Problem 1

The adversary in the question is deterministic and oblivious. So, the adversary has fixed a $N \times T$-dimensional cost table prior to time step 1. Here, N is the number of experts and T is the number of time steps for which the process is continued. As the algorithm is fixed, the adversary knows what the algorithm will do at each time step and hence it can rig the costs similarly.

Consider the situation where for time step t, the expert chosen by the deterministic algorithm incurs a loss of 1 and the rest of the experts incur a loss of 0. So, for T time steps, the algorithm suffers a total loss of T. Now, let $E_1, E_2, \ldots, E_N$ be the costs incurred by the N experts. Let the best expert suffer a loss of E^*. As, we have assumed that any time instant exactly one expert suffers a loss of 1, we can say that:

$$\sum_{i=1}^{N} E^* = NE^* \leq \sum_{i=1}^{N} E_i = T$$
$$\implies E^* \leq T/N$$

So, even if the best expert incurs at most loss of T/N, a deterministic algorithm can still incur a loss of T.

■

Problem 2

Let $\mathcal{F} = \{h_1, h_2, \ldots, h_{|\mathcal{F}|}\}$ be the set of classifiers. Let $\hat{y}_t, y_t \in \{0, 1\}, \forall t \in [T]$. So, total number of mistakes done over T time steps is given by:

$$M_T = \sum_{t=1}^{T} \mathbf{1}\left(\hat{y}_t \neq y_t\right) = \sum_{t=1}^{T} |\hat{y}_t - y_t| \tag{1}$$

Suppose the probability distribution over the $|\mathcal{F}|$ classifiers obtained from the Hedge algorithm be $p_t \in \mathbb{R}^{|\mathcal{F}|}$. Let the prediction of the ith classifier for $\boldsymbol{x}_t$ be $f_i(\boldsymbol{x}_t) \in \{0, 1\}$. So, the probability for predicting 1 at time step t is $q_t = \langle p_t, f_t \rangle$, where $f_t = [f_1(\boldsymbol{x}_t), \ldots, f_{|\mathcal{F}|}(\boldsymbol{x}_t)]^\top \in \{0, 1\}^{|\mathcal{F}|}$. So, total number of mistakes in expectation upto time T is given as:

$$\begin{aligned}
\mathbb{E}[M_T] &= \mathbb{E}\left[\sum_{t=1}^{T} |\hat{y}_t - y_t|\right] \\
&= \sum_{t=1}^{T} q_t|1 - y_t| + (1 - q_t)|y_t| \\
&= \sum_{t=1}^{T} q_t(1 - y_t) + y_t(1 - q_t) && y_t \in \{0, 1\} \\
&= \sum_{t=1}^{T} q_t + y_t - 2y_t q_t \\
&\overset{(a)}{=} \sum_{t=1}^{T} |q_t - y_t| && (2)
\end{aligned}$$

The equality (a) holds because if $y_t = 1, q_t + y_t - 2y_t q_t = 1 - q_t$ and if $y_t = 0, q_t + y_t - 2y_t q_t = q_t \implies q_t + y_t - 2y_t q_t = |q_t - y_t|$.

$$\begin{aligned}
|q_t - y_t| &= |\langle p_t, f_t \rangle - y_t| \\
&= \left| \sum_{i=1}^{|\mathcal{F}|} p_t[i] f_i(\boldsymbol{x}_t) - y_t \right| \\
&= \left| \sum_{i=1}^{|\mathcal{F}|} p_t[i] f_i(\boldsymbol{x}_t) - \sum_{i=1}^{|\mathcal{F}|} p_t[i] y_t \right| && \text{As } \sum_{i=1}^{|\mathcal{F}|} p_t[i] = 1 \\
&= \left| \sum_{i=1}^{|\mathcal{F}|} p_t[i] \left(f_i(\boldsymbol{x}_t) - y_t\right) \right| \\
&\leq \sum_{i=1}^{|\mathcal{F}|} p_t[i] \left| f_i(\boldsymbol{x}_t) - y_t \right| \\
&= \sum_{i=1}^{|\mathcal{F}|} p_t[i] l_t[i] = \langle p_t, l_t \rangle && \text{where } l_t[i] = |f_i(\boldsymbol{x}_t) - y_t| \qquad (3)
\end{aligned}$$

