Dynamic Programming | Set 31 (Optimal Strategy for a Game)

Problem statement: Consider a row of n coins of values v1 . . . vn, where n is even. We play a game against an opponent by alternating turns. In each turn, a player selects either the first or last coin from the row, removes it from the row permanently, and receives the value of the coin. Determine the maximum possible amount of money we can definitely win if we move first.

Note: The opponent is as clever as the user.

Let us understand the problem with few examples:

1. 5, 3, 7, 10 : The user collects maximum value as 15(10 + 5)
2. 8, 15, 3, 7 : The user collects maximum value as 22(7 + 15)

Does choosing the best at each move give an optimal solution?

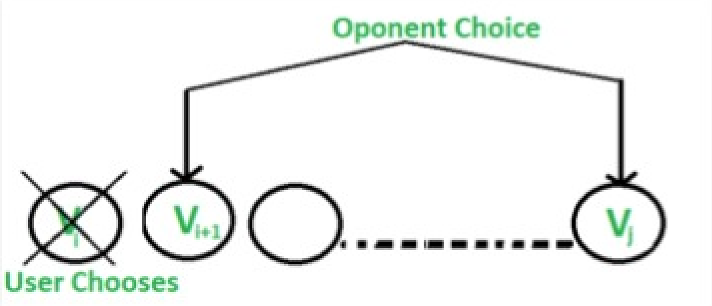
No. In the second example, this is how the game can finish:

1.
…….User chooses 8.
…….Opponent chooses 15.
…….User chooses 7.
…….Opponent chooses 3.
Total value collected by user is 15(8 + 7)

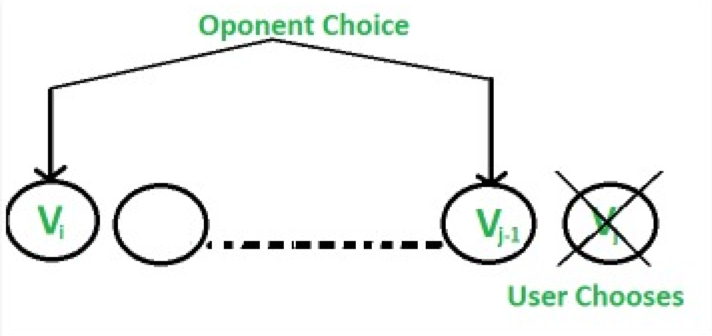
2.
…….User chooses 7.
…….Opponent chooses 8.
…….User chooses 15.
…….Opponent chooses 3.
Total value collected by user is 22(7 + 15)

So if the user follows the second game state, maximum value can be collected although the first move is not the best.

There are two choices:
1. The user chooses the ith coin with value Vi: The opponent either chooses (i+1)th coin or jth coin. The opponent intends to choose the coin which leaves the user with minimum value.
i.e. The user can collect the value Vi + min(F(i+2, j), F(i+1, j-1))



2. The user chooses the jth coin with value Vj: The opponent either chooses ith coin or (j-1)th coin. The opponent intends to choose the coin which leaves the user with minimum value.
i.e. The user can collect the value Vj + min(F(i+1, j-1), F(i, j-2))



Following is recursive solution that is based on above two choices. We take the maximum of two choices.

```
F(i, j)  represents the maximum value the user can collect from
         i'th coin to j'th coin.

    F(i, j)  = Max(Vi + min(F(i+2, j), F(i+1, j-1) ),
                   Vj + min(F(i+1, j-1), F(i, j-2) ))
Base Cases
    F(i, j)  = Vi           If j == i
    F(i, j)  = max(Vi, Vj)  If j == i+1
```

Why Dynamic Programming?

The above relation exhibits overlapping sub-problems. In the above relation, F(i+1, j-1) is calculated twice.

```c
// C program to find out maximum value from a given sequence of coins
#include <stdio.h>
#include <limits.h>

// Utility functions to get maximum and minimum of two intgers
int max(int a, int b)  {    return a > b ? a : b;  }
int min(int a, int b)  {    return a < b ? a : b;  }

// Returns optimal value possible that a player can collect from
// an array of coins of size n. Note than n must be even
int optimalStrategyOfGame(int* arr, int n)
{
    // Create a table to store solutions of subproblems
    int table[n][n], gap, i, j, x, y, z;

    // Fill table using above recursive formula. Note that the table
    // is filled in diagonal fashion (similar to http://goo.gl/PQqoS),
    // from diagonal elements to table[0][n-1] which is the result.
    for (gap = 0; gap < n; ++gap)
    {
        for (i = 0, j = gap; j < n; ++i, ++j)
        {
            // Here x is value of F(i+2, j), y is F(i+1, j-1) and
            // z is F(i, j-2) in above recursive formula
            x = ((i+2) <= j) ? table[i+2][j] : 0;
            y = ((i+1) <= (j-1)) ? table[i+1][j-1] : 0;
            z = (i <= (j-2))? table[i][j-2]: 0;

            table[i][j] = max(arr[i] + min(x, y), arr[j] + min(y, z));
        }
    }

    return table[0][n-1];
}

// Driver program to test above function
int main()
{
    int arr1[] = {8, 15, 3, 7};
    int n = sizeof(arr1)/sizeof(arr1[0]);
    printf("%d\n", optimalStrategyOfGame(arr1, n));

    int arr2[] = {2, 2, 2, 2};
    n = sizeof(arr2)/sizeof(arr2[0]);
    printf("%d\n", optimalStrategyOfGame(arr2, n));

    int arr3[] = {20, 30, 2, 2, 2, 10};
    n = sizeof(arr3)/sizeof(arr3[0]);
    printf("%d\n", optimalStrategyOfGame(arr3, n));

    return 0;
}
```

Run on IDE

Output:

```
22
4
42
```

Exercise

Your thoughts on the strategy when the user wishes to only win instead of winning with the maximum value. Like above problem, number of coins is even.

Can Greedy approach work quite well and give an optimal solution? Will your answer change if number of coins is odd?