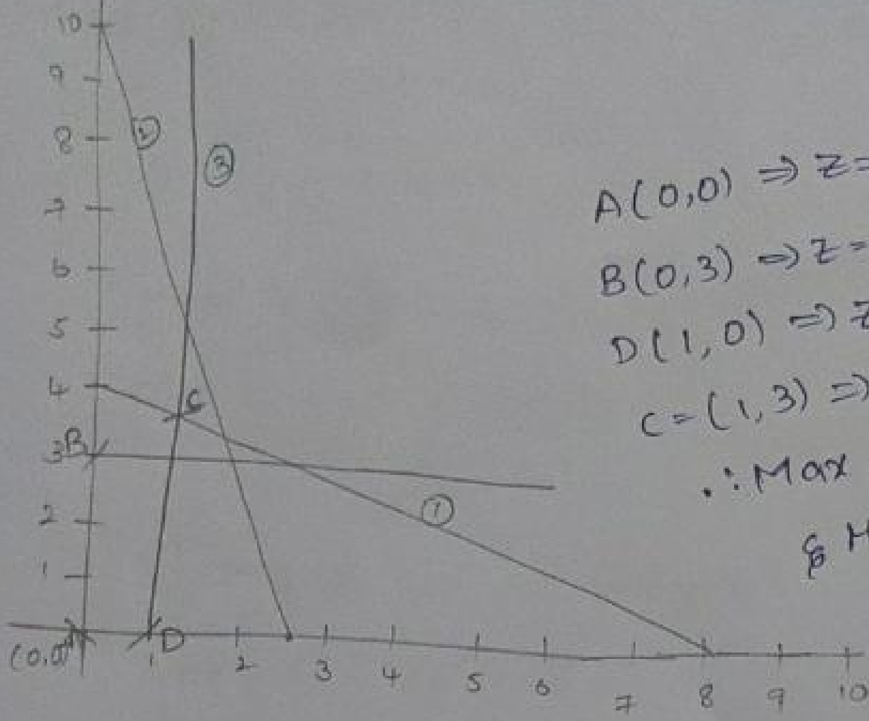


$A(0,0) \Rightarrow Z=0$

$B(0,3) \Rightarrow Z=24$

$D(1,0) \Rightarrow Z=5$

$C=(1,3) \Rightarrow Z=29$

$\therefore$ Max $Z=29$

& M.F.S are

$x_1=1$ & $x_2=3$

and are Integers.

Ar.

Unit-4 Game Theory & Simulation

Simulation: → Inventory Model.

Example:

The automobile company manufactures around 150 scotters. The daily Production varies 146 to 154 depending upon the availability of raw materials & other working materials.

Production/per day	146	147	148	149	150	151	152	153	154
Probability	0.04	0.09	0.12	0.14	0.11	0.10	0.20	0.12	0.08

The finished Scotters are transported in a specially arranged lorry accombinating 150 scotters. using the following randoms numbers 80, 81, 76, 75, 64, 43, 18, 26, 10, 12, 65, 68, 69, 61, 57. Simulation the process to find out

i) What will be the average number of scotter waiting in the factory

ii) What will be the average number of Empty Space on the lorry.

Sol.

Stage ① Random NO. C.I

Production per day	Probability	Cummulative Probability	C.I
146	0.04	0.04	0 - 3
147	0.09	0.13	4 - 12
148	0.12	0.25	13 - 24
149	0.14	0.39	25 - 38
150	0.11	0.50	39 - 49
151	0.10	0.60	50 - 59
152	0.20	0.80	60 - 79
153	0.12	0.92	80 - 91
154	0.08	1.00	92 - 99

S.NO	Random NO	Production per day	No. of ~~scotter~~ waiting in the factory	Empty Space in the lorry
1	80.	153	3	0
2	81	153	3	0
3	76	152	2	0
4	75	152	2	0
5	64	152	2	0
6	43	~~148~~ 150	0	0
7	18	148	0	2
8	26	149	0	1
9	10	147	0	3

10	12	~~152~~ 147	0	3
11	65	152	2	0
12	68	152	2	0
13	69	152	2	0
14	61	152	2	0
15	57	151	1	0
			21	9

① Average No. of seater waiting in the factory $\Rightarrow \frac{21}{15} \Rightarrow 1.9 \simeq 1$

② Average empty space in the lorry

$$\Rightarrow \frac{9}{15} \Rightarrow 0.6 \simeq 1$$

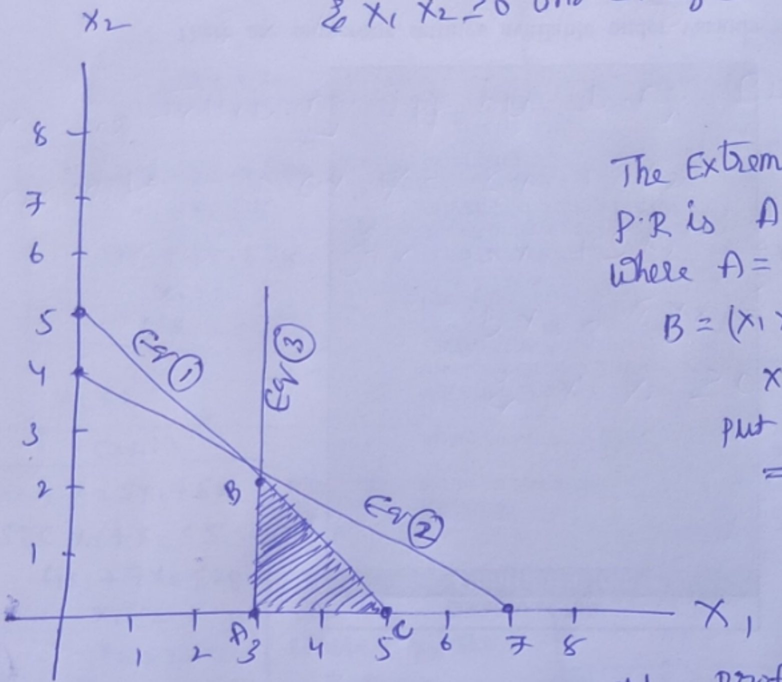
Case: ③ Max $Z = 5x_1 + 6x_2$ Shaded

STC $x_1 + x_2 \le 5 \longrightarrow (5,0)\ (0,5) \rightarrow$ Inside

$4x_1 + 7x_2 \le 28 \longrightarrow (7,0)\ (0,4) \rightarrow$ Inside

$x_1 \ge 3 \longrightarrow (3,0) \longrightarrow$ outside

& $x_1\ x_2 \ge 0$ and Integers



The Extreme points of the P·R is A, B, C

where $A = (3,0)$, $C = (5,0)$

$B = (x_1\ x_2)$ By solving ① & ③

$x_1 + x_2 = 5$

put $x_1 = 3$

$\Rightarrow 3 + x_2 = 5$

$x_2 = 5 - 3 = 2$

$\therefore B\ (x_1\ x_2) = (3,2)$

put A, B, C valleys in the profit function

Max $Z = 5x_1 + 6x_2$

at $A(3,0) \Rightarrow Z = 5(3) + 6(0) = 15$

at $B(3,2) \Rightarrow Z = 5(3) + 6(2) = 27$

at $C(5,0) \Rightarrow Z = 5(5) + 6(0) = 25$

$\therefore$ Max $Z = 27$ & Π·F·S are

$x_1 = 3$ & $x_2 = 2$ and are Integers

Conclusion

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
& $x_1 x_2 \ge 0$

→ Max $Z = 27.6$
M.F.S are
$x_1 = 7/3$ & $x_2 = 8/3$
$\Rightarrow x_1 = 2.33$ & $x_2 = 2.66$
where $x_1 x_2$ are Not Integers

case: ① $x_1 \le 2$

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
& $x_1 x_2 \ge 0$

→ Max $Z = 27.14$
MFS are $x_1 = 2$ & $x_2 = 20/7$
ie, $x_2 = 2.85$
where x_1 is Integer
But x_2 is Not a Integer

$x_1 \ge 3$ case ②

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \ge 3$
& $x_1 x_2 \ge 0$

→ Max $Z = 27$
& MFS are
$x_1 = 3$
$x_2 = 2$
where $x_1 x_2$ are Integers

Pb is Completed

$x_2 \le 2$

Case 1a

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
$x_2 \le 2$
& $x_1 x_2 \ge 0$

→ Max $Z = 22$
& M.F.S are
$x_1 = 2$
$x_2 = 2$
where $x_1 x_2$ are Integers

Pb is Completed

$x_2 \ge 3$

Case 1b

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
$x_2 \ge 3$
& $x_1 x_2 \ge 0$

→ Max $Z = 26.75$
& MFS are
$x_1 = 7/4$ & $x_2 = 3$
where $x_1 = 1.75$ is Not a Integer and
x_2 is a Integer

$x_1 \le 1$

Case 1b1

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
$x_2 \ge 3$
$x_1 \le 1$
& $x_1 x_2 \ge 0$

← Max $Z = 25.57$
& M.F.S are
$x_1 = 1$
$x_2 = 24/7 = 3.42$
when x_1 is Integer
But x_2 is Not a Integer

$x_2 \ge 2$

Case 1b2

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
$x_2 \ge 3$
$x_1 \ge 2$
& $x_1 x_2 \ge 0$

← The problem is a Infeasible Solution problem since $x_1 \le 2$ & $x_1 \ge 2$. There is No profit Area.
↓
No Solution

$x_2 \le 3$

Case 1b1a

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
$x_2 \ge 3$
$x_2 \le 3$
& $x_1 x_2 \ge 0$

→ The problem is a Infeasible solution problem since $x_2 \ge 3$ & $x_2 \le 3$
There is No profit Area.
↓
No solution

$x_2 \ge 4$

Case 1b1b

Max $= 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
$x_2 \ge 3$
$x_2 \ge 4$
& $x_1 x_2 \ge 0$

→ The problem is a Infeasible Solution pb since $x_2 \ge 3$ & $x_2 \ge 4$ the profit is outside of (0,4). So the prob is a Infeasible

$\therefore$ In case 1a the Max $Z = 22$ & In case ② the Max $Z = 27$

$\therefore$ Case ② is the best case for calculating Max profit

$\therefore$ Max $Z = 27$ & M.F.S are $x_1 = 3$ & $x_2 = 2$

② Solve the following Ipp by using Branch & Bound Technique

Max $Z = 10x_1 + 20x_2$

STC $6x_1 + 8x_2 \le 48$

$x_1 + 3x_2 \le 12$

& $x_1, x_2 \ge 0$ and Integers

③ Solve the following Ipp by using Branch & Bound Technique

Max $Z = 6x_1 + 8x_2$

STC $4x_1 + 5x_2 \le 22$

$5x_1 + 8x_2 \le 30$

& $x_1, x_2 \ge 0$ and Integers.

UNIT: 3

Integer programming problems [IPP]

Def:- A LPP is said to be an Integer programming problem [IPP] if some (or) all decision variables are restricted to take Integer values.

Ex:- Max (or) min $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

STC $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le = \ge b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le = \ge b_2$

$\vdots$

$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \le = \ge b_n$

Where $x_1, x_2 \dots x_n \ge 0$ and are integers.

This IPP can be solved by using the methods

→ Branch and Bound Technique

→ Gromory's cut (or) Cutting plane method

Branch & Bound Technique!

This method is like that of Graphical method only and using branches upto the Basic Feasible Solutions are Integers

Cutting plane Technique:-

This method is like that of Simplex method only and using branches up to the Basic Feasible Solutions are Integers.

Branch & Bound ~~Technique~~ Techniqlle

(1) Solve the following IPP by using Branch & Bound Techniqlle method:

$$\text{Max } Z = 5x_1 + 6x_2$$

STC $x_1 + x_2 \le 5$

$4x_1 + 7x_2 \le 28$

& $x_1 x_2 \ge 0$ and are integers

Sol:- Step: 1 Equating the Constraints

$\Rightarrow$ Max $Z = 5x_1 + 6x_2$

STC $x_1 + x_2 \cancel{\le} = 5 \longrightarrow 1$

$4x_1 + 7x_2 = 28 \longrightarrow 2$

& $x_1 x_2 \ge 0$

from 1 the points will be (0,5) & (5,0)

from 2 the points will be (0,4) & (7,0)

Step: 2 Now draw the Graph and Shade the regions Inside (or) outside

Take Constrain 1 $x_1 + x_2 \le 5$

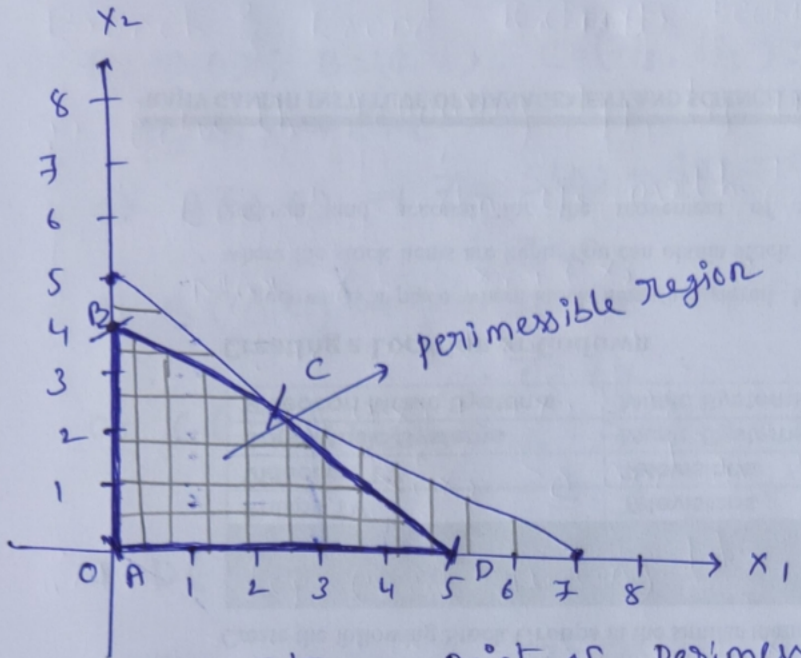
$0 \le 5$ is True

Inside only

Take Constrain 2 $4x_1 + 7x_2 = 28$

$0 \le 28$ is True.

Insideonly



The Extreme points of perimessible region is $A = (0,0)$ & $B = (0,4)$; $D = (5,0)$

and $C = (x_1, x_2)$ by solving ① & ②

Equations. ① $\rightarrow$ $x_1 + x_2 = 5$

② $\rightarrow$ $4x_1 + 7x_2 = 28$

$4 \times ① - ② \Rightarrow$
$$\begin{aligned} \cancel{4x_1} + 4x_2 &= 20 \\ \cancel{4x_1} + 7x_2 &= 28 \\ (-) \quad\quad (-) \quad\quad &(-) \\ \hline +3x_2 &= +8 \end{aligned} \Rightarrow \boxed{x_2 = 8/3}$$

put $x_2 = 8/3$ in ① $\Rightarrow$ $x_1 = 5 - x_2$
$= 5 - 8/3 = 7/3$
$\boxed{x_1 = 7/3}$

$\therefore C = [7/3, 8/3]$

Now Substitute A, B, C, D values in the profit function $Z = 5x_1 + 6x_2$

$A=(0,0)$; $B=(0,4)$; $C=(7/3, 8/3)$; $D=(5,0)$

$Z=5x_1+6x_2$

at $A(0,0) \Rightarrow Z = 5(0)+6(0)=0$

$\therefore Z=0$

at $B(0,4) \Rightarrow Z = 5(0)+6(4)=24$

$\therefore Z=24$

at $C(7/3, 8/3) \Rightarrow Z = 5(7/3)+6(8/3)$

$= \frac{35}{3}+\frac{48}{3}=\frac{83}{3}=27.6$

at $D(5,0) \Rightarrow Z = 5(5)+6(0)=25$

$\therefore Z=25$

$\therefore$ Max $Z=27.6$

& M.F.S are $x_1 = 7/3$ & $x_2=8/3$

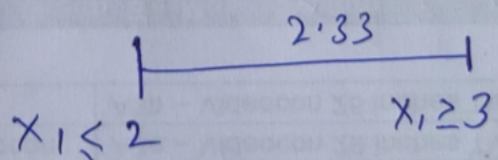
and are not a Integers.

NOW Apply Branches method for getting the Feasible Solutions are Integer.

NOW take $x_1 = 7/3 = 2.33$

NOW x_1 Can be split in two Cases.

i.e) $x_1 \le 2$ & $x_1 \ge 3$



NOW the reduced problem will be

2 Cases.

$$\text{Max } Z = 5x_1 + 6x_2$$
$$\text{STC. } x_1 + x_2 \le 5$$
$$4x_1 + 7x_2 \le 28$$
$$x_1\ x_2 \ge 0 \text{ and all Integers}$$

$x_1 \le 2$ | $x_1 \ge 3$

Case ①

$$\text{Max } Z = 5x_1 + 6x_2$$
$$\text{STC } x_1 + x_2 \le 5$$
$$4x_1 + 7x_2 \le 28$$
$$x_1 \le 2$$
$$\& \ x_1\ x_2 \ge 0 \text{ Integers}$$

Case ②

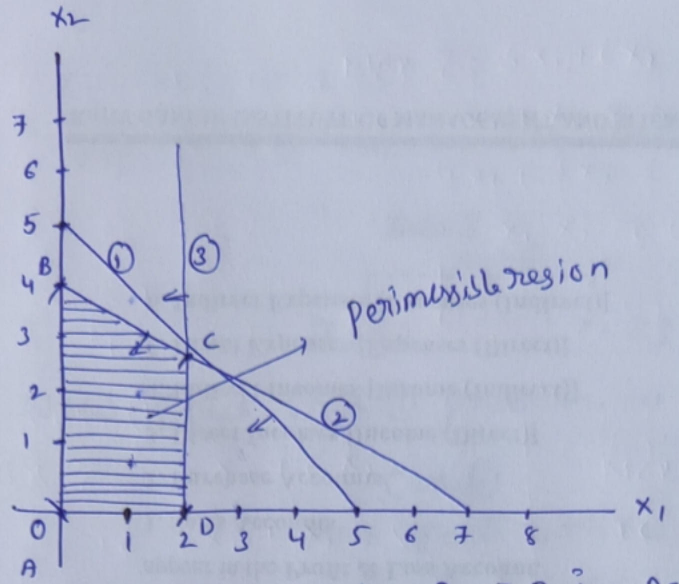
$$\text{Max } Z = 5x_1 + 6x_2$$
$$\text{STC } x_1 + x_2 \le 5$$
$$4x_1 + 7x_2 \le 28$$
$$x_1 \ge 3$$
$$\& \ x_1\ x_2 \ge 0 \text{ Integers}$$

Solution for Case ①

~~[illegible]~~ Equating the Constraints

$$\text{Max } Z = 5x_1 + 6x_2$$
$$\text{STC } x_1 + x_2 = 5 \longrightarrow (5,0)\ (0,5)$$
$$4x_1 + 7x_2 = 28 \longrightarrow (7,0)\ (0,4)$$
$$x_1 = 2 \longrightarrow (2,0)$$

x_2



The Extreme points of P.R is $A=(0,0)$, $B=(0,4)$
$D=(2,0)$ & $C=(x_1, x_2)$ by solving ② & ③

put $x_1=2$ in Eq ②

$4x_1+7x_2=28$

$4(2)+7x_2=28$

$7x_2=28-8=20$ $\therefore x_2=20/7$

$\therefore C=(2, 20/7)$

put A, B, C, D values in profit function

$Z=5x_1+6x_2$

at $A(0,0) \Rightarrow Z=0$

at $B(0,4) \Rightarrow Z=5(0)+6(4)=24$

at $C(2,20/7) \Rightarrow Z=5(2)+6(20/7)=10+\frac{120}{7}$
$=190/7$
$=27.14$

at $D(2,0) \Rightarrow Z=5(2)+6(0)=10$

$\therefore$ Max $Z=27.14$

& M.F.S are $x_1=2$

& $x_2=20/7$

x_1 is Integer & x_2 is Not a Integer.

$x_2 = 20/7 = 2.85$

NOW Split x_2 in two Sub Case.

~~2.85~~

2.85

$x_2 \le 2$ $x_2 \ge 3$

NOW Split Case ① problem in two Sub Cases.

Case ①

$$\text{Max } Z = 5x_1 + 6x_2$$

$$\text{STC } x_1 + x_2 \le 5$$

$$4x_1 + 7x_2 \le 28$$

$$x_1 \le 2$$

$$x_1, x_2 \ge 0$$

Case 1(a) $x_2 \le 2$

$x_2 \ge 3$ Case 1(b)

Case 1(a)

$$\text{Max } Z = 5x_1 + 6x_2$$

$$\text{STC } x_1 + x_2 \le 5$$

$$4x_1 + 7x_2 \le 28$$

$$x_1 \le 2$$

$$x_2 \le 2$$

$$\& \ x_1, x_2 \ge 0$$

Case 1(b)

$$\text{Max } Z = 5x_1 + 6x_2$$

$$\text{STC } x_1 + x_2 \le 5$$

$$4x_1 + 7x_2 \le 28$$

$$x_1 \le 2$$

$$x_2 \ge 3$$

$$\& \ x_1, x_2 \ge 0$$

Solution for Case 1(a)

$Max\ Z = 5x_1 + 6x_2$

STC $x_1 + x_2 \le 5 \longrightarrow (5,0)\ (0,5)$

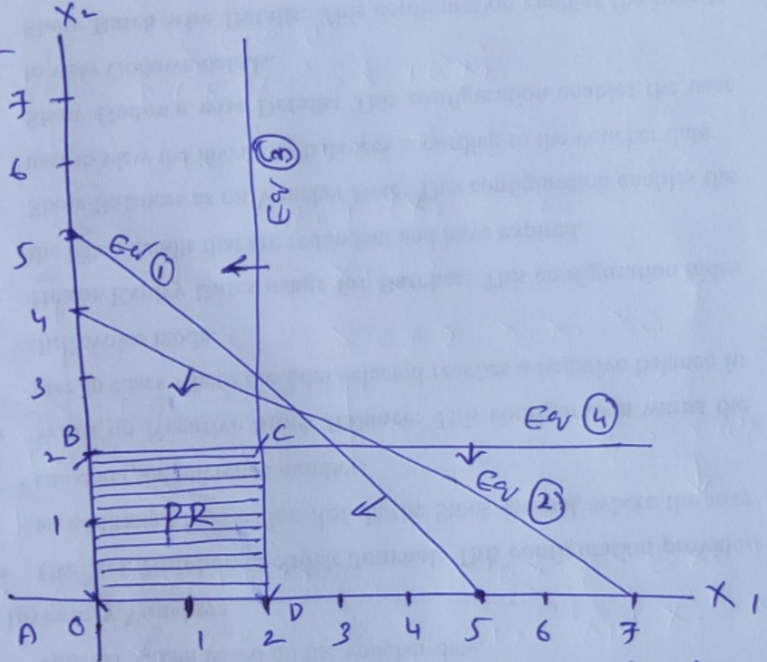
$4x_1 + 7x_2 \le 28 \longrightarrow (7,0)\ (0,4)$

$x_1 \le 2 \longrightarrow (2,0)$

$x_2 \le 2 \longrightarrow (0,2)$

& $x_1, x_2 \ge 0$

Graph :-



The Extreme points of P.R is A(0,0)

B(0,2) C = By solving (3) & (4) ie C = (2,2)

D = (2,0)

Substitute A, B, C, D values in profit function

$Z = 5x_1 + 6x_2$

at A(0,0) $\Rightarrow Z = 0$

at B(0,2) $\Rightarrow Z = 12$

at C(2,2) $\Rightarrow Z = 22$

at D(2,0) $\Rightarrow Z = 10$

$\therefore$ Max $Z = 22$ & M.F.S are $x_1 = 2, x_2 = 2$

and are Integers.

Solution For Case 1(b)

Max $Z = 5x_1 + 6x_2$

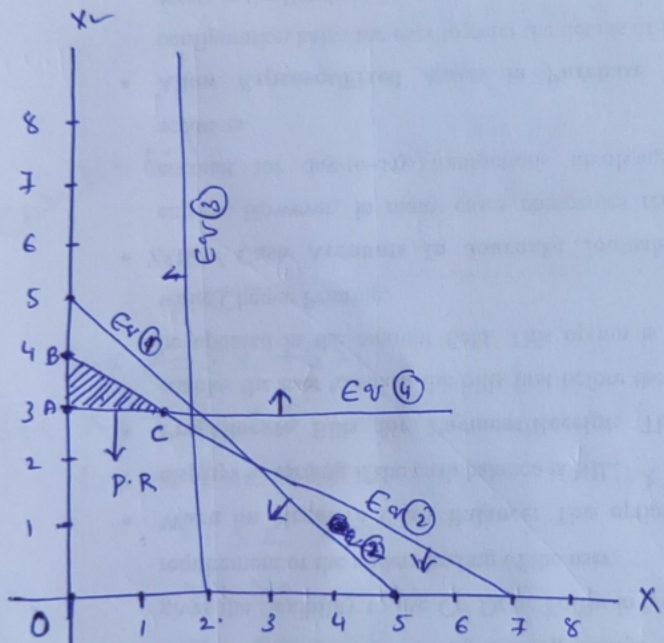
STC $x_1 + x_2 \le 5 \longrightarrow (5,0)\ (0,5)$

$4x_1 + 7x_2 \le 28 \rightarrow (7,0)\ (0,4)$

$x_1 \le 2 \longrightarrow (2,0)$

$x_2 \ge 3 \longrightarrow (0,3)$

& $x_1\ x_2 \ge 0$



Now the Extreme points of P.R is A, B, C

$A = (0,3)$; $B = (0,4)$; C = By solving ② & ④

$4x_1 + 7x_2 = 28$ $\therefore C = (7/4, 3)$

$4x_1 + 7(3) = 28$

$4x_1 = 28 - 21 = 7$ $\therefore x_1 = 7/4$

Put A, B, C values in profit function

$Z = 5x_1 + 6x_2$

at $A(0,3)$ $Z = 18$

at $B(0,4)$ $Z = 24$

at $C(7/4, 3)$ $Z = \frac{35}{4} + 18 = \frac{35+72}{4} = \frac{107}{4} = 26.75$

$\therefore$ Max $Z = 26.75$ & M.F.S are $x_1 = 7/4$ & $x_2 = 3$

x_1 is Not a Integer.

$x_1 = 7/4 = 1.75$

NOW split x_1 into two cases. $x_1 \le 1$ $\quad$ $x_1 \ge 2$ (1.75)

NOW case 1(b) will be split in two sub cases.

Case 1(b)

$\text{Max } Z = 5x_1 + 6x_2$

STC $x_1 + x_2 \le 5$

$4x_1 + 7x_2 \le 28$

$x_1 \le 2$

$x_2 \ge 3$

& $x_1\ x_2 \ge 0$

Case 1(b)-1 $\quad x_1 \le 1$

Case 1b-2 $\quad x_1 \ge 2$

$\text{Max } Z = 5x_1 + 6x_2$

STC $x_1 + x_2 \le 5$ — ①

$4x_1 + 7x_2 \le 28$ — ②

$x_1 \le 2$ — ③

$x_2 \ge 3$ — ④

$x_1 \le 1$ — ⑤

& $x_1\ x_2 \ge 0$

Solution for case 1-b-1

from ① $(5,0)\ (0,5)$

From ② $(7,0)\ (0,4)$

from ③ $(2,0)$

from ④ $(0,3)$

from ⑤ $(1,0)$

$\text{Max } Z = 5x_1 + 6x_2$

STC $x_1 + x_2 \le 5$

$4x_1 + 7x_2 \le 28$

$x_1 \le 2$

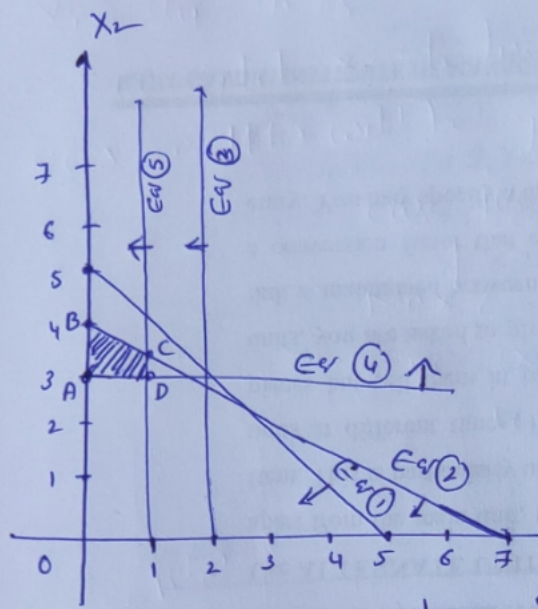
$x_2 \ge 3$

$x_1 \ge 2$

& $x_1\ x_2 \ge 0$

Case 1-b-2 is a Infeasible Solution problem since $x_1 \le 2$ & $x_1 \ge 2$

→ Infeasible of Case 1-b-2



The Extreme points of PR is

$A = (0,3)$ $B = (0,4)$

$D = (1,3)$ $C =$ by solving (2) & (5)

put $x_1 = 1$ in (2)

$4x_1 + 7x_2 = 28$

$7x_2 = 28 - 4 = 24$

$7x_2 = 24$

$x_2 = 24/7 = 3.42$

$\therefore C = (1, 24/7)$

Put A, B, C, D values in profit function

$Z = 5x_1 + 6x_2$

at $A(0,3) \rightarrow Z = 18$

at $B(0,4) \rightarrow Z = 24$

at $C(1, 24/7) \rightarrow Z = 5 + \frac{144}{7} = \frac{179}{7} = 25.57$

at $D(1,3) \rightarrow Z = 5 + 18 = 23$

$\therefore$ Max $Z = 25.57$ & M.F.S are $x_1 = 1$ & $x_2 = 24/7$

But $x_2 = 3.42$ is Not a Integer

split x_2 into two cases $\underbrace{\qquad}_{x_2 \le 3 \qquad x_2 \ge 4}^{3.42}$

Now split case 1-5-1 into 2 cases.

