Intro

RoadMap

TODO

Basic

Differential Equations

Definition: An equation involving derivatives of a dependent variable wrt one or more independent variables

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Examples:

1. $\frac{d^3y}{dx^3}^{101} + P(x)\frac{dy}{dx}^{202} = Q(y)$
2. $sin(\frac{dy}{dx}) = x^{10}$
3. $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2}^3$

Types of Differential Equations

- Ordinary DE: An eqn involving the derivatives of a dependent variable wrt a single independent variable as in example 1 and 2 above.
- Partial DE: An equation involving the derivatives of a dependent variable wrt more than one independent variable as in example 3 above.

Definition: The order of the highest order derivative involved in a DE is called the order of the DE. So example 1, 2 and 3 above are of order 3, 1 and 2 resp.

So an ODE of order n involving 2 variables is of the form: $f(1, x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}) = 0$

Definition: The degree (i.e. power) of the highest order derivative involved in a DE, when the DE satisfies the following:

- All derivatives have been made free from radicals (i.e. roots or fractional powers)

- There is no involvement of the derivatives in any denominator of a fraction.
- There shouldn't be involvement of highest order derivative as a transcendental function, trigonometric or exponential, etc. The coefficient of any term containing the highest order derivative should just be a function of x, y, or some lower order derivative.

So, example 1 above is of degree 101 whereas example 2 doesn't satisfy our conditions and example 3 has degree 3.

Examples:

- $c = \frac{(\sqrt{x}+(\frac{dy}{dx})^2)^{3/2}}{\frac{d^2y}{dx^2}}$ –> Order = degree = 2
- $(y^{'''})^{4/3} + sin(\frac{dy}{dx}) + xy = x$ –> Order = 3, degree = 4
- $(y^{'''})^{1/2} - 2(y^{'})^{1/4} + xy = 0$ –> Take $(y^{'})^{1/4}$ to one side and take to the 4th power on both sides and then lhs would have remaining radicals like $4a^3b + 4ab^3 = 4ab(a^2 + b^2)$ (can be seen by doing $(a + b)^4 = (a^2 + b^2 + 2ab)(a^2 + b^2 + 2ab)$) which can now be removed by squaring both sides. –> Order = 3, degree = 4
- $(y^{'''})^{4/3} + (y^{'})^{1/5} + 4 = 0$ ~~Since $GCD(3, 5) = 1$ that implies, order = 3, degree = 20 (simply take 1/5 power term to one side then raise to the 5th power then take 1/3 term common on one side and raise to the third power)~~ Tedious, yet to calculate.
- $(y^{'''})^{3/2} + (y^{'''})^{2/3} = 0$ Order = 3 but don't say degree = 9 yet as both the terms are of same order and in the end we will have $l^9 = l^4 \Rightarrow l^5 = 0$ so degree equals 5 (?) (although it is still a subjective answer and in my opinion answer should be 9).

Definition: A DE is said to be **linear** if:-

1. The dependent variable and all its derivatives occur in the first degree only.
2. No product of dependent variable or derivatives occur.

So, in general a linear differential equation involving two variables and of nth order is of the form:-

$y^{(n)} + P_1(x) * y^{(n-1)} + \cdots + P_n(x) * y = Q(x)$

Also if $Q(x) = 0$ then it is called as **Homogeneous Linear DE** o/w **Non Homogeneous Linear DE**.

Examples:

- $\frac{dy}{dx} = x + sin(x)$

- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y}$

Definition: A DE is said to be **non linear** if it is not linear.

Examples:

- $y = \sqrt{x}\frac{dy}{dx} + \frac{k}{\frac{dy}{dx}}$
- $\frac{dx}{dt}^3 + \frac{d^2x}{dt} = e^t$

Definition: (Soln to a DE) Any relation between the dependent and independent variables which when substituted in the DE reduces it to an identity is called a **soln** or **integral** or **primitive** of the DE.

Definition: (General Soln) The soln of a DE in which the number of arbitrary constants is equal to the order of the DE.

Example: $y = ce^{2x}$ is a GS of the DE $y^{'} = 2y$

Definition: (Particular Soln) A solution obtained by giving particular values to one or more of the n arbitrary constants in the general soln. So if we let $c = 1$ in the above example, we get a particular soln.

Definition: (Singular Soln)

An eqn $\Psi(x, y) = 0$ is called **singular soln** of the DE $F(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}) = 0$ if:-

1. $\Psi(x, y) = 0$ is a soln of the given DE.
2. $\Psi(x, y) = 0$ doesn't contain arbitrary constants.
3. $\Psi(x, y) = 0$ cannot be obtained by giving particular values to arbitrary constants in the general soln.

Example: $y = (x + c)^2$ is the general soln of $(dy/dx)^2 - 4y = 0$, notice that $y = 0$ is as well the soln of this eqn which cannot be obtained by any choice of c. Hence $y = 0$ is a singular soln.

Note: The complete soln to a DE of the nth order contains exactly n arbitrary constants.

Definition: (Family of plane curves) For each given set of real numbers $c_1, c_2, \dots, c_n$ the equation $\phi(x, y, c_1, \dots, c_n) = 0$ represents a curve in x-y plane.

For different sets of real values of $c_1, \dots, c_n$ the eqn $\phi(x, y, c_1, \dots, c_n) = 0$ represents infinitely many curves. The set of all these curves is called n parameter family of curves and $c_1, \dots, c_n$ are called parameters of the family.

Example: The set of circles defined by $(x - c_1)^2 + (y - c_2)^2 = c_3$ is three parameter family where $c_3 \geq 0$

Formation of DE

Working Rule

To form the DE from a given eqn in x and y, containing n arbitrary constants:

1. Write down the given eq., differentiate wrt x successively till the count reaches the number of arbitrary constants (n).
2. Eliminate the arbitrary constants from the $(n+1)$ eqn's obtained in above step.

Example: $y = ae^x + be^{-x} + c\cos x + d\sin x$ which arbitrary constants are (a, b, c, d)

Soln is $y^{(4)} = y$

Soln of DE of the first order and first degree

Definition: A DE of first order and first degree is an eqn of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ or $Mdx + Ndy = 0$ where M and N are functions of x and y.

