Probability Intro

By

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What do I mean?

Your odds of coming late to office is 40 %

There is 50% chance of rains tomorrow evening

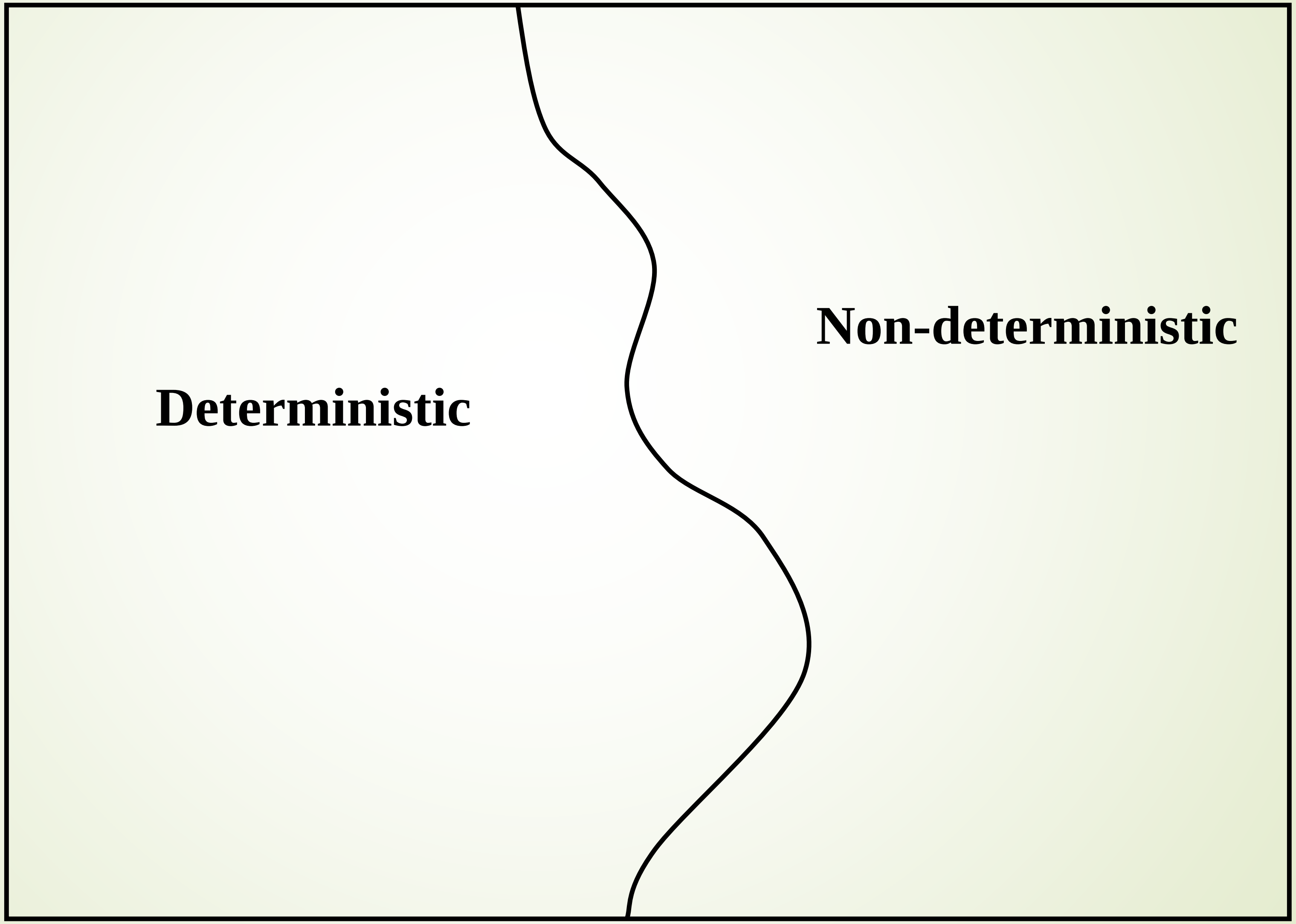
There in .33 probability of you failing

No one can be 100% sure about outcome of a lottery

The fair coin has a 50-50 % chance of either heads or tails

Team India has a solid chance of winning against Australia

Phenomena



Deterministic Phenomena

- There exists a mathematical model that allows "***perfect***" prediction the phenomena's outcome.
- Many examples exist in Physics, Chemistry (the exact sciences)

Non-deterministic Phenomena

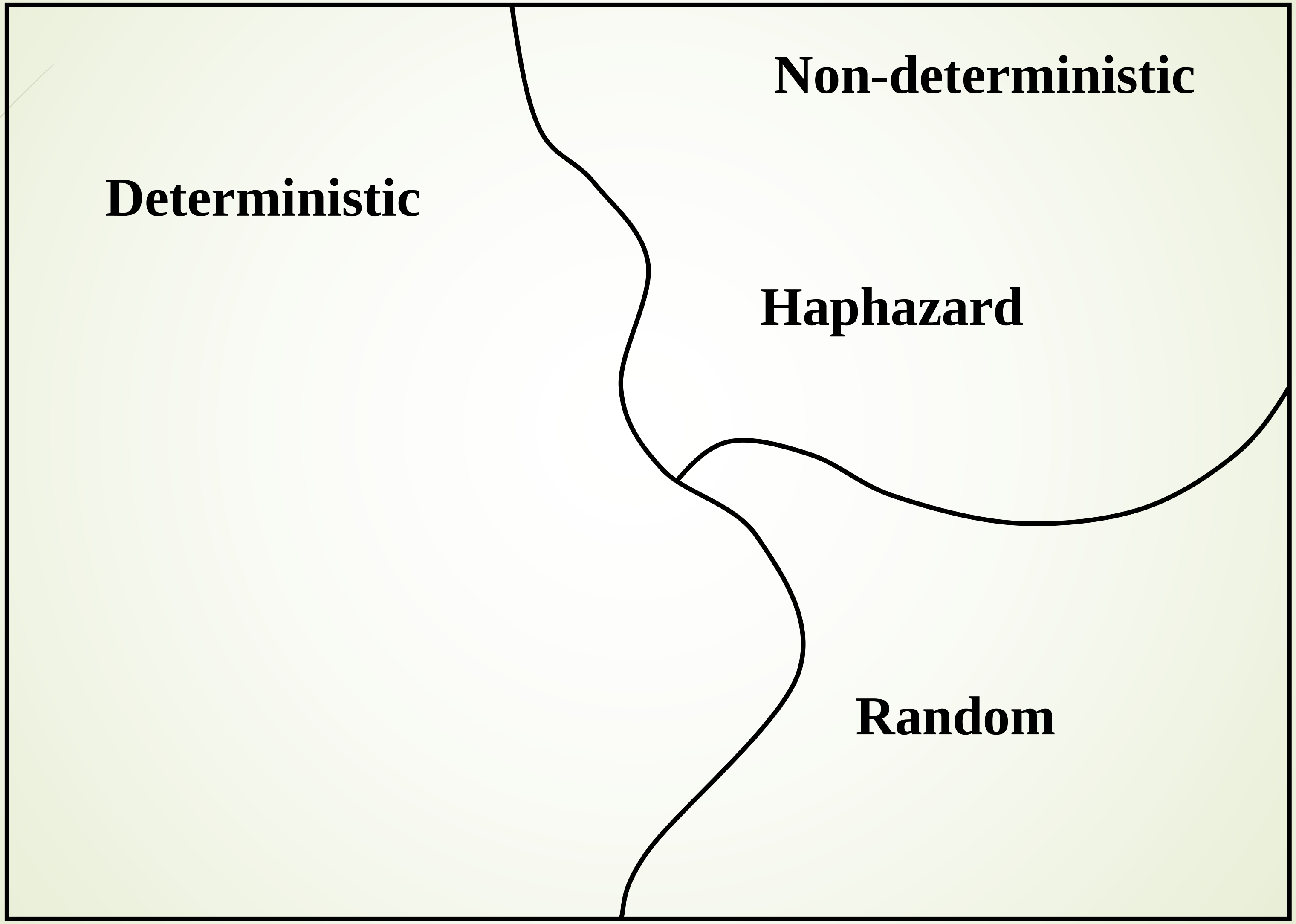
- **No** mathematical model exists that allows "***perfect***" prediction the phenomena's outcome.

Non-deterministic Phenomena

1. **Random (Stochastic) phenomena**
   - Unable to predict the outcomes, but in the long-run, the outcomes exhibit statistical regularity.

2. **Haphazard (Completely Random) phenomena**
   - unpredictable outcomes, but no long-run, exhibition of statistical regularity in the outcomes.

Phenomena



Stochastic phenomena

- Unable to predict the outcomes, but in the long-run, the outcomes exhibit statistical regularity.

Examples

1. Tossing a fair coin – outcomes S ={**Head, Tail**}

Unable to predict on each toss whether is Head or Tail.

In the long run can predict that 50% of the time heads will occur and 50% of the time tails will occur

2. Rolling a die – outcomes

$S = \{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\}$

Unable to predict outcome but in the long run can one can determine that each outcome will occur 1/6 of the time.

Use symmetry. Each side is the same. One side should not occur more frequently than another side in the long run. If the die is not balanced this may not be true.

Definitions

The sample Space, *S*

The **sample space**, *S*, for a random phenomena is the set of all possible outcomes.