2 Cases.

Case - 1-b-1

$$\text{Max } Z = 5x_1 + 6x_2$$

STC $x_1 + x_2 \le 5$

$4x_1 + 7x_2 \le 28$

$x_1 \le 2$

$x_2 \ge 3$

$x_1 \le 1$

& $x_1\ x_2 \ge 0$

$x_2 \le 3$ | $x_2 \ge 4$

Case 1-b-1-a

$$\text{Max } Z = 5x_1 + 6x_2$$

STC $x_1 + x_2 \le 5$

$4x_1 + 7x_2 \le 28$

$x_1 \le 2$

$x_2 \ge 3$

$x_1 \le 1$

$x_2 \le 3$

& $x_1\ x_2 \ge 0$

Now Case 1-b-1-a is a Infeasible solution problem since $x_2 \le 3$ & $x_2 \ge 3$

Case - 1-b-1-b

$$\text{Max } Z = 5x_1 + 6x_2$$

STC $x_1 + x_2 \le 5$

$4x_1 + 7x_2 \le 28$

$x_1 \le 2$

$x_2 \ge 3$

$x_1 \le 1$

$x_2 \ge 4$

& $x_1, x_2 \ge 0$

Now Case 1-b-1-b is a Infeasible solution since $4x_1 + 7x_2 \le 28$ & $x_2 \ge 4$

Case: ② Max $Z = 5x_1 + 6x_2$

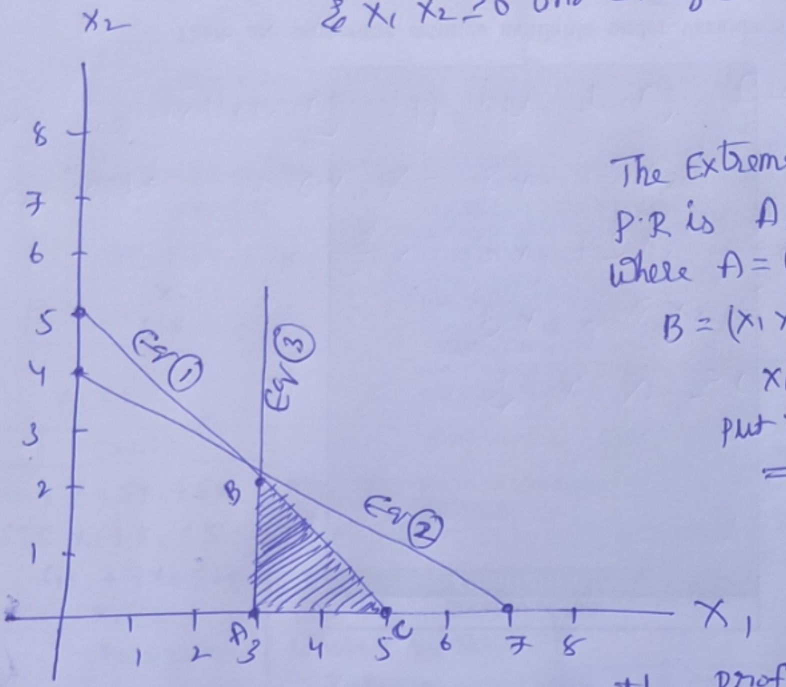
Shaded

STC $x_1 + x_2 \le 5 \longrightarrow (5,0)\ (0,5) \rightarrow$ Inside

$4x_1 + 7x_2 \le 28 \longrightarrow (7,0)\ (0,4) \rightarrow$ Inside

$x_1 \ge 3 \longrightarrow (3,0) \longrightarrow$ outside

& $x_1\ x_2 \ge 0$ and Integers



The Extreme points of the P.R is A, B, C

where $A = (3,0)$, $C = (5,0)$

$B = (x_1\ x_2)$ By solving ① & ③

$x_1 + x_2 = 5$

put $x_1 = 3$

$\Rightarrow 3 + x_2 = 5$

$x_2 = 5 - 3 = 2$

$\therefore B(x_1\ x_2) = (3,2)$

put A, B, C values in the profit function

Max $Z = 5x_1 + 6x_2$

at $A(3,0) \Rightarrow Z = 5(3) + 6(0) = 15$

at $B(3,2) \Rightarrow Z = 5(3) + 6(2) = 27$

at $C(5,0) \Rightarrow Z = 5(5) + 6(0) = 25$

$\therefore$ Max $Z = 27$ & I.F.S are

$x_1 = 3$ & $x_2 = 2$ and are Integers

Conclusion

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
& $x_1\, x_2 \ge 0$

→ Max $Z = 27.6$
M.F.S are
$x_1 = 7/3$ & $x_2 = 8/3$
$\Rightarrow x_1 = 2.33$ & $x_2 = 2.66$
where $x_1\, x_2$ are Not Integers

case: ① $x_1 \le 2$

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
& $x_1\, x_2 \ge 0$

→ Max $Z = 27.14$
MFS are $x_1 = 2$ & $x_2 = 20/7$
ie, $x_2 = 2.85$
where x_1 is Integer
But x_2 is Not a Integer

$x_1 \ge 3$ case ②

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \ge 3$
& $x_1\, x_2 \ge 0$

→ Max $Z = 27$
& MFS are
$x_1 = 3$
$x_2 = 2$
where $x_1\, x_2$ are Integers

Pb is Completed

$x_2 \le 2$

Case 1a

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
$x_2 \le 2$
& $x_1\, x_2 \ge 0$

→ Max $Z = 22$
& M.F.S are
$x_1 = 2$
$x_2 = 2$
where $x_1\, x_2$ are Integers

Pb is Completed

$x_2 \ge 3$

Case 1b

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
$x_2 \ge 3$
& $x_1\, x_2 \ge 0$

→ Max $Z = 26.75$
& MFS are
$x_1 = 7/4$ & $x_2 = 3$
where $x_1 = 1.75$ is Not a Integer and
x_2 is a Integer

$x_1 \le 1$

Case 1b1

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
$x_2 \ge 3$
$x_1 \le 1$
& $x_1\, x_2 \ge 0$

Max $Z = 25.57$
& M.F.S are
$x_1 = 1$
$x_2 = 24/7 = 3.42$
when x_1 is Integer
But x_2 is Not a Integer

$x_2 \ge 2$

Case 1b2

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
$x_2 \ge 3$
$x_1 \ge 2$
& $x_1\, x_2 \ge 0$

The problem is a Infeasible Solution problem since $x_1 \le 2$ & $x_1 \ge 2$. There is No profit Area.
↓
No solution

$x_2 \le 3$

Case 1b1a

Max $Z = 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
$x_2 \ge 3$
$x_2 \le 3$
& $x_1\, x_2 \ge 0$

→ The problem is a Infeasible solution problem since $x_2 \ge 3$ & $x_2 \le 3$
There is No profit Area.
↓
No solution

$x_2 \ge 4$

Case 1b1b

Max $= 5x_1 + 6x_2$
STC $x_1 + x_2 \le 5$
$4x_1 + 7x_2 \le 28$
$x_1 \le 2$
$x_2 \ge 3$
$x_2 \ge 4$
& $x_1\, x_2 \ge 0$

→ The problem is a Infeasible Solution pb since $x_2 \ge 3$ & $x_2 \ge 4$ the profit is outside of (0,4). So the probl is a Infeasible

$\therefore$ In case 1a the Max $Z = 22$ & In case ② the Max $Z = 27$
$\therefore$ Case ② is the best case for calculating Max profit
$\therefore$ Max $Z = 27$ & M.F.S are $x_1 = 3$ & $x_2 = 2$

(2) Solve the following Ipp by using Branch & Bound Technique

Max $Z = 10x_1 + 20x_2$

STC $6x_1 + 8x_2 \leq 48$

$x_1 + 3x_2 \leq 12$

& $x_1, x_2 \geq 0$ and Integers

(3) Solve the following Ipp by using Branch & Bound Technique

Max $Z = 6x_1 + 8x_2$

STC $4x_1 + 5x_2 \leq 22$

$5x_1 + 8x_2 \leq 30$

& $x_1, x_2 \geq 0$ and Integers.

GAME THEORY

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Aim

To teach the students the over view of game theory, classical examples of game theory and applications, history of game theory, classification of game theory, key elements in game theory, Mathematical concept used in game theory, Procedure for solving game theory problem and explain worked out example.

Objective

Students should get clear idea on what is game theory and the underlying concepts, students should get though knowledge on the mathematical concept used in game theory, they should understand the solving technique and know what method can be applied to the given problem for getting solution.

Outcome

At the end of the course student get the clear ides of the following , origin of game theory, Author who studied and developed game theory, Practical use of game theory in real life, types of game theory, Mathematic required for solving game theory, Technique of solving for different types of games.

Preview of lecture

- ✓ Overview of game theory
- ✓ Classic examples of game theory
- ✓ History of game theory
- ✓ Classification of game theory
- ✓ Key elements in game theory
- ✓ Mathematical concept in Game Theory
- ✓ Procedure to solve 2x2 game without saddle point
- ✓ Algebraic Method for solving games without saddle point
- ✓ Home work and Assignment Problem
- ✓ References

Overview of Game Theory

Game theory is an approach to modelling behaviour in situations where the outcome of your decisions *depends on the decisions of others*.

Game theory is the *study of strategic, interactive* decision making among rational individuals or organizations.

Game theory is a branch of *applied mathematics* that provides tools for analyzing situations in which parties (*called players*) make decisions that are interdependent.

This interdependence causes each player to consider the other player's possible decisions(*or strategies*) in formulating strategy.

In addition, a player *need not be an individual*; it may be a nation, a corporation, or a team comprising many people with shared interests.

A solution to a game describes the optimal decisions of the players, who may have *similar*, *opposed, or mixed interests*, and the outcomes that may result from these decisions.

Game theory is applied for determining different strategies in the *business world*. It offers valuable tools for solving strategy problems.

Classic examples of game theory

(i) The Prisoner's Dilemma; where two suspects are in police custody as accomplices for the same crime, but there is not enough evidence for a felony conviction. They are held and interrogated separately. If one prisoner testifies against the other while the other stays silent, the testifying prisoner goes free and the silent prisoner is convicted and serves ten years. If both prisoners stay silent, both are convicted on a minor charge and serve six years. If both prisoners testify against each other, each serves five years. How should the prisoners act? . The answer is that both prisoners should testify against the other, an outcome that is known as a *Nash equilibrium*. [i.e The players in each game depend on each other's rationality to make the optimal choices in every situation, thereby maximizing their own utility]

(ii) Conflict of sales; Let us assume that the client had a drug with dominant market share in a therapeutic area that is competitive, yet has few players. One of their competitors was behaving aggressively from a pricing perspective (e.g., continually matching and beating the client's price, even though the competitor gained little or no market share from such actions). The competitor was in the process of running a clinical trial to improve their product's label. The client wished to know how the competitor might behave depending on the client's next price move and the outcome of the clinical trial. The conclusion of the exercise was that engaging in a price war would not benefit any of the client or competitors. Therefore, keeping the price high and trading "market share for peace" would be the most profitable strategy.

(iii) Advertising War: Coke vs. Pepsi :- Without any advertising, each company earns $5b/year from Cola consumers. Each company can choose to spend $2b/year on advertising. Advertising does not increase total sales for Cola, but if one company advertises while the other does not, it captures $3b from the competitor

.

		Pepsi	
		No Ad	Ad
	No Ad	$5b,$5b	$2b,$6b
Coke	Ad	$6b,$2b	$3b,$3b

What will the Cola companies do? Is there a better feasible outcome

(iii) In your everyday life: Everything is a game, poker, chess, soccer, driving, dating, stock market advertising, setting prices, entering new markets, building a reputation bargaining, partnerships, job market search and screening designing contracts, auctions, insurance, environmental regulations international relations, trade agreements, electoral campaigns, Most modern economic research includes game theoretical elements.

(iv) Game theory has been used to analyze parlour games, but its applications are much broader.

History of game theory

The individual closely associated with the creation of the theory of games is *John von Neumann*, one of the greatest mathematicians of this century. Although others proceeded him in formulating a theory of games - notably *Emile Borel* - it was *von Neumann* who published in 1928 the paper that laid the foundation for the theory of two-person zero-sum games.

The theory of Games was born in 1944 with the publication of *Theory of Games and Economic Behaviour* by Hungarian-born American mathematician *John von Neumann* and his Princeton University colleague *Oskar Morgenstern*, a German-born American economist. In their book, . They observed that economics is much like a game, wherein players anticipate each other's moves, and therefore requires a new kind of mathematics, which they called game theory. Their choice of title was a little unfortunate, since it quickly got shortened to "*Game Theory*,"

Nobel Laureate and a **Father of Game Theory**, Lloyd S. Shapley(92), who shared the 2012 Nobel Memorial Prize in Economic Science for work on **game theory** that has been used to study subjects as diverse as matching couples and allocating costs.

Classification of game theory

It broadly classified into three main sub-categories of study

(1) Classical game theory

It focuses on optimal play in situations where *one or more people* must make a decision and the impact of that decision and the decisions of those involved is known.

Decisions may be made by use of a *randomizing device like piping a coin* .

It focuses on questions like, What is my best decision in a given economic scenario, where a reward function provides a way for me to understand how my decision will impact my result.

Examples:

Poker, Strategic military decision making, Negotiations.

(2) Combinatorial game theory

It focuses on optimal play in *two-player* games in which each player takes turns changing in *pre-defined ways*. In other word, combinatorial game theory does not consider games with chance (no randomness).

Generally two player strategic games played on boards. Moves change the structure of a game board.

Examples:

Chess, Checkers, Go.

(3)Dynamic game theory:

It focuses on the analysis of games in which players must make decisions over time and in which those decisions will affect the outcome at the next moment in time.

It often relies on differential equations to model the behaviour of players over time.

It can help optimize the behaviour of unmanned vehicles or it can help you capture your baby sister who has escaped from her playpen.

In general games with time, Games with motion or a dynamic component.

Examples:

Optimal play in a dog fight, Chasing your brother across a room.

Key elements in game theory

Player: who is interacting

Strategies: what are the options of each player? In what order do players act?

Payoffs: How do strategies translate into outcomes? What are players' preferences over possible outcomes?

Information/Beliefs: What do players know/believe about the situation and about one another? What actions do they observe before making decisions?

Rationality: How do players think?

Mathematical concept in Game Theory

The following the prerequisite required for game theory

Sum of gains and loss

If in a game sum of the gains to one player is exactly equal to the sum of losses to another player, so that sum of the gains and losses equal zero, the corresponding game is said to be zero sum game.

Types of games

Games can be classified according to certain significant features, the most obvious of which is the number of players. Thus, a game can be designated as being a *one-person, two-person, or n-person (with n greater than two) game*, games in each category having their own distinctive features.

One-person games

One-person games are also known as *games against nature*. With no opponents, the player only needs to list available options and then choose the optimal outcome. When chance is involved the game might seem to be more complicated, but in principle the decision is still relatively simple.

For example, a person deciding whether to carry an umbrella. While this person may make the wrong decision, there does not exist a conscious opponent. That is, nature is presumed to be completely indifferent to the player's decision, and the person can base his decision on simple probabilities. One-person games hold little interest for game theorists.

Two person zero sum game (with two players)

The game in which there are exactly *two player* and the interest of the players *completely opposed* are referred as two-person zero sum games. They are called zero-sum games because one player wins whatever the other player loses. In short it is denoted by TPZS game.

For example, All parlour game and sports, like Tic-tac-toe, chess, cribbage, backgammon, and tennis ect., are TPZS games

Two person zero sum game (with more than two players)

TPZS games with *more than two people involved* are

(i) Team sports with only two sides, but with more than one player in each side

(ii) Many people involved in *surrogates for military conflict*, so it should come as no surprise that many military problems can also be analyzed as TPZS games.

Games which are not TPZS

(i) Those parlour games in which the players cannot be clearly separated into two sides are not TPZS games

(ii) Those poker and Monopoly games when played by more than two people are not TPZS games.

(iii) Most real economic "games" are not TPZS, because there are too many players, and also the interests of the players are not completely opposed.

Positive-sum game

In game theory, a term positive sum refers to situations in which the total of gains and losses *is greater than zero*.

A positive sum occurs when resources are somehow increased and an approach is formulated such that the desires and needs of all concerned are satisfied.

Perfect games

Games of perfect information in which each player knows everything about the game at all times. It is called *perfect games*

For example, chess in which each player knows everything about the game at all times. In chess exactly one of three outcomes must occur if the players make optimal choices: (i) White wins (has a strategy that wins against any strategy of black), (ii) Black wins (iii) White and black draw.

Imperfect games

Games of imperfect information in which each player do not knows everything about the game at all times. It is also called *imperfect games*

For example, Poker in which players do not know all of their opponents' cards.

Finite games

Games in which each player has a finite number of options, the number of players is finite, and the game cannot go on indefinitely.

For example, chess, checkers, poker, and most parlour games are finite.

Cooperative games

In game theory, a cooperative game (*or coalitional game*) is a game with competition between groups of players *("coalitions")* due to the possibility of *external enforcement* of cooperative behaviour (e.g. through contract law).

Non cooperative games

Those are opposed to cooperative games in which there is either no possibility to forge alliances or all agreements need to be self-enforcing (e.g. through credible threats).

Pay off

The outcome of the game resulting from a particular decision (or strategy) is called pay off . It is assumed that pay off is also known to the player in advance.

It is expressed in times of numerical values such as money, percent of market share or utility.

Pay off matrix

The pay offs in terms of gains or losses, when players select their particular strategies, can be represented in the form of matrix is called pay off matrix.

Let $A_1, A_2, \ldots A_m$ are possible strategies for player A

Let $B_1, B_2, \ldots B_n$ are possible strategies for player B.

The total number of possible outcomes are m x n and it is assumed that each player knows not only his own list of possible course of action but also his opponent.

For our convenience, we assume that player A always a gainer whereas player B a looser.

Let a_{ij}=pay off which player A gain from player B, if player A choose strategy i and player B chooses strategy J.

Pay of matrix player A is represented in the form of table

Player A's strategies	Player B's strategies			
	B_1	B_2	....	B_n
A_1	a_{11}	a_{12}	....	a_{1n}
A_2	a_{21}	a_{22}	....	a_{2n}
....			....	
A_m	a_{m1}	a_{m2}	....	a_{mn}

Remarks

For the zero sum games, the gain of one player is equal to the loss of other and vice versa. i.e one player pay off table would contain the same amounts in pay off table of other player with the sign changed. Therefore it is enough to construct pay of table for one player.

Strategy

The strategy for a player is the list of all possible actions (moves or course of action) that he will take for every pay-off (outcome) that might arise. It is assumed that all course of possible actions are known in advance to the player.

Types of Strategy

Usually player in game theory uses two types of strategy namely pure strategy and mixed strategy .

(i) Pure strategy:

Particular course of action that are selected by player is called pure strategy (course of action). i.e each player knows in advance of all strategies out of which he always selects only one particular strategy regardless of the other players strategy, and objective of the player is to maximize gain or minimize loss

(ii) Mixed strategy:

Course of action that are to be selected on a particular occasion with some fixed probability are called mixed strategies. i.e there is a probabilistic situation and objective of the players is to maximize expected gain or minimize expected losses by making choice among pure strategy with fixed probabilities.

In mixed strategy, If there are 'n' number of pure strategies of the player, there exist a set $S=\{p_1,p_2,.....p_n\}$ where p_j is the probility with which the pure strategy, j would be selected and whose sum is unity.

i.e $p_1+p_2+.....+p_n=1$ and $p_j>=0$ for all $j=1,2,...n$.

Remark:

(i) If a player randomly chooses a pure strategy, we say that the player is using a "mixed strategy." In a pure strategy a player chooses an action for sure, whereas in a mixed strategy, he chooses a probability distribution over the set of actions available to him.

(ii) If a particular $p_j=1$ and all others are zero, then the player is said to select pure strategy J.

Optimal strategy

The particular strategy (or complete plan) by which a player optimizes his gains or losses without knowing the competitor's strategies is called optimal strategy.

Value of the game

The expected outcome when players follow their optimal strategy is called the value of the game, It is denoted by V

Basic assumptions of game

(i) Each player has available to him a finite number of possible strategies. The list may not be same for each player.

(ii) Player A attempts to maximize gains and player B minimize losses.

(iii) The decisions of both players are made individually prior to the play with no communication between them.

(iv) The decisions are made simultaneously and also announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player 's decision.

(v) Both the players know not only possible pay offs to themselves but also other.

Minmax-Maxmin principle

The selection of an optimal strategy by each player without the knowledge of the competitor's strategy is the basic problem of playing games. The objective of the study is to know how these players select their respective strategy so that they may optimize their pay off. Such a decision making criterion is referred to as the minmax -maxmin principle

Remarks

Minmax-Maxmin principle given the best possible selection of strategy for both players in pure strategy problem.

Saddle point

If the minmax value = maxmin value, then the game is said to have a saddle (equilibrium) point

Remarks

(i) The corresponding strategy at saddle point are called optimum strategy.

(ii) The amount of pay off at an saddle point is known as the value of the game.

(iii) A game may have more than one saddle point.

(iv) There are game without saddle point.

(v) Its name derives from its being the minimum of a row that is also the maximum of a column in a payoff matrix—to be illustrated shortly—which corresponds to the shape of a saddle.

Procedure to determine saddle point

Step:-01

Select the minimum (lowest) element in each row of the pay off matrix and write them under 'row minima' heading. Then select the largest element among these elements and enclose it in a rectangle.

Step:-02

Select the maximum (largest) element in each column of the pay of matrix and write them under 'column maxima' heading. Then select the lowest element among these elements and enclose it in a circle.

Step:-03

Find out the elements which is same in the circle as well as rectangle and mark the position of such elements in the matrix. This element represents the value of the game and is called the saddle point.

Games without saddle point

Suppose if there is no pure strategy solution for a game, then there is no saddle point exist. In these situations, to solve the game both the player must determine the *optimal mixtures of strategies* to find saddle point.

The optimum strategy mixture of each player may be determined by assigning each strategy *it probility of being chosen*. The optimal strategy so determined is called mixed strategy.

Fair Game

If the value of the game is zero (i.e. there is no loss or gain for any player), the game is called fair game. The simplest type of game is one where the best strategies for both players are pure strategies. This is the case if and only if, the pay-off matrix contains a saddle point.

Strictly determinable

A game is said to be strictly determinable if the maxmin and minmax values of the game are equal and both equal the value of the game.

Example:-01 (Games with sadle point)

Find the optimal plan for both the player

		Player-B			
		I	II	III	IV
Player-A	I	-2	0	0	5
	II	4	2	1	3
	III	-4	-3	0	-2
	IV	5	3	-4	2

Solution:-

We use maxmin-minmax principle for solving the game.

		Player-B				Row Minimum
		I	II	III	IV	
Player-A	I	-2	0	0	5	-2
	II	4	2	1	3	$\boxed{1}$
	III	-4	-3	0	-2	-4
	IV	5	3	-4	2	-6
Column Maximum		5	3	1	5	

Select minimum from the column maximum values.

ie. Minimax = 1, (marked as circle)

Select maximum from the row minimum values

ie. Maximin = 1, (marked as rectangle)

Player A will choose strategy II, which yields the maximum payoff of 1

Player B will choose strategy III.

The value of game is 1, which indicates that player A will gain 1 unit and player B will sacrifice 1 unit.

Since the maximin value = the minimax value =1, therefore the game has saddle point and the game is not fair game. (since value of the game is non zero)

Also maxmin=minimax=value of game, therefore the game is strictly determinable.

It is a pure strategy game and the saddle point is (A-II, B-III)

The optimal strategies for both players given by pure strategy , Player A must select strategy II and playerB must select strategy III.

Example:-02

For the game with payoff matrix

		Player B	
Player A	-1	2	-2
	6	4	-6

Determine the best strategies for players A and B and also the value of the game. Is this game (i) fair (ii) strictly determinable?

Solution:-

	Player B			Row minimum
Player A	-1	2	$\boxed{-2}$	$\boxed{-2}$
	6	4	-6	-6
Column maximum	6	4	-2	

Select minimum from the column maximum values.

ie. Minimax = 1, (marked as circle)

Select maximum from the row minimum values.

ie. Maximin = 1, (marked as rectangle)

Player A will choose strategy I, which yields the maximum payoff of -2

Player B will choose strategy III.

The value of game is -2, which indicates that player A will gain -2 unit and player B will sacrifice -2 unit.

Since the maximin value = the minimax value =-2, therefore the game has saddle point and the game is not fair game. (since value of the game is non zero)

Also maxmin=minimax=value of game, therefore the game is strictly determinable.

It is a pure strategy game and the saddle point is (A-I, B-III)

The optimal strategies for both players given by pure strategy , Player A must select strategy II and playerB must select strategy III.

Example:-03

Find the range of values of p and q which will render the entry (2,2) a saddle point for the game.

	Player B		
Player A	B_1	B_2	B_3
A_1	2	4	5
A_2	10	7	q
A_3	4	p	6

Solution:-

Let us ignore the values of p and q in pay off matrix and proceed to calculate maxmin and minmax values.

	Player B			
Player A	**B_1**	**B_2**	**B_3**	Row minimum
A_1	2	4	5	2
A_2	10	$\boxed{7}$	q	$\boxed{7}$
A_3	4	p	6	4
Column Maximum	10	7	6	

Here maxmin=7, minmax=6, i.e the value of the game may be between 6 and 7.

i.e maxmin is not equal to minmax, therefore there is no unique saddle point.

Games with no saddle point should solved using mixed strategy.

If the saddle point exist at (2,2), only if q>7 and p>=7.

Example:-04

For what value of a, the game with following pay-offs matrix is strictly determinable?

	Player B		
Player A	B_1	B_2	B_3
A_1	a	6	2
A_2	-1	a	-7
A_3	-2	4	a

Solution:-

Let us ignore the values of a in pay off matrix and proceed to calculate maxmin and minmax values.

		Player B			
Player A		B_1	B_2	B_3	Row minimum
	A_1	a	6	$\boxed{2}$	$\boxed{2}$
	A_2	-1	a	-7	-7
	A_3	-2	4	a	-2
Column Maximum		-1	6	2	

Here maxmin=2, minmax=-1, i.e the value of the game lies between -1 and 2.

i.e maxmin is not equal to minmax, therefore there is no unique saddle point.

Games with no saddle point should be solved using mixed strategy.

For strictly determinable game , we must have -1<=a<=2.

Procedure to solve 2x2 game without saddle point

If pay-off matrix for player A is given by

	Player B	
Player A	a_{11}	a_{12}
	a_{21}	a_{22}

then the following formulae are used to find the value of the game and optimal strategies.

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} \quad \& \quad p_2 = 1 - p_1$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} \quad \& \quad q_2 = 1 - q_1$$

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

Hence the optimal mixed strategy for player A and B given by

$S_A=\begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix}$ and $S_B=\begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$

Here

p_1=probability of player A choose strategy A_1

p_2=probability of player A choose strategy A_2

q_1=probability of player A choose strategy B_1

q_2=probability of player A choose strategy B_2

Example:-01

Two player A and B match coins. If the coins match, then A wins two units of value, if the coin do not match, then B win 2 units of value. Determine the optimum strategies for the players and the value of the game

Solution:-

Let us construct the pay off matrix for player A

	Player B		
Player A	H	T	Row minimum
H	2	-2	-2
T	-2	2	-2
Column Maximums	2	2	

Since maxmin=-2 and minmax=2. i.e the value of the game lies between -2 and 2. i.e maxmin is not equal to minmax, therefore there is no unique saddle point. Games with no saddle point should be solved using mixed strategy.

