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Homework 2

AA II
Mar. 1, 2019

Problem 1

Consider a homogeneous sphere of density ρ and total mass $M_{\rm tot}$.

1. Find the total binding energy of the sphere. (10)
2. Find the potential $\phi(r)$ at position r. (10)
3. What is the escape speed $v_{\rm esc}(r)$ for a test particle from this potential? (10)
4. What are the integrals of motion for this potential? Will orbits be confined to a plane in a potential like this? If so, why? (20)
5. Starting from a Lagrangian for the system, solve the equations of motion for a test particle in this potential. (40)
6. Write an expression for the turning points of an orbit in this potential. (10)
7. Show that all orbits in this potential are closed. (Hint: Find radial and azimuthal periods and infer whether orbits are always closed.) (10)
8. Draw orbits in this potential for an initial position-
   (a) $r < R$, (10)
   (b) $r > R$, (10)

   where, R is the size of the sphere.
9. What is the crossing time for a test particle in this potential? (10)

Problem 2

Consider an axisymmetric potential $\phi(r) = \frac{1}{2}v_0^2 \ln\left(R^2 + \frac{z^2}{q^2}\right)$, where $(R,\ \theta,\ z)$ constitute regular cylindrical coordinates, and q, and v_0 and constants.

1. What is the physical meaning of the constant v_0? (20)
2. What are the integrals of motion? (10)
3. Write down the expression for the turning points (also known as zero-velocity curve). You may need to define a quantity L_z = the angular momentum along the z axis. (50)
4. Numerically solve the equations of motion and plot the orbit of a test particle in the R–z plane using-
   (a) $q = 0.99$, $L_z = 0.2$, total energy $E = -0.8$, and $v_0 = 1$ (35)
   (b) $q = 0.1$, $L_z = 0.2$, total energy $E = -0.8$, and $v_0 = 1$ (35)

Problem 3

Assume that the fraction of stars with velocities between $\vec{v}$ and $\vec{v} + \Delta\vec{v}$ in a spherically symmetric system is given by-

$$f(\mathcal{E}) \propto \exp\left(\frac{\mathcal{E}}{k_b T}\right),$$

where $\mathcal{E} = m\phi(x) + \frac{mv^2}{2}$ is the total energy, k_B is the Boltzmann's constant.

1. Show that the velocity dispersion $\sigma^2 = \frac{3k_B T}{m}$, where m is the mass of the star. (20)

2. What fraction of stars escape the cluster? (30)

3. If the $t_{\rm evap}$ is the time it takes to completely evaporate the cluster due to random stellar motions, then show that $t_{\rm evap} \sim 136 t_{\rm relax}$, where $t_{\rm relax}$ is the relaxation time. (50)

4. Calculate the evaporation times (order of magnitude is OK) for systems with average densities $\overline{\rho}$ in the range 10–$10^7\,{\rm M_\odot pc^{-3}}$ and number of stars N in the range 10–10^9. Make a log-log plot showing the region in the N vs $\overline{\rho}$ plane that would completely evaporate within a Hubble time. (50)

5. Go to the Harris catalog of Milky Way globular clusters (GCs) at `http://physwww.mcmaster.ca/~harris/mwgc.dat`.
   Extract the average central density and the number of stars for the Milky Way GCs. Plot them on the same figure created as part of the above problem. [Hint: Be careful of units. You may need to convert a total V-band absolute magnitude to a bolometric luminosity ($\mathcal{L} = 2$), then from $\mathcal{L}$ to total mass ($M_{\rm tot}$) assuming $M_{rmtot}/\mathcal{L} = 2$, then you may need to assume an average stellar mass $< m > /M_\odot = 0.5$ to find N from $M_{\rm tot}$.] (50)