There are four methods to solve these eqn's:-

Variables separable

If in an eqn it is possible to get all the functions of x and dx to one side and all the functions of y and dy to another side then the variables are said to be separable.

Working rule

1. Consider the eqn $\frac{dy}{dx} = XY$
2. Modify to get $\frac{dy}{Y} = Xdx$ *(Now the variables have been seperated)*
3. Integrate both sides and don't forget to add arbitrary constant (only to one side as adding to both sides will ultimately lead to only one arbitrary constant), soln without this arbitrary const is wrong as it is not the general soln.

Example: $\frac{dy}{dx} = e^{x+y} + x^2e^y = e^y(e^x + x^2) \Rightarrow dye^{-y} = dx(e^x + x^2)$ solve and get $e^{-y} + e^x + x^3/3 + c = 0$

Second Form

Equations reducible to the form in which variables can be separated.

Equations of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to the form in which the variables are separated.

Put $ax + by + c = z \Rightarrow a + b\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{b}(\frac{dz}{dx} - a) \Rightarrow \frac{1}{b}(\frac{dz}{dx} - a) = f(z)$

Example: $\frac{dy}{dx} = cos(x + y) + sin(x + y)$

Sol: $x + y = z \Rightarrow 1 + y^{'} = z^{'}$

$\Rightarrow z^{'} - 1 = cos(z) + sin(z) \Rightarrow z^{'} = 2cos^2(z/2) + 2sin(z/2)cos(z/2) = 2cos(z/2)(sin(z/2) + cos(z/2))$

$\Rightarrow \frac{dzsec^2(z/2)}{2(tan(z/2)+1)} = dx$ integrate and get $log(1 + tan((x + y)/2)) = x + c$

Third Form

Differential eqns of the form

$\frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c_1}$ or $\frac{dy}{dx} = \frac{m(ax+by)+c}{ax+by+c_1}$ just put $ax + by = z$ and solve like second form.

Homogenous DE

Homogenous function

A function $f(x, y)$ is said to be a homogenous function of degree $'n'$ in x and y if $f(kx, ky) = k^n f(x, y)$

Examples:

- $f(x,y) = \frac{\sqrt[3]{x}+\sqrt[3]{y}}{x+y} \Rightarrow f(kx,ky) = k^{1/3-1}\frac{\sqrt[3]{x}+\sqrt[3]{y}}{x+y} \Rightarrow f(x,y)$ is homogenous of degree $-2/3$
- $f(x,y) = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$ is homogenous of degree 0.

:::tip Note If $f(x, y)$ is homogenous with degree 0 then $f(x, y)$ is a function of y/x or x/y :::

Homogenous DE

A DE is said to be homogenous if it can be written in the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f and g are homogenous functions of some degree in x & y

Working Rule

1. Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$
2. Use this to separate variables

Examples:

- $\frac{dy}{dx} = \frac{y}{x} + e^{\frac{y}{x}}$ just put $y = vx$ and solve.

Non Homogenous DE or Eqns Reducible To Homo. Form

Consider DE's of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a_1x + b_1y + c_1}$$

Case 1 $\frac{a}{a_1} \neq \frac{b}{b_1}$ i.e. $ab_1 \neq ba_1$

Working rule:

Put $x = X + h, y = Y + k$ where h and k are constants

$\Rightarrow dx = dX$ & $dy = dY \Rightarrow \frac{dy}{dx} = \frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a_1(X+h)+b_1(Y+k)+c_1}$

$$= \frac{aX + bY + (ah + bk + c)}{a_1X + b_1Y + (a_1h + b_1k + c_1)}$$

Choosing h and k such that $ah+bk+c = a_1h+b_1k+c_1 = 0 \Rightarrow \frac{dY}{dX} = \frac{aX+bY}{a_1X+b_1Y}$ which is homogenous.

Case 2 $\frac{a}{a_1} = \frac{b}{b_1}$ then it reduces to a third form of variable separable.

Exact Equations

Definition: The DE $M(x, y)dx+N(x, y)dy = 0$ is called an exact DE if $Mdx+Ndy = 0$ is an exact derivative of x and y, i.e. $Mdx+Ndy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ where u is a function of x and y, from $du = 0$ we get by integrating $u(x, y) = c$.

Working Rule:

- The DE $Mdx + Ndy = 0$ is an exact DE **iff** $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ from $(u_{xy} = u_{yx})$ (Note that this will only hold true if u is continuous and its first order derivatives are as well continuous) This equality condition is not only necessary but also sufficient (proof omitted)
- To obtain GS either do
    1. $u = \int Mdx + k(y)$ and then using $u_y = N$ get an eqn in $\frac{dk}{dy}$ to solve for k.
    2. $u = \int Ndy + l(x)$ and solve like above.

:::warning Warning As evident from this theory, we cannot solve by just doing $Mdx = -Ndy$ then integrating each side wrt x, y resp. Also eqn may not even be exact! (wanted to mention this point in variable separation section) :::

:::caution Caution The GS is **not** given by $\int Mdx$ (treat y as const) + $\int($ terms in N not containing x$)dy = c$

This might work in some cases only. :::

Integrating Factor

Sometimes $Mdx + Ndy = 0$ is not exact but can be made so by multiplying throughout by a suitable non zero $\mu(x, y)$. This multiplier is called the integrating factor.

