate $f_1(p)$ in $O(\log m)$ using binary exponentation algorithm. Similar for $f_2(q)$.

In the first step of the algorithm, we need to calculate f_1 for every possible values of p, and then sort them. Thus, this step has complexity:

$O(\lceil \frac{m}{n} \rceil (\log m + \log \lceil \frac{m}{n} \rceil)) = O(\lceil \frac{m}{n} \rceil \log m)$

In the second step of the algorithm, we need to calculate $f_2(q)$ for each possible value of q, and then do a binary search on the array of values of f_1, thus this step has complexity:

$O(n(\log m + \log \frac{m}{n})) = O(n \log m)$.

Now, when we add these two complexity, we would get $\log m$ multiplied by n and m/n, which has minimum value when $n = m/n$, which means, to achieve optimal performance, n should be chosen such that:

$n = \sqrt{m}$.

Then, the complexity of the algorithm becomes:

$O(\sqrt{m}\log m)$.

1.25.3 Implementation

The simplest implementation

In the following code, function powmod performs binary exponential $a^b \pmod m$, and function solve produces a proper solution to the problem. It will returns -1 if there is no solution, and returns one possible solution in case a solution exists.

```
int powmod (int a, int b, int m) {
  int res = 1;
  while (b > 0)
    if (b & 1) {
      res = (res * a) % m;
      --b;
    }
    else {
      a = (a * a) % m;
      b >>= 1;
    }
  return res % m;
}

int solve (int a, int b, int m) {
  int n = (int) sqrt (m + .0) + 1;
  map<int,int> vals;
  for (int i=n; i>=1; --i)
    vals[ powmod (a, i * n, m) ] = i;
  for (int i=0; i<=n; ++i) {
    int cur = (powmod (a, i, m) * b) % m;
    if (vals.count(cur)) {
      int ans = vals[cur] * n - i;
      if (ans < m)
        return ans;
    }
  }
  return -1;
}
```

In this code, we used map from C++ STL to store the values of $f_1(i)$. Internally, map uses red-black-tree to store values. This code is a little bit slower than if we uses array and binary search for $f1$, but is much easier to write.

Another thing to note is that, if there are multiple values of p that has same value of f_1, we only store one such value. This works in this case because we only want to return one possible solution. If we need to return all possible solutions, we need to change map¡int,int¿ to, say, map¡int, vector¡int¿ ¿. And we also need to change the second step accordingly.

1.25.4 Improved implementation

j

A possible improvement is to get rid of binary exponentiation in the second phase of the algorithm. This can be done by keeping a variable that multiplies by a each time we increase q. With this change, the complexity of the algorithm is still the same, but now the log part is only for *map*. Instead of *map*, we can also use hash table (unordered_map in GNU C++) which has complexity $O(1)$ for inserting and searching. And when the value of m is small enough, we can also get rid of *map*, and use a regular array to store and lookup values of f_1.

```
int solve (int a, int b, int m) {
  int n = (int) sqrt (m + .0) + 1;

  int an = 1;
  for (int i=0; i<n; ++i)
    an = (an * a) % m;

  map<int,int> vals;
  for (int i=1, cur=an; i<=n; ++i) {
    if (!vals.count(cur))
      vals[cur] = i;
    cur = (cur * an) % m;
  }

  for (int i=0, cur=b; i<=n; ++i) {
    if (vals.count(cur)) {
      int ans = vals[cur] * n - i;
      if (ans < m)
        return ans;
    }
    cur = (cur * a) % m;
  }
  return -1;
}
```

1.26 Chinese Remainder Theorem

Given pairwise coprime positive integers $n_1, n_2, \ldots, n_k$ and arbitrary integers $a_1, a_2, \ldots, a_k$ the system of simultaneous congruences

$$
\begin{aligned}
x &\equiv a_1(\mod n_1) && (15)\\
x &\equiv a_2(\mod n_2) && (16)\\
&\vdots && (17)\\
x &\equiv a_k(\mod n_k) && (18)
\end{aligned}
$$

has a soln and the soln is unique modulo $N = n_1 * n_2 * \cdots * n_k$

To compute that soln

1. Compute $N = n_1 * n_2 * \cdots * n_k$
2. for each $i = 1, 2, \ldots, k$ compute $y_i = N/n_i$
3. for each $i = 1, 2, \ldots, k$ compute $z_i = y_i^{-1} \mod n_i$ z_i exist since $n_1, n_2, \ldots, n_k$ are pairwise coprime.
4. The integer $x = \sum_{i=1}^{k} a_i * y_i * z_i$ is a soln to the system of congruences and $x \mod N$ is the unique soln modulo N.

Note:

- $a \leftrightarrow (a_1, a_2, \ldots, a_k)$ where $(a_i = a \mod n_i)$
- $b \leftrightarrow (b_1, b_2, \ldots, b_k)$ where $(b_i = b \mod n_i)$
- Then

$$(a+b) \mod N \leftrightarrow ((a_1+b_1) \mod n_1, \ldots, (a_k+b_k) \mod n_k)$$

$$(a-b) \mod N \leftrightarrow ((a_1-b_1) \mod n_1, \ldots, (a_k-b_k) \mod n_k)$$

$$(a*b) \mod N \leftrightarrow ((a_1*b_1) \mod n_1, \ldots, (a_k*b_k) \mod n_k)$$

Corollary if $n_1, n_2, \dots, n_k$ are pairwise coprime and $n = n_1 * n_2 * \dots * n_k$, then for all integers x and a, $x \equiv a(\mod n_i)$ fot $i = 1, 2, \dots, k$ iff $x \equiv a(mod n)$
Now if moduli $n_i's$ are not neccessarily pairwise coprime. Let $d_{i,j} = gcd(n_i, n_j)$ for $i \neq j$. Then the above system has a simultaneous soln iff $d_{i,j}$ divides $a_i - a_j$ for all $i \neq j$. Further such a soln is unique module $lcm(n_1, n_2, \dots, n_k)$.
Problem and solution You are given two pairs (main goal is to solve it for t pairs) of integers (a_1, n_1), (a_2, n_2). **There is no assumption that n_1 and n_2 are coprime**. Find an integer x that satisfies

$$\begin{cases} x \equiv a_1 \pmod{n_1} \\ x \equiv a_2 \pmod{n_2} \end{cases}$$

This system of congruences implies that

$$\begin{cases} x = a_1 + n_1 k_1 \\ x = a_2 + n_2 k_2 \end{cases}$$

for some integers k_1, k_2. Let's equate right sides of these equations. We get $a_1 + n_1 k_1 = a_2 + n_2 k_2$, which is the same as $n_1(-k_1) + n_2 k_2 = a_1 - a_2$. Since we know n_1, n_2, a_1, a_2, this is just linear diophantine equation. Let $d = GCD(n_1, n_2)$. It divides left-hand side of the equation, so for this equation to have solutions, d must also divide right-hand side which is $a_1 - a_2$. Now, thanks to Extended Euclidean Algorithm we can find (x', y') such that $n_1 x' + n_2 y' = d$ (by just calling $(x', y') = exGCD(n_1, n_2)$).

After multiplying both sides by $\frac{a_1 - a_2}{d}$ we get $n_1 x' \frac{a_1 - a_2}{d} + n_2 y' \frac{a_1 - a_2}{d} = a_1 - a_2$, so $k_1 = -x' \frac{a_1 - a_2}{d}$ and $k_2 = y' \frac{a_1 - a_2}{d}$. We can now substitute k_1 into $x = a_1 + n_1 k_1$ to get our solution: $x = a_1 + x' \frac{a_2 - a_1}{d} n_1$.

How many solutions are there? Now you may think: is this the only solution? Since we're dealing with congruences, you may guess that the answer is no. Let's say we have two different solutions x_1 and x_2. Now we have $x_1 \equiv a_1 \mod n_1$ and $x_2 \equiv a_1 \mod n_1$, so from transitivity of congruences we get $x_1 \equiv x_2 \mod n_1$. Doing the same thing for n_2 we get $x_1 \equiv x_2 \mod n_2$. These two congruences are equivalent to $x_1 \equiv x_2 \mod LCM(n_1, n_2)$. It means that any two solutions are congruent modulo $LCM(n_1, n_2)$. You can actualy check that every x given by our formula $\pm u * LCM(n_1, n_2)$ satisfies the original system of congruences. So there are infinitely many solutions.

Overflow issue Let's look at computational aspect of $x = a_1 + x' \frac{a_2 - a_1}{d} n_1$. Since the numbers can get quite big, we should perform our calculations modulo $LCM(n_1, n_2)$, so if $lcm = LCM(n_1, n_2)$ then $x = (a_1 + ((x' \cdot (a_2 - a_1)/d) \mod lcm) \cdot n_1 \mod lcm) \mod lcm$ (let's not worry about the amount of lcm's, we'll figure it out in code). But $LCM(n_1, n_2, ..., n_k) \leq n_1 n_2 ... n_k$, so if n_1 and n_2 are $\leq 10^9$ then $lcm \leq 10^{18}$, $x' \leq 10^9$, $\frac{a_2 - a_1}{d} \leq 10^9$. We can calculate $x' \frac{a_2 - a_1}{d}$, but since it is up to 10^{18} and lcm is also up to 10^{18} the final multiplication by n_1 will oveflow. How to handle this issue? Some 64bit compilers have special type int128_t that allows to store numbers up to $2^{127} - 1$, but 64bit compilers are not too common on competitive programming contests. Let's look at a formula for lcm. We know that $lcm = \frac{n_2}{d} \cdot n_1$. Modulo has a very nice property that says $ca \mod cb = c(a \mod b)$, so if we take $a = x' \frac{a_2 - a_1}{d}$, $b = \frac{n_2}{d}$, $c = n_1$ we get that $(x' \frac{a_2 - a_1}{d} \mod lcm) \cdot n_1 \mod lcm$ is equal to $(x' \frac{a_2 - a_1}{d} \mod \frac{n_2}{d}) \cdot n_1 \mod lcm$. Since $\frac{n_2}{d} \leq 10^9$, the overflow issue is solved.
Case of t congruences The last thing to consider is how to handle case of more than 2 congruences. Let's say we have t congruences $x \equiv a_i \mod n_i$ for $i = 1, 2, ..., t$. We can just merge equations one by one. After merging first two congruences we get something like $x \equiv s \mod LCM(n_1, n_2)$ and now we can merge it in the same way with $x \equiv a_3 \mod n_3$ and so on. The complexity of this algorithm is just $O(t \log LCM(n_1, n_2, ..., n_t))$.

```
#include<bits/stdc++.h>
using namespace std;
const int N = 20;
long long GCD(long long a, long long b) { return (b == 0) ? a :
GCD(b, a % b); }
inline long long LCM(long long a, long long b) { return a /
GCD(a, b) * b; }
inline long long normalize(long long x, long long mod) { x %=
mod; if (x < 0) x += mod; return x; }
struct GCD_type { long long x, y, d; };
GCD_type ex_GCD(long long a, long long b)
{
    if (b == 0) return {1, 0, a};
    GCD_type pom = ex_GCD(b, a % b);
    return {pom.y, pom.x - a / b * pom.y, pom.d};
}
int testCases;
int t;
long long a[N], n[N], ans, lcm;
int main()
{
    ios_base::sync_with_stdio(0);
    cin.tie(0);
    cin >> t;
    for(int i = 1; i <= t; i++) cin >> a[i] >> n[i],
    normalize(a[i], n[i]);
    ans = a[1];
    lcm = n[1];
    for(int i = 2; i <= t; i++)
    {
        auto pom = ex_GCD(lcm, n[i]);
        int x1 = pom.x;
        int d = pom.d;
        if((a[i] - ans) % d != 0) return cerr << "No solutions"
        << endl, 0;
        ans = normalize(ans + x1 * (a[i] - ans) / d % (n[i] /
        d) * lcm, lcm * n[i] / d);
        lcm = LCM(lcm, n[i]); // you can save time by replacing
        above lcm * n[i] /d by lcm = lcm * n[i] / d
    }
    cout << ans << " " << lcm << endl;

    return 0;
}
```

Remainders Game
Assume the answer of a test is *No*. There must exist a pair of integers x_1 and x_2 such that both of them have the same remainders after dividing by any c_i, but they differ in remainders after dividing by k. Find more facts about x_1 and x_2!

Solution

Consider the x_1 and x_2 from the hint part. We have $x_1 - x_2 \equiv 0 \pmod{c_i}$ for each $1 \leq i \leq n$.

So:

$$lcm(c_1, c_2, ..., c_n) \mid x_1 - x_2$$

We also have $x_1 - x_2 \not\equiv 0 \pmod{k}$. As a result:

$k \nmid lcm(c_1, c_2, ..., c_n)$ as k does not divide x1 - x2

We've found a necessary condition. And I have to tell you it's also sufficient!

Assume $k \nmid lcm(c_1, c_2, ..., c_n)$, we are going to prove there exists x_1, x_2 such that $x_1 - x_2 \equiv 0 \pmod{c_i}$ (for each $1 \leq i \leq n$), and $x_1 - x_2 \not\equiv 0 \pmod{k}$.

A possible solution is $x_1 = lcm(c_1, c_2, ..., c_n)$ and $x_2 = 2 \times lcm(c_1, c_2, ..., c_n)$, so the sufficiency is also proved.

So you have to check if $lcm(c_1, c_2, ..., c_n)$ is divisible by k, [redacted]

1.27 Josephus Problem

Given natural no. n and k, we write natural nos 1 to n along the circle. Then delete the kth no, (i..e no. k initially) and then from this position next kth no. is deleted and so on.
The process stops when there is only one no. remaining, problem asks us to find this no.

1.27.1 For k = 2

$$n = 1, 2, 3, 4, 5, 6 \quad (19)$$
$$f(n) = 1, 1, 3, 1, 3, 5 \quad (20)$$

Thm: if $n = 2^m + l$ where $0 \leq l < 2^m$ then $f(n) = 2l + 1$.

1.27.2 For general $k \geq 1$

We should note that after the kth person is killed we are left with a circle of n - 1 and we start the next count with the person whose no in the original problem was $(k \mod n) + 1$. The position of the survivior in the remaining circle would be $f(n-1, k)$ if we start counting at 1; shifting this to account for the fact that we are starting at $(k \mod n) + 1$ yilds the recurrence $f(n, k) = ((f(n-1, k) + k - 1) \mod n) + 1$ with $f(1, k) = 1$ which takes the simpler form $g(n, k) = (g(n-1, k) + k) \mod n$ with $g(1, k) = 0$. This approach has running time O(n).
But for small k and large n, there is another approach. The second apprach alaso uses dp but has running time O(klogn) and is based on cosidering killing kth, 2kth, ..., $(\lfloor n/k \rfloor * k)$th person as one step, then changing the numbering.

$$\begin{aligned} g(n, k) &= 0 \text{ if n = 1 (because of 0 indexing)} \\ &= (g(n-1, k) + k) \mod n \text{ if } 1 < n < k \\ &= \lfloor k * ((g(n_m, k) - n \mod k) \mod n_m)/(k-1) \rfloor \text{ where } n_m = n - \end{aligned}$$

1.28 Side Notes

1. relatively prime is same as coprime.
2. 2 nos a, b are said to be congruent modulo n, if there difference is an integer multiple of n, i.e. they give same remainder . We denote this as $a \equiv b(\text{mod n})$
3. For (a, b) three operations exist:
   (a) If a, b are even then (a/2, b/2).
   (b) (a + 1, b + 1)
   (c) If (a, b) exist and (b, c) exist then (a, c) exist.
   Every (x, y) pair s.t. $x < y$ can be transformed to (1, 1 + k x d) where k is any positive integer and d is maximal odd divisor of y - x. We want to generate $(1, a_1), (1, a_2), \dots, (1, a_n)$. So $d | gcd(a_1 - 1, a_2 - 1, \dots, a_n - 1)$. No. of pairs (x, y) which have d as their maximal odd divisors: these are cards with y - x = d, 2d, 4d, 8d, etc. as odd into odd is odd thus we can't multiply with any odd number as it will give bigger odd divisor. Don't forget that numbers x, y must no exceed m. Total no. of cards with some fixed difference t = y - x is exactly m - t. $1 \leq x < y \leq m$
   $1 - x \leq 0 < t \leq m - x$
   $1 \leq x \leq m - t$
   $m - t \geq 1$
   $t \leq m - 1$
   $2^l * d \leq m - 1$
   So sum up $m - 2^l * d$ where d is any odd divisor of gcd$(a_1 - 1, \dots)$ and l is such that $2^l * d \leq m - 1$.

4. People in cycle will commit suicide.
5. Every even no. greater than or equal to 4 can be expressed as a sum of 2 prime nos.
6. No. of digits in a no. $n = \lfloor(\log_{10} n)\rfloor + 1$
7. No. of digits in $\binom{n}{k} = \lfloor(\sum_{i=n-k+1}^{n} \log_{10} i - \sum_{i=1}^{k} \log_{10} i))\rfloor + 1$
8. No. of digits of a no. in some base b= $floor(1 + \log_b no. + eps)$. Also make sure that input no. is not 0.
9. for $\binom{n}{r}$ always do $r = min(r, n-r)$. Also to compute it either we can use dp or for a specific pair, if it is guarenteed that the final solution lies within data types limit then we can compute it as.

```
ll ncr(ll n, ll r) {
    r = min (r, n - r);
    ll res = 1;
    for (int k = 1; k <= r; k++, n--) {
        res *= n;
        res /= k;
    }
}
// Another way is
int c[maxn + 5][maxn + 5];
for (int n = 0; n <= maxn; n++) {
    c[n][0] = c[n][n] = 1;
    for (int k = 1; k < n; k++) {
        c[n][k] = c[n - 1][k - 1] + c[n - 1][k];
    }
}
// Or if long arithmetic is allowed, we can do fact[n] /
fact[n - k] / fact[k].
```

10. $(t^a - 1)/(t^b - 1)$ is not an integer with less than 100 digits if t = 1 or $a < b$ or $a \bmod b \neq 0$ or $(a - b) * \log_{10} t > 99.0$
11.
```
for (int j = 0; j < bigint_var.a.size (); j++) {
    int temp = bigint_var.a[j];
    while (temp > 9) {
        sum += temp % 10;
        temp /= 10;
    }
    sum += temp;
}
```

12. To get all divisors of a number n.

```
for (int i = 1; i * i <= n; i++) {
    if (n % i == 0) {
        d.push_back(i);
        if (i != n / i) d.push_back (n / i);
    }
}
```

13. We can find the nth root using pow func.
14. Some times we can generate first few terms and then look up at oeis.org
15.
```
// to compute (a * b) mod m when a * b can go above ll
ll bmodm;
ll compute (ll a, ll &b, ll &m) {
    if (a == 1) return bmodm;
    if (a & 1) return (((2 % m) * compute (a / 2, b, m) +
    bmodm) % m);
    else return ((2 % m) * compute (a/2, b, m)) % m;
}
```

16. Say $A = [a, b, c, d, e]$ and $P = [4, 3, 2, 0, 1] \rightarrow$ final array is $[e, d, c, a, b]$. but if we were to get P^k applied to A, we should first represent P by matrix.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And then compute it kth power and then do $A * P^k$

1.29 Important Problems

- UVA 11310 Prob: $dp[n] = dp[n-1] + 4 * dp[n-2] + 2 * dp[n-3]$
- UVA 11204 Prob: Tricky problem, it just asks *How many possible arrangements maximizing the assignment of the **first priority***. Thus only first priority instrument matters, so if 2 want A, 4 want B, 3 want C, then ans is $2 * 4 * 3$.
- UVA 10790 Prob: If there is an intersection that means we have a quadrilateral, hence answer is the number of quadrilaterals $= \binom{a}{2} * \binom{b}{2}$.
- last non zero digit of fact(n), Sol. If you know multiplicatioin limit, take mod with $(10^{\text{no. of digits}})$
- France 98, Sol.
- How many zeros and how many digits? Sol: then iterate through factors of base and get their powers in n!, take min. of all such powers divided by power of prime factor in that base. And for how many digits part, use that log formula.
- Is $n!$ divisible by m? Sol: Use prime factors
- Given n, maximize (find x) $n - p * x$ where $p * x \leq n < (p + 1) * x$ which somehow happens with $p = 1$
- **Prob:** Lenghts from 1 to n, max. no. of triangles?
  Sol:

```
void precal () {
    F[3] = P[3] = 0;
    ll var = 0;
    for (int i = 4; i <= 1000000; i++) {
        if (i % 2 == 0) {
            var++;
        }
        P[i] = P[i - 1] + var;
        F[i] = F[i - 1] + P[i];
    }
    // F[n] has ans
}
```

- Bottom right corner of a chess board must be white. If c == 0/1 bottom right corner of painting is black/white.

```
if (c == 0) ans = (n - 7) * (m - 7) / 2
else ans = ((n - 7) * (m - 7) + 1) / 2
```

- for 2 nos (a, b) output 2 nos (c, d) such that a = gcd(c, d) & b = lcm(c, d) and c is minimum. Sol: basically soln exist if $b \bmod a = 0$ and soln is c = a, d = b.

