ECMM450 Stochastic Processes: Assignment 1

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1 Consider the discrete random variable X that has the probability generating function

$$G_X(\theta) = 2(3-\theta)^{-1}$$

(a) Suppose that $Y = X^2$. Calculate $P(Y = k)$ for $0 \leq k \leq 10$ and the expectaion $E[Y]$

$G_X(\theta) = 2(3-\theta)^{-1}$
$= \frac{2}{3}(1 - \frac{\theta}{3})^{-1}$
$= \frac{2}{3}(1 + \frac{\theta}{3} + (\frac{\theta}{3})^2 + (\frac{\theta}{3})^3 + ...)$
$= \frac{2}{3}\theta^0 + \frac{2}{3^2}\theta^1 + \frac{2}{3^3}\theta^2 + \frac{2}{3^4}\theta^3 + ...$

We observe from the above equation that $P(X = 0) = \frac{2}{3}, P(X = 1) = \frac{2}{3^2}, P(X = 2) = \frac{2}{3^3}$ ....

$G_Y(\theta) = \sum_k^\infty \theta^k P(Y = k) = \sum_k^\infty \theta^k P(X^2 = k)$

$= \theta^0 P(X^2 = 0) + \theta^1 P(X^2 = 1) + ... + \theta^4 P(X^2 = 4) + ... + \theta^9 P(X^2 = 9) + \theta^{10} P(X^2 = 10)$

As X can only take on positive integer values, the only terms in the above equation which are non-zero are those associated where $X^2 \in \{0, 1, 4, 9\}$

Therefore, $G_Y(\theta) = \theta^0 P(X^2 = 0) + \theta^1 P(X^2 = 1) + \theta^4 P(X^2 = 4) + \theta^9 P(X^2 = 9)$
$G_Y(\theta) = \theta^0 P(X = 0) + \theta^1 P(X = 1) + \theta^4 P(X = 2) + \theta^9 P(X = 3)$

$G_Y(\theta) = \frac{2}{3}\theta^0 + \frac{2}{3^2}\theta^1 + \frac{2}{3^3}\theta^4 + \frac{2}{3^4}\theta^9$ **Ans**

Now, to calculate $E[Y]$, we can evaluate $E[X^2]$, which is as follows. $G''_X(\theta)|_{\theta=1} = E[X(X-1)] = E[X^2] - (E[X])$
$\implies G''_X(\theta)|_{\theta=1} = (-2) \cdot \frac{2}{(3-\theta)^3} \cdot (-1)|_{\theta=1} = \frac{4}{2^3} = 0.5$
$\implies E[X^2] - (E[X]) = 0.5$

To calculate $E[X]$, we have $G'_X(\theta)|_{\theta=1} = E[X]$
$\implies E[X] = \frac{2}{(3-\theta)^2}|_{\theta=1} = 0.5$

Therefore, going back to our earlier equation, we have $\implies E[X^2] - (0.5) = 0.5$
$\implies E[X^2] = E[Y] = 1$ **Ans**

(b) Suppose that Z is the random variable with probability generating function $G_Z(\theta) = G_X(G_X(\theta))$. Calculate $P(Z = 0), P(Z = 2)$, and $E[Z]$.

$G_Z(\theta) = G_X(2(3-\theta)^{-1}) = 2(3 - \frac{2}{3-\theta})^{-1} = \frac{2(3-\theta)}{3(3-\theta)-2} = \frac{2(3-\theta)}{7-3\theta}$

$\implies G_Z(\theta) = \frac{2}{3}\frac{9-3\theta}{7-3\theta} = \frac{2}{3}(1 + \frac{2}{7-3\theta}) = \frac{2}{3} + \frac{2}{3} \cdot 2 \cdot (7-3\theta)^{-1}$

$\implies G_Z(\theta) = \frac{2}{3} + \frac{4}{3} \cdot (7(1 - \frac{3}{7}\theta))^{-1} = \frac{2}{3} + \frac{4}{3} \cdot \frac{1}{7}(1 - \frac{3}{7}\theta)^{-1}$

$\implies G_Z(\theta) = \frac{2}{3} + \frac{4}{21} \cdot (1 + \frac{3}{7}\theta + (\frac{3}{7}\theta)^2 + (\frac{3}{7}\theta)^3 + ...)$

