ECMM450 Stochastic Processes: Assignment 1

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1 Consider the discrete random variable X that has the probability generating function

$$G_X(\theta) = 2(3-\theta)^{-1}$$

(a) Suppose that $Y = X^2$. Calculate $P(Y = k)$ for $0 \leq k \leq 10$ and the expectaion $E[Y]$

$G_X(\theta) = 2(3-\theta)^{-1}$
$= \frac{2}{3}(1 - \frac{\theta}{3})^{-1}$
$= \frac{2}{3}(1 + \frac{\theta}{3} + (\frac{\theta}{3})^2 + (\frac{\theta}{3})^3 + ...)$
$= \frac{2}{3}\theta^0 + \frac{2}{3^2}\theta^1 + \frac{2}{3^3}\theta^2 + \frac{2}{3^4}\theta^3 + ...$
We observe from the above equation that $P(X = 0) = \frac{2}{3}, P(X = 1) = \frac{2}{3^2}, P(X = 2) = \frac{2}{3^3}$ ....

$G_Y(\theta) = \sum_k^\infty \theta^k P(Y = k) = \sum_k^\infty \theta^k P(X^2 = k)$

$= \theta^0 P(X^2 = 0) + \theta^1 P(X^2 = 1) + ... + \theta^4 P(X^2 = 4) + ... + \theta^9 P(X^2 = 9) + \theta^{10} P(X^2 = 10)$
As X can only take on positive integer values, the only terms in the above equation which are non-zero are those associated where $X^2 \in \{0, 1, 4, 9\}$
Therefore, $G_Y(\theta) = \theta^0 P(X^2 = 0) + \theta^1 P(X^2 = 1) + \theta^4 P(X^2 = 4) + \theta^9 P(X^2 = 9)$
$G_Y(\theta) = \theta^0 P(X = 0) + \theta^1 P(X = 1) + \theta^4 P(X = 2) + \theta^9 P(X = 3)$

$G_Y(\theta) = \frac{2}{3}\theta^0 + \frac{2}{3^2}\theta^1 + \frac{2}{3^3}\theta^4 + \frac{2}{3^4}\theta^9$ **Ans**

Now, to calculate $E[Y]$, we evaluate $G'_Y(\theta)|_{\theta=1} = [\frac{2}{3^2} + 4\theta^3\frac{2}{3^3} + 9\theta^8\frac{2}{3^4}]|_{\theta=1}$
$= \frac{2}{9} + \frac{4 \cdot 2}{27} + \frac{9 \cdot 2}{81} = 0.7407$ **Ans**

(b) Suppose that Z is the random variable with probability generating function $G_Z(\theta) = G_X(G_X(\theta))$. Calculate $P(Z = 0), P(Z = 2)$, and $E[Z]$.

$$G_Z(\theta) = G_X(2(3-\theta)^{-1}) = 2(3 - \frac{2}{3-\theta})^{-1} = \frac{2(3-\theta)}{3(3-\theta)-2} = \frac{2(3-\theta)}{7-3\theta}$$

$$\implies G_Z(\theta) = \frac{2}{3}\frac{9-3\theta}{7-3\theta} = \frac{2}{3}(1 + \frac{2}{7-3\theta}) = \frac{2}{3} + \frac{2}{3} \cdot 2 \cdot (7-3\theta)^{-1}$$

$$\implies G_Z(\theta) = \frac{2}{3} + \frac{4}{3} \cdot (7(1 - \frac{3}{7}\theta))^{-1} = \frac{2}{3} + \frac{4}{3} \cdot \frac{1}{7}(1 - \frac{3}{7}\theta)^{-1}$$

$$\implies G_Z(\theta) = \frac{2}{3} + \frac{4}{21} \cdot (1 + \frac{3}{7}\theta + (\frac{3}{7}\theta)^2 + (\frac{3}{7}\theta)^3 + ...)$$

$$\implies G_Z(\theta) = \frac{2}{3} + \frac{4}{21} + \frac{4}{21} \cdot (\frac{3}{7}\theta + (\frac{3}{7}\theta)^2 + (\frac{3}{7}\theta)^3 + ...)$$

$\implies G_Z(\theta) = \frac{18}{21} + \frac{4}{21} \cdot (\frac{3}{7}\theta + (\frac{3}{7}\theta)^2 + (\frac{3}{7}\theta)^3 + ...)$

From the above equation, we see that $P(Z=0) = \frac{18}{21} = 0.8571$ **Ans**, $P(Z=2) = \frac{4}{21} \cdot \frac{3^2}{7^2} = 0.0349$ **Ans**

Since $G_Z(\theta) = G_X^2(\theta)$, we know that $E[Z] = \mu_X^2$, where $\mu_X = E[X]$.
To find this, we evaluate $G'_X(\theta)|_{\theta=1} = \frac{d}{d\theta}(2(3-\theta)^{-1}) = 2\frac{d}{d\theta}\frac{1}{3-\theta} = 2 \cdot (-1) \cdot \frac{1}{(3-\theta)^2} \cdot (-1) = \frac{2}{(3-\theta)^2}$