Examples

1. Tossing a coin – outcomes S ={**Head, Tail**}

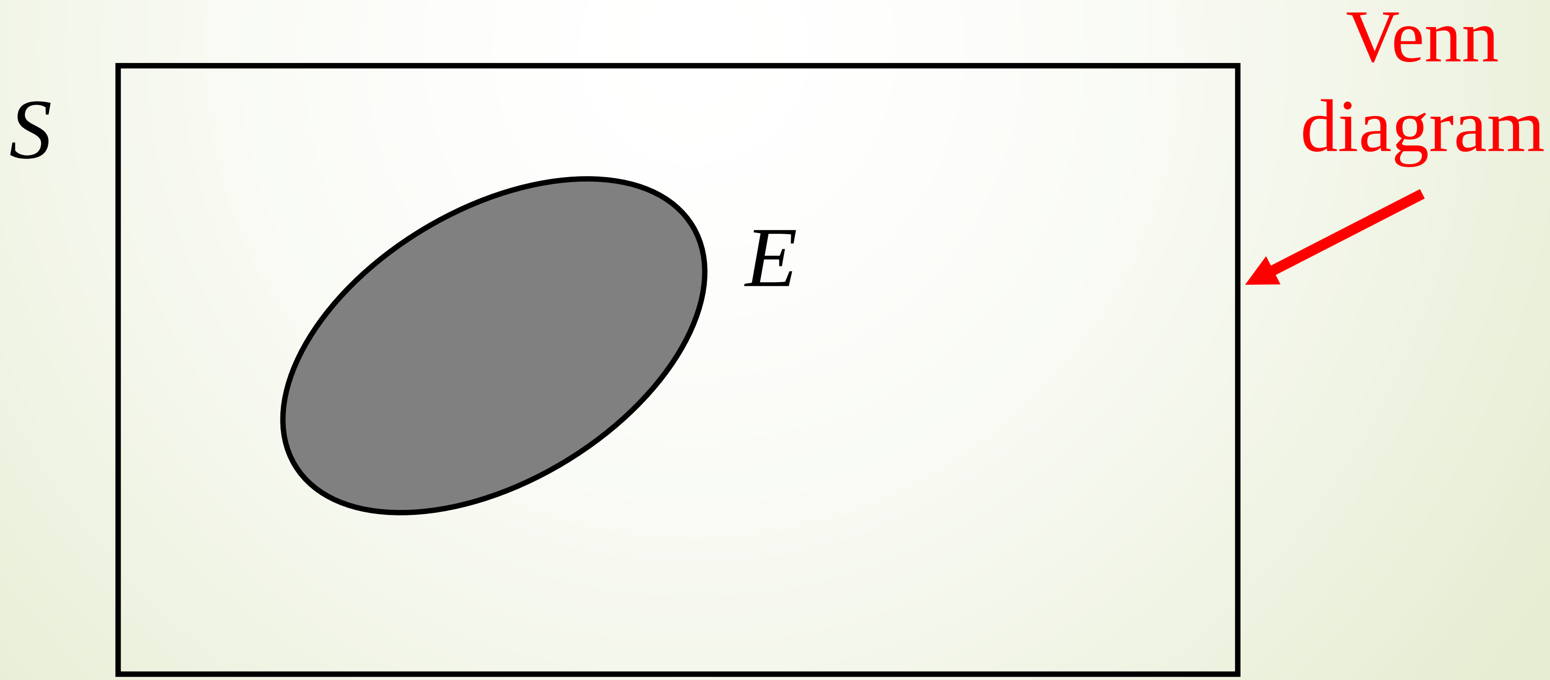
2. Rolling a die – outcomes

   S ={⚀, ⚁, ⚂, ⚃, ⚄, ⚅}

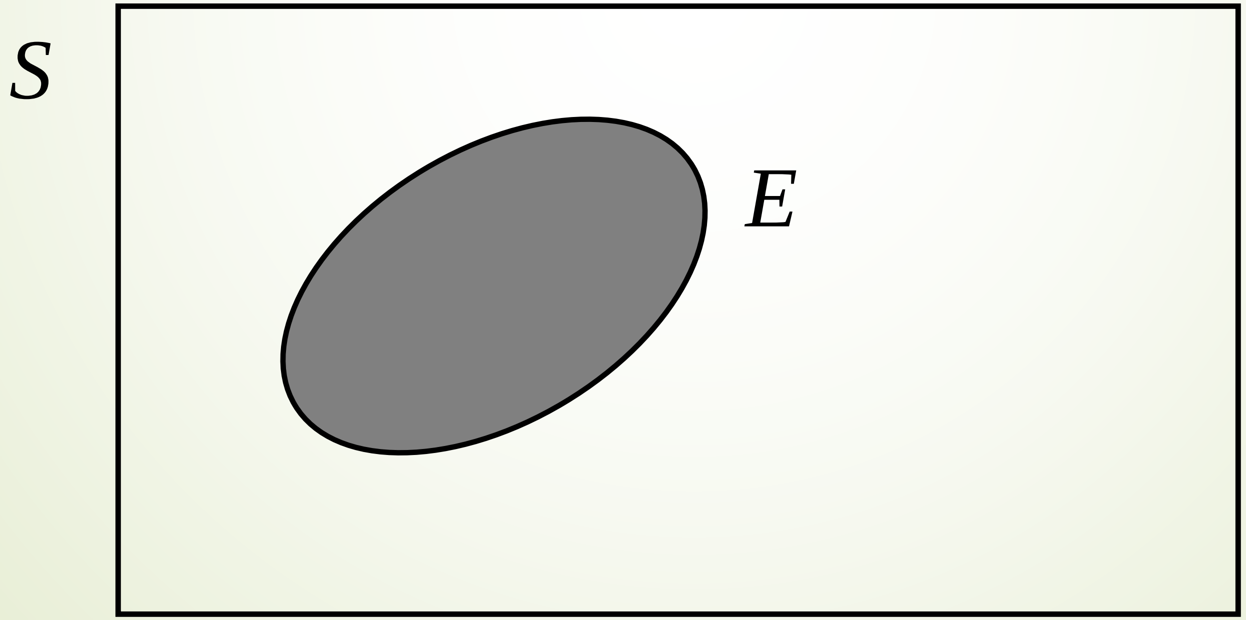
   ={1, 2, 3, 4, 5, 6}

Event , *E*

The **event**, *E*, is any subset of the **sample space**, *S*. i.e. any set of outcomes (not necessarily all outcomes) of the random phenomena

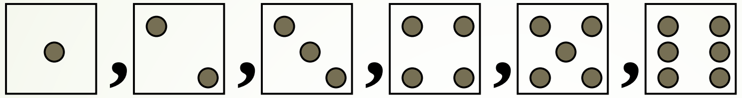


The **event**, *E*, is said to **have occurred** if after the outcome has been observed the outcome lies in *E*.



Examples

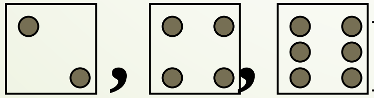
1. Rolling a die – outcomes

   $S = \{$  $\}$

   $= \{1, 2, 3, 4, 5, 6\}$

   E = the event that an even number is rolled

   $= \{2, 4, 6\}$

   $= \{$  $\}$

Special Events

The Null Event, The empty event - ϕ

$$\phi = \{\ \} = \text{the event that contains no outcomes}$$

The Entire Event, The Sample Space - S

$$S = \text{the event that contains all outcomes}$$

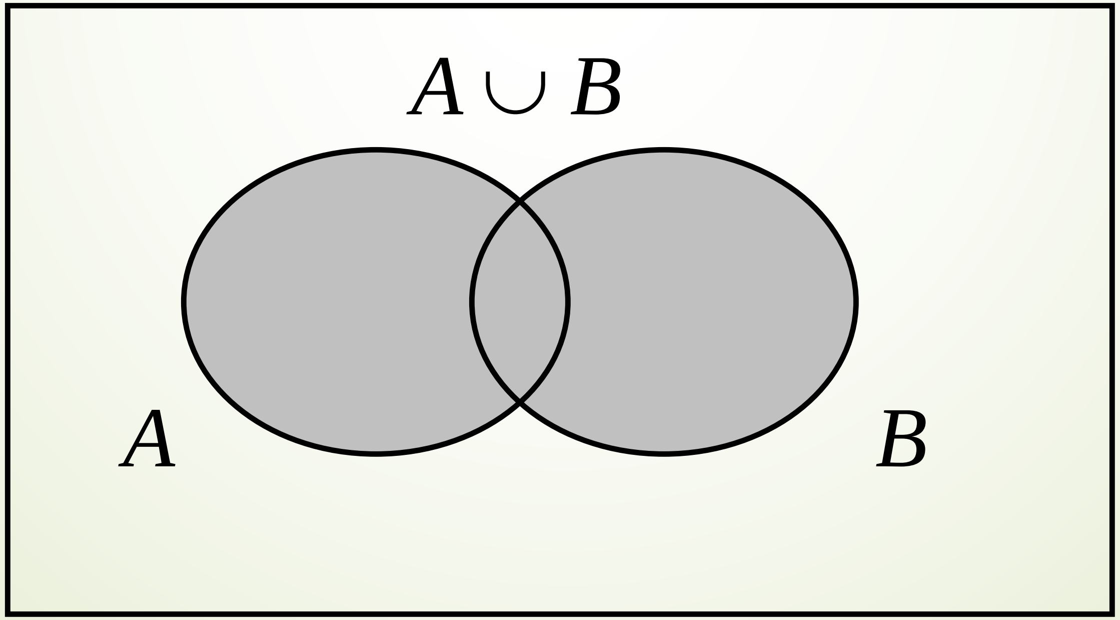
The empty event, ϕ , never occurs.

The entire event, S, always occurs.

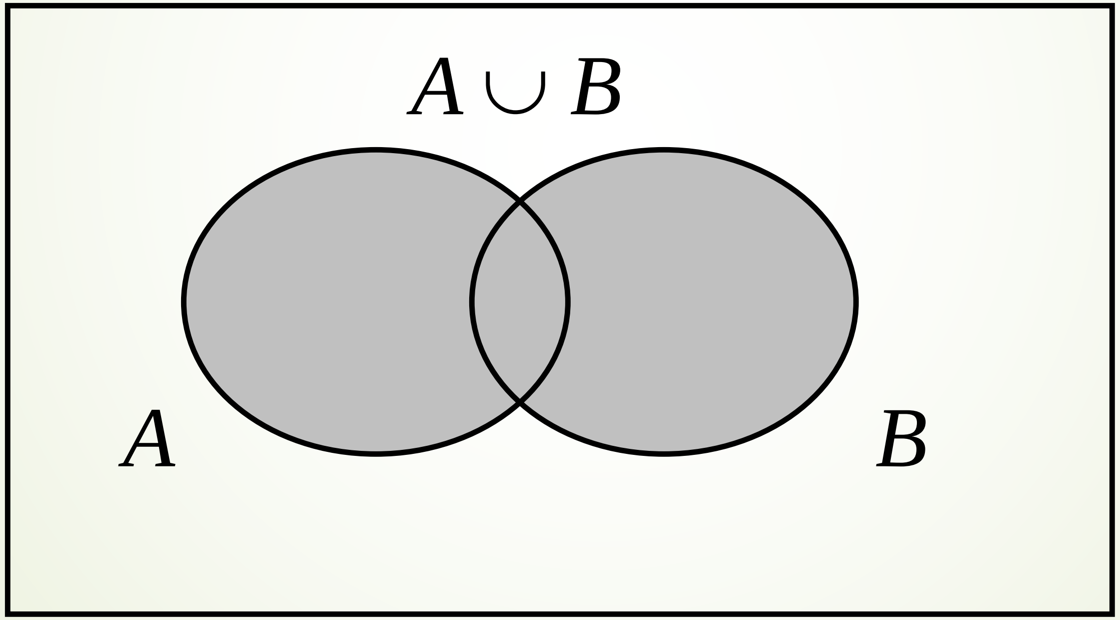
Union

Let A and B be two events, then the **union** of A and B is the event (denoted by $A \cup B$) defined by:

$A \cup B = \{e|\ e$ belongs to A **or** e belongs to $B\}$



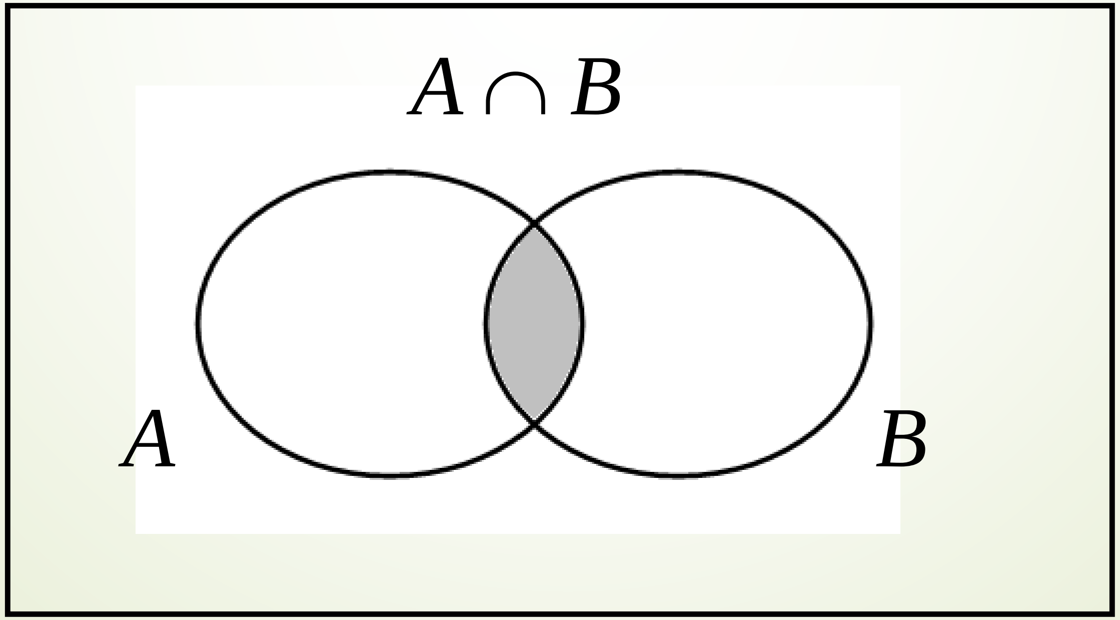
The event $A \cup B$ **occurs** if the event A **occurs or** the event and B **occurs** .



