Uncertain Knowledge

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Quantifying Uncertainty

Uncertainty

- Let action A_t = "leave for airport t minutes before flight". Will A_t get me there on time?
- Problems:
  1. partial observability (road state, other drivers' plans, etc.)
  2. noisy sensors (KCBS traffic reports)
  3. uncertainty in action outcomes (flat tire, etc.)
  4. immense complexity of modelling and predicting traffic
- Hence a purely logical approach either
  1. risks falsehood: "A_{25} will get me there on time" or
  2. leads to conclusions that are too weak for decision making: "A_{25} will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact etc etc."

  (A_{1440} might reasonably be said to get me there on time but I'd have to stay overnight in the airport ...)

Methods for handling uncertainty

- **Default** or **nonmonotonic logic**
  - Assume my car does not have a flat tire
  - Assume A_{25} works unless contradicted by evidence
  - **Issues**: What assumptions are reasonable? How to handle contradiction?
- **Rules with fudge factors**
  - $A_{25} \mapsto_{0.3} AtAirportOnTime$
  - $Sprinkler \mapsto_{0.99} WetGrass$
  - $WetGrass \mapsto_{0.7} Rain$
  - **Issues**: Problems with combination, e.g., *Sprinkler* causes *Rain*?
- **Probability**
  - Given the available evidence, A_{25} will get me there on time with probability 0.04
- **Fuzzy logic** handles *degree of truth* NOT uncertainty
  - e.g., *WetGrass* is true to degree 0.2

Probability

- Probabilistic assertions *summarize* effects of
  - **laziness**: failure to enumerate exceptions, qualifications, etc.
  - **ignorance**: lack of relevant facts, initial conditions, etc.
- **Subjective** or **Bayesian** probability
  - Probabilities relate propositions to one's own state of knowledge

    $$P(A_{25}|\text{no reported accidents}) = 0.06$$

    These are *not* claims of a "probabilistic tendency" in the current situation (but might be learned from past experience of similar situations)
  - Probabilities of propositions change with new evidence

    $$P(A_{25}|\text{no reported accidents}, 5 \text{ a.m.}) = 0.15$$

Uncertainty and rational decisions

- Suppose I believe the following:

$$
\begin{aligned}
P(A_{25} \text{ gets me there on time} | \ldots) &= 0.04 \\
P(A_{90} \text{ gets me there on time} | \ldots) &= 0.70 \\
P(A_{120} \text{ gets me there on time} | \ldots) &= 0.95 \\
P(A_{1440} \text{ gets me there on time} | \ldots) &= 0.9999
\end{aligned}
$$

  Which action to choose?
- Depends on my **preferences** for missing flight vs. airport cuisine, etc.
- **Utility theory** is used to represent and infer preferences

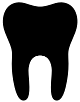
  Decision theory = utility theory + probability theory

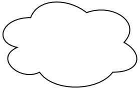
Uncertainty and rational decisions (cont.)

```
function DT-AGENT(percept) returns an action
persistent: belief_state, probabilistic beliefs about
                        the current state of the world
            action, the agent's action
  update belief_state based on action and percept
  calculate outcome probabilities for actions,
    given action descriptions and current belief_state
  select action with highest expected utility
    given probabilities of outcomes and utility information
return action
```

Examples

1. Diagnosing a dental patient's toothache with four random variables: *Toothache*, *Cavity*, *Catch*, *Weather*
2. Alarm problem has three variables: *Earthquake*, *Burglary*, *Alarm*





Syntax for propositions

- **Propositional** or **Boolean** random variables
  - e.g., *Cavity* (do I have a cavity?)
  - $\mathit{Cavity} = \mathit{true}$ is a proposition (also written *cavity*)
- **Discrete** random variables (**finite** or **infinite**)
  - e.g., *Weather* is one of $\langle \mathit{sunny}, \mathit{rain}, \mathit{cloudy}, \mathit{snow} \rangle$
  - $\mathit{Weather} = \mathit{rain}$ is a proposition
  - Values must be exhaustive and mutually exclusive
- **Continuous** random variables (**bounded** or **unbounded**)
  - e.g., $\mathit{Temp} = 21.6$; also allow, e.g., $\mathit{Temp} < 22.0$.
- Arbitrary Boolean combinations of basic propositions

Degrees of belief or Probability

Concept 1

- A **degree of belief** or **probability** in [0,1] is assigned to each world ω and denote it by $P(\omega)$
- The belief in, or probability of, a sentence α can then be defined as

$$P(\alpha) = \sum_{\omega:\omega\models\alpha} P(\omega)$$

Degrees of belief or Probability (cont.)

