Advanced Topics in Artificial Intelligence: EE6180
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Solution to PSET 2

- **Do not distribute these solutions outside the class!**

1. **(Performance of deterministic algorithms)** Consider a deterministic-oblivious adversary which, at every slot, incurs a cost of 1 for each choice made by the algorithm and set cost equal to 0 to the rest of the experts. This can be done as the algorithm is deterministic. Hence, it is clear that $\texttt{cost}(\text{ALG}) = T$. On the other hand, since all K experts taken together incurs a total cost of T, there must exist an expert with total cost of at most $\frac{T}{K}$.

2. **(Realizability and Hedge)** Using the small-loss bound for Hedge as derived in the class, we have

$$\mathcal{R}_T \leq \frac{1}{1-\eta}\bigg(\frac{\ln|\mathcal{F}|}{\eta} + \eta L_T(i^*)\bigg).$$

Using the realizability assumption, we have $L_T(i^*) = 0$. Hence, the result follows from the above upon taking $\eta = \frac{1}{2}$.

3. **(Hedge is an `FTPL`)**
(a)

$$\begin{aligned}
\mathbb{P}(i_t = j) &= \mathbb{P}(i_t = \arg\min(L_{t-1}(i) - L_0(i))) \\
&= \mathbb{P}(i_t = \arg\max \exp(-\eta(L_{t-1}(i) - L_0(i)) \\
&= \mathbb{P}\bigg(i_t = \arg\max_i \frac{\exp(-\eta L_{t-1}(i))}{\exp(-\eta L_0(i))}\bigg).
\end{aligned}$$

(b) We have

$$\begin{aligned}
\mathbb{P}(v(i) \leq x) &= \mathbb{P}(\exp(-\eta L_0(i)) \leq x) \\
&= \mathbb{P}(L_0(i) \geq \frac{1}{\eta}\ln(1/x)) \\
&= 1 - \exp(-\exp(\ln(x))) \\
&= 1 - \exp(-x).
\end{aligned}$$

(c)

$$\begin{aligned}\mathbb{P}\Big(j = \arg\max_i \frac{a(i)}{v(i)}\Big) &= \mathbb{P}\Big(v(i) \geq \frac{v(j)a(i)}{a(j)}, \forall i\Big)\\ &= \mathbb{E}_{v(j)} \exp\big(-\frac{v(j)}{a(j)}\sum_{i\neq j} a(i)\big)\\ &= \frac{a(j)}{\sum_{i=1}^N a(i)},\end{aligned}$$

where, in the above, we have used the independence of the random variables $\{v(i)\}_{i=1}^N$. Returning back to the `FTPL` algorithm with the above observation, at the round t, the algorithm chooses the arm j w.p. proportional to $\exp(-\eta L_{t-1}(j))$, which is what `Hedge` does.

4. **(Doubling Trick)** Recall the regret bound for Hedge for an arbitrary learning rate η :

$$\mathcal{R}_T \leq \frac{\ln N}{\eta} + T\eta.$$

Assume that $2^{k-1} < T \leq 2^k$ for some $k \geq 1$. Hence, we can bound the total regret by summing up the regret incurred at every sub intervals of the form $[2^{i-1}, 2^i), 1 \leq i \leq k$. This yields

$$\mathcal{R}_T \leq \ln N \sum_{i=1}^k \frac{1}{\eta_i} + \sum_{i=1}^k 2^i \eta_i$$

Recall that $\eta_i = \sqrt{\frac{\ln N}{2^i}}$. Substituting this value in the above summation, we obtain the result.

5. **(Generalizing the Fixed-Share algorithm)**

(a) To get an upper-bound on $|\mathcal{M}|$, divide the time-horizon T in S sub-intervals, each corresponding to the current competitor. Since there are only n possible competitors, we have $|\mathcal{M}| \leq n^S\binom{T}{S} \leq (\frac{nTe}{S})^S$. Using the regret upper bound for Hedge, we conclude that

$$\mathcal{R}_T(i_1, i_2, \ldots, i_T) \leq O(\sqrt{T \ln |\mathcal{M}|}) \leq O(\sqrt{ST \ln \frac{nTe}{S}}).$$