:::tip Note A DE can have more than one integrating factor like $1/x^2, 1/y^2, 1/xy, 1/(x^2+y^2), 1/(x^2 - y^2)$ are all integrating factors of $ydx - xdy = 0$ :::

:::note Remember If we have situation like this $l(x)dx+m(y)dy+d(N(x, y)) = 0$ then we can simply do $L(x) + M(y) + N(x, y) = c$ :::

Examples:

- $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$

  Sol: $\frac{\partial M}{\partial y} = h = \frac{\partial N}{\partial x}$, Answer: $ax^2/2 + hyx + gx + by^2/2 + fy = c$
- $\frac{xdy-ydx}{x^2+y^2} = xdx \Rightarrow d(tan^{-1}(y/x)) = xdx \Rightarrow tan^{-1}(y/x) = x^2/2 + c$
- $ydx-xdy+(1+x^2)dx+x^2sinydy = 0 \Rightarrow \frac{ydx-xdy}{x^2}+(1/x^2+1)dx+sinydy = 0$

  Clubbing dx and dy terms together (was of no use though), we get $-y/x+ -1/x+x+k(y) = u \Rightarrow -1/x+k^{'}y = -1/x+siny \Rightarrow k(y) = -cos(y)+c^*$

  Now using $u = c$ we get $-y/x - 1/x + x - cosy = c^{'}$
- $xdy = (y + xcos^2(y/x))dx \Rightarrow \frac{xdy-ydx}{x^2} = \frac{cos^2(y/x)}{x}dx$

  $\Rightarrow \frac{sec^2(y/x)(xdy-ydx)}{x^2} = dx/x \Rightarrow tan(y/x) = log(x) + c$
- $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$

  $\Rightarrow ydx + xdy + 2xy^2dx - x^2ydy = 0$

  $\Rightarrow \frac{ydx+xdy}{x^2y^2}$ (this is now in exact form,

  \$ can be verified thus it should be kept in mind that \$ \$ x^2y^2 is an integrating factor of ydx + xdy\$) $+2dx/x$ (wasn't able to do anything rather than making it free from y) $- dy/y = 0$

  Now getting answer is easy, which is $-1/xy + 2log(x) - log(y) = c$
- $xdx + ydy + (x^2 + y^2)dy = 0$

  $\Rightarrow \frac{xdx+ydy}{x^2+y^2} = -dy \Rightarrow log(x^2 + y^2)/2 + y = c$

- $x^2dy/dx + xy = \sqrt{1 - x^2y^2}$

  $\Rightarrow x\frac{xdy+ydx}{dx} = \sqrt{1-(xy)^2}$

  $\Rightarrow \frac{d(xy)}{\sqrt{1-(xy)^2}} = dx/x$

  $\Rightarrow sin^{-1}(xy) - log(x) = c$

Methods for finding Integrating Factor

Method 1 If $Mdx + Ndy = 0$ is homogenous and $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is an IF.

Example: $x^2ydx - (x^3 + y^3)dy = 0$, IF is $\frac{1}{x^3y-x^3y-y^4} = -1/y^4 \neq 0$ although this same problem can be solved with the homogenous theory discussed before.

Method 2 If $Mdx + Ndy = 0$ is such that $M = yf_1(x, y)$ and $N = xf_2(x, y)$ and $Mx - Ny \neq 0$ then $\frac{1}{Mx-Ny}$ is an IF.

Example: $y(xysin(xy) + cos(xy))dx + x(xysin(xy) - cos(xy))dy = 0$. IF is $1/xy(xysin(xy) + cos(xy) - xysin(xy) + cos(xy)) = 1/2xycos(xy)$

Method 3 If $Mdx + Ndy = 0$ is such that $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$. (as usual $f(x)$ can as well be a constant) then IF $= e^{\int f(x)dx}$

Example: $(y + y^3/3 + x^2/2)dx + (x + xy^2)dy/4 = 0$

$\Rightarrow f(x) = \frac{1+y^2-(1+y^2)/4}{x(1+y^2)/4} = 3/x$

$\Rightarrow IF = e^{3log(x)} = x^3$

Method 4 If $Mdx + Ndy = 0$ is such that $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$. ($f(y)$ can as well be a constant) then IF $= e^{\int f(y)dy}$

Method 5 If $Mdx + Ndy = 0$ can be put in the form of $x^{\alpha_1}y^{\beta_1}(m_1ydx + n_1xdy) + x^{\alpha_2}y^{\beta_2}(m_2ydx + n_2xdy) = 0$ where $\alpha_1, \alpha_2, \beta_1, \beta_2, m_1, m_2, n_1, n_2$ are constants then the IF is x^hy^k where h and k can be obtained by applying the condition that the given eqn must become exact after multiplying by x^hy^k.

Linear Equations

Here linear equation will of the form $\frac{dy}{dx} + P(x)y = Q(x)$.

Working Rule

1. Find IF $= e^{\int P(x)dx}$
2. GS is $y \cdot IF = \int (Q \cdot IF)dx + c$

Examples:

1. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 1 = 0$ Put $\frac{dy}{dx} = t$ we get

   $\frac{dt}{dx} + \frac{t}{x} = \frac{-1}{x}$ Now solve as mentioned.

2. $(1+y^2) = (tan^{-1}y - x)\frac{dy}{dx}$

   $\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{tan^{-1}y}{1+y^2}$ Now solve as mentioned.

Equations Reducible to Linear Form

1. $f'(y)dy/dx + P(x)f(y) = Q(x)$

   Put $f(y) = v \Rightarrow f'(y)dy/dx = dv/dx$

   $\Rightarrow dv/dx + Pv = Q$

2. $f'(x)dx/dy + P(y)f(x) = Q(y)$ Proceed like above

3. **Bernoulli's Equation:**

   $dy/dx + P(x)y = Q(x)y^n$ *Note:* $n \neq 0$ and $n \neq 1$

   (or)

   $dx/dy + P(y)x = Q(y)x^n$ with similar conditions

   Working Rule:

   $y^{-n}dy/dx + P(x)y^{1-n} = Q(x)$

   put $y^{1-n} = z$

   $\Rightarrow (1-n)y^{-n}dy/dx = dz/dx$

   $\Rightarrow dz/dx + (1-n)P(x)z = (1-n)Q(x)$ Which is linear.

   Examples:

   - $dz/dx + zlog(z)/x = z(log(z))^2/x^2$

     Put $log(z) = t \Rightarrow z = e^t$

     $\Rightarrow e^t dt/dx + e^t t/x = e^t t^2/x^2 \Rightarrow dt/dx + t/x = t^2/x^2$

     Now solve. Ans: $1/(xlog(z)) = 1/(2x^2) + c$

   - $\frac{dy}{dx}(x^2y^3 + xy) = 1$

     $\Rightarrow dx/dy = x^2y^3 + xy$

$\Rightarrow dx/dy - xy = x^2y^3$ Now solve.