2 Graphs

2.1 Basic

- The shortest path b/w vertices i and j after edge (u, v) is added is equal to $min(d_{i,j}, d_{i,u} + w_{u,v} + d_{v,j}, d_{i,v} + w_{v,u} + d_{u,j})$
- In BFS, in case of multiple sources, enqueue them all in queue and set dist[v] = 0. Some trick works in case of dijkstra.
- Also in case we are given both source and destination then we can terminate our while loop by checking whether the element we pop from queue is the destination or not. Same trick would work with weighted graph provided we have no -ve edges (dijkstra)
- Adjacency matrix equal to its transpose that implies undirected graph.
- To check whether the graph is bipartite, we can do bfs and color vertices 0/1.
- Graph Check

```
void graphcheck (int u) {
    dfs_num[u] = explored;
    for (auto &v : adjlist[u]) {
        if (dfs_num[v] == unvisited) {  // tree edge
            dfs_parent[v] = u;
            graphcheck (v);
        } else if (dfs_num[v] == explored) {  // back edge
        hence not DAG.
            if (v == dfs_parent[u]) cout << "two ways\n"
            else cout << "back edge\n"
        } else {  // dfs_num[v] == visited
            // forward/cross edge
            // [u [v v] u] this is tree/forward
            // [v [u u] v] back
            // [v v] [u u] cross
        }

    }
}
```

- Floodfill

```
int R, C;
string grid[100];
int dr[] = {1, 1, 0, -1, -1, -1, 0, 1};
int dc[] = {0, 1, 1, 1, 0, -1, -1, -1};
int floodfill (int r, int c, char c1, char c2) {
    if (r < 0 || r >= R || c < 0 || c >= C) return 0;
    if (grid[r][c] != c1) return 0;
    int ans = 0;
    grid[r][c] = c2;
    for (int d = 0; d < 8; d++) {
        ans += floodfill (r + dr[d], c + dc[d], c1, c2);
    }
    return ans;
}
```

- We can find the number of connected components by simply doing dfs (offline), and online using UFDS.

- Highest average is possible only by taking large

2.2 Articulation Points and Bridges (undirected graph)

- An articulation point is defined as a vertex in a graph G whose removal (all edges incident to this vertex are also removed) disconnects G. A graph without any articulatioin point is called "Biconnected" Similarly, a "Bridge" is defined as an edge in a graph G whose removal disconnects G.

```
        // dfs_low[u] will store the lowest iteration count
        vertex u can reach from u's current DFS Tree (i.e u
        can only reach vertices of lower iteration count
        only if a back-edge exists in one of its children)
vector<int> dfs_num;
vector<int> dfs_low;
vector<int> dfs_parent;
vector<bool> articulation_vertex;
int dfs_root, root_children;

void ArticulationPoint(int u)
{
    // Initially, dfs_num = dfs_low
    dfs_num[u] = dfs_low[u] = dfs_num_counter++;
    for(int i = 0; i < adj_list[u].size(); i++)
    {
        int v = adj_list[u][i];

        // v was not previously visited, i.e a normal tree
        edge
        if(dfs_num[v] == -1)
        {
            dfs_parent[v] = u;

            // special case if u is root
            if(u == dfs_root) root_children++;

            ArticulationPoint(v);

            // To detect articulation points
            if(dfs_low[v] >= dfs_num[u])
                articulation_vertex[u] = true;
            // To detect bridges
            if (dfs_low[v.first] > dfs_num[u])
                printf(" Edge (%d, %d) is a bridge\n", u,
                v.first);

            // update dfs_low[u] if dfs_low[v] is lower
            // i.e a back-edge exists in one of u's
            children
            dfs_low[u] = min(dfs_low[u], dfs_low[v]);
        }
        // if v was previously visited, and v is not the
        parent of u
        // then a back-edge certainly exists, not a direct
        cycle
        // update dfs_low[u] to store dfs_num of ancestor
        is lower than
        // current dfs_low
        else if(v != dfs_parent[u])
            dfs_low[u] = min(dfs_low[u], dfs_num[v]);
    }
}

// Example main usage
int main()
{
    dfs_num_counter = 0;
    dfs_num.clear(); dfs_num.resize(N, -1);
    dfs_low.clear(); dfs_low.resize(N, 0);
    dfs_parent.clear(); dfs_parent.resize(N, 0);
    articulation_vertex.clear();
    articulation_vertex.resize(N, false);

    for(int i = 0; i < N; i++)
        if (dfs_num[i] == -1)
        {
            dfs_root = i; root_children = 0;
            ArticulationPoint(i);

            // special case for root
            articulation_vertex[dfs_root] = (root_children
            > 1);
        }
}
/* Variation

A slight variation to this problem is how many disconnected
components would result as a direct consequence of removing
a vertex u

Another variation is to find the articulation vertex whose
removal would cause a greater amount of components to be
disconnected.
General Idea of Variation

Instead of keeping track of whether or not a node is an
articulation point using vector<bool> articulation_vertex,
we'll use a vector<int> articulation_vertex to keep track
of how many components will be connected after the removal
of vertex u.

To achieve this, we'll first assume that each node in our
graph G is not an articulation vertex. In other words, the
removal of any node u in G will result in there being only
one connected component (G will remain one entity and not
be disconnected).

We'll then use the same algorithm we've used before,
however, this around around, whenever we find that u is an
articulation vertex relative to one of its children, we'll
increment articulation_vertex[u].

So, if we have a vertex u with say, 3 child components with
no back-edges, the removal of u will result in G being cut
into four connected components.
*/
vector<int> dfs_num;
vector<int> dfs_low;
vector<int> dfs_parent;
int dfs_root, root_children;


// vector<int> instead of vector<bool>
vector<int> articulation_vertex;

void ArticulationPoint(int u)
{
    dfs_num[u] = dfs_low[u] = dfs_num_counter++;
    for(int i = 0; i < adj_list[u].size(); i++)
    {
        int v = adj_list[u][i];

        if(dfs_num[v] == -1)
        {
            dfs_parent[v] = u;
            if(u == dfs_root) root_children++;

            ArticulationPoint(v);

            // we increment articulation_vertex here
            if(dfs_low[v] >= dfs_num[u])
                articulation_vertex[u]++;
            if (dfs_low[v.first] > dfs_num[u])
                printf(" Edge (%d, %d) is a bridge\n", u,
                v.first);

            dfs_low[u] = min(dfs_low[u], dfs_low[v]);
        }
        else if(v != dfs_parent[u])
            dfs_low[u] = min(dfs_low[u], dfs_num[v]);
    }
}

int main()
{
    dfs_num_counter = 0;
    dfs_num.clear(); dfs_num.resize(N, -1);
    dfs_low.clear(); dfs_low.resize(N, 0);
    dfs_parent.clear(); dfs_parent.resize(N, 0);

    // articulation_vertex initialized to 1 here
    articulation_vertex.clear();
    articulation_vertex.resize(N, 1);
```

```
        for(int i = 0; i < N; i++)
            if (dfs_num[i] == -1)
            {
                dfs_root = i; root_children = 0;
                ArticulationPoint(i);

                // special case for root
                // number of connected components after the
                removal of root
                // is equal to how many children root has
                articulation_vertex[dfs_root] = root_children;
            }
    }
```

- Two way to one way UVA 610

2.3 Tree

Undirected, acyclic, connected, $|V|-1$ edges.
All edges are bridges, and internal vertices (degree >1) are articulation points.
It is as well a bipartite graph.
SSSP: Simply take the sum of edge weights of that unique path. $O(|V|)$
APSP: Simply do SSSP from all vertices. $O(|V^2|)$

```
void preorder (v) {
  visit (v);
  preorder (left (v));
  preorder (right (v));
}
void inorder (v) {
  inorder (left (v));
  visit (v);
  inorder (right (v));
}
void postorder (v) {
  postorder (left (v));
  postorder (right (v));
  visit (v);
}
```

It is **impossible** to construct binary tree with just Preorder traversal.
It is **impossible** to construct binary tree with just Inorder traversal.
It is **impossible** to construct binary tree with just Postorder traversal.

2.3.1 LCA

- Jammie and Tree, Sol: One stop soln to understand LCA. How to find the LCA of u and v using the precomputed LCA table that assumes the root is vertex 1? Let's separate the situation into several cases. If both u and v are in the subtree of r, then query the LCA directly is fine. If exactly one of u and v is in the subtree of r, the LCA must be r. If none of u and v is in the subtree of r, we can first find the lowest nodes p and q such that p is an ancestor of both u and r, and q is an ancestor of both v and r. If p and q are different, we choose the deeper one. If they are the same, then we query the LCA directly. Combining the above cases, one may find the LCA is the lowest vertex among $lca(u,v)$, $lca(u,r)$, $lca(v,r)$.

  After we have found the origin w of update (for query, it is given), how to identify the **subtree of a vertex** and carry out updates/queries on it? Again, separate the situation into several cases. If $w=r$, update/query the whole tree. If w is in the subtree of r, or w isn't an ancestor of r, update/query the subtree of w. Otherwise, update/query the whole tree, then undo update/exclude the results of the subtree of w', such that w' is a child of w and the subtree of w' contains r.

2.3.2 Important Problems

- UVA 11695 Sol: Problem Desc: Find which edge to remove and add so as to minimise the number of hops to travel between flights. Problem Sol: Just link the center of diameters. Brute force which edge to remove.
- UVA 112 Sol, UVA 112 Prob: Just see how I processed the input.
- UVA 10029 Sol, UVA 10029 Prob: Edit steps, (lexicographic sequence of words)
- UVA 536 Sol, UVA 536 Prob: Construct binary tree with preorder and inorder
- UVA 10459 Sol, UVA 10459 Prob: Centers of diameters are best where as corners are worst.
- Tree Destruction, Sol:

2.3.3 MVC on Tree

```
int mvc(int at, int flag, int parent) { //You can start this
from any node, i.e. in main: int ans =  min(mvc(0, 0, -1),
mvc(0, 1, -1)); and handle the case n == 1 seperately
   if(memo[at][flag] != -1) {
       return memo[at][flag];
   }
   if(glist[at].size() == 1 and parent != -1) { //leaf node
       return memo[at][flag] = flag;
   }
   int ans = flag;
   if(flag) // to take this
   {
       for(auto to : glist[at]) {
           if(to != parent)
               ans += min(mvc(to, 0, at), mvc(to, 1, at));
       }
   } else { //we must take its neighbours
       for(auto to : glist[at]) {
           if(to != parent)
               ans += mvc(to, 1, at);
       }
   }
   return memo[at][flag] = ans;
}  // Similar code can be written to find MWIS.
```

2.3.4 MWIS on Tree

```
int mwis(int at, int flag, int parent) { //You can start this
from any node, i.e. in main: int ans =  max(mwis(0, 0, -1),
mwis(0, 1, -1)); and handle the case n == 1 seperately
   if(memo[at][flag] != -1) {
       return memo[at][flag];
   }
   if(glist[at].size() == 1 and parent != -1) { //leaf node
        flag ? ans = weight[at] : 0;
        return ans;
   }
   if (flag) {
       ans = weight[v];
       for (auto to : glist[at]) {
           if (to != parent)
            ans += mwis (to, 0, at);
       }
   } else {
        ans = 0;
        for (auto &to : glist[at]) {
            ans += max (mwic(to, 1, at), mwic(to, 0, at));
        }
   }
   return memo[at][flag] = ans;
}  // Similar code can be written to find MWIS.
```

2.4 Terminology

- A **vertex cover** is a subset of vertices S, such that for each edge (u, v) in graph, either u or v (or both) are in S.
- An **independent set** is a subset of vertices S, such that no two vertices u, v in S are adjacent in graph.
- A subset of vertices is a vertex cover iff the complement of the set is an independent set. I.e. $MinVC + MaxIS = V$.
- It is easy to see why it is the case, firstly for vertices in the complement of MinVC, they can't have any edge between them as then our MinVC isnt VC. Also if we add to this IS any more vertex then it wont be an IS. Suppost it is not the maximum then there occurs an MIS with size more than this, clearly its complement is VC which would have size smaller than our MinVC which leads to contradiction.
- Complement of VC is an IS and vice versa (easy to see)
- Given an undirected graph G = (V, E) and weighting function defined on the vertex set, the minimum weighted vertex cover problem is to find a vertex set S ⊆ V whose total weight is minimum subject to every edge of G has at least one end point in S.
- In the maximum-weight independent set problem, the input is an undirected graph with weights on its vertices and the output is an independent set with maximum total weight.
- Again it happens to be the case that size of MWVC + MWIS = Total weight = V.
- A matching is a subset of edges such that each vertex is adjacent to at most one edge in the subset. Clearly Matching edges can be atmost $|V|/2$ as each edge joins two vertices and now no other matched edge can touch them.
- Note: This MVC, MIS, MM, is defined for undirected, unweighted graph.
- Once we have maximum matching. Clearly since these matching edges are aswell edges of the graph and minimum vertex cover should have vertices that are adjacent to these edges. But since matching edges have no vertex in common, size of minimum vertex cover is atleast the size of maximum matching.
- Maximum matchings can be found in polynomial time for any graph, while minimum vertex cover is NP complete. Thus, finding maximum independent sets is another NP-complete problem.
- The equivalence between matching and covering articulated in Kőnig's theorem allows minimum vertex covers and maximum independent sets to be computed in polynomial time for bipartite graphs, despite the NP-completeness of these problems for more general graph families.

2.5 Konigs Theorem

Size of Min VC in a bipartite graph is equal to the size of Max Matching in that graph.

Kőnig's theorem can be proven in a way that provides additional useful information beyond just its truth: the proof provides a way of constructing a minimum vertex cover from a maximum matching.

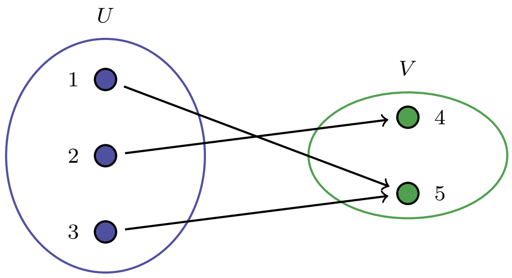
Proof: Let $G = (V, E)$ be a bipartite graph, and let the vertex set V V be partitioned into left set L and right set R. Suppose that M is a maximum matching for G.

let U be the set of unmatched vertices in L (possibly empty), and let Z be the set of vertices that are either in U or are connected to U by alternating paths. Let $K = (L \setminus Z) \cup (R \cap Z)$.

Every edge e in E either belongs to an alternating path (and has a right endpoint in K, or it has a left endpoint in K. For, if e is matched but not in an alternating path, then its left endpoint cannot be in an alternating path (for such a path would have had to have included e) and thus belongs to $L \setminus Z$. Alternatively, if e is unmatched but not in an alternating path, then its left endpoint cannot be in an alternating path, for such a path could be extended by adding e to it. Thus, K forms a vertex cover.

Additionally, every vertex in K is an endpoint of a matched edge. For, every vertex in $L \setminus Z$ is matched because Z is a superset of U. And every vertex in $R \cap Z$ must also be matched, for if there existed an alternating path to an unmatched vertex then changing the matching by removing the matched edges from this path and adding the unmatched edges in their place would increase the size of the matching. However, no matched edge can have both of its endpoints in K. Thus, K is a vertex cover of cardinality equal to M, and must be a minimum vertex cover.

Small diagram to understand proof well.



Matched edges are (2, 4) and (3, 5).

$U = \{1\}$, $Z = \{1, 5, 3\}$, $L = \{1, 2, 3\}$, $K = \{2, 5\}$

2.6 Bipartite Matching

2.6.1 Hopcroft Karp

1. **Free node or vertex:** Given a matching M, a node that is not a part of mathing is called a free node. Initially all vertices are free.
2. **Matching and not matching edges:** Given a matching M, edges that are part of matching are called matching edges and edges that are not part of M (or connect free nodes) are called non matching edges.
3. **Alternating Paths:** Given a matching M, an alternating path is a path in which edges belong alternatively to the matching and not matching.
4. **Augmenting path:** Given a matching M, an augmenting path is an alternating path that starts from and ends on free vertices.
5. The Hopcroft karp algorithm is based on below concept:
6. A matching M is not maximum if there exist an augmenting path. It is also true other way, i.e., a matching is maximum if no augmenting path exists.
7. Hopcroft Karp Algo $O(\sqrt{V} * E)$:
   (a) Initialize maximal matching M as empty.
   (b) While there exists an augmenting path P, remove matching edges of P from M and add not matching edges of P to M. (This increases size of M by 1 as P starts and ends with a free vertex).
   (c) Return M.