Table 1: A state of belief

world/model	**Earthquake**	**Burglary**	**Alarm**	**P**
ω_1	true	true	true	.0190
ω_2	true	true	false	.0010
ω_3	true	false	true	.0560
ω_4	true	false	false	.0240
ω_5	false	true	true	.1620
ω_6	false	true	false	.0180
ω_7	false	false	true	.0072
ω_8	false	false	false	.7128

Properties of beliefs

1. $0 \leq P(\alpha) \leq 1$ for any sentence α
2. $P(\alpha) = 0$ when α is inconsistent
3. $P(\alpha) = 1$ when α is valid
4. $P(\alpha) + P(\neg\alpha) = 1$ for any sentence α

Entropy

Concept 2

Entropy is an uncertainty quantification about a random variable or several random variables X

$$Entropy(X) = -\sum_{x} P(x) \times \log_2 P(x)$$

where $0 \log_2 0 = 0$ by convention

	Earthquake	**Burglary**	**Alarm**
true	.1	.2	.2442
false	.9	.8	.7558
$Entropy(\cdot)$	.469	.722	.802

Entropy (cont.)

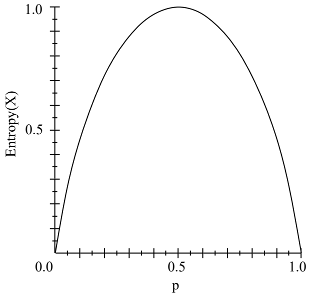


Figure 1: The entropy for a binary variable X with $P(X) = p$

Syntax for probability

- **Probabilities** of propositions

$$P(Cavity = true) = 0.1 \text{ and } P(Weather = sunny) = 0.72$$

- **Probability distribution** gives values for all possible assignments:

$$\mathbf{P}(Weather) = \langle 0.72, 0.1, 0.08, 0.1 \rangle$$

- **Joint probability distribution** for a set of r.v.s gives the probability of every atomic event on those r.v.s

$$\mathbf{P}(Weather, Cavity) = \begin{array}{lcccc} Weather = & sunny & rain & cloudy & snow \\ Cavity = true & 0.144 & 0.02 & 0.016 & 0.02 \\ Cavity = false & 0.576 & 0.08 & 0.064 & 0.08 \end{array}$$

Bayes' Rule

Concept 3

- **Prior** or **unconditional probabilities** are corresponded to belief prior to arrival of any (new) evidence
- **Posterior probability** or conditional probability (Bayes' rule)

$$P(a \mid b) = \frac{P(a \wedge b)}{P(b)} \text{ if } P(b) \neq 0$$

- **Product rule** gives an alternative formulation:

$$P(a \wedge b) = P(a \mid b)P(b) = P(b \mid a)P(a)$$

Bayes' Rule (cont.)

- **Chain rule** is derived by successive application of product rule:

$$
\begin{aligned}
\mathbf{P}(X_1, \ldots, X_n) &= \mathbf{P}(X_1, \ldots, X_{n-1})\mathbf{P}(X_n \mid X_1, \ldots, X_{n-1}) \\
&= \mathbf{P}(X_1, \ldots, X_{n-2})\mathbf{P}(X_{n-1} \mid X_1, \ldots, X_{n-2})\mathbf{P}(X_n \mid X_1, \ldots, X_{n-1}) \\
&= \cdots \\
&= \prod_{i=1}^{n} \mathbf{P}(X_i \mid X_1, \ldots, X_{i-1})
\end{aligned}
$$

Inference by enumeration

Enumeration

For **any proposition** α, sum the **atomic events** where it is true

$$P(\alpha) = \sum_{\omega:\omega \models \alpha} P(\omega)$$

- Start with the sample joint distribution:

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	**.108**	**.012**	**.072**	**.008**
$\neg$ *cavity*	**.016**	**.064**	**.144**	**.576**

Inference by enumeration (cont.)

	$toothache$		$\neg\ toothache$	
	$catch$	$\neg\ catch$	$catch$	$\neg\ catch$
$cavity$	**.108**	**.012**	**.072**	**.008**
$\neg\ cavity$	**.016**	**.064**	**.144**	**.576**

$$\begin{aligned} P(toothache) &= 0.108 + 0.012 + 0.016 + 0.064 \\ &= 0.2 \end{aligned}$$

Inference by enumeration (cont.)

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	**.108**	**.012**	**.072**	**.008**
$\neg$ *cavity*	**.016**	**.064**	**.144**	**.576**

$$\begin{aligned} P(\mathit{cavity} \vee \mathit{toothache}) &= 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 \\ &= 0.28 \end{aligned}$$

Inference by enumeration (cont.)