- $dy/dx + 1/x = e^y/x^2$

  $\Rightarrow e^{-y}dy/dx + e^{-y}/x = 1/x^2$ Put $e^{-y} = t$ and solve.

  Ans: $e^{-y}/x = 1/2x^2 + c$

Linear Equations With Const. Coeff.

Definition: Linear DE of order n of the form: $d^n y/dx^n + a_1 d^{n-1}y/dx^{n-1} + \cdots + a_{n-1}dy/dx + a_n y = Q(x)$ *(or)* X

This eqn can be written as $f(D)y = Q \ldots (1)$ where

$f(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n$

Note that these coefficients can be complex (imaginary).

Homogeneous Eqn

If $Q = 0$ then the eqn obtained is called **Homogeneous Eqn** with constant coeff.

Solving Linear Eqn

Solving this linear eqn is divided into two parts,

1. First we find the general solution of the corresponding Homogeneous eqn which is called as Complementary Function (CF) (*Note:* It must contain as many arbitrary constants as the order of the eqn)
2. Next find a particular soln of original eqn which does not contain any arbitrary constant. This is called the Particular Integral (PI).
3. GS = CF + PI.

Auxiliary Eqn

$f(m) = 0$ is called auxiliary eqn of (1). Clearly it will must have n roots (which can be complex).

Find CF

Consider AE $f(m) = 0$

Case 1

All roots are real and distinct.

Let $m_1, m_2, \dots, m_n$ be those n roots.

Then $y = e^{m_1 x}, y = e^{m_2 x}, \dots, y = e^{m_n x}$ are independent solns of homogeneous eqn.

Hence the GS is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$ where $c_1, c_2, \dots, c_n$ are our n constants.

Example: $(D - m_1)y = 0$

$\Rightarrow m - m_1 = 0, i.e., m = m_1$

$\Rightarrow GS = ce^{m_1 x}$

Which was evident as

$dy/dx = m_1 y \Rightarrow dy/y = m_1 x$

$\Rightarrow log(y) = m_1 x + c$

Case 2

Same as Case 1 but now two roots are equal.

Let $m_1, m_1, m_3, \dots, m_n$ be those n roots, then GS is

$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$

Example: $(D - m_1)^2 y = 0$

$\Rightarrow GS = (c_1 + c_2 x)e^{m_1 x}$

Case 3

Same as Case 2 but now three roots are equal.

GS is $y = (c_1 + c_2 x + c_3 x^2)e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$

Similar is the treatment for further equality of roots.

Case 4

Same as Case 3 but all roots are equal, then GS is

$y = (c_1 + c_2x + c_3x^2 + \cdots + c_nx^{n-1})e^{m_1x}$

Case 5

If in Case 1 we have $\alpha \pm i\beta$ as a pair of complex roots then GS is

$y = e^{\alpha x}[Acos(\beta x) + Bsin(\beta x)]$ (Ignoring other terms for now)

Here A and B can be complex. This is simplified form, putting soln in form of Case 1 is as well valid.

Derivation:

$y = c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x} + c_3e^{m_3x} + ...$

$= e^{\alpha x}(c_1e^{i\beta x} + c_2e^{-i\beta x}) + ...$

$= e^{\alpha x}(c_1(cos(\beta x) + isin(\beta x)) + c_2(cos(\beta x) + -sin(\beta x)) + ...$

$= e^{\alpha x}(Acos(\beta x) + Bsin(\beta x)) + ...$

Where $A = c_1 + c_2$ and $B = i(c_1 - c_2)$

If the imaginary roots are repeated, say, $\alpha \pm i\beta$ occur twice then the soln would be

$y = e^{\alpha x}[(A + Bx)cos(\beta x) + (C + Dx)sin(\beta x)]$

:::tip Note

1. Expression $e^{\alpha x}[Acos(\beta x) + Bsin(\beta x)]$ can as well be written as

   $Ae^{\alpha x}cos(\beta x + B)$

2. If AE has $\alpha \pm \sqrt{\beta}$ as a pair of roots, then GS is

   $y = e^{\alpha x}[Acosh(\sqrt{\beta}x) + Bsinh(\sqrt{\beta}x)]$

   which may be written as $y = Ae^{\alpha x}cosh(\sqrt{\beta}x + B)$

   If these roots are repeated then the GS is

   $y = e^{\alpha x}[(A + Bx)cosh(\sqrt{\beta}x) + (C + Dx)sinh(\sqrt{\beta}x)]$

:::

Find PI

Inverse Operator

$\frac{1}{f(D)}X$ is that function of x, not containing arbitrary constants which when operated upon by $f(D)$ gives X

i.e. $f(D)(\frac{1}{f(D)}X) = X$

Thus $\frac{1}{f(D)}X$ is a PI.

Note: Say $X = c_1 g_1(x) + c_2 g_2(x)$ where c_1, c_2 are constants. Then $PI = c_1 \frac{g_1(x)}{f(D)} + c_2 \frac{g_2(x)}{f(D)}$ and each of these can be solved independently of the other as if other doesn't exist.

$\frac{1}{D}X = \int X dx$

Proof: Let $\frac{1}{D}X = y$

$\Rightarrow Dy = X \Rightarrow dy/dx = X$

$\Rightarrow y = \int X dx$ no constant being added because y doesn't contain any constant.

Corollary: $\frac{1}{D^2}X = \int [\int X dx] dx$

$\frac{1}{D-a}X = e^{ax} \int X e^{-ax} dx$

Proof: Let $\frac{1}{D-a}X = y$

$\Rightarrow (D-a)y = X$

$\Rightarrow dy/dx - ay = X$

$\Rightarrow y = e^{ax} \int X e^{-ax} dx$

Case 1

To find PI when $Q = e^{ax}$ and $f(a) \neq 0$

Since $D^k e^{ax} = a^k e^{ax}$ therefore PI is $y = e^{ax}/f(a)$

:::note If $f(a) = 0$ Since a is the root of AE, therefore $(D-a)$ is a factor of $f(D)$. Suppose $f(D) = (D-a)\phi(D)$, where $\phi(a) \neq 0$. Then

$\frac{1}{f(D)}e^{ax} = \frac{1}{D-a}\frac{1}{\phi(D)}e^{ax} = \frac{1}{\phi(a)}\frac{1}{D-a}e^{ax}$