So, we have the following relation:

$$\mathbb{E}[M_T] \overset{(2)}{=} \sum_{t=1}^{T} |q_t - y_t| \overset{(3)}{\leq} \sum_{t=1}^{T} \langle p_t, l_t \rangle \tag{4}$$

Now, $0 \leq l_t[i] = |f_i(\boldsymbol{x}_t) - y_t| \leq 1$,i.e., $l_t[i]$ is bounded. As we have a perfect classifier, the regret and total loss is same as $l_t(i^*) = f_{i^*}(\boldsymbol{x}_t) - y_t = 0, \forall t \in [T]$. So, using the regret bounds for Hedge algorithm, we get:

$$\begin{aligned}
R_T &= \sum_{t=1}^{T} \langle p_t, l_t \rangle \\
&\leq \frac{\ln |\mathcal{F}|}{\eta} + \eta \sum_{t=1}^{T} \langle p_t, l_t^2 \rangle \\
&\leq \frac{\ln |\mathcal{F}|}{\eta} + \eta \sum_{t=1}^{T} \langle p_t, l_t \rangle && \text{As } 0 \leq l_t[i] \leq 1 \\
\implies (1-\eta) \sum_{t=1}^{T} \langle p_t, l_t \rangle &\leq \frac{\ln |\mathcal{F}|}{\eta} \\
\implies \sum_{t=1}^{T} \langle p_t, l_t \rangle &\leq \frac{\ln |\mathcal{F}|}{\eta(1-\eta)} = 4 \ln |\mathcal{F}| && \text{As } \eta = \frac{1}{2} \qquad (5)
\end{aligned}$$

Combining (4) and (5), we get:

$$\mathbb{E}[M_T] \leq \sum_{t=1}^{T} \langle p_t, l_t \rangle \leq 4 \ln |\mathcal{F}|$$

It is to be noted that even if we assumed the total number of mistakes upto time T, the upper bound is independent of T.

■

Problem 3

The strategy of FTPL is given as:

$$i_t = \underset{i}{\text{argmin}}\ (L_{t-1}(i) - L_0(i)), \tag{6}$$

where $L_0(i), 1 \leq i \leq N$ are N i.i.d. random variables from the *Gumbel* distribution, i.e., $P(L_0(i) \leq x) = \exp(-\exp(-\eta x))), \eta > 0, \forall i$

(a) From (6),

$$\begin{aligned}
i_t &= \underset{i}{\text{argmin}}\ (L_{t-1}(i) - L_0(i)) \\
&= \underset{i}{\text{argmin}}\ \exp((\eta L_{t-1}(i) - \eta L_0(i))) \qquad f(x) = e^{\eta x} \text{ is non-decreasing for } x \in \mathbb{R}, \eta > 0 \\
&= \underset{i}{\text{argmin}}\ \frac{\exp(\eta L_{t-1}(i))}{\exp(\eta L_0(i)} \\
&= \underset{i}{\text{argmax}}\ \frac{\exp(-\eta L_{t-1}(i))}{\exp(-\eta L_0(i)}
\end{aligned} \tag{7}$$

So, we have $P(i_t = j) = P(j = \text{argmax}_i\ \frac{\exp(-\eta L_{t-1}(i))}{\exp(-\eta L_0(i)})$.