$\implies G_Z(\theta) = \frac{2}{3} + \frac{4}{21} + \frac{4}{21} \cdot (\frac{3}{7}\theta + (\frac{3}{7}\theta)^2 + (\frac{3}{7}\theta)^3 + ...)$

$\implies G_Z(\theta) = \frac{18}{21} + \frac{4}{21} \cdot (\frac{3}{7}\theta + (\frac{3}{7}\theta)^2 + (\frac{3}{7}\theta)^3 + ...)$

From the above equation, we see that $P(Z=0) = \frac{18}{21} = 0.8571$ **Ans**, $P(Z=2) = \frac{4}{21} \cdot \frac{3^2}{7^2} = 0.0349$ **Ans**

Since $G_Z(\theta) = G_X^2(\theta)$, we know that $E[Z] = \mu_X^2$, where $\mu_X = E[X]$.

To find this, we evaluate $G'_X(\theta)|_{\theta=1} = \frac{d}{d\theta}(2(3-\theta)^{-1}) = 2\frac{d}{d\theta}\frac{1}{3-\theta} = 2 \cdot (-1) \cdot \frac{1}{(3-\theta)^2} \cdot (-1) = \frac{2}{(3-\theta)^2}$

Putting the value of $\theta = 1$, we have, $E[X] = \frac{2}{(3-1)^2} = \frac{2}{2^2} = 0.5$ Therefore, $E[Z] = 0.5^2 = 0.25$ **Ans**

(c) $W = X_1 + X_2 + X_3$, where X_1, X_2, X_3 are independent

Therefore, $G_W(\theta) = \{2(3-\theta)^{-1}\}^3 = 2^3(3-\theta)^{-3} = 8(3-\theta)^{-3} = \frac{8}{27}(1 - \frac{\theta}{3})^{-3}$

Using binomial expansion for negative coefficients, we have

$G_W(\theta) = \frac{8}{27}(1 + (-3)(-\frac{\theta}{3}) + \frac{(-3)(-3-1)}{2!}\frac{\theta^2}{3^2} + ...)$

$G_W(\theta) = \frac{8}{27}(1 + \theta + \frac{2}{3}\theta^2 + ...)$

From the above equation, it can be observed that

$P(W=0) = \frac{8}{27} = 0.2962$ **Ans** , $P(W=2) = \frac{8*2}{27*3} = 0.1975$ **Ans**

For $E[W]$, we have $G'_W(\theta)|_{\theta=1} = 2^3(-3)(3-\theta)^{-4}(-1)|_{\theta=1} = \frac{24}{(3-1)^4} = \frac{24}{16} = 1.5$ **Ans**

2 Suppose that X is a discrete random variable with probability generating function: $G_X(\theta) = \frac{1-2\lambda}{1-\lambda\theta-\lambda\theta^2}$, where $\lambda > 0$ is a constant. Consider a branching process $\{S_n\}_{n\geq 0}$ denoting the number of individuals at stage n (notation as in lectures), with $S_0 = 1$. Assume the number of offspring X arising from any individual in any generation has the probability distribution governed by $G_X(\theta)$, and that all individuals evolve independently of all others.

(a) Write down the range of values of λ for which G_X is a valid probability generating function.

$G_X(\theta) = (1-2\lambda)(1 - \lambda\theta^2 - \lambda\theta)^{-1} = (1-2\lambda)(1 - \lambda\theta(\theta+1))^{-1}$

Expanding, we have

$G_X(\theta) = (1-2\lambda)[1 + \lambda\theta(\theta+1) + (\lambda\theta(\theta+1))^2 + (\lambda\theta(\theta+1))^3 + ...]$

It is clear from the above equation that $P(X=0)$ or the coefficient of θ^0 is $(1-2\lambda)$

For $G_X(\theta)$ to be a valid PGF, $0 < (1-2\lambda) \leq 1 \implies 0 \leq \lambda < \frac{1}{2} \implies \lambda \in [0, \frac{1}{2})$ **Ans**

(b) For $\lambda = \frac{1}{4}$, $G_X(\theta) = \frac{1-2\lambda}{1-\lambda\theta-\lambda\theta^2} = \frac{1-2\cdot\frac{1}{4}}{1-\frac{1}{4}\theta-\frac{1}{4}\theta^2} = \frac{2}{4-\theta-\theta^2}$