It is a 2x2 game without saddle point, we use the following formulae

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{(2) - (-2)}{(2+2) - (-2-2)} = 4/8 = 1/2$$

$$p_2 = 1 - p_1 = 1/2$$

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{2 - (-2)}{2 + 2 - (-2-2)} = 4/8 = 1/2$$

$$q_2 = 1 - q_1 = 1/2$$

Therefore it is a fair game.

Hence the optimal mixed strategy for player A and B given by

$S_A = \begin{pmatrix} H & T \\ 1/2 & 1/2 \end{pmatrix}$ and $S_B = \begin{pmatrix} H & T \\ 1/2 & 1/2 \end{pmatrix}$

Example:-02

Consider a modified form of " matching biased coins" game problem. The matching player is paid Rs. 8.00 if the two coins turn both heads and Rs. 1.00 if the coins turn both tails. The non-matching player is paid Rs. 3.00 when the two coins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy?

Solution:-

Let us construct the pay off matrix for matching player

	Non matching player		
Matching Player	H	T	Row minimum
H	8	-3	-3
T	-3	1	-3
Column Maximums	8	1	

Since maxmin=-3 and minmax=1. i.e the value of the game lies between -3 and 1.

i.e maxmin is not equal to minmax, therefore there is no unique saddle point.

Games with no saddle point should be solved using mixed strategy.

It is a 2x2 game without saddle point, we use the following formulae

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{(1) - (-3)}{(8+1) - (-3-3)} = 4/15$$

$p_2 = 1 - p_1$ =11/15

$$q_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})} = \frac{(1) - (-3)}{(8+1) - (-3-3)} = 4/15$$

$q_2 = 1 - q_1$ =11/15

Therefore it is a fair game.

Hence the optimal mixed strategy for player A and B given by

$S_A = \begin{pmatrix} H & T \\ 4/15 & 11/15 \end{pmatrix}$ and $S_B = \begin{pmatrix} H & T \\ 4/15 & 11/15 \end{pmatrix}$

Algebraic Method for solving games without saddle point

Let $p_1, p_2, \ldots, p_m$ be the probability that the player A choose his strategy $A_1, A_2, \ldots A_m$ respectively. where $p_1+p_2+\ldots+p_m=1$

Let $q_1, q_2, \ldots, q_n$ be the probability that the player B choose his strategy $B_1, B_2, \ldots B_n$ respectively. where $q_1+q_2+\ldots+q_n=1$

Player A's strategies	Player B's strategies				Probability
	B_1	B_2	....	B_n	p_1
A_1	a_{11}	a_{12}	....	a_{1n}	p_2
A_2	a_{21}	a_{22}	....	a_{2n}	p_3
....			....		
A_m	a_{m1}	a_{m2}	....	a_{mn}	p_m
Probability	q_1	q_2		q_n	

Let V be the value of the game.

To find S_A

The expected gain to player A when player B selects strategies $B_1, B_2, \ldots B_n$ respectively

Since player A is gainer player and he expects at least V, therefore we have

$a_{11}p_1+a_{12}p_2+\ldots\ldots+a_{m1}p_m>=V$

$a_{12}p_1+a_{22}p_2+\ldots+a_{m1}p_m>=V$

......

$a_{1n}p_1+a_{22}p_2+\ldots+a_{mn}p_m>=V$

To find the values of p_i's , above inequalities are considered as equations and are then solved for given unknowns.

To find S_B

The expected loss to player B when player A selects strategies $A_1, A_2, \ldots a_m$ respectively

Since player B is loser player , therefore we have

$a_{11}q_1+a_{12}q_2+\ldots\ldots+a_{1n}q_n<=V$

$a_{12q1}+a_{22}q_2+\ldots+a_{m1}q_m>=V$

......

$a_{1n}q_1+a_{2n}q_2+\ldots+a_{mn}q_m>=V$

To find the values of q_i's , above inequalities are considered as equations and are then solved for given unknowns.

By substituting the values of a_i 's and q_i's in any one of the above equation give the value of the game.

Remarks:-

This method becomes quite lengthy when number of strategies for both the players are more than two.

Example:-01

Two player A and B match coins. If the coins match, then A wins two units of value, if the coin do not match, then B win 2 units of value. Determine the optimum strategies for the players and the value of the game

Solution:-

Let us construct the pay off matrix for player A

	Player B			
Player A	H	T	Row minimum	Probability
H	2	$\boxed{-2}$	$\boxed{-2}$	p_1
T	$\boxed{-2}$	2	$\boxed{-2}$	p_2
Column Maximums	2	2		
Probability	q_1	q_2		

Since maxmin=-2 and minmax=2. i.e the value of the game lies between -2 and 2. i.e maxmin is not equal to minmax, therefore there is no unique saddle point. Games with no saddle point should be solved using mixed strategy.

It is a 2x2 game without saddle point, we use the algebraic method

To find S_A

$2p_1-2p_2=V$ ----(1) and $-2p_1+2p_2=V$ ----(2)

Therefore we have $2p_1-2p_2=-2p_1+2p_2$

$\Rightarrow 2p_1-2(1-p_1)=-2p_1+2(1-p_1)$ $\qquad [p_1+p_2=1]$

$\Rightarrow 2p_1-2+2p_1=-2p_1+2-2p_1$

$\Rightarrow 2p_1-2+2p_1=-2p_1+2-2p_1$

$\Rightarrow 8p_1=4$

$\Rightarrow p_1=1/2$

Thus $p_2=1-p_1=1-1/2=1/2$

To find S_B

$2q_1-2q_2=V$ ----(3) and $-2q_1+2q_2=V$ ----(4)

Therefore we have $2q_1-2q_2 =-2q_1+2q_2$

$\Rightarrow 2q_1-2(1-q_1) =-2q_1+2(1-q_1)$ $[q_1+q_2=1]$

$\Rightarrow 2q_1-2+2q_1 =-2q_1+2-2q_1$

$\Rightarrow 2q_1-2+2q_1 =-2q_1+2-2q_1$

$\Rightarrow 8q_1=4$

$\Rightarrow q_1=1/2$

Thus $q_2=1-q_1=1-1/2=1/2$

Hence the optimal mixed strategy for player A and B given by

$S_A=\begin{pmatrix} H & T \\ 1/2 & 1/2 \end{pmatrix}$ and $S_B=\begin{pmatrix} H & T \\ 1/2 & 1/2 \end{pmatrix}$

Value of the game

$(1)\Rightarrow 2p_1-2p_2=V$

$\Rightarrow 2(1/2)-2(1/2)=V$

$\Rightarrow V=0$

Remark:-

To find the value of the game in the above problem we can also use equations (2), (3) and (4), we will get the same value for V. i.e V=0 (verify yourself)

Example:-02

Solve the following 2x2 game

	Player A	
Player B	2	5
	7	3

Solution:-

	Player B			
Player A	H	T	Row minimum	Probability
H	$\boxed{2}$	5	$\boxed{2}$	p_1
T	7	$\boxed{3}$	$\boxed{3}$	p_2
Column Maximums	7	5		
Probability	q_1	q_2		

Since maxmin=3 and minmax=5. i.e the value of the game lies between 3 and 5.

i.e maxmin is not equal to minmax, therefore there is no unique saddle point.

Games with no saddle point should be solved using mixed strategy.

It is a 2x2 game without saddle point, we use the algebraic method

To find S_B

$2q_1+5q_2=V$ ---(1) and $7q_1+3q_2=V$----(2)

Therefore we have $2q_1+5q_2 =7q_1+3q_2$

$\Rightarrow 2q_1+5(1-q_1) =7q_1+3(1-q_1)$ $\qquad [q_1+q_2=1]$

$\Rightarrow 2q_1+5-5q_1 =7q_1+3-3q_1$

$\Rightarrow 5-3q_1 =4q_1+3$

$\Rightarrow -7q_1=-2$

$\Rightarrow q_1=2/7$

Thus $q_2=1-q_1=1-2/7=5/7$

To find S_A

$2p_1+7p_2=V$ ---(3) and $5p_1+3p_2=V$---(4)

Therefore we have $2p_1+7p_2 =5p_1+3p_2$

$\Rightarrow 2p_1+7(1-p_1) = 5p_1+3(1-p_1)$ $[p_1+p_2=1]$

$\Rightarrow 2p_1+7-7p_1 = 5p_1+3-3p_1$

$\Rightarrow 7-5p_1 = 2p_1+3$

$\Rightarrow -7q_1=-4$

$\Rightarrow p_1=4/7$

Thus $p_2=1-p_1=1-4/7=3/7$.

Hence the optimal mixed strategy for player A and B given by

$S_A=\begin{pmatrix} H & T \\ 4/7 & 3/7 \end{pmatrix}$ and $S_B=\begin{pmatrix} H & T \\ 2/7 & 5/7 \end{pmatrix}$

Value of the game

$(4) \Rightarrow 5p_1+3p_2=V$

$\Rightarrow 5(4/7)+3(3/7)=V$

$\Rightarrow 20/7+9/7=V$

$\Rightarrow 29/7=V$

Home work and Assignment Problem

Solve the following game

1. Pay off matrix is

	Player B		
Player A	20	15	22
	35	45	40
	18	20	25

2. Pay off matrix is

	Company B		
Company A	40	45	50
	20	45	60
	25	30	30

3. Pay off matrix is

	Player B	
Player A	2	5
	4	1

4. Pay off matrix is

	Player B	
Player A	6	9
	8	4

5. Pay off matrix is

	Player B	
Player A	1	1
	4	-3

6. Pay off matrix is

	Player B		
Player A	1	3	1
	0	-4	-3
	1	5	-1

7. Pay off matrix is

	Player B				
Player A	9	3	1	8	0
	6	5	4	6	7
	2	4	3	3	8
	5	6	2	2	1

8. Pay off matrix is

	Player B	
Player A	2	-1
	-1	0

9. Pay off matrix is

	Player B	
Player A	5	1
	3	4

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Some useful links

https://www.britannica.com/science/game-theory

https://www.google.com/search?client=firefox-b-d&q=game+theory+history

https://plato.stanford.edu/entries/game-theory/

21/12/2020

Unit-4

Definition:-

A Competition b/w the two Competitors is called as game.

Form of the game

player B

player A	B_1	B_2	...	B_n
A_1				
A_2				
⋮				
A_n				

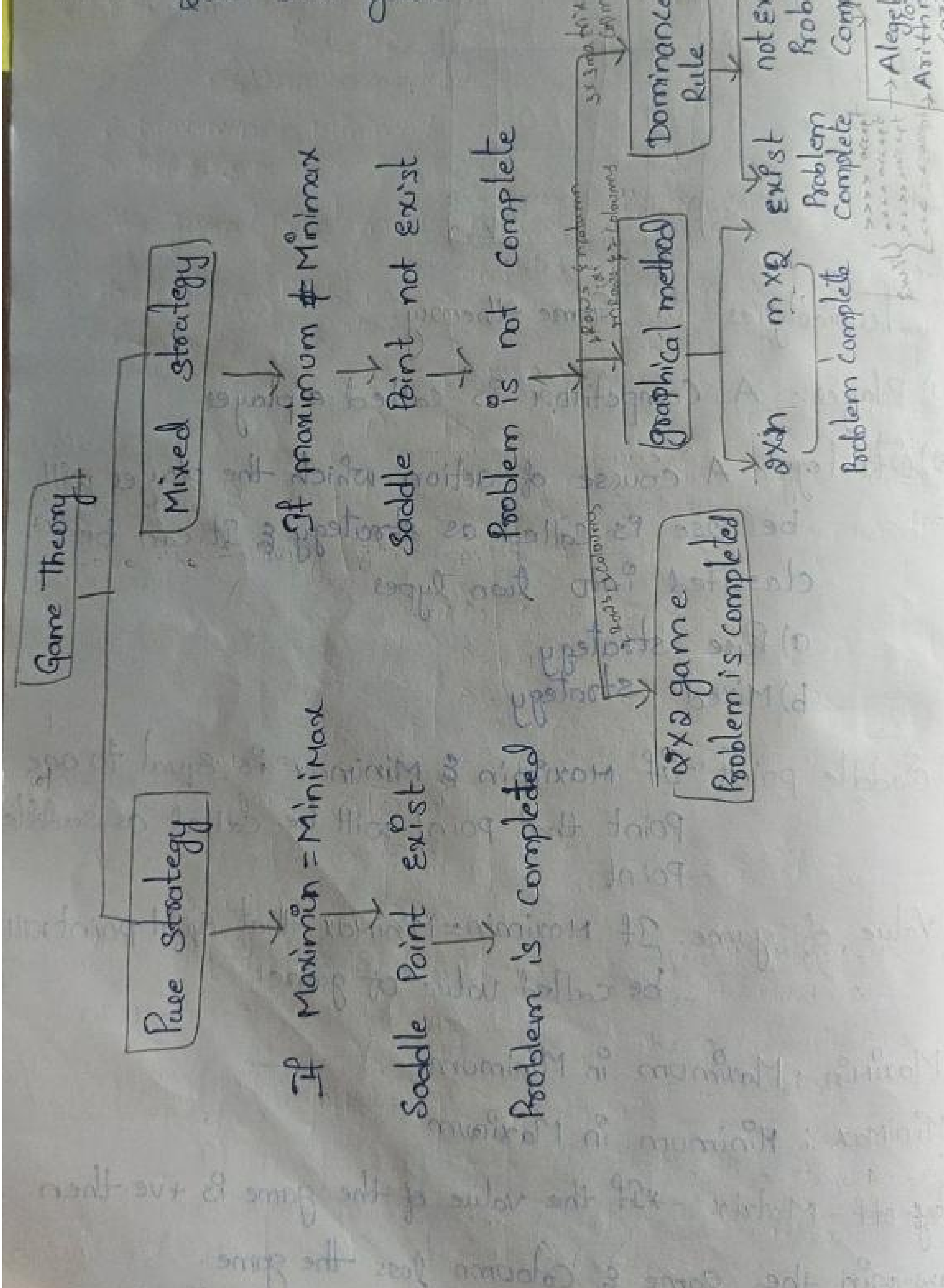
Unit -4

Terminolgies in Game theory:

1) Player:- A Competitor is called a player.
2) Strategy:- A course of action which the player will be use is called as strategy & it can be classifed into two types
   a) Pure strategy
   b) Mixed strategy
3) Saddle point:- If Maximin & Minimax is equal to one point that point will be called as saddle point
4) Value of game: If Maximin = MiniMax that equal point will be called value of game.
5) MaxiMin : Maximum in Minimum
6) MiniMax : Minimum in Maximum
7) Pay-off-Matrix:- *If the value of the game is +ve then row win the game & Coloumn loss the game.

*If the value of the game -ve then coloumn wins the game & row loss the game.

8) Two person zero sum game:- If the value of the game equal to the zero then It will be called as two person zero sum game.



Unit-4 Game Theory

Date: 10/Mar/2020

- Game
  - Pure strategy
    - If Maximini = Minimax
      - Saddle point
        - Exist
  - Mixed strategy
    - 2×2 game
    - Graphical Method
      - 2×n
      - m×2
    - Dominance Rule
      - Exist
      - Not Exist
        - Algebraic Method
        - Arithmetic Method
        - Game by LPP

Conditions:-

1) Value of the game v=?
2) strategy for A & B
3) Win/Loss.

Example:-

1) Solve the following Game

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 2 & 4 & 7 \\ A_2 & 10 & 7 & 9 \\ A_3 & 4 & 3 & 8 \end{array}$$

player 'B' (columns), player 'A' (rows)

Sol:-

player B (columns), player A (rows)

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{Row Min} \\ \hline A_1 & 2 & 4 & 7 & 2 \\ A_2 & \boxed{10} & \boxed{7} & \boxed{9} & 7 \\ A_3 & 4 & 3 & 8 & 3 \\ \hline \text{Col Max} & 10 & 7 & 9 & \end{array}$$

Maximin = 7

MiniMax = 7

$\therefore$ Maximin = Minmax

$\Rightarrow 7 = 7$

Saddle point Exist

The given Plb is pure strategy Problem.

(1) value of game $V = 7$

(2) strategy for player $A = \{A_2\}$

strategy for player $B = \{B_2\}$

(3) win / loss.

$\Rightarrow$ player A win / B loss.

Example : 2

For the game with the following Pay off matrix, Determine the optim strategy and the value of the game

$$
\begin{array}{c c c}
 & & \text{Player B} \\
 & B_1 & B_2 \\
\text{player A} \quad A_1 & 1 & 2 \\
A_2 & 3 & 4
\end{array}
$$

Sol:-

$$
\begin{array}{c c c c}
 & \text{player B} & & \\
 & B_1 & B_2 & \text{Row Min} \\
\text{player A} \quad A_1 & 1 & 2 & 1 \\
A_2 & 3 & 4 & 3 \\
\text{col max} & 3 & 4 &
\end{array}
$$

Maxi Min = 3.

$\underbrace{3 \quad 4}_{3}$

Minimax = 3

∴ Maxmin = Minimax.

Saddle point exist

The given plb is pure strategy Problem.

1) $V = 3$

2) A win | B loss.

3) S-A = $\{A_1\}$

S-B = $\{B_1\}$

Example: 3

player-B

$$\text{player A}\quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \end{array}$$

Sol:-

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{RowMin} \\ \hline A_1 & -1 & 2 & -1 \\ A_2 & 3 & -4 & -4 \\ \hline \text{colMax} & 3 & 2 & \end{array}$$

Maximin = −1

Minimax = 2

∴ Maximin ≠ Minimax

$-1 \neq 2$

Saddle point not exist

The given Problem is a mixed strategy Problem

Now apply 2×2 game.

$$\begin{array}{c|cc|l} & B_1 & B_2 & \\ A_1 & -1 & 2 & 3-(-4)=7 \rightarrow 7/10 \\ A_2 & 3 & -4 & 2-(-1)=3 \rightarrow 3/10 \end{array}$$

$2-(-4) \Rightarrow 6 \downarrow 6/10$ $\quad$ $3-(-1) \Rightarrow 4 \downarrow 4/10$ $\quad$ $\boxed{10}$

1) Value of game: $V = -1 \times 6/10 + 2 \times 4/10$

$\Rightarrow -6/10 + 8/10$

$\Rightarrow 2/10 = 1/5$

2) A win / B loss

3) strategy for player $A = \{7/10, 3/10\}$ (A_1, A_2)

$B = \{6/10, 4/10\}$ (B_1, B_2)

Example: 3

Solve the following 2x2 game graphically

player B

$$\text{player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{bmatrix} \end{array}$$

Sol:-

$$\begin{array}{c|cccc|c} & B_1 & B_2 & B_3 & B_4 & \text{Rowmin} \\ A_1 & 2 & 1 & 0 & -2 & -2 \\ A_2 & 1 & 0 & 3 & 2 & 0 \\ \text{col max} & 2 & 1 & 3 & 2 & \end{array}$$

Maximin = 0

Minimax = 1

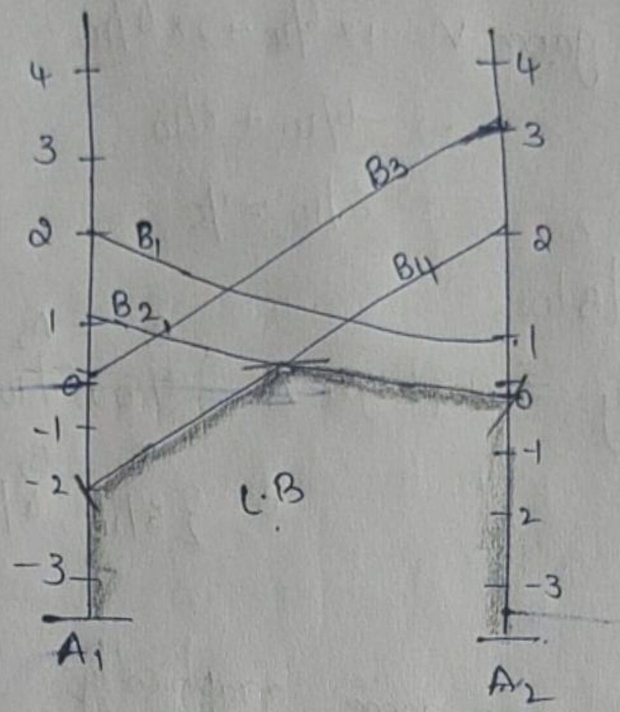
Minimax & Mo

Maximin ≠ Minimax

Saddle point not exist

The given Problo is Mixed strategy Problem

Now apply graphical method.



Note:-1

If the given matrix form is 2xn We have to taken lower bounder in that upper point

Note:-2

If the given matrix form is mx2 We have to take upper bounder in that lower point

$$\begin{array}{c|cc} & B_2 & B_4 \\ \hline A_1 & 1 & -2 \\ A_2 & 0 & 2 \end{array}$$

$$
\begin{array}{c|cc}
 & B_2 & B_4 \\
A_1 & 1 & -2 \\
A_2 & 0 & 2
\end{array}
$$

$2-0=2 \rightarrow 2/5$

$1-(-2)=3 \rightarrow 3/5$

$\boxed{5}$

$2-(-2)=4 \rightarrow 4/5$ $\qquad$ $1-0=1 \rightarrow 1/5$

1) Value of the game v=

$1(4/5) + -2(1/5) \Rightarrow 4/5 - \frac{2}{5} = 2/5$

2) A win/B loss

3) $S-A = \{A_1 = 2/5 ; A_2 = 3/5\}$

$S-B = \{B_1 = 0 ; B_2 = 4/5 ; B_3 = 0 ; B_4 = 1/5\}$

Example: 4

~~Obtain~~ obtain the optimal strategy for both persones and the value of the game for zero sum to person game who's pay of Matrix is as follows?.

$$
\begin{array}{c|cc}
 & B_1 & B_2 \\
A_1 & 1 & -3 \\
A_2 & 3 & 5 \\
A_3 & -1 & 6 \\
A_4 & 4 & 1 \\
A_5 & 2 & 2 \\
A_6 & -5 & 0
\end{array}
$$

Sol:-

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ \text{colmax} \end{array}
\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix} \\ \underbrace{\begin{array}{cc} 4 & 6 \end{array}}_{\text{Minimax}=4} \end{array}
\left.\begin{array}{c} \\ -3 \\ 3 \\ -1 \\ 1 \\ 2 \\ -5 \\ \\ \end{array}\right\} \text{Maximin} = 3$$

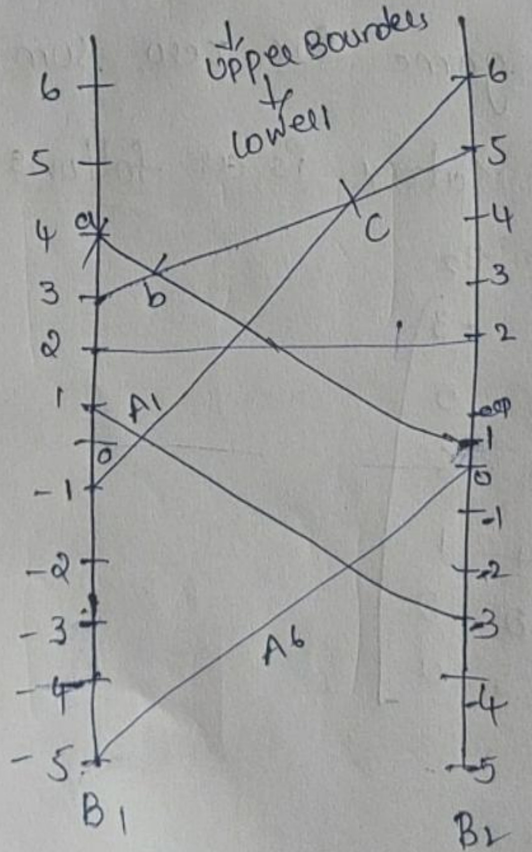
$\therefore$ Maximin $\neq$ Minimax

$\Rightarrow 3 \neq 4$

No saddle point

The given Problem is mixed Strategy Problem.

$\therefore$ Now apply Graphical method.



$$\begin{array}{c|cc|l} & B_1 & B_2 & \\ \hline A_2 & 3 & 5 & 3 \rightarrow 3/5 \\ A_4 & 4 & 1 & 2 \rightarrow 2/5 \\ \hline & \downarrow & \downarrow & \boxed{5} \\ & 4 & 1 & \\ & \downarrow & \downarrow & \\ & 4/5 & 1/5 & \end{array}$$

$$\therefore V = 3\left[4/5\right] + 5\left[1/5\right] = \frac{12+5}{5} = 17/5$$

A win / B loss

$S\text{-}A = \{A_1 = 0,\ A_2 = 3/5,\ A_3 = 0,\ A_4 = 2/5,$
$A_5 = 0,\ A_6 = 0\}$

$S\text{-}B = \{B_1 = 4/5,\ B_2 = 1/5\}.$

Dominance Rule :-

Exist	Not-Exist
>>>>	
<<<<	><
>=>>	
==<<	

Example : 1

solve the following 3×3 game

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 7 & 1 & 7 \\ A_2 & 9 & -1 & 1 \\ A_3 & 5 & 7 & 6 \end{array}$$

Sol:-

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \\ A_1 & 7 & 1 & 7 & 1 \\ A_2 & 9 & -1 & 1 & -1 \\ A_3 & 5 & 7 & 6 & 5 \\ \hline & 9 & 7 & 7 & \end{array}$$

Mainimax = 5

Maxmin = 7

∴ Maximin ≠ Minimax

$7 \neq 5$

No saddle point

The given Pblm is mixed strategy Problem.

Now apply Dominance Rule for A_1 & A_2

$$\begin{array}{c} A_1 \\ A_2 \end{array}\begin{bmatrix} 7 & 1 & 7 \\ \wedge & \vee & \vee \\ 9 & -1 & 1 \end{bmatrix}$$

No Dominance b/w A_1 & A_2.

A_1 & A_3

$$\begin{array}{c} A_1 \\ A_3 \end{array}\begin{bmatrix} 7 & 1 & 7 \\ \vee & \wedge & \vee \\ 5 & 7 & 6 \end{bmatrix} \rightarrow \text{No}$$

A_2 & A_3

$$\begin{array}{c} A_2 \\ A_3 \end{array}\begin{bmatrix} 9 & -1 & 1 \\ 5 & \wedge & \wedge \\ 5 & 7 & 6 \end{bmatrix} \rightarrow \text{No.}$$

B_1 & B_2

$$\begin{array}{cc} B_1 & B_2 \end{array}$$
$$\begin{bmatrix} 7 > 1 \\ 9 > -1 \\ 5 < 7 \end{bmatrix}$$

NO

B_1 & B_3

$$\begin{array}{cc} B_1 & B_3 \end{array}$$
$$\begin{bmatrix} 7 = 7 \\ 9 > 1 \\ 5 < 6 \end{bmatrix}$$

NO

B_2 & B_3

$$\begin{array}{cc} B_2 & B_3 \end{array}$$
$$\begin{bmatrix} 1 < 7 \\ -1 < 1 \\ 7 > 6 \end{bmatrix}$$

NO

$$\begin{array}{c} A_1+A_2 \\ \\ A_3 \end{array} \begin{bmatrix} 16 & 0 & 8 \\ \vee & \wedge & \vee \\ 5 & 7 & 6 \end{bmatrix} \rightarrow \text{NO}$$

$$\begin{array}{c} A_1+A_3 \\ \\ A_2 \end{array} \begin{bmatrix} 12 & 8 & 13 \\ \vee & \vee & \vee \\ 9 & -1 & 1 \end{bmatrix}$$

There is a dominance b/w A_1+A_3 & A_2

Now delete A_2 in the given Problem.

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 7 & 1 & 7 \\ A_3 & 5 & 7 & 6 \end{array}$$

B_1 & B_2

$$\begin{bmatrix} 7 > 1 \\ 5 < 7 \end{bmatrix}$$

NO

B_1 & B_3

$$\begin{bmatrix} 7 = 7 \\ 5 < 6 \end{bmatrix}$$

Exist

Now Delete B_3 in the reduced matrix.

$$\begin{array}{c|cc|l} & B_1 & B_2 & \\ A_1 & 7 & 1 & 2 \rightarrow 2/8 \\ A_3 & 5 & 7 & 6 \rightarrow 6/8 \\ \hline & 6 & 2 & \boxed{8} \\ & \downarrow & \downarrow & \\ & 6/8 & 2/8 & \end{array}$$

$$\therefore V = 7\left[6/8\right] + 1\left[2/8\right]$$

$$= \frac{42}{8} + \frac{2}{8} = \frac{44}{8} = \frac{11}{2}$$

2) A wins / B loss

3) strategy for player A = $\{ \overset{A_1}{2/8}, \overset{A_3}{6/8} \}$

player B = $\{ 6/8, 2/8 \}$ //

Example:- Solve the following game.

