ECMM450 Stochastic Processes

Coursework Assignment 2

March 2023

1. Consider customers arriving independently at a bar that has two bartenders serving independently. Customers are assigned to an available bartender on a first come, first served basis but walk away if there are more than $K > 1$ customers waiting or being served at the bar. You may assume that arrivals are a Poisson process with rate λ and each of the bartenders serves customers according to a Poisson process with rate μ.

(a) The Kendall notation for the above Markov chain is M/M/2

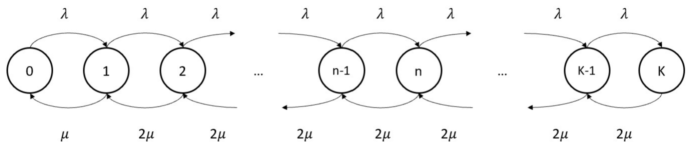

Figure 1: Diagram for Markov Chain. Numbers represent size of the system, i.e. how many people there are in the cafe

(b) Let $\rho = \frac{\lambda}{2\mu}$. The steady state probability equations are as follows:

$$\lambda P_0 = \mu P_1 \implies P_1 = \frac{\lambda}{\mu} P_0 = 2\rho P_0$$
$$\lambda P_1 = 2\mu P_2 \implies P_2 = \frac{\lambda}{2\mu} P_1 = 2\rho^2 P_0$$
$$\ldots$$
$$\lambda P_{n-1} = 2\mu P_n \implies P_n = \frac{\lambda}{2\mu} P_{n-1} = 2\rho^n P_0$$
$$\ldots$$
$$\lambda P_{K-1} = 2\mu P_K \implies P_K = \frac{\lambda}{2\mu} P_{K-1} = 2\rho^K P_0$$

We know that the sum of probabilities must add up to 1.

$$P_0 + P_1 + \ldots + P_K = 1$$
$$\implies P_0(1 + 2\rho + 2\rho^2 + \ldots + 2\rho^K) = 1$$
$$\implies P_0(2 + 2\rho + 2\rho^2 + \ldots + 2\rho^K - 1) = 1$$
$$\implies P_0(2(1 + \rho + \rho^2 + \ldots + \rho^K) - 1) = 1$$
$$\implies P_0(2 \cdot \frac{1-\rho^{K+1}}{1-\rho} - 1) = 1, \text{ using given hint}$$
$$\implies P_0(2 - 2\rho^{K+1} - 1 + \rho) = 1 - \rho$$
$$\implies P_0(1 + \rho - 2\rho^{K+1}) = 1 - \rho$$
$$\implies P_0 = \frac{1-\rho}{1+\rho-2\rho^{K+1}}$$
$$\implies P_n = 2\rho^n \frac{1-\rho}{1+\rho-2\rho^{K+1}}, \text{ for } n \geq 1$$

(c) At steady state, expected number of customers at the bar is given by E[N].

$$E[N] = \sum_{n=0}^{K} nP_n$$

$$E[N] = \sum_{n=1}^{K} n \cdot 2\rho^n \frac{1-\rho}{1+\rho-2\rho^{K+1}}$$

$$E[N] = \frac{2(1-\rho)}{1+\rho-2\rho^{K+1}} \sum_{n=0}^{K} n\rho^n$$

$$E[N] = \frac{2(1-\rho)}{1+\rho-2\rho^{K+1}} [\rho + 2\rho^2 + 3\rho^3 + ... + K\rho^K]$$

$$E[N] = \frac{2(1-\rho)}{1+\rho-2\rho^{K+1}} [\rho \cdot \frac{d}{d\rho}(1+\rho+\rho^2+\rho^3+...+\rho^K)]$$

$$E[N] = \frac{2\rho(1-\rho)}{1+\rho-2\rho^{K+1}} \frac{d}{d\rho}(\frac{1-\rho^{K+1}}{1-\rho}), \text{ using given hint as K is a positive integer}$$

$$E[N] = \frac{2\rho(1-\rho)}{1+\rho-2\rho^{K+1}} \frac{(1-\rho)(-(K+1)\rho^K) - (1-\rho^{K+1})(-1)}{(1-\rho)^2}$$

$$E[N] = \frac{2\rho(1-\rho)}{1+\rho-2\rho^{K+1}} \frac{-(K+1)\rho^K + (K+1)\rho^{K+1} + 1 - \rho^{K+1}}{(1-\rho)^2}$$

$$E[N] = \frac{2\rho}{1+\rho-2\rho^{K+1}} \frac{1-(K+1)\rho^K + K\rho^{K+1}}{(1-\rho)}$$