$\mu_X = G'_X(\theta)|_{\theta=1} = \frac{2}{(4-\theta-\theta^2)^2} \cdot (-1) \cdot (-1-2\theta) = \frac{2}{(4-\theta-\theta^2)^2} \cdot (1+2\theta)$

Putting the value of θ, $\mu_X = \frac{2}{(4-1-1^2)^2}(1+2\cdot 1) = \frac{6}{4} = 1.5$

$E[S_n] = \mu_X^n = (1.5)^n$ **Ans**

$G_{S1}(\theta) = G_X^1(\theta) = G_X(\theta) = (1-2\lambda)[1+\lambda\theta(\theta+1)+(\lambda\theta(\theta+1))^2+(\lambda\theta(\theta+1))^3+...]$

$\implies G_{S1}(\theta) = (1-2\lambda)[1+\lambda\theta(\theta+1)+\lambda^2\theta^2(\theta+1)^2+...]$

$\implies G_{S1}(\theta) = (1-2\lambda)[1+\lambda\theta^2+\lambda\theta+\lambda^2\theta^2(\theta^2+2\theta+1)+...]$

$\implies G_{S1}(\theta) = (1-2\lambda)[1+\lambda\theta^2+\lambda\theta+\lambda^2\theta^2(\theta^2+2\theta+1)+...]$

$\implies G_{S1}(\theta) = (1-2\lambda)[1+\lambda\theta^2+\lambda\theta+\lambda^2\theta^4+2\lambda^2\theta^3+\lambda^2\theta^2+...]$

Here, $P(S_1 = 2)$ is the coefficient of θ^2, which is $(1-2\lambda)(\lambda^2+\lambda) = (1-\frac{2}{4})(\frac{1}{16}+\frac{1}{4}) = 0.5 * 0.3125 = 0.15625$ **Ans**

Probability of extinction is given by the smallest root of the equation $G_X(e) = e$

$G_X(e) = e \implies \frac{1-2\lambda}{1-\lambda e-\lambda e^2} = e \implies 1-2\lambda = e\cdot(1-\lambda e-\lambda e^2) \implies 1-2\lambda = e-\lambda e^2-\lambda e^3 \implies \lambda e^3+\lambda e^2-e-2\lambda+1 = 0$

We know that (e-1) must always be a factor as the probability of extinction is upper-bounded by 1.

Factorizing,

$(e-1)(\lambda e^2+2\lambda e-1+2\lambda) = 0$

Now, either $e = 1$ or $(\lambda e^2+2\lambda e-1+2\lambda) = 0$

Using the Sridharacharya formula for solving quadratics, we have

$e = \frac{-2\lambda\pm\sqrt{4\lambda^2-4\lambda(2\lambda-1)}}{2\lambda}$

$e = \frac{-2\lambda\pm\sqrt{4\lambda^2-4\lambda^2\cdot\frac{1}{\lambda}(2\lambda-1)}}{2\lambda}$

$e = \frac{-2\lambda\pm\sqrt{4\lambda^2(1-\frac{1}{\lambda}(2\lambda-1))}}{2\lambda}$

$e = \frac{-2\lambda\pm 2\lambda\sqrt{(1-\frac{1}{\lambda}(2\lambda-1))}}{2\lambda}$

$e = -1\pm\sqrt{(1-\frac{1}{\lambda}(2\lambda-1))}$

$e = -1\pm\sqrt{(1-2+\frac{1}{\lambda})}$

$e = -1\pm\sqrt{(\frac{1}{\lambda}-1)}$

Using $\lambda = \frac{1}{4}$, we have $e = -1\pm\sqrt{3}$. Since the negative solution is nonsensical, $e = -1+\sqrt{3} = 0.732$ **Ans**

(c) Compute the range of values of λ for which the population is guaranteed to become extinct

To achieve this, we need $\mu_X < 1$

$\mu_X = G'_x(\theta)|_{\theta=1} = \frac{1-2\lambda}{(1-\lambda\theta-\lambda\theta^2)^2} \cdot (-1) \cdot (-\lambda-2\lambda\theta) < 1$

Substituting $\theta = 1$, we have $\mu_X = \frac{1-2\lambda}{(1-\lambda-\lambda)^2}(\lambda+2\lambda) = \frac{1-2\lambda}{(1-2\lambda)^2}3\lambda < 1$