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 0 & 1 & 2 \\ A_2 & 2 & 0 & 1 \\ A_3 & 1 & 2 & 0 \end{array}$$

Sol:-

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{Rowmin} \\ \hline A_1 & 0 & 1 & 2 & 0 \\ A_2 & 2 & 0 & 1 & 0 \\ A_3 & 1 & 2 & 0 & 0 \\ \hline \text{ColMax} & 2 & 2 & 2 & \end{array}$$

MiniMax = 0

Maximin = 2

$\therefore$ Maximin $\neq$ Minimax

$2 \neq 0$

No saddle point

The given Pblm is mixed strategy Problem

Now apply Dominance Rule for A_1 & A_2

$$\begin{array}{c} A_1 \\ \\ A_2 \end{array} \begin{bmatrix} 0 & 1 & 2 \\ \wedge & \vee & \vee \\ 2 & 0 & 1 \end{bmatrix}$$

No Dominance b/w A_1 & A_2

A_2 & A_3

$$\begin{array}{c} A_2 \\ \\ A_3 \end{array} \begin{bmatrix} 2 & 0 & 1 \\ \vee & \wedge & \vee \\ 1 & 2 & 0 \end{bmatrix}$$

No Dominance b/w A_2 & A_3

A_1 & A_3

$$\begin{array}{c} A_1 \\ \\ A_3 \end{array} \begin{bmatrix} 0 & 1 & 2 \\ \wedge & \wedge & \vee \\ 1 & 2 & 0 \end{bmatrix}$$

No Dominance b/w A_1 & A_3

B_1 & B_2 $\qquad$ B_1 & B_3 $\qquad$ B_2 & B_3

$$\begin{array}{cc} B_1 & B_2 \end{array} \begin{bmatrix} 0 \le 1 \\ 2 > 0 \\ 1 \le 2 \end{bmatrix} \to NO \qquad \begin{array}{cc} B_1 & B_3 \end{array} \begin{bmatrix} 0 < 2 \\ 2 > 1 \\ 1 > 0 \end{bmatrix} \to NO \qquad \begin{array}{cc} B_2 & B_3 \end{array} \begin{bmatrix} 1 < 2 \\ 0 < 1 \\ 2 > 0 \end{bmatrix} \to NO$$

$$\begin{array}{c} A_1 + A_2 \\ \\ A_3 \end{array} \begin{bmatrix} 2 & 1 & 3 \\ \vee & \wedge & \vee \\ 1 & 2 & 0 \end{bmatrix} \to NO \qquad \begin{array}{c} A_1 + A_3 \\ \\ A_2 \end{array} \begin{bmatrix} 1 & 3 & 2 \\ \wedge & \vee & \vee \\ 2 & 0 & 1 \end{bmatrix} \to NO$$

$$\begin{array}{cc} B_1 + B_2 & B_3 \end{array} \begin{bmatrix} 1 < 2 \\ 2 > 1 \\ 3 > 0 \end{bmatrix} \to NO \qquad \begin{array}{cc} B_1 + B_3 & B_2 \end{array} \begin{bmatrix} 2 > 1 \\ 3 > 0 \\ 1 < 2 \end{bmatrix} \to NO$$

Dominance does not Exist

Now this Problem Can be solved by using either Algebric (or) Arithmetic (or) Game by LPP

Arithmetic Method:-

Construction of Row matrix

$$R = \begin{array}{c} C_1 - C_2 \\ C_2 - C_3 \end{array} \begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$R_1 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = |4-1| = 3$$

$$R_2 = \begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} = |-2-1| = 3$$

$$R_3 = \begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix} = |1+2| = 3$$

Construction of coloumn matrix.

$$C = \begin{array}{cc} R_1 - R_2 & R_2 - R_3 \end{array} \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$C_1 = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = |1+2| = 3$$

$$C_2 = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = |-2-1| = 3$$

$$C_3 = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = |4-1| = 3$$

$$\begin{array}{c|ccc|c} & C_1 & C_2 & C_3 & \\ A_1 & 0 & 1 & 2 & \rightarrow 3 \rightarrow 3/9 \rightarrow 1/3 \\ A_2 & 2 & 0 & 1 & \rightarrow 3 \rightarrow 3/9 \rightarrow 1/3 \\ A_3 & 1 & 2 & 0 & \rightarrow 3 \rightarrow 3/9 \rightarrow 1/3 \\ & \downarrow & \downarrow & \downarrow & \\ & 3 & 3 & 3 & \\ & \downarrow & \downarrow & \downarrow & \\ & 3/9 & 3/9 & 3/9 & \\ & \downarrow & \downarrow & \downarrow & \\ & 1/3 & 1/3 & 1/3 & \end{array}$$

$\boxed{9}$

① $V = 0(1/3) + 1(1/3) + 2(1/3) = 3/3 = 1$

② Row win / col loss

③ S - $A = \{1/3, 1/3, 1/3\}$

$B = \{1/3, 1/3, 1/3\}$

Unit-5 Replacement:-

Example:-

A farm is consider in replacement of a machin, whose cost Price is 12200/- and the Scrape value is 200/-. The running Cost (maintains & operating) in Ruppes are found from Experience to be as follows

Years	1	2	3	4	5	6	7	8
Running cost	200	500	800	1200	1800	2500	3200	4000

When should the machine will replaced

Sol:-
We are given the running cost $f(n)$

scrape value $S = 200/-$

And the cost of the Machine is $C = 12200$

In order to determine the optimal time n When the machine be replaced.

We calculate the average total cost per year during the life of the machine. as shown in the table.

Years of Service (n) (1)	Running cost f(n) (2)	Cumulative Running cost $\Sigma f(n)$ (3)	Depreciation (C-S) (4)	Total cost TC = 3+4	Average cost A(n) (6) = (5)/(1)
1	200	200	12000	12200	12200/1 = 12200
2	500	700	12000	12700	12700/2 = 6350
3	800	1500	12000	13500	13500/3 = 4500
4	1200	2700	12000	14700	14700/4 = 3675
5	1800	4500	12000	16500	16500/5 = 3300
6	2500	7000	12000	19000	19000/6 = 3,166
7	3200	10200	12000	22200	22200/7 = 3171
8	4000	14200	12000	26200	26200/8 = 3275

from the above table it is noted that the average total cost per year A(n) is minimum in the 6th year at the Rupees 3166 & also the average cost in the 7th year Rupees 3171 is more than the cost in the 6th year.

Hence the machine should be replace after Every 6 years.

2.1 Introduction to CPM / PERT Techniques

CPM/PERT or Network Analysis as the technique is sometimes called, developed along two parallel streams, one industrial and the other military.

CPM (Critical Path Method) was the discovery of M.R.Walker of E.I.Du Pont de Nemours & Co. and J.E.Kelly of Remington Rand, circa 1957. The computation was designed for the UNIVAC-I computer. The first test was made in 1958, when CPM was applied to the construction of a new chemical plant. In March 1959, the method was applied to maintenance shut-down at the Du Pont works in Louisville, Kentucky. Unproductive time was reduced from 125 to 93 hours.

PERT (Project Evaluation and Review Technique) was devised in 1958 for the POLARIS missile program by the Program Evaluation Branch of the Special Projects office of the U.S.Navy, helped by the Lockheed Missile Systems division and the Consultant firm of Booz-Allen & Hamilton. The calculations were so arranged so that they could be carried out on the IBM Naval Ordinance Research Computer (NORC) at Dahlgren, Virginia.

The methods are essentially **network-oriented techniques** using the same principle. PERT and CPM are basically time-oriented methods in the sense that they both lead to determination of a time schedule for the project. The significant difference between two approaches is that the time estimates for the different activities in CPM were assumed to be **deterministic** while in PERT these are described **probabilistically**. These techniques are referred as **project scheduling** techniques.

In **CPM** activities are shown as a network of precedence relationships using activity-on-node network construction

- Single estimate of activity time
- Deterministic activity times

USED IN: Production management - for the jobs of repetitive in nature where the activity time estimates can be predicted with considerable certainty due to the existence of past experience.

In **PERT** activities are shown as a network of precedence relationships using activity-on-arrow network construction

- Multiple time estimates
- Probabilistic activity times

USED IN: Project management - for non-repetitive jobs (research and development work), where the time and cost estimates tend to be quite uncertain. This technique uses probabilistic time estimates.

Benefits of PERT/CPM

- Useful at many stages of project management
- Mathematically simple

- Give critical path and slack time
- Provide project documentation
- Useful in monitoring costs

Limitations of PERT/CPM

- Clearly defined, independent and stable activities
- Specified precedence relationships
- Over emphasis on critical paths

2.2 Applications of CPM / PERT

These methods have been applied to a wide variety of problems in industries and have found acceptance even in government organizations. These include

- Construction of a dam or a canal system in a region
- Construction of a building or highway
- Maintenance or overhaul of airplanes or oil refinery
- Space flight
- Cost control of a project using PERT / COST
- Designing a prototype of a machine
- Development of supersonic planes

2.3 Basic Steps in PERT / CPM

Project scheduling by PERT / CPM consists of four main steps

1. **Planning**

- The planning phase is started by splitting the total project in to small projects. These smaller projects in turn are divided into activities and are analyzed by the department or section.
- The relationship of each activity with respect to other activities are defined and established and the corresponding responsibilities and the authority are also stated.
- Thus the possibility of overlooking any task necessary for the completion of the project is reduced substantially.

2. Scheduling

- The ultimate objective of the scheduling phase is to prepare a time chart showing the start and finish times for each activity as well as its relationship to other activities of the project.
- Moreover the schedule must pinpoint the critical path activities which require special attention if the project is to be completed in time.
- For non-critical activities, the schedule must show the amount of slack or float times which can be used advantageously when such activities are delayed or when limited resources are to be utilized effectively.

3. Allocation of resources

- Allocation of resources is performed to achieve the desired objective. A resource is a physical variable such as labour, finance, equipment and space which will impose a limitation on time for the project.
- When resources are limited and conflicting, demands are made for the same type of resources a systematic method for allocation of resources become essential.
- Resource allocation usually incurs a compromise and the choice of this compromise depends on the judgment of managers.

4. Controlling

- The final phase in project management is controlling. Critical path methods facilitate the application of the principle of management by expectation to identify areas that are critical to the completion of the project.
- By having progress reports from time to time and updating the network continuously, a better financial as well as technical control over the project is exercised.
- Arrow diagrams and time charts are used for making periodic progress reports. If required, a new course of action is determined for the remaining portion of the project.

2.4 <u>The Framework for PERT and CPM</u>

Essentially, there are six steps which are common to both the techniques. The procedure is listed below:

I. Define the Project and all of its significant activities or tasks. The Project (made up of several tasks) should have only a single start activity and a single finish activity.

II. Develop the relationships among the activities. Decide which activities must precede and which must follow others.

III. Draw the "Network" connecting all the activities. Each Activity should have unique event numbers. Dummy arrows are used where required to avoid giving the same numbering to two activities.

IV. Assign time and/or cost estimates to each activity

V. Compute the longest time path through the network. This is called the critical path.

VI. Use the Network to help plan, schedule, and monitor and control the project.

The Key Concept used by CPM/PERT is that a small set of activities, which make up the longest path through the activity network control the entire project. If these "critical" activities could be identified and assigned to responsible persons, management resources could be optimally used by concentrating on the few activities which determine the fate of the entire project.

Non-critical activities can be replanned, rescheduled and resources for them can be reallocated flexibly, without affecting the whole project.

Five useful questions to ask when preparing an activity network are:

- Is this a Start Activity?
- Is this a Finish Activity?
- What Activity Precedes this?
- What Activity Follows this?
- What Activity is Concurrent with this?

2.5 Network Diagram Representation

In a network representation of a project certain definitions are used

1. Activity

Any individual operation which utilizes resources and has an end and a beginning is called activity. An arrow is commonly used to represent an activity with its head indicating the direction of progress in the project. These are classified into four categories

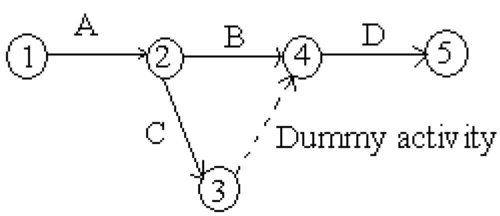
1. **Predecessor activity** – Activities that must be completed immediately prior to the start of another activity are called predecessor activities.

2. **Successor activity** – Activities that cannot be started until one or more of other activities are completed but immediately succeed them are called successor activities.
3. **Concurrent activity** – Activities which can be accomplished concurrently are known as concurrent activities. It may be noted that an activity can be a predecessor or a successor to an event or it may be concurrent with one or more of other activities.
4. **Dummy activity** – An activity which does not consume any kind of resource but merely depicts the technological dependence is called a dummy activity.

The dummy activity is inserted in the network to clarify the activity pattern in the following two situations

- To make activities with common starting and finishing points distinguishable
- To identify and maintain the proper precedence relationship between activities that is not connected by events.

For example, consider a situation where A and B are concurrent activities. C is dependent on A and D is dependent on A and B both. Such a situation can be handled by using a dummy activity as shown in the figure.



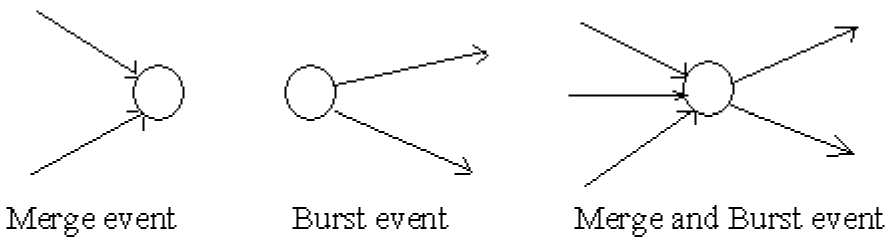
2. Event

An event represents a point in time signifying the completion of some activities and the beginning of new ones. This is usually represented by a circle in a network which is also called a node or connector.

The events are classified in to three categories

1. **Merge event** – When more than one activity comes and joins an event such an event is known as merge event.

2. **Burst event** – When more than one activity leaves an event such an event is known as burst event.
3. **Merge and Burst event** – An activity may be merge and burst event at the same time as with respect to some activities it can be a merge event and with respect to some other activities it may be a burst event.



3. Sequencing

The first prerequisite in the development of network is to maintain the precedence relationships. In order to make a network, the following points should be taken into considerations

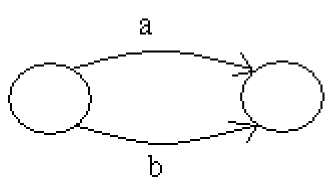
- What job or jobs precede it?
- What job or jobs could run concurrently?
- What job or jobs follow it?
- What controls the start and finish of a job?

Since all further calculations are based on the network, it is necessary that a network be drawn with full care.

2.6 Rules for Drawing Network Diagram

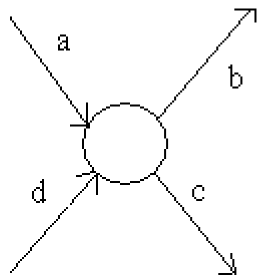
Rule 1

Each activity is represented by one and only one arrow in the network



Rule 2

No two activities can be identified by the same end events



Rule 3

In order to ensure the correct precedence relationship in the arrow diagram, following questions must be checked whenever any activity is added to the network

- What activity must be completed immediately before this activity can start?
- What activities must follow this activity?
- What activities must occur simultaneously with this activity?

In case of large network, it is essential that certain good habits be practiced to draw an easy to follow network

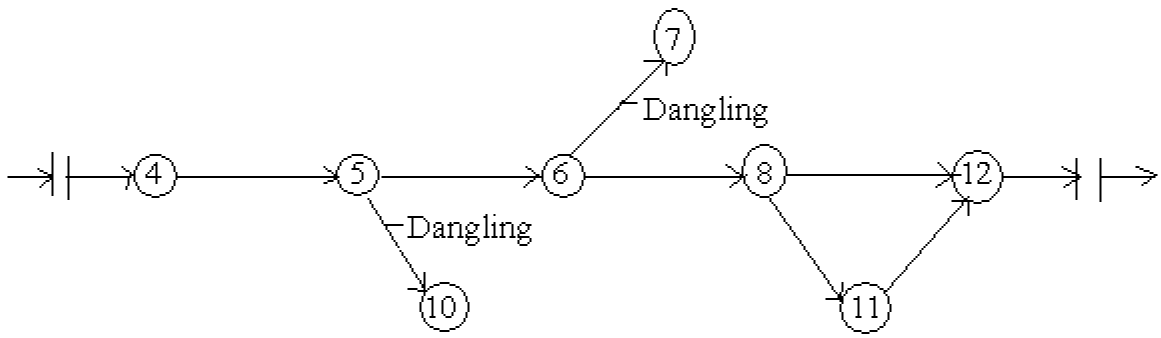
- Try to avoid arrows which cross each other
- Use straight arrows
- Do not attempt to represent duration of activity by its arrow length
- Use arrows from left to right. Avoid mixing two directions, vertical and standing arrows may be used if necessary.
- Use dummies freely in rough draft but final network should not have any redundant dummies.
- The network has only one entry point called start event and one point of emergence called the end event.

2.7 Common Errors in Drawing Networks

The three types of errors are most commonly observed in drawing network diagrams

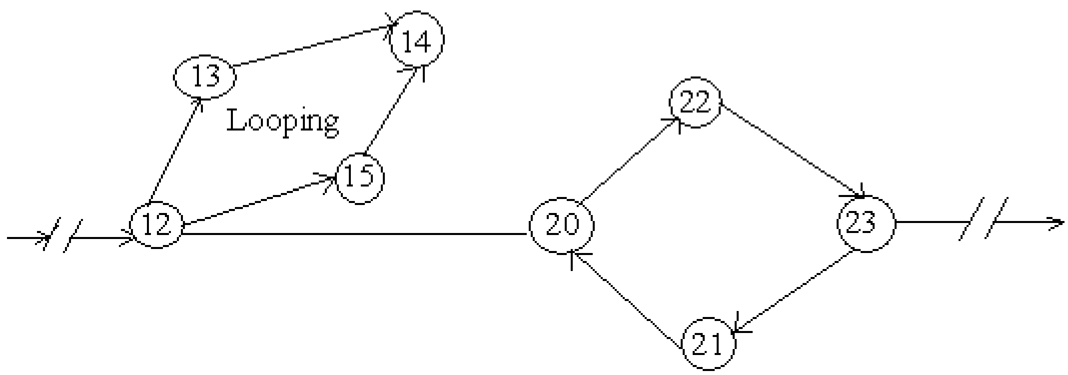
1. Dangling

To disconnect an activity before the completion of all activities in a network diagram is known as dangling. As shown in the figure activities (5 – 10) and (6 – 7) are not the last activities in the network. So the diagram is wrong and indicates the error of dangling



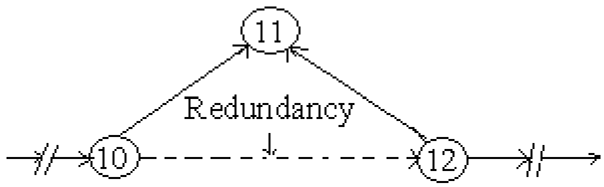
2. Looping or Cycling

Looping error is also known as cycling error in a network diagram. Drawing an endless loop in a network is known as error of looping as shown in the following figure.



3. Redundancy

Unnecessarily inserting the dummy activity in network logic is known as the error of redundancy as shown in the following diagram



2.8 Advantages and Disadvantages

PERT/CPM has the following advantages

- A PERT/CPM chart explicitly defines and makes visible dependencies (precedence relationships) between the elements,
- PERT/CPM facilitates identification of the critical path and makes this visible,
- PERT/CPM facilitates identification of early start, late start, and slack for each activity,
- PERT/CPM provides for potentially reduced project duration due to better understanding of dependencies leading to improved overlapping of activities and tasks where feasible.

PERT/CPM has the following disadvantages:

- There can be potentially hundreds or thousands of activities and individual dependency relationships,
- The network charts tend to be large and unwieldy requiring several pages to print and requiring special size paper,
- The lack of a timeframe on most PERT/CPM charts makes it harder to show status although colours can help (e.g., specific colour for completed nodes),
- When the PERT/CPM charts become unwieldy, they are no longer used to manage the project.

2.9 Critical Path in Network Analysis

Basic Scheduling Computations

The notations used are

(i, j) = Activity with tail event i and head event j

E_i = Earliest occurrence time of event i

L_j = Latest allowable occurrence time of event j

D_{ij} = Estimated completion time of activity (i, j)

$(Es)_{ij}$ = Earliest starting time of activity (i, j)

$(Ef)_{ij}$ = Earliest finishing time of activity (i, j)

$(Ls)_{ij}$ = Latest starting time of activity (i, j)

$(Lf)_{ij}$ = Latest finishing time of activity (i, j)

The procedure is as follows

1. **Determination of Earliest time (E_j): Forward Pass computation**

- **Step 1**

  The computation begins from the start node and move towards the end node. For easiness, the forward pass computation starts by assuming the earliest occurrence time of zero for the initial project event.

- **Step 2**

  i. Earliest starting time of activity (i, j) is the earliest event time of the tail end event i.e. $(Es)_{ij} = E_i$

  ii. Earliest finish time of activity (i, j) is the earliest starting time + the activity time i.e. $(Ef)_{ij} = (Es)_{ij} + D_{ij}$ or $(Ef)_{ij} = E_i + D_{ij}$

iii. Earliest event time for event j is the maximum of the earliest finish times of all activities ending in to that event i.e. E_j = max [$(Ef)_{ij}$ for all immediate predecessor of (i, j)] or E_j =max [$E_i + D_{ij}$]

2. **Backward Pass computation (for latest allowable time)**

- **Step 1**

  For ending event assume E = L. Remember that all E's have been computed by forward pass computations.

- **Step 2**

  Latest finish time for activity (i, j) is equal to the latest event time of event j i.e. $(Lf)_{ij} = L_j$

- **Step 3**

  Latest starting time of activity (i, j) = the latest completion time of (i, j) – the activity time or $(Ls)_{ij} = (Lf)_{ij} - D_{ij}$ or $(Ls)_{ij} = L_j - D_{ij}$

- **Step 4**

  Latest event time for event 'i' is the minimum of the latest start time of all activities originating from that event i.e. L_i = min [$(Ls)_{ij}$ for all immediate successor of (i, j)] = min [$(Lf)_{ij} - D_{ij}$] = min [$L_j - D_{ij}$]

3. **Determination of floats and slack times**

There are three kinds of floats

- **Total float** – The amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time without affecting the overall project duration time.

  Mathematically

  $(Tf)_{ij}$ = (Latest start – Earliest start) for activity (i – j)

  $(Tf)_{ij} = (Ls)_{ij} - (Es)_{ij}$ or $(Tf)_{ij} = (L_j - D_{ij}) - E_i$

- **Free float –** The time by which the completion of an activity can be delayed beyond the earliest finish time without affecting the earliest start of a subsequent activity.

  Mathematically

  $(Ff)_{ij}$ = (Earliest time for event j – Earliest time for event i) – Activity time for (i, j)

  $(Ff)_{ij} = (E_j - E_i) - D_{ij}$

- **Independent float** – The amount of time by which the start of an activity can be delayed without effecting the earliest start time of any immediately following activities, assuming that the preceding activity has finished at its latest finish time.

  Mathematically

  $(If)_{ij} = (E_j - L_i) - D_{ij}$

  The negative independent float is always taken as zero.

- **Event slack** - It is defined as the difference between the latest event and earliest event times.

  Mathematically

  Head event slack = $L_j – E_j$, Tail event slack = $L_i - E_i$

4. Determination of critical path

- **Critical event** – The events with zero slack times are called critical events. In other words the event i is said to be critical if $E_i = L_i$
- **Critical activity** – The activities with zero total float are known as critical activities. In other words an activity is said to be critical if a delay in its start will cause a further delay in the completion date of the entire project.
- **Critical path** – The sequence of critical activities in a network is called critical path. The critical path is the longest path in the network from the starting event to ending event and defines the minimum time required to complete the project.

Exercise

1. What is PERT and CPM?
2. What are the advantages of using PERT/CPM?
3. Mention the applications of PERT/CPM
4. Explain the following terms
    a. Earliest time
    b. Latest time
    c. Total activity slack
    d. Event slack
    e. Critical path
5. Explain the CPM in network analysis.
6. What are the rules for drawing network diagram? Also mention the common errors that occur in drawing networks.
7. What is the difference between PERT and CPM/

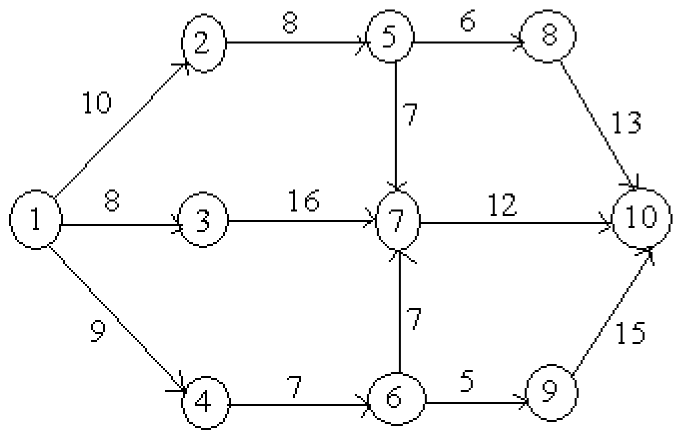
8. What are the uses of PERT and CPM?
9. Explain the basic steps in PERT/CPM techniques.
10. Write the framework of PERT/CPM.

Unit 3

3.1 Worked Examples on CPM

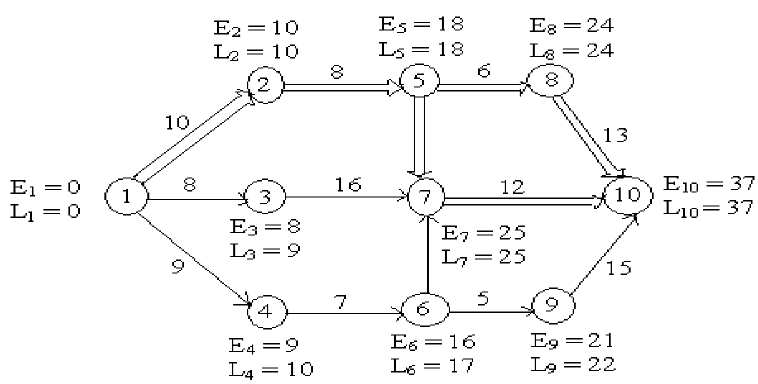
Example 1

Determine the early start and late start in respect of all node points and identify critical path for the following network.



