Loan Repayment Calculations

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1 Definitions

1.1 Notation

A loan is a fixed value of money borrowed by an entity and usually repaid over a series of instalments. The following notation is used for each of the components of the loan:

- L: Loan amount (decimal).
- R: Periodic interest rate (decimal).
- N: Total number of repayments (integer).
- P: Total periodic repayment value (decimal). If this changes over time, the value at period n is denoted P_n.
- b: Whether the interest is applied before or after the repayment (boolean).
- B_n: The balance on the loan at period n (decimal). Note that $B_0 = L$.

The periodic repayment for a loan usually has (at least) 2 components:

- $P_{P,n}$: The principal part of the periodic repayment value at period n, which is paying off the original money that was borrowed.
- $P_{I,n}$: The interest part of the periodic repayment value at period n, which is paying off the interest applied on the loan.

In 'real life', a loan can have other components such as fees. These are outside the scope of these calculations.

1.2 Helpful Formulae

The finite sum of a geometric series:

$$\sum_{i=0}^{n-1} a^i = \frac{a_n - 1}{a - 1}$$

(See `https://en.wikipedia.org/wiki/Geometric_series#Finite_series`)

1.3 Loan Types

There are 3 types of loans discussed here:

1. Fixed Repayment Loans

2. Fixed Principal Loans

3. Interest Only Loans

2 Fixed Repayment Loan

2.1 Balance at a period

The formula for B_{k+1} for some k is:

$$\begin{aligned} B_{k+1} &= \begin{cases} B_kR - P + B_k & \text{if interest is applied before, } b = 0 \\ (B_k - P)R + B_k & \text{if interest is applied after, } b = 1 \end{cases} \\ &= B_kR - PR^b + B_k \\ &= B_k(R+1) - PR^b \end{aligned}$$

By definition, we know that $B_0 = L$. Therefore, we can expand the above formula and express it in terms of $B_0 = L$:

$$\begin{aligned} B_{k+1} &= B_k(R+1) - PR^b \\ &= \left[B_{k-1}(R+1) - PR^b\right](R+1) - PR^b \\ &= B_{k-1}(R+1)^2 - PR^b(R+1) - PR^b \\ &= \left[B_{k-2}(R+1) - PR^b\right](R+1)^2 - PR^b(R+1) - PR^b \\ &= B_{k-2}(R+1)^3 - PR^b(R+1)^2 - PR^b(R+1) - PR^b \end{aligned}$$

Continuing this iteratively leads to the expression

$$B_{k-i}(R+1)^{i+1} - PR^b\left[(R+1)^i + (R+1)^{i-1} + ... + (R+1)^1 + (R+1)^0\right]$$

so that

$$B_{k+1} = B_{k-i}(R+1)^{i+1} - PR^b\sum_{x=0}^{i}(R+1)^x$$

Let $i = k$ and then $k = n + 1$ so that:

$$\begin{aligned} B_{k+1} &= B_0(R+1)^{k+1} - PR^b\sum_{x=0}^{k}(R+1)^x \\ B_n &= B_0(R+1)^n - PR^b\sum_{x=0}^{n-1}(R+1)^x \end{aligned}$$

Using the formula for the sum of a geometric series, the summation can be replaced with its corresponding quotient:

$$\begin{aligned} B_n &= B_0(R+1)^n - PR^b\frac{(R+1)^n - 1}{R+1-1} \\ &= B_0(R+1)^n - PR^{b-1}\left[(R+1)^n - 1\right] \\ &= B_0(R+1)^n - PR^{b-1}(R+1)^n + PR^{b-1} \\ &= (R+1)^n\left[B_0 - PR^{b-1}\right] + PR^{b-1} \end{aligned}$$

Expressing this using L instead of B_0 looks like either of the following:

$$B_n = L(R+1)^n - PR^{b-1}\left[(R+1)^n - 1\right]$$
$$B_n = (R+1)^n\left[L - PR^{b-1}\right] + PR^{b-1}$$

2.2 Periodic Repayment Amount

To find P, we can solve for $B_N = 0$ which corresponds to the balance on the load being 0 at the completion of its term.

$$\begin{aligned} B_N = 0 &\implies L(R+1)^N - PR^{b-1}\left[(R+1)^N - 1\right] = 0 \\ &\implies L(R+1)^N = PR^{b-1}\left[(R+1)^N - 1\right] \\ &\implies \frac{L(R+1)^N}{(R+1)^N - 1} = PR^{b-1} \\ &\implies P = \frac{LR^{1-b}(R+1)^N}{(R+1)^N - 1} \end{aligned}$$