(b) Let $\tilde{p}_{t+1}(i) = \frac{p_t(i)\exp(-\eta l_t(i))}{K}$, where $K = \sum_i p_t(i)\exp(-\eta l_t(i))$ is the normalizing factor. We compute

$$\begin{aligned} D(q_t||p_t) - D(q_t||\tilde{p}_{t+1}) &= \mathbb{E}_{q_t} \log \frac{\tilde{p}_{t+1}}{p_t} \\ &= \mathbb{E}_{q_t} \log \frac{\exp(-\eta l_t)}{K} \\ &= -\eta\langle q_t, l_t\rangle - \log K \\ &\geq -\eta\langle q_t, l_t\rangle - \log \mathbb{E}_{p_t}(1 - \eta l_t + \eta^2 l_t^2) \\ &\geq \eta\langle p_t - q_t, l_t\rangle - \eta^2, \end{aligned} \tag{1}$$

where the last inequality follows from the fact that $\log(1+x) \leq x, e^{-x} \leq 1-x+x^2, \forall x \geq 0$, and $0 \leq l_t(i) \leq 1$.
Next, we lower bound $D(q_t||\tilde{p}_{s_t+1}) - D(q_t||p_t)$. Due to mixing from the past, note that, we have:

$$p_t(i) \geq \alpha_t(s_t+1)\tilde{p}_{s_t+1}(i), \quad \forall i.$$

We have

$$D(q_t||\tilde{p}_{s_t+1}) - D(q_t||p_t) = \mathbb{E}_{q_t} \log \frac{p_t}{\tilde{p}_{s_t+1}} \geq \log(\alpha_t(s_t+1)). \tag{2}$$

Thus, combining (1) and (2), we have

$$\langle p_t - q_t, l_t\rangle \leq \frac{\ln\left(\frac{1}{\alpha_t(s_t+1)}\right) + D(q_t||\tilde{p}_{s_t+1}) - D(q_t||\tilde{p}_{t+1})}{\eta} + \eta.$$

(c) Let the the competitor distribution q_i appears at the time instants $\{t_j^i\}_j$. Then, we have:

$$\begin{aligned} & \sum_{t=1}^{T} D(q_t||\tilde{p}_{s_t+1}) - D(q_t||\tilde{p}_{t+1}) \\ = & \sum_{i=1}^{n}\sum_{j}\Big(D(q_i||\tilde{p}_{t_{j-1}^i+1}) - D(q_i||\tilde{p}_{t_j^i+1})\Big) \\ \overset{(a)}{\leq} & \sum_{i=1}^{n} D(q_i||\tilde{p}_{t_0^i+1}) \\ \overset{(b)}{=} & \sum_{i=1}^{n}\big(-H(q_i) + \mathbb{E}_{q_i}\ln N\big) \\ \leq & \; n\ln N. \end{aligned}$$

where (a) follows from telescoping and the fact that KL-divergences are non-negative, and the equality (b) follows from the fact that the initial distribution is uniform over all experts. Hence, from part (a), we conclude that

$$\mathcal{R}_T(q_1, q_2, \ldots, q_T) \leq \frac{1}{\eta}\sum_{t=1}^{T} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) + \frac{n \ln N}{\eta} + \eta T. \tag{3}$$

(d) [**Uniform Past Mixing**] For $t \geq 2$, consider the uniform mixing sequence $\alpha_t(t) = 1 - \alpha$ and $\alpha_t(\tau) = \frac{\alpha}{t-1}, \forall \tau < t$. With these mixing coefficients, we now upper bound the following summation:

$$\sum_{t=1}^{T} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right)$$

Note that, between any two switches, the value of $\alpha_t(s_t+1)$ remains constant and equals $1-\alpha$. On the other hand, at the beginning of any switch, the value of $\alpha_t(s_t+1)$ is lower bounded by α/T. Hence,

$$\sum_{t=1}^{T} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) \leq (T-S)\ln\frac{1}{1-\alpha} + S\ln\frac{T}{\alpha}.$$

Setting the derivative of the RHS to zero, we find that the RHS above is minimized at $\alpha^* = S/T$. Choosing this value of α, we have

$$\sum_{t=1}^{T} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) \leq S\ln T + TH(S/T). \tag{4}$$