$(1-2\lambda)(3\lambda) < (1-2\lambda)^2$

$(1-2\lambda)(3\lambda)-(1-2\lambda)^2 < 0$

$(1-2\lambda)(3\lambda-(1-2\lambda)) < 0$

$(1-2\lambda)(5\lambda-1) < 0$

$(2\lambda - 1)(5\lambda - 1) > 0$

$(\lambda - \frac{1}{2})(\lambda - \frac{1}{5}) > 0$

This inequality yields $\lambda \in (-\infty, \frac{1}{5}) \cap (\frac{1}{2}, \infty)$ Combining this with the earlier bounded region for λ where $(\lambda \in [0, \frac{1}{2}))$ for $G_X(\theta)$ to be a valid PGF, we have $\lambda \in [0, \frac{1}{5})$ **Ans**

(d) For ultimate extinction as a function of λ, we refer to part (b). It is $-1 + \sqrt{\frac{1}{\lambda} - 1}$ **Ans**

3 At the University of Exeter, the frequency of cars passing the Harrison Building are recorded. It is noticed that Mercedes, BMWs and Ferraris pass by at a combined rate of 8 cars per hour. Out of these cars, it is noticed that 40% are Mercedes, 50% are BMWs and the remaining 10% are Ferraris. You may assume that all passing car types are independent Poisson processes.

(a) Find the probability that the next car observed is a Mercedes.

Since all of the passing cars follow independent Poisson processes, P(next Mercedes) = 0.4 **Ans**

(b) Given that the last car observed was a BMW, find the probability that the next car to be observed passing the Harrison Building is a Ferrari.

We have independence of events in the future from events in the past. Therefore P(next car Ferrari) = 0.1 **Ans**

(c) Calculate the probability that exactly 2 BMWS pass by the Harrison Building in 30 minutes.

The combined rate of passing is known to be $\lambda_{combined} = 8/hr$. Since BMWs account for 50% of the volume, $\lambda_{BMW} = 0.5 * 8 = 4/hr$

Recall that formula for Poisson process is $P((N = t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$ Hence, $P(N(t = 1/2) = 2) = \frac{(4 \cdot \frac{1}{2})^2 \cdot e^{-2}}{2!} = 2e^{-2} = 0.2706$ **Ans**

(d) Calculate the probability that at least three cars (of any of those three types) pass by the Harrison building in 45 minutes.

$\lambda = 8$ and $t = 3/4$, which implies $\lambda t = 8 * \frac{3}{4} = 6$

$P(N(t = \frac{3}{4}) \geq 3) = 1 - P(N(t = \frac{3}{4}) < 3) = 1 - P(N(t = \frac{3}{4}) \leq 2) = 1 - \sum_{k=0}^{2} P(N(t = \frac{3}{4}) = k)$

$= 1 - \frac{(6)^2 e^{-6}}{2!} - \frac{(6)^1 e^{-6}}{1!} - \frac{(6)^0 e^{-6}}{0!} = 1 - 0.04461 - 0.0148 - 0.00247 = 0.9380312$ **Ans**

(e) Calculate the probability that the time between seeing the first Ferrari and the fifth Ferrari exceeds 2 hours.

This can be interpreted as the time taken for 4 sightings of Ferraris is above 2 hours. Another way to look at it is that in 2 hours, 3 or fewer Ferraris are observed to pass. We know that the combined rate $\lambda_{combined} = 8$. Since Ferraris account for 10% of that, we can say the rate of Ferraris passing is $\lambda_F = 8 * 0.1 = 0.8$ and $t = 2$. Therfore, $\lambda_F \cdot t = 0.8 * 2 = 1.6$

$P(S_4 > 2) = P(N(t = 2) \leq 3) = \sum_{k=0}^{3} P(N(t = 2) = k) = \sum_{k=0}^{3} \frac{(\lambda_F \cdot t)^k e^{(-\lambda_F \cdot t)}}{k!}$

$= \sum_{k=0}^{3} \frac{1.6^k e^{-1.6}}{k!} = 0.9212$ **Ans**

(f) What is the probability that it takes between 1 and 2 hours to see the 3rd BMW pass by the Harrison Building?

From (c), we know that rate of BMWs passing Harrison building is $4/hr$. We need to calculate $P(1 \leq S_3 \leq 2)$ where S_3 is the time taken to see the 3rd BMW pass.