2.3 Loan Amount

To find L, we can re-arrange the formula for P:

$$P = \frac{LR^{1-b}(R+1)^N}{(R+1)^N - 1} \implies L = \frac{PR^{b-1}\left[(R+1)^N - 1\right]}{(R+1)^N}$$

2.4 Loan Term

To get N, we can re-arrange the formula for either P or L:

$$\begin{aligned} &L(R+1)^N = PR^{b-1}\left[(R+1)^N - 1\right] \\ \implies &L(R+1)^N = PR^{b-1}(R+1)^N - PR^{b-1}] \\ \implies &PR^{b-1} = PR^{b-1}(R+1)^N - L(R+1)^N \\ \implies &PR^{b-1} = \left[PR^{b-1} - L\right](R+1)^N \\ \implies &(R+1)^N = \frac{PR^{b-1}}{PR^{b-1} - L} \\ \implies &(R+1)^N = \frac{1}{1 - LP^{-1}R^{1-b}} \\ \implies &(R+1)^N = (1 - LP^{-1}R^{1-b})^{-1} \\ \implies &N\ln(R+1) = -\ln(1 - LP^{-1}R^{1-b}) \\ \implies &N = -\frac{\ln(1 - LP^{-1}R^{1-b})}{\ln(R+1)} \end{aligned}$$

Note that we need:

$$P - LR^{1-b} > 0 \implies P > LR^{1-b}$$

Rather than using the natural log, ln, we can express this using an explicit base:

$$
\begin{aligned}
N &= -\frac{\ln\left(1 - LP^{-1}R^{1-b}\right)}{\ln\left(R+1\right)} \\
\implies N &= -\frac{\ln\frac{P-LR^{1-b}}{P}}{\ln\left(R+1\right)} \\
\implies N &= \frac{\ln\frac{P}{P-LR^{1-b}}}{\ln\left(R+1\right)} \\
\implies N &= \log_{R+1}\left(\frac{P}{P-LR^{1-b}}\right)
\end{aligned}
$$

2.5 Periodic Interest Rate

We don't have a formula for R yet.

3 Fixed Principal Loan

3.1 Balance at a period

Recall that $P = P_P + P_I$. Extend the subscript notion so that:

- P_n is the total payment made for period n.
- $P_{P,n}$ is the principal payment made for period n.
- $P_{I,n}$ is the interest payment made for period n.

For a fixed principal loan, the value $P_{P,n} = P_P$ will be fixed for all n but $P_{I,n}$ will vary. In particular:

$$P_{I,n} = RB_{n-1} \implies P_n = P_P + RB_{n-1}$$

We saw above that the formula for B_n is:

$$B_n = L(R+1)^n - PR^{b-1}\left[(R+1)^n - 1\right]$$

Combining the two formulae above, we see that:

$$\begin{aligned} P_{I,n} &= RB_{n-1} \\ &= R\left[L(R+1)^{n-1} - PR^{b-1}\left[(R+1)^{n-1} - 1\right]\right] \\ &= LR(R+1)^{n-1} - PR^{b}\left[(R+1)^{n-1} - 1\right] \end{aligned}$$

Additionally, it follows that:

$$\begin{aligned} P_n &= P_P + RB_{n-1} \\ &= P_P + LR(R+1)^{n-1} - PR^{b}\left[(R+1)^{n-1} - 1\right] \end{aligned}$$

4 From Confluence

P: the constant regular repayment amount each period

R: the interest rate per period

S: the starting loan amount

B_n: the balance on the loan at period n, with $B_0 = S$

4.1 Repayment Amount

$$P = \frac{SR(R+1)^n}{[(R+1)^n - 1]}$$

4.2 Repayment Length

4.2.1 Log Form

$$n = \log_{R+1} \frac{P}{P - SR}$$

4.2.2 Natural Log Form

$$n = \frac{\ln P/(P - SR)}{\ln(R+1)}$$

4.3 Balance at Period n

$$B_n = S(R+1)^n - \frac{P((R+1)^n - 1)}{R}$$

4.4 Sum of Interest by Period n

$$I_n = PR + \frac{(SR - P)((R+1)^n - 1)}{R}$$