Solution

Calculation of E and L for each node is shown in the network



Activity(i, j)	Normal Time (D_{ij})	Earliest Time		Latest Time		Float Time (L_i - D_{ij}) - E_i
		Start (E_i)	Finish (E_i + D_{ij})	Start (L_i - D_{ij})	Finish (L_i)	
(1, 2)	10	0	10	0	10	0
(1, 3)	8	0	8	1	9	1
(1, 4)	9	0	9	1	10	1
(2, 5)	8	10	18	10	18	0
(4, 6)	7	9	16	10	17	1

(3, 7)	16	8	24	9	25	1
(5, 7)	7	18	25	18	25	0
(6, 7)	7	16	23	18	25	2
(5, 8)	6	18	24	18	24	0
(6, 9)	5	16	21	17	22	1
(7, 10)	12	25	37	25	37	0
(8, 10)	13	24	37	24	37	0
(9, 10)	15	21	36	22	37	1

Network Analysis Table

From the table, the critical nodes are (1, 2), (2, 5), (5, 7), (5, 8), (7, 10) and (8, 10)

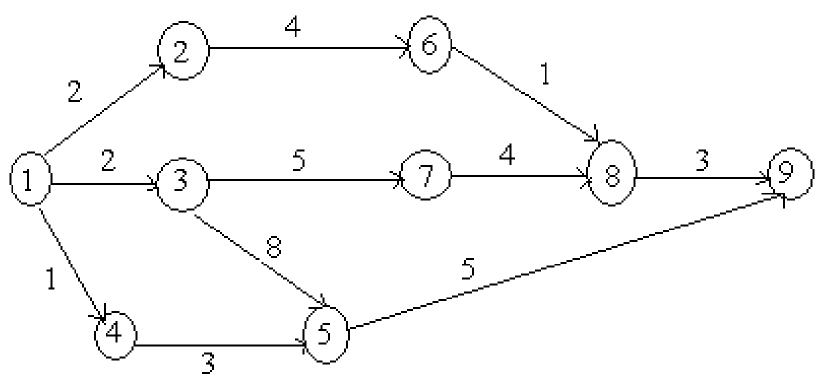
From the table, there are two possible critical paths

i. $1 \rightarrow 2 \rightarrow 5 \rightarrow 8 \rightarrow 10$

ii. $1 \rightarrow 2 \rightarrow 5 \rightarrow 7 \rightarrow 10$

Example 2

Find the critical path and calculate the slack time for the following network

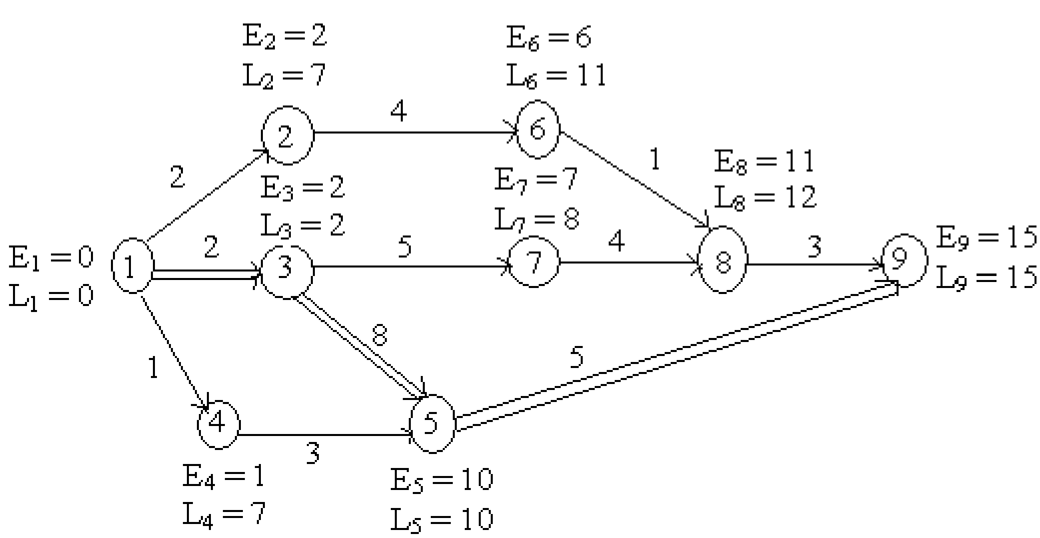


Solution

The earliest time and the latest time are obtained below

Activity(i, j)	Normal Time (D_{ij})	Earliest Time Start (E_i)	Earliest Time Finish ($E_i + D_{ij}$)	Latest Time Start ($L_i - D_{ij}$)	Latest Time Finish (L_i)	Float Time ($L_i - D_{ij}$) - E_i
(1, 2)	2	0	2	5	7	5
(1, 3)	2	0	2	0	2	0
(1, 4)	1	0	1	6	7	6
(2, 6)	4	2	6	7	11	5
(3, 7)	5	2	7	3	8	1
(3, 5)	8	2	10	2	10	0
(4, 5)	3	1	4	7	10	6
(5, 9)	5	10	15	10	15	0
(6, 8)	1	6	7	11	12	5
(7, 8)	4	7	11	8	12	1
(8, 9)	3	11	14	12	15	1

From the above table, the critical nodes are the activities (1, 3), (3, 5) and (5, 9)



The critical path is 1 → 3 → 5 → 9

Example 3

A project has the following times schedule

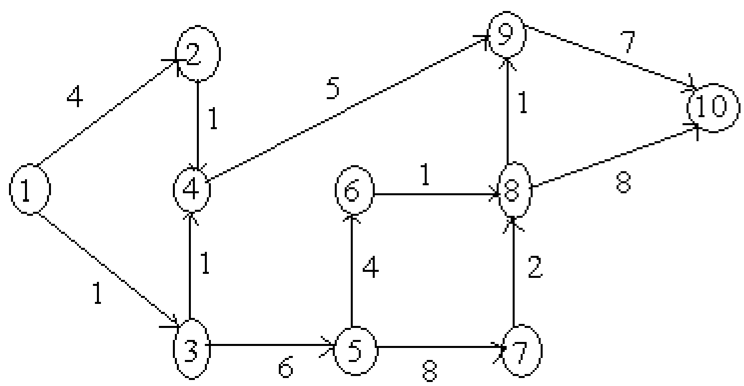
Activity	Times in weeks	Activity	Times in weeks
(1 – 2)	4	(5 – 7)	8
(1 – 3)	1	(6 – 8)	1
(2 – 4)	1	(7 – 8)	2
(3 – 4)	1	(8 – 9)	1
(3 – 5)	6	(8 – 10)	8
(4 – 9)	5	(9 – 10)	7
(5 – 6)	4		

Construct the network and compute

1. T_E and T_L for each event
2. Float for each activity
3. Critical path and its duration

Solution

The network is

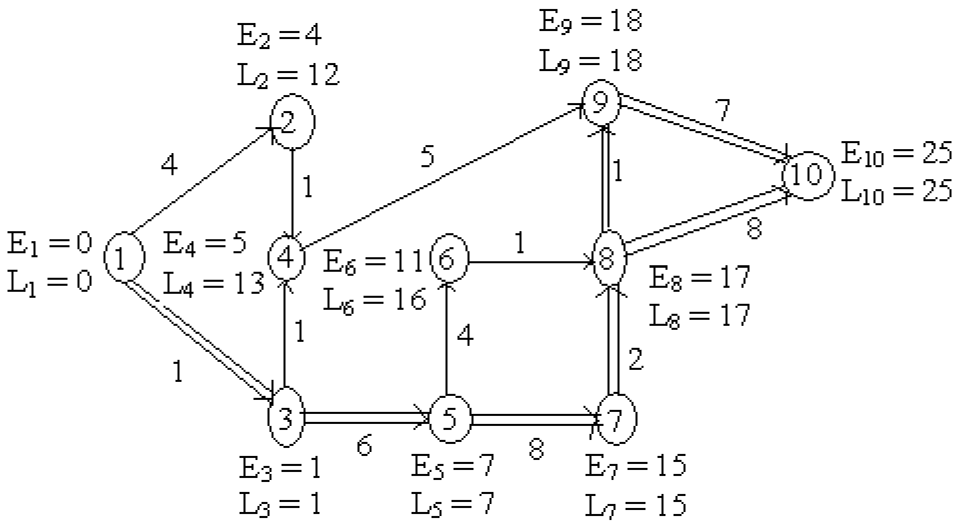


Event No.:	1	2	3	4	5	6	7	8	9	10
T_E:	0	4	1	5	7	11	15	17	18	25
T_L:	0	12	1	13	7	16	15	17	18	25

Float = T_L (Head event) – T_E (Tail event) – Duration

Activity	Duration	T_E (Tail event)	T_L (Head event)	Float
(1 – 2)	4	0	12	8
(1 – 3)	1	0	1	0
(2 – 4)	1	4	13	8
(3 – 4)	1	1	13	11
(3 – 5)	6	1	7	0
(4 – 9)	5	5	18	8
(5 – 6)	4	7	16	5
(5 – 7)	8	7	15	0
(6 – 8)	1	11	17	5
(7 – 8)	2	15	17	0
(8 – 9)	1	17	18	0
(8 – 10)	8	17	25	0
(9 – 10)	7	18	25	0

The resultant network shows the critical path



The two critical paths are

i. $1 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10$

ii. $1 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 8 \rightarrow 10$

3.2 Project Evaluation and Review Technique (PERT)

The main objective in the analysis through PERT is to find out the completion for a particular event within specified date. The PERT approach takes into account the uncertainties. The three time values are associated with each activity

1. **Optimistic time** – It is the shortest possible time in which the activity can be finished. It assumes that every thing goes very well. This is denoted by t_0.
2. **Most likely time** – It is the estimate of the normal time the activity would take. This assumes normal delays. If a graph is plotted in the time of completion and the frequency of completion in that time period, then most likely time will represent the highest frequency of occurrence. This is denoted by t_m.
3. **Pessimistic time** – It represents the longest time the activity could take if everything goes wrong. As in optimistic estimate, this value may be such that

only one in hundred or one in twenty will take time longer than this value. This is denoted by t_p.

In PERT calculation, all values are used to obtain the percent expected value.

1. **Expected time** – It is the average time an activity will take if it were to be repeated on large number of times and is based on the assumption that the activity time follows Beta distribution, this is given by

$$t_e = (t_0 + 4\ t_m + t_p) / 6$$

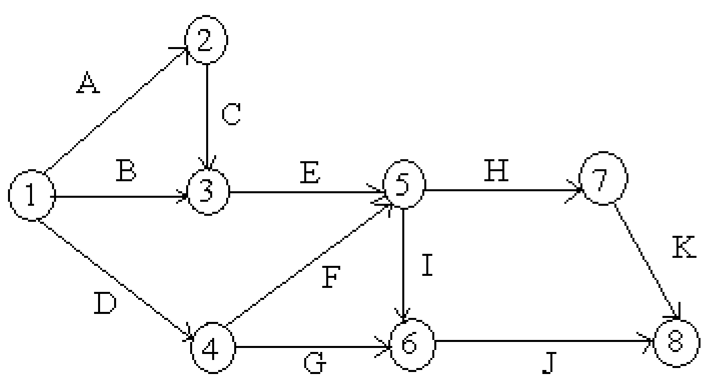
2. The **variance** for the activity is given by

$$\sigma^2 = [(t_p - t_o) / 6]^2$$

3.3 Worked Examples

Example 1

For the project



Task:	A	B	C	D	E	F	G	H	I	J	K
Least time:	4	5	8	2	4	6	8	5	3	5	6

Greatest time:	8	10	12	7	10	15	16	9	7	11	13
Most likely time:	5	7	11	3	7	9	12	6	5	8	9

Find the earliest and latest expected time to each event and also critical path in the network.

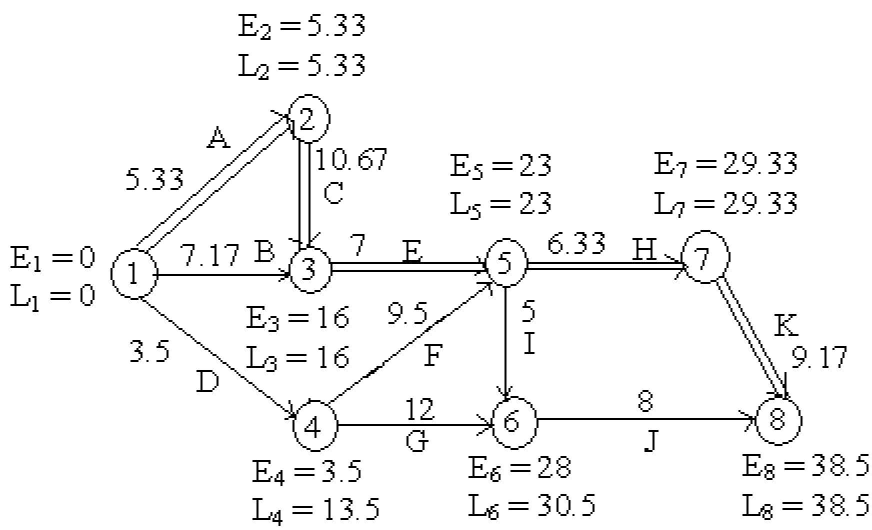
Solution

Task	Least time(t_0)	Greatest time (t_p)	Most likely time (t_m)	Expected time (to + t_p + $4t_m$)/6
A	4	8	5	5.33
B	5	10	7	7.17
C	8	12	11	10.67
D	2	7	3	3.5
E	4	10	7	7
F	6	15	9	9.5
G	8	16	12	12
H	5	9	6	6.33
I	3	7	5	5
J	5	11	8	8
K	6	13	9	9.17

Task	Expected time (t_e)	Start		Finish		Total float
		Earliest	Latest	Earliest	Latest	
A	5.33	0	0	5.33	5.33	0
B	7.17	0	8.83	7.17	16	8.83
C	10.67	5.33	5.33	16	16	0
D	3.5	0	10	3.5	13.5	10
E	7	16	16	23	23	0

F	9.5	3.5	13.5	13	23	10
G	12	3.5	18.5	15.5	30.5	15
H	6.33	23	23	29.33	29.33	0
I	5	23	25.5	28	30.5	2.5
J	8	28	30.5	36	38.5	2.5
K	9.17	29.33	29.33	31.5	38.5	0

The network is



The critical path is A →C →E → H → K

Example 2

A project has the following characteristics

Activity	Most optimistic time (a)	Most pessimistic time (b)	Most likely time (m)
(1 – 2)	1	5	1.5
(2 – 3)	1	3	2
(2 – 4)	1	5	3

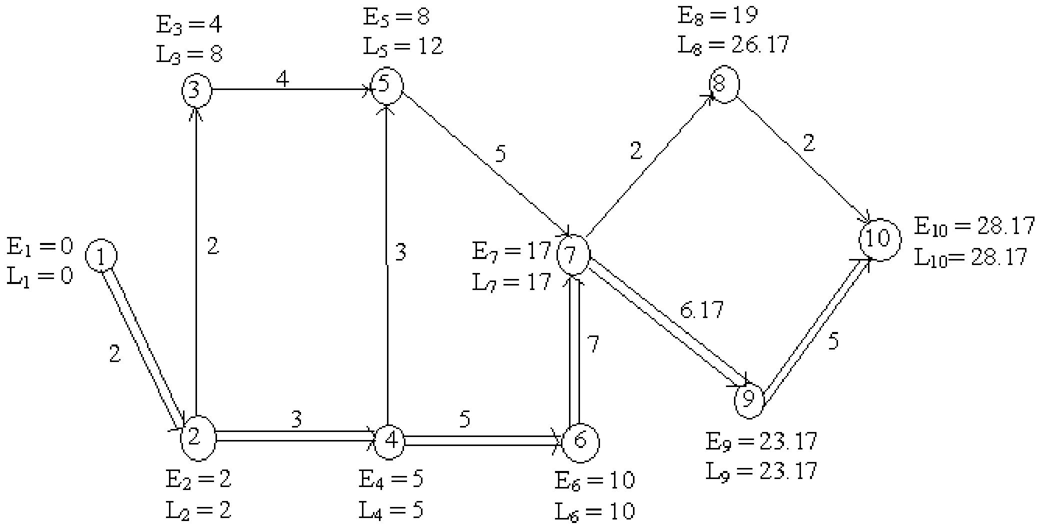
(3 – 5)	3	5	4
(4 – 5)	2	4	3
(4 – 6)	3	7	5
(5 – 7)	4	6	5
(6 – 7)	6	8	7
(7 – 8)	2	6	4
(7 – 9)	5	8	6
(8 – 10)	1	3	2
(9 – 10)	3	7	5

Construct a PERT network. Find the critical path and variance for each event.

Solution

Activity	(a)	(b)	(m)	(4m)	t_e (a + b + 4m)/6	v $[(b - a) / 6]^2$
(1 – 2)	1	5	1.5	6	2	4/9
(2 – 3)	1	3	2	8	2	1/9
(2 – 4)	1	5	3	12	3	4/9
(3 – 5)	3	5	4	16	4	1/9
(4 – 5)	2	4	3	12	3	1/9
(4 – 6)	3	7	5	20	5	4/9
(5 – 7)	4	6	5	20	5	1/9
(6 – 7)	6	8	7	28	7	1/9
(7 – 8)	2	6	4	16	4	4/9
(7 – 9)	5	8	6	24	6.17	1/4
(8 – 10)	1	3	2	8	2	1/9
(9 – 10)	3	7	5	20	5	4/9

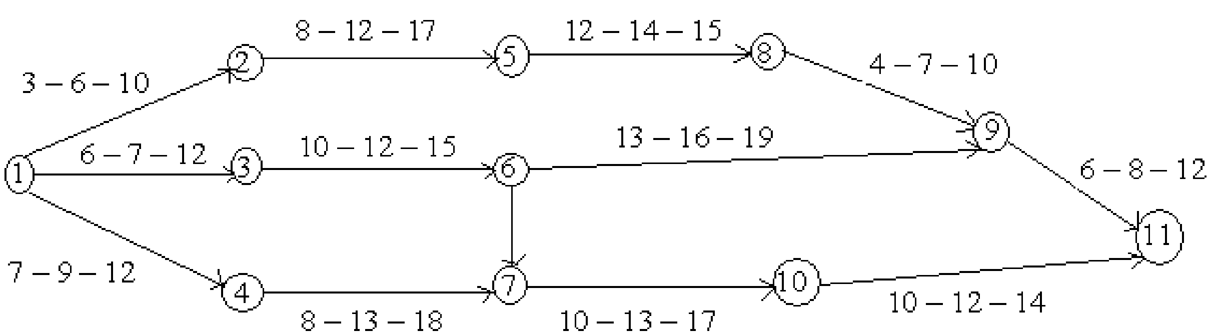
The network is constructed as shown below



The critical path = 1 → 2 → 4 → 6 → 7 →9 →10

Example 3

Calculate the variance and the expected time for each activity



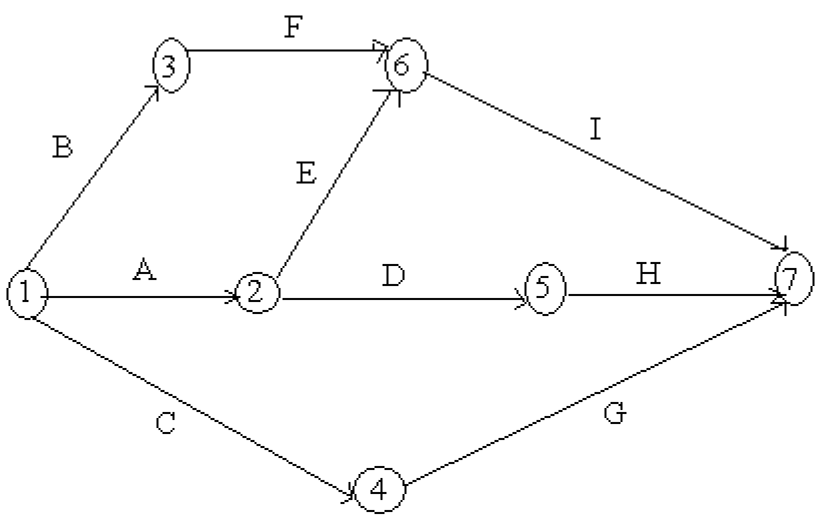
Solution

Activity	(t_o)	(t_m)	(t_p)	t_e $(t_o + t_p + 4t_m)/6$	v $[(t_p – t_o) / 6]^2$
(1 – 2)	3	6	10	6.2	1.36
(1 – 3)	6	7	12	7.7	1.00
(1 – 4)	7	9	12	9.2	0.69

(2 – 3)	0	0	0	0.0	0.00
(2 – 5)	8	12	17	12.2	2.25
(3 – 6)	10	12	15	12.2	0.69
(4 – 7)	8	13	19	13.2	3.36
(5 – 8)	12	14	15	13.9	0.25
(6 – 7)	8	9	10	9.0	0.11
(6 – 9)	13	16	19	16.0	1.00
(8 – 9)	4	7	10	7.0	1.00
(7 – 10)	10	13	17	13.2	1.36
(9 – 11)	6	8	12	8.4	1.00
(10 – 11)	10	12	14	12.0	0.66

Example 4

A project is represented by the network as shown below and has the following data



Task:	A	B	C	D	E	F	G	H	I
Least time:	5	18	26	16	15	6	7	7	3

Greatest time: 10 22 40 20 25 12 12 9 5

Most likely time: 15 20 33 18 20 9 10 8 4

Determine the following

1. Expected task time and their variance
2. Earliest and latest time

Solution

1.

Activity	Least time (t_0)	Greatest time (t_p)	Most likely time (t_m)	Expected time $(to + t_p + 4t_m)/6$	Variance (σ^2)
(1-2)	5	10	8	7.8	0.69
(1-3)	18	22	20	20.0	0.44
(1-4)	26	40	33	33.0	5.43
(2-5)	16	20	18	18.0	0.44
(2-6)	15	25	20	20.0	2.78
(3-6)	6	12	9	9.0	1.00
(4-7)	7	12	10	9.8	0.69
(5-7)	7	9	8	8.0	0.11
(6-7)	3	5	4	4.0	0.11

2.

Earliest time

$E_1 = 0$

$E_2 = 0 + 7.8 = 7.8$

$E_3 = 0 + 20 = 20$

$E_4 = 0 + 33 = 33$

$E_5 = 7.8 + 18 = 25.8$

$E_6 = \max [7.8 + 20, 20 + 9] = 29$

$E_7 = \max [33 + 9.8, 25.8 + 8, 29 + 4] = 42.8$

Latest time

$L_7 = 42.8$

$L_6 = 42.8 – 4 = 38.8$

$L_5 = 42.8 – 8 = 34.3$

$L_4 = 42.8 – 9.8 = 33$

$L_3 = 38.8 – 9 = 29.8$

$L_2 = \min [34.8 – 18, 38.8 – 20] = 16.8$

$L_1 = \min [16.8 – 7.8, 29.8 – 20, 33 - 33] = 0$

Exercise

1. What is PERT?

2. For the following data, draw network. Find the critical path, slack time after calculating the earliest expected time and the latest allowable time

Activity	Duration	Activity	Duration
(1 – 2)	5	(5 – 9)	3
(1 – 3)	8	(6 – 10)	5
(2 – 4)	6	(7 – 10)	4
(2 – 5)	4	(8 – 11)	9
(2 – 6)	4	(9 – 12)	2
(3– 7)	5	(10 – 12)	4
(3 – 8)	3	(11 – 13)	1
(4 – 9)	1	(12 – 13)	7

[Ans. Critical path: 1 → 3 → 7 → 10 → 12 →13]

3. A project schedule has the following characteristics

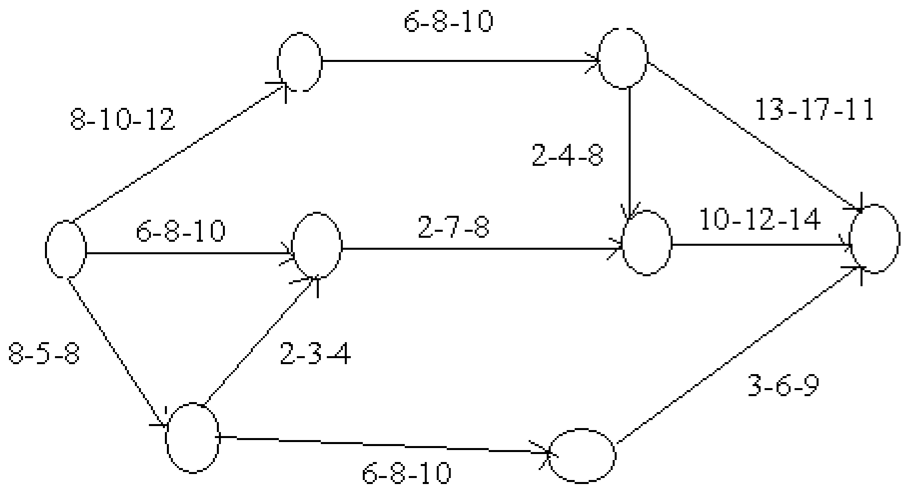
Activity	Most optimistic time	Most likely time	Most pessimistic time
(1 – 2)	1	2	3
(2 – 3)	1	2	3
(2 – 4)	1	3	5
(3 – 5)	3	4	5
(4 – 5)	2	5	4
(4 – 6)	3	5	7
(5 – 7)	4	5	6
(6 – 7)	6	7	8
(7 – 8)	2	4	6

(7 – 9)	4	6	8
(8 – 10)	1	2	3
(9 – 10)	3	5	7

Construct a PERT network and find out

a. The earliest possible time
b. Latest allowable time
c. Slack values
d. Critical path

4. Explain the following terms
    a. optimistic time
    b. Most likely time
    c. Pessimistic time
    d. Expected time
    e. Variance
5. Calculate the variance and the expected time for each activity



Course: BBA Part III

Paper: XVIII

Topic: Difference Between PERT and CPM

Teacher's Name: Prof. (Dr.) Reyazuddin

School: Commerce and Management

Date: 28/08/2020

Difference Between PERT and CPM

Project management can be understood as a systematic way of planning, scheduling, executing, monitoring, controlling the different aspects of the project, so as to attain the goal made at the time of project formulation. PERT and CPM are the two network-based project management techniques, which exhibit the flow and sequence of the activities and events. Program (Project) Management and Review Technique (PERT) is appropriate for the projects where the time needed to complete different activities are not known. On the other hand, the Critical Path Method or CPM is apt for the projects which are recurring in nature.

The two scheduling methods use a common approach for designing the network and for ascertaining its critical path. They are used in the successful completion of a project and hence used in conjunction with each other. Nevertheless, the truth is that CPM is different from PERT in a way that the latter concentrates on time while the former stresses on the time-cost trade-off.

The most important differences between PERT and CPM are provided below:

1. PERT is a project management technique, whereby planning, scheduling, organising, coordinating and controlling uncertain activities are done. CPM is a statistical technique of project management in which planning, scheduling, organising, coordination and control of well-defined activities take place.
2. PERT is a technique of planning and control of time. Unlike CPM, which is a method to control costs and time.
3. While PERT is evolved as a research and development project, CPM evolved as a construction project.
4. PERT is set according to events while CPM is aligned towards activities.
5. A deterministic model is used in CPM. Conversely, PERT uses a probabilistic model.
6. There are three times estimates in PERT, i.e. optimistic time (to), most likely time ™, pessimistic time (tp). On the other hand, there is only one estimate in CPM.

7. PERT technique is best suited for a high precision time estimate, whereas CPM is appropriate for a reasonable time estimate.
8. PERT deals with unpredictable activities, but CPM deals with predictable activities.
9. PERT is used where the nature of the job is non-repetitive. In contrast to, CPM involves the job of repetitive nature.
10. There is a demarcation between critical and non-critical activities in CPM, which is not in the case of PERT.
11. PERT is best for research and development projects, but CPM is for non-research projects like construction projects.
12. Crashing is a compression technique applied to CPM, to shorten the project duration, along with the least additional cost. The crashing concept is not applicable to PERT.

UNIT IV: NETWORK MANAGEMENT – CRASHING OF A PROJECT

COST CONSIDERATION IN PERT/CPM

PROJECT COST

In order to include the cost factors in project scheduling, we must first define the cost duration relationship for various activities in the project. The total cost of any project comprises of direct and indirect costs.

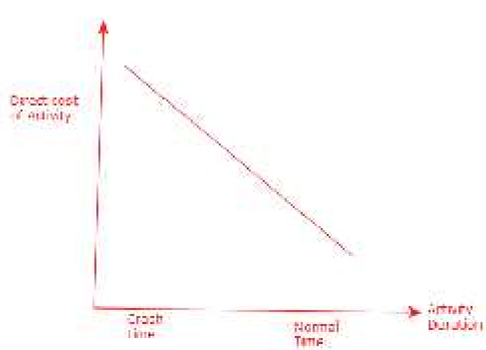
PROJECT COST ANALYSIS

So far we have dealt with how to find project completion time in PERT and CPM networks. In CPM network, when the time required by an activity is deterministic in nature, we may come across a situation that we may have to reduce the activity duration. This is not possible in PERT activity; because activity duration is probabilistic in nature and we have three time estimates. Which time (either to, tm or tp) is to be reduced is a question. Hence activity time crashing is possible in critical path network only.

Before crashing the activity duration, we must understand the costs associated with an activity.

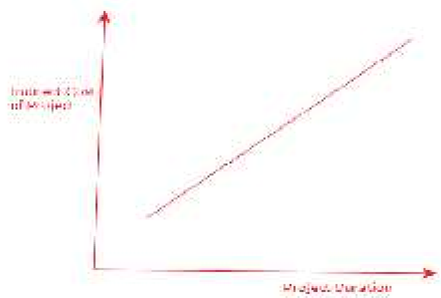
Direct Cost

Direct costs are the costs that can be identified with activity. For example, labour costs, material cost etc. When an activity whose duration is to be reduced (crashed), we have to supply extra resources, specially manpower. Let us say an activity takes 7 days with 2 men. If 4 men works it can be done in 4 days. The cost of 2 workmen increases. As we go on reducing the activity time, cost goes on increasing as shown in figure



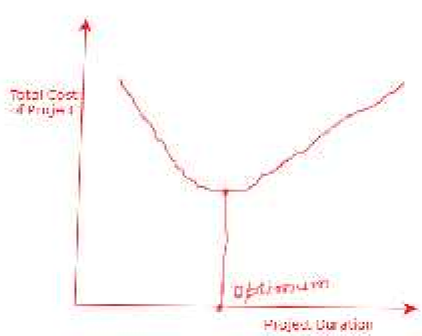
Indirect Cost

These are the costs, which cannot be identified with the activity. Say the salary of a manager, who is in-charge of many projects. Exact amount of his salary that should be charged to a particular project cannot be estimated correctly as it is very difficult to say how much time he has spent on each project. We can express all indirect costs put together in terms of an amount per time period, for example say Rs. 100/- per day, as the indirect costs are expressed as so much of amount per time period, as the duration of project goes on reducing the indirect cost also goes on decreasing as in figure



Total cost

The total cost which is the sum of direct cost and indirect cost is shown in figure. As the project duration goes on reducing the total cost reduces from B to C and if duration is still crashed the total cost increases to A. Hence our problem here is to find out the optimal duration of the project and optimal cost.