(b) It is known as $L_0(i)$ are random variables from the *Gumbel* distribution. So, for $\eta > 0$,

$$\begin{aligned}
P(L_0(i) \leq x) &= P(-\eta L_0(i) \geq -\eta x) \\
&= P(\exp(-\eta L_0(i)) \geq \exp(-\eta x)) \\
&= \exp(-\exp(-\eta x)))
\end{aligned}$$

Let $v(i) = L_0(i)$ and $t = \exp(-\eta x)$, we get:

$$\begin{aligned}
P(\exp(-\eta L_0(i)) \geq \exp(-\eta x)) &= \exp(-\exp(-\eta x))) \\
\implies P(v(i) \geq t) &= \exp(-t) \\
\implies \text{CDF}(v(i)) &= 1 - \exp(-t)
\end{aligned}$$

As the CDF follows that of the standard exponential distribution, we conclude $v(i) \sim \exp(1)$.

(c) We begin this proof by proving two lemmas.

Lemma 1. *If* $v(i) \sim \exp(1) \implies \frac{v(i)}{a(i)} \sim \exp(a(i))$ *for* $a(i) \geq 0$.

Proof. If $v(i) \sim \exp(1)$,

$$\begin{aligned}
P(v(i) \geq t) &= \exp(-t) \\
\implies P(v(i) \geq a(i)t) &= \exp(-a(i)t) \qquad \forall a(i) \geq 0 \\
\implies P\left(\frac{v(i)}{a(i)} \geq t\right) &= \exp(-a(i)t) \\
\implies \text{CDF}\left(\frac{v(i)}{a(i)}\right) &= 1 - \exp(-a(i)t) \implies \frac{v(i)}{a(i)} \sim \exp(a(i))
\end{aligned}$$

□

Lemma 2. *If $X_i \sim \exp(a_i), a_i \geq 0, i \in [N]$, and $X_1, \ldots, X_n$ are i.i.d., then*

$$P(j = \text{argmin}\{X_1, \ldots, X_n\}) = \frac{a_j}{\sum_{i=1}^N a_i}$$

Proof.

$$\begin{aligned}
P(j = \underset{i}{\text{argmin}}\, X_i) &= \int_0^\infty P(X_j = x) P(X_{i \neq j} \geq x) dx \\
&= \int_0^\infty a_j \exp(-a_j x) \left(\prod_{i \neq j} \exp(-a_i x) \right) dx \\
&= a_j \int_0^\infty \exp\left(-\sum_{i=1}^N a_i x \right) dx \\
&= \frac{a_j}{\sum_{i=1}^N a_i} \qquad (8)
\end{aligned}$$

□

Using Lemma 1, we get that $\frac{v(i)}{a(i)} \sim \exp(a(i)), a(i) \geq 0, i \in [N]$. Then, by using Lemma 2, we get that $P\left(j = \text{argmin}_i \frac{v(i)}{a(i)}\right) = P\left(j = \text{argmax}_i \frac{a(i)}{v(i)}\right) = \frac{a_j}{\sum_{i=1}^N a_i}$.

If we assign $a(i) = \exp(-\eta L_{t-1}(i))$, then from part (a), we get:

$$\begin{aligned}
P(i_t = j) &\overset{(7)}{=} P\left(j = \underset{i}{\text{argmax}}\ \frac{\exp(-\eta L_{t-1}(i))}{\exp(-\eta L_0(i)}\right) \\
&\overset{(8)}{=} \frac{\exp(-\eta L_{t-1}(j))}{\sum_{i=1}^N \exp(-\eta L_{t-1}(i))}
\end{aligned}$$

Hence, the probability of picking the *ith* expert while using FTPL with Gumbel noise is the same as the weightage given to the *ith* expert in the Hedge algorithm, i.e., FTPL with Gumbel noise is equivalent to sample an expert using Hedge's prediction.