Putting the value of $\theta = 1$, we have, $E[X] = \frac{2}{(3-1)^2} = \frac{2}{2^2} = 0.5$ Therefore, $E[Z] = 0.5^2 = 0.25$ **Ans**

(c) $W = X_1 + X_2 + X_3$, where X_1, X_2, X_3 are independent

Therefore, $G_W(\theta) = \{2(3-\theta)^{-1}\}^3 = 2^3(3-\theta)^{-3} = 8(3-\theta)^{-3} = \frac{8}{27}(1-\frac{\theta}{3})^{-3}$

Using binomial expansion for negative coefficients, we have
$G_W(\theta) = \frac{8}{27}(1 + (-3)(-\frac{\theta}{3}) + \frac{(-3)(-3-1)}{2!}\frac{\theta^2}{3^2} + ...)$

$G_W(\theta) = \frac{8}{27}(1 + \theta + \frac{2}{3}\theta^2 + ...)$

From the above equation, it can be observed that
$P(W=0) = \frac{8}{27} = 0.2962$ **Ans** , $P(W=2) = \frac{8*2}{27*3} = 0.1975$ **Ans**

For $E[W]$, we have $G'_W(\theta)|_{\theta=1} = 2^3(-3)(3-\theta)^{-4}(-1)|_{\theta=1} = \frac{24}{(3-1)^4} = \frac{24}{16} = 1.5$ **Ans**

2 Suppose that X is a discrete random variable with probability generating function: $G_X(\theta) = \frac{1-2\lambda}{1-\lambda\theta-\lambda\theta^2}$, where $\lambda > 0$ is a constant. Consider a branching process $\{S_n\}_{n\geq 0}$ denoting the number of individuals at stage n (notation as in lectures), with $S_0 = 1$. Assume the number of offspring X arising from any individual in any generation has the probability distribution governed by $G_X(\theta)$, and that all individuals evolve independently of all others.

(a) Write down the range of values of λ for which G_X is a valid probability generating function.

$G_X(\theta) = (1-2\lambda)(1-\lambda\theta^2-\lambda\theta)^{-1} = (1-2\lambda)(1-\lambda\theta(\theta+1))^{-1}$

Expanding, we have

$G_X(\theta) = (1-2\lambda)[1 + \lambda\theta(\theta+1) + (\lambda\theta(\theta+1))^2 + (\lambda\theta(\theta+1))^3 + ...]$

It is clear from the above equation that $P(X=0)$ or the coefficient of θ^0 is $(1-2\lambda)$

For $G_X(\theta)$ to be a valid PGF, $0 \leq (1-2\lambda) \leq 1 \implies 0 < \lambda \leq \frac{1}{2} \implies \lambda \in [0, \frac{1}{2})$ **Ans**

(b) For $\lambda = \frac{1}{4}$, $G_X(\theta) = \frac{1-2\lambda}{1-\lambda\theta-\lambda\theta^2} = \frac{1-2\cdot\frac{1}{4}}{1-\frac{1}{4}\theta-\frac{1}{4}\theta^2} = \frac{2}{4-\theta-\theta^2}$

$\mu_X = G'_X(\theta)|_{\theta=1} = \frac{2}{(4-\theta-\theta^2)^2} \cdot (-1) \cdot (-1-2\theta) = \frac{2}{(4-\theta-\theta^2)^2} \cdot (1+2\theta)$

Putting the value of θ, $\mu_X = \frac{2}{(4-1-1^2)^2}(1+2\cdot 1) = \frac{6}{4} = 1.5$

$E[S_n] = \mu_X^n = (1.5)^n$ **Ans**

$G_{S1}(\theta) = G_X^1(\theta) = G_X(\theta) = (1-2\lambda)[1 + \lambda\theta(\theta+1) + (\lambda\theta(\theta+1))^2 + (\lambda\theta(\theta+1))^3 + ...]$

$\implies G_{S1}(\theta) = (1-2\lambda)[1 + \lambda\theta(\theta+1) + \lambda^2\theta^2(\theta+1)^2 + ...]$

$\implies G_{S1}(\theta) = (1-2\lambda)[1 + \lambda\theta^2 + \lambda\theta + \lambda^2\theta^2(\theta^2 + 2\theta + 1) + ...]$

$\implies G_{S1}(\theta) = (1-2\lambda)[1 + \lambda\theta^2 + \lambda\theta + \lambda^2\theta^2(\theta^2 + 2\theta + 1) + ...]$

$\implies G_{S1}(\theta) = (1-2\lambda)[1 + \lambda\theta^2 + \lambda\theta + \lambda^2\theta^4 + 2\lambda^2\theta^3 + \lambda^2\theta^2 + ...]$

Here, $P(S_1 = 2)$ is the coefficient of θ^2, which is $(1-2\lambda)(\lambda^2+\lambda) = (1-\frac{2}{4})(\frac{1}{16}+\frac{1}{4}) = 0.5 * 0.3125 = 0.15625$ **Ans**

Probability of extinction is given by the smallest root of the equation $G_X(e) = e$

$G_X(e) = e \implies \frac{1-2\lambda}{1-\lambda e-\lambda e^2} = e$ $1-2\lambda = e\cdot(1-\lambda e-\lambda e^2)$ $1-2\lambda = e-\lambda e^2-\lambda e^3$ $\lambda e^3+\lambda e^2-e-2\lambda+1=0$

We know that (e-1) must always be a factor as the probability of extinction is upper-bounded by 1.

