Advanced Topics in Artificial Intelligence: EE6180
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Solution to PSET 4

• **Do not** distribute the solutions outside the class.

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1. **(A PAC-Bayesian Theorem)** (a) We have

$$
\begin{aligned}
(m-1)D(\hat{L}_S(Q)||L(Q)) &= (m-1)D\Big(\mathbb{E}_{h\sim Q}\hat{L}_S(h)||\mathbb{E}_{h\sim Q}L(h)\Big) \\
&\overset{\text{Jensen}}{\leq} (m-1)\mathbb{E}_{h\sim Q}\Big(D(\hat{L}_S(h)||L(h))\Big) \\
&\overset{\text{Donsker-Varadhan}}{\leq} D(Q||P) + \ln \mathbb{E}_{h\sim P}[e^{((m-1)D(\hat{L}_S(h)||L(h))}].
\end{aligned} \tag{1}
$$

(b)

I Since f is non-negative and non-increasing, we have

$$\mathbb{P}(f(X) \geq f(\epsilon)) \leq \mathbb{P}(X \leq \epsilon) \leq e^{-mf(\epsilon)}.$$

Thus, for any $t \geq 0$, we have:

$$\mathbb{P}(f(X) \geq t) \leq \min(1, e^{-mt}).$$

Hence,

$$
\begin{aligned}
\mathbb{E}[e^{(m-1)f(X)}] &= \int_0^\infty \mathbb{P}(e^{(m-1)f(X)} \geq z)dz \\
&= \int_0^\infty \mathbb{P}(f(X) \geq \ln z^{\frac{1}{m-1}})dz \\
&\leq \int_1^\infty z^{-\frac{m}{m-1}}dz + \int_0^1 1dz \\
&= m
\end{aligned}
$$

II Note that, the r.v. $\hat{L}_S(h)$ follows a binomial distribution with mean $L(h)$. Hence, using the standard Chernoff estimate for Binomial distribution, we have

$$\mathbb{P}(\hat{L}_S(h) \leq \epsilon) \leq e^{-mD^+(\epsilon||L(h))}.$$

Hence, using the result from part I., we have

$$\mathbb{E}_{S\sim D^m}[e^{(m-1)D^+(\hat{L}_S(h)||L(h))}] \leq m.$$

III Taking expectation of the above w.r.t. prior distribution $h \sim P$, we have:

$$\mathbb{E}_{h\sim P}\mathbb{E}_{S\sim D^m}[e^{(m-1)D^+(\hat{L}_S(h)||L(h))}] = \mathbb{E}_{S\sim D^m}\mathbb{E}_{h\sim P}[e^{(m-1)D^+(\hat{L}_S(h)||L(h))}] \le m$$

The result follows from an application of Markov's inequality.

IV The PAC-Bayes bound follows upon combining the above result with Eqn. (1).

2. **(Non-Parametric Least Square Estimation)** We first prove the following basic inequality:

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\hat{f}(x_i) - f^*(x_i)\Big)^2 \le \frac{2}{n}\sum_{i=1}^{n}\epsilon_i\Big(\hat{f}(x_i) - f^*(x_i)\Big). \tag{2}$$

This follows immediately from the definition of the least-square estimator. In particular, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\Big(Y_i - \hat{f}(x_i)\Big)^2 &= \frac{1}{n}\sum_{i=1}^{n}\Big(f^*(x_i) + \epsilon_i - \hat{f}(x_i)\Big)^2 \\ &\overset{(a)}{\le} \frac{1}{n}\sum_{i=1}^{n}\Big(f^*(x_i) + \epsilon_i - f^*(x_i)\Big)^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}\epsilon_i^2,\end{aligned}$$

where the inequality (a) follows from the least-square optimality of $\hat{f}$. The above inequality implies that

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\hat{f}(x_i) - f^*(x_i)\Big)^2 \le \frac{2}{n}\sum_{i=1}^{n}\epsilon_i\Big(\hat{f}(x_i) - f^*(x_i)\Big),$$

which proves (2). The RHS of (2) may be bounded further as follows:

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\hat{f}(x_i) - f^*(x_i)\Big)^2 \le \frac{4}{n}\Big(\sup_{f\in S_{\alpha,\gamma}(C_{\max},L)}\sum_{i=1}^{n}\epsilon_i f(x_i)\Big). \tag{3}$$

Taking expectation (w.r.t. $\{\epsilon_i\}_{i=1}^n$) of both sides, we have

$$\mathbb{E}\Big[\frac{1}{n}\sum_{i=1}^{n}\Big(\hat{f}(x_i) - f^*(x_i)\Big)^2\Big] \le \frac{4}{n}\mathcal{G}\big(S_{\alpha,\gamma}(C_{\max},L);\boldsymbol{x}\big),$$

where the RHS denotes the Gaussian complexity of the function class $S_{\alpha,\gamma}(C_{\max},L)$ induced by the points $\boldsymbol{x}$.
Note that, the diameter of the class $S_{\alpha,\gamma}(C_{\max},L)$ is $\delta = \sup_{x\in[0,1]}|f(x)-g(x)| \le 2C_{\max}$.