   The following is the sol to problem UVA 11419 where we were just required to find minimum vertex cover.

```
#include <bits/stdc++.h>
#define FOR(i, a, b) for (int i = a; i <= b; i++)
#define REP(i, n) for (int i = 0; i < n; i++)
#define pb push_back
#define INF 500000000
#define maxN 1010
using namespace std;

int n, m, matchX[maxN], matchY[maxN];
int dist[maxN];
vector<int> adj[maxN];
bool Free[maxN];

bool bfs() {
    queue<int> Q;
    FOR (i, 1, n)
        if (!matchX[i]) {  // only free vertices are pushed
        in queue and have their distance set to 0. Thus
        already matched vertices in X will have their
        distance set to INF.
            dist[i] = 0;
            Q.push(i);
        }
        else dist[i] = INF;
    dist[0] = INF;  // 0 is nil
    // Thus we would always start from free vertices
    traverse then alternating path and if in end from Y
    there is no match i.e. its a free vertice, we found an
    augmenting path.
    // Side Notes: If we popped an already matched vertex
    from queue then it wont go to its matching edges
    neighbor as its matchY is popped vertex itself and
    hence it wont have distance set to INF.
    while (!Q.empty()) {
        int i = Q.front(); Q.pop();
        REP(k, adj[i].size()) {
            int j = adj[i][k];
            if (dist[matchY[j]] == INF) {
                dist[matchY[j]] = dist[i] + 1;
                Q.push(matchY[j]);
            }
        }
    }
    return dist[0] != INF;
}

bool dfs(int i) {
    if (!i) return true; // to handle nil.
    REP(k, adj[i].size()) {
        int j = adj[i][k];
        if (dist[matchY[j]] == dist[i] + 1 &&
        dfs(matchY[j])) {
            matchX[i] = j;
            matchY[j] = i;
            return true;
        }
    }
    dist[i] = INF;
    return false;
}

int hopcroft_karp() {
    int matching = 0;
    while (bfs())
        FOR (i, 1, n)
            if (!matchX[i] && dfs(i))
                matching++;
    return matching;
}

void dfs_konig(int i) {
    Free[i] = false;
    REP(k, adj[i].size()) {
        int j = adj[i][k];
        if (matchY[j] && matchY[j] != INF) {
            int x = matchY[j];
            matchY[j] = INF;  // as we have undirected
            edge, we dont want to traverse that same edge
            again, so its just a way of noting that.
            if (Free[x]) dfs_konig(x);
        }
    }
}

void solve() {
    printf("%d", hopcroft_karp());
    FOR (i, 1, n)
        if (!matchX[i])
            dfs_konig(i);  // finding Z.
    FOR (i, 1, n)
        if (matchX[i] && Free[i])  // i.e. in L but not in
        Z.
            printf(" r%d", i);
    FOR (j, 1, m)
        if (matchY[j] == INF)  // i.e. we traversed this
        edge i.e. its in R intersection Z.
```

```
        printf(" c%d", j);
    putchar('\n');
}

void initialize() {
    FOR (i, 1, n) {
        adj[i].clear();
        matchX[i] = 0;
        Free[i] = true;
    }
    memset(matchY, 0, (m + 1) * sizeof(int));
}

int ar[5];
char buff[20];
void read_line() {
    gets(buff);
    int len = strlen(buff), i = 0, m = 0;
    while (i < len)
        if (buff[i] != ' ') {
            ar[m] = 0;
            while (i < len && buff[i] != ' ')
                ar[m] = ar[m] * 10 + buff[i++] - 48;
            m++;
        }
        else i++;
}

main() {
    int k, u, v;
    while (scanf(" %d %d %d ", &n, &m, &k) != EOF) {
        if (!n && !m && !k) break;
        initialize();
        while (k--) {
            read_line();
            adj[ar[0]].pb(ar[1]);
        }
        solve();
    }
}
```

2.6.2 Using max flow algo

Our MM problem can be reduced to max flow problem by assigning a dummy source vertex s connected to all vertices in set 1 and all vertices in set 2 are connected to dummy sink vertex t. The edges are directed (s to u, u to v, v to t) where u belongs to set 1 and v belongs to set 2). By setting capacities of all edges in this flow graph to 1, we satisfy the criteria of matching. Thus, this max flow will be equal to the max. no. of matchings on the original graph.

2.7 Paths

- An **euler path** is defined as a path in a graph which visits each edge of the graph exactly once. Similarly and **euler tour/cycle** is an euler path which starts & ends on the same vertex. A graph which has either an euler path or an euler tour is called **eulerian**.
- For *undirectedgraph* euler tour exist iff all vertices have even deg.
- For *undirectedgraph* euler path exists iff all except 2 vertices have even deg. This euler path will start from one of thee odd deg vertices and end in the other.
- For *directedgraph*, euler tour exists iff every verte has equal indeg & outdeg.
- For *directedgraph*, euler path exists iff at most one vertex has $(outdeg)-(indeg)=1$, atmost one vertex has $(indeg)-(outdeg)=1$, every other vertex has $indeg = outdeg$.

```
// Code to find euler tour (will be able to euler path provided
we start with correct vertex) for an undirected graph.
list<int> cyc; // we need list for fast insertion in the middle
void EulerTour(list<int>::iterator i, int u) {
    for (int j = 0; j < (int)AdjList[u].size(); j++) {
        ii v = AdjList[u][j];
        if (v.second) { // if this edge can still be used/not
        removed
            v.second = 0; // make the weight of this edge to be
            0 ('removed')
            for (int k = 0; k < (int)AdjList[v.first].size();
            k++) {
                ii uu = AdjList[v.first][k]; // remove
                bi-directional edge
                if (uu.first == u && uu.second) {
                    uu.second = 0;
                    break;
                }
            }
            EulerTour(cyc.insert(i, u), v.first);
        }
    }
}
// inside int main()
cyc.clear();
EulerTour(cyc.begin(), A); // cyc contains an Euler tour
starting at A
for (list<int>::iterator it = cyc.begin(); it != cyc.end();
it++)
    printf("%d\n", *it); // the Euler tour
```

2.8 SCC

Strongly Connected Components. A directed graph is strongly connected if there is a path between all pairs of vertices. A strongly connected component (SCC) of a directed graph is a maximal strongly connected subgraph

2.8.1 Tarjan

```
vi dfs_num, dfs_low, S, visited; // global variables
void tarjanSCC(int u) {
    dfs_low[u] = dfs_num[u] = dfsNumberCounter++; // dfs_low[u]
    <= dfs_num[u]
    S.push_back(u); // stores u in a vector based on order of
    visitation
    visited[u] = 1;
    for (int j = 0; j < (int)AdjList[u].size(); j++) {
        ii v = AdjList[u][j];
        if (dfs_num[v.first] == UNVISITED)
            tarjanSCC(v.first);
        if (visited[v.first]) // condition for update
            dfs_low[u] = min(dfs_low[u], dfs_low[v.first]);
    }
    if (dfs_low[u] == dfs_num[u]) { // if this is a root
    (start) of an SCC
        printf("SCC %d:", ++numSCC); // this part is done after
        recursion
        while (1) {
            int v = S.back(); S.pop_back(); visited[v] = 0;
            printf(" %d", v);
            if (u == v) break;
        }
        printf("\n");
    }
}
// inside int main()
dfs_num.assign(V, UNVISITED); dfs_low.assign(V, 0);
visited.assign(V, 0);
dfsNumberCounter = numSCC = 0;
for (int i = 0; i < V; i++)
    if (dfs_num[i] == UNVISITED)
        tarjanSCC(i);
```

2.8.2 Kosaraju

1. It is obvious that strongly connected components do not intersect each other, i.e. this is a partition of all graph vertices. Thus we can give a definition of condensation graph G^{SCC} as a graph containing every strongly connected component as one vertex. Each vertex of the condensation graph corresponds to the strongly connected component of the graph G. There is a direccted edge b/w two vertice C_i and C_j of the condensation graph iff there are 2 vertices $u \in C_i$ and $v \in C_j$ such that there is an edge in initial graph, i.e. $(u, v) \in E$. The most important property of the condensation grph is that it is acyclic.
2. Let $C \& C^{`}$ be 2 diff SCC & there is an edge $(C, C^{`})$ in a condensation graph then $tout[C] > tout[C^{`}]$, note: $tout[C] = max_{v_i \in C}(tout[v_i])$.

```
vector < vector<int> > g, gr;
vector<bool> used;
vector<int> order, component;

void dfs1 (int v) {
    used[v] = true;
    for (size_t i=0; i<g[v].size(); ++i)
        if (!used[ g[v][i] ])
            dfs1 (g[v][i]);
    order.push_back (v);
}

void dfs2 (int v) {
    used[v] = true;
    component.push_back (v);
    for (size_t i=0; i<gr[v].size(); ++i)
        if (!used[ gr[v][i] ])
            dfs2 (gr[v][i]);
```

```
}

int main() {
    int n;
    ... reading n ...
    for (;;) {
        int a, b;
        ... reading next edge (a,b) ...
        g[a].push_back (b);
        gr[b].push_back (a);
    }

    used.assign (n, false);
    for (int i=0; i<n; ++i)
        if (!used[i])
            dfs1 (i);
    used.assign (n, false);
    for (int i=0; i<n; ++i) {
        int v = order[n-1-i];
        if (!used[v]) {
            dfs2 (v);
            ... printing next component ...
            component.clear();
        }
    }
}
```

2.9 SAT

2.9.1 1 SAT

$f = x_1 \wedge x_2 \wedge \cdots \wedge x_n$ is satisfiable iff there isnt both $x_i \& \bar{x_i}$ in f.

2.9.2 2 SAT

$f = (x_1 \vee y_1) \wedge \cdots \wedge (x_n \vee y_n)$ is satisfiable iff both $x_i \& \bar{x_i}$ are not in same SCC as one them has to be true and in SCC one value is true all others must be true. For each $(x_i \vee y_i)$ add 2 edges $\bar{x_i} \rightarrow y_i$ and $\bar{y_i} \rightarrow x_i$.
After seeing whether the soln exists or not, soln can be constructed with the help of Kosaraju's algo, let comp[v] denote the index of strongly connected component to which the vertex v belongs. Then, if $comp[x] < comp[\bar{x}]$ we assign x with false and true otherwise.

2.10 DAG

topological sort or topological ordering of a directed graph is a linear ordering of its vertices such that for every directed edge uv from vertex u to vertex v, u comes before v in the ordering.
A topological ordering is possible if and only if the graph has no directed cycles, that is, if it is a directed acyclic graph (DAG). Any DAG has at least one topological ordering.

```
adjmat[i][j] |= (adjmat[i][k] & adjmat[k][j])
```

2.10.1 SSSP

Do topological sort then relax edges according to this order.

2.10.2 SSLP

Simply negate all the edge weights and run SSSP as above.
Pay attention on the word "we are not allowed to go back" so dont use normal SSSP algo but relax edges according to the required order.

2.10.3 Counting Paths in DAG

find toposort. Set numPaths[firstElement] = 1. Then we process the remaining vertices one by one acc. to toposort. When processing a vertex u, we update each neighbour v of u by setting numPaths[v] += numPaths[u].

2.10.4 Min Path cover on DAG

This is described as a problem of finding the min. no. of paths to cover each vertex on DAG. The start of each path can be arbitrary, we are just interested in min. no. of paths.
Construct a bipartite graph $G' = (V_out \cup V_in, E')$ from G where $V_{out/in} = \{v \in V : vhaspoitiveout/indegree\}$
$E' = \{(u, v) \in (V_out, V_in) : (u, v) \in E\}$
G' is a bipartite graph, do max. matching on it. Say answer obtained is m that means ans is $|V| - m$ as initially —V— vertices can be convered with —V— paths of length of length 0 (the vertices themselves). One matching b/w vertex a and b using edge (a, b) says that we can use one less path as edge (a, b) in E' can cover path $a \in V_out \& b \in V_in$

2.11 APSP Floyd Warshalls

```
for (int k = 0; k < V; k++) {
    for (int i = 0; i < V; i++) {
        for (int j = 0; j < V; j++) {
            if (adjmat[i][j] > adjmat[i][k] + adjmat[k][j]) {
                adjmat[i][j] = adjmat[i][k] + adjmat[k][j];
                path[i][j] = path[k][j];
            }
        }
    }
}
void printPath (int u) {
    if (u == i) cout << i << " ";
    else {
        printPath (path[i][u]);
        cout << u << " ";
    }
}
// in main printPath(j).
// Initially path[i][j] = i for edge (i, j);
// in APSP problems, always do
cin >> u >> v >> wt;
adjmat[u][v] = min (graph[u][v], wt);
// very initially
if (i != j) adjmat[i][j] = inf;
else adjmat[i][j] = 0;
```

- If diagonal (initially 0) becomes negative = negative cycle
- If diagonal (initially inf) becomes finite = cycle
- The smallest non negative adjmat[i][i] for all i is the cheapest cycle
- Transitive closure to determine if i is connected to j or not.

```
// For negative edge weights provided we have no negative
cycles.
// Idea: Shortest path must have atmost |V| - 1 edges.
// Thus if we relax each each edge |V| - 1 times then we
would have got the answer as in first relaxation
edge(start, neighbour) will be correct and so on...
vi dist (V, inf);
dist[s] = 0;
bool modified = true;
for (int i = 0; i < V - 1 and modified; i++) {
    modified = false;
    for (int u = 0; u < V; u++) {
        for (int j = 0; j < adjlist[u].size (); j++) {
            ii v = adjlist[u][j];
            if (dist[v.first] > dist[u] + v.second) {
                dist[v.first] = dist[u] + v.second;
                p[v.first] = u;
                modified = true;
            }
        }
    }
}
// to check for negative cycle
void solve()
{
    vector<int> d(n);
    vector<int> p(n, -1);
    int x;
    for (int i = 0; i < n; ++i) {
        x = -1;
        for (Edge e : edges) {
            if (d[e.a] + e.cost < d[e.b]) {
                d[e.b] = d[e.a] + e.cost;
                p[e.b] = e.a;
                x = e.b;
            }
        }
    }

    if (x == -1) {
        cout << "No negative cycle found.";
    } else {
        for (int i = 0; i < n; ++i)
            x = p[x];

        vector<int> cycle;
        for (int v = x;; v = p[v]) {
            cycle.push_back(v);
            if (v == x && cycle.size() > 1)
                break;
        }
        reverse(cycle.begin(), cycle.end());

        cout << "Negative cycle: ";
        for (int v : cycle)
            cout << v << ' ';
        cout << endl;
    }
}
// To check for negative cycle, run this one more time
int x = -1;
for (int u = 0; u < V; u++) {
    for(int j = 0; j < adjlist[u].size (); j++) {
```

```
            ii v = adjlist[u][j];
            if (dist[v.first] > dist[u] + v.second) {
                dist[v.first] = dist[u] + v.second;
                p[v.first] = u;
                x = v.first;
            }
        }
    }
    if (x != -1) {  // negative cycle
        int y = x;
        for (int i=0; i<n; ++i)
            y = p[y];
        vector<int> path;
        for (int cur=y; ; cur=p[cur])
        {
            path.push_back (cur);
            if (cur == y && path.size() > 1)
                break;
        }
        reverse (path.begin(), path.end());
        cout << "Negative cycle: ";
        for (size_t i=0; i<path.size(); ++i)
            cout << path[i] << ' ';
    }
```

- Diameter of a graph is maximum shortest path distance between any pair of vertices of that graph. So do simply max (adjmat[i][j] for all i, j) after doing APSP.
- SCC of a directed graph (aliter): first do transitive closure then to find all members of an SCC that contains vertex i, check all vertices j, if (adjmat[i][j] && adjmat[j][i]) is true then vertex i and j belong to same SCC.
- Minimax: Minimax path problem of finding the minimum of maximum edge weight among all posssible paths between two vertices i to j. The reverse problem maximin is defined similarly.

```
adjmat[i][j] = min (adjmat[i][j], max (adjmat[i][k],
adjmat[k][j]));
```

2.12 MST (Kruskal)

```
// O (ElogV)
// Connected, undirected weighted graph
vector<pair<int, ii> > edgelist;
for (int i = 0; i < E; i++) {
    cin >> u >> v >> w;
    edgelist.pb (make_pair(w, ii (u, v)));
}
sort(edgelist.begin (), edgelist.end ());
int mstCost = 0;
UFDS uf (V);
for (int i = 0; i < E and uf.numSets > 1; i++) {
    auto front = edgelist[i];
    if (!uf.isSameSet (front.second.first,
    front.second.second)) {
        mstCost += front.first;
        uf.unionSet (front.second.first, front.second.second);
    }
}
cout << mstCost;
```

- If maximum instead of minimum, simply sort by dec. edge weights
- Minimum spanning subgraph: first union all these fixed edges, then run MST as normal
- Minimum spanning forest (we want to form a forest of k connected components (k subtrees)) in the least cost way; soln: run kruskal until no. of connected components (numsets) equals k.
- Second best ST O(VE). While doing this also check that in resultant ST, numdisjointset == 1 or not.

2.13 SSSP

2.13.1 Dijkstra

```
// Subpaths of shortest paths from u to v are shortest paths
// This implementation would work even if the graph has
negative edge provided there is no negative cycle
// O(ElogV)
struct node {
    int cost, vertex;
    node () {}
    node (int n, int c) {
        vertex = n; cost = c;
    }
    bool operator < (const node &node) const {
        return cost > node.cost;  // as priority queue is max
        heap
    }
}
int dijkstra (int s, int e) {
    memset (dist, inf, sizeof (dist));
    dist[s] = 0;
    priority_queue<node> pq;
    pq.push (node (s, 0));
    int from, to, wt, cost;
    while (!pq.empty ()) {
        from = pq.top ().vertex;
        cost = pq.top ().cost;
        pq.pop ();
        if (from == e) return dist[e];
        if (cost == dist[from]) {  // lazily deleting
            for (int i = 0; i < adjlist[from].size (); i++) {
                to = adjlist[form][i].first;
                wt = adjlist[from][i].second;
                if (dist[to] > dist[from] + wt) {
                    dist[to] = dist[from] + wt;
                    p[to] = from;
                    pq.push (node (to, dist[to]));
                }
            }
        }
    }
}
```

- Hotel Booking Problem, sol
- Fuel Tank problem, sol
- Logical Expression, Sol: pr denotes from what grammer it is derived. Also number of functions on n variables = 2^{2^n}.