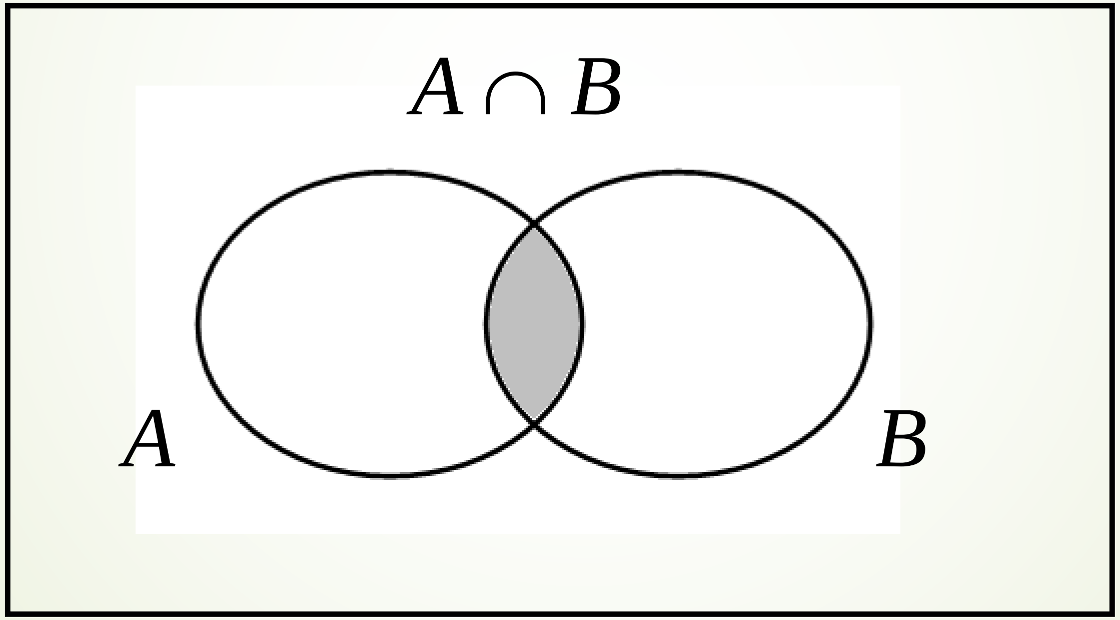
Intersection

Let A and B be two events, then the **intersection** of A and B is the event (denoted by $A \cap B$) defined by:

$$A \cap B = \{e \mid e \text{ belongs to } A \textbf{ and } e \text{ belongs to } B\}$$



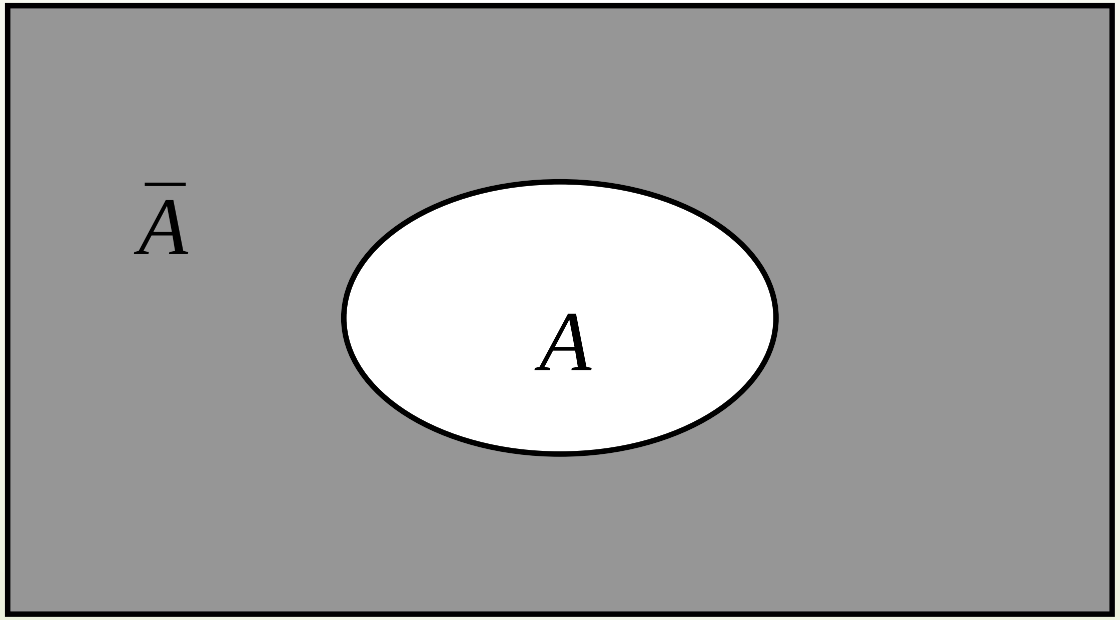
The event $A \cap B$ **occurs** if the event A **occurs and** the event and B **occurs** .



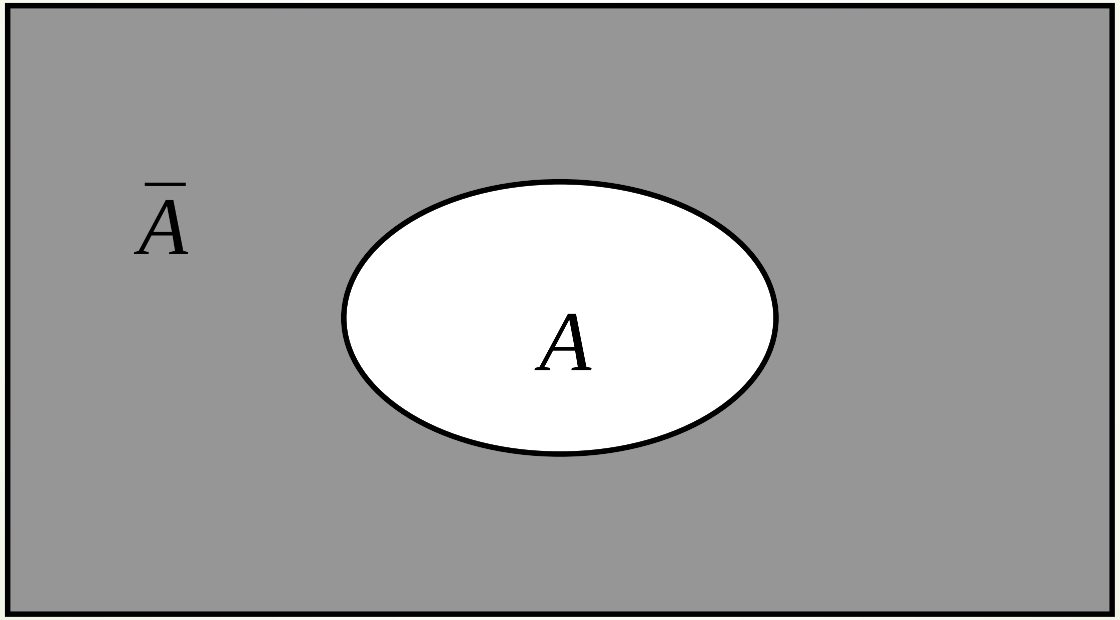
Complement

Let A be any event, then the **complement** of A (denoted by $\bar{A}$) defined by:

$$\bar{A} = \{e |\ e \textbf{ does not} \text{ belongs to } A\}$$



The event $\bar{A}$ **occurs** if the event A **does not occur**



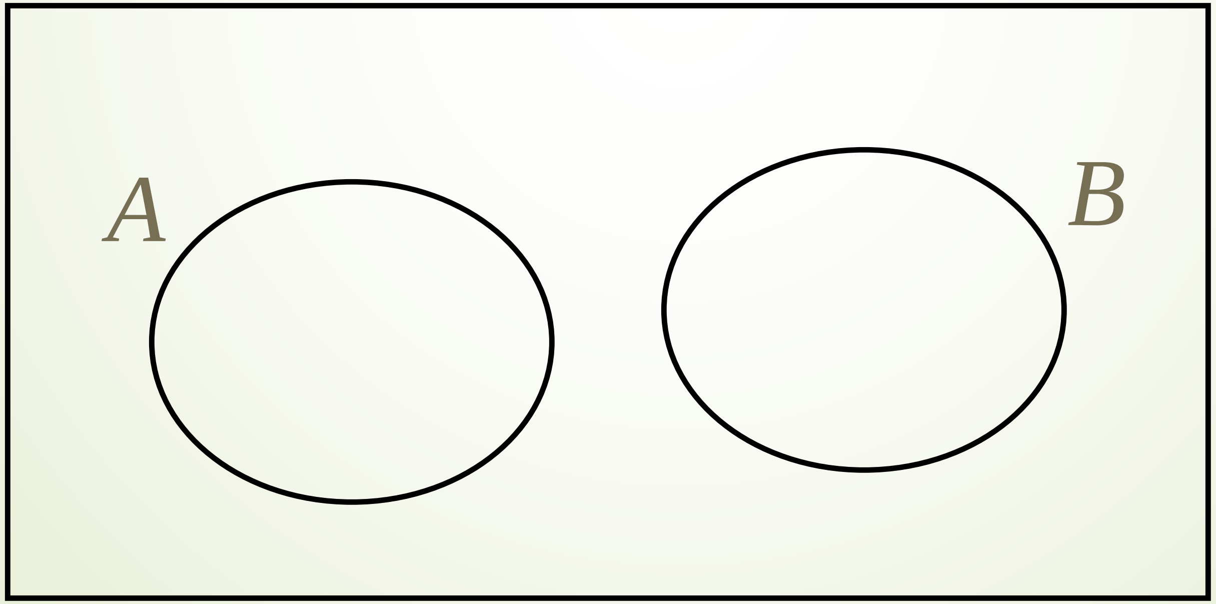
In problems you will recognize that you are working with:

1. **Union** if you see the word **or**,
2. **Intersection** if you see the word **and**,
3. **Complement** if you see the word **not**.

Definition: mutually exclusive

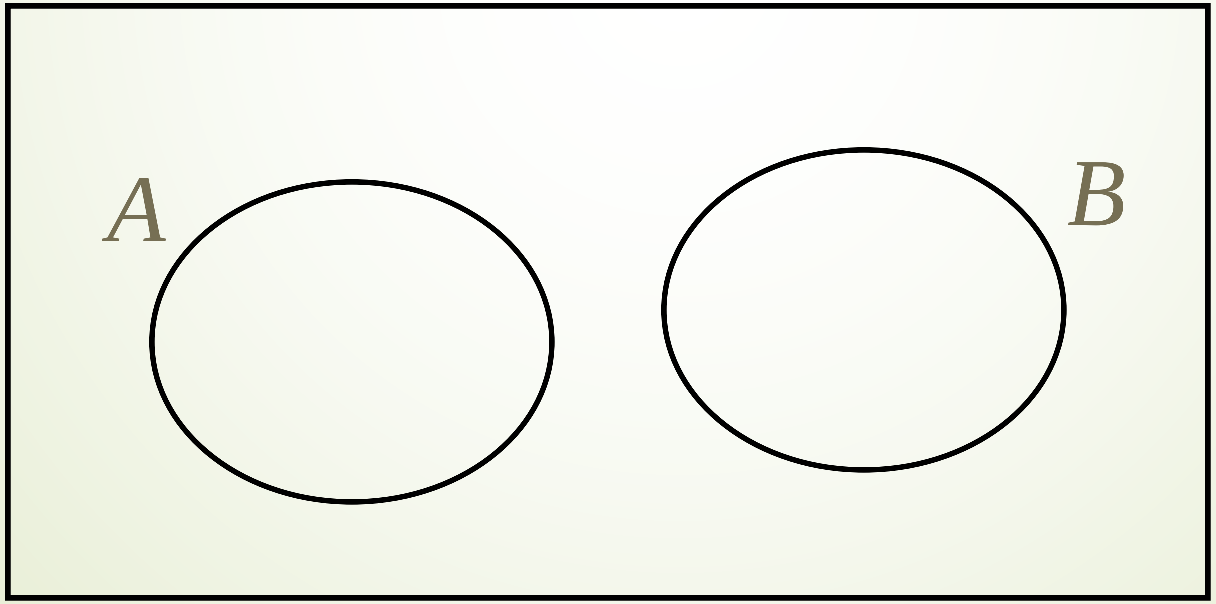
Two events A and B are called **mutually exclusive** if:

$$A \cap B = \phi$$



If two events *A* and *B* are are **mutually exclusive** then:

1. They have no outcomes in common.
   They can't occur at the same time. The outcome of the random experiment can not belong to both *A* and *B*.