Define the Gaussian process, $X_f = \sum_{i=1}^n \epsilon_i f(x_i), \forall f \in S_{\alpha,\gamma}(C_{\max}, L)$. For any two $f, g \in S_{\alpha,\gamma}(C_{\max}, L)$, we have

$$X_f - X_g = \sum_{i=1}^{n} \epsilon_i \big(f(x_i) - g(x_i)\big).$$

The variance of $X_f - X_g$ is bounded as

$$\mathrm{Var}(X_f - X_g) = \sum_{i=1}^{n} \big(f(x_i) - g(x_i)\big)^2 \leq n||f - g||_{\infty}^2.$$

Hence, for each $f, g \in S_{\alpha,\gamma}(C_{\max}, L)$, $X_f - X_g \sim \mathsf{SG}\big(0, n||f - g||_{\infty}^2\big)$. Hence, using **Dudley's entropy integral**, we have

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\hat{f}(x_i) - f^*(x_i)\right)^2\right] \leq \frac{48}{\sqrt{n}} \int_0^{C_{\max}} \sqrt{C\left(\frac{1}{u}\right)^{1/(\alpha+\gamma)}} du = O\left(\frac{C_{\max}^{1-\frac{1}{2(\alpha+\gamma)}}}{\sqrt{n}}.\right)$$

3. **(Online Mirror Descent)** (a) Let $w^* \in \arg\min D_\psi(w, w'_{t+1})$ Setting the gradient of the objective function $D_\psi(w, w'_{t+1})$ to zero, we have

$$0 = \nabla D_\psi(w^*, w'_{t+1}) = \nabla\psi(w^*) - \nabla\psi(w'_{t+1}) = \nabla\psi(w^*) - \nabla\psi(w_t) + \eta\nabla f_t(w_t).$$

On the other hand, by definition,

$$0 = \nabla f_t(w_t) + \frac{1}{\eta}(\nabla\psi(w_{t+1}) - \nabla\psi(w_t)).$$

Combining the above two equations, we conclude that

$$\nabla\psi(w^*) = \nabla\psi(w_{t+1}).$$

Assuming that the mapping $\nabla\psi : \Omega \to \Omega$ is invertible, we have

$$w_{t+1} = w^*.$$

(b) The result follows upon simplifying the RHS using the definition of Bregman divergence and using the fact that

$$\nabla\psi(w'_{t+1}) = \nabla\psi(w_t) - \eta\nabla f_t(w_t).$$

(c) Since $w_{t+1} = \arg\min D_\psi(w, w'_{t+1})$, using the first order optimality criterion for convex function $D_\psi(\cdot, w'_{t+1})$, we have for any $u \in \Omega$:

$$\langle\nabla\psi(w_{t+1}) - \nabla\psi(w'_{t+1}), u - w_{t+1}\rangle \geq 0. \tag{4}$$

Hence,

$$\begin{aligned}
& D_\psi(u, w'_{t+1}) - D_\psi(u, w_{t+1}) \\
= \;& \underbrace{\psi(w_{t+1}) - \psi(w'_{t+1}) - \langle \nabla\psi(w'_{t+1}), w_{t+1} - w'_{t+1}\rangle}_{\geq 0 \text{ (as } \psi \text{ is convex)}} + \underbrace{\langle \nabla\psi(w_{t+1}) - \nabla\psi(w'_{t+1}), u - w_{t+1}\rangle}_{\geq 0 \text{ from eqn.(4)}} \\
\geq \;& 0
\end{aligned}$$

(d) The result follows from (b) and (c) upon telescoping the summation.
(e) In the experts setting, we have for any $w_t \in \Delta_N$:

$$f_t(w_t) = \langle w_t, l_t\rangle.$$

Take $\psi : \Omega \to \mathbb{R}$ to be the negative entropy function, i.e., $\psi(w) = \sum_i w_i \ln w_i$. Hence,

$$D_\psi(w, u) = \sum_i w_i \ln \frac{w_i}{u_i} - \sum_i (w_i - u_i).$$

Also, the mirror action w'_{t+1} satisfies the equation:

$$1 + \ln w'_{t+1,i} = 1 + \ln w_{t,i} - \eta l_{t,i},$$

i.e, $w'_{t+1,i} = w_{t,i}\exp(-\eta l_{t,i})$, which leads to the `Hedge` algorithm. Thus,

$$\begin{aligned}
D_\psi(w_t, w'_{t+1}) &= \sum_i w_{t,i} \ln \frac{w_{t,i}}{w'_{t+1,i}} - 1 + \sum_i w'_{t+1,i} \\
&= \sum_i w_{t,i}\big(e^{-\eta l_{t,i}} - \eta l_{t,i} - 1\big) \\
&\leq \eta^2,
\end{aligned}$$

where in the last step, we have used the inequality $e^{-x} \leq 1 - x + x^2, \forall x \geq 0$ and the fact that $w_t \in \Delta_N$. Also, taking w_1 to be uniform over $[N]$, we have $D_\psi(u, w_1) \leq \log N$. Hence, using the regret bound for OMD from part (d), we have

$$R_T^{\texttt{Hedge}} \leq \frac{\log N}{\eta} + T\eta.$$

Choosing $\eta = \sqrt{\frac{\log N}{T}}$, we have $R_T^{\texttt{Hedge}} \leq 2\sqrt{T \log N}$.