2.13.2 Bellman ford

```
// For negative edge weights provided we have no negative
cycles.
// Idea: Shortest path must have atmost |V| - 1 edges.
// Thus if we relax each each edge |V| - 1 times then we would
have got the answer as in first relaxation edge(start,
neighbour) will be correct and so on...
vi dist (V, inf);
dist[s] = 0;
bool modified = true;
for (int i = 0; i < V - 1 and modified; i++) {
    modified = false;
    for (int u = 0; u < V; u++) {
        for (int j = 0; j < adjlist[u].size (); j++) {
            ii v = adjlist[u][j];
            if (dist[v.first] > dist[u] + v.second) {
                dist[v.first] = dist[u] + v.second;
                p[v.first] = u;
                modified = true;
            }
        }
    }
}
// to check for negative cycle
void solve()
{
    vector<int> d(n);
    vector<int> p(n, -1);
    int x;
    for (int i = 0; i < n; ++i) {
        x = -1;
        for (Edge e : edges) {
            if (d[e.a] + e.cost < d[e.b]) {
                d[e.b] = d[e.a] + e.cost;
                p[e.b] = e.a;
                x = e.b;
            }
        }
    }

    if (x == -1) {
        cout << "No negative cycle found.";
    } else {
        for (int i = 0; i < n; ++i)
            x = p[x];

        vector<int> cycle;
        for (int v = x;; v = p[v]) {
            cycle.push_back(v);
            if (v == x && cycle.size() > 1)
                break;
```

```
        }
        reverse(cycle.begin(), cycle.end());

        cout << "Negative cycle: ";
        for (int v : cycle)
            cout << v << ' ';
        cout << endl;
    }
}
// To check for negative cycle, run this one more time
int x = -1;
for (int u = 0; u < V; u++) {
    for(int j = 0; j < adjlist[u].size (); j++) {
        ii v = adjlist[u][j];
        if (dist[v.first] > dist[u] + v.second) {
            dist[v.first] = dist[u] + v.second;
            p[v.first] = u;
            x = v.first;
        }
    }
}
if (x != -1) {  // negative cycle
    int y = x;
    for (int i=0; i<n; ++i)
        y = p[y];
    vector<int> path;
    for (int cur=y; ; cur=p[cur])
    {
        path.push_back (cur);
        if (cur == y && path.size() > 1)
            break;
    }
    reverse (path.begin(), path.end());
    cout << "Negative cycle: ";
    for (size_t i=0; i<path.size(); ++i)
        cout << path[i] << ' ';
}
```

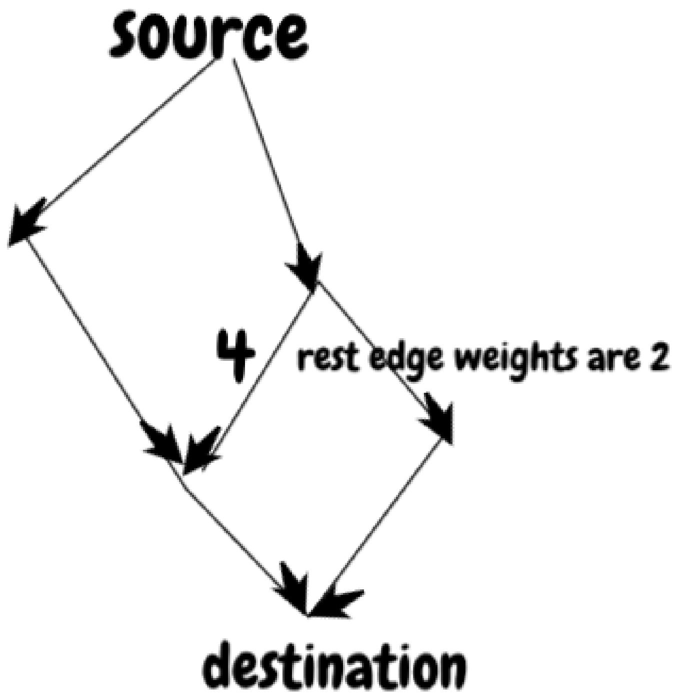
- Hopeless/winnable UVA 10557: Basically check for positive cycle and see if it is connected from that.
- Stop problem UVA 11280.

2.14 Max Flow

2.14.1 Edmond karps

```
// O (V * E^2)
void augment(int v, int minEdge) { // traverse BFS spanning tree from s->t
    if (v == s) { f = minEdge; return; } // record minEdge in a global var f
    else if (p[v] != -1) { augment(p[v], min(minEdge, res[p[v]][v]));
    res[p[v]][v] -= f; res[v][p[v]] += f; }
}
    // in main
    mf = 0; // mf stands for max_flow
    while (1) { // O(VE^2) (actually O(V^3 E) Edmonds Karp's algorithm
        f = 0;
        // run BFS
        vi dist(MAX_V, INF); dist[s] = 0; queue<int> q;
        q.push(s);
        p.assign(MAX_V, -1); // record the BFS spanning tree, from s to t!
        while (!q.empty()) {
            int u = q.front(); q.pop();
            if (u == t) break; // immediately stop BFS if we already reach sink t
            for (int v = 0; v < MAX_V; v++) // note: this part is slow
                if (res[u][v] > 0 && dist[v] == INF)
                    dist[v] = dist[u] + 1, q.push(v), p[v] = u;
                    // 3 lines in 1!
        }
        augment(t, INF); // find the min edge weight 'f' in this path, if any
        if (f == 0) break; // we cannot send any more flow ('f' = 0), terminate
        mf += f; // we can still send a flow, increase the max flow!
    }
```

- Reason why we need to consider back flow: so that if by back flow we reach destination then that means we could have done it in straight way.



- VIMP Note: if you are traversing through all 'v' like in the code above then its fine but otherwise if you do glist[u].pb(v) you **must** also do glist[v].pb (u) but only set res[u][v].
- Lets define an s-t cut C = (s-component, t-component) as a partition of V s.t. source s belongs to s-component and sink t belongs to t-component. Lets also define a cut-set to be set $\{(u, v) \in E | u \in s-comp, v \in t-comp\}$ such that if all edges in cut-set of C are removed, max flow from s to t is 0. (i.e. s and t are disconnected). The cost of an s-t cut C is defined by the sum of capacities of edges in cut-set of C. The min cut problem (min cut) is to minimize the amount of capacity of an s-t cut. Sol: After running max flow algo, do bfs/dfs from s, all vertices reachable from s using positive weighted edges in the residual graph belong to s-comp. and remaining belong to t-comp. and mf = min cut value. This is max-flow min-cut theorem.
- Multisource/multisink: Create a supersource ss and super sing st. Connect ss with all s with infinite capacity and also connect all t with st with infinte capacity, then run edmonds karps as per normal.
- Vertex capacities: Network flow variant where capacities are not just defined along the edges but also on the vertices. Split each v to vin and vout, reassigning its incomint/outgoing edges to vin/vout resp. and finally putting the original vertex v's weight as the weight of edge vin -> vout. Note that this will double the number of vertices
- Max Independent paths: two paths are said to be independent if they do not share any vertex apart from s and t. Soln: construct a flow network N = (V, E) from G with vertex capacitites, where N is the carbon copy of G except that the capacity of each v in V is 1 (i.e. each vertex can only be used once) and the capacity of each edge e in V is also 1. Then run edmonds karps algo as per normal.
- Max edge disjoint paths: finding the maximum number of edge disjoint paths rom s to t is simlar to finding max independent paths. The only difference is that this time we do not have any vertex capacity (i.e. 2 edge disjoint paths can still share the same vertex)
- Crimewave, sol
- **MWIS on a bipartite graph**
  Problem is equivalent to finding the minimum weight vertex cover in the graph. The latter can be solved using maximum flow techniques: Introduce a super-source S and a super-sink T. connect the nodes on the left side of the bipartite graph to S, via edges that have their weight as capacity. Do the same thing for the right side and sink T. Assign infinite capacity to the edges of the original graph.
  Now find the minimum S-T cut in the constructed network. The value of the cut is the weight of the minimum vertex cover. To see why this is true, think about the cut edges: They can't be the original edges, because those have infinite capacity. If a left-side edge is cut, this corresponds to taking the corresponding left-side node into the vertex cover. If we do not cut a left-side edge, we need to cut all the right-side edges from adjacent vertices on the right side.
  Thus, to actually reconstruct the vertex cover, just collect all the vertices that are adjacent to cut edges, or alternatively, the left-side nodes not reachable from S and the right-side nodes reachable from

S.

2.15 Minimum Cost Flow

Given a network G consisting of n vertices and m edges. For each edge (generally speaking, oriented edges, but see below), the capacity (a non-negative integer) and the cost per unit of flow along this edge (some integer) are given. Also the source s and the sink t are marked.

For a given value K, we have to find a flow of this quantity, and among all flows of this quantity we have to choose the flow with the lowest cost. This task is called minimum-cost flow problem.

Sometimes the task is given a little differently: you want to find the maximum flow, and among all maximal flows we want to find the one with the least cost. This is called the minimum-cost maximum-flow problem.

Both these problems can be solved effectively with the algorithm of sucessive shortest paths.

First we only consider the simplest case, where the graph is oriented, and there is at most one edge between any pair of vertices (e.g. if (i, j) is an edge in the graph, then (j, i) cannot be part in it as well).

Let U_{ij} be the capacity of an edge (i, j) if this edge exists. And let C_{ij} be the cost per unit of flow along this edge (i, j). And finally let $F_{i,j}$ be the flow along the edge (i, j). Initially all flow values are zero.

We modify the network as follows: for each edge (i, j) we add the reverse edge (j, i) to the network with the capacity $U_{ji} = 0$ and the cost $C_{ji} = -C_{ij}$. Since, according to our restrictions, the edge (j, i) was not in the network before, we still have a network that is not a multigraph (graph with multiple edges). In addition we will always keep the condition $F_{ji} = -F_{ij}$ true during the steps of the algorithm.

We define the residual network for some fixed flow F as follow (just like in the Ford-Fulkerson algorithm): the residual network contains only unsaturated edges (i.e. edges in which $F_{ij} < U_{ij}$), and the residual capacity of each such edge is $R_{ij} = U_{ij} - F_{ij}$.

Now we can talk about the algorithms to compute the minimum-cost flow. At each iteration of the algorithm we find the shortest path in the residual graph from s to t. In contrary to Edmonds-Karp we look for the shortest path in terms of the cost of the path, instead of the number of edges. If there doesn't exists a path anymore, then the algorithm terminates, and the stream F is the desired one. If a path was found, we increase the flow along it as much as possible (i.e. we find the minimal residual capacity R of the path, and increase the flow by it, and reduce the back edges by the same amount). If at some point the flow reaches the value K, then we stop the algorithm (note that in the last iteration of the algorithm it is necessary to increase the flow by only such an amount so that the final flow value doesn't surpass K).

It is not difficult to see, that if we set K to infinity, then the algorithm will find the minimum-cost maximum-flow. So both variations of the problem can be solved by the same algorithm.

The case of an undirected graph or a multigraph doesn't differ conceptually from the algorithm above. The algorithm will also work on these graphs. However it becomes a little more difficult to implement it.

An undirected edge (i, j) is actually the same as two oriented edges (i, j) and (j, i) with the same capacity and values. Since the above-described minimum-cost flow algorithm generates a back edge for each directed edge, so it splits the undirected edge into 4 directed edges, and we actually get a multigraph.

How do we deal with multiple edges? First the flow for each of the multiple edges must be kept separately. Secondly, when searching for the shortest path, it is necessary to take into account that it is important which of the multiple edges is used in the path. Thus instead of the usual ancestor array we additionally must store the edge number from which we came from along with the ancestor. Thirdly, as the flow increases along a certain edge, it is necessary to reduce the flow along the back edge. Since we have multiple edges, we have to store the edge number for the reversed edge for each edge.

There are no other obstructions with undirected graphs or multigraphs.
Sample Prob: UVA 10594, sol

2.16 More Problems

- UVA 928: just bfs with dp
- That worst feast problem UVA 10246, sol
- Atcoder
- df

3 Some Basic

- ```
  #pragma GCC optimize("Ofast")  // tells the compiler to optimize the code for speed to make it as fast as possible (and not look for space)
  #pragma GCC optimize ("unroll-loops")  // normally if we have a loop there is a "++i" instruction somewhere. We normally dont care because code inside the loop requires much more time but in this case there is only one instruction inside the loop so we want the compiler to optimize this.
  #pragma GCC
  target("sse,sse2,sse3,ssse3,sse4,popcnt,abm,mmx,avx,tune=native")
  // tell the compiler that our cpu has simd instructions and allow him to vectorize our code

  // Backtracking
  void bktk () {
      if (state == complete ) {
          // process it
      } else {
          for each possible next move P {
              apply move P;
              bktk ();
              undo move P;
          }
      }
  }
  // ex1: generating permutations O(n! * n)
  // better use next_permutation.
  // think of doing such things for n <= 11. 11! ~ 4 * 10^7
  void bktk () {
      if (perm.size () == n) {
          // process permutation
      } else {
          for (int i = 0; i < n; i++) {
              if (!chosen[i]) {
                  chosen[i] = true;
                  perm.push_back (i);
                  bktk ();
                  perm.pop_back ();
                  chosen[i] = false;
              }
          }
      }
  }
  // ex2: generating subsets
  void bktk (int k) {
      if (k == n) {
          // process subset
      } else {
          // Move one is to not push it and move 2 is to consider it.
          bktk (k + 1);
          subset.push_back(k);
          bktk (k + 1);
          subset.pop_back;
      }
  }
  // better way to generate subsets O(2^n) so valid for n <= 25. So whenever you see this, think of iterating through subsets
  // 1 << 0 = 1.
  for (int i = 0; i < (1 << n); i++) {
      for (int b = 0; b < n; b++) {
          if (i & (1 << b)) {
              subset.pb (b);
          }
          // process subset
      }
  }
  ```
- Remember that just like we could have solved this using binary search (other than simple DP), we can actually solve many problems which ask to find optimal value using binary search.
- Outputing a double or even int my have an exponential form so it is better to do ¡¡ fixed ¡¡ s...(0).
- Range of double is 10^{15}
- Range of int64 is $9 * 10^{18}$
- Inbuilt swap can be used to basically swap anything.
- $ceil(m/a) = (m + a - 1)/m$
- Consider $m * n$ surface, you have a sq. of size $a * a$. Min. no. of squares to cover completely the surface? Ans: $ceil(m/a) * ceil(n/a)$
- Not neccessary to cover the border cells? Ans: $m/a * n/a$
- Note: Sides of the square must be parallel to grid.
- lower_bound Returns an iterator pointing to the first element in the range [first,last) which does not compare less than val. Unlike upper_bound, the value pointed by the iterator returned by this function may also be equivalent to val, and not only greater. If all the element in the range compare less than val, the function returns last.
- upper_bound Returns an iterator pointing to the first element in the range [first,last) which compares greater than val. If no element in the range compares greater than val, the function returns last.
  ```

- ```
  /* we need to do binary search for range [st + 1, en -
  1], i.e. [st + 1, en). */
      if (binary_search (arr.begin() + st + 1,
      arr.begin() + en, leftover_))
  ```
- If you want to iterate through subsets of fixed size k, you may want to use k for loops.
- UVA 410 Load Balancing, sol
- ferry loading, sol, just know that best time is last cars arrival + t/2, so just achieve it.

  ```
  One can write i = (m % n == 0 ? n - 1 : m % n - 1) as
  i = (m + n - 1) % n.
  ```

  ```
  We can use cin.peek () to see the next available
  character
  ```
- ```
  1011101010000000 >>= 8 => num is 10111010 similarly num
  <<= 8 implies num is 10000000
  ```
- tictactoe game starts by placing 'X' so if in between $O_cnt = X_cnt$ and X has win that means invalid game. And if $O_cnt+1 = X_cnt$ and O has win that means not a valid game.
-

$$
\begin{aligned}
24_{hours} &= 10_{decimalHours} && (24)\\
24*60*60*100_{normalCC} &= 10*(100^3)_{decimalCC} && (25)\\
1_{decimalCC} &= 0.864_{normalCC} && (26)
\end{aligned}
$$

that means $1_{nCC} * (1_{dCC}/0.864_{nCC})$ Would give us dCC

- XOR is associative and commutative
- x^x = 0, x^0 = x.
- x^(string of 1's, i.e. (~0)) = ~x
- x^y = x^z implies y = z. (how? just take xor with x on both sides)
- Swapping 2 nos with XOR

  ```
  x = A, y = B;
  x = x ^ y;
  y = x ^ y; // i.e. (A ^ B) ^ B = A
  x = x ^ y; // i.e. (A ^ B) ^ A = B
  ```
- UVA 10309 Lights off:

  ```
  This problem is solved using complete search technique.
  Notice that each bulb can be toggled by 3 switches on
  the same row and 1 switch on the upper row and another
  one on the lower row, keep this in mind as we will use
  it later. Our solution has three steps: 1. For the
  first row try all 2^10 possible combinations of
  switches. We expect to have some switches turned on
  after this step. 2. For the rest of the rows, toggle
  switch (i, j) if (i-1, j) is switched on, this means
  toggle any switch if the bulb above it (in the previous
  row is turned on). This step insures that all bulbs
  will be turned off using the switch in the next row,
  except the last row, as there is no next row. 3. Check
  the bulbs in the last row, if all bulbs are turned off,
  then the current solution if valid, if any of the bulbs
  is on, this solution is not valid. Keep track of the
  number of switches toggled in each solution, and the
  final result is their minimum value.
  ```
- `while (first || cin >> temp) {  // something }`
- **Interval Covering:** Tell the minimum no. of intervals to cover the entire big interval.