Cost Slope

Cost slope is the slope of the direct cost curve, approximated as a straight line. It is given by

Cost slope = (Crash cost – Normal cost)/(Normal time – Crash time) = $(C_c - C_n)/(T_n - T_c)$

i.e., it represents the rate of increase in the cost of performing the activity per unit reduction in time and is called cost/time trade off. It varies from activity to activity. The total cost of project is the sum total of the project's direct and indirect costs.

Time – Cost Optimization Algorithm

Step 1: Find the normal critical path and identify the critical activities.

Step 2: calculate the cost slope for the different activities by using the formula:
Cost slope = (Crash cost – Normal cost)/(Normal time – Crash time) = $(C_c - C_n)/(T_n - T_c)$

Step 3: Rank the activities. The activity whose cost slope is minimum is to be ranked 1, the next minimum as rank 2 and so on, i.e., the ranking takes place in ascending order of cost slope.

Step 4: By crashing the activities on the critical path, other paths also become critical and are called parallel paths.
In such case, project duration can be reduced by crashing activities simultaneously on the parallel critical path.

Step 5: find the total cost of the project at each step.

Step 6: Continue the process until all the critical activities are fully crashed or no further crashing is possible.

In the case of indirect cost, the process of crashing is repeated until the total cost is minimum, beyond which it may increase.

This minimum cost is called the optimum project cost and the corresponding time, the optimum project time.

Solved Example of Crashing of a Project

Q1. The normal and crash times and direct costs for the activities of a project are shown below:

Activities	Time		Cost	
	Normal	Crash	Normal	Crash
1 – 2	8	4	3000	6000
1– 3	5	3	4000	8000
2 – 4	9	6	4000	5500
3 – 5	7	5	2000	3200
2 – 5	5	1	8000	12000
4 – 6	3	2.5	10000	11200
5 – 6	6	2	4000	6800
6 – 7	10	7	6000	8700
5 – 7	9	5	4200	9000

(a) Draw the network diagram

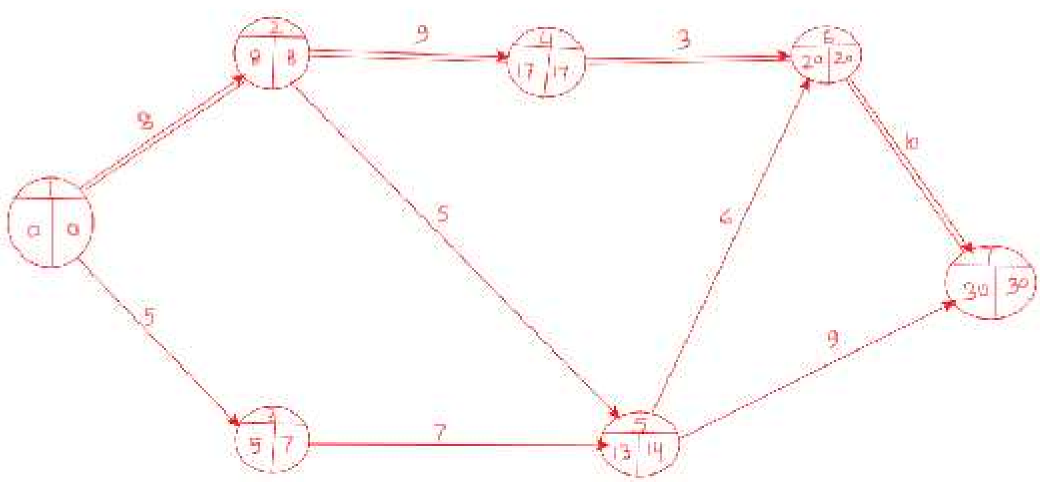
(b) Determine the critical path

(c) Find the minimum cost project schedule if the indirect costs are Rs2000 per week.

Step 1:

Network Diagram & critical Path

Figure – 1



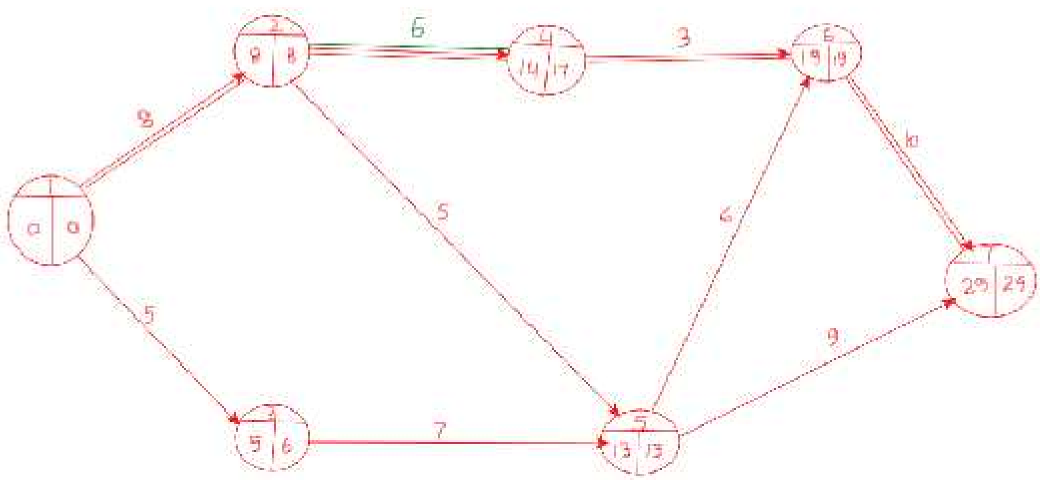
Critical path is (1 -2-4-6-7). The Project duration is 30 weeks and total direct cost is Rs.45200

Step 2 & 3: Cost Slope & Ranking

Activities	Time		Cost		Cost Slope	Rank
	Normal	Crash	Normal	Crash		
1 – 2	8	4	3000	6000	**750**	**IV**
1– 3	5	3	4000	8000	**2000**	**VIII**
2 – 4	9	6	4000	5500	**500**	**I**
3 – 5	7	5	2000	3200	**600**	**II**
2 – 5	5	1	8000	12000	**1000**	**VI**
4 – 6	3	2.5	10000	11200	**2400**	**IX**
5 – 6	6	2	4000	6800	**700**	**III**
6 – 7	10	7	6000	8700	**900**	**V**
5 – 7	9	5	4200	9000	**1200**	**VII**

Examining the time cost slope of activities on the critical path we find that activity (2-4) has the lowest cost slope; in other word cost to expedite per week is the lowest for activity (2-4). Hence, activity (2-4) is crashed. Project network after such crashing is shown below:

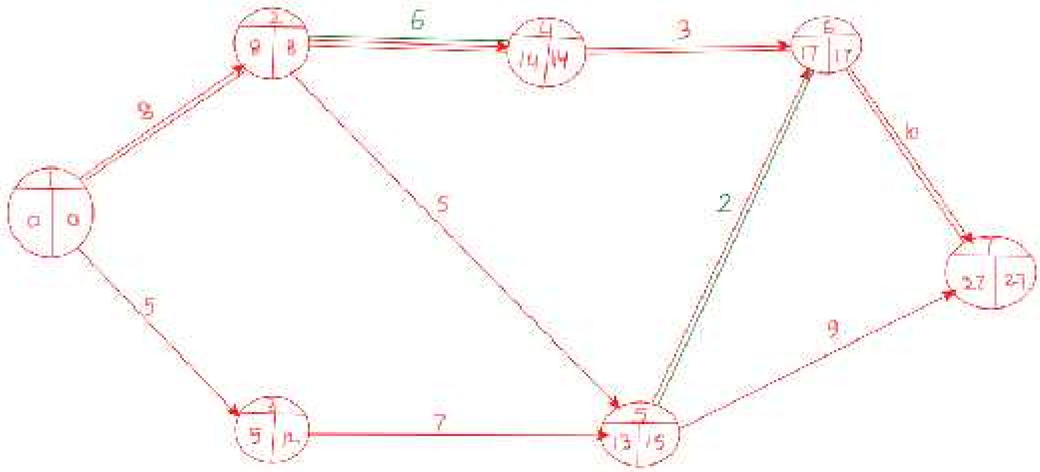
Figure – 2



The new critical path is (1-2-5-6-7), with length of 29 weeks, and the total direct cost is Rs.46700

Examining the time cost slope of activities on the new critical path (1-2-5-6-7) we find that activity (5-6) has the lowest cost slope; in other word cost to expedite per week is the lowest for activity (5-6). Hence, activity (5-6) is crashed. Project network after such crashing is shown below:

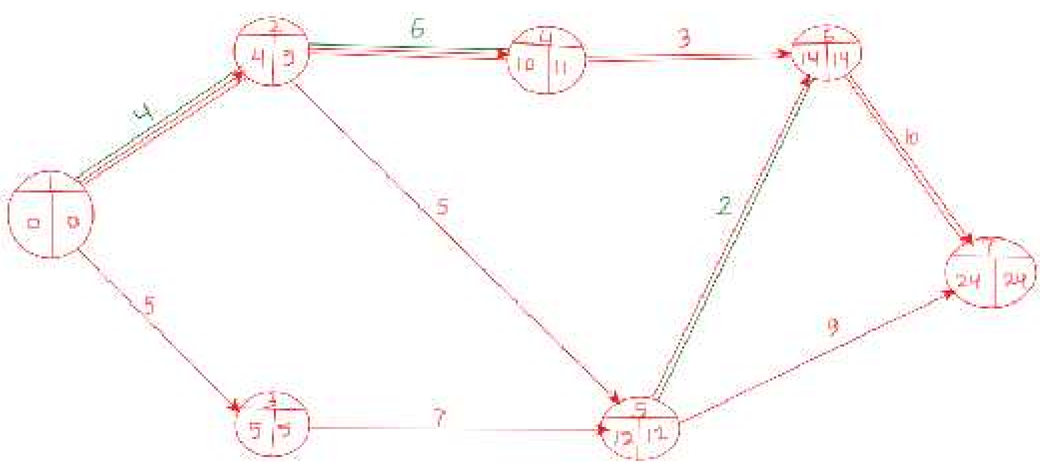
Figure – 3



The new critical path is (1-2-4-6-7), with length of 27 weeks, and the total direct cost is Rs.49500

Examining the time cost slope of activities on the new critical path (1-2-4-6-7) we find that activity (1-2) has the lowest cost slope. Hence, activity (1-2) is crashed. Project network after such crashing is shown below:

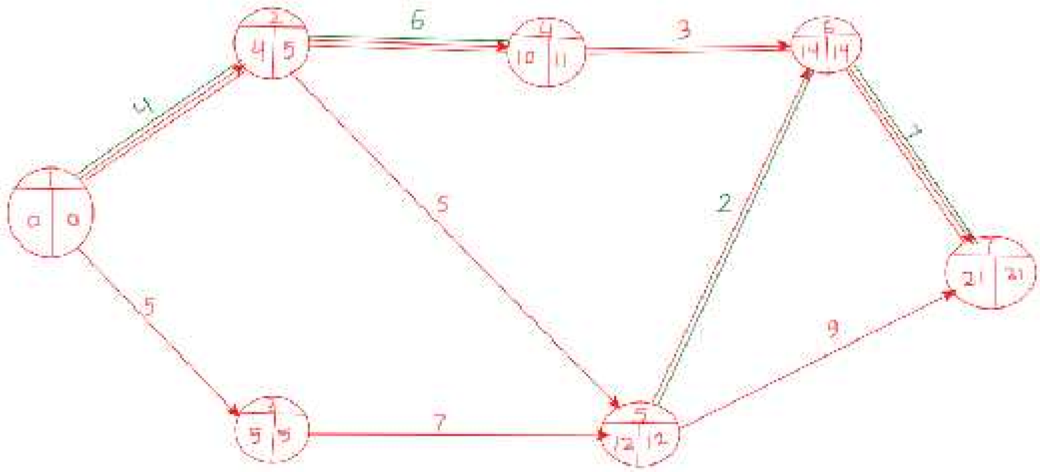
Figure – 4



The new critical path is (1-3-5-6-7), with length of 24 weeks, and the total direct cost is Rs.52500

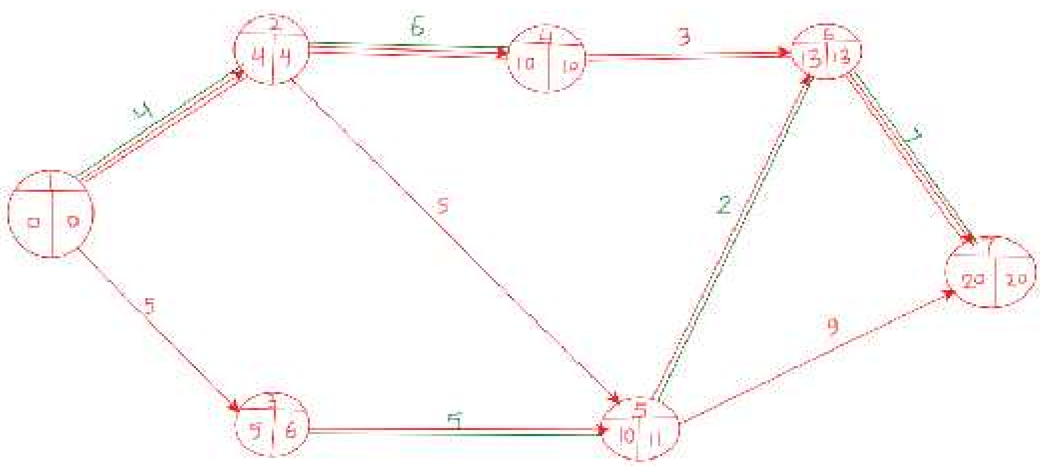
Examining the time cost slope of activities on the new critical path (1-3-5-6-7) we find that activity (6-7) has the lowest cost slope. Hence, activity (6-7) is crashed. Project network after such crashing is shown below:

Figure – 5



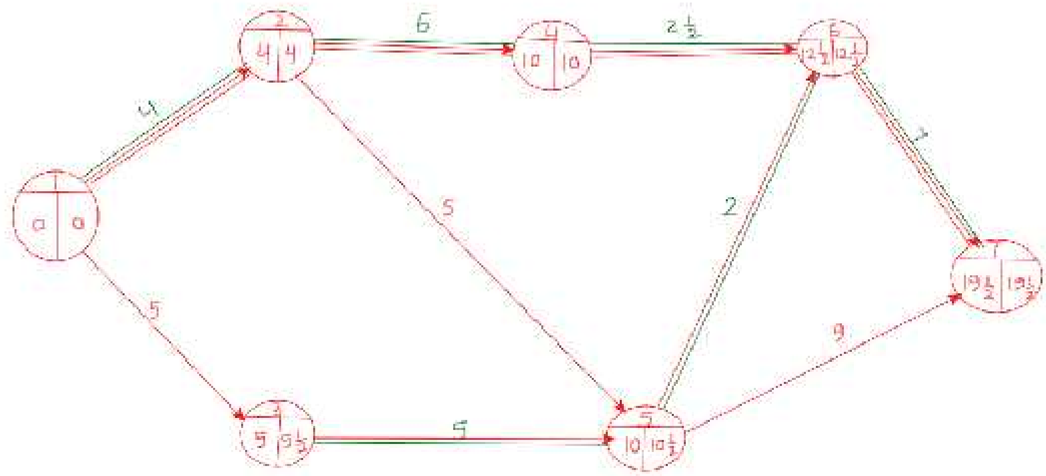
The new critical path is (1-3-5-6-7), and (1-3-5-7), both with length of 21 weeks, and the total direct cost is Rs.55200. Examining the time cost slope of activities on the new critical path (1-3-5-6-7), and (1-3-5-7) we find that activity (3-5) has the lowest cost slope. Hence, activity (3-5) is crashed. Project network after such crashing is shown below:

Figure – 6



The new critical path is (1-2-4-6-7), with length of 20 weeks, and the total direct cost is Rs.56400. Examining the time cost slope of activities on the new critical path (1-2-4-6-7) we find that activity (4-6) has the lowest cost slope. Hence, activity (4-6) is crashed. Project network after such crashing is shown below:

Figure – 7



The new critical path is (1-2-4-6-7), with length of 19 ½ weeks, and the total direct cost is Rs.57600. Since all the activities on the critical path (1-2-4-6-7) are crashed, there is no possibility of further time reduction.

Project duration and Total Cost

Activities crashed	Project duration in weeks	Total direct cost	Total Indirect Cost	Total Cost
None	30	45200	60000	105200
(2-4)	29	46700	58000	104700
(2-4) and (5-6)	27	49500	54000	103500
(1-2), (2-4) and (5-6)	24	52500	48000	100500
(1-2), (2-4), (5-6) and (6-7)	21	55200	42000	97200
(1-2), (2-4), (3-5), (5-6) and (6-7)	20	56400	40000	96400
(1-2), (2-4), (3-5), (5-6), (4-6) and (6-7)	19.5	57600	39000	96600

If the objective is to minimize the total cost of the project, the pattern of crashing suggested by figure – 6 is optimal. If the objective is to minimize the total duration of the project, the pattern of crashing suggested by figure – 7 is optimal. In real life situations, however, both the factors may be important. In addition, factors like strain on resources and degree of manageability are also important. The final decision would involve a careful weighing and balancing of these diverse factors, some quantitative, some qualitative.

Q2.

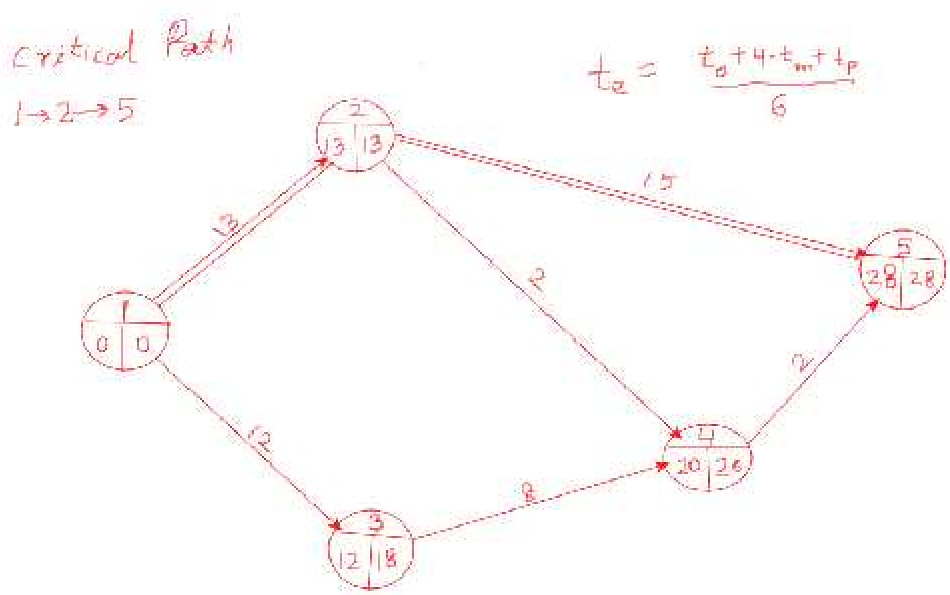
Activities	Optimistic time (in days)	Most likely time (in days)	Pessimistic time (in days)
1 – 2	9	12	21
1 – 3	6	12	18
2 – 4	1	1.5	5
3 – 4	4	8.5	10
2 – 5	10	14	24
4 – 5	1	2	3

a) Draw the network diagram
b) Determine the critical path
c) Calculate event slacks and activity floats

Solution: first of find the estimated time (t_e) by using the formula given below:

$$t_e = \frac{(t_o + 4t_m + t_p)}{6}$$

Activities	Optimistic time	Most likely time	Pessimistic time	Estimated time
1 – 2	9	12	21	13
1 – 3	6	12	18	12
2 – 4	1	1.5	5	2
3 – 4	4	8.5	10	8
2 – 5	10	14	24	15
4 – 5	1	2	3	2



Steps:

(1) Calculate the earliest occurrence time (EOT) for each event.
(2) Calculate the latest occurrence time (LOT) for each event.
(3) Obtain the critical and slack path
(4) Compute the activity floats

Event slack in days

Events	LOT_i = Min($LOT_j - d_{ij}$)	EOT_i = Max($EOT_j - d_{ij}$)	Slack = LOT - EOT
5	28	28	0
4	26	20	6
3	18	12	6
2	13	13	0
1	0	0	0

Activity Float

Activity	Duration	$EST_{ij}=EST_i$	$EFT_{ij}=EFT_j$	$LST_{ij}=LST_i$	$LFT_{ij}=LFT_j$	Total float	Free float	Independent float
1 – 2	13	0	13	0	13	0	0	0
1 – 3	12	0	12	6	18	6	0	0
2 – 4	2	13	15	24	26	11	5	5
3 – 4	8	12	20	18	26	6	0	-6
2 – 5	15	13	28	13	28	0	0	0
4 – 5	2	20	22	26	28	6	6	0

Total float: $TF_{ij} = LOT_j - EOT_i - d_{ij}$ = LS – ES OR LF – EF
Free float: $FF_{ij} = EOT_j - EOT_i - d_{ij}$ = TF – (Head event slack)
Independent float: $IF_{ij} = EOT_j - LOT_i - d_{ij}$ = FF – Tail event slack

Q3. A small Project is composed of seven activities, whose time estimates (in weeks) are listed in table as follows:

Activity	T_o	T_m	T_p
1 to 2	**1**	**1**	**7**
1 to 3	**1**	**4**	**7**
2 to 4	**2**	**2**	**8**
2 to 5	**1**	**1**	**1**
3 to 5	**2**	**5**	**14**
4 to 6	**2**	**5**	**8**
5 to 6	**3**	**6**	**15**

you are required to:

1. Draw the project network

2. Find the expected duration and variance of each activity

3. Calculate the earliest and latest occurrence of each event and the expected project length

4. Calculate the variance and standard deviation of project length.

5. what is the probability that the project will be completed,

(a) 4 weeks earlier than expected?

(b) Not more than 4 weeks later than expected?

(c) If the project's due date is 19 weeks, what is the probability of meeting the due date?

Solution:

Activity	to	tm	tp	te	σ^2
1 to 2	1	1	7	2	1
1 to 3	1	4	7	4	1
2 to 4	2	2	8	3	1
2 to 5	1	1	1	1	0
3 to 5	2	5	14	6	4
4 to 6	2	5	8	5	1
5 to 6	3	6	15	7	4

expected duration = 17
critical path = 1 -3 - 5 - 6
project length variance = 9
S.D. = 3

5(a) **the probability that the project will be completed in 13 weeks = P(Z=<13))**
Z = (Ts - Te)/σ = -1.33333
area under the normal curve for the region z=< -1.33 9.18%

5(b) **the probability that the project will be completed in 21 weeks = P(Z=<21))**
Z = (Ts - Te)/σ = 1.333333
area under the normal curve for the region z=< 1.33 90.82%

the probability that the project will be completed in 19 weeks = P(Z=<19))
5(c) **Z = (Ts - Te)/σ = 0.666667**
area under the normal curve for the region z=< 0.67 74.86%

Exercise:

Q1. A building project consists of the following activities:
A = Lay foundation
B = Erect framework
C = Install millwork
D = Install wiring
E = Install plumbing
F = plaster walls
G = install siding
H = Decorate the interior
I = Finish the exterior
The interrelationship among these activities is as follows:

1. A should precede B
2. B should precede C, D, E, F, and G

3. C, D, E, and F should precede H
4. G should precede I

Draw the network diagram for the above activities for a building project

Q2.

Activities	Optimistic time	Most likely time	Pessimistic time
1 – 2	4	6	10
1 – 3	3	7	12
1 – 4	5	6	9
1 – 7	2	4	6
2 – 4	6	10	20
2 – 6	3	4	7
2 -7	5	9	15
3 – 4	3	7	12
4 – 5	2	4	5
5 – 6	1	3	6
3 – 7	2	5	8
6 – 7	1	2	6

(a) Draw the network diagram
(b) Determine the critical path
(c) Calculate event slacks and activity floats
(d) Find the standard deviation of the critical path duration
(e) Compute the probability of completing the project in 30 weeks

Q3. A small Project is composed of seven activities, whose time estimates (in weeks) are listed in table as follows:

job	to	tm	tp
1 to 2	1	7	13
1 to 6	2	5	14
2 to 3	2	14	26
2 to 4	2	5	8
3 to 5	7	10	19
4 to 5	5	5	17
6 to 7	5	8	29
5 to 8	3	3	9
7 to 8	8	17	32

you are required to:
1. Draw the project network
2. Find the expected duration and variance of each activity
3. Calculate the earliest and latest occurrence of each event and the expected project length
4. Calculate the variance and standard deviation of project length.
5. what is the probability that the project will be completed in 40weeks

Q4. The normal and crash times and direct costs for the activities of a project are shown below:

Activities	Time		Cost	
	Normal	Crash	Normal	Crash
1 – 2	5	2	6000	9000
2– 4	6	3	7000	10000
1 – 3	4	2	1000	2000
3 – 4	7	4	4000	8000
4 – 7	9	5	6000	9200
3 – 5	12	3	16000	19600
4 – 6	10	6	15000	18000
6 – 7	7	4	4000	4900
7 – 9	6	4	3000	4200
5 – 9	12	7	4000	8500

(a) Draw the network diagram
(b) Determine the critical path
(c) Find the minimum cost project schedule if the indirect costs are Rs1000 per week.

Q5. The normal and crash times and direct costs for the activities of a project are shown below:

Activities	Time		Cost	
	Normal	Crash	Normal	Crash
1 – 2	8	6	100	200
1 – 3	4	2	150	350
2– 4	2	1	50	90
2 – 5	10	5	100	400
3 – 4	5	1	100	200
4 – 5	1	1	80	100

(a) Draw the network diagram
(b) Determine the critical path
(c) Find the minimum cost project schedule if the indirect costs are Rs70 per day.

18

Replacement Problem and System Reliability

"Conduct is wise or foolish only in reference to its results"

18:1. INTRODUCTION

The study of *replacement* is concerned with situations that arise when some items such as machines, men, electric-light bulbs, etc., need replacement due to their deteriorating efficiency, failure or breakdown. The deteriorating efficiency or complete breakdown may be either gradual or all of a sudden. For example, a machine becomes more and more expensive to maintain after a number of years, a railway time-table gradually becomes more and more out of date, an electric-light bulb fails all of a sudden, pipeline is blocked, or an employee loses his job, and so like. In all such situations, there is a need to formulate a most economic replacement policy for replacing faulty units or to take some remedial special action to restore the efficiency of deteriorating units.

Following are the situations when the replacement of certain items needs to be done :

(*i*) An old item has failed and does not work at all, or the old item is expected to fail shortly.

(*ii*) The old item has deteriorated and works badly or requires expensive maintenance.

(*iii*) A better design of equipment has been developed.

Replacement problems can be broadly classified into the following two categories :

(*a*) When the equipment/assets deteriorate with time and the value of money

(*i*) does not change with time.

(*ii*) changes with time.

(*b*) When the items/units fail completely all of a sudden.

18:2. REPLACEMENT OF EQUIPMENT/ASSET THAT DETERIORATES GRADUALLY

Generally, the cost of maintenance and repair of certain items (equipments) increases with time and a stage may come when these costs become so high that it is more economical to replace the item by a new one.

At this point, a replacement is justified.

18:2.1. Replacement Policy when Value of Money does not change with time

The aim here is to determine the optimum replacement age of an equipment/item whose running/maintenance cost increases with time and the value of money remains static during that period. Let

C : capital cost of equipment, S : scrap value of equipment,
n : number of years that equipment would be in use,
$f(t)$: maintenance cost function, and $A(n)$: Average total annual cost.