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	**.108**	**.012**	**.072**	**.008**
¬ *cavity*	**.016**	**.064**	**.144**	**.576**

$$
\begin{aligned}
P(\neg cavity \mid toothache) &= \frac{P(\neg cavity \land toothache)}{P(toothache)} \\
&= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} \\
&= 0.4
\end{aligned}
$$

Inference by enumeration (cont.)

Problem

Let $\boldsymbol{X}$ be all the variables. We want the posterior joint distribution of the **query variables** $\boldsymbol{Y}$ given specific values $\boldsymbol{e}$ for the **evidence variables** $\boldsymbol{E}$

Solution

- General idea: compute distribution on query variable by fixing **evidence variables** and summing over **hidden variables**
- Let the **hidden variables** be $\boldsymbol{H} = \boldsymbol{X} - \boldsymbol{Y} - \boldsymbol{E}$ and denominator can be viewed as a **normalization constant** α

$$\boldsymbol{P}(\boldsymbol{Y} \mid \boldsymbol{E} = \boldsymbol{e}) = \alpha\, \boldsymbol{P}(\boldsymbol{Y}, \boldsymbol{E} = \boldsymbol{e}) = \alpha \sum_{\boldsymbol{h}} \boldsymbol{P}(\boldsymbol{Y}, \boldsymbol{E} = \boldsymbol{e}, \boldsymbol{H} = \boldsymbol{h})$$

■

Inference by enumeration (cont.)

- Worst-case time complexity $O(d^n)$ where d is the largest arity
- Space complexity $O(d^n)$ to store the joint distribution

Inference by enumeration (cont.)

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	**.108**	**.012**	**.072**	**.008**
$\neg$ *cavity*	**.016**	**.064**	**.144**	**.576**

$$\begin{aligned}
& \mathbf{P}(Cavity|toothache) \\
= \ & \alpha\, \mathbf{P}(Cavity, toothache) \\
= \ & \alpha\, [\mathbf{P}(Cavity, toothache, catch) + \mathbf{P}(Cavity, toothache, \neg catch)] \\
= \ & \alpha\, [\langle 0.108, 0.016\rangle + \langle 0.012, 0.064\rangle] \\
= \ & \alpha\, \langle 0.12, 0.08\rangle \\
= \ & \langle 0.6, 0.4\rangle
\end{aligned}$$

Independence

Concept 4

A and B are **independent** $(A \perp B)$ iff

$$\boldsymbol{P}(A \mid B) = \boldsymbol{P}(A)$$

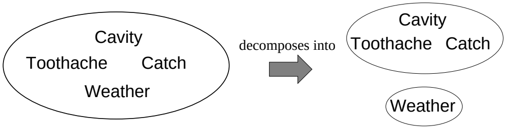
or

$$\boldsymbol{P}(B \mid A) = \boldsymbol{P}(B)$$

or

$$\boldsymbol{P}(A, B) = \boldsymbol{P}(A) P(B)$$

Independence (cont.)



$$\mathbf{P}(Toothache, Catch, Cavity, Weather) = \mathbf{P}(Toothache, Catch, Cavity)\mathbf{P}(Weather)$$

- 32 entries reduced to 12; for n independent biased coins, $2^n \to n$
- Absolute independence powerful but rare
- Dentistry is a large field with hundreds of variables, none of which are independent. What to do?

Conditional independence

Concept 5

A and B are **independent** given C ($A \perp B \mid C$) iff

$$\boldsymbol{P}(A \mid B, C) = \boldsymbol{P}(A \mid C)$$

or

$$\boldsymbol{P}(B \mid A, C) = \boldsymbol{P}(B \mid C)$$

or

$$\boldsymbol{P}(A, B \mid C) = \boldsymbol{P}(A \mid C)\boldsymbol{P}(B \mid C)$$

Conditional independence (cont.)

- $\mathbf{P}(\mathit{Toothache}, \mathit{Cavity}, \mathit{Catch})$ has $2^3 - 1 = 7$ independent entries
- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
$$P(\mathit{catch} \mid \mathit{toothache}, \mathit{cavity}) = P(\mathit{catch} \mid \mathit{cavity})$$
- The same independence holds if I haven't got a cavity:
$$P(\mathit{catch} \mid \mathit{toothache}, \neg \mathit{cavity}) = P(\mathit{catch} \mid \neg \mathit{cavity})$$
- *Catch* is **conditionally independent** of *Toothache* given *Cavity*:
$$\mathbf{P}(\mathit{Catch} \mid \mathit{Toothache}, \mathit{Cavity}) = \mathbf{P}(\mathit{Catch} \mid \mathit{Cavity})$$

Conditional independence (cont.)