$= \frac{1}{\phi(a)}e^{ax} \int e^{ax} e^{-ax} dx$

$= xe^{ax}/phi(a) = xe^{ax}/f'(a)$

$\because f'(D) = (D-a)\phi'(D) + 1 \cdot \phi(D)$

$\therefore f'(a) = \phi(a)$

If $f'(a) = 0$ then applying this procedure again we get $\frac{1}{f(D)}e^{ax} = x^2e^{ax}/f''(a)$

Instead of taking $f''(a)$ one can take $2\phi(a)$ as:-

$f(D) = (D-a)^2\phi(D)$

$\Rightarrow f'(D) = 2(D-a)\phi(D) + (D-a)^2\phi'(D)$

$\Rightarrow f''(a) = 2\phi(a)$

Similarly $f^{(k)}(a) = k!\,\phi(a)$ :::

Examples:

- $(D+2)(D-1)^2y = e^{-2x} + 2sinhx$

  $ = e^{-2x} + e^x - e^{-x}$

  $\Rightarrow y = xe^{-2x}/f'(-2) + x^2e^x/f''(1) - e^{-x}/f(-1)$

  $= xe^{-2x}/9 + x^2e^x/6 - e^{-x}/4$

Case 2

$X = sin(ax+b)$ or $cos(ax+b)$

$\Rightarrow (D^2)^r sin(ax+b) = (-a^2)^r sin(ax+b)$

$\therefore f(D^2)sin(ax+b) = f(-a^2)sin(ax+b)$

Operating on both sides by $\frac{1}{f(D^2)}$ and dividing both sides by $f(-a^2)$ we get

$\frac{1}{f(D^2)}sin(ax+b) = \frac{1}{f(-a^2)}sin(ax+b)$ provided $f(-a^2) \neq 0$

If $f(-a^2) = 0$ [Note that if $f(a) = 0$ then it doesn't mean $f(-a^2) = 0$ thus be careful when checking this] then we proceed further.

$\frac{1}{D^2}sin(ax+b) =$ I.P. of $\frac{1}{D^2}e^{i(ax+b)}$

$=$ IP of $\frac{x}{f'(D^2)}e^{i(ax+b)}$

$= x\frac{1}{f'(-a^2)}sin(ax+b)$

If $f'(-a^2) = 0$, $\frac{1}{D^2}sin(ax+b) = x^2\frac{1}{f''(-a^2)}sin(ax+b)$ provided $f''(-a^2) \neq 0$ and so on.

Similarly,

- $\frac{1}{f(D^2)}cos(ax+b) = \frac{1}{f(-a^2)}cos(ax+b)$ provided $f(-a^2) \neq 0$

- If $f(-a^2) = 0$ then, $\frac{1}{f(D^2)}cos(ax+b) = x\frac{1}{f'(-a^2)}cos(ax+b)$ provided $f'(-a^2) \neq 0$
- If $f'(-a^2) = 0$ then, $\frac{1}{f(D^2)}cos(ax+b) = x^2\frac{1}{f''(-a^2)}cos(ax+b)$ provided $f''(-a^2) \neq 0$

Examples:

- $(D^3+1)y = cos(2x-1)$

  PI = $\frac{1}{D^3+1}cos(2x-1)$

  $= \frac{1}{-4D+1}cos(2x-1)$

  $= \frac{1+4D}{(1-4D)(1+4D)}cos(2x-1)$

  $(1+4D)\frac{1}{1-16(-4)}cos(2x-1)$

  $= \frac{1}{65}[cos(2x-1)+4Dcos(2x-1)]$

  $= \frac{1}{65}[cos(2x-1)+-8sin(2x-1)]$

- $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = sin(2x)$

  PI = $\frac{1}{D(D^2+4)}sin(2x)$

  $= x\frac{1}{3D^2+4}sin(2x)$

  Note: We couldn't have just did $D(2D)$... as derivative should be applied to whole.

  $= x\frac{1}{3(-4)+4}sin(2x)$

  $= -\frac{x}{8}sin(2x)$

Case 3

$X = x^m$

PI = $\frac{1}{f(D)}x^m = [f(D)]^{-1}x^m$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since $(m+1)th$ and higher derivatives of x^m are zero, we need not consider terms beyond D^m

Examples:

- $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2+2x+4$

  PI = $\frac{1}{D(D+1)}(x^2+2x+4)$

  $= \frac{1}{D}(1-D+D^2+...)(x^2+2x+4)$

  $= \frac{1}{D}(x^2+2x+4-(2x+2)+2)$

$= \int (x^2+4)dx = x^3/3 + 4x$

- $\frac{1}{(D-2)^2}x^2 = \frac{1}{4}(1-D/2)^{-2}x^2$

  $= \frac{1}{4}(1 + D + \frac{(-2)(-3)}{2!}(-D/2)^2 + ...)x^2$

  $= \frac{1}{4}(x^2 + 2x + 3/2)$

- $\frac{1}{D^2-2D+2}(x)$

  $= \frac{1}{(D-(1+i))(D-(1-i))}$

  $= \frac{1}{2(1-(D/(1+i))(1-(D/(1-i))))}x$

  $= \frac{1}{2(1-D/(1+i))}(1 + D/(1-i))x$

  $= \frac{1}{2}(x + 1/(1+i))(1 + D/(1-i))$

  $= \frac{1}{2}(x + 1/(1-i) + 1/(1+i))$

  $= (x+1)/2$

  Aliter

  $= \frac{1}{2}(1 - (D - D^2/2))^{-1}x$

  $= \frac{1}{2}(1 + D - D^2/2)x$

:::caution Caution In $\frac{x}{(D+1)^2-4} = -\frac{1}{4}\frac{x}{1-(\frac{D+1}{2})^2}$

Is not equal to $-\frac{1}{4}(1 + (\frac{D+1}{2})^2)x$

Solve it the way done in aliter above, i.e. in $(1+x)^n$, x should only contain $D's$ (not sure why $D's$ would be valid). :::

Case 4

$X = e^{ax}V(x)$

It can be proven

$\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V$

Examples:

- $(D^2 - 2D + 4)y = e^x cosx$

  PI $= e^x \frac{cosx}{f(D+1)}$

  $= e^x \frac{cosx}{D^2+3}$

  $= e^x \frac{cosx}{2}$

Case 5

When X is any other function of x.

