Marginal and conditional distributions

Let a random vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is partitioned, $x_1 \in R^p$

$x \in R^n$, $x_1 \in R^p$, $x_2 \in R^q$ $\qquad$ $x_2 \in R^q$

$p+q = n$ $\qquad$ $\therefore p+q = n$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix}$$

$\Rightarrow x_1, x_2$ are sub-vectors of dimension 'p' and q resp.

Joint dist $= f(x) = f(x_1, x_2)$

$$f(x_1, x_2) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

Now for the partitioned case,

$$f(x_1, x_2) = \frac{1}{(2\pi)^{P/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2} Q(x_1, x_2)\right)$$

where, $$Q(x_1, x_2) = \begin{bmatrix} (x_1 - \mu_1)^T & (x_2 - \mu_2)^T \end{bmatrix} \begin{bmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{bmatrix} \begin{bmatrix} (x_1 - \mu_1) \\ (x_2 - \mu_2) \end{bmatrix}$$

$$I_n = \Sigma \Sigma^{-1}$$

$$= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} \begin{bmatrix} \Sigma^{11} & \Sigma^{12} \\ (\Sigma^{12})^T & \Sigma^{22} \end{bmatrix}$$

or $$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{pmatrix} \begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ (\Sigma^{12})^T & \Sigma^{22} \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$$

or, $$\Sigma_{11}\Sigma^{11} + \Sigma_{12} \cdot (\Sigma^{12})^T = I$$

or, $$\boxed{\Sigma^{11} = \Sigma_{11}^{-1} - \Sigma_{11}^{-1} \Sigma_{12} (\Sigma^{12})^T} \quad - (1)$$

$$\boxed{\Sigma^{12} = - \Sigma_{11}^{-1} \Sigma_{12} \Sigma^{22}} \quad - (2)$$

$$\Sigma_{12}^T \Sigma^{11} + \Sigma_{22} (\Sigma^{12})^T = 0$$

or, $$\boxed{(\Sigma^{12})^T = - \Sigma_{22}^{-1} \Sigma_{12}^T \Sigma^{11}} \quad -$$

Finally $$\Sigma_{12}^T \Sigma^{12} + \Sigma_{22} \cdot \Sigma^{22} = I$$

or, $$\boxed{\Sigma^{22} = \Sigma_{22}^{-1} - \Sigma_{22}^{-1} \Sigma_{12}^T \Sigma^{12}}$$

Now solving for Σ^{11}, putting value of $(\Sigma^{12})^T$ from eq 3 to eq 2

$$\Sigma^{11} = \Sigma_{11}^{-1} + \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T\Sigma^{11}$$

or, $$\left(I - \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T\right)\Sigma^{11} = \Sigma_{11}^{-1}$$

or, $$\left(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T\right)\Sigma^{11} = I$$

or, $$\boxed{\Sigma^{11} = \left(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T\right)^{-1}} \quad - 5$$

Similarly putting value of Σ^{12} from eq 1 in eq 4

$$\Sigma^{22} = \Sigma_{22}^{-1} + \Sigma_{22}^{-1}\Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}\Sigma^{22}$$

or, $$\left(I - \Sigma_{22}^{-1}\Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}\right)\Sigma^{22} = \Sigma_{22}^{-1}$$

or, $$\left(\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}\right)\Sigma^{22} = I$$

or, $$\boxed{\Sigma^{22} = \left(\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}\right)^{-1}} \quad - 6$$

$$\boxed{\Sigma^{12} = -\Sigma_{11}^{-1}\Sigma_{12}\left(\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}\right)^{-1}} \quad -$$

..nally

$$\boxed{(\Sigma^{12})^T = -\Sigma_{22}^{-1}\Sigma_{12}^T\left(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T\right)^{-1}}$$

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Now,

$$Q(x_1, x_2) = \begin{pmatrix} (x_1-\mu_1)^T & (x_2-\mu_2)^T \end{pmatrix} \begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ (\Sigma^{12})^T & \Sigma^{22} \end{pmatrix} \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}$$

or, $\cancel{Q(x_1, x_2) = \begin{pmatrix} (x_1-\mu_1)^T & (x_2-\mu_2)^T \end{pmatrix} \begin{pmatrix} (x_1-\mu_1)\Sigma^{11} + (x_2-\mu_2)\Sigma^{12} \\ x_1- \end{pmatrix}}$

or $$Q(x_1, x_2) = \begin{pmatrix} (x_1-\mu_1)^T, & (x_2-\mu_2)^T \end{pmatrix} \begin{pmatrix} \Sigma^{11}(x_1-\mu_1) + \Sigma^{12}(x_2-\mu_2) \\ (\Sigma^{12})^T(x_1-\mu_1) + \Sigma^{22}(x_2-\mu_2) \end{pmatrix}$$

or $$Q(x_1, x_2) = (x_1-\mu_1)^T \Sigma^{11} (x_1-\mu_1) + (x_1-\mu_1)^T \Sigma^{12} (x_2-\mu_2) + (x_2-\mu_2)^T (\Sigma^{12})^T (x_1-\mu_1) + (x_2-\mu_2)^T \Sigma^{22} (x_2-\mu_2)$$

$$\boxed{= (x_1-\mu_1)^T \Sigma^{11} (x_1-\mu_1) + 2(x_1-\mu_1)^T \Sigma^{12} (x_2-\mu_2) + (x_2-\mu_2)^T \Sigma^{22} (x_2-\mu_2)} \rightarrow (10)$$