■

Problem 4

Let us break down the total time horizon T into fragments of $2^k, k \geq 1$ and calculate the total regret, R_T, as the sum of regrets of the individual fragments, R_{T_i}. Let $K \geq 2$ be the smallest integer such that:

$$\sum_{i=1}^{K-1} 2^i \geq T \implies (2^K - 2) \geq T \implies K = \lceil \log_2(T+2) \rceil \leq \log_2(T+2) + 1 \tag{9}$$

$$\begin{aligned}
R_T &\leq \sum_{i=1}^{K-1} R_{T_i} \\
&= \sum_{i=1}^{K-1} \sqrt{2T_i \ln N} \\
&= \sqrt{2 \ln N} \sum_{i=1}^{K-1} 2^{i/2} \\
&= \sqrt{2 \ln N} \cdot \frac{\sqrt{2}(2^{K/2} - 1)}{\sqrt{2} - 1} \\
&\overset{(9)}{\leq} \frac{\sqrt{4 \ln N}}{\sqrt{2} - 1} \cdot (\sqrt{2(T+2)} - 1) \\
&\leq \frac{\sqrt{4 \ln N}}{\sqrt{2} - 1} \cdot \sqrt{2(T+2)} \\
&= \mathcal{O}(\sqrt{T \ln N})
\end{aligned}$$

$\blacksquare$

Problem 5

(a) We apply the Hedge algorithm on the set of meta experts, $\mathcal{M}$. We need to estimate $|\mathcal{M}|$. The $S-1$ switches can take place in any of T time steps and in each switch any one out of the n competitors can be chosen. So, we have:

$$|\mathcal{M}| \leq \binom{T}{S-1} n^{S-1} \leq \binom{T}{S} n^S \leq \left(\frac{nTe}{S}\right)^S \tag{10}$$

The last inequality is obtained using $\binom{n}{k} \leq (\frac{ne}{k})^k$. Using the regret guarantees for Hedge algorithm, we obtain:

$$\begin{aligned}
\mathcal{R}_T^{Hedge}(e_1, \ldots, e_T) &\leq 2\sqrt{T \ln |\mathcal{M}|} \\
&\overset{(10)}{\leq} 2\sqrt{T \ln \left(\frac{nTe}{S}\right)^S} \\
&= 2\sqrt{TS \ln \frac{nTe}{S}}
\end{aligned}$$

(b) We are given the following conditions of the Fixed-Share algorithm.

$$p_t = \sum_{\tau=1}^{t} \alpha_t(\tau) \tilde{p}_\tau \tag{11}$$

$$\tilde{p}_{t+1} \propto p_t(i) \exp(-\eta l_t(i)), \forall i \tag{12}$$

We start by analysing $D(q_t || \tilde{p}_{t+1})$.

$$\begin{aligned}
D(q_t || \tilde{p}_{t+1}) &= \sum_i q_t(i) \ln \frac{q_t(i)}{\tilde{p}_{t+1}(i)} \\
&\overset{(12)}{=} \sum_i q_t(i) \ln \frac{q_t(i) \sum_{j=1}^N p_t(j) \exp(-\eta l_t(j))}{p_t(i) \exp(-\eta l_t(i))} \\
&= \sum_i q_t(i) \left(\ln \frac{q_t(i)}{p_t(i)} + \ln \left(\sum_{j=1}^N p_t(j) \exp(-\eta l_t(j)) \right) + \eta l_t(i) \right) \\
&= \sum_i q_t(i) \ln \frac{q_t(i)}{p_t(i)} + \ln \left(\sum_{j=1}^N p_t(j) \exp(-\eta l_t(j)) \right) + \eta \langle q_t, l_t \rangle \\
&\overset{(a)}{\leq} \sum_i q_t(i) \ln \frac{q_t(i)}{p_t(i)} + \ln \left(\sum_{j=1}^N p_t(j)(1 - \eta l_t(i) + \eta^2 l_t(i)^2) \right) + \eta \langle q_t, l_t \rangle \\
&= \sum_i q_t(i) \ln \frac{q_t(i)}{p_t(i)} + \ln \left(1 - \eta \langle p_t, l_t \rangle + \eta^2 \langle p_t, l_t^2 \rangle \right) + \eta \langle q_t, l_t \rangle \\
&\overset{(b)}{\leq} \sum_i q_t(i) \ln \frac{q_t(i)}{p_t(i)} - \eta \langle p_t, l_t \rangle + \eta^2 \langle p_t, l_t^2 \rangle + \eta \langle q_t, l_t \rangle
\end{aligned}$$