PI = $\frac{1}{f(D)}X$

If $f(D) = (D - m_1)(D - m_2) \dots (D - m_n)$

$\Rightarrow 1/f(D) = \frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \dots + \frac{A_n}{D-m_n}$

$\therefore PI = A_1 \frac{1}{D-m_1} X + A_2 \frac{1}{D-m_2} X + \dots + A_2 \frac{1}{D-m_n} X$

$= A_1 e^{m_1 x} \int X e^{-m_1 x} dx + A_2 e^{m_2 x} \int X e^{-m_2 x} dx + \dots + A_n e^{m_n x} \int X e^{-m_n x} dx$

Examples:

- $(D^2 + D + 1)y = (1 - e^x)^2 = 1 + e^{2x} - 2e^x$

  $\Rightarrow y = CF + e^{0x}/(0 + 0 + 1) + e^{2x}/(4 + 2 + 1) - 2e^x/3$

  $y = CF + 1 + e^{2x}/7 - 2e^x/3$

- $(D^4 + 2D^2 + 1)y = x^2 cos(x)$

  $AE = (m^2 + 1)^2$

  $\Rightarrow m = \pm i, \pm i$

  $\Rightarrow CF = (c_1 + c_2 x)cosx + (c_3 + c_4 x)sinx$

  $PI = Re[e^{ix} \frac{x^2}{((D+i)^2+1)^2}]$

  $= Re[e^{ix}(-\frac{1}{4D^2}(1 - iD/2)^{-2} x^2)]$ $(Since\ \frac{1}{i} = -i)$

  $= Re[-\frac{e^{ix}}{4} \frac{1}{D^2}(1 + 2iD/2 + 3(iD/2)^2)x^2]$

  $= Re[-\frac{e^{ix}}{4} \frac{1}{D^2}(x^2 + 2ix - 3/2)]$

  $= Re[-\frac{e^{ix}}{4} \frac{1}{D}(x^3/3 + ix^2 - 3x/2)]$

  $= Re[-\frac{e^{ix}}{4}(x^4/12 + ix^3/3 - 3x^2/4)]$

  $= \frac{-1}{48} Re[(cosx + isinx)(x^4 + 4ix^3 - 9x^2)]$

  $= \frac{-1}{48}[(x^4 - 9x^2)cosx - 4x^3 sinx]$

- $(D^2 - 4D + 4)y = 8x^2 e^{2x} sin(2x)$

  $y = CF + IM[8e^{2x(1+i)} \frac{x^2}{(D+2i)^2}]$

  Now after a bit long calculation, you will arrive at answer. The better way is to always factorize $f(D)$ first so that we can see some structure. For example, in this case $f(D) = (D - 2)^2$, thus, $y = CF + 8e^{2x} \frac{x^2 sin(2x)}{D^2}$ which is comparatively easy to compute.

- $(D^2 + a^2)y = sec(ax)$

  $\rightarrow D = \pm ai$

  $CF = c_1 cosax + c_2 sinax$

  $PI = \frac{1}{D^2+a^2} secax = \frac{1}{2ia}[\frac{1}{D-ia} - \frac{1}{D+ia}]secax$

  Now $\frac{1}{D-ia} secax = e^{iax} \int secaxe^{-iax} dx$

  $= e^{iax} \int \frac{cosax - isinax}{cosax} dx = e^{iax} \int (1 - itanax)dx$

  $= e^{iax}(x + \frac{i}{a} log(cosax))$

  Changing i to $-i$ we get

  $\frac{1}{D+ia} secax = e^{-iax}(x - \frac{i}{a} log(cosax))$

  $PI = \frac{1}{2ia}[e^{iax}(x + \frac{i}{a} log(cosax)) - e^{-iax}(x - \frac{i}{a} log(cosax))]$

Cauchy Euler Equation

Is an eqn of the form:

$(x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots + a_{n-1} x^1 D + a_n)y = X$

where a_i are constants.

Method Of Solution

Reduce this linear equation into linear equation with constant coefficients.

Put $x = e^z \rightarrow z = log(x)(x > 0)$

$\rightarrow \frac{dz}{dx} = 1/x$

$dy/dx = dy/dx \cdot dz/dx = \frac{dy}{dz} \cdot \frac{1}{x}$

$x\frac{dy}{dx} = D_z y$ where $D_z = \frac{d}{dz}$

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}[\frac{1}{x}\frac{dy}{dz}]$

$= \frac{1}{x}\frac{d}{dx}(\frac{dy}{dz}) + \frac{dy}{dz}(\frac{d}{dx}(1/x))$

$= \frac{1}{x}\frac{d}{dz}(\frac{dy}{dx}) + \frac{dy}{dz}(-1/x^2)$

$= \frac{1}{x}\frac{d}{dz}(\frac{1}{x}\frac{dy}{dz}) + \frac{dy}{dz}(\frac{-1}{x^2})$

$= \frac{1}{x^2}(D_z^2 - D_z)y$

$= \frac{1}{x^2} D_z(D_z - 1)y$

Similarly, $x^n \frac{d^n y}{dx^n} = D_z(D_z - 1)(D_z - 2) \ldots (D_z - (n-1))y$

Thus using this we can reduce our eqn to linear eqn with constant coefficients for y in terms of z. If $y = F(z)$ is its soln then putting $z = logx$ we get the required soln $y = F(logx)$

Examples:

- $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = x + sinx$

  $\rightarrow (D(D-1) + 4D + 2)y = e^z + sin(e^z)$

  $\rightarrow PI = e^z/(D^2 + 3D + 2) + sin(e^z)/(D^2 + 3D + 2)$

  $\rightarrow PI = e^z/6 + \frac{sin(e^z)}{D+1} - \frac{sin(e^z)}{D+2})$ Solving we get $y = c_1 e^{-2z} + c_2 e^{-z} + e^z/6 - e^{-2z} sin(e^z)$

  $\rightarrow y = c_1 x^{-2} + c_2 x^{-1} + x/6 - x^{-2} sin(x)$

Legendre's Linear Equations

Is an eqn of the form

$(ax+b)^n \frac{d^ny}{dx^n} + k_1(ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + k_n y = X$

where a, b, k_i are constants.