$P(1 \leq S_3 \leq 2) = P(S_3 \leq 2) - P(S_3 \leq 1) = P(N(t = 2) \geq 3) - P(N(t = 1) \geq 3)$

$= 1 - P(N(t = 2) < 3) - [1 - P(N(t = 1) < 3)] = P(N(t = 1) < 3) - P(N(t = 2) < 3)$

$= P(N(t = 1) \leq 2) - P(N(t = 2) \leq 2)$

$= \sum_{k=0}^{2} P(N(t=1)=k) - P(N(t=2)=k)$

$= \sum_{k=0}^{2} \frac{(4*1)^k e^{-4*1}}{k!} - \frac{(4*2)^k e^{-4*2}}{k!}$

$= \sum_{k=0}^{2} \frac{(4*1)^k e^{-4}}{k!}(1 - 2^k e^{-4})$

$= \frac{4^0 e^{-4}}{0!}(1 - 2^0 e^{-4}) + \frac{4^1 e^{-4}}{1!}(1 - 2^1 e^{-4}) + \frac{4^2 e^{-4}}{2!}(1 - 2^2 e^{-4}) = 0.2243$ **Ans**

```
k <- c(0, 1, 2)
sum(4^k/factorial(k) * exp(-4) * (1 - 2^k * exp(-4)))
```

```
## [1] 0.2243493
```

4 Consider the random variable X with a zero-truncated Poisson distribution given by

$P(X=0)=0$, and $P(X=k) = c_\lambda \frac{\lambda^k e^{-\lambda}}{k!}, \ \lambda > 0, k \geq 1$

(a) Find the constant c_λ in terms of λ

For this to be a valid probability distribution, $\sum_{k=0}^{\infty} P(X=k) = 1$

$\implies P(X=0) + \sum_{k=1}^{\infty} P(X=k) = 1$

$\implies 0 + \sum_{k=1}^{\infty} P(X=k) = 1$

$\implies \sum_{k=1}^{\infty} c_\lambda \frac{\lambda^k e^{-\lambda}}{k!} = 1$

$\frac{c_\lambda}{e^\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} = 1$

$\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} = \frac{e^\lambda}{c_\lambda}$

$1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} = 1 + \frac{e^\lambda}{c_\lambda}$

$e^\lambda = \frac{e^\lambda}{c_\lambda} + 1$

$\frac{e^\lambda - 1}{e^\lambda} = \frac{1}{c_\lambda}$

$c_\lambda = \frac{e^\lambda}{e^\lambda - 1}$

(b) Derive the probability generating function for X and use it to find E[X] and Var(X)

$G_X(\theta) = \sum_{k=0}^{\infty} \theta^k P(X=k) = \theta^0 P(X=0) + \sum_{k=1}^{\infty} \theta^k P(X=k)$

$\implies G_X(\theta) = \sum_{k=1}^{\infty} \theta^k P(X=k) = \sum_{k=1}^{\infty} \theta^k \frac{c_\lambda \lambda^k e^{-\lambda}}{k!}$

$\implies G_X(\theta) = c_\lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{(\lambda\theta)^k}{k!} = c_\lambda e^{-\lambda} [\sum_{k=0}^{\infty} \frac{(\lambda\theta)^k}{k!} - 1] = c_\lambda e^{-\lambda}(e^{\lambda\theta} - 1)$

$\implies G_X(\theta) = \frac{e^\lambda}{e^\lambda - 1} e^{-\lambda}(e^{\lambda\theta} - 1) = \frac{e^{\lambda\theta} - 1}{e^\lambda - 1}$ **Ans**

$E[X] = G'_X(\theta)|_{\theta=1} = \frac{1}{e^\lambda - 1} \lambda e^{\lambda\theta}|_{\theta=1} = \frac{\lambda e^{\lambda\theta}}{e^\lambda - 1}|_{\theta=1} = \frac{\lambda e^\lambda}{e^\lambda - 1}$

For variance, $Var[X] = E[X^2] - (E[X])^2$

We know that $G''_X(\theta)|_{\theta=1} = E[X(X-1)] = E[X^2] - E[X]$

$\implies E[X^2] - E[X] = \frac{d^2}{d\theta^2}(G_X(\theta))|_{\theta=1} = \frac{d}{d\theta}(\frac{\lambda e^{\lambda\theta}}{e^\lambda - 1}) = \frac{\lambda^2 e^{\lambda\theta}}{e^\lambda - 1} = \frac{\lambda^2 e^\lambda}{e^\lambda - 1}$