Case 1. ***When t is a continuous variable.*** If the equipment is used for 'n' years, then the total cost incurred during this period is as under :

$$TC = \text{Capital cost} - \text{Scrap value} + \text{Maintenance cost} = C - S + \int_0^n f(t)\,dt.$$

Average annual total cost, therefore is : $$A(n) = \frac{1}{n}\,TC = \frac{C-S}{n} + \frac{1}{n}\int_0^n f(t)\,dt.$$

For minimum cost, we must have $\frac{d}{dn}[A(n)] = 0$. This implies that

$$\frac{-(C-S)}{n^2} - \frac{1}{n^2}\int_0^n f(t)\,dt + \frac{1}{n}f(n) = 0 \quad \text{or} \quad f(n) = \frac{C-S}{n} + \frac{1}{n}\int_0^n f(t)\,dt \equiv A(n).$$

Clearly, $$\frac{d^2}{dn^2}[A(n)] > 0 \quad \text{at} \quad f(n) = A(n).$$

This suggests that the *equipment should be replaced when maintenance cost equals the average annual total cost.*

Case 2. ***When t is a discrete variable.*** Here, the resale value at the end of nth year is denoted by $S(n)$, where $n = 1, 2, 3, \ldots$. Let $d(n)$ be the change in the resale value in the nth year and $f(n)$ be the maintenance cost during the nth year. Then

$$d(1) = S(0) - S(1),\quad d(2) = S(1) - S(2),\quad d(3) = S(2) - S(3),$$
$$d(4) = S(3) - S(4),\quad \ldots,\quad d(n) = S(n-1) - S(n).$$

So, $$\sum_{t=1}^{n} d(t) = d(1) + d(2) + d(3) + \ldots + d(n)$$
$$= S(0) - S(n) = C - S(n),\quad \text{where } S(0) = C.$$

Total cost during n years is given as

$$TC = C - S(n) + \sum_{t=1}^{n} f(t)$$
$$= \sum_{t=1}^{n} d(t) + \sum_{t=1}^{n} f(t) = \sum_{t=1}^{n} [d(t) + f(t)].$$

Let $e(t) = d(t) + f(t)$ be the effective maintenance cost in the nth year. Then

$$TC = \sum_{t=1}^{n} e(t)$$

and the average cost during the nth year is given as under :

$$A(n) = \frac{TC}{n} = \frac{1}{n}\sum_{t=1}^{n} e(t).$$

Now, $A(n)$ will be minimum for that value of n, for which
$$A(n+1) \geq A(n) \quad \text{and} \quad A(n-1) \geq A(n).$$

For this, we write

$$A(n+1) = \frac{1}{n+1}\sum_{t=1}^{n+1} e(t) = \frac{1}{n+1}\sum_{t=1}^{n} e(t) + \frac{1}{n+1}e(n+1) = \frac{nA(n)}{n+1} + \frac{e(n+1)}{n+1}.$$

$$\therefore \quad A(n+1) - A(n) = \frac{nA(n)}{n+1} + \frac{e(n+1)}{n+1} - A(n) = \frac{e(n+1) - A(n)}{n+1}.$$

So, $$A(n+1) - A(n) \geq 0 \Rightarrow e(n+1) \geq A(n).$$
Similarly, $$A(n) - A(n-1) \leq 0 \Rightarrow e(n) \leq A(n-1).$$

This suggests the optimal replacement policy :

Replace the equipment at the end of n years, if the effective maintenance cost in the (n + 1)th year is more than the average total cost in the nth year and the nth year's effective maintenance cost is less than the previous year's average total cost.

SAMPLE PROBLEMS

1801. *A firm is considering replacement of a machine, whose cost price is Rs. 12,200 and the scrap value, Rs. 200. The running (maintenance and operating) cost in rupees are found from experience to be as follows :*

Year :	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Running cost :	*200*	*500*	*800*	*1,200*	*1,800*	*2,500*	*3,200*	*4,000*

When should the machine be replaced? **[Meerut M.Sc. (Math.) 2001; Madras M.B.A. 2004]**

Solution. We are given the running cost, $f(n)$, the scrap value S = Rs. 200 and the cost of machine, C = Rs. 12,200. In order to determine the optimal time 'n' when the machine should be replaced, we calculate an average total cost per year during the life of the machine as shown in the table given below :

Year of service n (1)	*Running cost (Rs.)* $f(n)$ (2)	*Resale value (Rs.)* $s(n)$ (3)	*Change in resale value (Rs.)* $d(n)$ (4)	*Effective running cost (Rs.)* $e(n)$ (5)	*Total cost (Rs.)* TC $\Sigma e(n)$ (6)	*Average cost (Rs.)* $A(n)$ (6)/(1) (7)
1	200	200	12,200 – 200 = 12,000	12,200	12,200	12,200
2	500	200	0	500	12,700	6,350
3	800	200	0	800	13,500	4,500
4	1,200	200	0	1,200	14,700	3,675
5	1,800	200	0	1,800	16,500	3,300
6	2,500	200	0	2,500	19,000	3,167
7	3,200	200	0	3,200	22,200	3,171
8	4,000	200	0	4,000	26,200	3,275

From the table, we observe that the average total cost per year, $A(n)$ is minimum in the 6th year (Rs. 3,167). Also, the average total cost in 7th year (Rs. 3,171) is more than the cost in the 6th year. Hence, the machine should be replaced after every 6 years.

1802. (*a*) *Machine A costs Rs. 9,000. Annual operating costs are Rs. 200 for the first year, and then increase by Rs. 2,000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine?* **[Madras B.E. (Comp. Sc.) 2002]**

(*b*) *Machine B costs Rs. 10,000. Annual operating costs are Rs. 400 for the first year, and then increase by Rs. 800 every year. You now have a machine of type A which is one-year old, should you replace it with B; if so when?* **[Delhi M.Com. 2006]**

Solution. (*a*) Let the machine have no resale value when replaced. Then, for machine A, the average total annual cost $ATC(n)$ is computed as follows :

Year (n)	$f(n)$	$s(n)$	$d(n)$	$e(n)$	TC	$A(n)$
1	200	0	9,000 – 0 = 9,000	9,200	9,200	9,200
2	2,200	0	0	2,200	11,400	5,700
3	4,200	0	0	4,200	15,600	5,200
4	6,200	0	0	6,200	21,800	5,450
5	8,200	0	0	8,200	30,000	6,000

This table shows that the best age for the replacement of machine A is 3rd year. The average yearly cost of owning and operating for this period is Rs. 5,200.

(*b*) For machine B, the average cost per year can similarly be computed as given in the following table :

Year (n)	$f(n)$	$s(n)$	$d(n)$	$e(n)$	TC	$A(n)$
1	400	0	10,000	10,400	10,400	10,400
2	1,200	0	0	1,200	11,600	5,800
3	2,000	0	0	2,000	13,600	4,533
4	2,800	0	0	2,800	16,400	4,100
5	3,600	0	0	3,600	20,000	4,000
6	4,400	0	0	4,400	24,400	4,066

Since, the minimum average cost for machine B is lower than that for machine A, machine A should be replaced.

To decide the time of replacement, we should compare the minimum average cost for B (Rs. 4,000) with yearly cost of maintaining and using the machine A. Since, there is no salvage value of the machine, we shall consider only the maintenance cost. We would keep the machine A so long as the yearly maintenance cost is lower than Rs. 4,000 and replace when it exceeds Rs. 4,000.

On the one-year old machine A, Rs. 2,200 would be required to be spent in the next year; while Rs. 4,200 would be needed in the year following. Thus, we should keep machine A for one year and replace it thereafter.

1803. *The data collected in running a machine, the cost of which is Rs. 60,000, are given below :*

Year		*1*	2	3	4	5
Resale value (Rs.)	:	42,000	30,000	20,400	14,400	9,650
Cost of spares (Rs.)	:	4,000	4,270	4,880	5,700	6,800
Cost of labour (Rs.)	:	14,000	16,000	18,000	21,000	25,000

Determine the optimum period for replacement of the machine. ***[Lucknow B.M.S. 2008]***

Solution. The operating or maintenance cost of machine in successive years is as follows :

Year	:	*1*	2	3	4	5
Operating cost (Rs.)	:	18,000	20,270	22,880	26,700	31,800

(The cost of spares and labour together determine the operating or running or maintenance cost.)

The average total annual cost is computed below :

Year of service n	*Operating cost (Rs.)* $f(n)$	*Resale value (Rs.)* $s(n)$	*Change in resale value (Rs.)* $d(n)$	*Effective op. cost (Rs.)* $e(n)$	*Total cost (Rs.)* TC	*Average cost (Rs.)* $A(n)$
1	18,000	42,000	18,000	36,000	36,000	36,000.00
2	20,270	30,000	12,000	32,270	68,270	34,135.00
3	22,880	20,400	9,600	32,480	1,00,750	33,583.30
4	26,700	14,400	6,000	32,700	1,33,450	33,362.50
5	31,800	9,650	4,750	36,550	1,70,000	34,000.00

The calculations in the above table shows that the average cost is *lowest* during the *fourth year*. Hence, the machine should be replaced after every fourth year.

PROBLEMS

1804. The cost of a machine is Rs. 6,100 and its scrap value is Rs. 100. The maintenance costs found from experience are as follows :

Year	:	*1*	2	3	4	5	6	7	8
Maintenance cost (Rs.)	:	100	250	400	600	900	1,200	1,600	2,000

When should the machine be replaced? ***[Kerala M.Com. 1996; Panjab M.B.A. (June) 2003]***

1812. A new tempo costs Rs. 80,000 and may be sold at the end of any year at the following prices :

Year (end)	1	2	3	4	5	6
Selling price (in Rs.) (at present value)	50,000	33,000	20,000	11,000	6,000	1,000

The corresponding annual operating costs are :

Year (end)	1	2	3	4	5	6
Cost/year (in Rs.) (at present value)	10,000	12,000	15,000	20,000	30,000	50,000

It is not only possible to sell the tempo after use but also to buy a second hand tempo.

It may be cheaper to do so than to replace by a new tempo.

Age of tempo	0	1	2	3	4	5
Purchase price (in Rs.) (at present value)	80,000	58,000	40,000	26,000	16,000	10,000

What is the age to buy and to sell tempo so as to minimize average annual cost?

1813. (*a*) A transport manager finds from his past records that the costs per year of running a truck whose purchase price is Rs. 6,000 are as given below :

Year	1	2	3	4	5	6	7	8
Running cost (Rs.)	1,000	1,200	1,400	1,800	2,300	2,800	3,400	4,000
Resale value (Rs.)	3,000	1,500	750	375	200	200	200	200

Determine at what age is replacement due? *[IAS 1993; Kerala M.Com. 1991; Jodhpur M.Sc. (Math.) 1992]*

(*b*) Let the owner of a fleet have three trucks, two of which are two years old and the third one year old. The cost price, running cost and resale vale of these trucks are same as given in (*a*). Now he is considering a new type of truck with 50% more capacity than one of the old ones at a unit price of Rs. 8,000. He estimates that the running costs and resale price for the truck will be as follows :

Year	1	2	3	4	5	6	7	8
Running costs (Rs.)	1,200	1,500	1,800	2,400	3,100	4,000	5,000	6,100
Resale price (Rs.)	4,000	2,000	1,000	500	300	300	300	300

Assuming that the loss of flexibility due to fewer trucks is of no importance, and that he will continue to have sufficient work for three of the old trucks, what should his policy be? *[Poona M.B.A. 1992]*

1814. Machine *A* costs Rs. 3,600. Annual operating costs are Rs. 40 for the first year and then increase by Rs. 360 every year. Assuming that machine *A* has no resale value, determine the best replacement age.

Another machine *B*, which is similar to machine *A*, costs Rs. 4,000. Annual running costs are Rs. 200 for the first year and then increase by Rs. 200 every year. It has resale value of Rs. 1,500, Rs. 1,000 and Rs. 500, if replaced at the end of first, second and third years respectively. It has no resale value during fourth year and onwards.

Which machine would you prefer to purchase? Future costs are not to be discounted. *[Guwahati M.C.A. 1992]*

1815. Machine *A* costs Rs. 45,000 and the operating costs are estimated at Rs. 1,000 for the first year, increasing by Rs. 10,000 per year in the second and subsequent years. Machine *B* costs Rs. 50,000 and operating costs are Rs. 2,000 for the first year, increasing by Rs. 4,000 in the second and subsequent years. If we now have a machine of type *A*, should we replace it with *B* ? If so when? Assume that both machines have no resale value and future costs are not discounted. *[Madras M.B.A. (Nov.) 2006; Lucknow B.M.S. 2008]*

18:2.2 Replacement Policy when Value of Money changes with time

When the time value of money is taken into consideration, we shall assume that (*i*) the equipment in question has no salvage value, and (*ii*) the maintenance costs are incurred in the beginning of the different time periods.

Since, it is assumed that the maintenance cost increases with time and each cost is to be paid just in the start of the period, let the money carry a rate of interest r per year. Thus, a rupee invested now will be worth $(1 + r)$ after a year, $(1 + r)^2$ after two years, and so on. In this way a rupee invested today will be worth $(1+r)^n$, n years hence, or, in other words, if we have to make a payment of one rupee in n years time, it is equivalent to making a payment of $(1+r)^{-n}$ rupees today. The quantity $(1+r)^{-n}$ is called the *present worth* factor (*Pwf*) of one rupee spent in n years time from now onwards. The expression $(1+r)^n$ is known as the *payment compound amount factor* (*Caf*). of one rupee spent in n years time.

Let the initial cost of the equipment be C and let R_n be the operating cost in year n. Let v be the rate of interest in such a way that $v = (1+r)^{-1}$ is the discount rate (present worth factor). Then the present value of all future discounted costs V_n associated with a policy of replacing the equipment at the end of each n years is given by

$$V_n = \{(C+R_0) + vR_1 + v^2R_2 + \ldots + v^{n-1}R_{n-1}\} + \{(C+R_0)v_n + v^{n+1}R_1 + v^{n+2}R_2 + \ldots + v^{2n-1}R_{n-1}\} + \ldots$$

$$= \left[C + \sum_{k=0}^{n-1} v^k R_k\right] \times \sum_{k=0}^{\infty} (v^n)^k = \left[C + \sum_{k=0}^{n-1} v^k R_k\right] \Big/ (1-v^n)^{-1}$$

Now, V_n will be a minimum for that value of n, for which

$$V_{n+1} - V_n > 0 \quad \text{and} \quad V_{n-1} - V_n > 0.$$

For this, we write

$$V_{n+1} - V_n = \left[C + \sum_{k=0}^{n} v^k R_k\right](1 - v^{n+1})^{-1} - V_n$$

$$= v^n [R_n - (1-v) V_n] / (1 - v^{n+1})$$

and similarly $$V_n - V_{n-1} = V^{n-1}[R_{n-1} - (1-v) V_n] / (1 - v^{n-1})$$

Since v is the depreciation value of money, it will always be less than 1, and therefore $1-v$ will always be positive. This implies that $v^n/(1-v^{n+1})$ will always be positive.

Hence, $\quad V_{n+1} - V_n > 0 \Rightarrow R_n > (1-v) V_n$ and $V_n - V_{n-1} < 0 \Rightarrow R_{n-1} < (1-v) V_n$

Thus, $$R_{n-1} < (1-v) V_n < R_n$$

or $$R_{n-1} < \frac{C + R_0 + vR_1 + v^2R_2 + \ldots + v^{n-1}R_{n-1}}{1 + v + v^2 + \ldots + v^{n-1}} < R_n,$$

since $$(1 - v^n)(1-v)^{-1} = \sum_{k=0}^{n-1} v^k.$$

The expression which lies between R_{n-1} and R_n is called the "*weighted average cost*" of all the previous n years with weights $1, v, v^2, \ldots, v^{n-1}$ respectively.

Hence, the optimal replacement policy of the equipment after n periods is :

(*a*) Do not replace the equipments if the next period's operating cost is less than the weighted average of previous costs.

(*b*) Replace the equipments if the next period's operating cost is greater than the weighted average of previous costs.

Remark. Procedure for determining the weighted average of costs (annualized cost) may be summarized in the following steps :

Step 1. Find the present value of the maintenance cost for each of the years,

i.e., $$\Sigma R_{n-1} v^{n-1} \; (n = 1, 2, \ldots); \text{ where } v = (1+r)^{-1}.$$

Step 2. Calculate cost plus the accumulated present values obtained in *step 1*, *i.e.*, $C + \Sigma Rv^{n-1}$.

Step 3. Find the cumulative present value factor up to each of the years 1, 2, 3, ..., *i.e.*, Σv^{n-1}.

Step 4. Determine the annualized cost $W(n)$, by dividing the entries obtained in *step 2* by the corresponding entries obtained in *step 3*, *i.e.*, $[C + \Sigma R_{n-1} v^{n-1}] / \Sigma v^{n-1}$.

Corollary. When the time value of money is not taken into consideration, the rate of interest becomes zero and hence v approaches unity. Therefore, as $v \to 1$, we get

$$R_{n-1} < \frac{C + R_0 + R_1 + \ldots + R_{n-1}}{1 + 1 + \ldots + n \text{ times}} < R_n$$

or

$$R_{n-1} < W(n) < R_n$$

Note. It may be noted that the above result is in complete agreement with the result that was obtained in 18 : 2.1.

Selection of the Best Equipment Amongst Two

Following is the procedure for determining a policy for the selection of an economically best item amongst the available equipments :

Step 1. Considering the case of two equipments, say A and B, we first find the best replacement age for both the equipments by making use of

$$R_{n-1} < (1 - v) V_n < R_n.$$

Let the optimum replacement age for A and B comes out to be n_1 and n_2 respectively.

Step 2. Next, compute the fixed annual payment (or weighted average cost) for each equipment by using the formula

$$W(n) = \frac{C + R_0 + vR_1 + v^2R_2 + \ldots + v^{n-1}R_{n-1}}{1 + v + \ldots + v^{n-1}}$$

and substitute $n = n_1$ for equipment A and $n = n_2$ for equipment B in it.

Step 3. (*i*) If $W(n_1) < W(n_2)$, choose equipment A.

(*ii*) If $W(n_1) > W(n_2)$, choose equipment B.

(*iii*) If $W(n_1) = W(n_2)$, both equipments are equally good.

SAMPLE PROBLEMS

1816. *Let the value of money be assumed to be 10% per year and suppose that machine A is replaced after every 3 years whereas machine B is replaced after every six years. The yearly costs of both the machines are given below :*

Year	1	2	3	4	5	6
Machine A	1,000	200	400	1,000	200	400
Machine B	1,700	100	200	300	400	500

Determine which machine should be purchased. **[Bharathidasan B.Com. 1999]**

Solution. Since the money carries the rate of interest, the present worth of the money to be spent over in a period of one year is

$$v = \frac{100}{100 + 10} = \frac{10}{11} = 0.9091$$

∴ The total discounted cost (present worth) of A for 3 years is

$$1000 + 200 \times (0.9091) + 400 \times (0.9091)^2 = \text{Rs. } 1512 \text{ approx.}$$

Again, the total discounted cost of B for six years is

$$1{,}700 + 100 \times (0.9091) + 200 \times (0.9091)^2 + 300 \times (0.9091)^3 + 400 \times (0.9091)^4 + 500 \times (0.9091)^5 = \text{Rs. } 2{,}765.$$

Average yearly cost of machine A = Rs. 1,512/3 = Rs. 504.

Average yearly cost of machine B = Rs. 2,765/6 = Rs. 461.

This shows that the apparent advantage is with machine B. But, the comparison is unfair since the periods for which the costs are considered are different. So, if we consider 6 years period for machine A also, then the total discounted cost of A will be

$$1{,}000 + 200 \times (0.9091) + 400 \times (0.9091)^2 + 1{,}000 \times (0.909)^3 + 200 \times (0.9091)^4 + 400 \times (0.9091)^5.$$

After simplification this comes out to be Rs. 2,647 which is Rs. 118 less costlier than machine B over the same period.

Hence, machine A should be purchased.

1817. *A pipeline is due for repairs. It will cost Rs. 10,000 and last for 3 years. Alternatively, a new pipeline can he laid at a cost of Rs. 30,000 and lasts for 10 years. Assuming cost of capital to be 10% and ignoring salvage value, which alternative should be chosen?*

Solution. Consider the two types of pipeline for infinite replacement cycles of 10 years for the new pipeline, and 3 years for the existing pipeline.

Since, the discount rate of money per year is 10%, theretore the present worth of money to be spent over in a period of one year is

$$v = \frac{100}{100 + 10} = 0.9091$$

Let k_n denote the discounted value of all future costs associated with a policy of replacing the equipment after n years. Then, if we designate the initial outlay by C,

$$k_n = C + Cv^n + Cv^{2n} + \ldots + \infty = C(1 + v^n + v^{2n} + \ldots + \infty) = C/(1 - v^n)$$

Making use of values of C, v and n for two types of pipelines, the discounted value, therefore, yields

$$k_3 = \frac{10{,}000}{1 - (0.9091)^3} = \text{Rs. } 4{,}021 \text{ for the existing pipeline,}$$

and

$$k_{10} = \frac{30{,}000}{1 - (0.9091)^{10}} = \frac{30{,}000}{1 - 0.3855} = \text{Rs. } 48{,}820 \text{ for the new pipeline.}$$

Since $k_3 < k_{10}$, the existing pipeline should be continued. Alternatively, the comparison may be made over $3 \times 10 = 30$ years.

1818. *A person is considering to purchase a machine for his own factory. Relevant data about alternative machines are as follows :*

	Machine A	*Machine B*	*Machine C*
Present investment (Rs.)	10,000	12,000	15,000
Total annual cost (Rs.)	2,000	1,500	1,200
Life (years)	10	10	10
Salvage value (Rs.)	500	1,000	1,200

As an adviser to the buyer, you have been asked to select the best machine, considering 12% normal rate of return.

You are given that :

(a) Single payment present worth factor (pwf) at 12% interest for 10 years (= 0.322).

(b) Annual series present worth factor (Pwf) at 12% interest for 10 years (= 5.650).

Solution. The present value of total cost of each of the three machines for a period of ten years is computed in the following table :

Machine	*Present investment*	*Present value of total annual cost*	*Present value of salvage value*	*Net cost (Rs.)*
(1)	(2)	(3)	(4)	(5) = (2) + (3) − (4)
A	10,000	$2000 \times 5.65 = 11{,}300$	$500 \times 0.322 = 161.00$	21,139.00
B	12,000	$1500 \times 5.65 = 8{,}475$	$1000 \times 0.322 = 322.00$	**20,153.00**
C	15,000	$1200 \times 5.65 = 6{,}780$	$1200 \times 0.322 = 386.40$	21,393.60

From the information in the table, we observe that the present value of total cost for machine B is the least. Hence, machine B should be purchased.

1819. *The cost of a new machine is Rs. 5,000. The maintenance cost of nth year is given by $C_n = 500(n-1)$; $n = 1, 2, \ldots$ Suppose that the discount rate per year is 0.5. After how many years it will be economical to replace the machine by a new one?*

[Madras M.B.A. 1996; Meerut M.Sc. (Math.) 1996]

Solution. Since the discount rate of money per year is 0.05, the present worth of the money to be spent over a period of one year is

$$v = (1 + 0.05)^{-1} = 0.9523.$$

The optimum replacement time is determined in the following table :

Year (n)	R_{n-1}	v^{n-1}	$R_{n-1}v^{n-1}$	$C+\sum_k R_{k-1}v^{k-1}$	$\sum_k v^{k-1}$	$W(n)$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0	1.0000	0	5,000	1.0000	5,000
2	500	0.9523	476	6,476	1.9523	2,805
3	1,000	0.9070	907	6,383	2.8593	2,232
4	1,500	0.8638	1,296	7,679	3.7231	2,063
5	2,000	0.8227	1,645	9,324	4.5458	2,051*
6	2,500	0.7835	1,959	11,283	5.3293	2,117

Since, $W(n)$ is minimum for $n = 5$ and R_4 (= 1,500) < $W(5)$ as well as $W(5) > R_6$ (= 2,500); it is economical to replace the machine by a new one at the end of five years.

1820. *A manufacturer is offered two machines A and B. A is priced at Rs. 5,000, and running costs are estimated at Rs. 800 for each of the first five years, increasing by Rs. 200 per year in the sixth and subsequent years. Machine B, which has the same capacity as A, costs Rs. 2,500 but will have running costs of Rs. 1,200 per year for six years, increasing by Rs. 200 per year thereafter.*

If money is worth 10% per year, which machine should be purchased? (Assume that the machine will eventually be sold for scrap at a negligible price.)

[Annamalai M.B.A. 2009]

Solution. Since the money is worth 10% per year, the discount rate for both the machines is given by

$$v = \frac{1}{1 + 0.10} = 0.9091$$

For the solution of this problem, we compute the following tables for machines *A* and *B* separately, by using *Pwf* table given at the end of the book.

For Machine A

Year (n)	R_{n-1}	v^{n-1}	$v^{n-1}R_{n-1}$	$C+\sum_k v^{k-1}R_{k-1}$	$\sum_k v^{k-1}$	$W(n)$
(1)	(2)	(3)	(4)	(5)	(6)	(7) = (5)/(6)
1	800	1.0000	800	5,800	1.0000	5,800.00
2	800	0.9091	727	6,527	1.9091	3,418.88
3	800	0.8264	661	7,188	2.7355	2,627.67
4	800	0.7513	601	7,789	3.4868	2,233.85
5	800	0.6830	546	8,335	4.1698	1,998.89
6	1,000	0.6209	621	8,956	4.7907	1,869.45
7	1,200	0.5645	677	9,633	5.3552	1,798.81
8	1,400	0.5132	718	10,351	5.8684	1,763.85
9	1,600	0.4665	746	11,097	6.3349	1,751.72
10	1,800	0.4241	763	11,860	6.7590	1,754.70

For Machine B

Year (n)	R_{n-1}	v^{n-1}	$v^{n-1} R_{n-1}$	$C + \sum_k v^{k-1} R_{k-1}$	$\sum_k v^{k-1}$	$W(n)$
(1)	(2)	(3)	(4)	(5)	(6)	(7) = (5)/(6)
1	1,200	1.0000	1,200.00	3,700.00	1.0000	3,700.00
2	1,200	0.9091	1,090.91	4,790.91	1.9091	2,509.51
3	1,200	0.8264	991.98	5,782.59	2.7353	2,113.91
4	1,200	0.7513	901.56	6,684.15	3.4868	1,916.99
5	1,200	0.6830	819.60	7,503.75	4.1698	1,799.55
6	1,200	0.6209	745.08	8,248.83	4.7907	1,721.84
7	1,400	0.5645	790.30	9,039.13	5.3552	1,687.92
8	1,600	0.5132	821.12	9,860.25	5.8684	1,680.23
9	1,800	0.4665	839.70	10,699.95	6.3349	1,689.05
10	2,000	0.4241	848.20	11,548.15	6.7590	1,708.56

From the above tables we observe that for machine A, 1,600 < 1,751.72 < 1,800.

Now, since the running cost of 9th year is Rs. 1,600 and that of 10th year is Rs. 1,800 and since 1,800 > 1,751.72, it is better to replace the machine A after 9th year.

Similarly, for machine B since 1,800 > 1,680.23, it is better to replace the machine B after 8th year.

Further since the weighted average cost in 9 years of machine A is Rs. 1751.72 and the weighted average cost in 8 years of machine B is Rs. 1,680.23, it is advisable to purchase machine B.

PROBLEMS

1821. Let $v = 0.9$ and initial price is Rs. 5,000. Running cost varies as follows :

Year :	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Running cost (in Rs.) :	400	500	700	1,000	1,300	1,700	2,100

What would be the optimum replacement interval?

1822. The initial cost of an item is Rs. 15,000 and maintenance or running costs for different years are given below :

Year :	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Running cost (in Rs.) :	2,500	3,000	4,000	5,000	6,500	8,000	10,000

What is the replacement policy to be adopted if the capital is worth 10% and there is no salvage value?

1823. The yearly cost of 2 machines A and B when the money value is neglected is as follows :

Year :	*1*	*2*	*3*	*4*	*5*
Machine A :	1,800	1,200	1,400	1,600	1,000
Machine B :	2,800	200	1,400	1,100	600

Find their cost patterns if money value is 10% per year and hence find which machine is most economical.

[Madras B.E. (Mech.) 1999]

1824. A manual stamper currently valued at Rs. 1,000 is expected to last 2 years and costs Rs. 4,000 per year to operate. An automatic stamper which can be purchased for Rs. 3,000 will last 4 years and can be operated at an annual cost of Rs. 3,000. If money carries the rate of interest 10% per annum, determine which stamper should be purchased.

1825. A manufacturer is offered two machines A and B. A is priced at Rs. 5,000 and running costs are estimated at Rs. 800 for each of the first five years, increasing by Rs. 200 per year in the sixth and subsequent years. Machine B, which has the same capacity as A, costs Rs. 2,500 but will have running costs of Rs. 1,200 per year for six years, increasing by Rs. 200 per year thereafter.

