EE 6180: Advanced Topics in Artificial Intelligence October 22, 2020

Lecture 14

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1 Follow the Perturbed Leader (FTPL)

Recall the FTPL algorithm as described in the last lecture.

Algorithm 1 Follow the Perturbed Leader

procedure FTPL
Input: $\eta > 0, \mathcal{X}$
  $\Theta_1 \leftarrow 0$
  for $t = 1, \dots, T$ **do**
    sample $\gamma_t \sim \mathcal{N}(\mathbf{0}, I_d)$
    predict: $x_t \leftarrow \text{argmin}_x \langle x, \Theta_t + \eta\gamma_t \rangle$
    $\Theta_{t+1} = \Theta_t + \theta_t$

Theorem 1 (Cohen and Hazan (2015))**.** *Assuming that $\sum_{i=1}^d x_i \le K, \forall x \in \mathcal{X}$, using FTPL algorithm, the following bound is obtained for expectation of total regret R_T upto time T:*

$$\mathbb{E}_{\gamma_1^T}[R_T] \le \sqrt{2K \ln |\mathcal{X}|} \left(\frac{KT}{\eta} + \eta \right), \tag{1}$$

Using $\eta^* = \sqrt{KT}$, we get the strictest upper bound:

$$\mathbb{E}_{\gamma_1^T}[R_T] \le 2K\sqrt{T \ln |\mathcal{X}|}, \tag{2}$$

Let $\boldsymbol{z} = \theta + \eta\gamma$, where $\theta \in \mathbb{R}^d$, $\gamma_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1), \forall i \in [d]$. So, $\boldsymbol{z} \sim \mathcal{N}(\Theta, \eta^2 I)$. Using Stein's Lemma,

$$\begin{aligned}
\eta^2 \mathbb{E}\left[\frac{\partial g(\boldsymbol{z})}{\partial \boldsymbol{z}_i}\right] &= \mathbb{E}[(\boldsymbol{z}_i - \theta_i) g(\boldsymbol{z})] \\
&= \eta \mathbb{E}[\gamma_i g(\boldsymbol{z})] \\
\implies \mathbb{E}\left[\frac{\partial g(\boldsymbol{z})}{\partial \boldsymbol{z}_i}\right] &= \frac{1}{\eta}\mathbb{E}[\gamma_i g(\boldsymbol{z})] \qquad (3)
\end{aligned}$$

Lemma 2. *Let $g(\theta) = \min_{x \in \mathcal{X}} \langle x, \theta + \eta\gamma \rangle$, $\Phi(\theta) = \mathbb{E}[g(\theta)]$ and $H = \nabla^2 \Phi(\theta)$, then:*

$$H_{ij} = [\nabla^2 \Phi(\theta)]_{ij} = \frac{1}{\eta}\mathbb{E}[\gamma_i \widehat{x}_j] \tag{4}$$

where $\widehat{x} = \text{argmin}_{x \in \mathcal{X}} \langle x, \theta + \eta\gamma \rangle$.

Proof.

$$\begin{aligned}
\frac{\partial \Phi(\theta)}{\partial \theta_i} &= \mathbb{E}\left[\frac{\partial g(\theta)}{\partial \theta_i}\right] \overset{(3)}{=} \frac{1}{\eta}\mathbb{E}[\gamma_i g(\theta)] \\
&= \frac{1}{\eta}\mathbb{E}[\gamma_i \min_{x \in \mathcal{X}} \langle x, \theta + \eta\gamma \rangle] \\
\implies H_{ij} = \frac{\partial^2 \Phi(\theta)}{\partial \theta_i \partial \theta_j} &= \frac{1}{\eta}\frac{\partial \mathbb{E}[\gamma_i \min_{x \in \mathcal{X}} \langle x, \theta + \eta\gamma \rangle]}{\partial \theta_j} = \frac{1}{\eta}\mathbb{E}[\gamma_i \widehat{x}_j]
\end{aligned}$$

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Proposition 3.

$$-\theta_t^\top \nabla^2 \Phi(\tilde{\Theta}_t)\theta_t \leq \frac{2K}{\eta}\sqrt{2K \ln |\mathcal{X}|}$$

Proof. We omit the detailed proof for this proposition. The proof uses Lemma 2 as an essential component. An interested reader is encouraged to refer (Cohen and Hazan, 2015, Lemma 2) for the proof of this proposition. ■

Now we proceed to prove the regret bound of FTPL.