Definition: probability of an Event *E*.

Suppose that the sample space $S = \{o_1, o_2, o_3, \dots o_N\}$ has a finite number, N, of oucomes.

Also each of the outcomes is equally likely (because of symmetry).

Then for any event *E*

$$P[E] = \frac{n(E)}{n(S)} = \frac{n(E)}{N} = \frac{\text{no. of outcomes in } E}{\text{total no. of outcomes}}$$

$$\textbf{Note}: \text{ the symbol } n(A) = \text{ no. of elements of } A$$

$$P[E] = \frac{n(E)}{n(S)} = \frac{n(E)}{N} = \frac{\text{no. of outcomes in } E}{\text{total no. of outcomes}}$$

Applies only to the special case when

1. The sample space has a finite no.of outcomes, and
2. Each outcome is equi-probable

   If this is not true a more general definition of probability is required.

Rules of Probability

Rule The additive rule (Mutually exclusive events)

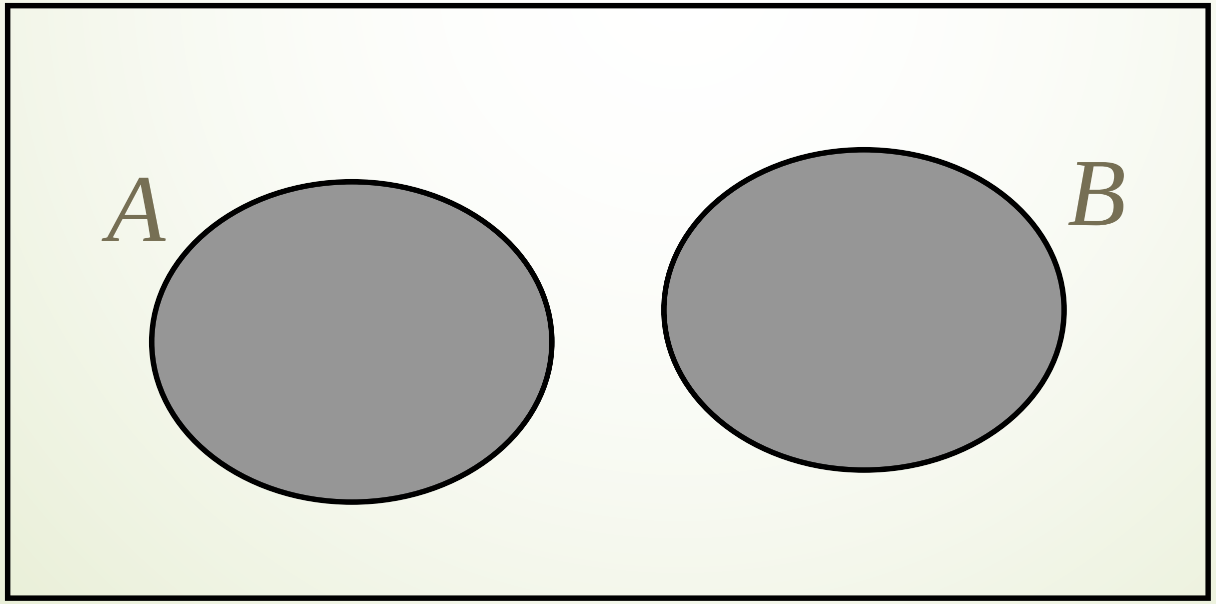
$$\boxed{\begin{array}{l} P[A \cup B] = P[A] + P[B] \\ \text{i.e.} \\ P[A \text{ or } B] = P[A] + P[B] \end{array}}$$

$$\text{if } A \cap B = \phi$$

(A and B mutually exclusive)

If two events *A* and *B* are are **mutually exclusive** then:

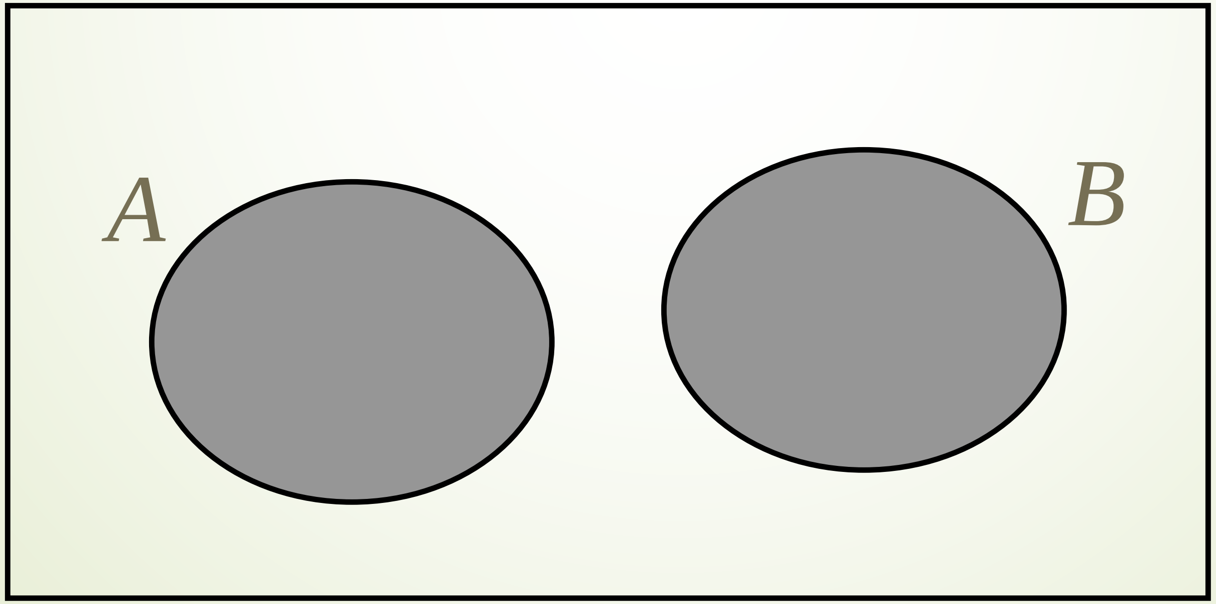
1. They have no outcomes in common.
   They can’t occur at the same time. The outcome of the random experiment can not belong to both *A* and *B*.



$$P[A \cup B] = P[A] + P[B]$$

i.e.

$$P[A \text{ or } B] = P[A] + P[B]$$



Rule The additive rule

(In general)

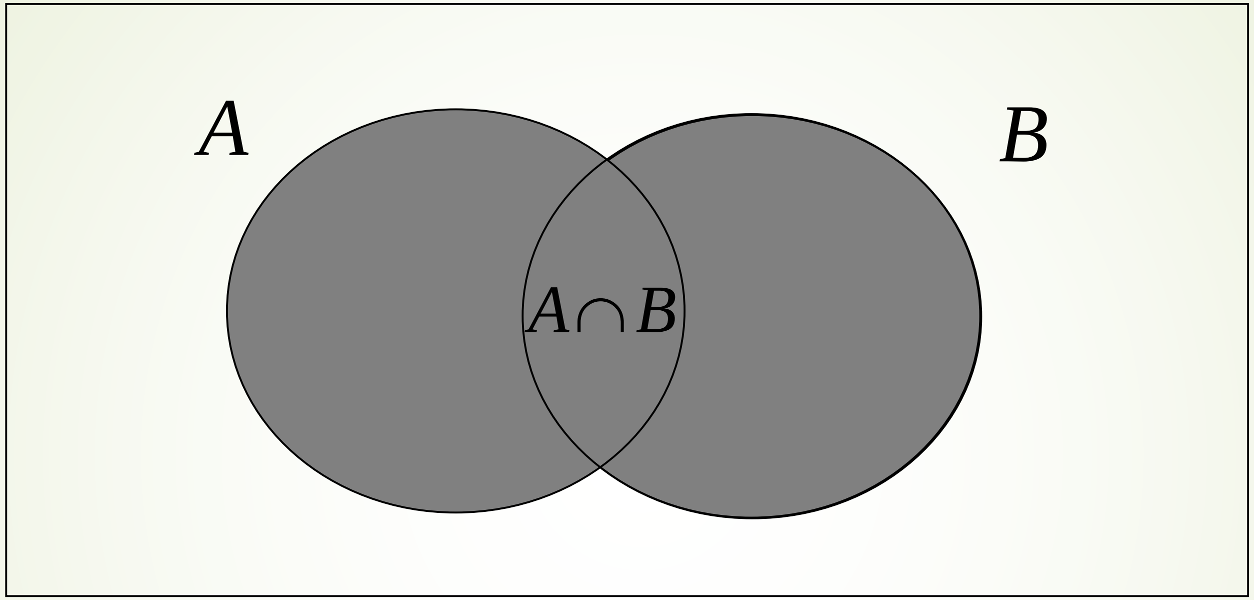
$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

or

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B]$$

Logic

$A \cup B$



When $P[A]$ is added to $P[B]$ the outcome in $A \cap B$ are counted twice

hence

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

Example:

Saskatoon and Moncton are two of the cities competing for the World university games. (There are also many others). The organizers are narrowing the competition to the **final 5 cities.**

There is a 20% chance that Saskatoon will be amongst the **final 5**. There is a 35% chance that Moncton will be amongst the **final 5** and an 8% chance that both Saskatoon and Moncton will be amongst the **final 5**.

What is the probability that Saskatoon or Moncton will be amongst the **final 5.**

Solution:

Let A = the event that Saskatoon is amongst the **final 5**.

Let B = the event that Moncton is amongst the **final 5**.

Given $P[A] = 0.20$, $P[B] = 0.35$, and $P[A \cap B] = 0.08$

What is $P[A \cup B]$?