  ```
  void solve() {
      // Greedy Algorithm
      sort (data.begin (), data.end ());
      for (; i < data.size(); i = j) {
          if (data[i].first > rightmost) break;
          for (j = i + 1; j < data.size() and data[j].first <=
          rightmost; j++) {
              if (data[j].second > data[i].second) {
                  i = j;
              }
          }
          ans.push_back(data[i]);
          rightmost = data[i].second;
          if (rightmost >= m) break;

      }
      if (rightmost < m) {
          cout << "0\n";
      }
  }
  ```
- **Prob:** We have a stack of turtles and we have some final permutation of them, each turtule can crawl out of its position and move to top. Determine a minimal sequence of operations to obtain the final permutation.
  Sol:

  ```
  int mvc(int at, int flag, int parent) { //You can start
  this from any node, i.e. in main: int ans =  min(mvc(0, 0,
  -1), mvc(0, 1, -1)); and handle the case n == 1 seperately
      if(memo[at][flag] != -1) {
          return memo[at][flag];
      }
      if(glist[at].size() == 1 and parent != -1) { //leaf node
          return memo[at][flag] = flag;
      }
      int ans = flag;
      if(flag) // to take this
      {
          for(auto to : glist[at]) {
              if(to != parent)
                  ans += min(mvc(to, 0, at), mvc(to, 1, at));
          }
      } else { //we must take its neighbours
          for(auto to : glist[at]) {
              if(to != parent)
                  ans += mvc(to, 1, at);
          }
      }
      return memo[at][flag] = ans;
  } // Similar code can be written to find MWIS.
  ```
- ```
  #define MAX_N 2 // Fibonacci matrix,
  increase/decrease this value as needed
  struct Matrix { int mat[MAX_N][MAX_N]; }; // we will return
  a 2D array
  Matrix matMul(Matrix a, Matrix b) { // O(n^3)
      Matrix ans; int i, j, k;
      for (i = 0; i < MAX_N; i++)
          for (j = 0; j < MAX_N; j++)
              for (ans.mat[i][j] = k = 0; k < MAX_N; k++) //
              if necessary, use
                  ans.mat[i][j] += a.mat[i][k] * b.mat[k][j];
                  // modulo arithmetic
      return ans;
  }
  Matrix matPow(Matrix base, int p) { // O(n^3 log p)
      Matrix ans; int i, j;
      for (i = 0; i < MAX_N; i++) for (j = 0; j < MAX_N; j++)
              ans.mat[i][j] = (i == j); // prepare identity
              matrix
      while (p) { // iterative version of Divide & Conquer
      exponentiation
          if (p & 1) ans = matMul(ans, base); // if p is odd
          (last bit is on)
          base = matMul(base, base); // square the base
          p >>= 1; // divide p by 2
      }
      return ans;
  }
  ```
- ```
  #define MAX_N 2 // Fibonacci matrix,
  increase/decrease this value as needed
  struct Matrix { int mat[MAX_N][MAX_N]; }; // we will return
  a 2D array
  Matrix matMul(Matrix a, Matrix b) { // O(n^3)
      Matrix ans; int i, j, k;
      for (i = 0; i < MAX_N; i++)
          for (j = 0; j < MAX_N; j++)
              for (ans.mat[i][j] = k = 0; k < MAX_N; k++) //
              if necessary, use
                  ans.mat[i][j] += a.mat[i][k] * b.mat[k][j];
                  // modulo arithmetic
      return ans;
  }
  Matrix matPow(Matrix base, int p) { // O(n^3 log p)
      Matrix ans; int i, j;
      for (i = 0; i < MAX_N; i++) for (j = 0; j < MAX_N; j++)
              ans.mat[i][j] = (i == j); // prepare identity
              matrix
      while (p) { // iterative version of Divide & Conquer
      exponentiation
          if (p & 1) ans = matMul(ans, base); // if p is odd
          (last bit is on)
          base = matMul(base, base); // square the base
  ```

```
        p >>= 1; // divide p by 2
    }
    return ans;
}
```

- **Algorithm to convert from infix to postifx:**
  1. Scan the infix expression from left to right.
  2. If the scanned character is an operand, output it.
  3. Else,
     (a) If the precedence of the scanned operator is greater than the precedence of the operator in the stack (or the stack is empty or the stack contains a '('), push it.
     (b) Else, Pop all the operators from the stack which are greater than or equal to in precedence than that of the scanned operator. After doing that Push the scanned operator to the stack. (If you encounter parenthesis while popping then stop there and push the scanned operator in the stack.)
  4. If the scanned character is an '(', push it to the stack.
  5. If the scanned character is an ')', pop the stack and and output it until a '(' is encountered, and discard both the parenthesis.
  6. Repeat steps 2-6 until infix expression is scanned.
  7. Print the output
  8. Pop and output from the stack until it is not empty.
- **Algorithm to convert from infix to prefix:**
  1. Properly reverse the infix exp.
  2. Gets its postfix as above
  3. Reverse postfix and output it.
- **Algorithm to convert from postfix to infix:**
  1. If the symbol is an operand, push it onto stack
  2. Else, if there are fewer than two values in stack, show error. Else, pop top 2 expressions from stack (say e1, e2), put the operator (op) between them and push to stack ((e1 op e2))
  3. After reading postfix expression, Stack should have only one item which is our answer
- **Merge Sort**

```
    // IMP NOTE: In both bubble sort and merge sort we are
    getting minimum no. of swaps to sort an array (i.e. by
    swapping adjacent elements)
void merge(int arr[], int l, int m, int r)
{
    int i, j, k;
    int n1 = m - l + 1;
    int n2 =  r - m;

    /* create temp arrays */
    int L[n1], R[n2];

    /* Copy data to temp arrays L[] and R[] */
    for (i = 0; i < n1; i++)
        L[i] = arr[l + i];
    for (j = 0; j < n2; j++)
        R[j] = arr[m + 1+ j];

    /* Merge the temp arrays back into arr[l..r]*/
    i = 0; // Initial index of first subarray
    j = 0; // Initial index of second subarray
    k = l; // Initial index of merged subarray
    while (i < n1 && j < n2)
    {
        if (L[i] <= R[j])
        {
            arr[k] = L[i];
            i++;
        }
        else///i.e we need to swap
        {
            arr[k] = R[j];
            swaps+=n1-i;//Most important line. basically
            once we are doing arr[k]=R[j] that means we are
            ///putting R[j] before each of n1-i elements
            thus there are that many swaps.
            j++;
        }
        k++;
    }

    /* Copy the remaining elements of L[], if there
       are any */
    while (i < n1)
    {
        arr[k] = L[i];
        i++;
        k++;
    }

    /* Copy the remaining elements of R[], if there
       are any */
    while (j < n2)
    {
        arr[k] = R[j];
        j++;
        k++;
    }
}

/* l is for left index and r is right index of the
   sub-array of arr to be sorted */
void mergeSort(int arr[], int l, int r)
{
    if (l < r)
    {
        // Same as (l+r)/2, but avoids overflow for
        // large l and h
        int m = l+(r-l)/2;

        // Sort first and second halves
        mergeSort(arr, l, m);
        mergeSort(arr, m+1, r);

        merge(arr, l, m, r);
    }
}
```

- set is like min heap. Only unique elements are present.
- On a line you are given the x coordinates of various houses, tell the house of vito (h) such that $\sum |h_i - h|$ is minimised. **Obs1:** h could be any of h_i so $O(n^2)$ algo. will work. **Obs2:** Taking derivative we get $i - j = 0$ i.e. $i = j = n/2$, that means simply sort and output the middlemost house.
  Note: Some times the math become cumbersome, in such cases, use **ternary search**
- **Prob:** n people have to cross the bridge, one torch, atmost 2 can travel
  Sol: if $n = 3 \Rightarrow$ time $= x + y + z$, if $n \geq 4$ let A, B, a, b be the fastest, second fastest, slowest, second slowest resp. **Goal:** Get the slowest members to the other side. So choose the best among the two options.
  option 1: Fastest member does back and forth.
  option 2: The two fastest members go, allowing the two slowest two to go together.
- **Inversions:** From a permutation, parity of number of swaps needed to get to the identical permutation is same as parity of inversion count of this permutation.
  Parity of inversions can be calculated in $O(n)$ by finding the number of cycles.
  Exact value of number of inverions can be calculated in $(nlog(n))$ by using segment trees.
- **Prob:** You are given two positive integer numbers a and b. Permute (change order) of the digits of a to construct maximal number not exceeding b.
  Sol: Take the number as string. sort string a, then for each $i \in [1, n]$ swap it with j trying from $ntoi + 1$ such that it is $\leq b$ (normal string comparison can be used).
- **Prob:** From a digraph, remove atmost one edge so that it becomes DAG.
  Sol: Get any one cycle the iteratively try to remove each edge and see if it makes it DAG or not.
- **UFDS**

```
struct UFDS {
    vector<int> p, rank, setSizes;
    int numSets;
    UFDS(int N) {
        numSets = N;
        rank.assign(N, 0);
        p.assign(N, 0);
        for (int i = 0; i < N; i++)
            p[i] = i;
        setSizes.assign(N, 1);
    }
    int findSet(int i) {
        return (p[i] == i) ? i : p[i] = findSet(p[i]);
    }
    bool isSameSet(int i, int j) {
        return findSet(i) == findSet(j);
```

```
    }
    void unionSet(int i, int j) {
        if (!isSameSet(i, j)) {
            int x = findSet(i), y = findSet(j);
            if (rank[x] > rank[y]) {
                setSizes[findSet(x)] +=
                setSizes[findSet(y)];
                p[y] = x;
            } else {
                setSizes[findSet(y)] +=
                setSizes[findSet(x)];
                p[x] = y;
                if (rank[x] == rank[y])
                    rank[y]++;
            }
            numSets--;
        }
    }
    int setSize(int i) {
        return setSizes[findSet(i)];
    }
    int numDisjointSets() {
        return numSets;
    }
};
```

- UVA 10158 Prob, UVA 10158 Sol
- Imbalance of a tree, Sol, summation(max - min) is same as summation(max) - summation(min).
- Party Lemonade, Sol
- Jamie and binary sequence, Sol.
- Prime gift, Sol. awesome 2 pointers problem.
- Fishes, Sol.
- **Max 1D Range Sum (kadane)**

```
vi dfs_num, dfs_low, S, visited; // global variables
void tarjanSCC(int u) {
    dfs_low[u] = dfs_num[u] = dfsNumberCounter++; //
    dfs_low[u] <= dfs_num[u]
    S.push_back(u); // stores u in a vector based on order
    of visitation
    visited[u] = 1;
    for (int j = 0; j < (int)AdjList[u].size(); j++) {
        ii v = AdjList[u][j];
        if (dfs_num[v.first] == UNVISITED)
            tarjanSCC(v.first);
        if (visited[v.first]) // condition for update
            dfs_low[u] = min(dfs_low[u], dfs_low[v.first]);
    }
    if (dfs_low[u] == dfs_num[u]) { // if this is a root
    (start) of an SCC
        printf("SCC %d:", ++numSCC); // this part is done
        after recursion
        while (1) {
            int v = S.back(); S.pop_back(); visited[v] = 0;
            printf(" %d", v);
            if (u == v) break;
        }
        printf("\n");
    }
}
// inside int main()
dfs_num.assign(V, UNVISITED); dfs_low.assign(V, 0);
visited.assign(V, 0);
dfsNumberCounter = numSCC = 0;
for (int i = 0; i < V; i++)
    if (dfs_num[i] == UNVISITED)
        tarjanSCC(i);
```

- **Max 2D range sum, algo1**

```
// grid need not be square
// O(n^4)
// Commented part shows for torus
cin >> n;
for (int i = 0; i < n; i++) { // < 2n
    for (int j = 0; j < n; j++) { // <2n
        cin >> A[i][j];
        /*
        if (i < n and j < n) {
            cin >> A[i][j];
            A[i + n][j] = A[i][j + n] = A[i + n][j + n] =
A[i][j];
        }
        */
        if (i) A[i][j] += A[i - 1][j];
        if (j) A[i][j] += A[i][j - 1];
        if (i and j) A[i][j] -= A[i - 1][j - 1];
    }
    int maxSubRect = -127 * 100 * 100;
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n; j++) {
            for (int k = i; k < n; k++) {  // < i + n
                for (int l = j; l < n; l++) {  // < j + n
                    subRect = A[k][l];
                    if (i) subRect -= A[i - 1][l];
                    if (j) subRect -= A[k][j - 1];
                    if (i and j) subRect += A[i - 1][j -
                    1];
                    maxSubRect = max (maxSubRect, subRect);
                }
            }
        }
    }
// "No tree" => make tree (1) = -inf
// no tree (0) = 1.
}
```

- **Max 2D range sum, algo1**

```
// O(n^3)
int maxSum2D () {
    int maxsum = INT_MIN, finalleft, finalright, finaltop,
    finalbottom;
    for (int leftc = 0; leftc < COL; leftc++) {
        vector<int> temp (ROW, 0);
        for (int rightc = leftc; rightc < COL; rightc++) {
            for (int i = 0; i < ROW; i++) {
                temp[i] += M[i][rightc]
            }
            int rstart, rend;
            sum = kadane (temp, rstart, rend);
            // kadane will give us rstart and rend
            if (sum > maxsum) {
                maxsum = sum;
                finalleft = left;
                finalright = right;
                finaltop = rstart;
                finalbottom = rend;
            }
        }
    }
}
```

- Given an $n*m$ graph with each cell containing either 0 or 1, count how many rectangles can be formed by using the 1s (UVA 10502), Precalculate array sum[i][j],the numbers of 1 from graph[1][1] to graph[i][j] (index starting from 1 would be easier to deal with around edges) using $sum[i][j] = sum[i-1][j] + sum[i][j-1] - sum[i-1][j-1] + graph[i][j] -' 0'$. Thus use algo1 of max 2d range. Then we could iterate through all possible rectangle in the given size in $O(n^4)$
- A problem is said to have optimal substructure if an optimal solution can be constructed from optimal solutions of its subproblems.
- A problem must exhibit these two properties in order for a greedy algorithm to work.
  - It has optimal substructures
  - It has the greedy property (if we make a choice that seems like the best at the moment and proceed to solve the remaining subproblem, we reach the optimal soln. We will never have to reconsider our previous choices)
- **Stack Sorting:** If there exist a way to perform the operations such that array b is sorted in non descending order in the end then array a is called stack sortable. Operations availabe:
  1. Take the first element of a, push it into S and remove it from a.
  2. Take the top element from S, append it to the end of array b and remove it from S.

  If you think about it you can see that problem occurs if we have sequence (x, y, z) where z ¡ x ¡ y. So after an element we want either all elements which are smaller than it to come first than elements which are bigger than it or all elements bigger than it. So say if we are given prefix of some permutation as [6, 3, 1] and asked to find lexicographic sequence to append such that it is stack sortable then we can read prefix number one by one and do
  $6 + A(1,5) + A(7,7)$
  $6 + 3 + A(1,2) + A(4,5) + A(7,7)$
  $6 + 3 + 1 + 2 + 5 + 4$ i.e. reverse each A(l, r) $+7$
  And if at any time we couldn't proceed as desired that means soln does not exist.

- stoi, stol, stoll, stod
- to_string
- ```
  int __builtin_clz(int x);// number of leading zero
  int __builtin_ctz(int x);// number of trailing zero
  int __builtin_clzll(long long x);// number of leading zero
  int __builtin_ctzll(long long x);// number of trailing zero
  int __builtin_popcount(int x);// number of 1-bits in x
  int __builtin_popcountll(long long x);// number of 1-bits
  in x

  lsb(n): (n & -n); // last bit (smallest)
  floor(log2(n)): 31 - __builtin_clz(n | 1);
  floor(log2(n)): 63 - __builtin_clzll(n | 1);
  // Suppose we have a pattern of N bits set to 1 in an
  integer and we want the next permutation of N 1 bits in a
  lexicographical sense. For example, if N is 3 and the bit
  pattern is 00010011, the next patterns would be 00010101,
  00010110, 00011001,00011010, 00011100, 00100011, and so
  forth. The following is a fast way to compute the next
  permutation.

  unsigned int v; // current permutation of bits
  unsigned int w; // next permutation of bits

  unsigned int t = v | (v - 1); // t gets v's least
  significant 0 bits set to 1
  // Next set to 1 the most significant bit to change,
  // set to 0 the least significant ones, and add the
  necessary 1 bits.
  w = (t + 1) | (((~t & -~t) - 1) >> (__builtin_ctz(v) + 1));
  ```

3.1 Meet in the middle

Given a set of n integers where $n \le 40$. Each of them is at most 10^{12}, determine the maximum sum subset having sum less than or equal S where S ¡= 10^{13}.

Example:

Input : set[] = 45, 34, 4, 12, 5, 2 and S = 42 Output : 41 Maximum possible subset sum is 41 which can be obtained by summing 34, 5 and 2.

Input : Set[] = 3, 34, 4, 12, 5, 2 and S = 10 Output : 10 Maximum possible subset sum is 10 which can be obtained by summing 2, 3 and 5.

A Brute Force approach to solve this problem would be find all possible subset sums of N integers and check if it is less than or equal S and keep track of such a subset with maximum sum. The time complexity using this approach would be $O(2^n)$ and n is at most 40. 240 will be quite large and hence we need to find more optimal approach.

Meet in the middle is a search technique which is used when the input is small but not as small that brute force can be used. Like divide and conquer it splits the problem into two, solves them individually and then merge them. But we can't apply meet in the middle like divide and conquer because we don't have the same structure as the original problem.

Split the set of integers into 2 subsets say A and B. A having first n/2 integers and B having rest. Find all possible subset sums of integers in set A and store in an array X. Similarly calculate all possible subset sums of integers in set B and store in array Y. Hence, Size of each of the array X and Y will be at most $2^{n/2}$. Now merge these 2 subproblems. Find combinations from array X and Y such that their sum is less than or equal to S. One way to do that is simply iterate over all elements of array Y for each element of array X to check the existence of such a combination. This will take $O((2^{n/2})^2)$ which is equivalent to $O(2^n)$. To make it less complex, first sort array Y and then iterate over each element of X and for each element x in X use binary search to find maximum element y in Y such that $x + y \le S$. Binary search here helps in reducing complexity from 2^n to $2^{n/2} \log 2^{n/2}$ which is equivalent to $2^{n/2} * n$.

```
// C++ program to demonstrate working of Meet in the
// Middle algorithm for maximum subset sum problem.
#include <bits/stdc++.h>
using namespace std;
typedef long long int ll;
ll X[2000005],Y[2000005];

// Find all possible sum of elements of a[] and store
// in x[]
void calcsubarray(ll a[], ll x[], int n, int c)
{
    for (int i=0; i<(1<<n); i++)
    {
        ll s = 0;
        for (int j=0; j<n; j++)
            if (i & (1<<j))
                s += a[j+c];
        x[i] = s;
    }
}

// Returns the maximum possible sum less or equal to S
ll solveSubsetSum(ll a[], int n, ll S)
{
    // Compute all subset sums of first and second
    // halves
    calcsubarray(a, X, n/2, 0);
    calcsubarray(a, Y, n-n/2, n/2);

    int size_X = 1<<(n/2);
    int size_Y = 1<<(n-n/2);

    // Sort Y (we need to do doing binary search in it)
    sort(Y, Y+size_Y);

    // To keep track of the maximum sum of a subset
    // such that the maximum sum is less than S
    ll max = 0;

    // Traverse all elements of X and do Binary Search
    // for a pair in Y with maximum sum less than S.
    for (int i=0; i<size_X; i++)
    {
        if (X[i] <= S)
        {
            // lower_bound() returns the first address
            // which has value greater than or equal to
            // S-X[i].
            int p = lower_bound(Y, Y+size_Y, S-X[i]) - Y;

            // If S-X[i] was not in array Y then decrease
            // p by 1
            if (p == size_Y || Y[p] != (S-X[i]))
                p--;

            if ((Y[p]+X[i]) > max)
                max = Y[p]+X[i];
        }
    }
    return max;
}

// Driver code
int main()
{
    ll a[] = {3, 34, 4, 12, 5, 2};
    int n=sizeof(a)/sizeof(a[0]);
    ll S = 10;
    printf("Largest value smaller than or equal to given "
           "sum is %lld\n", solveSubsetSum(a,n,S));
    return 0;
}

Output: Largest value smaller than or equal to given sum is 10
```

3.2 To find subarray (contingous) with maximum average and of length k

It doesnt make sense to introduce average here as our length is fixed, so it just asks to find maximum sum subarray of length k. Which is easy to compute by simply using sliding window.

