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Consulting books, notes, are OK. Discussing among your class mates about ideas is OK. However, write the solutions by yourselves without anyone else's help.

Problem 1

Consider a star cluster with N stars and that some star has the total energy per mass $E = v^2/2 + \phi$, where $\vec{v}$ is its velocity and ϕ is the potential.

1. What does $E > 0$ mean? [5]
2. Is it possible for a star to achieve $E > 0$ <u>for a time</u>, but still not escape from the system? [5]
3. What is the order-of-magnitude probability of retention (if possible) for stars that ever achieved $E > 0$ in a star cluster with N stars? Argue for your answer. [10]

   Hint: The probability of retention would likely be proportional to a ratio of two relevant timescales.

Problem 2

Consider a spherical stellar system that has uniform density (ρ) initially. Also assume that the stellar system consists of two mass species, m_1 and m_2, with $m_1 > m_2$. The total mass in the two species are M_1 and M_2, and the total number of stars $N = N_1 + N_2$, where N_i is the number of stars in i-th species. Assume that $N_1 < N_2$, such that it is alright to consider that the heavier species is moving in the potential created by the lighter species. Also assume that initially both species are uniformly mixed in phase-space.

1. Argue that species m_1 will preferentially sink in the potential created by the species m_2. Also find order of magnitude the RMS speed that stars of mass m_1 will aim to achieve. [20]
2. Show that uniform density leads to orbits that are similar to a harmonic oscillator. [30]
3. Find the spatial turning points in this potential. [20]
4. Assuming equipartition in energy, show that the heavier species will be confined within a volume $\propto R(m_2/m_1)^{1/2}$, where R is the size of the full system. [20]
5. Show that $M_1 \geq M_2(m_2/m_1)^{3/2}$ is needed for the assumption that species 1 is moving in the potential of species 2 (made at the beginning of this problem). This is also order-of-magnitude the condition for whether the heavier species may ever achieve equipartition. You may need to assume that the density of the lighter species is not changing. [20]

6. In a real star cluster with a stellar mass function given by $dn/dm_\star \propto m_\star^{-2.3}$, check this condition by defining the two species as stars above and below $20\,M_\odot$ and assume that the stellar mass function goes from 0.1–100 $M_\odot$. [20]

7. What physical processes can make this simple estimate complicated? Do these effects help or hinder achivement of energy equipartition? [20]

Problem 3

Consider a polytropic ergodic distribution function of the form

$$\begin{aligned} f(\mathcal{E}) &= F\mathcal{E}^{n-3/2} && (\mathcal{E} > 0) \\ &= 0 && (\mathcal{E} \le 0), \end{aligned} \tag{1}$$

where, $\Psi \equiv -\phi + \phi_0$, $\mathcal{E} \equiv -H + \phi_0 = \Psi - \frac{1}{2}v^2$, ϕ is the potential, ϕ_0 is an arbitrary constant for the potential, and H is the Hamiltonian. Further assume that the spatial density $\rho \propto r^{-\alpha}$.

1. Show that mass enclosed within some distance r [20]

$$M(r) \propto r^{(n-3)/(n-1)}.$$

What type of mass distribution is represented by

(a) $n = 3$ [10]

(b) $n \to \infty$ [10]

2. Rescale variables such that $s \equiv \frac{r}{b}$ and $\psi \equiv \frac{\Psi}{\Psi_0}$, where, $b \equiv (\frac{4}{3}\pi G \Psi_0^{n-1} c_n)^{-1/2}$, c_n is a constant such that $\rho = c_n \Psi^n$ is a solution for the DF given in Equation 1, and $\Psi_0 = \Psi(0)$. Show that,

(a) The Poisson's equation takes the form [20]

$$\begin{aligned} \frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi}{ds}\right) &= -3\psi^n && (\psi > 0) \\ &= 0 && (\psi \le 0) \end{aligned} \tag{2}$$

(b) Show that the Plummer sphere is a solution of Equation 2 with $n = 5$. [20]
Hint: Plummer sphere potential is $\phi = -\frac{GM}{\sqrt{r^2+b^2}}$, where M is the total mass.

(c) Show that $n = 5$ marks a boundary such that for $n > 5$ the total mass as well as the extent of the stellar system are infinite, for $n < 5$ both the total mass and the extent are finite, and for $n = 5$, the total mass is finite, but $\rho > 0$ everywhere (extent is infinite). [20]

Problem 4

Assume that all star clusters have density profiles like Plummer spheres, given by,

$$\rho(r) = \frac{3M}{4\pi b^3}\left(1 + \frac{r^2}{b^2}\right)^{-5/2}, \tag{3}$$

where, b is the scale length and have the same meaning as in Problem 3.