Now substituting the values of Σ^{11}, Σ^{22} and Σ^{12} in eq ⓐ

$$(x_1-\mu_1)^T\left[\left(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T\right)^{-1}\right](x_1-\mu_1) \cdots$$

Now, $(A+CBD)^{-1} = A^{-1} - A^{-1}C\left(B^{-1}+DA^{-1}C\right)^{-1}DA^{-1}$

ie, $\left(\underset{A}{\Sigma_{11}}-\underset{C}{\Sigma_{12}}\underset{B}{\Sigma_{22}^{-1}}\underset{D}{\Sigma_{12}^T}\right)^{-1}$ can be written as

$$= \Sigma_{11}^{-1} + \Sigma_{11}^{-1}\Sigma_{12}\Big(\underbrace{\Sigma_{22}-\Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}}_{A}\Big)^{-1}\Sigma_{12}^T\,\Sigma_{11}^{-1}$$

let $\boxed{A = \Sigma_{22}-\Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}}$

2) $(x_1-\mu_1)^T\left(\Sigma_{11}^{-1}+\Sigma_{11}^{-1}\Sigma_{12}A^{-1}\Sigma_{12}^T\Sigma_{11}^{-1}\right)(x_1-\mu_1)$

$$-2(x_1-\mu_1)^T\left[\Sigma_{11}^{-1}\Sigma_{12}\left(\Sigma_{22}-\Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}\right)^{-1}\right](x_2-\mu_2)$$

$$-2(x_1-\mu_1)^T\left[\Sigma_{11}^{-1}\Sigma_{12}A^{-1}\right](x_2-\mu_2)$$

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$$=-2(x_1-\mu_1)^T\left[\Sigma_{11}^{-1}\Sigma_{12}A^{-1}\right](x_2-\mu_2)$$

3) $(x_2-\mu_2)^T\left[\left(\Sigma_{22}-\Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}\right)^{-1}\right](x_2-\mu_2)$

$$= (x_2 - \mu_2)^T \left[\left(\underset{A}{\Sigma_{22}} - \underset{C}{\Sigma_{12}^T} \, \underset{D}{\Sigma_{11}^{-1}} \, \underset{B}{\Sigma_{12}} \right)^{-1} \right] (x_2 - \mu_2)$$

$$\otimes = (x_2 - \mu_2)^T \left[\Sigma_{22}^{-1} + \Sigma_{22}^{-1} \Sigma_{12}^T \left(\Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}^T \right)^{-1} \Sigma_{12} \cdot \Sigma_{22}^{-1} \right. (x_2 - \mu_2)$$

$$= \cancel{(x_2 - \mu_2)^T} \left[\cancel{\Sigma_{22}^{-1}} + \cancel{\Sigma_{22}^{-1}} \cancel{\Sigma_{12}^T} \right.$$

$$= (x_2 - \mu_2)^T \left[\left(\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12} \right)^{-1} \right] (x_2 - \mu_2)$$

Now, replacing these values in eq (1)

$$Q(x_1, x_2) = (x_1 - \mu_1)^T \left[\left(\Sigma_{11}^{-1} + \Sigma_{11}^{-1} \Sigma_{12} \left(\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12} \right)^{-1} \Sigma_{12}^T \Sigma_{11}^{-1} \right) \right] (x_1 - \mu_1)$$

$$- 2 (x_1 - \mu_1)^T \left[\Sigma_{11}^{-1} \Sigma_{12} \left(\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12} \right)^{-1} \right] (x_2 - \mu_2)$$

$$+ (x_2 - \mu_2)^T \left[\left(\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12} \right)^{-1} \right] (x_2 - \mu_2)$$