```
// C++ program to find maximum average subarray
// of given length.
#include<bits/stdc++.h>
using namespace std;

// Returns beginning index of maximum average
// subarray of length 'k'
int findMaxAverage(int arr[], int n, int k)
{
    // Check if 'k' is valid
    if (k > n)
        return -1;

    // Compute sum of first 'k' elements
    int sum = arr[0];
    for (int i=1; i<k; i++)
        sum += arr[i];

    int max_sum = sum, max_end = k-1;
```
```

```
    // Compute sum of remaining subarrays
    for (int i=k; i<n; i++)
    {
        int sum = sum + arr[i] - arr[i-k];
        if (sum > max_sum)
        {
            max_sum = sum;
            max_end = i;
        }
    }

    // Return starting index
    return max_end - k + 1;
}

// Driver program
int main()
{
    int arr[] = {1, 12, -5, -6, 50, 3};
    int k = 4;
    int n = sizeof(arr)/sizeof(arr[0]);
    cout << "The maximum average subarray of "
         "length "<< k << " begins at index "
         << findMaxAverage(arr, n, k);
    return 0;
}
```

3.3 To find subarray (contiguous) with maximum average and length more than k

Clearly we could have done this in $O(n^2)$ but i'll describe $O(n \log n)$ soln below.

1. Binary search for the maximum average (let it be x)
2. Subtract x from every element of array
   Now the problem reduces to finding longest subarray with sum ≥ 0 as then average would be ≥ 0.
3. Replace a_i by the partial sums $S_i = \sum_{j=1}^{i} a_j$
   Now we are looking for a pair $0 \leq l \leq r \leq N$ s.t. $S_r - S_{l-1} \geq 0$ and $r - (l-1) \geq k$.
4. Mark the positions of mimimas of S_i from left to right in array A and position for maximums S_i from right to left in array B of an array containing S.
5. Do 2 pointers technique now.

```
// This is the code for steps 2-5.
int maxIndexDiff(int arr[], int n)
{
    int maxDiff;
    int i, j;

    int LMin[n], RMax[n];

    // Construct LMin[] such that LMin[i]
    // stores the minimum value
    // from (arr[0], arr[1], ... arr[i])
    LMin[0] = arr[0];
    for (i = 1; i < n; ++i)
        LMin[i] = min(arr[i], LMin[i - 1]);

    // Construct RMax[] such that RMax[j]
    // stores the maximum value
    // from (arr[j], arr[j+1], ..arr[n-1])
    RMax[n - 1] = arr[n - 1];
    for (j = n - 2; j >= 0; --j)
        RMax[j] = max(arr[j], RMax[j + 1]);

    // Traverse both arrays from left to right
    // to find optimum j - i
    // This process is similar to merge()
    // of MergeSort
    i = 0, j = 0, maxDiff = -1;
    while (j < n && i < n) {
        if (LMin[i] < RMax[j]) {
            maxDiff = max(maxDiff, j - i);
            j = j + 1;
        }
        else
            i = i + 1;
    }

    return maxDiff + 1;
}
```

```
// utility Function which subtracts X from all
// the elements in the array
void modifyarr(int arr[], int n, int x)
{
    for (int i = 0; i < n; i++)
        arr[i] = arr[i] - x;
}

// Calculating the prefix sum array
// of the modified array
void calcprefix(int arr[], int n)
{
    int s = 0;
    for (int i = 0; i < n; i++) {
        s += arr[i];
        arr[i] = s;
    }
}

// Function to find the length of the longest
// subarray with average >= x
int longestsubarray(int arr[], int n, int x)
{
    modifyarr(arr, n, x);
    calcprefix(arr, n);

    return maxIndexDiff(arr, n);
}

// Driver code
int main()
{
    int arr[] = { 1, 1, 2, -1, -1, 1 };
    int x = 1;
    int n = sizeof(arr) / sizeof(int);
    cout << longestsubarray(arr, n, x) << endl;

    return 0;
}
```

3.4 Find subarray with given sum, elements are non negative

```
/* A simple program to print subarray with sum as given sum */
#include<stdio.h>

/* Returns true if the there is a subarray of arr[] with sum
equal to 'sum'
   otherwise returns false.  Also, prints the result */
int subArraySum(int arr[], int n, int sum)
{
    int curr_sum, i, j;

    // Pick a starting point
    for (i = 0; i < n; i++)
    {
        curr_sum = arr[i];

        // try all subarrays starting with 'i'
        for (j = i+1; j <= n; j++)
        {
            if (curr_sum == sum)
            {
                printf ("Sum found between indexes %d and %d",
                i, j-1);
                return 1;
            }
            if (curr_sum > sum || j == n)
                break;
           curr_sum = curr_sum + arr[j];
        }
    }

    printf("No subarray found");
    return 0;
}

// Driver program to test above function
int main()
{
    int arr[] = {15, 2, 4, 8, 9, 5, 10, 23};
    int n = sizeof(arr)/sizeof(arr[0]);
```

```
    int sum = 23;
    subArraySum(arr, n, sum);
    return 0;
}
```

3.5 Largest subarray with gcd one

if any two elements have GCD equals to one, then whole array has GCD one. So the output is either -1 or length of array.

3.6 Smallest subarray with given gcd

The idea is to use Segment Tree and Binary Search to achieve time complexity O(n (logn)2).

1. If we have any number equal to 'k' in the array then the answer is 1 as GCD of k is k. Return 1.
2. If there is no number which is divisible by k, then GCD doesn't exist. Return -1.
3. If none of the above cases is true, the length of minimum subarray is either greater than 1 or GCD doesn't exist. In this case, we follow following steps.
   (a) Build segment tree so that we can quicky find GCD of any sub-array using the approach discussed here
   (b) After building Segment Tree, we consider every index as starting point and do binary search for ending point such that the subarray between these two points has GCD k

```
// Returns size of smallest subarray of arr[0..n-1]
// with GCD equal to k.
int findSmallestSubarr(int arr[], int n, int k)
{
    // To check if a multiple of k exists.
    bool found = false;

    // Find if k or its multiple is present
    for (int i=0; i<n; i++)
    {
        // If k is present, then subarray size is 1.
        if (arr[i] == k)
            return 1;

        // Break the loop to indicate presence of a
        // multiple of k.
        if (arr[i] % k == 0)
            found = true;
    }

    // If there was no multiple of k in arr[], then
    // we can't get k as GCD.
    if (found == false)
        return -1;

    // If there is a multiple of k in arr[], build
    // segment tree from given array
    constructSegmentTree(arr, n);

    // Initialize result
    int res = n+1;

    // Now consider every element as starting point
    // and search for ending point using Binary Search
    for (int i=0; i<n-1; i++)
    {
        // a[i] cannot be a starting point, if it is
        // not a multiple of k.
        if (arr[i] % k != 0)
            continue;

        // Initialize indexes for binary search of closest
        // ending point to i with GCD of subarray as k.
        int low = i+1;
        int high = n-1;

        // Initialize closest ending point for i.
        int closest = 0;

        // Binary Search for closest ending point
        // with GCD equal to k.
        while (true)
        {
            // Find middle point and GCD of subarray
            // arr[i..mid]
            int mid = low + (high-low)/2;
            int gcd = findRangeGcd(i, mid, arr, n);

            // If GCD is more than k, look further
            if (gcd > k)
                low = mid;

            // If GCD is k, store this point and look for
            // a closer point
            else if (gcd == k)
            {
                high = mid;
                closest = mid;
                break;
            }

            // If GCD is less than, look closer
            else
                high = mid;

            // If termination condition reached, set
            // closest
            if (abs(high-low) <= 1)
            {
                if (findRangeGcd(i, low, arr, n) == k)
                    closest = low;
                else if (findRangeGcd(i, high, arr, n)==k)
                    closest = high;
                break;
            }
        }

        if (closest != 0)
            res = min(res, closest - i + 1);
    }

    // If res was not changed by loop, return -1,
    // else return its value.
    return (res == n+1) ? -1 : res;
}
```

This same idea can be extended to find largest subarray with given gcd

3.7 LIS

```
// O(n^2)
int LIS () {
    vi L (n, 1);
    for (int i = 0; i < n; i++) {
        for (int j = i + 1; j < n; j++) {
            if (sequence[j] > sequence[i]) {
                L[j] = max (L[j], L[i] + 1);
            }
        }
    }
    return *max_element(L.begin (), L.end ());
}
vi LIS (int ans) {
    vi lis;
    for (int i = n - 1; i >= 0; i--) {
        if (L[i] == ans) {
            lis.pb (sequence[i]);
            ans--;
        }
    }
    reverse (lis.begin (), lis.end ());
    return lis;
}
//------------------------
// O(nlogk)
int LIS (vi &seq) {
    vi L(n, 1);
    vi I;
    for (int i = 0; i < seq.size (); i++) {
        int pos = lower_bound (I.begin (), I.end (), seq[i]) -
        I.begin ();
        if (pos == I.size ()) {
            I.pb (seq[i]);
        } else {
            I[pos] = num;
        }
        L[i] = pos + 1;
        ans = max (ans, L[i]);
    }
    return ans;
}
```

LIS of reverse sequence LDS starting from pos after reversing L.
LIS of reverse negative sequence gives LIS stating from pos after reversing L.

4 Data Structures

4.1 Segment Tree

- Construction of segment tree is O(n) (by masters theorem) and height of segment tree is O(logn). Also time complexity of each query is O(logn).
- freq, sol
- simple update, sol
- lazy range update, sol
- In similar way segment tree can be used to answer queries regarding gcd/lcm.
- Sereja and brackets, sol
- Addition on Segments, sol
- Jamie and to do list, Sol: Just basic application of Persistent segment tree. When updating some element, at most O(logn) nodes in the segment tree get changed: the nodes along the path from root to the updated leaf. For each timepoint, instead of creating a copy of the entire segment tree, copy only nodes on the path to be updated and update them. Therefore total storage is O(n + tlogn).

5 DP

Following are the two main properties of a problem that suggests that the given problem can be solved using Dynamic programming.

1. Overlapping Subproblems: Like Divide and Conquer, Dynamic Programming combines solutions to sub-problems. Dynamic Programming is mainly used when solutions of same subproblems are needed again and again.
2. Optimal Substructure

5.1 Coin Change

```
int mvc(int at, int flag, int parent) { //You can start this
from any node, i.e. in main: int ans =  min(mvc(0, 0, -1),
mvc(0, 1, -1)); and handle the case n == 1 seperately
   if(memo[at][flag] != -1) {
       return memo[at][flag];
   }
   if(glist[at].size() == 1 and parent != -1) { //leaf node
       return memo[at][flag] = flag;
   }
   int ans = flag;
   if(flag) // to take this
   {
       for(auto to : glist[at]) {
           if(to != parent)
               ans += min(mvc(to, 0, at), mvc(to, 1, at));
       }
   } else { //we must take its neighbours
       for(auto to : glist[at]) {
           if(to != parent)
               ans += mvc(to, 1, at);
       }
   }
   return memo[at][flag] = ans;
}  // Similar code can be written to find MWIS.
```

5.2 0/1 Knapsack

Given weights and values of n items, put these items in a knapsack of capacity W to get the maximum total value in the knapsack. You cannot break an item, either pick the complete item, or don't pick it (thus we cannot use greedy algorithm)

```
vector1.insert (vector1.end (), vector2.begin (), vector2.end
());
```

5.3 Balanced Bracket Sequence

A Balanced bracket sequence is a string consisting of only brackets, such that this sequence, when certain numbers and + is inserted gives a valid mathematical expression.

5.3.1 One type of bracket

Let depth be the current no. of open brackets, initially depth = 0. We iterate over all character of the string; if the current bracket character is an opening bracket then we increment depth, o/w we decrement it. I f at any time the variable depth gets negative, or at the end it is different from 0, then the string is not a balanced sequence otherwise it is.

5.3.2 MultiType

Maintain a stack, in chich we will store all opening brackets that we meet. If the current bracket character is an opening one, we put it onto the stack. If it is a closing one, then we check if the stack is non empty, and if the top element is of the same type as the current closing bracket, if both conditions are fulfilled, then we remove the opening bracker from the stack. If at any time one of the ocnditions is not fulfilled or at the end the stack is non empty, then the string is not balanced otherwise it is.

5.3.3 No. of balanced Sequences

The number of balanced bracket sequences with only one bracket type can be calculated using the Catalan numbers. The number of balanced bracket sequences of length $2n$ (n pairs of brackets) is:

$$\frac{1}{n+1}\binom{2n}{n}$$

If we allow k types of brackets, then each pair be of any of the k types (independently of the others), thus the number of balanced bracket sequences is:

$$\frac{1}{n+1}\binom{2n}{n}k^n$$

On the other hand these numbers can be computed using dynamic programming. Let $d[n]$ be the number of regular bracket sequences with n pairs of bracket. Note that in the first position there is always an opening bracket. And somewhere later is the corresponding closing bracket of the pair. It is clear that inside this pair there is a balanced bracket sequence, and similarly after this pair there is a balanced bracket sequence. So to compute $d[n]$, we will look at how many balanced sequences of i pairs of brackets are inside this first bracket pair, and how many balanced sequences with $n-1-i$ pairs are after this pair. Consequently the formula has the form:

$$d[n] = \sum_{i=0}^{n-1} d[i] \cdot d[n-1-i]$$

The initial value for this recurrence is $d[0] = 1$.

5.3.4 Lexicographically next balanced sequence

```
  // Idea: "dep" indicates the imbalance in the string s[0...i
  - 1]. Now after replacing s[i] with ')', dep dec. and we want
  to add the lexicographically least string having 'dep - 1'
  closing brackets reserved.
  bool next_balanced_sequence(string & s) {
    int n = s.size();
    int depth = 0;
    for (int i = n - 1; i >= 0; i--) {
        if (s[i] == '(')
            depth--;
        else
            depth++;

        if (s[i] == '(' && depth > 0) {
            depth--;
            int open = (n - i - 1 - depth) / 2;
            int close = n - i - 1 - open;
            string next = s.substr(0, i) + ')' + string(open,
            '(') + string(close, ')');
            s.swap(next);
            return true;
        }
    }
    return false;
}
```

If it is required to find and output all balanced bracket sequences of a specific length n.

To generate them, we can start with the lexicographically smallest sequence $((\dots(())\dots))$, and then continue to find the next lexicographically sequences with the algorithm described above.

5.3.5 Sequence Index

Given a balanced bracket sequence with n pairs of brackets. We have to find its index in the lexicographically ordered list of all balanced sequences with n bracket pairs.

Let's define an auxiliary array $d[i][j]$, where i is the length of the bracket sequence (semi-balanced, each closing bracket has a corresponding opening bracket, but not every opening bracket has necessarily a corresponding closing one), and j is the current balance (difference between opening and closing brackets). $d[i][j]$ is the number of such sequences that fit the parameters. We will calculate these numbers with only one bracket type.

For the start value $i = 0$ the answer is obvious: $d[0][0] = 1$, and $d[0][j] = 0$ for $j > 0$. Now let $i > 0$, and we look at the last character in the sequence. If the last character was an opening bracket (, then the state before was $(i-1, j-1)$, if it was a closing bracket), then the previous state was $(i-1, j+1)$. Thus we obtain the recursion formula:

$$d[i][j] = d[i-1][j-1] + d[i-1][j+1]$$

$d[i][j] = 0$ holds obviously for negative j. Thus we can compute this array in $O(n^2)$.

Now let us generate the index for a given sequence.

First let there be only one type of brackets. We will us the counter depth which tells us how nested we currently are, and iterate over the characters of the sequence. If the current character $s[i]$ is equal to (, then we increment depth. If the current character $s[i]$ is equal to), then we must add $d[2n-i-1][\text{depth}+1]$ to the answer, taking all possible endings starting with a (into account (which are lexicographically smaller sequences), and then decrement depth.

New let there be k different bracket types.

Thus, when we look at the current character $s[i]$ before recomputing depth, we have to go through all bracket types that are smaller than the current character, and try to put this bracket into the current position (obtaining a new balance ndepth $=$ depth ± 1), and add the number of ways to finish the sequence (length $2n-i-1$, balance $ndepth$) to the answer:

$$d[2n-i-1][\text{ndepth}] \cdot k^{\frac{2n-i-1-ndepth}{2}}$$

This formula can be derived as follows: First we "forget" that there are multiple bracket types, and just take the answer $d[2n-i-1][\text{ndepth}]$. Now we consider how the answer will change is we have k types of brackets. We have $2n-i-1$ undefined positions, of which ndepth are already predetermined because of the opening brackets. But all the other brackets ($(2n-i-i-\text{ndepth})/2$ pairs) can be of any type, therefore we multiply the number by such a power of k.

5.3.6 Finding the kth sequence

Let n be the number of bracket pairs in the sequence. We have to find the k-th balanced sequence in lexicographically sorted list of all balanced sequences for a given k.

As in the previous section we compute the auxiliary array $d[i][j]$, the number of semi-balanced bracket sequences of length i with balance j.

First, we start with only one bracket type.

We will iterate over the characters in the string we want to generate. As in the previous problem we store a counter depth, the current nesting depth. In each position we have to decide if we use an opening of a closing bracket. To put an opening bracket character, it $d[2n-i-1][\text{depth}+1] \geq k$. We increment the counter depth, and move on to the next character. Otherwise we decrement k by $d[2n-i-1][\text{depth}+1]$, put a closing bracket and move on.

```
string kth_balanced(int n, int k) {
  vector<vector<int>> d(2*n+1, vector<int>(n+1, 0));
  d[0][0] = 1;
  for (int i = 1; i <= 2*n; i++) {
      d[i][0] = d[i-1][1];
      for (int j = 1; j < n; j++)
          d[i][j] = d[i-1][j-1] + d[i-1][j+1];
      d[i][n] = d[i-1][n-1];
  }

  string ans;
  int depth = 0;
  for (int i = 0; i < 2*n; i++) {
      if (depth + 1 <= n && d[2*n-i-1][depth+1] >= k) {
          ans += '(';
          depth++;
      } else {
          ans += ')';
          if (depth + 1 <= n)
              k -= d[2*n-i-1][depth+1];
          depth--;
      }
  }
  return ans;
}
```

Now let there be k types of brackets. The solution will only differ slightly in that we have to multiply the value $d[2n-i-1][\text{ndepth}]$ by $k^{(2n-i-1-\text{ndepth})/2}$ and take into account that there can be different bracket types for the next character.