1. Calculate the velocity dispersion (v_σ) for stars in a Plummer sphere as a function of the total mass M, the scale length b, and the density ρ. [20]

2. Find an expression for the rate of binary-single interactions per binary $\Gamma_{BS}(r, m_1, m_2, m, b, M, v_\infty, n)$, where m is the mass of the single star coming from infinity with relative speed v_∞ for a hyperbolic encounter with impact parameter b, m_1 and m_2 are the masses of the components of the binary, and n is the number density of stars. [10]

3. Write down an expression for the escape speed from a cluster like this in the form $v_{\rm esc}(r, \rho, M, b)$. [10]

4. The dynamical process of forming binary black holes (BBH) is quite simple conceptually. BHs or their progenitor stars sink in the cluster potential. BHs then interact preferrantially with themselves to form new binaries. Once a BBH is formed, it keeps interacting with other stars (and BHs) in the cluster. These interactions result in 'exchange' (which preferentially ejects the lightest star and inserts the other two in the binary), or 'fly-by' which shrinks the binary. The binary keeps interacting until one of two things happen: (i) the binary is ejected from the cluster, i.e., $v_b > v_{\rm esc}$, where v_b and $v_{\rm esc}$ denote the post-encounter speed of the binary center-of-mass (CM) and escape speed from the cluster; (ii) while $v_b < v_{\rm esc}$, the orbits shrink so much that before the next strong encounter in the cluster the BBH merges due to gravitational wave (GW) radiation. Assume,

    - binary-single interactions is the dominant type of strong encounters.
    - all encounters have a relative speed at infinity $v_\infty \sim v_\sigma$.
    - binaries with orbital speed $v_{\rm orb} < v_\infty$ break due to strong encounters ('soft') and only the other ('hard') binaries take part in this process.
    - recoil speed of the binary CM after encounter is $v_{\rm recoil} \sim v_{\rm orb}$.
    - scale length $b \sim 1\,{\rm pc}$.
    - typical mass of BHs is $\sim 10\,M_\odot$.
    - BBH eccentricities are thermal, i.e., $\frac{dn}{de} = 2e$, hence, the mean $<e>= \frac{2}{3}$. You can simply use $<e>$ as typical eccentricity for the dynamically formed BBHs.

- Average stellar mass is $< m_\star > /M_\odot = 0.5$. You may need it for the calculation of the relaxation time $t_{\rm relax}$.
- Assume that in each cluster the number of BHs formred is $N_{\rm BH} \sim 10^{-3}N$, where N is the total number of stellar objects (in the counting of N, binaries are not counted as two stars, but one stellar object). Also assume that supernova does not eject any BHs from the clusters.
- Binary fraction is $f_b \equiv N_b/N = 5\%$, where, N_b is the total number of binaries.
- The binary semimajor axis is distributed as $dN_b/d\log a \propto$ constant, where, a is the semi-major axis.

(a) Using everything you have derived earlier, the code given below that calculates the GW inspiral time $t_{\rm GW}$ in Gyr, estimate the region in cluster mass which dominantly produce BBHs that would merge within a Hubble time. [50]

(b) Plot the typical $t_{\rm GW}$ (y axis) for BBHs created in star clusters of some total mass M (x axis). Use a range in mass 10^3–$10^8\, M_\odot$. [10]

```
import scipy
def inspiral_time_peters(a0,e0,m1,m2):
        ''''''
        Computes the inspiral time, in Gyr, for a binary
            with a0 in Au, and masses in solar masses using the Peters' equat
        """

        coef = 6.086768e-11
        #G^3 / c^5 in au, gigayear, solar mass units
        beta = (64./5.) * coef * m1 * m2 * (m1+m2)

        if e0 == 0:
                return a0**4 / (4*beta)

        c0 = a0 * (1.-e0**2.) * e0**(-12./19.) *

                        (1.+(121./304.)*e0**2.)**(-870./2299.)

        time_integrand = lambda e:

                 e**(29./19.)*(1.+(121./304.)*e**2.)**
                             (1181./2299.) / (1.-e**2.)**1.5

        integral,abserr = scipy.integrate.quad(time_integrand,0,e0)

        return integral * (12./19.) * c0**4. / beta
```