Note: "and" $\equiv \cap$, "or" $\equiv \cup$.

$$P\left[A \cup B\right] = P\left[A\right] + P\left[B\right] - P\left[A \cap B\right]$$

$$= 0.20 + 0.35 - 0.08 = 0.47$$

Rule for complements

2. $$P\left[\bar{A}\right]=1-P\left[A\right]$$

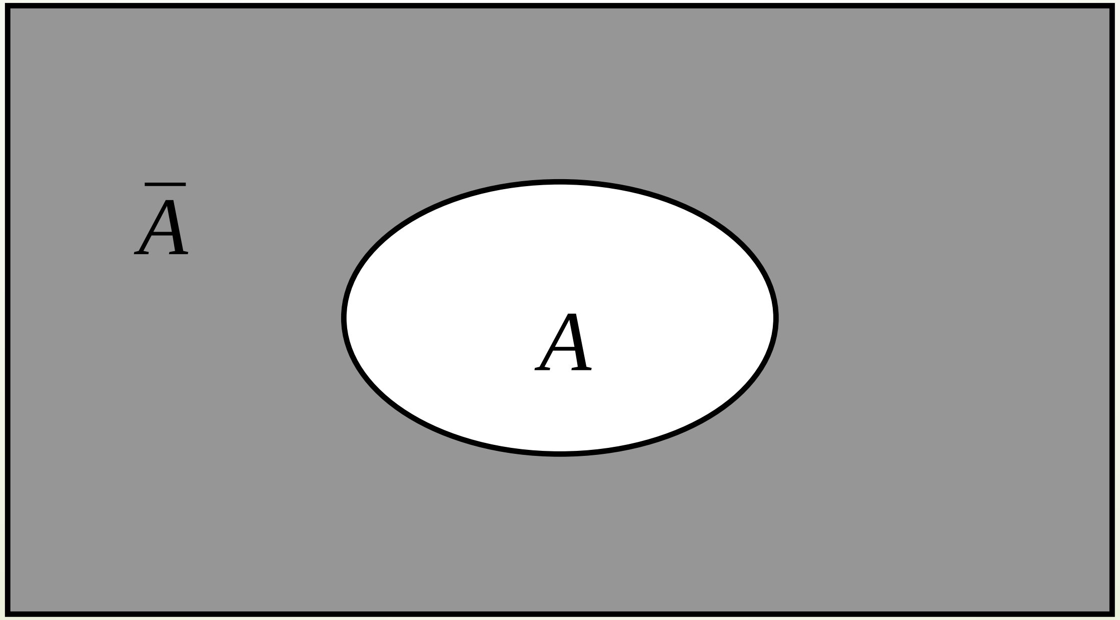
or

$$P\left[\text{not } A\right]=1-P\left[A\right]$$

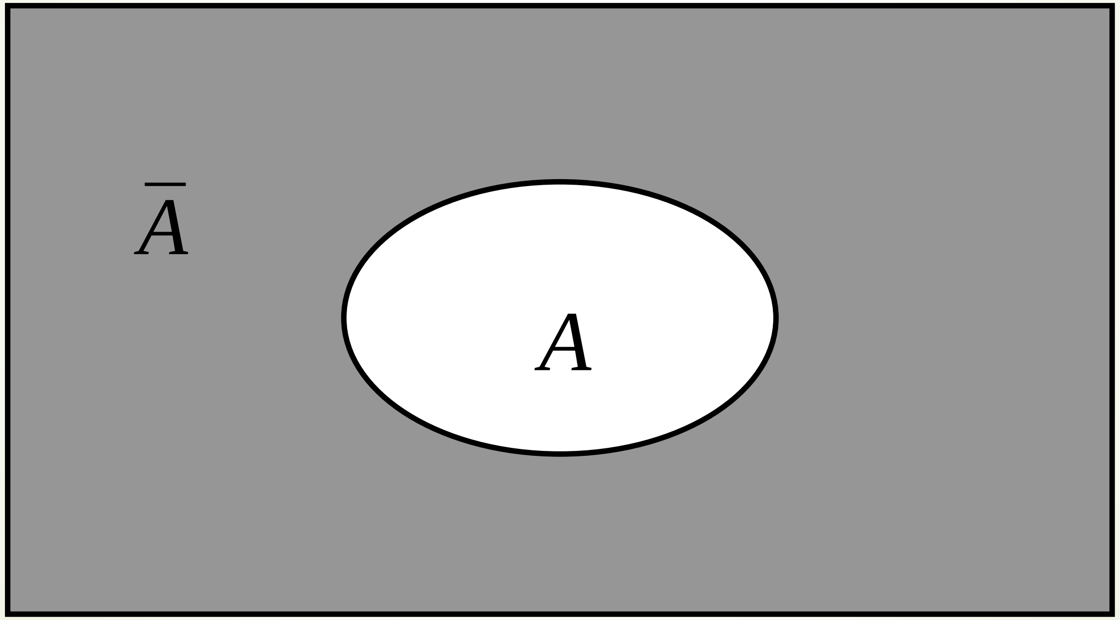
Complement

Let A be any event, then the **complement** of A (denoted by $\bar{A}$) defined by:

$\bar{A}$ = {e| e **does not** belongs to A}



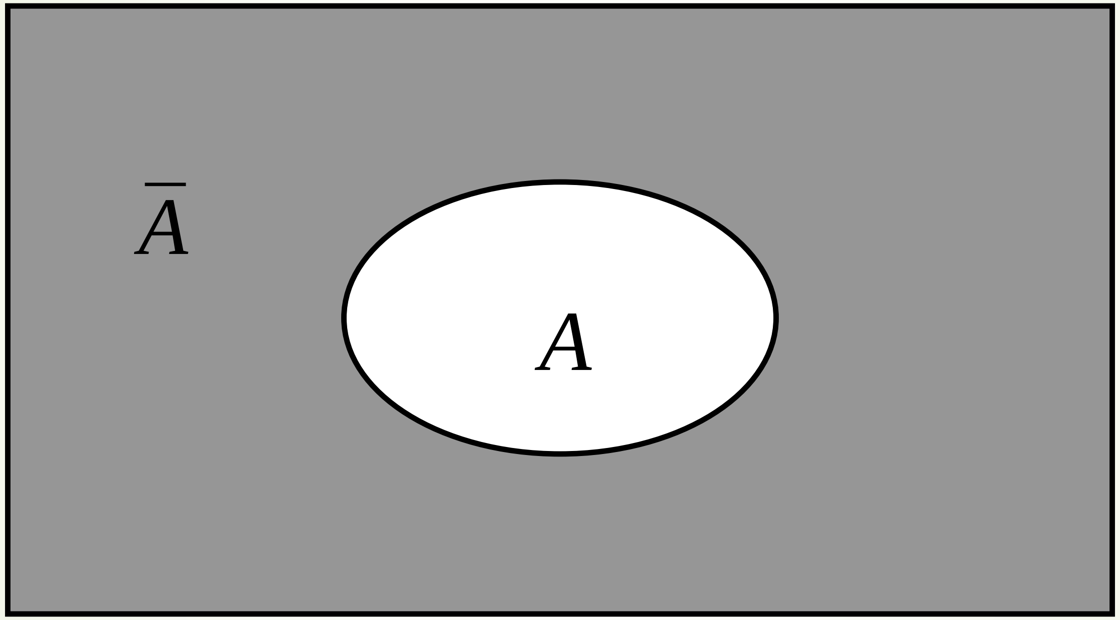
The event $\bar{A}$ **occurs** if the event A **does not occur**



Logic:

$\bar{A}$ and A are **mutually exclusive**.

and $S = A \cup \bar{A}$



thus $1 = P[S] = P[A] + P[\bar{A}]$

and $P[\bar{A}] = 1 - P[A]$

Conditional Probability

Conditional Probability

- Frequently before observing the outcome of a random experiment you are given information regarding the outcome
- How should this information be used in prediction of the outcome.
- Namely, how should probabilities be adjusted to take into account this information
- Usually the information is given in the following form: You are told that the outcome belongs to a given event. (i.e. you are told that a certain event has occurred)

Definition

Suppose that we are interested in computing the probability of event *A* and we have been told event *B* has occurred.

Then the conditional probability of *A* given *B* is defined to be:

$$P\left[A\middle|B\right]=\frac{P\left[A\cap B\right]}{P\left[B\right]}\quad \text{if}\quad P\left[B\right]\neq 0$$

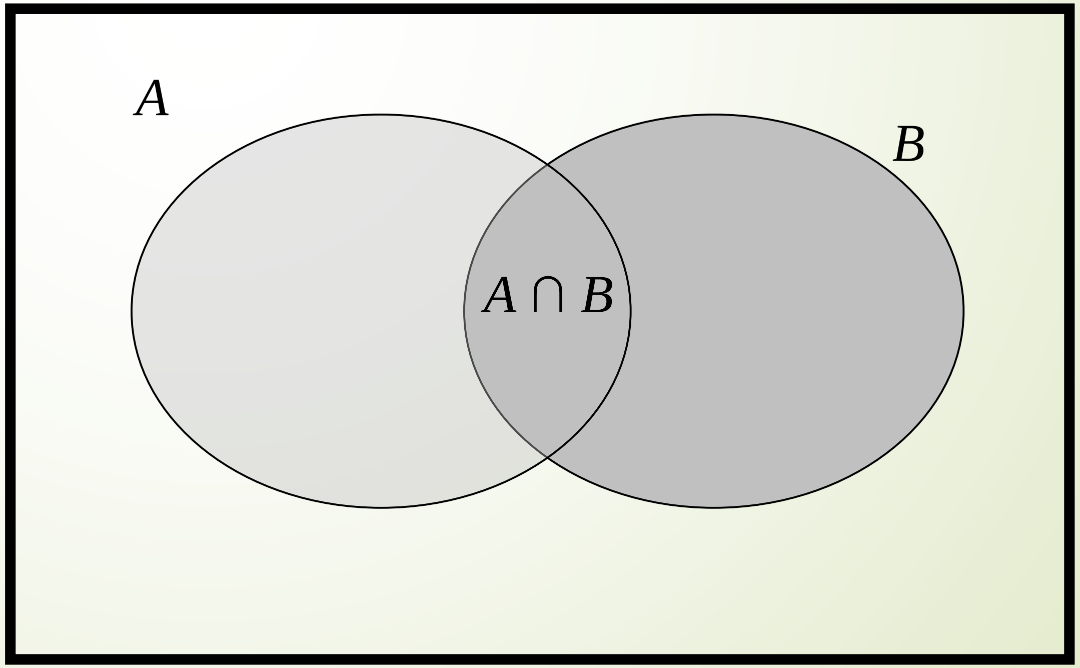
Rationale:

If we're told that event *B* has occurred then the sample space is restricted to *B*.

The probability within *B* has to be normalized, This is achieved by dividing by *P*[*B*]

The event *A* can now only occur if the outcome is in of *A* ∩ *B*. Hence the new probability of *A* is:

$$P\left[A\middle|B\right]=\frac{P\left[A\cap B\right]}{P\left[B\right]}$$



An Example

The academy awards is soon to be shown.

For a specific married couple the probability that the husband watches the show is 80%, the probability that his wife watches the show is 65%, while the probability that they both watch the show is 60%.

If the husband is watching the show, what is the probability that his wife is also watching the show

Solution:

The academy awards is soon to be shown.

Let B = the event that the husband watches the show

$P[B] = 0.80$

Let A = the event that his wife watches the show

$P[A] = 0.65$ and $P[A \cap B] = 0.60$

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{0.60}{0.80} = 0.75$$

Independence

Definition

Two events *A* and *B* are called **independent** if

$$P[A \cap B] = P[A]P[B]$$

Note if $P[B] \neq 0$ and $P[A] \neq 0$ then

$$P\left[A \middle| B\right] = \frac{P[A \cap B]}{P[B]} = \frac{P[A]P[B]}{P[B]} = P[A]$$

and $$P\left[B \middle| A\right] = \frac{P[A \cap B]}{P[A]} = \frac{P[A]P[B]}{P[A]} = P[B]$$

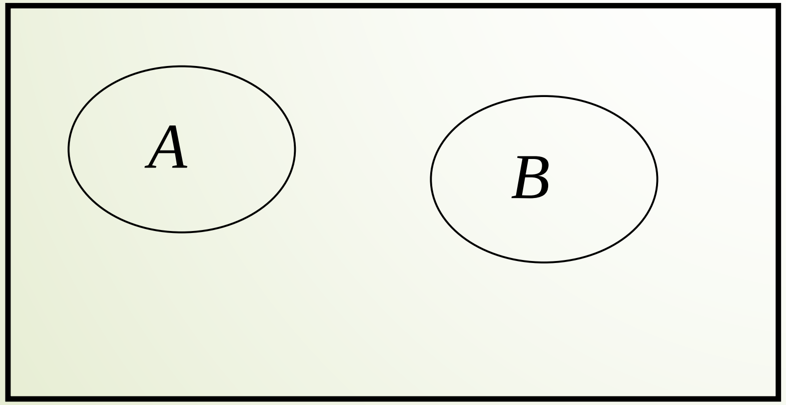
Thus in the case of independence the conditional probability of an event is not affected by the knowledge of the other event

Difference between **independence** and **mutually exclusive**

mutually exclusive

Two mutually exclusive events are independent only in the special case where

$$P[A]=0 \text{ and } P[B]=0. \text{ (also } P[A\cap B]=0$$



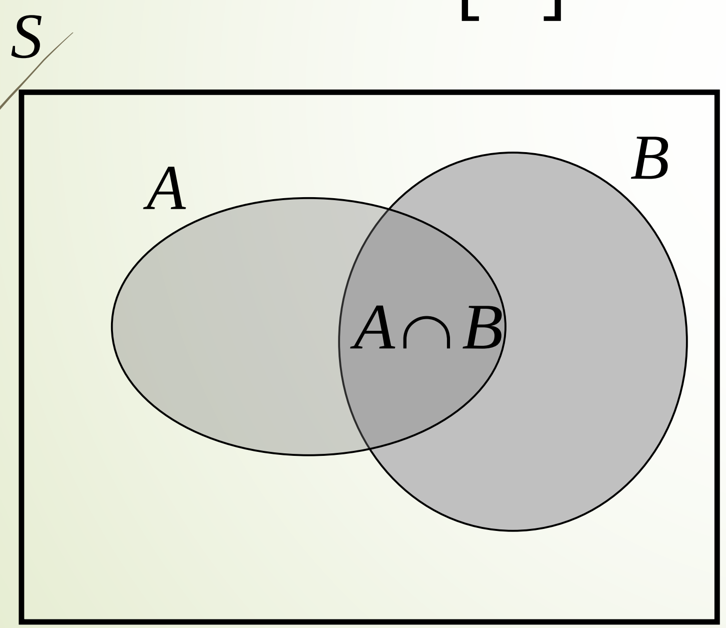
Mutually exclusive events are highly dependent otherwise. A and B **cannot** occur simultaneously. If one event occurs the other event does not occur.