- Equivalent statements:

$$\mathbf{P}(Toothache \mid Catch, Cavity) = \mathbf{P}(Toothache \mid Cavity)$$

$$\mathbf{P}(Toothache, Catch \mid Cavity) = \mathbf{P}(Toothache \mid Cavity)\mathbf{P}(Catch \mid Cavity)$$

- Write out full joint distribution using chain rule:

$$\begin{aligned}
& \mathbf{P}(Toothache, Catch, Cavity) \\
= \; & \mathbf{P}(Toothache \mid Catch, Cavity)\mathbf{P}(Catch, Cavity) \\
= \; & \mathbf{P}(Toothache \mid Catch, Cavity)\mathbf{P}(Catch \mid Cavity)\mathbf{P}(Cavity) \\
= \; & \mathbf{P}(Toothache \mid Cavity)\mathbf{P}(Catch \mid Cavity)\mathbf{P}(Cavity)
\end{aligned}$$

I.e., $2 + 2 + 1 = 5$ independent numbers

Conditional independence (cont.)

- **In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in n to linear in n.**
- *Conditional independence is our most basic and robust* form of knowledge about uncertain environments.

Bayes' rule and conditional independence

- Bayes' rule

$$\mathbf{P}(Y \mid X) = \frac{\mathbf{P}(X \mid Y)\mathbf{P}(Y)}{\mathbf{P}(X)} = \alpha\, \mathbf{P}(X \mid Y)\mathbf{P}(Y)$$

Useful for assessing **diagnostic** probability from **causal** probability

$$\mathbf{P}(Cause \mid Effect) = \frac{\mathbf{P}(Effect \mid Cause)\mathbf{P}(Cause)}{\mathbf{P}(Effect)}$$

$$\begin{aligned} & \mathbf{P}(Cavity \mid toothache \wedge catch) \\ = \quad & \alpha\, \mathbf{P}(toothache \wedge catch \mid Cavity)\mathbf{P}(Cavity) \\ = \quad & \alpha\, \mathbf{P}(toothache \mid Cavity)\mathbf{P}(catch \mid Cavity)\mathbf{P}(Cavity) \end{aligned}$$

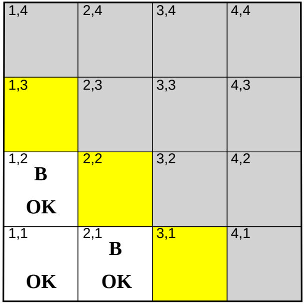
Bayes' rule and conditional independence (cont.)

- This is an example of a **naive Bayes** model

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause) \prod_i \mathbf{P}(Effect_i \mid Cause)$$

- Total number of parameters is *linear* in n

The Wumpus World Revisitted



- $P_{ij} = \mathit{true}$ iff $[i,j]$ contains a pit
- $B_{ij} = \mathit{true}$ iff $[i,j]$ is breezy
- Include only $B_{1,1}, B_{1,2}, B_{2,1}$ in the probability model

Specifying the probability model

- The full joint distribution is $\boldsymbol{P}(P_{1,1}, \ldots, P_{4,4}, B_{1,1}, B_{1,2}, B_{2,1})$
- Apply product rule: $\boldsymbol{P}(B_{1,1}, B_{1,2}, B_{2,1} \mid P_{1,1}, \ldots, P_{4,4})\boldsymbol{P}(P_{1,1}, \ldots, P_{4,4})$ (Do it this way to get $P(Effect \mid Cause)$.)
- First term: 1 if pits are adjacent to breezes, 0 otherwise
- Second term: pits are placed randomly, probability 0.2 per square:

$$\boldsymbol{P}(P_{1,1}, \ldots, P_{4,4}) = \prod_{i,j=1,1}^{4,4} \boldsymbol{P}(P_{i,j}) = 0.2^n \times 0.8^{16-n}$$

for n pits.

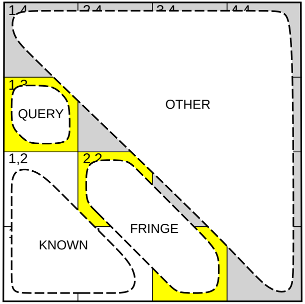
Observations and query

- We know the following facts:
  $b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$
  $known = \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$
  Define $Unknown = P_{ij}$s other than $P_{1,3}$ and $Known$
- Query is $\mathbf{P}(P_{1,3} \mid known, b)$
- For inference by enumeration, we have

$$\mathbf{P}(P_{1,3} \mid known, b) = \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, known, b)$$

Grows exponentially with number of squares!