$$
\begin{aligned}
&\overset{(c)}{\leq} \sum_i q_t(i) \ln \frac{q_t(i)}{p_t(i)} - \eta \langle p_t - q_t, l_t \rangle + \eta^2 \\
&= \sum_i q_t(i) \left(\ln \frac{q_t(i)}{\tilde{p}_{s_t+1}(i)} + \ln \frac{\tilde{p}_{s_t+1}(i)}{p_t(i)} \right) - \eta \langle p_t - q_t, l_t \rangle + \eta^2 \\
&= D(q_t || \tilde{p}_{s_t+1}) + \sum_i q_t(i) \ln \frac{\tilde{p}_{s_t+1}(i)}{p_t(i)} - \eta \langle p_t - q_t, l_t \rangle + \eta^2 \\
&\overset{(d)}{\leq} D(q_t || \tilde{p}_{s_t+1}) + \sum_i q_t(i) \ln \left(\frac{1}{\alpha_t(s_t+1)} \right) - \eta \langle p_t - q_t, l_t \rangle + \eta^2 \\
&= D(q_t || \tilde{p}_{s_t+1}) + \ln \left(\frac{1}{\alpha_t(s_t+1)} \right) - \eta \langle p_t - q_t, l_t \rangle + \eta^2
\end{aligned}
$$

Rearranging,

$$
\langle p_t - q_t, l_t \rangle \leq \frac{\ln\left(\frac{1}{\alpha_t(s_t+1)}\right) + D(q_t||\tilde{p}_{s_t+1}) - D(q_t||\tilde{p}_{t+1})}{\eta} + \eta
$$

The inequality (a) follows because $e^{-x} \leq 1 - x + x^2, x \geq 0$,
(b) follows as $\ln(1+x) \leq x, x \in \mathbb{R}$,
(c) holds as $\langle p_t, l_t^2 \rangle = \sum_{i=1}^N p_t(i) l_t(i)^2 \leq \sum_{i=1}^N p_t(i) = 1$ as $0 \leq l_t(i) \leq 1$,
and (d) holds because, using (11), we have $p_t(i) = \sum_{\tau=1}^t \alpha_t(\tau) \tilde{p}_\tau(i) \geq \alpha_t(s_t+1) \tilde{p}_{s_t+1}(i) \implies \frac{\tilde{p}_{s_t+1}(i)}{p_t(i)} \leq \frac{1}{\alpha_t(s_t+1)}$ as all the terms are non-negative.

(c) To get the total regret, we need aggregate the regrets at each time step t. Let $\mathcal{E}$ be the set of n competitors that can be chosen at any time. Let t_k^f be the final time step when kth competitor appears.

$$
\begin{aligned}
\mathcal{R}_T(q_1, \ldots, q_T) &= \sum_{t=1}^T \langle p_t - q_t, l_t \rangle \\
&= \sum_{t=1}^T \left(\frac{\ln\left(\frac{1}{\alpha_t(s_t+1)}\right) + D(q_t||\tilde{p}_{s_t+1}) - D(q_t||\tilde{p}_{t+1})}{\eta} + \eta \right) \\
&= \frac{1}{\eta} \sum_{t=1}^T \ln \left(\frac{1}{\alpha_t(s_t+1)} \right) + \eta T + \frac{1}{\eta} \sum_{t=1}^T \left(D(q_t||\tilde{p}_{s_t+1}) - D(q_t||\tilde{p}_{t+1}) \right) \\
&= \frac{1}{\eta} \sum_{t=1}^T \ln \left(\frac{1}{\alpha_t(s_t+1)} \right) + \eta T + \frac{1}{\eta} \sum_{i \in \mathcal{E}} \sum_{q_t = i} \left(D(q_t||\tilde{p}_{s_t+1}) - D(q_t||\tilde{p}_{t+1}) \right) \\
&= \frac{1}{\eta} \sum_{t=1}^T \ln \left(\frac{1}{\alpha_t(s_t+1)} \right) + \eta T + \frac{1}{\eta} \sum_{i \in \mathcal{E}} \left(D(q_t||\tilde{p}_1) - D(q_t||\tilde{p}_{t_i^f+1}) \right) \\
&\leq \frac{1}{\eta} \sum_{t=1}^T \ln \left(\frac{1}{\alpha_t(s_t+1)} \right) + \eta T + \frac{1}{\eta} \sum_{i \in \mathcal{E}} D(q_t||\tilde{p}_1)
\end{aligned}
$$