When $K \to \infty$ and $\rho < 1$,

$$\lim_{K\to\infty} E[N] = \lim_{K\to\infty} \frac{2\rho}{1+\rho-2\rho^{K+1}} \frac{1-(K+1)\rho^K + K\rho^{K+1}}{(1-\rho)}$$

$$\lim_{K\to\infty} E[N] = \frac{2\rho}{1+\rho-0} \frac{1-0+0}{(1-\rho)}$$

Note: We see terms with ρ^K or ρ^{K+1} becoming 0 as such terms drop to 0 exponentially fast thus offsetting their coefficients, which tend to inifinity. Thus, we see that for $K \to \infty$ and $\rho < 1$, $E[N] \to \frac{2\rho}{1-\rho^2}$

(d) Using Little's Law, we find the average waiting time as $W_s = \frac{E[N]}{\lambda_{eff}}$. To derive an expression for λ_{eff}, we have

$$\lambda_{eff} = \sum_{n=0}^{K} \lambda_n P_n = \sum_{n=0}^{K-1} \lambda_n P_n = \lambda(1-P_K), \text{ (as } \lambda_K = 0)$$

$$\lambda_{eff} = \lambda \left[1 - \frac{2\rho^K(1-\rho)}{1+\rho-2\rho^{K+1}}\right]$$

$$\lambda_{eff} = \lambda \left[\frac{1+\rho-2\rho^{K+1} - 2\rho^K + 2\rho^{K+1}}{1+\rho-2\rho^{K+1}}\right]$$

$$\lambda_{eff} = \lambda \left[\frac{1+\rho-2\rho^K}{1+\rho-2\rho^{K+1}}\right]$$

(e) Probability at steady state of having 2 or more customers at the bar is found as

$$
\begin{aligned}
P(N \geq 2) &= P(N=2) + P(N=3) + ... + P(N=k) \\
&= P_2 + P_3 + ... + P_K \\
&= P_0(2\rho^2 + 2\rho^3 + ... + 2\rho^K) \\
&= \frac{(1-\rho)}{1+\rho-2\rho^{K+1}}(2\rho^2 + 2\rho^3 + ... + 2\rho^K) \\
&= \frac{(1-\rho)}{1+\rho-2\rho^{K+1}} \cdot 2\rho^2(1+\rho+...+\rho^{K-2}) \\
&= \frac{2(1-\rho)\rho^2}{1+\rho-2\rho^{K+1}} \cdot \left[\frac{1-\rho^{K-1}}{1-\rho}\right] \\
&= \frac{2\rho^2}{1+\rho-2\rho^{K+1}} \cdot \left[1-\rho^{K-1}\right]
\end{aligned}
$$

When $K \to \infty$,

$$
\begin{aligned}
\lim_{K\to\infty} P(N \geq 2) = \frac{2\rho^2}{1+\rho} &< \frac{1}{3} \\
\implies 6\rho^2 &< 1+\rho \\
\implies 6\rho^2 - \rho - 1 &< 0
\end{aligned}
$$

Solving the above, we get $\frac{-1}{3} < \rho < \frac{1}{2}$. However, the first part of this is non-sensical as ρ cannot be less than 0 (both λ and μ are positive). Taking that into account, the possible range of values for ρ is given as $0 < \rho < \frac{1}{2}$.

2. Consider the following Markov chain with transition matrix:

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1-\alpha & 0 & \alpha & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\beta & 0 & 0 & 0 & 1-\beta \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
$$

Here $T_{ij} = P(X_n = j | X_{n-1} = i)$ for all $i, j \in \{1,2,3,4,5\}$, and α and β do not depend on n.