Using conditional independence



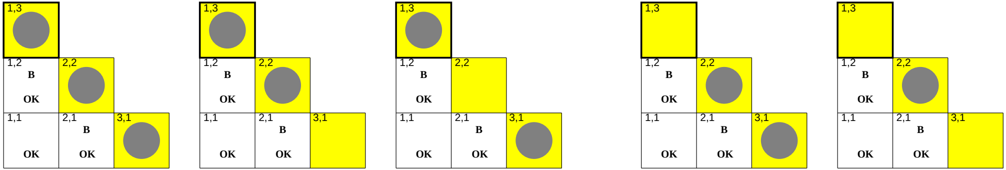
- Basic insight: observations are conditionally independent of other hidden squares given neighbouring hidden squares
- Define $Unknown = Fringe(\text{or } Frontier) \cup Other$
  $\boldsymbol{P}(b \mid P_{1,3}, Known, Unknown) = \boldsymbol{P}(b \mid P_{1,3}, Known, Fringe)$

Using conditional independence (cont.)

- Manipulate query into a form where we can use this!

$$
\begin{aligned}
& \mathbf{P}(P_{1,3} \mid known, b) \\
= \quad & \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, known, b) \\
= \quad & \alpha \sum_{unknown} \mathbf{P}(b \mid P_{1,3}, known, unknown) \mathbf{P}(P_{1,3}, known, unknown) \\
= \quad & \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b \mid known, P_{1,3}, fringe, other) \mathbf{P}(P_{1,3}, known, fringe, other) \\
= \quad & \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b \mid known, P_{1,3}, fringe) \mathbf{P}(P_{1,3}, known, fringe, other) \\
= \quad & \alpha \sum_{fringe} \mathbf{P}(b \mid known, P_{1,3}, fringe) \sum_{other} \mathbf{P}(P_{1,3}, known, fringe, other) \\
= \quad & \alpha \sum_{fringe} \mathbf{P}(b \mid known, P_{1,3}, fringe) \sum_{other} \mathbf{P}(P_{1,3}) \mathbf{P}(known) \mathbf{P}(fringe) \mathbf{P}(other) \\
= \quad & \alpha \, \mathbf{P}(known) P(P_{1,3}) \sum_{fringe} \mathbf{P}(b \mid known, P_{1,3}, fringe) \mathbf{P}(fringe) \sum_{other} \mathbf{P}(other) \\
= \quad & \alpha' \, \mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b \mid known, P_{1,3}, fringe) \mathbf{P}(fringe)
\end{aligned}
$$

Using conditional independence (cont.)



0.2 x 0.2 = 0.04 0.2 x 0.8 = 0.16 0.8 x 0.2 = 0.16 0.2 x 0.2 = 0.04 0.2 x 0.8 = 0.16

$$
\begin{aligned}
P(P_{1,3} \mid known, b) &= \alpha' \langle 0.2(0.04 + 0.16 + 0.16), 0.8(0.04 + 0.16) \rangle \\
&\approx \langle 0.31, 0.69 \rangle \\
P(P_{2,2} \mid known, b) &\approx \langle 0.86, 0.14 \rangle
\end{aligned}
$$

Bayesian Networks

Bayesian networks

Concept 6

A simple, graphical notation for conditional independence assertions and hence for compact specification of full joint distributions

Syntax

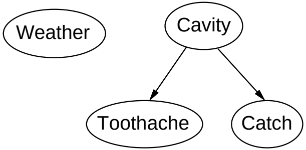
- A set of nodes, one per variable
- A directed, acyclic graph (link $\approx$ "directly influences")
- A conditional distribution for each node given its parents:

$$\boldsymbol{P}(X_i \mid parents(X_i))$$

- In the simplest case, conditional distribution represented as a **conditional probability table** (CPT) giving the distribution over X_i for each combination of parent values

Example

- Topology of network encodes conditional independence assertions:



  - *Weather* is independent of the other variables
  - *Toothache* and *Catch* are conditionally independent given *Cavity*

Example (cont.)

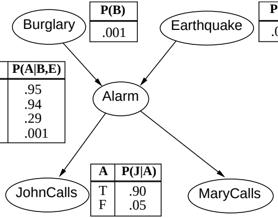
Problem

I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

- Variables: *Burglar*, *Earthquake*, *Alarm*, *JohnCalls*, *MaryCalls*

Network topology reflects "causal" knowledge:

- A burglar can set the alarm off
- An earthquake can set the alarm off
- The alarm can cause Mary to call
- The alarm can cause John to call



P(B)
.001

P(E)
.002

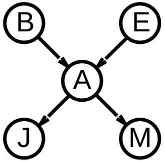
B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01