Factorizing,

$(e-1)(\lambda e^2+2\lambda e-1+2\lambda)=0$

Now, either $e=1$ or $(\lambda e^2+2\lambda e-1+2\lambda)=0$

Using the Sridharacharya formula for solving quadratics, we have

$e = \frac{-2\lambda\pm\sqrt{4\lambda^2-4\lambda(2\lambda-1)}}{2\lambda}$

$e = \frac{-2\lambda\pm\sqrt{4\lambda^2-4\lambda^2\cdot\frac{1}{\lambda}(2\lambda-1)}}{2\lambda}$

$e = \frac{-2\lambda\pm\sqrt{4\lambda^2(1-\frac{1}{\lambda}(2\lambda-1))}}{2\lambda}$

$e = \frac{-2\lambda\pm 2\lambda\sqrt{(1-\frac{1}{\lambda}(2\lambda-1))}}{2\lambda}$

$e = -1\pm\sqrt{(1-\frac{1}{\lambda}(2\lambda-1))}$

$e = -1\pm\sqrt{(1-2+\frac{1}{\lambda})}$

$e = -1\pm\sqrt{(\frac{1}{\lambda}-1)}$

Using $\lambda = \frac{1}{4}$, we have $e = -1\pm\sqrt{3}$. Since the negative solution is nonsensical, $e = -1+\sqrt{3} = 0.732$ **Ans**

(c) Compute the range of values of λ for which the population is guaranteed to become extinct

To achieve this, we need $\mu_X < 1$

$\mu_X = G'_x(\theta)|_{\theta=1} = \frac{1-2\lambda}{(1-\lambda\theta-\lambda\theta^2)^2}\cdot(-1)\cdot(-\lambda-2\lambda\theta) < 1$

Substituting $\theta = 1$, we have $\mu_X = \frac{1-2\lambda}{(1-\lambda-\lambda)^2}(\lambda+2\lambda) = \frac{1-2\lambda}{(1-2\lambda)^2}3\lambda < 1$

$(1-2\lambda)(3\lambda) < (1-2\lambda)^2$

$(1-2\lambda)(3\lambda)-(1-2\lambda)^2 < 0$

$(1-2\lambda)(3\lambda-(1-2\lambda)) < 0$

$(1-2\lambda)(5\lambda-1) < 0$

$(2\lambda-1)(5\lambda-1) > 0$

$(\lambda-\frac{1}{2})(\lambda-\frac{1}{5}) > 0$

This inequality yields $\lambda\in(-\infty,\frac{1}{5})\cap(\frac{1}{2},\infty)$ Combining this with the earlier bounded region for λ where $(\lambda\in[0,\frac{1}{2}))$ for $G_X(\theta)$ to be a valid PGF, we have $\lambda\in[0,\frac{1}{5})$ **Ans**

(d) For ultimate extinction as a function of λ, we refer to part (b). It is $-1+\sqrt{\frac{1}{\lambda}-1}$ **Ans**

3 At the University of Exeter, the frequency of cars passing the Harrison Building are recorded. It is noticed that Mercedes, BMWs and Ferraris pass by at a combined rate of 8 cars per hour. Out of these cars, it is noticed that 40% are Mercedes, 50% are BMWs and the remaining 10% are Ferraris. You may assume that all passing car types are independent Poisson processes.

(a) Find the probability that the next car observed is a Mercedes.

Since all of the passing cars follow independent Poisson processes, P(next Mercedes) = 0.4 **Ans**

(b) Given that the last car observed was a BMW, find the probability that the next car to be observed passing the Harrison Building is a Ferrari.

We have independence of events in the future from events in the past. Therefore P(next car Ferrari) = 0.1 **Ans**

(c) Calculate the probability that exactly 2 BMWS pass by the Harrison Building in 30 minutes.

The combined rate of passing is known to be $\lambda_{combined} = 8/hr$. Since BMWs account for 50% of the volume, $\lambda_{BMW} = 0.5 * 8 = 4/hr$

Recall that formula for Poisson process is $P((N = t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$ Hence, $P(N(t = 1/2) = 2) = \frac{(4 \cdot \frac{1}{2})^2 \cdot e^{-2}}{2!} = 2e^{-2} = 0.2706$ **Ans**

(d) Calculate the probability that at least three cars (of any of those three types) pass by the Harrison building in 45 minutes.

$P(N(t = \frac{3}{4}) \geq 3) = 1 - P(N(t = \frac{3}{4}) < 3) = 1 - P(N(t = \frac{3}{4}) \leq 2) = 1 - \sum_{k=0}^{2} P(N(t = \frac{3}{4}) = k)$