Method Of Soln

Put $ax + b = e^z i.e., z = log(ax + b)$

Now by similar procedure, we get:

$(ax+b)^n \frac{d^ny}{dx^n} = a^3 D_z(D_z - 1)(D_z - 2) \ldots (D_z - (n-1))$

Examples:

- $[(3x+2)^2 D^2 + 3(3x+2)D - 36]y = 3x^2 + 4x + 1$

  $[9D(D-1) + 9D - 36]y = 3((e^z - 2)/3)^2 + 4(e^z - 2)/3 + 1$

  Now easily solve and get the answer.

Linear Equations Of Second Order With Variable Coeff.

Is an eqn of the form

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$

can be solved by following methods:

Change of the dependent variable when a part of the CF is known

Method for solving

Method for Finding one integral (soln) in CF by inspection i.e. one soln $u(x)$ of $(D^2 + P(x)D + Q(x))y = 0$

Condition Satisfied	one soln of CF
$a^2 + aP + Q = 0$	$u = e^{ax}$
$1 + P + Q = 0$	$u = e^{x}$
$1 - P + Q = 0$	$u = e^{-x}$
$m(m-1) + Pmx + Qx^2$	$u = x^m (m \geq 2)$
$P + Qx = 0$	$u = x$
$2 + 2Px + Qx^2$	$u = x^2$

Now assume the GS of given eqn is of the form $y = uv$ where u is obtained as above, now v can be obtained by solving:

$\frac{d^2v}{dx^2} + (P + \frac{2}{u}\frac{du}{dx})\frac{dv}{dx} = \frac{R(x)}{u}$

Examples:

- $xy'' - (2x-1)y' + (x-1)y = 0$

  $\rightarrow y'' - (2 - 1/x)y' + (1 - 1/x)y = 0$

  $\rightarrow u = e^x$

$\rightarrow \frac{d^2v}{dx^2} + (-2 + 1/x + 2e^{-x}e^x)\frac{dv}{dx} = 0$

$\rightarrow \frac{dt}{dx} + t/x = 0$

$\rightarrow log(t) = -log(x) + c$

$\rightarrow tx = c_1$

$\rightarrow v = c_1 log(x) + c_2$

:::warning Warning Here from "part of soln" means that $u(x)$ is a soln of the corresponding homogeneous eqn. Thus if in general we are given $y = u(x)v(x)$ where $u(x)$ is given, we **cannot** apply this method unless corresponding homogeneous eqn turns out to be zero when substituting $y = u(x)$ in it. :::

Changing the dependent variable and removal of the first order derivative

i.e. Reduce $y'' + P(x)y' + Q(x)y = R(x)$ to the form $\frac{d^2v}{dx^2} + Iv = S$ which is called as the **normal form** of the given eqn.

Method for solving

1. Write the given eqn in the standard form $y'' + P(x)y' + Q(x)y = R(x)$
2. To remove the first order derivative we choose $u = e^{\frac{-1}{2}\int P dx}$
3. Assume the GS is $y = uv$, where v is given by the normal form $\frac{d^2v}{dx^2} + Iv = S$ where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$ and $S = \frac{R}{u}$

Examples:

- $y'' - 4xy' + (4x^2 - 1)y = -3e^{x^2} sin(2x)$

  $\Rightarrow u = e^{x^2}$

  $\Rightarrow I = 4x^2 - 1 - \frac{1}{4}16x^2 - \frac{1}{2}(-4) = 1$

  $\Rightarrow S = -3sin(2x)$

  $\Rightarrow \frac{d^2v}{dx^2} + v = -3sin(2x)$

  $\Rightarrow PI = -3sin(2x)/(D^2 + 1) = sin(2x)$

  $\Rightarrow CF = c_1 cos(x) + c_2 sin(x)$

  $\Rightarrow y = e^{x^2}(CF + PI)$

- Make use of the transformation $y(x) = v(x)sec(x)$ to obtain the soln of $y'' - 2tanxy' + 5y = 0$, where $y(0) = 0, y'(0) = \sqrt{6}$

  Here note that $e^{\frac{-1}{2}\int -2tanxdx} = e^{-log(cosx)} = secx$ which is our given u. Thus we can apply our method.

Soln by changing independent variable

Let $z = f(x)$ then after a bit of work,

$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$ where:

$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{(\frac{dz}{dx})^2}$

$Q_1 = \frac{Q}{(\frac{dz}{dx})^2}$

$R_1 = \frac{R}{(\frac{dz}{dx})^2}$

Case 1

Choose z to make $P_1 = 0$ i.e., $\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0$

$\rightarrow z = \int e^{-\int Pdx}dx$

Now the eqn reduces to $\frac{d^2y}{dz^2} + Q_1y = R_1$

which can be easily solved if Q_1 turns out to be a constant or a constant multiplied by $\frac{1}{z^2}$

Case 2

Choose z such that $Q_1 = a^2$

$\rightarrow a\int dz = \int \sqrt{\pm Q}dx$

Take appropriate sign to make expression under radical positive.

Now the eqn reduces to $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + a^2y = R_1$

which can be easily solved provided P_1 comes out to be a constant.

Examples:

- $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3sin(x^2)$

  $\Rightarrow \frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2sin(x^2)$

  $\Rightarrow z = \int e^{-\int Pdx}dx = x^2/2$

  $\Rightarrow Q_1 = -4, R_1 = 8sin(x^2) = 8sin(2z)$

  $\Rightarrow PI = \frac{8sin(2z)}{D^2-4} = -sin(2z) = -sin(x^2)$

  $\Rightarrow y = c_1e^{x^2} + c_2e^{-x^2} - sin(x^2)$

- Transform the DE $xy'' - y' + 4x^3y = x^5$ into z as independent variable where $z = x^2$ and solve it.