Independent events

$$P[A \cap B] = P[A]P[B]$$

or

$$\frac{P[A \cap B]}{P[B]} = P[A] = \frac{P[A]}{P[S]}$$



The ratio of the probability of the set *A* within *B* is the same as the ratio of the probability of the set *A* within the entire sample *S*.

The multiplicative rule of probability

$$P[A \cap B] = \begin{cases} P[A]P\left[B \middle| A\right] & \text{if } P[A] \neq 0 \\ P[B]P\left[A \middle| B\right] & \text{if } P[B] \neq 0 \end{cases}$$

and

$$P[A \cap B] = P[A]P[B]$$

if A and B are **independent**.

Summary of the Rules of Probability

The additive rule

$$\boxed{P[A \cup B] = P[A] + P[B] - P[A \cap B]}$$

and

$$P[A \cup B] = P[A] + P[B] \quad \text{if } A \cap B = \phi$$

The Rule for complements

for any event E

$$\boxed{P\left[\bar{E}\right] = 1 - P\left[E\right]}$$

Conditional probability

$$P\left[A\middle|B\right] = \frac{P\left[A \cap B\right]}{P\left[B\right]}$$

The multiplicative rule of probability

$$P[A\cap B]=\begin{cases} P[A]P\left[B\middle|A\right] & \text{if } P[A]\neq 0 \\ P[B]P\left[A\middle|B\right] & \text{if } P[B]\neq 0 \end{cases}$$

and

$$P[A\cap B]=P[A]P[B]$$

if A and B are **independent**.

This is the definition of independent

Counting techniques

Finite uniform probability space

Many examples fall into this category

1. Finite number of outcomes
2. All outcomes are equally likely
3. $$P[E]=\frac{n(E)}{n(S)}=\frac{n(E)}{N}=\frac{\text{no. of outcomes in } E}{\text{total no. of outcomes}}$$

$$\textbf{Note}: \quad n(A)= \text{ no. of elements of } A$$

To handle problems in case we have to be able to count. Count $n(E)$ and $n(S)$.

Rule 1

Suppose we carry out have a sets A_1, A_2, A_3, … and that any pair are mutually exclusive

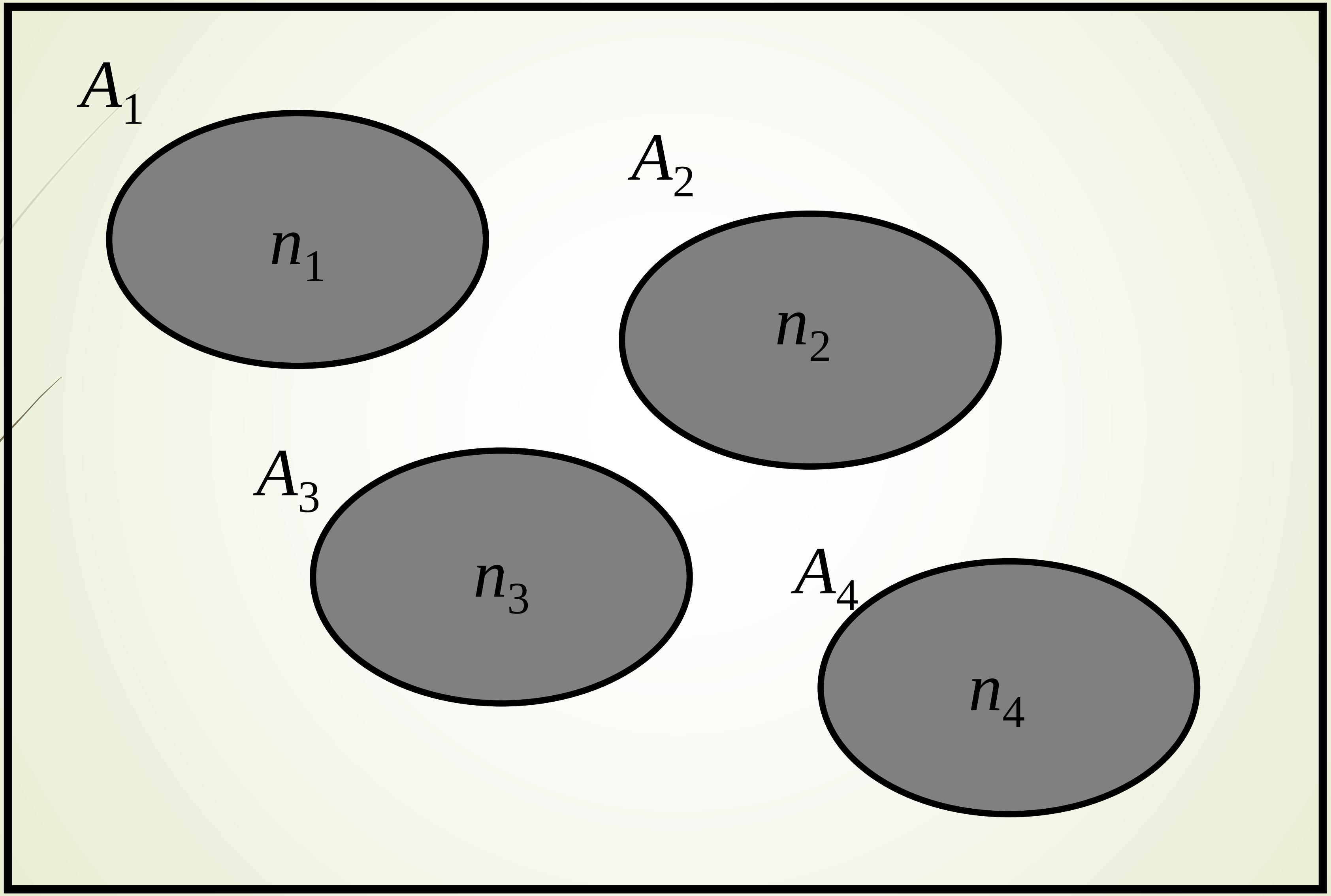
(i.e. $A_1 \cap A_2 = \phi$) Let

$$n_i = n\,(A_i) = \text{the number of elements in } A_i.$$

$$\text{Let } A = A_1 \cup A_2 \cup A_3 \cup \;\ldots.$$

$$\text{Then } N = n(\,A\,) = \text{the number of elements in } A$$

$$= n_1 + n_2 + n_3 + \ldots$$


A_1
n_1
A_2
n_2
A_3
n_3
A_4
n_4

Rule 2

Suppose we carry out two operations in sequence

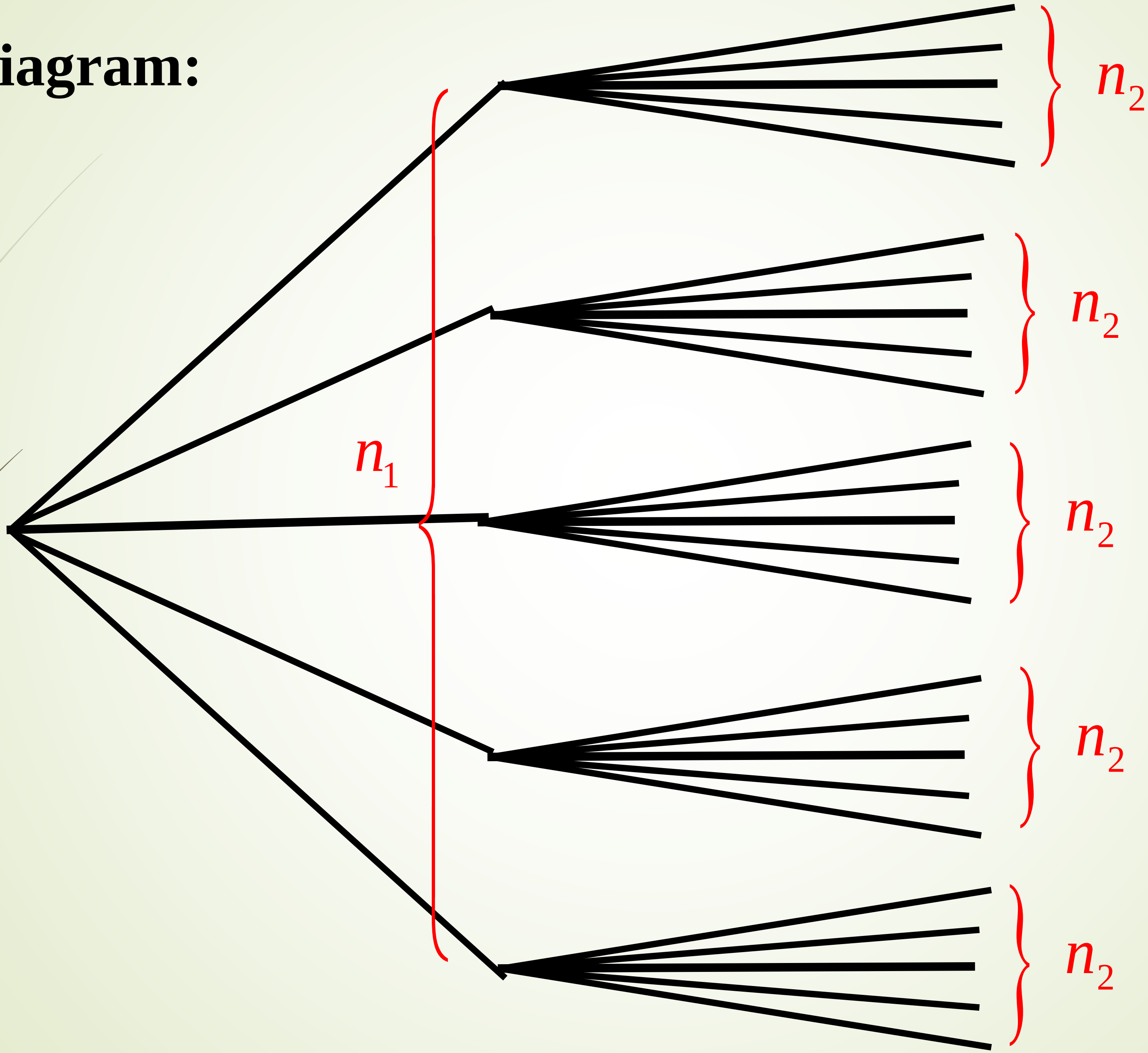
Let

n_1 = the number of ways the first operation can be performed

n_2 = the number of ways the second operation can be performed once the first operation has been completed.