Here is an implementation using two types of brackets: round and square:

```
string kth_balanced2(int n, int k) {
  vector<vector<int>> d(2*n+1, vector<int>(n+1, 0));
  d[0][0] = 1;
  for (int i = 1; i <= 2*n; i++) {
      d[i][0] = d[i-1][1];
      for (int j = 1; j < n; j++)
          d[i][j] = d[i-1][j-1] + d[i-1][j+1];
      d[i][n] = d[i-1][n-1];
  }

  string ans;
  int depth = 0;
  stack<char> st;
  for (int i = 0; i < 2*n; i++) {
      // '('
      if (depth + 1 <= n) {
          int cnt = d[2*n-i-1][depth+1] << ((2*n-i-1-depth-1)
          / 2);
          if (cnt >= k) {
              ans += '(';
              st.push('(');
              depth++;
              continue;
          }
          k -= cnt;
      }

      // ')'
      if (depth && st.top() == '(') {
          int cnt = d[2*n-i-1][depth-1] << ((2*n-i-1-depth+1)
          / 2);
          if (cnt >= k) {
              ans += ')';
              st.pop();
              depth--;
              continue;
          }
          k -= cnt;
      }

      // '['
      if (depth + 1 <= n) {
          int cnt = d[2*n-i-1][depth+1] << ((2*n-i-1-depth-1)
          / 2);
          if (cnt >= k) {
              ans += '[';
              st.push('[');
              depth++;
              continue;
          }
          k -= cnt;
      }

      // ']'
      ans += ']';
      st.pop();
      depth--;
  }
  return ans;
}
```

5.4 Important Problems

- dividing coins sol
- TSP, Sol
- Square, Sol
- Pebble Solitare, sol
- Forming Quiz Teams, sol
- Simple tree coloring, sol

6 Strings

To map keyboard etc, it is better to create 2 strings then loop through and map.
To transform complete string to lowercase:

```
transform (word.begin (), word.end (), word.begin (),
::tolower);
```

To concatenate two vectors:

```
string kth_balanced2(int n, int k) {
  vector<vector<int>> d(2*n+1, vector<int>(n+1, 0));
  d[0][0] = 1;
  for (int i = 1; i <= 2*n; i++) {
      d[i][0] = d[i-1][1];
      for (int j = 1; j < n; j++)
          d[i][j] = d[i-1][j-1] + d[i-1][j+1];
      d[i][n] = d[i-1][n-1];
  }

  string ans;
  int depth = 0;
  stack<char> st;
  for (int i = 0; i < 2*n; i++) {
      // '('
      if (depth + 1 <= n) {
          int cnt = d[2*n-i-1][depth+1] << ((2*n-i-1-depth-1)
          / 2);
          if (cnt >= k) {
```

```
                ans += '(';
                st.push('(');
                depth++;
                continue;
            }
            k -= cnt;
        }

        // ')'
        if (depth && st.top() == '(') {
            int cnt = d[2*n-i-1][depth-1] << ((2*n-i-1-depth+1)
            / 2);
            if (cnt >= k) {
                ans += ')';
                st.pop();
                depth--;
                continue;
            }
            k -= cnt;
        }

        // '['
        if (depth + 1 <= n) {
            int cnt = d[2*n-i-1][depth+1] << ((2*n-i-1-depth-1)
            / 2);
            if (cnt >= k) {
                ans += '[';
                st.push('[');
                depth++;
                continue;
            }
            k -= cnt;
        }

        // ']'
        ans += ']';
        st.pop();
        depth--;
    }
    return ans;
}

string.substr (startposn, length); // Where startposn is 0
indexed.
str.erase (posn); // erase single character
str.erase (posn, length);
str.insert (pos, anotherString_orCharacter);

int pos1 = line.find ("U=");
if (pos1 != -1) { // process }
line.replace (pos, len, newString); // pos = line.find (f), len
= f.size ()
```

We can iterate through all substrings of string $O(n^2)$ and see which all of them are palindromes in $O(n^3)$ or in $O(n^2)$ by using dp $(dp[startpos][endpos] = (s[startpos] == s[endpos] \&\& dp[startpos+1][endpos-1])$ or hash.

6.1 Minimum Edit Distance

```
void fillmem() {
   for (int j = 0; j <= a.size(); j++) mem[0][j] = j;
   for (int i = 0; i <= b.size(); i++) mem[i][0] = i;
   for (int i = 1; i <= b.size(); i++) {
      for (int j = 1; j <= a.size(); j++) {
         if (a[j - 1] == b[i - 1]) mem[i][j] = mem[i - 1][j -
         1];
         else mem[i][j] = min(mem[i - 1][j - 1], min(mem[i -
         1][j], mem[i][j - 1])) + 1;
      }
   }
    // mem[b.size ()][a.size ()] contains the answer
}
void print() {
   int i = b.size(), j = a.size();
   while (i || j) {
      if (i and j and a[j - 1] == b[i - 1]) { i--; j--;
      continue; }
      if (i and j and mem[i][j] == mem[i - 1][j - 1] + 1) {
         cout << "C" << b[i - 1]; if (j <= 9) cout << "0";
         cout << j;
         i--; j--;
         continue;
      }
      if (i and mem[i][j] == mem[i - 1][j] + 1) {
         cout << "I" << b[i - 1];
         if (j <= 9) cout << "0";
         cout << j + 1;
         i--;
         continue;
      }
      else if (j) {
         cout << "D" << a[j - 1];
         if (j <= 9) cout << "0";
         cout << j;
         j--;
      }
   }
   cout << "E\n";
}
```

6.2 Length of longest Palindrome possible by removing 0 or more characters

```
dp[startpos][endpos] = s[startpos] == s[endpos] ? 2 +
dp[startpos + 1][endpos - 1] : max (dp[startpos + 1][endpos],
dp[startpos][endpos - 1])
```

6.3 Longest Common Subsequence

```
memset (mem, 0, sizeof (mem));
for (int i = 1; i <= b.size (); i++) {
  for (int j = 1; j <= a.size (); j++) {
    if (b[i - 1] == a[j - 1]) mem[i][j] = mem[i - 1][j - 1] +
    1;
    else mem[i][j] = max (mem[i - 1][j], mem[i][j - 1])
  }
}
void printsol (int ui, int li) {
  ui--; li--;
  vector<string> ans;
  while (ui || li) {
    if (a[ui] == b[li]) {
      ans.push_back (a[ui]);
      ui--; li--;
      continue;
    }
    if (ui and mem[ui][li] == mem[ui - 1][li]) {
      ui--;
      continue;
    }
    if (li and mem[ui][li] == mem[ui][li - 1]) {
      li--;
      continue;
    }
  }
  reverse (ans.begin (), ans.end ());
  cout << ans << "\n";
}
```

6.4 Prefix Function and KMP

6.4.1 Prefix Function

The prefix function for this string is defined as an array π of length n, where $\pi[i]$ is the length of the longest proper prefix of the substring $s[0 \dots i]$ which is also a suffix of this substring. A proper prefix of a string is a prefix that is not equal to the string itself. By definition, $\pi[0] = 0$. Example:
$abcabchejfabcabca$
00012300001234564
Note: $\pi[i+1] \leq \pi[i] + 1$ as if $\pi[i+1] > \pi[i] + 1$ then consider this suffix ending at position i + 1 & having length $\pi[i+1]$ - removing the last character we get a suffix ending in position i & having length $\pi[i+1] - 1$ that is better than $\pi[i]$. Should be able to reason the following code.

```
vector<int> prefix_function(string &s) {  // O(n)
    int n = (int)s.length();
    vector<int> pi(n, 0);
    for (int i = 1; i < n; i++) {
        int j = pi[i-1];
        while (j > 0 && s[i] != s[j])
            j = pi[j-1];
        if (s[i] == s[j])
            j++;
        pi[i] = j;
    }
    return pi;
}
```

6.4.2 KMP

Given a text t and a string s, we want to find and display the positions of all occurrences of the string s in the text t.
For convenience we denote with n the length of the string s and with m the length of the text t.
We generate the string s+#+t, where # is a separator that doesn't appear in s and t. Let us calculate the prefix function for this string. Now think about the meaning of the values of the prefix function, except for the first n+1 entries (which belong to the string s and the separator). By definition the value π[i] shows the longest length of a substring ending in position i that coincides with the prefix. But in our case this is nothing more than the largest block that coincides with s and ends at position i. This length cannot be bigger than n due to the separator. But if equality π[i]=n is achieved, then it means that the string s appears completely in at this position, i.e. it ends at position i. Just do not forget that the positions are indexed in the string s+#+t.
Thus if at some position i we have π[i]=n, then at the position $i`(n+1)`n+1=i`2n$ in the string t the string s appears.
As already mentioned in the description of the prefix function computation, if we know that the prefix values never exceed a certain value, then we do not need to store the entire string and the entire function, but only its beginning. In our case this means that we only need to store the string s+# and the values of the prefix function for it. We can read one character at a time of the string t and calculate the current value of the prefix function.

```
    // IMP NOTE: In both bubble sort and merge sort we are
    getting minimum no. of swaps to sort an array (i.e. by
    swapping adjacent elements)
void merge(int arr[], int l, int m, int r)
{
   int i, j, k;
   int n1 = m - l + 1;
   int n2 =  r - m;

   /* create temp arrays */
   int L[n1], R[n2];

   /* Copy data to temp arrays L[] and R[] */
   for (i = 0; i < n1; i++)
       L[i] = arr[l + i];
   for (j = 0; j < n2; j++)
       R[j] = arr[m + 1+ j];

   /* Merge the temp arrays back into arr[l..r]*/
   i = 0; // Initial index of first subarray
   j = 0; // Initial index of second subarray
   k = l; // Initial index of merged subarray
   while (i < n1 && j < n2)
   {
       if (L[i] <= R[j])
       {
           arr[k] = L[i];
           i++;
       }
       else///i.e we need to swap
       {
           arr[k] = R[j];
           swaps+=n1-i;//Most important line. basically once we
           are doing arr[k]=R[j] that means we are
           ///putting R[j] before each of n1-i elements thus
           there are that many swaps.
           j++;
       }
       k++;
   }

   /* Copy the remaining elements of L[], if there
      are any */
   while (i < n1)
   {
       arr[k] = L[i];
       i++;
       k++;
   }

   /* Copy the remaining elements of R[], if there
      are any */
   while (j < n2)
   {
       arr[k] = R[j];
       j++;
       k++;
   }
}

/* l is for left index and r is right index of the
  sub-array of arr to be sorted */
void mergeSort(int arr[], int l, int r)
{
   if (l < r)
   {
       // Same as (l+r)/2, but avoids overflow for
       // large l and h
       int m = l+(r-l)/2;

       // Sort first and second halves
       mergeSort(arr, l, m);
       mergeSort(arr, m+1, r);

       merge(arr, l, m, r);
   }
}
```

6.4.3 Counting number of occurrences of each prefix

```
  // Idea: "dep" indicates the imbalance in the string s[0...i
  - 1]. Now after replacing s[i] with ')', dep dec. and we want
  to add the lexicographically least string having 'dep - 1'
  closing brackets reserved.
  bool next_balanced_sequence(string & s) {
    int n = s.size();
    int depth = 0;
    for (int i = n - 1; i >= 0; i--) {
        if (s[i] == '(')
            depth--;
        else
            depth++;

        if (s[i] == '(' && depth > 0) {
            depth--;
            int open = (n - i - 1 - depth) / 2;
            int close = n - i - 1 - open;
            string next = s.substr(0, i) + ')' + string(open,
            '(') + string(close, ')');
            s.swap(next);
            return true;
        }
    }
    return false;
}
```

6.5 Notes

- In case of hashing a string, we follow polynomial rolling hash function, with p as a prime number roughly equal to the size of character domain and m as a huge prime number.
- If s is palindrome and if $s[0...n-2]$ is palindrome, that means all characters are same thus if all characters are not same then the longest non palindromic substring is $s[0...n-2]$ or $s[1...n-1]$

6.6 SAM

A suffix automaton for a given string s is a minimal DFA that accepts all the suffixes of the string s.

- A suffix automaton is an oriented acyclic graph.
- One of the states t_0 is the initial state
- All transitions originating from a state must have different labels
- One or multiple states are marked as terminal states. If we start from the initial state t_0 and move along transitions to a terminal state, then the labels of the passed transitions must spell one of the suffixes of the string s. Each of the suffixes of s must be spellable using a path from t_0 to a terminal state.

Consider any non-empty substring t of the string s. We will denote with endpos(t) the set of all positions in the string s, in which the occurrences of t end. For instance, we have endpos("bc")= {2,4} for the string "abcbc". We will call two substrings t1 and t2 endpos-equivalent, if their ending sets coincide i.e. endpos(t1) = endpos(t2). Thus all non-empty substrings of the string s can be decomposed into several equivalence classes according to their sets endpos.
It turns out, that in a suffix machine endpos-equivalent substrings correspond to the same state. In other words the number of states in a suffix automaton is equal to the number of equivalence classes among all substrings, plus the initial state.
Lemma 1: Two non-empty substrings u and w (with length(u) ≤ length(w)) are endpos-equivalent, if and only if the string u occurs in s only in the form of a suffix of w. (Proof is obvious)
Lemma 2: Consider two non-empty substrings u and w (with length(u)

≤ length(w)). Then their sets endpos either don't intersect at all, or endpos(w) is a subset of endpos(u). And it depends on if u is a suffix of w or not. (Proof is obvious)
Lemma 3: Consider an endpos-equivalence class. Sort all the substrings in this class by non-increasing length. Then in the resulting sequence each substring will be one shorter than the previous one, and at the same time will be a suffix of the previous one. In other words the substrings in the same equivalence class are actually each others suffixes, and take all possible lengths in a certain interval [x;y].
Consider some state v ≠ t_0 in the automaton. As we know, the state v corresponds to the class of strings with the same endpos values. And if we denote by w the longest of these strings, then all the other strings are suffixes of w. **suffix link** *link(v)* leads to the state that corresponds to the longest suffix of w that is another endpos-equivalent class.
Lemma 4: Suffix links form a tree with the root t_0.
Lemma 5: If we build a endpos tree from all the existing sets (according to the principle "the set-parent contains as subsets of all its children"), then it will coincide in structure with the tree of suffix references. **Note:** $endpos(t_0) = \{-1, 0, \ldots, length(s) - 1\}$
Note: For each state v one or multiple substrings match. We denote by longest(v) the longest such string, and through len(v) its length. We denote by shortest(v) the shortest such substring, and its length with minlen(v). Then all the strings corresponding to this state are different suffixes of the string longest(v) and have all possible lengths in the interval [minlength(v);len(v)]. For each state v ≠ t_0 a suffix link is defined as a link, that leads to a state that corresponds to the suffix of the string longest(v) of length $minlen(v)`1$. minlen(v) = len(link(v))+1
Number of states in suffix automaton of the string s of length n doesn't exceed $2n - 1$ (for n ≥ 2)
Number of transitions $\leq 3n - 4$.

```
#include<bits/stdc++.h>

using namespace std;

typedef pair<int, int> ii;
typedef long long int int;
//Learning in depth about suffix automaton.
struct state {
    int len, link;
    map<char,int> next;
    int cnt;
    int firstpos;
    bool is_clon;
    vector<int> inv_link;
};
const int MAXLEN = 250005;
vector<state> st;
int sz, last;
vector<vector<int> > tcntdata;
vector<int> nsubs, d, lw;
vector<bool> isterminal;
void sa_init(unsigned int size) {
    nsubs.assign(2 * size, 0);
    isterminal.assign(2 * size, false);
    tcntdata.clear();
    tcntdata.resize(2 * size);
    lw.assign(2 * size, 0);
    d.assign(2 * size, 0);
    st.clear();
    st.resize(2 * size);
    sz = last = 0;
    st[0].len = 0;
    st[0].cnt = 0;
    st[0].link = -1;
    st[0].firstpos = -1;
    st[0].is_clon = false;
    ++sz;
    tcntdata[0].push_back(0);
}
void sa_extend (char c) {
    int cur = sz++;
    st[cur].cnt = 1;
    st[cur].len = st[last].len + 1;
    st[cur].firstpos = st[cur].len - 1;
    st[cur].is_clon = false;
    tcntdata[st[cur].len].push_back(cur);
    int p;
    for (p=last; p!=-1 && !st[p].next.count(c); p=st[p].link)
        st[p].next[c] = cur;
    if (p == -1) // In case we came to the root, every
    non-empty suffix of string sc is accepted by state cur
    hence we can make link(cur) = t0 and finish our work on
    this step.
        st[cur].link = 0;
    else {  // Otherwise we found such state q`, which already
    has transition by character c. It means that all suffixes
    of length ≤ len(q`) + 1 are already accepted by some state
    in automaton hence we don't need to add transitions to
    state new anymore. But we also have to calculate suffix
    link for state new.
        int q = st[p].next[c];
        if (st[p].len + 1 == st[q].len)  // The largest string
        accepted by this state will be suffix of sc of length
        len(q`) + 1. It is accepted by state t at the moment,
        in which there is transition by character c from state
        q`. But state t can also accept strings of bigger
        length. So, if len(t) = len(q`) + 1, then t is the
        suffix link we are looking for. We make link(cur) = t
        and finish algorithm.
            st[cur].link = q;
        else {
            int clone = sz++;
            st[clone].len = st[p].len + 1;
            st[clone].next = st[q].next;
            st[clone].link = st[q].link;
            st[clone].cnt = 0;
            st[clone].firstpos = st[q].firstpos;
            st[clone].is_clon = true;
            tcntdata[st[clone].len].push_back(clone);
            for (; p!=-1 && st[p].next[c]==q; p=st[p].link)
                st[p].next[c] = clone;
            st[q].link = st[cur].link = clone;
        }
    }
    last = cur;
}
// A state v will correspond to set of endpos equivalent
strings, cnt[v] will give the number of occurences of such
strings
void processcnt() {
    int maxlen = st[last].len;
    for(int i = maxlen; i >= 0; i--) {
        for(auto v : tcntdata[i]) {
            st[st[v].link].cnt += st[v].cnt;
        }
    }
}

// Clearly suffixes should be marked as terminal
void processterminal() {
    isterminal[last] = true;
    int p = st[last].link;
    while(p != -1) {
        isterminal[p] = true;
        p = st[p].link;
    }
}

// Gives the number of substrings (not necessarily distinct).
Clearly it should return n.(n+1)/2
int processnumsubs(int at) {
    if(nsubs[at] != 0) return nsubs[at];
    nsubs[at] = st[at].cnt;
    for(auto to : st[at].next) {
        nsubs[at] += processnumsubs(to.second);
    }
    return nsubs[at];
}

void constructSA(string ss) {
    sa_init(ss.size());
    for(int i = 0; i < ss.size(); i++) {
        sa_extend(ss[i]);
    }
    processterminal();
    processcnt();
    for (int v = 1; v < sz; ++v)
        st[st[v].link].inv_link.push_back(v);
    processnumsubs(0);
}
// --------------------------------------After SA Construction
```

```
//
int getcorrstate(string &tosearch) {
    int at = 0;
    for (int i = 0; i < tosearch.size(); i++) {
        if (!st[at].count (tosearch[i])) return -1;
        at = st[at].next[tosearch[i]];
    }
    return at;
}

bool exist(string &tosearch) {
    int at = getcorrstate (tosearch);
    return at == -1 ? false : true;
}

// Returns number of different substrings = number of paths in
DAG. And number of paths is clearly not a function of number of
states in DAG.
// d[v] = 1 + summation (d[w])
int numdiffsub(int at) {
    if(d[at] != 0) return d[at];
    d[at] = 1;
    for(auto to : st[at].next) {
        d[at] += numdiffsub(to.second);
    }
    return d[at];
}

// Returns total length of all distinct substrings =
summation_path (number of edges constituting that path) in DAG.
// ans[v] = summation (d[w] + ans[w]) basically, once we know
ans[w], we know that we have number of paths starting from that
node + ans[w] // as we know that in each of the contributing
strings we should add 1 for this character transition as this
character occurs in path for reaching this state. Plus 1 as to
consider this character on its own.
int totlength(int at) {
    if(lw[at] != 0) return lw[at];
    for(auto to : st[at].next) {
        lw[at] += d[to.second] + totlength(to.second);
    }
    return lw[at];
}

// Find Lexicographically K-th Substring (here repeated
substring is allowed):
void kthlexo(int at, int k, string &as) {
    if(k <= 0) return;
    for(auto to : st[at].next) {
        if(nsubs[to.second] >= k) {
            as.push_back(to.first);
            kthlexo(to.second, k - st[to.second].cnt, as);
            break;
        } else {
            k -= nsubs[to.second];
        }
    }
}
// Repeated substring not allowed
void kthlexo2(int at, int k, string &as) {
    if(k <= 0) return;
    for(auto to : st[at].next) {
        if(d[to.second] >= k) {
            as.push_back(to.first);
            kthlexo2(to.second, k - 1, as);
            break;
        } else {
            k -= d[to.second];
        }
    }
}
// Returns true is the given string is the suffix of T
bool issuffix(string &tosearch) {
    int at = getcorrstate (tosearch);
    return isterminal[at];
}

// Returns how many times P enters in T (occurences can
overlap)
/* for each state v of the machine calculate a number 'cnt[v]'
which is equal to the
 * size of the set endpos(v). In fact, all the strings
corresponding to the same state
 * enter the T same number of times which is equal to the
number of positions in the set
 * endpos. */
int numoccur(string &tosearch) {
    int at = getcorrstate (tosearch);
    return at == -1 ? 0 : st[at].cnt;
}

// Return position of the first occurrence of substring in T
int firstpos(string &tosearch) {
    int at = getcorrstate (tosearch);
    return st[at].firstpos - tosearch.size() + 1;
}

// Returns Positions of all occurrences of substring in T
void output_all_occurences (int v, int P_length) {
    if (!st[v].is_clon)
        cout << st[v].firstpos - P_length + 1 << "\n";
    for (size_t i=0; i<st[v].inv_link.size(); ++i)
        output_all_occurences(st[v].inv_link[i], P_length);
}

void smallestcyclicshift(int n) {
    int at = 0;
    string anss;
    int length = 0;
    while(length != n) {
        for (auto it : st[at].next) {
            anss.push_back(it.first);
            at = it.second;
            length++;
            break;
        }
    }
    cout << anss << "\n";
    // cout << st[at].firstpos - n + 1 << "\n"; may give the
    index for that shift.
}


int main() {
    string s;
    cin >> s;
    constructSA(s);
    int choice;
    cout << "Choose your option:\n1: Substring exist in T or
    not\n2: Number of different substring of T\n";
    cout << "3: To find total length of distinct substrings\n";
    cout << "4: To check whether the given string is suffix or
    not\n";
    cout << "5(5.1): To print the K-th lexicographic substring
    (Repeated substrings allowed)\n";
    cout << "6: To see how many times, given string occurs in
    T\n";
    cout << "7: To find the position of the first occurrence of
    substring in T\n";
    cout << "8: To find position of all the occurences of
    substring in T\n";
    cout << "9(5.2): To print the K-th lexicographic substring
    (Repeated substrings not allowed)\n";
    cout << "10: To find the smallest cyclic shift\n";

    cout << "15: to exit\n";
    cin >> choice;
    if(choice == 15) break;
    string ss, ns;
    int k, v;
    switch(choice) {
        case 1:
            cout << "Enter your string\n";
            cin >> ss;
            if (exist(ss)) {
                cout << "yes it exist\n";
            } else {
                cout << "no it does not exist\n";
            }
            //cout << "Enter new to string to search for\n";
            break;
        case 2:
            cout << numdiffsub(0) - 1 << "\n";
```

```
        break;
    case 3:
        numdiffsub(0);
        cout << totlength(0) << "\n";
        break;
    case 4:
        cout << "Enter the string\n";
        cin >> ss;
        if(issuffix(ss)) cout << "yes\n";
        else cout << "no\n";
        break;
    case 5:
        cin >> k;
        ss.clear();
        kthlexo(0, k, ss);
        if(ss.empty()) {
            ss = "No such line.";
        }
        cout << ss << "\n";
        break;
    case 6:
        cout << "Enter string\n";
        cin >> ss;
        cout << numoccur(ss) << "\n";
        break;
    case 7:
        cout << "Enter string\n";
        cin >> ss;
        cout << firstpos(ss) << "\n";
        break;
    case 8:
        cout << "Enter string\n";
        cin >> ss;
        /*for(v = 0; v < s.size(); v++) {
            cout << setw(2) << v;
        }
        cout << "\n";
        for(v = 0; v < s.size(); v++) {
            cout << setw(2) << s[v];
        }
        cout << "\n";*/
        v = getcorrstate(ss);
        if(v != -1) {
            output_all_occurences(v, ss.size());
        }
        break;
    case 9:
        cin >> k;
        numdiffsub(0);
        kthlexo2(0, k, ss);
        if(ss.empty()) {
            ss = "No such line.";
        }
        cout << ss << "\n";
        break;
    case 10:
        cout << "Enter S\n";
        cin >> ss;
        s = ss + ss;
        constructSA(s);
        smallestcyclicshift(ss.size ());
        break;
    }
    return 0;
}
```