Compactness

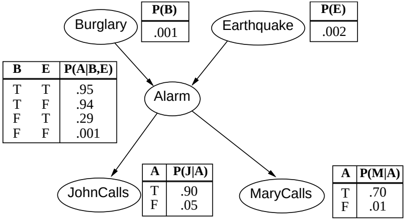
- A CPT for Boolean X_i with k Boolean parents has 2^k rows for the combinations of parent values
- Each row requires one number p for $X_i = true$ (the number for $X_i = false$ is just $1 - p$)
- If each variable has no more than k parents, the complete network requires $O(n \times 2^k)$ numbers
  - I.e., grows linearly with n, vs. $O(2^n)$ for the full joint distribution
- For burglary net, $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5 - 1 = 31$)



Global semantics

- **Global semantics** defines the full joint distribution as the product of the local conditional distributions:

$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i \mid parents(X_i))$$



- For example,

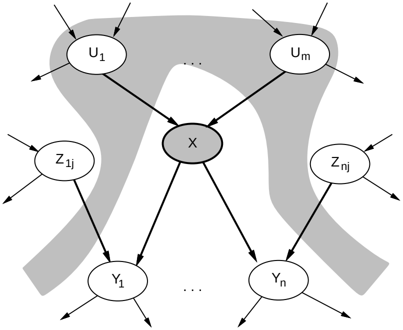
$$\begin{aligned} P(j \wedge m \wedge a \wedge \neg b \wedge \neg e) &= P(j \mid a)P(m \mid a)P(a \mid \neg b, \neg e)P(\neg b)P(\neg e) \\ &= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 \\ &\approx 0.00063 \end{aligned}$$

Local semantics

Local semantics: each node is conditionally independent of its nondescendants given its parents

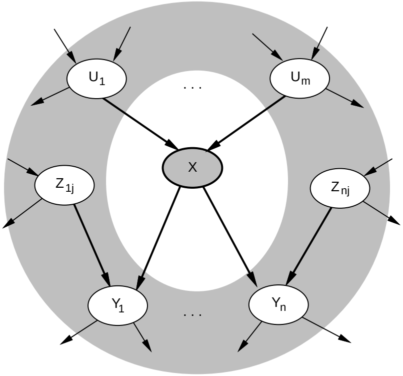
Theorem 1

Local semantics ≡ global semantics



Markov blanket

- Each node is conditionally independent of all others given its
- **Markov blanket**: parents + children + children's parents



Constructing Bayesian networks

- Need a method such that a series of locally testable assertions of conditional independence guarantees the required global semantics

1. Choose an ordering of variables $X_1, \ldots, X_n$
2. For $i = 1$ to n
   add X_i to the network
   select parents from $X_1, \ldots, X_{i-1}$ such that

$$\mathbf{P}(X_i \mid parents(X_i)) = \mathbf{P}(X_i \mid X_1, \ldots, X_{i-1})$$

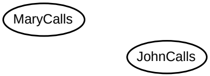
This choice of parents guarantees the global semantics:

$$\begin{aligned} \mathbf{P}(X_1, \ldots, X_n) &= \prod_{i=1}^{n} \mathbf{P}(X_i \mid X_1, \ldots, X_{i-1}) && \text{(chain rule)} \\ &= \prod_{i=1}^{n} \mathbf{P}(X_i \mid parents(X_i)) && \text{(by construction)} \end{aligned}$$

Example: Burglary alarm

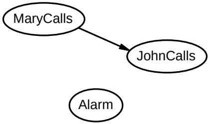
- Suppose we choose the ordering M, J, A, B, E

$$P(J \mid M) = P(J)?$$



Example: Burglary alarm (cont.)

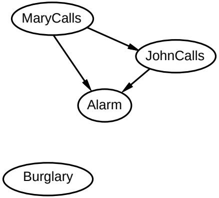
- Suppose we choose the ordering M, J, A, B, E



$P(J \mid M) = P(J)$? No
$P(A \mid J, M) = P(A \mid J)$? $P(A \mid J, M) = P(A)$?

Example: Burglary alarm (cont.)

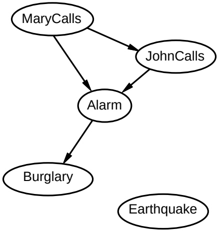
- Suppose we choose the ordering M, J, A, B, E



$$P(J \mid M) = P(J)? \text{ No}$$
$$P(A \mid J, M) = P(A \mid J)? \; P(A \mid J, M) = P(A)? \text{ No}$$
$$P(B \mid A, J, M) = P(B \mid A)?$$
$$P(B \mid A, J, M) = P(B)?$$

Example: Burglary alarm (cont.)