(a) The state transition diagram is given below

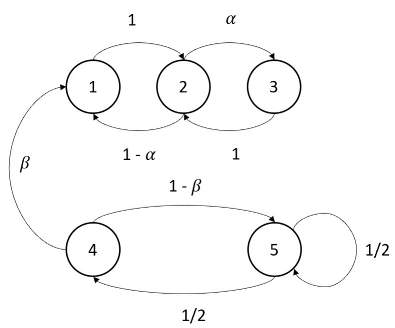

Figure 2: State transition diagram

State 2 To prove that state 2 is positively recurrent, we need to prove that the mean recurrence time $\mu = \sum_{n=1}^{\infty} n f_i^{(n)} < \infty$. First, we check the times of first return.

$$\begin{aligned} f_2^{(1)} &= 0 \\ f_2^{(2)} &= 1 - \alpha + \alpha = 1 \\ f_2^{(3)} &= 0 \\ f_2^{(4)} &= 0 \\ f_2^{(5)} &= 0 \\ &\vdots \end{aligned}$$

Note that the process cannot return to state 2 for the *first* time after more than 2 steps - by the 3rd or 4th time (and indeed any step more than that), it will already be *revisiting* state 2. Calculate the mean recurrence time $\mu = 1 \cdot f_2^{(1)} + 2 \cdot f_2^{(2)} + 3 \cdot f_2^{(3)} + 4 \cdot f_2^{(4)} + \cdots$
$\implies \mu = 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 0 + 4 \cdot 0 + \cdots$
$\implies \mu = 2 < \infty$, Therefore, state 2 is positively recurrent.

State 5 To see if state 5 is transient, we check if sum of times of first return is < 1.

$$\begin{aligned} f_5^{(1)} &= \frac{1}{2} \\ f_5^{(2)} &= \frac{1}{2}(1 - \beta) \\ f_5^{(3)} &= 0 \\ f_5^{(4)} &= 0 \\ f_5^{(5)} &= 0 \\ &\vdots \\ \sum_{n=1}^{\infty} f_5^{(n)} &= \frac{1}{2}(1 - \beta) \end{aligned}$$

It is not dependent on α, but for $0 < \beta < 1$, the sum is lesser than 1, which means that state 5 is transient. We notice that similar to state 2, the process cannot return to state 5 for the *first* time after more than 2 steps - it will have already visited it before (on the first or second step).

(b) For fixed $\alpha, \beta > 0$, it can be seen that there are two sub-chains: 1-2-3 and 4-5.

Following are the classifications of the states:

State 1: It is periodic as for $k = 2$, $(T^n)_{11} > 0$ for all $n = k, 2k, 3k...$ and $(T^n)_{11} = 0$ for all $n \neq k, 2k, 3k....$ To elaborate, starting at state 1, the system can return to state 1 after 2,4,6... steps, but cannot return to it after 1,3,5... steps.

State 2: Following the above logic, it can be seen that state 2 is recurrent. The difference from state 1 is that, for state 1, the system can return to it after any even step size (2,4,6,8....), whereas, for state 2, the system must return to it every 2 steps. Mean recurrence time for state 2 is therefore $\mu = 2$, as seen earlier in part (a).

State 3: Following exactly the same logic as state 1, it can be seen that state 3 is also periodic with the system returning to it after any even-numbered time step(s).

States 4 & 5: There is a probability of β of the system leaving state 4 (and thus the subchain) and never being able to return to it. So states 4 and 5 area transient.

The criteria for a subchain to be *ergodic* is for it to be both aperiodic and positively recurrent. Suchain 1-2-3 violates the condition for aperiodicity. Suchain 4-5 violates the condition of positively recurrent. Thus, neither subchain is ergodic.

(c) To find the steady state probability vector in terms of α and β., we can solve the eigenvector equation $\tilde{P} = \tilde{P}T$.

$$\implies [P_1, P_2, P_3, P_4, P_5] = [P_1, P_2, P_3, P_4, P_5] \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1-\alpha & 0 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 & 1-\beta \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

From the above equation, the probabilities can be arranged as

$$P_1 = P_2(1-\alpha) + P_4\beta \tag{1}$$

$$P_2 = P_1 + P_3 \tag{2}$$

$$P_3 = P_2\alpha \tag{3}$$

$$P_4 = \frac{P_5}{2} \tag{4}$$

$$P_5 = P_4(1-\beta) + \frac{P_5}{2} \tag{5}$$

Equation 5 can be rewritten as $P_5 = \frac{P_5}{2}(1-\beta) + \frac{P_5}{2}$, solving which we get $P_5 = 0$ for $\beta > 0$. It follows from 4 that $P_4 = \frac{P_5}{2} = 0$. This also reduces 1 to $P_1 = P_2(1-\alpha)$.