Proof. Consider the potential function $\Phi(\theta) = \mathbb{E}[\min_{x\in\mathcal{X}}\langle x, \theta + \eta\gamma\rangle]$. Also recall that $\Theta_t = \sum_{i=1}^{t-1}\theta_i$ where θ_i are the adversary outputs. Consider:

$$\begin{aligned}
\Phi(\Theta_t) &= \mathbb{E}[\min_{x\in\mathcal{X}}\langle x, \Theta_t + \eta\gamma\rangle] \\
\implies \nabla_{\Theta_t}\Phi(\Theta_t) &= \nabla\mathbb{E}[\min_{x\in\mathcal{X}}\langle x, \Theta_t + \eta\gamma\rangle] \\
&= \mathbb{E}[x_t] \qquad \text{By definition of } x_t \qquad (5)
\end{aligned}$$

Using residual form of Taylor series expansion,

$$\begin{aligned}
\Phi(\Theta_{t+1}) &= \Phi(\Theta_t + \theta_t) \\
&= \Phi(\Theta_t) + \langle\nabla\Phi(\Theta_t), \theta_t\rangle + \frac{1}{2}\theta_t^\top\nabla^2\Phi(\tilde{\Theta}_t)\theta_t \\
&\overset{(5)}{=} \Phi(\Theta_t) + \mathbb{E}[\langle x_t, \theta_t\rangle] + \frac{1}{2}\theta_t^\top\nabla^2\Phi(\tilde{\Theta}_t)\theta_t
\end{aligned}$$

Summing up from $t = 1$ to $t = T$,

$$\Phi(\Theta_{T+1}) = \Phi(\Theta_1) + \sum_{t=1}^{T}\mathbb{E}[\langle x_t, \theta_t\rangle] + \frac{1}{2}\sum_{t=1}^{T}\theta_t^\top\nabla^2\Phi(\tilde{\Theta}_t)\theta_t \qquad (6)$$

Now, consider the following:

$$\begin{aligned}
\Phi(\Theta_{T+1}) &= \mathbb{E}[\min_{x\in\mathcal{X}}\langle x, \Theta_{T+1} + \eta\gamma\rangle] \\
&\leq \min_{x\in\mathcal{X}}\mathbb{E}[\langle x, \Theta_{T+1} + \eta\gamma\rangle] \qquad \text{Jensen's inequality} \\
&= \min_{x\in\mathcal{X}}\langle x, \Theta_{T+1}\rangle \\
&= \min_{x\in\mathcal{X}}\sum_{t=1}^{T}\langle x, \theta_t\rangle = L_T^* \qquad (7)
\end{aligned}$$

Using (6) and (7), and definition of expected regret, i.e, $\mathbb{E}[R_T] = \sum_{t=1}^{T}\mathbb{E}[\langle x_t, \theta_t\rangle] - L_T^*$

$$\begin{aligned}
\sum_{t=1}^{T}\mathbb{E}[\langle x_t, \theta_t\rangle] &\leq L_T^* - \Phi(\Theta_1) - \frac{1}{2}\sum_{t=1}^{T}\theta_t^\top\nabla^2\Phi(\tilde{\Theta}_t)\theta_t \\
\implies \mathbb{E}[R_T] &\leq -\Phi(\Theta_1) - \frac{1}{2}\sum_{t=1}^{T}\theta_t^\top\nabla^2\Phi(\tilde{\Theta}_t)\theta_t \qquad (8)
\end{aligned}$$

Now, we need to find upper bound on the two terms in RHS.

$$-\Phi(\Theta_1) = -\eta\mathbb{E}[\min_{x\in\mathcal{X}}\langle x, \gamma\rangle] = \eta\mathbb{E}[\max_{x\in\mathcal{X}}\langle x, \gamma\rangle] \leq \eta\sqrt{2K \ln |\mathcal{X}|} \qquad (9)$$

The last inequality follows because notice that $\langle x, \gamma \rangle$ is (sub) Gaussian. $\mathrm{Var}(\langle x, \gamma \rangle) = \mathrm{Var}(\sum_{i=1}^{d} x_i \gamma_i) = \sum_{i=1}^{d} x_i^2 \mathrm{Var}(\gamma_i) = \sum_{i=1}^{d} x_i^2 \leq \sum_{i=1}^{d} x_i \leq K$. The last result follows from the assumption $x_i \in [0, 1]$. Once we establish that $\mathrm{Var}(\langle x, \gamma \rangle) \leq K$, the rest follows from Massart's Lemma, i.e, $\mathbb{E}[\max_i X_i] \leq \sqrt{2\sigma^2 \ln |\mathcal{X}|}$ for $X_i \in \mathcal{X}, X_i \sim \mathrm{subG}(\sigma^2)$. Continuing from (8), using (9) and Proposition 3, we get:

$$\mathbb{E}[R_T] \leq \sqrt{2K \ln |\mathcal{X}|} \left(\frac{KT}{\eta} + \eta \right)$$

■

References

Alon Cohen and Tamir Hazan. Following the perturbed leader for online structured learning. volume 37 of *Proceedings of Machine Learning Research*, pages 1034–1042, Lille, France, 07–09 Jul 2015. PMLR. URL http://proceedings.mlr.press/v37/cohena15.html.