Note that, for $0 \leq x < 0.5$, we have

$$\begin{aligned} x\ln(1/x) - (1-x)\ln(1/(1-x)) &= x\ln\frac{1}{x(1-x)} + \ln(1-x) \\ &\geq 2x\ln(2) + \ln(1-x), \end{aligned}$$

where the first term follows from the fact that $x(1-x) \leq \frac{1}{4}, \forall 0 \leq x \leq 1$. Now note that the RHS of the above bound is concave in x. Hence, its minimum value over an interval $[0, 1/2]$ is obtained at the endpoints. Evaluating the RHS at the points 0 and $1/2$, we conclude that for $0 \leq x \leq 1/2$:

$$(1-x)\ln(1/(1-x)) \leq x\ln(1/x). \tag{5}$$

We now use (5) with $x = S/T \leq \frac{1}{2}$ to conclude:

$$H(S/T) = x\ln\frac{1}{x} + (1-x)\ln\frac{1}{1-x} \leq 2x\ln\frac{1}{x} = 2\frac{S}{T}\ln(\frac{T}{S}).$$

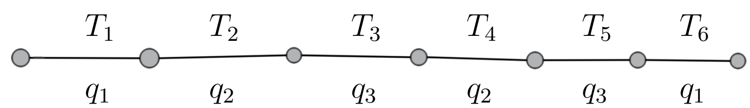

Figure 1: Illustrating the switching and different competitor distributions for $n = 3$ and $S = 5$. We have $\Delta_1 = T_1, \Delta_2 = T_1 + T_2, \Delta_3 = T_3, \Delta_4 = T_4, \Delta_5 = T_2 + T_3 + T_4 + T_5$.

Hence, from equation (4):

$$\sum_{t=1}^{T} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) \leq 3S\ln T.$$

Thus, from part (b), we have that

$$\mathcal{R}_T(q_1, q_2, \ldots, q_T) \leq \frac{3S\ln T + n\ln N}{\eta} + \eta T.$$

Choosing η which minimizes the RHS, we obtain:

$$\mathcal{R}_T(q_1, q_2, \ldots, q_T) \leq 2\sqrt{T(3S\ln T + n\ln N)}.$$

(e) **(Decaying Past Mixing)** Next we consider the time-decaying mixing sequence $\alpha_t(t) = 1-\alpha$ and $\alpha_t(\tau) = \frac{\alpha}{(t-\tau)Z_t}, \forall \tau < t$, with $Z_t = \sum_{\tau=1}^{t-1}\frac{1}{t-\tau} \leq \ln(t)$. Let Δ_i be the time difference before which the same competitor distribution as the one after the j^{th} switch appeared (See Figure 1 for an illustration). With these mixing coefficients, we have

$$\begin{aligned}\sum_{t=1}^{T} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) &\leq (T-S)\ln\frac{1}{1-\alpha} + \sum_{j=1}^{S}\ln\frac{\Delta_j \ln T}{\alpha} \\ &\overset{(a)}{\leq} (T-S)\ln\frac{1}{1-\alpha} + S\ln(\frac{1}{\alpha}) + S\ln(\ln T) + S\ln\frac{\sum_{j=1}^{S}\Delta_j}{S},\end{aligned}$$

where in (a), we have used Jensen's inequality in the last term. Now notice that, since there are only n distinct competitor distribution, the length of each constant competitor interval (i.e., T_i's in the figure 1) appears at most n times in the summation $\sum_j \Delta_j$, i.e., $\sum_j \Delta_j \leq n \sum_i T_i = nT$. Hence, we have

$$\sum_{t=1}^{T} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) \leq (T-S)\ln\frac{1}{1-\alpha} + S\ln(\frac{1}{\alpha}) + S\ln(\ln T) + S\ln\frac{nT}{S}.$$

As before, we choose $\alpha = S/T$, which minimizes the RHS and we obtain for $S/T < 0.3$:

$$\begin{aligned}\sum_{t=1}^{T} \ln\left(\frac{1}{\alpha_t(s_t+1)}\right) &\leq TH(S/T) + S\ln(\ln T) + S\ln\frac{nT}{S} \\ &\leq 3S\ln\frac{nT}{S} + S\ln(\ln T).\end{aligned}$$

Hence, using the regret bound (3) and optimally tuning the learning rate η, we obtain:

$$\mathcal{R}_T(q_1, q_2, \ldots, q_T) \leq 2\sqrt{T\left(S\ln(\ln T) + 3S\ln\frac{nT}{S} + n\ln N\right)}.$$