4. **(Foresight and Hindsight Regret for the IID Cost Model)**

(1) Consider the clean event $\mathcal{E}$:

$$\mathcal{E} = \{|\bar{\mu}_T(a) - \mu(a)| \leq r_T, \forall a\},$$

where $r_T = \sqrt{\frac{\log(KT)}{T}}$. Using Hoeffding's inequality and union bound, we have $\mathbb{P}(\mathcal{E}) \geq 1 - \frac{1}{KT^2}$. Assuming the clean event, we have for each arm $a \in \mathcal{A}$:

$$\mathbb{E}(\texttt{cost}(a)) \leq \texttt{cost}(a) + \sqrt{T \log(KT)}.$$

Hence,

$$\min_a \mathbb{E}(\texttt{cost}(a)) \leq \min_a \texttt{cost}(a) + \sqrt{T \log(KT)}$$

On the other hand, on $\mathcal{E}^c$, we have the trivial lower bound, $\min_a \texttt{cost}(a) \geq 0$. Hence,

$$\begin{aligned} & \mathbb{E}(\min_a \texttt{cost}(a)) \\ \geq \;& (1 - \frac{1}{KT^2})\Big(\min_a \mathbb{E}(\texttt{cost}(a)) - \sqrt{T \log(KT))}\Big). \end{aligned}$$

Noting that $\min_a \mathbb{E}(\texttt{cost}(a)) \leq T$, the above equation implies that

$$\min_a \mathbb{E}(\texttt{cost}(a)) \leq \mathbb{E}(\min_a \texttt{cost}(a)) + O(\sqrt{T \log(KT)}).$$

(2) Fix any algorithm. We consider an ensemble of problem instances where all-arms have $0-1$ costs with mean $\frac{1}{2}$. Hence, irrespective of the choice that the algorithm makes, we have

$$\mathbb{E}(\texttt{cost(ALG)}) = \frac{T}{2}.$$

Further, from Hoeffding's inequality, we have

$$\mathbb{E}[\min_a \texttt{cost}(a)] \leq \frac{T}{2} - \Omega(\sqrt{T \log K}).$$

Combining the above two results, we conclude that

$$\mathbb{E}[\texttt{cost(ALG)} - \min_a \texttt{cost}(a)] \geq \Omega(\sqrt{T \log K}),$$

where the expectation is over the random ensemble of problem instances and the randomness of the algorithm. Hence, for any algorithm, there must exist a problem instant for which the above lower bound holds.

(3) Under the clean event, which happens w.h.p., we have for all $a \in \mathcal{A}$:

$$|\texttt{cost}(a) - T\mu(a)| \leq \sqrt{T \log T}. \tag{5}$$

Next, divide the set of arms in two categories:

(a) **Category-I** ($a \in \mathcal{A} : T\Delta(a) > 3\sqrt{T \log T}$): Clearly, $\arg\min_a \texttt{cost}(a) \equiv a^{**}$ does not belong to this category.

(b) **Category-II** ($a \in \mathcal{A} : T\Delta(a) \leq 3\sqrt{T \log T}$): Clearly, a^{**} belongs to this category. Hence, we have

$$\begin{aligned} & \texttt{cost}(a^{**}) \geq T\mu(a^{**}) - \sqrt{T \log T} \\ = \; & T\Delta(a^{**}) + T\mu(a^{*}) - \sqrt{T \log T} \\ = \; & \mathbb{E}(\texttt{cost}(a^{*})) + \log(T)\left(\Delta(a^{**})\frac{T}{\log T} - \sqrt{\frac{T}{\log T}}\right). \end{aligned}$$

Next, we lower bound the term in the bracket. Define $x = \sqrt{\frac{T}{\log T}}$. Using calculus, we see that

$$\Delta(a^{**})x^2 - x \geq -\frac{1}{4\Delta(a^{**})}.$$

This yields,

$$\texttt{cost}(a^{**}) \geq \mathbb{E}(\texttt{cost}(a^{*})) - \frac{\log T}{4\Delta(a^{**})}.$$

Since, the above event (5) happens w.h.p., we have

$$\mathbb{E}(\min_a \texttt{cost}(a)) \geq \min_a \mathbb{E}(\texttt{cost}(a)) - O(\log(T)/\Delta(a^{**})).$$

Thus, we see that the instance dependent logarithmic regret bound carries over for hindsight regret for `UCB` and Successive Elimination algorithms.