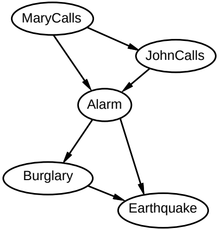
- Suppose we choose the ordering M, J, A, B, E



$P(J \mid M) = P(J)$? No
$P(A \mid J, M) = P(A \mid J)$? $P(A \mid J, M) = P(A)$? No
$P(B \mid A, J, M) = P(B \mid A)$? Yes
$P(B \mid A, J, M) = P(B)$? No
$P(E \mid B, A, J, M) = P(E \mid A)$?
$P(E \mid B, A, J, M) = P(E \mid A, B)$?

Example: Burglary alarm (cont.)

- Suppose we choose the ordering M, J, A, B, E



$P(J \mid M) = P(J)$? No
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$P(B \mid A, J, M) = P(B \mid A)$? Yes
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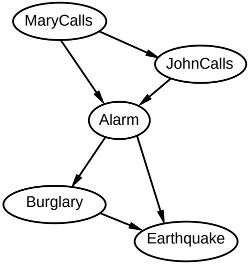
$P(E \mid B, A, J, M) = P(E \mid A, B)$? Yes

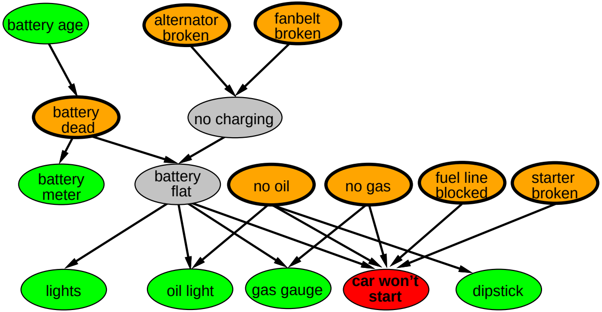
Example: Burglary alarm (cont.)



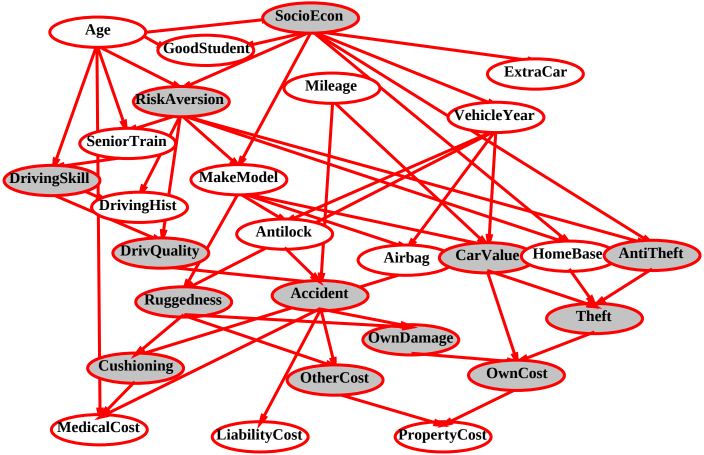
- Deciding conditional independence is hard in noncausal directions (Causal models and conditional independence seem hardwired for humans!)
- Assessing conditional probabilities is hard in noncausal directions
- Network is less compact: $1 + 2 + 4 + 2 + 4 = 13$ numbers needed

Example: Car diagnosis

- Initial evidence: car won't start
- Testable variables (green), "broken, so fix it" variables (orange)
- Hidden variables (gray) ensure sparse structure, reduce parameters



Example: Car insurance



Compact conditional distributions

Problem

- CPT grows exponentially with number of parents
- CPT becomes infinite with continuous-valued parent or child

Solution: **canonical** distributions that are defined compactly

- **Deterministic** nodes are the simplest case:

$$X = f(parents(X)) \text{ for some function } f$$

  - Boolean functions

$$NorthAmerican \equiv Canadian \vee US \vee Mexican$$

  - Numerical relationships among continuous variables

$$\frac{\partial Level}{\partial t} = \text{ inflow + precipitation - outflow - evaporation}$$

Compact conditional distributions (cont.)

- **Noisy-OR** distributions model multiple noninteracting causes
  1. Parents $U_1 \ldots U_k$ include all causes (can add **leak node**)
  2. Independent failure probability q_i for each cause alone

$$P(X \mid U_1 \ldots U_j, \neg U_{j+1} \ldots \neg U_k) = 1 - \prod_{i=1}^{j} q_i$$

Number of parameters *linear* in number of parents

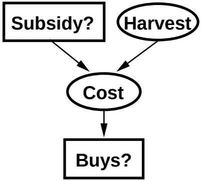
Compact conditional distributions (cont.)