$$= \frac{1}{\eta}\sum_{t=1}^{T}\ln\left(\frac{1}{\alpha_t(s_t+1)}\right) + \eta T + \frac{n\ln N}{\eta}$$

The last step follows from the fact that $\tilde{p}_1$ is uniformly distributed and q_t has unit mass in exactly one coordinate $\implies D(q_t||\tilde{p}_1) = \ln N$ and $|\mathcal{E}| = n$.

(d) We prove an important result that would be instrumental in this proof.

Lemma 3. *For* $0 \le x \le \frac{1}{2}$,

$$(1-x)\ln\frac{1}{1-x} \le x\ln\frac{1}{x}$$

Proof. Consider the function $f(t) = t\ln\frac{1}{t} = -t\ln t$.

$$\begin{aligned} f(t) &= -t\ln t \\ f'(t) &= -1-\ln t \\ f''(t) &= -\frac{1}{t^2} \end{aligned}$$

As $f''(t) < 0$ for $\forall t \in \mathbb{R}$, $f(t)$ is concave. To find the maximum of $f(t)$, we equate $f'(t) = 0 \implies t = 1/e < 1/2$. Let us consider two cases:

(a) When $0 \le x \le \frac{1}{e}$:

As $0 \le x \le \frac{1}{e} \implies e \le 1/x < \infty \implies 1 \le \ln(1/x) < \infty$. Since $f(t)$ is concave, using first-order conditions for concavity, we have:

$$\begin{aligned} & f(y) \le f(x) + f'(x)(y-x) \\ \implies & f(1-x) \le f(1) + f'(1)(1-x-1) \\ \implies & (1-x)\ln\frac{1}{1-x} \le x && f(1)=0, f'(1)=-1 \\ \implies & (1-x)\ln\frac{1}{1-x} \le x\ln\frac{1}{x} && \ln(1/x)\in[1,\infty) \end{aligned}$$

(b) When $\frac{1}{e} \le x \le \frac{1}{2}$:

If $\frac{1}{e} \le x \le \frac{1}{2} \implies \frac{1}{e} \le x \le \frac{1}{2} \le 1-x \le 1-\frac{1}{e}$. Further more for $x \ge \frac{1}{e}, f'(x) < 0$. So, we have the result:

$$\begin{aligned} & f(1-x) \le f(x) && \text{As } 1-x \ge x, \forall x \in [1/e, 1/2] \\ \implies & (1-x)\ln\frac{1}{1-x} \le x\ln\frac{1}{x} \end{aligned}$$

So, $\forall x \in [0, 1/2]$, we have

$$(1-x)\ln\frac{1}{1-x} \le x\ln\frac{1}{x}$$

□

We consider the uniform mixing case, i.e., $\alpha_t(t) = 1-\alpha$ and $\alpha_t(\tau) = \frac{\alpha}{t-1}, \forall \tau < t$. Let the $X \subset [T]$ be the set of time steps when there a switching took place. So, we have $|X| = S$. If there is no switch at a time step t, then by definition, $s_t = t-1 \implies s_t+1 = t \implies \alpha_t(s_t+1) = 1-\alpha$. Similarly, if there is a switch at time step t, then, $s_t < t-1 \implies \alpha_t(s_t+1) = \frac{\alpha}{t-1}$.