We can now say that $P_1 + P_2 + P_3 = 1$ as the sum of the steady state probabilities must be 1. Solving,

$$\begin{gathered} P_1 + P_2 + P_3 = 1 \\ P_2(1-\alpha) + P_2 + P_2\alpha = 1 \\ P_2(1-\alpha+1-\alpha) = 1 \\ P_2 = \frac{1}{2} \\ \implies P_1 = \frac{1-\alpha}{2} \ \& \ P_3 = \frac{\alpha}{2} \end{gathered}$$

(d) Suppose now $\alpha := \alpha_n = \frac{n}{n+1}$, so that now $T \equiv T(n)$ is a time dependent transition matrix. This amends the state transition diagram to

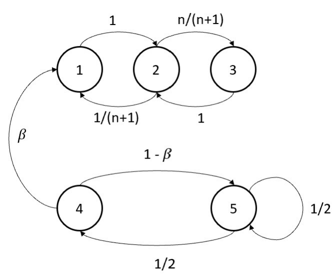

Figure 3: State transition diagram for time-dependent Markov chain

To find the classification of state 1, we need to compute $\sum_{n=1}^{\infty} f_1^{(n)}$ and $\mu = \sum_{n=1}^{\infty} n f_1^{(n)} < \infty$.

The times of first return are

$$f_1^{(1)} = 0$$
$$f_1^{(2)} = \frac{1}{n_2+1} = \frac{1}{3}$$
$$f_1^{(3)} = 0$$
$$f_1^{(4)} = \frac{n_2}{n_2+1}\frac{1}{n_4+1} = \frac{2}{3}\cdot\frac{1}{5}$$
$$f_1^{(5)} = 0$$
$$f_1^{(6)} = \frac{n_2}{n_2+1}\frac{n_4}{n_4+1}\frac{n_6}{n_6+1} = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{7}$$
$$f_1^{(7)} = 0$$
$$\vdots$$

Unsurpisingly, the probability of every odd-step return is 0.

$$\sum_{n=1}^{\infty} f_5^{(n)} = \frac{1}{3} + (\frac{2}{3}\cdot\frac{1}{5}) + (\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{7}) + (\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{1}{9}) + \cdots$$
$$\sum_{n=1}^{\infty} f_5^{(n)} = \frac{1}{3} + (\frac{2}{3}\cdot(1-\frac{4}{5})) + (\frac{2}{3}\cdot\frac{4}{5}\cdot(1-\frac{6}{7})) + (\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot(1-\frac{8}{9})) + \cdots$$
$$\sum_{n=1}^{\infty} f_5^{(n)} = \frac{1}{3} + (\frac{2}{3} - \frac{2}{3}\cdot\frac{4}{5}) + (\frac{2}{3}\cdot\frac{4}{5} - \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}) + (\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7} - \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}) + \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9} + \cdots$$

After the second term, every subsequent pair of terms cancels each other out, leaving behind $\frac{1}{3}+\frac{2}{3} = 1$. Therefore, state 1 is recurrent. To further investigate if it is positively recurrent or null recurrent, we compute mean recurrence time μ. Recall that every odd time-step probability of first return is 0, so we will only consider the even terms.

$$\mu = \sum_{n=1}^{\infty} n f_1^{(n)} = 2\cdot\frac{1}{3} + 4\cdot(\frac{2}{3}\cdot\frac{1}{5}) + 6\cdot(\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{7}) + 8\cdot(\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{1}{9}) + \cdots$$
$$\mu = \sum_{n=1}^{\infty} n f_1^{(n)} = \frac{2}{3} + (\frac{4}{3}\cdot\frac{1}{5}) + (\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}) + (\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{8}{9}) + \cdots$$

The nth term can be represented as $t_n = \prod_{i=1}^{n}(1-a_n)$, where $a_n = (2n+1)^{-1}$

As a series, we can rewrite t_n

$$t_n = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots(1-a_n) = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots(1-\frac{1}{2n+1}) = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n+1}$$
$$\implies t_n = 2^n\frac{1}{3}\cdot\frac{2}{5}\cdot\frac{3}{7}\cdots\frac{n}{2n+1} = \frac{2^n n!}{3\cdot 5\cdot 7\cdots(2n+1)} = \frac{2^n n!\cdot 2\cdot 4\cdot 6\cdots 2n}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdots 2n\cdot(2n+1)}$$
$$\implies t_n = \frac{2^n n!\cdot 2^n(1\cdot 2\cdot 3\cdot 4\cdots n)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdots 2n\cdot(2n+1)} = \frac{2^n n!\cdot 2^n n!}{(2n+1)!} = \frac{(2^n n!)^2}{(2n+1)!}$$