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$$f(x_1 x_2) = (x_1-\mu_1)^T \Sigma_{11}^{-1} (x_1-\mu_1) + \Big\{ (x_1-\mu_1)^T \left[\Sigma_{11}^{-1} \Sigma_{12} A^{-1} \Sigma_{12}^T \Sigma_{11}^{-1} \right] (x_1-\mu_1)$$

$$- 2(x_1-\mu_1)^T \left[\Sigma_{11}^{-1} \Sigma_{12} A^{-1} \right] (x_2-\mu_2)$$

$$+ (x_2-\mu_2)^T \left[\left(\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12} \right)^{-1} \right] (x_2-\mu_2) \Big\}$$

q_2

$$q_2 = (x_1-\mu_1)^T \left[\Sigma_{11}^{-1} \Sigma_{12} A^{-1} \Sigma_{12}^T \Sigma_{11}^{-1} \right] (x_1-\mu_1)$$

$$- 2 (x_1-\mu_1)^T \left[\Sigma_{11}^{-1} \Sigma_{12} A^{-1} \right] (x_2-\mu_2)$$

$$+ (x_2-\mu_2)^T \left[\underbrace{\left(\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12} \right)}_{A}^{-1} \right] (x_2-\mu_2)$$

[illegible] $= (x_1-\mu_1)^T \left[\Sigma_{11}^{-1} \Sigma_{12} A^{-1} \Sigma_{12}^T \Sigma_{11}^{-1} \right] (x_1-\mu_1)$

$- 2 (x_1-\mu_1)^T (\Sigma_{11}^{-1} \Sigma_{12} A^{-1}) (x_2-\mu_2)$

$+ (x_2-\mu_2)^T A^{-1} (x_2-\mu_2)$

合并 $= (x_1-\mu_1)^T \left[\Sigma_{11}^{-1} \Sigma_{12} A^{-1} \Sigma_{12}^T \Sigma_{11}^{-1} \right] (x_1-\mu_1)$

$- (x_1-\mu_1)^T (\Sigma_{11}^{-1} \Sigma_{12} A^{-1}) (x_2-\mu_2) - (x_1-\mu_1)^T (\Sigma_{11}^{-1} \Sigma_{12} A^{-1}) (x_2-\mu_2)$

$+ (x_2-\mu_2)^T A^{-1} (x_2-\mu_2)$ 2019

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$$(x_1-\mu_1)^T \left[\Sigma_{11}^{-1}\Sigma_{12} A^{-1} \Sigma_{12}^T \Sigma_{11}^{-1}\right] (x_1-\mu_1)$$

$$- (x_1-\mu_1)^T \cdot \left[\Sigma_{11}^{-1}\Sigma_{12} \cdot A^{-1}\right] (x_2-\mu_2)$$

$$- (x_1-\mu_1)^T \left[\Sigma_{11}^{-1}\Sigma_{12} \cdot A^{-1}\right] (x_2-\mu_2)$$

$$+ (x_2-\mu_2)^T A^{-1} (x_2-\mu_2)$$

$$(x_1-\mu_1)^T \Sigma_{11}^{-1}\Sigma_{12} A^{-1} \left[\Sigma_{12}^T \Sigma_{11}^{-1}(x_1-\mu_1) - (x_2-\mu_2)\right]$$

$$- \left[(x_1-\mu_1)^T \Sigma_{11}^{-1}\Sigma_{12} - (x_2-\mu_2)^T\right] A^{-1} (x_2-\mu_2)$$

$$= (x_1-\mu_1)^T \Sigma_{11}^{-1} \Sigma_{12} A^{-1} \left[\Sigma_{12}^T \Sigma_{11}^{-1} (x_1-\mu_1) - (x_2-\mu_2) \right]$$

$$- (x_2-\mu_2)^T A^{-1} \left[(x_1-\mu_1)^T \Sigma_{11}^{-1} \Sigma_{12} - (x_2-\mu_2)^T \right]^T$$

$$\left(\because B = B^T, \text{if } B \to 1\times 1 \right)$$

$$(x_1-\mu_1)^T \Sigma_{11}^{-1} \Sigma_{12} A^{-1} \left(\Sigma_{12}^T \Sigma_{11}^{-1} (x_1-\mu_1) - (x_2-\mu_2) \right)$$

$$- (x_2-\mu_2)^T A^{-1} \left(\Sigma_{12}^T \Sigma_{11}^{-1} (x_1-\mu_1) - (x_2-\mu_2) \right)$$

Now, taking $\left(\Sigma_{12}^T \Sigma_{11}^{-1} (x_1-\mu_1) - (x_2-\mu_2) \right)$ as common