$$\begin{aligned}
\sum_{t=1}^{T} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) &= \sum_{t\in X} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) + \sum_{t\in T\setminus X} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) \\
&= \sum_{t\in X} \ln\left(\frac{t-1}{\alpha}\right) + \sum_{t\in T\setminus X} \ln\left(\frac{1}{1-\alpha}\right) \\
&\leq \sum_{t\in X} \ln\left(\frac{T}{\alpha}\right) + \sum_{t\in T\setminus X} \ln\left(\frac{1}{1-\alpha}\right) \\
&= S\ln\left(\frac{T}{\alpha}\right) + (T-S)\ln\left(\frac{1}{1-\alpha}\right) \\
&= S\ln T + S\ln\left(\frac{1}{\alpha}\right) + (T-S)\ln\left(\frac{1}{1-\alpha}\right)
\end{aligned}$$

The expression involving α attain its maxima at $\alpha^* = S/T$. So, we get,

$$\begin{aligned}
RHS &\leq S\ln T + S\ln\left(\frac{T}{S}\right) + (T-S)\ln\left(\frac{1}{1-\frac{S}{T}}\right) \\
&= S\ln T + T\left(\frac{S}{T}\ln\left(\frac{T}{S}\right) + \left(1-\frac{S}{T}\right)\ln\left(\frac{1}{1-\frac{S}{T}}\right)\right) \\
&\leq S\ln T + T\cdot\frac{2S}{T}\ln\left(\frac{T}{S}\right) && \text{Lemma 3} \\
&\leq S\ln T + 2S\ln T = 3S\ln T && (13)
\end{aligned}$$

To get an upper bound on the total regret,

$$\begin{aligned}
\mathcal{R}_T(q_1,\ldots,q_T) &= \frac{1}{\eta}\sum_{t=1}^{T} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) + \eta T + \frac{n\ln N}{\eta} \\
&\overset{(13)}{\leq} \frac{3S\ln T + n\ln N}{\eta} + \eta T \\
&= 2\sqrt{T(3S\ln T + n\ln N)}
\end{aligned}$$

The last step follows by minimizing the terms involving η to get the tightest regret bound which is attained at $\eta^* = \sqrt{\frac{3S\ln T + n\ln N}{T}}$.

(e) We consider the decaying mixing case, i.e., $\alpha_t(t) = 1-\alpha$ and $\alpha_t(\tau) = \frac{\alpha}{(t-\tau)Z_t}$, where $Z_t = \sum_{\tau=1}^{t-1}\frac{1}{t-\tau} \leq \ln t, \forall \tau < t$. Let the $X \subset [T]$ be the timesteps when there a switching took place. So, we have $|X| = S$. Furthermore, as there are exactly n competitors, we can partition X into n partitions $X_1, \ldots, X_n$ such that partition X_i contains the time steps when the adversary switched to the $ith, i \in [n]$ competitor. If there is no switch at a time step t, then by definition,

$s_t = t-1 \implies s_t+1 = t \implies \alpha_t(s_t+1) = 1-\alpha$. Similarly, if there is a switch at time step t, then, $s_t < t-1 \implies \alpha_t(s_t+1) = \frac{\alpha}{(t-\tau)Z_t} \geq \frac{\alpha}{(t-\tau)\ln t} \geq \frac{\alpha}{(t-\tau)\ln T}$.