Then $\boxed{N = n_1 \, n_2}$ = the number of ways the two operations can be performed in sequence.

Diagram:



Examples

1. We have a committee of 10 people. We choose from this committee, a chairman and a vice chairman. How may ways can this be done?

Solution:

Let n_1 = the number of ways the chairman can be chosen = 10.

Let n_2 = the number of ways the vice-chairman can be chosen once the chair has been chosen = 9.

Then $N = n_1 n_2 = (10)(9) = 90$

2. In **Black Jack** you are dealt 2 cards. What is the probability that you will be dealt a 21?

Solution:

The number of ways that two cards can be selected from a deck of 52 is $N = (52)(51) = 2652$.

A “21” can occur if the first card is an ace and the second card is a face card or a ten {10, J, Q, K} **or** the first card is a face card or a ten and the second card is an ace.

The number of such hands is $(4)(16) + (16)(4) = 128$

Thus the probability of a “21” = $128/2652 = 32/663$

The Multiplicative Rule of Counting

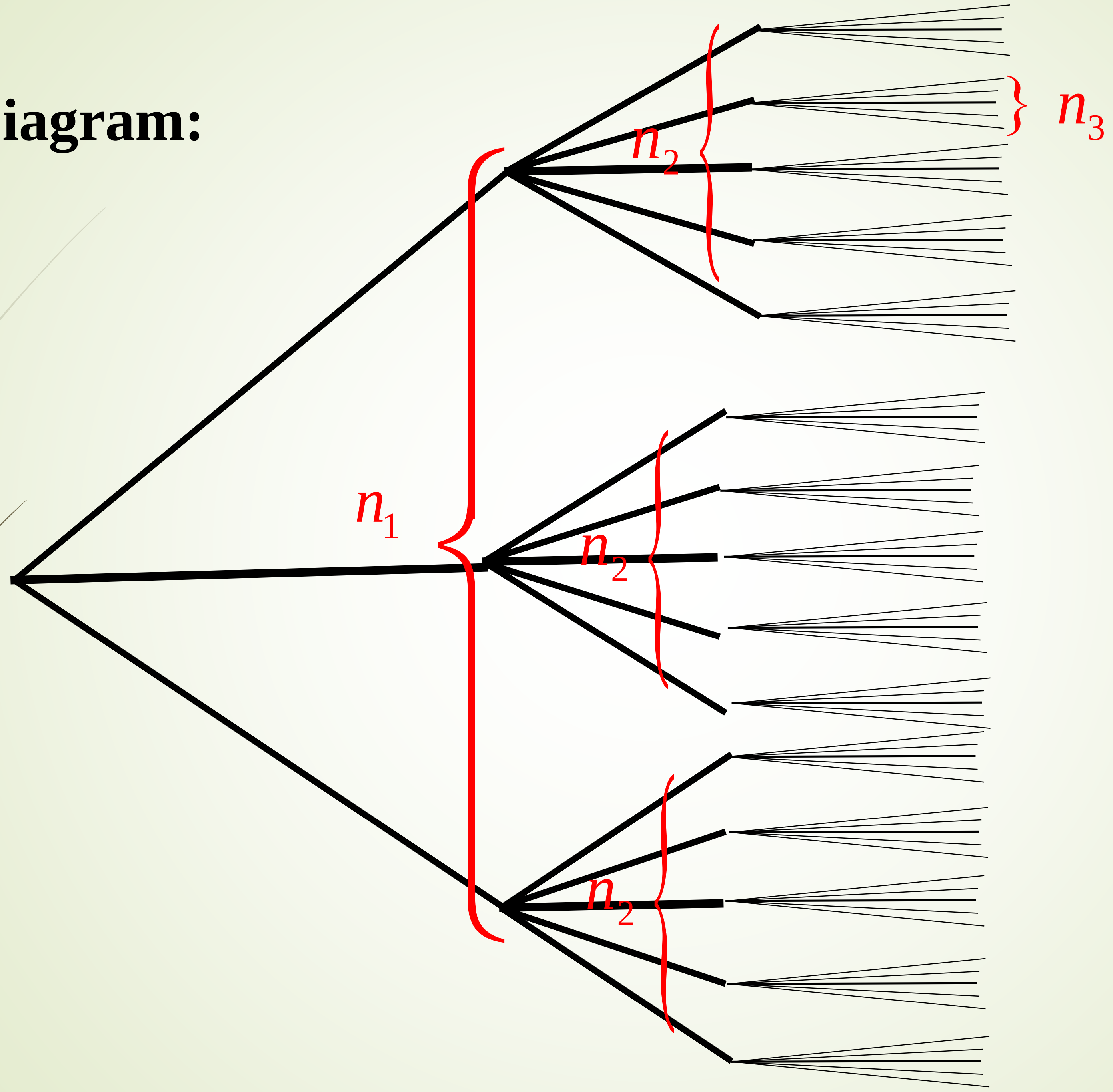
Suppose we carry out *k* operations in sequence

Let n_1 = the number of ways the first operation can be performed

n_i = the number of ways the i^{th} operation can be performed once the first $(i - 1)$ operations have been completed. $i = 2, 3, \ldots, k$

Then $\boxed{N = n_1 n_2 \ldots n_k}$ = the number of ways the *k* operations can be performed in sequence.

Diagram:



Examples

1. **Permutations:** How many ways can you order *n* objects

Solution:

Ordering *n* objects is equivalent to performing *n* operations in sequence.

1. Choosing the first object in the sequence ($n_1 = n$)
2. Choosing the 2^{nd} object in the sequence ($n_2 = n$ -1).

…

k. Choosing the k^{th} object in the sequence ($n_k = n - k + 1$)

…

n. Choosing the n^{th} object in the sequence ($n_n = 1$)

The total number of ways this can be done is:

$$\boxed{N = n(n-1)\ldots(n-k+1)\ldots(3)(2)(1) = n!}$$

Example How many ways can you order the 4 objects $\{A, B, C, D\}$

Solution:

$N = 4! = 4(3)(2)(1) = 24$

Here are the orderings.

ABCD	*ABDC*	*ACBD*	*ACDB*	*ADBC*	*ADCB*
BACD	*BADC*	*BCAD*	*BCDA*	*BDAC*	*BDCA*
CABD	*CADB*	*CBAD*	*CBDA*	*CDAB*	*CDBA*
DABC	*DACB*	*DBAC*	*DBCA*	*DCAB*	*DCBA*

Examples - continued

2. **Permutations of size *k* (< *n*):** How many ways can you choose *k* objects from *n* objects in a specific order

Solution: This operation is equivalent to performing *k* operations in sequence.

1. Choosing the first object in the sequence ($n_1 = n$)
2. Choosing the 2^{nd} object in the sequence ($n_2 = n - 1$).

…

k. Choosing the k^{th} object in the sequence ($n_k = n - k + 1$)

The total number of ways this can be done is:

$$N = n(n-1)\dots(n-k+1) = n!/(n-k)!$$

This number is denoted by the symbol

$$_nP_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

Definition: $0! = 1$

This definition is consistent with

$$ {}_nP_k = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!} $$

for $k = n$

$$ {}_nP_n = \frac{n!}{0!} = \frac{n!}{1} = n! $$

Example How many permutations of size 3 can be found in the group of 5 objects {*A*, *B*, *C*, *D*, *E*}

Solution: $${}_5P_3 = \frac{5!}{(5-3)!} = 5(4)(3) = 60$$

ABC	*ABD*	*ABE*	*ACD*	*ACE*	*ADE*	*BCD*	*BCE*	*BDE*	*CDE*
ACB	*ADB*	*AEB*	*ADC*	*AEC*	*AED*	*BDC*	*BEC*	*BED*	*CED*
BAC	*BAD*	*BAE*	*CAD*	*CAE*	*DAE*	*CBD*	*CBE*	*DBE*	*DCE*
BCA	*BDA*	*BEA*	*CDA*	*CEA*	*DEA*	*CDB*	*CEB*	*DEB*	*DEC*
CAB	*DAB*	*EAB*	*DAC*	*EAC*	*EAD*	*DBC*	*EBC*	*EBD*	*ECD*
CAB	*DBA*	*EBA*	*DCA*	*ECA*	*EDA*	*DCB*	*ECB*	*EDB*	*EDC*

Example We have a committee of n = 10 people and we want to choose a **chairperson**, **a vice-chairperson** and a **treasurer**

Solution: Essentually we want to select 3 persons from the committee of 10 in a specific order. (Permutations of size 3 from a group of 10).

$$ {}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10(9)(8) = 720 $$

Example We have a committee of n = 10 people and we want to choose a **chairperson**, **a vice-chairperson** and a **treasurer**. Suppose that 6 of the members of the committee are male and 4 of the members are female. What is the probability that the three executives selected are all male?

Solution: Again we want to select 3 persons from the committee of 10 in a specific order. (Permutations of size 3 from a group of 10).The total number of ways that this can be done is:

$$ {}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10(9)(8) = 720 $$

This is the size, $N = n(S)$, of the sample space S. Assume all outcomes in the sample space are equally likely.

Let E be the event that all three executives are male

$$ n(E) = {}_{6}P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 6(5)(4) = 120 $$

Hence

$$P[E] = \frac{n(E)}{n(S)} = \frac{120}{720} = \frac{1}{6}$$

Thus if all candidates are equally likely to be selected to any position on the executive then the probability of selecting an all male executive is:

$$\frac{1}{6}$$

Examples - continued

3. **Combinations of size *k* (≤ *n*):** A combination of size *k* chosen from *n* objects is a subset of size *k* where the order of selection is irrelevant. How many ways can you choose a combination of size *k* objects from *n* objects (order of selection is irrelevant)

Here are the combinations of size 3 selected from the 5 objects $\{A, B, C, D, E\}$

$\{A,B,C\}$	$\{A,B,D\}$	$\{A,B,E\}$	$\{A,C,D\}$	$\{A,C,E\}$
$\{A,D,E\}$	$\{B,C,D\}$	$\{B,C,E\}$	$\{B,D,E\}$	$\{C,D,E\}$

Important Notes

1. In **combinations** ordering is **irrelevant**. Different orderings result in the same combination.
2. In **permutations** order is **relevant**. Different orderings result in the different permutations.

How many ways can you choose a combination of size *k* objects from *n* objects (order of selection is irrelevant)

Solution: Let n_1 denote the number of combinations of size k.
One can construct a permutation of size k by:

1. Choosing a combination of size k (n_1 = unknown)
2. Ordering the elements of the combination to form a permutation ($n_2 = k!$)

Thus $${}_nP_k = \frac{n!}{(n-k)!} = n_1 k!$$

and $$n_1 = \frac{{}_nP_k}{k!} = \frac{n!}{(n-k)!k!} = \text{ the \# of combinations of size } k.$$

The number:

$$n_1 = \frac{{}_nP_k}{k!} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots(1)}$$

is denoted by the symbol

$${}_nC_k \quad \text{or} \quad \binom{n}{k} \quad \text{read "}n\text{ choose }k\text{"}$$

It is the number of ways of choosing k objects from n objects (order of selection irrelevant).

${}_nC_k$ is also called a **binomial coefficient**.