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$$= (x_2 - \mu_2)^T A^{-1} \left((x_2 - \mu_2) - \Sigma_{12}^T \Sigma_{11}^{-1} (x_1 - \mu_1) \right)$$

$$- (x_1 - \mu_1)^T \Sigma_{11}^{-1} \Sigma_{12} A^{-1} \left((x_2 - \mu_2) - \Sigma_{12}^T \Sigma_{11}^{-1} (x_1 - \mu_1) \right)$$

$$= \Big[\cancel{(x_2 - \mu_2)^T A}$$

$$= \left[(x_2 - \mu_2)^T - (x_1 - \mu_1)^T \Sigma_{11}^{-1} \Sigma_{12} \right] A^{-1} \left[(x_2 - \mu_2) - \Sigma_{12}^T \Sigma_{11}^{-1} (x_1 - \mu_1) \right.$$

$$\left((x_2 - \mu_2) - \Sigma_{12}^T \Sigma_{11}^{-1} (x_1 - \mu_1) \right)^T A^{-1} \left((x_2 - \mu_2) - \Sigma_{12}^T \Sigma_{11}^{-1} (x_1 - \mu_1) \right)$$

$$\therefore \; Q(x_1, x_2) = Q_1(x_1) + Q_2(x_1, x_2)$$

where $Q_1(x_1) = (x_1 - \mu_1)^T \Sigma_{11}^{-1} (x_1 - \mu_1)$

$$Q_2(x_1, x_2) = \left[(x_2 - \mu_2) - \Sigma_{12}^T \Sigma_{11}^{-1} (x_1 - \mu_1)\right]^T A^{-1} \left[(x_2 - \mu_2) - \Sigma_{12}^T \Sigma_{11}^{-1} (x_1 - \mu_1)\right]$$

or, $Q_2(x_1, x_2) = \cancel{x_2 - \mu_2} \; (x_2 - b)^T A^{-1} (x_2 - b)$

where
$$\boxed{\begin{aligned} b &= \mu_2 + \Sigma_{12}^T \Sigma_{11}^{-1} (x_1 - \mu_1) \\ A &= \Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12} \end{aligned}} \quad (12)$$

Again, $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix}$, can be written as

$$= \cancel{\begin{bmatrix} \Sigma_{11} & 0 \\ \Sigma_{12}^T & I \end{bmatrix} \begin{bmatrix} I & \Sigma_{11}^{-1}\Sigma_{12} \\ 0 & -\Sigma_{12}^T\Sigma_{11}^{-1} \end{bmatrix}}$$

$$\begin{pmatrix} \Sigma_{11} & 0 \\ \Sigma_{12}^T & I \end{pmatrix} \begin{pmatrix} I & \Sigma_{11}^{-1}\Sigma_{12} \\ 0 & \Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12} \end{pmatrix}$$

Now, $|AB| = |A||B|$

Now,

$$|\Sigma| = |\Sigma_{11}| \cdot |\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12}|$$

or

$$\boxed{|\Sigma| = |\Sigma_{11}| \cdot |A|} \quad (3)$$

Now, replacing the equation in the joint pdf

$$f(x_1, x_2) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp\left[-\frac{1}{2} Q(x_1, x_2)\right]$$

or, $$f(x_1, x_2) = \frac{1}{(2\pi)^{\frac{p+q}{2}} |\Sigma_{11}| |A|} \exp\left[-\frac{1}{2}\left(Q_1(x_1) + Q_2(x_1, x_2)\right)\right]$$

or $$f(x_1, x_2) = \frac{1}{(2\pi)^{p/2} \cdot |\Sigma_{11}|} \exp\left(-\frac{1}{2} Q_1(x_1)\right) \cdot \frac{1}{(2\pi)^{q/2} |A|} \exp\left[-\frac{1}{2} Q_2(x_1, x_2)\right]$$

$$f(x_1, x_2) = \frac{1}{(2\pi)^{p/2} \cdot |\Sigma_{11}|^{1/2}} \cdot \exp\left[-\frac{1}{2}(x_1 - \mu_1)^T \Sigma_{11}^{-1} (x_1 - \mu_1)\right] \cdot$$

$$\frac{1}{(2\pi)^{q/2} |A|^{1/2}} \exp\left[-\frac{1}{2}(x_2 - b)^T A^{-1} (x_2 - b)\right]$$

or $$\boxed{f(x_1, x_2) = N(x_1, \mu_1, \Sigma_{11}) \cdot N(x_2, b, A)}$$

<u>Marginal distribution</u>

$$f(x_1) = \int_{x_2} f(x_1, x_2)\, dx_2$$

$$= \int_{x_2} N(x_1, \mu_1, \Sigma_{11}) \cdot N(x_2, b, A)\, dx_2$$

$$= N(x_1, \mu_1, \Sigma_{11})$$

$$f(x_1) = \frac{1}{(2\pi)^{n/2} |\Sigma_{11}|^{1/2}} \exp\left[-\frac{1}{2} (x_1 - \mu_1)^T \Sigma_{11}^{-1} (x_1 - \mu_1)\right.$$

Conditional dist. of 2/1

$$f(x_2/x_1) = \frac{f(x_1, x_2)}{f(x_1)}$$

or, $$f(x_2/x_1) = \frac{\cancel{N(x_1, \mu_1, \Sigma_{11})} \cdot N(x_2, b, A)}{\cancel{N(x_1, \mu_1, \Sigma_{11})}}$$

or, $$f(x_2/x_1) = N(x_2, b, A)$$