$$
\begin{aligned}
\sum_{t=1}^{T} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) &= \sum_{t\in X} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) + \sum_{t\in T\setminus X} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) \\
&\leq \sum_{t\in X} \ln\left(\frac{(t-(s_t+1))\ln T}{\alpha}\right) + \sum_{t\in T\setminus X} \ln\left(\frac{1}{1-\alpha}\right) \\
&= \sum_{t\in X} \ln\left(\frac{t-(s_t+1)}{\alpha}\right) + \sum_{t\in X} \ln(\ln T) + \sum_{t\in T\setminus X} \ln\left(\frac{1}{1-\alpha}\right) \\
&\leq \sum_{t\in X} \ln\left(\frac{t-s_t}{\alpha}\right) + S\ln(\ln T) + (T-S)\ln\left(\frac{1}{1-\alpha}\right) \\
&= S\cdot\frac{1}{S}\sum_{t\in X} \ln(t-s_t) + S\ln\left(\frac{1}{\alpha}\right) + S\ln(\ln T) + (T-S)\ln\left(\frac{1}{1-\alpha}\right) \\
&\overset{(a)}{\leq} S\ln\left(\frac{1}{S}\sum_{t\in X}(t-s_t)\right) + S\ln\left(\frac{1}{\alpha}\right) + S\ln(\ln T) + (T-S)\ln\left(\frac{1}{1-\alpha}\right) \\
&= S\ln\left(\frac{1}{S}\sum_{X_i,\dots,X_n}\sum_{t\in X_i}(t-s_t)\right) + S\ln\left(\frac{1}{\alpha}\right) + S\ln(\ln T) + (T-S)\ln\left(\frac{1}{1-\alpha}\right) \\
&\overset{(b)}{\leq} S\ln\left(\frac{1}{S}\sum_{X_i,\dots,X_n} t_i^f\right) + S\ln(\ln T) + S\ln\left(\frac{1}{\alpha}\right) + (T-S)\ln\left(\frac{1}{1-\alpha}\right) \\
&\overset{(c)}{\leq} S\ln\left(\frac{nT}{S}\right) + S\ln(\ln T) + S\ln\left(\frac{1}{\alpha}\right) + (T-S)\ln\left(\frac{1}{1-\alpha}\right)
\end{aligned}
$$

The inequality (a) follows from the Jensen's inequality,
(b) follows by telescoping the sum inside the second summation and t_i^f is the final time step when the ith competitor appears. The sum telescopes because the cancelling terms are the time steps when a particular ith competitor appears, i.e.,

$$
\begin{aligned}
\sum_{t\in X_i}(t-s_t) &= t_i^f - s_{t_i^f} + s_{t_i^f} - s_{s_{t_i^f}} + \cdots - 1 \\
&= t_i^f - 1 < t_i^f
\end{aligned}
$$

(c) holds as $t_i^f \leq T, \forall i \in [n]$. The expression involving α attain its maxima at $\alpha^* = S/T$. So,

$$
\begin{aligned}
RHS &\leq S\ln\left(\frac{nT}{S}\right) + S\ln(\ln T) + S\ln\left(\frac{T}{S}\right) + (T-S)\ln\left(\frac{1}{1-\frac{S}{T}}\right) \\
&= S\ln\left(\frac{nT}{S}\right) + S\ln(\ln T) + T\left(\frac{S}{T}\ln\left(\frac{T}{S}\right) + \left(1-\frac{S}{T}\right)\ln\left(\frac{1}{1-\frac{S}{T}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq S\ln(\ln T) + S\ln\left(\frac{nT}{S}\right) + T\cdot\frac{2S}{T}\ln\left(\frac{T}{S}\right) && \text{Lemma 3} \\
&\leq S\ln(\ln T) + 3S\ln\left(\frac{nT}{S}\right) && n \geq 1 \quad (14)
\end{aligned}
$$

To get an upper bound on the total regret,

$$
\begin{aligned}
\mathcal{R}_T(q_1,\ldots,q_T) &= \frac{1}{\eta}\sum_{t=1}^{T}\ln\left(\frac{1}{\alpha_t(s_t+1)}\right) + \eta T + \frac{n\ln N}{\eta} \\
&\overset{(14)}{\leq} \frac{S\ln(\ln T) + 3S\ln\left(\frac{nT}{S}\right) + n\ln N}{\eta} + \eta T \\
&= 2\sqrt{T\left(S\ln(\ln T) + 3S\ln\left(\frac{nT}{S}\right) + n\ln N\right)}
\end{aligned}
$$

The last step follows by minimizing the terms involving η to get the tightest regret bound which is attained at $\eta^* = \sqrt{\frac{S\ln(\ln T)+3S\ln\left(\frac{nT}{S}\right)+n\ln N}{T}}$.

$\blacksquare$