  $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = 2x\frac{dy}{dz}$

  $\frac{d^2y}{dx^2} = 2\frac{dy}{dz} + 2x\frac{d^2y}{dz^2}\frac{dz}{dx}$

  $= 2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2}$

  Now using this, it will reduce in a good solvable form.

Method of variation of parameters

1. Write the given equation in the standard form $y'' + Py' + Qy = R$.
2. Find the soln of corresponding homogeneous eqn. Let it be $y_c = c_1u(x) + c_2v(x)$ by using methods discussed before.
3. Let the PI of the given eqn be $y_p = A(x)u + B(x)v$ where $A = -\int \frac{vR}{W(u,v)}dx$ and $B = \int \frac{uR}{W(u,v)}dx$ are functions of x.
4. GS of the given eqn is $y = y_c + y_p$

Examples:

- $((x-1)D^2 - xD + 1)y = (x-1)^2$

  $\rightarrow (D^2 - \frac{x}{x-1}D + \frac{1}{x-1})y = x - 1$

  It can be seen by inspection that $y = e^x$ and $y = x$ are soln of corresponding homogeneous eqn. Therefore

  $\rightarrow y_c = c_1e^x + c_2x$

  And after some calculation

  $y_p = -(1 + x + x^2)$

  $GS = y_c + y_p$

Soln of DE of the first order but not of first degree

Is of the form (for degree $= n$) $P^n + A_1 P^{n-1} + A_2 P^{n-2} + \cdots + A_{n-1} P + A_n y = 0$ where A_i are functions of x and $P = \frac{dy}{dx}$

Solvable for p

Such equations can be resolved into linear factors of first degree.

i.e., it reduces to $(P - f_1(x,y))(P - f_2(x,y)) \ldots (P - f_n(x,y) = 0$

i.e., $P = f_i(x,y)$, these equations on integration gives $F_i(x,y,c_i) \rightarrow y = F_1(x,y,c_1) \ldots F_n(x,y,c_n)$

But our DE is of first order and thus should have only one arbitrary constant, therefore let $c_i = c$

Examples:

- $(P - xy)(P - x^2)(P - y^2) = 0$

  $(log(y) - x^2/2 - c)(y - x^3/3 - c)(-1/y - x - c) = 0$

Solvable for x

i.e., it can be put in the form

$$x = f(y, P) \tag{1}$$

Differentiating wrt y we get

$$1/P = F(y, P, \frac{dP}{dy}) \tag{2}$$

Let the soln of (2) be

$$\phi(y, P, c) = 0 \tag{3}$$

Eliminating P between (1) and (3) gives the soln for the given eqn.

If it is not possible to eliminate P then the values of x and y in terms of P i.e. $x = f_1(P, c)$ & $y = f_2(P, c)$ together constitute the soln.

Examples:

- $xP^3 = a + bP$

  $\rightarrow x = a/P^3 + b/P^2$

  $\rightarrow 1/P = (-3a/P^4 - 2b/P^3)\frac{dP}{dy}$

  $\rightarrow dy = (-3a/P^3 - 2b/P^2)dP$

  $\rightarrow y = 3a/2P^2 + 2b/P + c$

  Here it is not possible to eliminate P from 1 and 3. GS therefore is $x = a/P^3 + b/P^2, y = 3a/2P^2 + 2b/P + c$

- $P^3 - 4xyP + 8y^2 = 0$

  $\rightarrow x = P^2/4y + 2y/P$

  $\rightarrow 1/P = 2P/4y\frac{dP}{dy} - P^2/4y^2 + 2/P - 2y/P^2\frac{dP}{dy}$

  $\rightarrow (\frac{P}{2y} - \frac{2y}{P^2})(\frac{dP}{dy} - \frac{P}{2y}) = 0$

  Omitting the first factor which leads to a singular soln, we get

  $\frac{dP}{dy} - \frac{P}{2y} = 0$

  $\rightarrow p = y^{1/2}c$

  Eliminating p from 1 and 3 we get

  $x = \frac{y}{4yc^2} + \frac{2y}{y^{1/2}c}$

  $\rightarrow x = \frac{1}{4c^2} + \frac{2y^{1/2}}{c}$

:::tip Note The factor which does not involve a derivative of P wrt x or y will be omitted as it will always lead to a singular soln :::

Solvable for y

i.e. of the form $y = f(x, P)$

differentiating both sides wrt x, we get $p = F(x, P, \frac{dP}{dx})$

proceed similar to Solvable for x.

Example:

- $y = 3x + logP$

  $P = 3 + \frac{1}{P}\frac{dP}{dx}$

  $\frac{1}{3}[\frac{1}{P-3} - \frac{1}{P}]dP = dx$

$\frac{P-3}{P} = Ae^{3x}$

$P = \frac{3}{1-Ae^{3x}}$

Eliminate P between 1 and 3 to get

$y = 3x + log(\frac{3}{1-Ae^{3x}})$

Clairaut's Eqn

Is of the form $y = xP + f(P)$. It is solved as in Solvable for y

Differentiating wrt x we get, $P = P + x\frac{dP}{dx} + f'(P)\frac{dP}{dx}$

$\rightarrow (x - f'(P))\frac{dP}{dx} = 0$

$\rightarrow \frac{dP}{dx} = 0$ ($x + f'(P)$ is discarded)

$\rightarrow P = c$

Therefore the GS is $y = xc + f(c)$

Examples:

- $P = log(Px - y)$

  $\rightarrow y = xc + e^c$

- $sin(y - Px) = P$

  $y = Pc + sin^{-1}(c)$

Equations reducible to Clairaut's form

Form 1 $y^2 = pxy + f(\frac{py}{x}$

Intuition tells that we must do $f(\frac{py}{x}) = f(P)$

Let $X = x^2, Y = y^2 \rightarrow dX = 2xdx, dY = 2ydy$

$\rightarrow dY/dX = P = \frac{y}{x}p \rightarrow p = \frac{x}{y}P$

$\rightarrow Y = XP + f(P)$

$\rightarrow y^2 = x^2c + f(c)$

Examples:

- $x^2(y - px) = yp^2$

  $\rightarrow x^2y^2 - x^3py = y^2p^2$

  $\rightarrow y^2 = pxy + (\frac{py}{x})^2$