It arises when we expand $(x + y)^n$ **(the binomial theorem)**

The Binomial theorem:

$$(x+y)^n = {}_nC_0 x^0 y^n + {}_nC_1 x^1 y^{n-1} + {}_nC_2 x^2 y^{n-2} +$$

$$\ldots + {}_nC_k x^k y^{n-k} + \ldots + {}_nC_n x^n y^0$$

$$= \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} +$$

$$\ldots + \binom{n}{k} x^k y^{n-k} + \ldots + \binom{n}{n} x^n y^0$$

Proof: The term $x^k y^{n-k}$ will arise when we select x from *k* of the factors of $(x + y)^n$ and select y from the remaining *n* – *k* factors. The no. of ways that this can be done is: $\binom{n}{k}$

Hence there will be $\binom{n}{k}$ terms equal to $x^k y^{n=k}$ and

$$(x+y)^n = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} +$$

$$\ldots + \binom{n}{k} x^k y^{n-k} + \ldots + \binom{n}{n} x^n y^0$$

Pascal’s triangle – a procedure for calculating binomial coefficients

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

- The two edges of Pascal's triangle contain 1's
- The interior entries are the sum of the two nearest entries in the row above
- The entries in the n^{th} row of Pascals triangle are the values of the binomial coefficients

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{3} \quad \binom{n}{4} \qquad \cdots \qquad \binom{n}{k} \cdots \binom{n}{n-1} \binom{n}{n}$$

Pascal's triangle

1

$1 \quad 1$ $\qquad \binom{1}{k}$

$1 \quad 2 \quad 1$ $\qquad \binom{2}{k}$

$1 \quad 3 \quad 3 \quad 1$ $\qquad \binom{3}{k}$

$1 \quad 4 \quad 6 \quad 4 \quad 1$ $\qquad \binom{4}{k}$

$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$ $\qquad \binom{5}{k}$

$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$ $\qquad \binom{6}{k}$

$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$ $\qquad \binom{7}{k}$

The Binomial Theorem

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^3$$

$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

$$(x+y)^6 = x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$$

$$(x+y)^7 = x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7$$

Summary of counting rules

Rule 1

$$n(A_1 \cup A_2 \cup A_3 \cup \ldots.) = n(A_1) + n(A_2) + n(A_3) + \ldots$$

if the sets $A_1, A_2, A_3, \ldots$ are pairwise mutually exclusive

(i.e. $A_i \cap A_j = \phi$)

Rule 2

$N = n_1 n_2$ = the number of ways that two operations can be performed in sequence if

n_1 = the number of ways the first operation can be performed

n_2 = the number of ways the second operation can be performed once the first operation has been completed.

Rule 3 $N = n_1 n_2 \ldots n_k$

= the number of ways the k operations can be performed in sequence if

n_1 = the number of ways the first operation can be performed

n_i = the number of ways the i^{th} operation can be performed once the first $(i - 1)$ operations have been completed. $i = 2, 3, \ldots, k$

Basic counting formulae

1. Orderings

$n! =$ the number of ways you can order n objects

2. Permutations

$${}_{n}P_{k} = \frac{n!}{(n-k)!} =$$ The number of ways that you can choose k objects from n in a specific order

3. Combinations

$$\binom{n}{k} = {}_{n}C_{k} = \frac{n!}{k!(n-k)!} =$$ The number of ways that you can choose k objects from n (order of selection irrelevant)

Applications to some counting problems

- The trick is to use the basic counting formulae together with the **Rules**
- We will illustrate this with examples
- Counting problems are not easy. The more practice better the techniques

Application to Lotto 6/49

Here you choose 6 numbers from the integers 1, 2, 3, …, 47, 48, 49.

Six **winning** numbers are chosen together with a **bonus** number.

How many choices for the 6 winning numbers

$$\binom{49}{6} = {}_{49}C_6 = \frac{49!}{6!43!} = \frac{49(48)(47)(46)(45)(44)}{6(5)(4)(3)(2)(1)}$$

$$= 13,983,816$$

You can lose and win in several ways

1. No winning numbers – lose
2. One winning number – lose
3. Two winning numbers - lose
4. Two + bonus – win $5.00
5. Three winning numbers – win $10.00
6. Four winning numbers – win approx. $80.00
7. 5 winning numbers – win approx. $2,500.00
8. 5 winning numbers + bonus – win approx. $100,000.00
9. 6 winning numbers – win approx. $4,000,000.00

Counting the possibilities

1. No winning numbers – lose

All **six** of your numbers have to be chosen from the losing numbers and the bonus.

$$\binom{43}{6} = 6{,}096{,}454$$

2. One winning numbers – lose

One number is chosen from the six winning numbers and the remaining **five** have to be chosen from the losing numbers and the bonus.

$$\binom{6}{1}\binom{43}{5} = 6(\ 962{,}598\ ) = 5{,}775{,}588$$

3. Two winning numbers – lose

Two numbers are chosen from the six winning numbers and the remaining **four** have to be chosen from the losing numbers (bonus not included)

$$\binom{6}{2}\binom{42}{4} = 15(111{,}930) = 1{,}678{,}950$$

4. Two winning numbers + the bonus – win $5.00

Two numbers are chosen from the six winning numbers, the **bonus** number is chose and the remaining **three** have to be chosen from the losing numbers.

$$\binom{6}{2}\binom{1}{1}\binom{42}{3} = 15(1)(11{,}480) = 172{,}200$$

5. Three winning numbers – win $10.00

Three numbers are chosen from the six winning numbers and the remaining **three** have to be chosen from the losing numbers + the bonus number

$$\binom{6}{3}\binom{43}{3} = 20(12{,}341) = 246{,}820$$

6. four winning numbers – win approx. $80.00

Four numbers are chosen from the six winning numbers and the remaining **two** have to be chosen from the losing numbers + the bonus number

$$\binom{6}{4}\binom{43}{2} = 15(903) = 13{,}545$$

7. five winning numbers (no bonus) – win approx. $2,500.00

Five numbers are chosen from the six winning numbers and the remaining **number** has to be chosen from the losing numbers (excluding the bonus number)

$$\binom{6}{5}\binom{42}{1} = 6(42) = 252$$

8. five winning numbers + bonus – win approx. $100,000.00

Five numbers are chosen from the six winning numbers and the remaining **number** is chosen to be the bonus number

$$\binom{6}{5}\binom{1}{1} = 6(1) = 6$$

9. six winning numbers (no bonus) – win approx. $4,000,000.00

Six numbers are chosen from the six winning numbers,

$$\binom{6}{6} = 1$$

Summary

	n	*Prize*	Prob
0 winning	6,096,454	nil	0.4359649755
1 winning	5,775,588	nil	0.4130194505
2 winning	1,678,950	nil	0.1200637937
2 + bonus	172,200	$ 5.00	0.0123142353
3 winning	246,820	$ 10.00	0.0176504039
4 winning	13,545	$ 80.00	0.0009686197
5 winning	252	$ 2,500.00	0.0000180208
5 + bonus	6	$ 100,000.00	0.0000004291
6 winning	1	$ 4,000,000.00	0.0000000715
Total	13,983,816		

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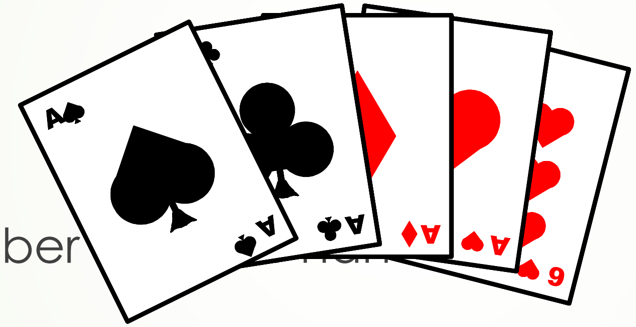
Another Example
counting poker hands

A **poker hand** consists of five cards chosen at random from a deck of 52 cards.

The total number



$$N = \binom{52}{5} = 2,598,960$$

Counting poker hands

1. Nothing Hand $\{x, y, z, u, v\}$
   - Not all in sequence or not all the same suit
2. Pair $\{x, x, y, z, u\}$
3. Two pair $\{x, x, y, y, z\}$
4. Three of a kind $\{x, x, x, y, z\}$
5. Straight $\{x, x+1, x+2, x+3, x+4\}$
   - 5 cards in sequence
   - Not all the same suit
6. Flush $\{x, y, z, u, v\}$
   - Not all in sequence but **all the same suit**

7. Full House $\{x, x, x, y, y\}$

8. Four of a kind $\{x, x, x, x, y\}$

9. Straight Flush $\{x, x+1, x+2, x+3, x+4\}$
   - 5 cards in sequence but not {10, J, Q, K, A}
   - **all the same suit**

10. Royal Flush {10, J, Q, K, A}
    - **all the same suit**

counting the hands

2. Pair $\{x, x, y, z, u\}$

We have to:

- Choose the value of x $\binom{13}{1} = 13$
- Select the suits for the for x. $\binom{4}{2} = 6$
- Choose the denominations $\{y, z, u\}$ $\binom{12}{3} = 220$
- Choose the suits for $\{y, z, u\}$ - $4\times4\times4 = 64$

Total # of hands of this type = $13 \times 6 \times 220 \times 64 = 1{,}098{,}240$

3. Two pair $\{x, x, y, y, z\}$

We have to:

- Choose the values of x, y $\binom{13}{2} = 78$
- Select the suits for the for x and y. $\binom{4}{2} \times \binom{4}{2} = 36$
- Choose the denomination z $\binom{11}{1} = 11$
- Choose the suit for z - 4

Total # of hands of this type = $78 \times 36 \times 11 \times 4 = 123{,}552$

4. Three of a kind $\{x, x, x, y, z\}$

We have to:

- Choose the value of x $\binom{13}{1} = 13$
- Select the suits for the for x. $\binom{4}{3} = 4$
- Choose the denominations $\{y, z\}$ $\binom{12}{2} = 66$
- Choose the suits for $\{y, z\}$ - $4 \times 4 = 16$

Total # of hands of this type = $13 \times 4 \times 66 \times 16 = 54{,}912$

7. Full House $\{x, x, x, y, y\}$

We have to:

- Choose the value of x then y ${}_{13}P_2 = 13(12) = 156$
- Select the suits for the for x. $\binom{4}{3} = 4$
- Select the suits for the for y. $\binom{4}{2} = 6$

Total # of hands of this type = $156 \times 4 \times 6 = 3{,}696$

8. Four of a kind $\{x, x, x, x, y\}$

We have to:

- Choose the value of x $\binom{13}{1} = 13$
- Select the suits for the for x. $\binom{4}{4} = 1$
- Choose the denomination of y. $\binom{12}{1} = 12$
- Choose the suit for y - 4

Total # of hands of this type = 13 × 1 × 12 × 4 = 624

10. Royal Flush {10, J, Q, K, A}

- **all the same suit**

Total # of hands of this type = 4 (no. of suits)

9. Straight Flush $\{x, x+1, x+2, x+3, x+4\}$

- 5 cards in sequence but not {10, J, Q, K, A}
- **all the same suit**

Total # of hands of this type = 9×4 = 36 (no. of suits)

The hand could start with {A, 2, 3, 4, 5, 6, 7, 8, 9}

5. Straight $\{x, x+1, x+2, x+3, x+4\}$

- 5 cards in sequence
- Not all the same suit

We have to:

- Choose the starting value of the sequence, x.

  Total of 10 possibilities {A, 2, 3, 4, 5, 6, 7, 8, 9, 10}
- Choose the suit for each card

  $4 \times 4 \times 4 \times 4 \times 4 = 1024$

Total # of hands of this type = $1024 \times 10 - 36 - 4 = 10200$

We would have also counted straight flushes and royal flushes that have to be removed

6. Flush $\{x, y, z, u, v\}$

- Not all in sequence but **all the same suit**

We have to:

- Choose the suit 4 choices
- Choose the denominations $\{x, y, z, u, v\}$

$$\binom{13}{5} = 1287$$

Total # of hands of this type = 1287 × 4 - $\underline{36 - 4}$ = 5108

We would have also counted straight flushes and royal flushes that have to be removed

Summary

	Frequency	Prob.
nothing	1,302,588	0.50119586
pair	1,098,240	0.42256903
two pair	123,552	0.04753902
3 of a kind	54,912	0.02112845
straight	10,200	0.00392465
flush	5,108	0.00196540
full house	3,696	0.00142211
4 of a kind	624	0.00024010
straight flush	36	0.00001385
royal flush	4	0.00000154
Total	2,598,960	1.00000000