$\implies E[X^2] = \frac{\lambda^2 e^\lambda}{e^\lambda - 1} + E[X]$

$\implies E[X^2] = \frac{\lambda^2 e^\lambda}{e^\lambda - 1} + \frac{\lambda e^\lambda}{e^\lambda - 1}$

$\implies Var[X] = E[X^2] - (E[X])^2 = \frac{\lambda e^\lambda}{e^\lambda - 1}(\lambda + 1) - (\frac{\lambda e^\lambda}{e^\lambda - 1})^2$

$\implies Var[X] = \frac{\lambda e^\lambda}{e^\lambda - 1}(\lambda + 1 - \frac{\lambda e^\lambda}{e^\lambda - 1})$

$\implies Var[X] = \frac{\lambda e^\lambda}{(e^\lambda - 1)^2}(e^\lambda - (\lambda + 1))$ **Ans**

(c) Suppose that X and Y are two independent random variables each having a zero-truncated Poisson distribution with parameters λ and μ, respectively. Find the probability generating function for Z where Z = X + Y . Hence show that for $k \geq 2$: $P(Z = k) = C(\lambda, \mu, k)[(\lambda + \mu)^k - \lambda^k - \mu^k]$

$G_X(\theta) = c_\lambda e^{-\lambda}[e^{\lambda\theta} - 1]$

$G_Y(\theta) = c_\mu e^{-\mu}[e^{\mu\theta} - 1]$

$G_Z(\theta) = G_X(\theta)G_Y(\theta) = \frac{c_\lambda c_\mu}{e^{(\lambda+\mu)}}[e^{\lambda\theta} - 1][e^{\mu\theta} - 1]$

$\implies G_Z(\theta) = \frac{e^\lambda}{e^\lambda - 1}\frac{1}{e^\lambda}\frac{e^\mu}{e^\mu - 1}\frac{1}{e^\mu}[e^{\lambda\theta} - 1][e^{\mu\theta} - 1]$

$\implies G_Z(\theta) = \frac{1}{e^\lambda - 1}\frac{1}{e^\mu - 1}[e^{\lambda\theta} - 1][e^{\mu\theta} - 1]$

$\implies G_Z(\theta) = \frac{e^{(\lambda+\mu)\theta} - e^{\lambda\theta} - e^{\mu\theta} + 1}{(e^\lambda - 1)(e^\mu - 1)}$

$\implies G_Z(\theta) = \frac{1}{(e^\lambda - 1)(e^\mu - 1)}[1 + \frac{(\lambda+\mu)\theta}{1!} + \frac{(\lambda+\mu)^2\theta^2}{2!}\frac{(\lambda+\mu)^3\theta^3}{3!} + ... - (1 + \frac{(\lambda)\theta}{1!} + \frac{(\lambda)^2\theta^2}{2!}\frac{(\lambda)^3\theta^3}{3!} + ...) - (1 + \frac{(\mu)\theta}{1!} + \frac{(\mu)^2\theta^2}{2!}\frac{(\mu)^3\theta^3}{3!} + ...) + 1]$

$\implies G_Z(\theta) = \frac{1}{(e^\lambda - 1)(e^\mu - 1)}[\theta\frac{(\lambda+\mu) - \lambda - \mu}{1!} + \theta^2\frac{(\lambda+\mu)^2 - \lambda^2 - \mu^2}{2!} + \theta^3\frac{(\lambda+\mu)^3 - \lambda^3 - \mu^3}{3!} + ...]$

$\implies G_Z(\theta) = \frac{1}{(e^\lambda - 1)(e^\mu - 1)}[\theta^2\frac{(\lambda+\mu)^2 - \lambda^2 - \mu^2}{2!} + \theta^3\frac{(\lambda+\mu)^3 - \lambda^3 - \mu^3}{3!} + ...]$

Following the above pattern, we can say that $P(Z = k)$ is the coefficient of θ^k. Therefore,

$P(Z = k) = \frac{1}{k!(e^\lambda - 1)(e^\mu - 1)}[(\lambda + \mu)^k - \lambda^k - \mu^k]$

This can be written as required in the question.

$P(Z = k) = C(\lambda, \mu, k)[(\lambda + \mu)^k - \lambda^k - \mu^k]$, where $C(\lambda, \mu, k) = \frac{1}{k!(e^\lambda - 1)(e^\mu - 1)}$ **Ans**