6.7 Important Problems

Review: cf 631D

- UVA 10739 Sol, UVA 10739 Prob: String to palindrome, just see the minimum edit distance between this string and its reverse but need to divide by 2 later as both strings are it itself.
- Queries for the number of palindromic substrings within given range, **See this soln to see power of hashing.**:

Note: Strings and arrays are considered 0-based in the following solution.

Let $isPal[i][j]$ be 1 if $s[i...j]$ is palindrome, otherwise, set it 0. Let's define $dp[i][j]$ to be number of palindrome substrings of $s[i...j]$. Let's calculate $isPal[i][j]$ and $dp[i][j]$ in $O(|S|^2)$. First, initialize $isPal[i][i] = 1$ and $dp[i][i] = 1$. After that, loop over len which states length of substring and for each specific len, loop over $start$ which states starting position of substring. $isPal[start][start + len - 1]$ can be easily calculated by the following formula:

```
isPal[start][start+len-1] = isPal[start+1][start+len-2] & (s[start] == s[start+len-1])
```

After that, $dp[start][start + len - 1]$ can be calculated by the following formula which is derived from Inc-Exc Principle.

```
dp[start][start+len-1] = dp[start][start+len-2] + dp[start+1][start+len-1] - dp[start+1][start+len-2] + isPal[start][start+len-1]
```

After preprocessing, we get queries l_i and r_i and output $dp[l_i - 1][r_i - 1]$. Overall complexity is $O(|S|^2)$.

- UVA 11107 Sol - simple, UVA 11107 Sol - complicated but more powerful: Problem is to find the longest substring shared by more than half of given strings.
- UVA 10459 Sol, UVA 10029 Prob: Edit steps, (lexicographic sequence of words)

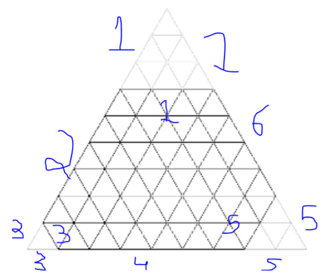
7 Geometry

- for given 3 points of a valid parallelogram, there are 3 possible locations for the 4th point.
- we have a hexagon with integral sides and all interior angles equal to 120 deg. we draw lines parallel to the side of the hexagon, such that we get equilateral triangles of side 1 unit, how many equilateral triangles did we got? Sol:

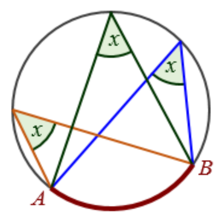
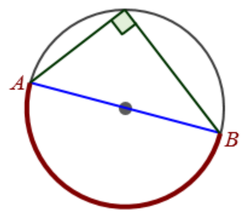
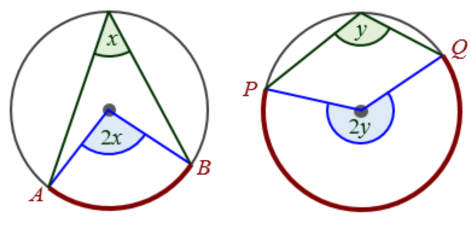
Let's consider regular triangle with sides of k Let's split it to regular triangles with sides of 1 by lines parallel to the sides. Big triange area k^2 times larger then small triangles area and therefore big triangle have splitted by k^2 small triangles.

If we join regular triangles to sides a_1, a_3 and a_5 of hexagon we get a triangle sides of $a_1 + a_2 + a_3$. Then ~~hexagon area~~ is equals to $(a_1 + a_2 + a_3)^2 - a_1{}^2 - a_3{}^2 - a_5{}^2$.

Δ_5



- An inscribed angle is an angle formed by 2 chords in a circle which have a common endpoint. This common endpoint form the vertex o the inscribed angle. The other 2 endpoints define what we call an intercepted arc.
- Measure of intercepted arc of a unit circle is $2/\pi$ that of inscribed angle.

Angles In A Circle	
Inscribed angles subtended by the same arc are equal.	
Angles subtended by the diameter (or semi-circle) is 90°.	
Central angle is twice any inscribed angle subtended by the same arc.	

-

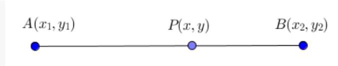
- Circles will certainly not touch or intersect iff the dist. between their center is greater than the sum of their radii.
- A circle 'a' contains a circle 'b' iff the distance between their centers is less than the absolute value of their radii difference.
- let the bottom left/top right corner point be denoted as a/c resp. Then rectangles intersect iff $max(a1.x, a2.x) < min(c1.x, c2.x)$ and $max(a1.y, a2.y) < min(c1.y, c2.y)$.
- Similarly in case of 3d, vol of intersection of all cuboids is given by $(ux - lx)(uy - ly)(uz - lz)$ where lx = max(x1, x2, ..., xn) and ux = min(...).
- To get unique points

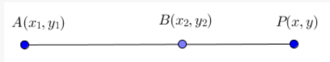
```
sort(cops.begin(), cops.end());
cops.resize (distance(cops.begin (), unique
(cops.begin(), cops.end())))
```

- Remember that we can apply bisection method ($while(abs(hi - lo) < eps$) and ternary search in geometry.
- for a quadrilateral drawn inside circle sum of opposite angles is 180 deg.
- Sum of all angles of a quadrilateral is always 360 deg.
- Center of mass of pts $= \sum m_i \vec{r_i} / \sum m_i$. This is same as centre of mass of union of mutually exclusive objects where each $\vec{r_i}$ is that objects COM and m_i is that objects mass.
- COM of a line is its midpoint
- COM of $\triangle = (\vec{r_1} + \vec{r_2} + \vec{r_3})/3$ but this is not the case for general polygon.
- For a general convex polygon, we may triangulate it, find that triangles area and COM and the combine to get COM of original figure.
- Similarly in case of 3D, COM of tetrahadron $= \sum_{i=1}^{4} r_i/4$ and a general 3d object can be again divided into tetrahedrons.
- For general polygon we can do $\vec{r_c} = (\sum_i r_{z,p_i,\vec{p_{i+1}}} * S_{z,p_i,p_{i+1}}) / \sum_i S_{z,p_i,p_{i+1}}$ where S term denotes triangles area with sign.
- **Some properties of triangles**
  - $s = p/2$
  - $A = \sqrt{s * (s - a) * (s - b) * (s - c)}$
  - $a/\sin A = b/\sin B = c/\sin C = 2 * R$
  - $R = abc/(4 * A)$
  - $c^2 = a^2 + b^2 - 2 * a * b * \cos(C)$
  - Inscribed circle (incircle), $r = A/s$
  - Center of incircle is the meeting point of angle bisectors.
  - Medians divide a triangle into 6 triangles of equal area and area of original triangle is $= 4/3 * \sqrt{s * (s - a) * (s - b) * (s - c)}$, here a, b, c is the length of medians.
  - For valid $\triangle$ sum of any 2 sides should be greater than third. If the three lengths are sorted, we can simply check whether $a + b > c$. For quadrangle sum of any 3 sides should be greater than 4th.
  - The center of circumcircle is the meeting point of $]triangle$'s perpendicular bisector.
  - Triangle angle bisector property: $|AB|/|AC| = |BD|/|DC|$ where AD is the angle bisector of angle BAC.
  - Given sides of triangle, sort them, then see 3 consecutive sides, if the area is positive (using herons formula), they form a valid triangle, mx = max (mx, area).
- Kite is a quadrilateral which has two pair of sides of same length which are adjacent to each other. The area of kits is $diagonal_1 * diagonal_2/2$. Diagonals of kite are perpendicular.
- Rhombus is a special parallelogram where every side has equal length. It is also a special case of kits where every side has equal length.
- Convex Polygon: All interior angles should be less than 180 deg. Polygon which is not Convex is Concave
- Concave polygon has critical point (point from which entire polygon is not visible).
- **Pick's Theorem**. $A = i + \frac{b}{2} - 1$, where: P is a simple polygon whose vertices are grid points, A is area of P, i is # of grid points in the interior of P, and b is # of grid points on the boundary of P.
  If h is # of holes of P ($h + 1$ simple closed curves in total), $A = i + \frac{b}{2} + h - 1$.

```
// way to get boundary points

ll getb (vector<point> &poly) {
    ll b = 0;
    int n = P.size () - 1;
    for (int i = 0; i < n ;i++) {
        int j = i + 1;
        ll ret = gcd (abs(poly[i].x - poly[j].x), abs
        (poly[i].y - poly[j].y));
        // for point to be on lattice its x and y
        coordinate has to be a multiple of gcd.
        b += ret;
    }
    return b;
}
```

- To check whether a point is on or inside a polygon that area method is best.
- Centroid of a polygon, $C_x = 1/(6*A) * \sum_{i=0}^{n-1} (x_i + x_{i+1}) * (x_i * y_{i+1} - x_{i+1} * y_i)$
  $C_y = 1/(6 * A) * \sum_{i=0}^{n-1} (y_i + y_{i+1}) * (x_i * y_{i+1} - x_{i+1} * y_i)$
  Here: $(x_n, y_n) = (x_0, y_0)$. And dont given absolute value to A.
- $$A = 1/2 \begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} = 1/2 * ((x_0 * y_1 + x_1 * y_2 + \cdots + x_{n-2} * y_{n-1}) - (x_1 * y_0 \ldots)$$
- 

  If point $P(x, y)$ lies on line segment $\overline{AB}$ (between points A and B) and satisfies $AP : PB = m : n$, then we say that P divides $\overline{AB}$ internally in the ratio $m : n$. The point of division has the coordinates
  $$P = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right).$$
- 

  If $P = (x, y)$ lies on the extention of line segment $\overline{AB}$ (not lying between points A and B) and satisfies $AP : PB = m : n$, then we say that P divides $\overline{AB}$ externally in the ratio $m : n$. The point of division is
  $$P = \left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right).$$
- Area of union of triangles
- Finding common tangents to two circles
- Minimum Bounding Rectangle, Sol: basically use atan2 and rotate wrt that point.
- Given a set of points, to determine whether a point lies inside a triangle formed by any 3 points, it is enough to check whether the given point lies inside the convex hull of given data points.

7.1 Fast application of a set of geometric operations to a set of points

- Given n points $p_i = [x_i, y_i, z_i]$, apply m transformations to each of these points. Each transformation can be a shift, a scaling or a rotation around a given axis by a given angle. There is also a "loop" operation which applies a given list of transformations k times. ("loop" operations can be nested) All the operations should be applied faster than $O(n * length)$, where length is the total no. of transformations to be applied (after unrolling "loop" operations).
  Sol. Each of the transformation can be written as a linear combination of old coordinates giving us a 4x4 matrix.
- point is represented as $[x, y, z, 1]$ and here I am multiplying operation to the right of the point (actually I should have done it as a transpose of whole thing)
- Shift x by 5, y by 7, z by 9.
  $$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 7 & 9 & 1 \end{bmatrix}$$
- Scale x by 10 other two by 5.
  $$\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
- Rotate θ deg. around x axis.
  $$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
- Now, once every transformation is described as a matrix, the sequence of transformations can be described as a product of these matrices and a "loop" of k repetitions can be described as the matrix raised to the power k (which can be calc. using binary exponentiation). Time complexity $O(n + m * \log k)$.

7.2 Klee's Algo

Given n segments on a line, each described by a pair of coordinates (a_{i1}, a_{i2}). We have to find the length of their union.
It works in $O(n \log n)$ and has been proven to be the asymptotically optimal.

```
// Returns sum of lengths covered by union of given
// segments
int segmentUnionLength(const vector <pair <int,int> > &seg) {
    int n = seg.size();
    // Create a vector to store starting and ending
    // points
    vector <pair <int, bool> > points(n * 2);
    for (int i = 0; i < n; i++)
    {
        points[i*2]     = make_pair(seg[i].first, false);
        points[i*2 + 1] = make_pair(seg[i].second, true);
    }

    // Sorting all points by point value
    sort(points.begin(), points.end());

    int result = 0; // Initialize result

    // To keep track of counts of current open segments
    // (Starting point is processed, but ending point
    // is not)
    int Counter = 0;

    // Trvaerse through all points
    for (int i=0; i<n*2; i++)
    {
        // If there are open points, then we add the
        // difference between previous and current point.
        // This is interesting as we don't check whether
        // current point is opening or closing,
        if (Counter)
            result += (points[i].first - points[i-1].first);

        // If this is an ending point, reduce, count of
        // open points.
        (points[i].second)? Counter-- : Counter++;
    }
    return result;
}
```

7.3 Closest Pair Problem

To be **revised**.

```
// First sort the points by their x coordinates. Do whatever if
there is tie.
// I have to write the correct implementation following the
idea mentioned in cormen and use it to solve codejams prob.
// Commented section shows how to solve the problem:
// Find out the maximum size such that if you draw such size
quared around each point (that point will be at the center of
the square) and no two squared will intersct each other (cna
touch but not intersect). To make the problem simple the sides
of the square will be parallel to X and Y axis.
double dvac(int low, int high) {
   if(low < high) {
       if(low == high - 1) {
           return dist(data[low], data[high]);  // return max
           (data[high].x - data[low].x, abs (data[high].y -
           data[low].y));
       }
       int mid = (low + high) / 2;
       double d1 = dvac(low, mid);
       double d2 = dvac(mid + 1, high);
       double dp = min(d1, d2);
       double d3 = 10000;
       // It is guarenteed that there can be atmost 6 points
       for(int i = mid; i >= low; i--) {
           double temp = dist(point(data[i].x, 0),
           point(data[mid].x, 0));
           if(temp > dp - EPS) break;
           for(int j = mid + 1; j <= high; j++) {
               double temp2 = dist(point(data[i].x, 0),
               point(data[j].x, 0));
               if(temp2 > dp - EPS) break;
               d3 = min(d3, dist(data[i], data[j]));
               // d3 = min (d3, max (data[j].x - data[i].x,
               abs(...));)
           }
       }
       return min(dp, d3);
   }
   return 10000;
}
```