If money is worth 10% per year, which machine should be purchased? (Assume that the machines will eventually be sold for scrap at a negligible price.) *[Madras B.E. 1999]*

1826. An engineering company is offered two types of material handling equipment A and B. A is priced at Rs. 60,000 including cost of installation, and the costs for operation and maintenance are estimated to be Rs. 10,000 for each of the first five years, increasing by Rs. 3,000 per year in the sixth and subsequent year. Equipment B with a rated capacity same as A, requires an initial investment of Rs. 30,000 but in terms of operation and maintenance costs more than A. These costs for B are estimated to be Rs. 13,000 per year for the first six years, increasing by Rs. 4,000 per year for each year from the 7th year onwards. The company expects a return of 10 per cent on all its investments. Neglecting the scrap value of the equipment at the end of its economic life, determine which equipment the company should buy.

1827. An individual is planning to purchase a car. A new car will cost Rs. 1,20,000. The resale value of the car at the end of the year is 85% of the previous year value. Maintenance and operation costs during the first year are Rs. 20,000 and they increase by 15% every year. The minimum resale value of car can be Rs. 40,000.

(*i*) When should the car be replaced to minimise average annual cost (ignore interest)?

(*ii*) If interest of 12% is assumed, when should the car be replaced? *[Kerala M.Com. 1990]*

18:3. REPLACEMENT OF EQUIPMENT THAT FAILS SUDDENLY

It is difficult to predict that a particular equipment will fail at a particular time. This difficulty can be overcome by determining the probability distribution of failures. Here it is assumed that the failures occur only at the end of the period, say t. Thus the objective becomes to find the value of t which minimizes the total cost involved for the replacement.

We shall consider the following two types of replacement policies :

Individual Replacement Policy. Under this policy, an item is replaced immediately after its failure.

Group Replacement Policy. Under this policy, we take decisions as to when all the items must be replaced, irrespective of the fact that items have failed or have not failed, with a provision that if any item fails before the optimal time, it may be individually replaced.

Mortality Tables. These are used to derive the probability distribution of the life span of an equipment. Let

$$M(t) = \text{number of survivors at any time } t,$$
$$M(t-1) = \text{number of survivors at any time } t-1, \text{ and}$$
$$N = \text{initial number of equipments}$$

Then the probability of failure during time period t is given by

$$p(t) = [M(t-1) - M(t)]/N$$

The probability that an equipment survived till age $(t-1)$, will fail during the interval $(t-1)$ to t can be defined as the *conditional probability* of failure. It is given by

$$p_c(t) = [M(t-1) - M(t)]/M(t-1)$$

The probability of survival till age t is given by

$$p_s(t) = M(t)/N.$$

Theorem 18-1 **(Mortality).** *A large population is subject to a given mortality law for a very long period of time. All deaths are immediately replaced by births and there are no other entries or exists. Then the age distribution ultimately becomes stable and that the number of deaths per unit time becomes constant (which is equal to the size of the total population divided by the mean age at death).*

Proof. Let k be a constant such that no item of the system can survive upto and beyond time $k+1$, *i.e.*, the life span of any item lies between $t = 0$ and $t = k$. We define

$f(t)$: the number of births at time t, and

$p(x)$: the probability that an equipment will die (fail) just before achieving the age $x+1$, *i.e.*, at age x.

It is easy to note that $\sum_{x=0}^{k} p(x) = 1$.

Now $f(t-x)$ births take place at time $t-x$, $t = k, k+1, \ldots$ Such newly born items attain the age x at time t. Therefore, the expected number of deaths of such alive numbers at time t is $p(x) f(t-x)$. Thus the mathematical expectation of the number of deaths before time $t+1$ is $\Sigma f(t-x)\, p(x)$, $t = k, k+1, \ldots$ Moreover, since deaths are immediately replaced by births, we must have

$$f(t+1) = \sum_{x=0}^{k} f(t-x)\, p(x), \qquad t = k, k+1, \ldots$$

The solution to this difference equation in t can be obtained by making use of

$$f(t) = A\alpha^t, \text{ where } A \text{ is some constant and } |\alpha| < 1.$$

$$\therefore \quad A\alpha^{t+1} = \sum_{x=0}^{k} A\alpha^{t-x} p(x) = A\,[\alpha^t p(0) + \alpha^{t-1} p(1) + \ldots + \alpha^{t-k} p(k)]$$

or

$$\alpha^{k+1} = \alpha^k \Big[\sum_{x=0}^{k} \alpha^{-x} p(x)\Big] = \alpha^k\,[p(0) + \alpha^{-1} p(1) + \ldots + \alpha^{-k} p(k)]$$

Thus

$$\alpha^{k+1} - [\alpha^k p(0) + \alpha^{k-1} p(1) + \ldots + p(k)] = 0.$$

This is a linear homogeneous difference equation of degree $k+1$ and thus has exactly $k+1$ roots. Let the roots be $\alpha_0, \alpha_1, \ldots, \alpha_k$.

For $\alpha = 1$, the equation yields

$$L.H.S. = 1 - \sum_{x=0}^{k} p(x) = 1 - 1 = 0 = R.H.S.$$

Thus, $\alpha = 1$ is a root of the above equation. Let us denote it by $\alpha_0 = 1$. The most general solution of the difference equation will be of the form

$$f(t) = A_0\alpha_0^t + A_1\alpha_1^t + \ldots + A_k\alpha_k^t$$
$$= A_0 + A_1\alpha_1^t + \ldots + A_k\alpha_k^t$$

where $A_0, A_1, \ldots, A_k$ are constants whose values are to be determined. We observe that, since $|\alpha_t| < 1$ as $t \to \infty$, $\lim_{t\to\infty} f(t) = A_0$. Thus, under our assumption for a long period t, the number of deaths per unit time is equal to A_0.

Now, the problem is to determine the value of the constant A_0.

Let $\quad g(x)$ = Probability of surviving for more than x years

or $\quad g(x) = 1 - P$ (survivor will die before attaining the age x)

$$= 1 - \{p(0) + p(1) + \ldots + p(x-1)\}$$

Obviously, it can be assumed that $g(0) = 1$.

Since, the number of births as well as deaths have become constant, each equal to A_0, therefore expected number of survivors of age x is given by $A_0 \cdot g(x)$.

As deaths are immediately replaced by births and therefore size N of the population remains constant. Thus, we must have

$$N = A_0 \sum_{x=0}^{k} g(x) \quad \text{or} \quad A_0 = N \Big/ \sum_{x=0}^{k} g(x).$$

The denominator represents the average age at death. This can also be proved as follows :

From finite differences, we know that

$$\Delta(x) = (x+1) - x = 1$$

and $$\sum_{x=a}^{b} f(x)\,\Delta h(x) = f(b+1)\,h(b+1) - f(a)\,h(a) - \sum_{x=a}^{b} h(x+1)\,\Delta f(x)$$

Therefore, we can write,

$$\sum_{x=0}^{k} g(x) = \sum_{x=0}^{k} g(x)\,\Delta(x) = \left[g(x)\cdot x \right]_0^{k+1} = (k+1)\,g(k+1) - 0\times g(0) - \sum_{x=0}^{k} (x+1)\,\Delta g(x).$$

$$= (k+1)\,g(k+1) - \sum_{x=0}^{k} (x+1)\,\Delta g(x).$$

But $g(k+1) = 1 - \{p(0) + p(1) + \ldots + p(k)\} = 0,$ since no one can survive for more than $(k+1)$ years of age.

and
$$\Delta g(x) = g(x+1) - g(x)$$
$$= [1 - p(0) - p(1) - \ldots - p(x)] - [1 - p(0) - \ldots - p(x-1)]$$
$$= -p(x).$$

$$\therefore \quad \sum_{x=0}^{k} g(x) = (k+1)\,g(k+1) - \sum_{x=0}^{k} (x+1)\,[-p(x)]$$

$$= \sum_{x=0}^{k} (x+1)\,p(x)\,; \text{ since } g(k+1) = 0.$$

This happens to be the mean (expected age at death).

Hence, $$A_0 = N\,/\text{Mean age at death.}$$

***Theorem 18-2* (Group Replacement).** *Let all the items in a system be replaced after a time interval 't' with provisions that individual replacements can be made if and when any item fails during this time period. Then*

(a) Group replacement must be made at the end of t^{th} period if the cost of individual replacement for the period is greater than the average cost per unit time period through the end of t periods.

(b) Group replacement is not advisable at the end of period t if the cost of individual replacements at the end of period $t-1$ is less than the average cost per unit period through the end of period t.

Proof. Let,

N = total number of items in the system,
C_2 = cost of replacing an individual item,
C_1 = cost of replacing an item in group,
$C(t)$ = total cost of group replacement after time period t,
$f(t)$ = number of failures during time period t.

Then, clearly

$$C(t) = NC_1 + C_2 \sum_{x=1}^{t-1} f(x)$$

The average cost of group replacement per unit period of time during a period t, is thus given by

$$A(t) = \frac{C(t)}{t} = \left[NC_1 + C_2 \sum_{x=1}^{t-1} f(x) \right] \Big/ t.$$

We shall determine the optimum t so as to minimize $C(t)/t$.

Note that whenever $\frac{C(t-1)}{t-1} > \frac{C(t)}{t}$ and $\frac{C(t+1)}{t+1} > \frac{C(t)}{t}$, it is better to replace all the items after time period t.

Now,

$$\frac{C(t+1)}{t+1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t) > C(t)/t\,;$$

and
$$\frac{C(t-1)}{t-1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t-1) < C(t)/t$$

$\therefore$
$$t \cdot C_2 f(t-1) < C(t) < t \cdot C_2 f(t).$$

or
$$t f(t-1) - \sum_{x=1}^{t-1} f(x) < \frac{NC_1}{C_2} < t f(t) - \sum_{x=1}^{t-1} f(x)$$

SAMPLE PROBLEMS

1828. *The following failure rates have been observed for a certain type of transistors in a digital computer :*

End of the week :	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Probability of failure to date :	0.05	0.13	0.25	0.43	0.68	0.88	0.96	1.00

The cost of replacing an individual failed transistor is Rs. 1.25. The decision is made to replace all these transistors simultaneously at fixed intervals, and to replace the individual transistors as they fail in service. If the cost of group replacement is 30 paise per transistor, what is the best interval between group replacements? At what group replacement price per transistor would a policy of strictly individual replacement become preferable to the adopted policy? **[Guwahati MCA 1992]**

Solution. Suppose there are 1,000 transistors in use. Let p_i be the probability that a transistor, which was new when placed in position for use, fails during the ith week of its life. Thus, we have

$$p_1 = 0.05, \qquad p_2 = 0.13 - 0.5 = 0.08,$$
$$p_3 = 0.25 - 0.13 = 0.12, \qquad p_4 = 0.43 - 0.25 = 0.18,$$
$$p_5 = 0.68 - 0.43 = 0.25, \qquad p_6 = 0.88 - 0.68 = 0.20,$$
$$p_7 = 0.96 - 0.88 = 0.08, \qquad p_8 = 1.00 - 0.96 = 0.04.$$

Let N_i denote the number of replacements made at the end of the ith week. Then, we have

N_o = number of transistors in the beginning $= 1{,}000$
$$N_1 = N_o p_1 = 1{,}000 \times 0.05 = 50$$
$$N_2 = N_o p_2 + N_1 p_1 = 1{,}000 \times 0.08 + 50 \times 0.05 = 82$$
$$N_3 = N_o p_3 + N_1 p_2 + N_2 p_1 = 1{,}000 \times 0.12 + 50 \times 0.08 + 82 \times 0.05 = 128$$
$$N_4 = N_o p_4 + N_1 p_3 + N_2 p_2 + N_3 p_1 = 199$$
$$N_5 = N_o p_5 + N_1 p_4 + N_2 p_3 + N_3 p_2 + N_4 p_1 = 289$$
$$N_6 = N_o p_6 + N_1 p_5 + N_2 p_4 + N_3 p_3 + N_4 p_2 + N_5 p_1 = 272$$
$$N_7 = N_o p_7 + N_1 p_6 + N_2 p_5 + N_3 p_4 + N_4 p_3 + N_5 p_2 + N_6 p_1 = 194$$
$$N_8 = N_o p_8 + N_1 p_7 + N_2 p_6 + N_3 p_5 + N_4 p_4 + N_5 p_3 + N_6 p_2 + N_7 p_1 = 195$$

From the above calculations, we observe that the expected number of transistors failing each week increases till 5th week and then starts decreasing and later again increasing from 8th week.

Thus, N_i will oscillate till the system acquires a steady state. The expected life of each transistor is
$$1 \times 0.5 + 2 \times .08 + 3 \times 0.12 + 4 \times 0.18 + 5 \times 0.25 + 6 \times 0.2 + 7 \times 0.08 + 8 \times 0.04$$
$$= 4.62 \text{ weeks.}$$

Average number of failures per week
$$= 1{,}000/4.62 = 216 \text{ approximately.}$$

Therefore, the cost of individual replacement
$$= 216 \times 1.25 = \text{Rs. } 270.00 \text{ per week.}$$

Now, since the replacement of all the 1,000 transistors simultaneously cost 30 paise per transistors and the replacement of an individual transistor on failure cost Rs. 1.25, the average cost for different group replacement policies is given as under :

End of week	Individual replacement	Total cost (Rs.) Individual + Group	Average cost (Rs.)
1	50	$50 \times 1.25 + 1{,}000 \times 0.30 = 363$	363
2	132	$132 \times 1.25 + 1{,}000 \times 0.30 = 465$	232.50
3	260	$260 \times 1.25 + 1{,}000 \times 0.30 = 625$	208.30
4	459	$459 \times 1.25 + 1{,}000 \times 0.30 = 874$	218.50

Since, the average cost is lowest against week 3, the optimum interval between group replacements is 3 weeks. Further, since the average cost is less than Rs. 270 (for individual replacement), the policy of group replacement is better.

1829. *A computer has a large number of electronic tubes. They are subject to mortality as given below :*

Period	*Age of failure (hours)*	*Probability of failure*
1	0—200	0.10
2	201—400	0.26
3	401—600	0.35
4	601—800	0.22
5	801—1000	0.07

If the tubes are group replaced, the cost of replacement is Rs. 15 per tube. Group replacement can be done at fixed intervals in the night shift when the computer is not normally used. Replacement of individual tubes which fail in service costs Rs. 60 per tube. How frequently should the tubes be replaced?

Solution. Consider each block of 200 hours as one period and assume that there are 1000 tubes initially.

Let N_i be the number of replacements made at the end of the ith period, if all the 1000 tubes are new initially. Then the expected number of failures at different weeks can be calculated as shown below :

$$N_1 = N_o p_1 = 1000 \times 0.10 = 100$$
$$N_2 = N_o p_2 + N_1 p_1 = 1000 \times 0.26 + 100 \times 0.10 = 270$$
$$N_3 = N_o p_3 + N_1 p_2 + N_2 p_1 = 1000 \times 0.35 + 100 \times 0.26 + 270 \times 0.10 = 403$$
$$N_4 = N_o p_4 + N_1 p_3 + N_2 p_2 + N_3 p_1$$
$$= 1000 \times 0.22 + 100 \times 0.35 + 270 \times 0.26 + 403 \times 0.10 = 365$$
$$N_5 = N_o p_5 + N_1 p_4 + N_2 p_3 + N_3 p_2 + N_4 p_1$$
$$= 1000 \times 0.07 + 100 \times 0.22 + 270 \times 0.35 + 403 \times 0.26 + 365 \times 0.10 = 328$$

From the above calculations, we observe that the number of tubes failing in each period increases till the third period, and then starts decreasing. Thus the value of N_i will oscillate till the system settles down to a steady state. In the steady state, the proportion of tubes failing during each period is the reciprocal of their average life.

Expected life of a tube $= 1 \times 0.10 + 2 \times 0.26 + 3 \times 0.35 + 4 \times 0.22 + 5 \times 0.07 = 2.90$ periods.

Expected number of failures per period $= \dfrac{1000}{2.90} = 345$

$\therefore$ Cost of individual replacements per period $= 345 \times 60 =$ Rs. 20,700.

Now, since the replacement of all the 1,000 tubes simultaneously cost Rs. 15 per tube, and the replacement of an individual tube on failure cost Rs. 60; the average cost for different group replacement policies is given as follows :

End of period (hours)	Individual replacement	Total cost (Rs.) Individual + Group	Average cost (Rs.)
0—200	100	$100 \times 60 + 1000 \times 15 = 21{,}000$	21,000
201—400	370	$370 \times 60 + 1000 \times 15 = 37{,}200$	18,600
401—600	773	$773 \times 60 + 1000 \times 15 = 61{,}380$	20,460
601—800	1138	$1138 \times 60 + 1000 \times 15 = 83{,}280$	20,820

Since, the average cost is lowest against period 2, the optimum interval between group replacements is two periods, *i.e.*, after 400 hours. Further, since the average cost of group replacement (which is Rs. 18,600 is less Rs. 20,700 (cost of individual replacement), the policy of group replacement is better.

1830. *At time zero all items in a system are new. Each item has a probability p of failing immediately before the end of the first month of life, and a probability q = (1 − p) of failing immediately before the end of the second month (i.e., all items fail by the end of the second month). If all items are replaced as they fail, show that the expected number of failures f(x) at the end of month x is given by*

$$f(x) = \frac{N}{1+q}\,[1 - (-q)^{x+1}].$$

where N is the number of items in the system.

If the cost per item of individual replacement is C_1, and the cost per item of group replacement is C_2, find the condition under which

(*a*) *A group replacement policy at the end of each month is the most profitable.*

(*b*) *No group replacement policy is better than a policy of pure individual replacement.*

[Delhi B.Sc. (Stat.) 1994]

Solution. Let

N = number of items in the system in the beginning

N_1 = number of items expected to fail at the end of 1st month

$= N_0 p = N(1-q)$, since $p = 1-q$.

N_2 = number of items expected to fail at the end of 2nd month

$= N_0 q + N_1 p = Nq + N(1-q)^2 = N(1-q+q^2)$,

N_3 = number of items expected to fail at the end of 3rd month

$= N_1 q + N_2 p = N(1-q)\,q + N(1-q+q^2)(1-q) = N(1-q+q^2+q^3)$,

and so on. In general,

$$N_k = N[1 - q + q^2 - q^3 + \ldots + (-q)^k].$$

$$\therefore \quad N_{k+1} = N_{k-1} q + N_k p$$

$$= N[1 - q + q^2 + \ldots + (-q)^{k-1}]\,q + N[1 - q + q^2 + \ldots + (-q)^k]\,(1-q)$$

$$= N[1 - q + q^2 + \ldots + (-q)^{k+1}]$$

Hence, by mathematical induction, the expected number of failures at the end of month x will be given by

$$f(x) = N[1 - q + q^2 + \ldots + (-q)^x] = N[1 - (-q)^{x+1}]/(1+q).$$

The value of $f(x)$ at the end of month x will vary for different values of $(-q)^{x+1}$ and it will reach the steady-state as $x \to \infty$.

Hence, in the steady state case, the expected number of failures will be

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} N[1 - (-q)^{x+1}]/(1+q)$$

$= N/(1+q)$; since $q < 1$ and $(-q)^{x+1} \to 0$ as $x \to \infty$.

= Total number of items in the system / Mean age, Now, since C_1 is the cost of replacement per item individually and C_2 is the cost of an item in group, therefore

(*i*) If we have a group replacement at the end of each month, then the cost of replacement is NC_2.

(*ii*) If we have a group replacement policy at the end of every other month, then the cost is $NC_2 + NpC_1$.

The average cost per month, therefore, is $(NC_2 + NpC_1)/2$, and

Average life of an item $= 1 \times p + 2 \times q = 1 \times (1-q) + 2q = 1 + q$.

Therefore, the average number of failures is $N/(1 + q)$ and hence the cost of individual replacement is $NC_1/(1+q)$.

(*a*) A group replacement at the end of first month will be better than individual replacement, if total cost of group replacement is less than the average monthly cost of individual replacement.

Thus, $$N(1-q)C_1 + NC_2 < NC_1/(1+q), \text{ i.e., } C_2 < c_1q^2/(1+q).$$

For a group replacement at the end of every second month, the total cost of replacement will be

$$(N_1 + N_2)C_1 + NC_2 = N(2 - 2q + q^2)C_1 + NC_2.$$

$\therefore$ Average monthly cost of group replacement at the end of second month is

$$[N(2 - 2q + q^2)C_1 + NC_2]/2$$

In this case, the group replacement policy will be better than the individual replacement policy, if

$$\text{Average monthly cost of group replacement} < \text{Average monthly cost of individual replacement}$$

or $$[N(2 - 2q + q^2)C_1 + NC_2]/2 < NC_1/(1+q),$$

or $$C_2 < q^2(1-q)C_1/(1+q).$$

(*b*) For the individual replacement policy to be better than any of the group replacement policies discussed above, we must have

$$C_2 > C_1q^2/(q+1) \quad \text{and} \quad C_2 > C_1q^2(1-q)/(q+1)$$

or $$C_1 < C_2(1+q)/q^2 \quad \text{and} \quad C_1 < C_2(1+q)/[q^2(1-q)]$$

But $q < 1$, therefore $(1+q)/q^2 < (1+q)/q^2(1-q)$

Hence, $C_1 < (1+q)C_2/q^2$.

PROBLEMS

1831. In a machine shop, a particular cutting tool costs Rs. 6 to replace. If a tool breaks on the job, the production disruption and associate costs amount to Rs. 30. The past life of a tool is given as follows :

Job No.	1	2	3	4	5	6	7
Proportion of broken tools on job	0.01	0.03	0.09	0.13	0.25	0.55	0.95

After how many jobs, should the shop replace a tool before it breaks down? *[IAS 1989]*

1832. A group of process plants in an oil refinery are fitted with valves. Over a period of time, the failure pattern of these 400 valves has been observed and it is as follows :

Month	1	2	3	4	5	6	7	8	*Total*
Number of valves failing	8	20	48	104	120	56	32	12	400

It costs Rs. 100 to replace each valve individually. If all the valves are replaced at a time, it costs Rs. 50 per valve.

The maintenance department is considering following replacement policies :

(*a*) Replace all valves simultaneously at fixed intervals, in addition to replacing valves as and when they fail.

(*b*) Replace valves as and when they fail.

Suggest the optimal replacement policy. *[Bangalore M.B.A. (June) 1998]* from the production line before replacing a sprinkler at the end of the shift or on weekends. Due to the high setup cost for this task, management is considering a policy of group replacement. What replacement policy should the management adopt when the information regarding sprinkler breakdowns and cost is as given below :

Run Time (months)	:	*1*	*2*	*3*	*4*	*5*	*6*
Probability of failure	:	0.05	0.05	0.10	0.10	0.30	0.40

REPLACEMENT COST

	Purchase	*Installation*	*Per unit*
Individual	Rs. 50	Rs. 175	Rs. 225
Group	Rs. 50	Rs. 25	Rs. 75

[Delhi M.B.A. (Nov.) 1998]

18:4. RECRUITMENT AND PROMOTION PROBLEM

Like industrial replacement problems principles of replacement can also be applied to the problems of recruitment and promotion of staff. For this we assume that life distribution of the service of staff in the organization is already known.

SAMPLE PROBLEM

1840. *An airline requires 200 assistant hostesses, 300 hostesses, and 50 supervisors. Girls are recruited at age 21 and if still in service, retire at the age of 60. Given the following Life Table, determine (i) How many girls should be recruited each year? (ii) At what age promotion should take place?*

Life Table for Airline Hostesses

Age	*21*	22	23	24	25	26	27	28
No. in service	1,000	600	480	384	307	261	228	206
Age	29	*30*	*31*	*32*	*33*	*34*	*35*	*36*
No. in Service	190	181	173	167	161	155	150	146
Age	*37*	*38*	*39*	*40*	*41*	*42*	*43*	*44*
No. in service	141	136	131	125	119	113	106	99
Age	*45*	*46*	*47*	*48*	*49*	*50*	*51*	*52*
No. in service	93	87	80	73	66	59	53	46
Age	*53*	*54*	*55*	*56*	*57*	*58*	*59*	—
No. in service	39	33	27	22	18	14	11	—

Solution. The total number of girls recruited at age 21 and those serving up to the age of 59 will be equal to 6,480. We require 200 + 300 + 50 = 550 girls in all in the airline.

The recruitment every year is 1,000 when total number of girls is 6,480 after 59 years. Therefore, in order to maintain a strength of 550 hostesses we should recruit

$$550 \times 1{,}000/6{,}480 = 85 \text{ (nearly) every year}$$

If we want to promote the assistant hostesses at the age x, then up to age $x-1$ we need 200 assistant hostesses. Among 550, there are 200 assistant hostesses. Therefore, out of a strength of 1,000 there will be

$$200 \times 1{,}000/550 = 364 \text{ assistant hostesses,}$$

and from the life Table this number is available up to the age of 24 years. Hence the promotion of assistant hostesses will be due in 25th year.

Also, out of 550 recruitments, we need only 300 hostesses.

Therefore, if we recruit 1,000 girls, then we shall require

$$1{,}000 \times 300/550 = 545 \text{ hostesses.}$$

Hence, the number of hostesses and assistant hostesses in a recruitment of 1,000 will be

$$364 + 545 = 909.$$

So, we shall need only 1,000 − 909 = 91 supervisors, whereas at the age of 46 only 86 will survive. Hence promotion of hostesses to supervisors will be due in 46th year.

PROBLEMS

1841. Calculate the probability of a staff resignation in each year from the following survival table :

Year	0	1	2	3	4	5	6	7	8	9	10
Staff at year end	1,000	940	820	580	400	280	190	130	70	30	0

1842. A research team is planned to raise to a strength of 50 chemists and then to remain at that level. The number of recruits depends on their length of service which is as follows :

Year	1	2	3	4	5	6	7	8	9	10
Percentage left at year ends	5	36	56	63	68	73	79	87	97	100

What is the recruitment per year necessary to maintain the required strength? There are 8 senior posts for which the length of service is the main criterion. What is the average length of service after which new entrant expects his promotion to one of these posts?
[Bharthidasan B.Com. 1999]

1843. A public sector bank requires 400 clerks, 250 officers, and 50 managers for a certain state. The employees are recruited at the age of 21 years as clerks. The promotions take place only from within on the basis of the length of service. You are given the following life table :

Age (years)	21	22	23	24	25	26	27	28	29	30
No. of employees in service	1,000	640	560	450	400	355	300	205	190	182
Age (years)	31	32	33	34	35	36	37	38	39	40
No. of employees in service	175	170	162	155	150	148	145	142	140	132
Age (years)	41	42	43	44	45	46	47	48	49	50
No. of employees in service	125	118	112	105	101	100	95	90	88	85
Age (years)	51	52	53	54	55	56	57	58	59	60
No. of employees in service	80	72	70	63	55	40	32	25	20	0

Determine :

(*a*) how many employees should be recruited each year?
(*b*) at what age should the promotions take place?

18:5. EQUIPMENT RENEWAL PROBLEM

Knowledge of probability distribution of failure time of certain equipments enables us to solve many practical problems of replacement. In statistical terminology, by the term renewal we mean either to insert a new equipment in place of an old one or to repair the old equipment, so that the *p.d.f.* of its future life-time is that of a new equipment. Thus the problem of replacement can be dealt with what is called the *Renewal Theory*, where the life span of an equipment is considered to be a random variable.

The probability that a renewal occurs during the small time interval $(t, t+dt)$ is called the renewal rate at time t, where t is measured from the instant the first equipment was started. It is generally denoted by $h(t)\,dt$ and $h(t)$ is called the *renewal density function.*

Theorem 18-3. *The renewal rate of a machine is asymptotically reciprocal of the mean life of the machine.*

Proof. Let the *p.d.f.* of failure-time of a machine be $f(x)$. Also let the life-time of all the items in the system follow the same probability distribution, say $f(x)$. that is, if X_i be the life-span for ith machine $(i = 1, 2, \ldots)$ then $X_1, X_2, \ldots$ each has $f(x)$ as if its *p.d.f.*

Let the machine fail $(n-1)$ times during the period $(0, t)$, and at each failure an immediate replacement being made with a similar machine. Then, if at the end of this period, n the machine is in service, we have

$$X_1 + X_2 + \ldots + X_{n-1} < t \quad \text{and} \quad X_1 + X_2 + \ldots + X_n > t.$$

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18BMA46S : Unit: IV: Replacement Problem
[18 : 18.1 to 18.4]

(i) Introduction
(ii) Replacement of equipment/asset that deteriorates gradually
(iii) Replacement of equipment that fails suddenly

(iv) Recruitment and Promotion problem.