Expanding t_n, we can write

$$t_n = \frac{(2^n n!)^2}{(2n+1)!} = 2^{2n}\frac{(n!)^2}{(2n+1)!} = 4^n\frac{(n!)^2}{(2n+1)!}$$

$$\implies t_n = 4^n \frac{n^2}{(2n+1)(2n)} \cdot \frac{(n-1)^2}{(2n-1)(2n-2)} \cdot \frac{(n-2)^2}{(2n-3)(2n-4)} \cdots \frac{2^2}{5\cdot 4} \cdot \frac{1^2}{3\cdot 2}$$

$$\implies t_n = 4^n \frac{n^2}{(2n+1)(2n)} \cdot \frac{(n-1)^2}{(2n-1)(2(n-1))} \cdot \frac{(n-2)^2}{(2n-3)(2(n-2))} \cdots \frac{2^2}{5\cdot 2\cdot 2} \cdot \frac{1^2}{3\cdot 2\cdot 1}$$

$$\implies t_n = 4^n \frac{n}{(2n+1)(2)} \cdot \frac{(n-1)}{(2n-1)(2)} \cdot \frac{(n-2)}{(2n-3)(2)} \cdots \frac{2}{5\cdot 2} \cdot \frac{1}{3\cdot 2}$$

$$\implies t_n = 4^n \frac{n}{(2(n+\frac{1}{2}))(2)} \cdot \frac{(n-1)}{(2(n-\frac{1}{2}))(2)} \cdot \frac{(n-2)}{(2(n-\frac{3}{2}))(2)} \cdots \frac{2}{2\cdot\frac{5}{2}\cdot 2} \cdot \frac{1}{2\cdot\frac{3}{2}\cdot 2}$$

$$\implies t_n = \frac{4^n}{4^n}\frac{n}{(n+\frac{1}{2})} \cdot \frac{(n-1)}{(n-\frac{1}{2})} \cdot \frac{(n-2)}{(n-\frac{3}{2})} \cdots \frac{2}{\frac{5}{2}} \cdot \frac{1}{\frac{3}{2}}$$

$$\implies t_n = \frac{n}{(n+\frac{1}{2})} \cdot \frac{(n-1)}{(n-\frac{1}{2})} \cdot \frac{(n-2)}{(n-\frac{3}{2})} \cdots \frac{2}{\frac{5}{2}} \cdot \frac{1}{\frac{3}{2}}$$

Subtracting $\frac{1}{2}$ from all the numerators inevitably leaves us with a term smaller than t_n. Therefore,

$$t_n \geq \frac{n-\frac{1}{2}}{(n+\frac{1}{2})} \cdot \frac{(n-\frac{3}{2})}{(n-\frac{1}{2})} \cdot \frac{(n-\frac{5}{2})}{(n-\frac{3}{2})} \cdots \frac{\frac{3}{2}}{\frac{5}{2}} \cdot \frac{\frac{1}{2}}{\frac{3}{2}}$$

$$\implies t_n \geq \frac{\frac{1}{2}}{n+\frac{1}{2}}$$

The last line follows from alternate numerators and denominators cancelling out. We now have $t_n \geq \frac{1}{2n+1}$, which can be compared against the harmonic series to showcase divergence. By the limit comparison test, for two given series $\sum a_n$ and $\sum b_n$, if $\lim_{n\to\infty}\frac{a_n}{b_n} = c$ for $0 < c < \infty$, where a_n, b_n are respectively the *nth* terms of the series, then either both series converge or both series diverge.

We have $a_n = \frac{1}{2n+1}$, $b_n = \frac{1}{n}$, the latter denoting the harmonic series.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2}, \text{L'Hospital rule}$$

As we know that the harmonic series diverges, the series with *nth* term $a_n = \frac{1}{2n+1}$ also diverges. Therefore the series with nth term t_n, which is greater than a_n, must also diverge. Hence, the mean recurrence time μ diverges. This proves that the state 1 is null recurrent.