$$
\begin{aligned}
q_{cold} &= P(\neg fever \mid cold, \neg flu, \neg malaria) = 0.6 \\
q_{flu} &= P(\neg fever \mid \neg cold, flu, \neg malaria) = 0.2 \\
q_{malaria} &= P(\neg fever \mid \neg cold, \neg flu, malaria) = 0.1
\end{aligned}
$$

Cold	*Flu*	*Malaria*	$P(Fever)$	$P(\neg Fever)$
F	F	F	**0.0**	1.0
F	F	T	0.9	**0.1**
F	T	F	0.8	**0.2**
F	T	T	0.98	$0.02 = 0.2 \times 0.1$
T	F	F	0.4	**0.6**
T	F	T	0.94	$0.06 = 0.6 \times 0.1$
T	T	F	0.88	$0.12 = 0.6 \times 0.2$
T	T	T	0.988	$0.012 = 0.6 \times 0.2 \times 0.1$

Bayesian nets with continuous variables

Discrete (*Subsidy*? and *Buys*?); continuous (*Harvest* and *Cost*)



- Option 1: discretization – possibly large errors, large CPTs
- Option 2: finitely parameterized canonical families
  1. Continuous variable, discrete+continuous parents (e.g., *Cost*)
  2. Discrete variable, continuous parents (e.g., *Buys*?)

Continuous child variables

- Need one **conditional density** function for child variable given continuous parents, for each possible assignment to discrete parents
- Most common is the **linear Gaussian** (LG) model, e.g.,:

$$P(Cost = c | Harvest = h, Subsidy? = true)$$
$$= N(a_t h + b_t, \sigma_t)(c) = \frac{1}{\sigma_t \sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{c - (a_t h + b_t)}{\sigma_t}\right)^2\right)$$

- Mean *Cost* varies linearly with *Harvest*, variance is fixed
- Linear variation is unreasonable over the full range but works OK if the *likely* range of *Harvest* is narrow

Continuous child variables (cont.)

- All-continuous network with LG distributions $\Longrightarrow$ full joint distribution is a multivariate Gaussian
- Discrete+continuous LG network is a **conditional Gaussian** network i.e., a multivariate Gaussian over all continuous variables for each combination of discrete variable values

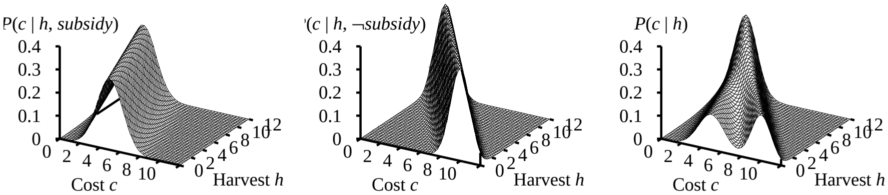
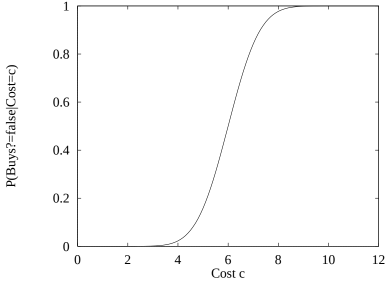


Figure 2: The graphs in (1) and (2) show the probability distribution over *Cost* as a function of *Harvest* size, with *Subsidy* true and false, respectively. Graph (3) shows the distribution $P(Cost \mid Harvest)$, obtained by summing over the two subsidy cases.

Discrete variable given continuous parents

- Probability of *Buys*? given *Cost* should be a “soft” threshold:



- **Probit** distribution uses integral of Gaussian:

$$\Phi(x) = \int_{-\infty}^{x} N(0,1)(x)dx$$

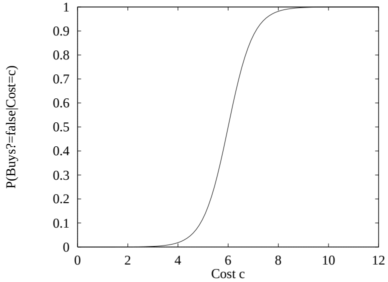
$$P(Buys? = true \mid Cost = c) = \Phi((-c+\mu)/\sigma)$$

Discrete variable given continuous parents

- **Sigmoid** (or **logit**) distribution also used in neural networks:

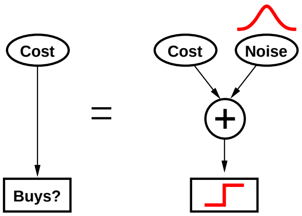
$$P(Buys? = true \mid Cost = c) = \frac{1}{1 + exp(-2\frac{-c+\mu}{\sigma})}$$

- Sigmoid has similar shape to probit but much longer tails:



Why the probit?

1. It's sort of the right shape
2. Can view as hard threshold whose location is subject to noise



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