# Handout

## Finite Difference Methods

MIE 1624H October 3, 2023

### Finite difference methods:

Finite difference methods are based on the idea of approximating each partial derivative by a difference quotient. Taylor series expansion states that a function $f(x)$ may be represented as

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x)$$

If we neglect the term of order $h^2$, we get

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

This is the *forward* approximation for the derivative; the derivative is defined as a limit of the difference quotient above as $h \to 0$. There are alternative ways to approximate first-order derivatives. By similar reasoning, we may write

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x)$$

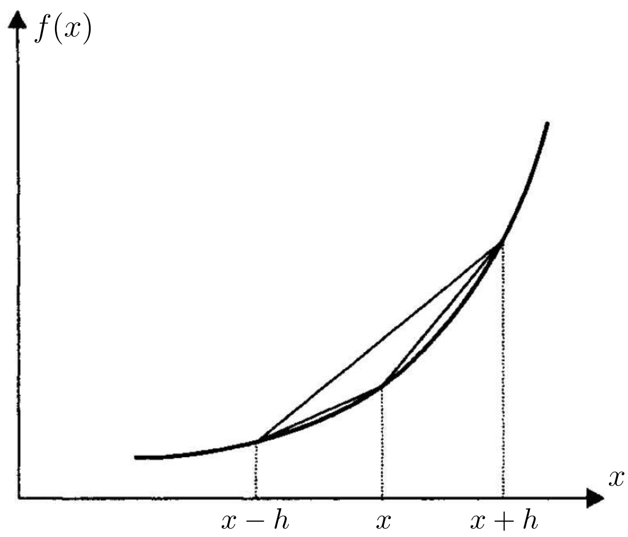
from which we obtain the *backward* approximation,

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

In both cases we get a truncation error of order $O(h)$ (why?). A better approximation can be obtained by subtracting expression for $f(x-h)$ from expression for $f(x+h)$ and rearranging:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}.$$

This is the *central* or *symmetric* approximation, and for small $h$ it is a better approximation, since the truncation error is $O(h^2)$ (why?). Figure below illustrates the computations.




The reasoning may be extended to second-order derivatives. This is obtained by adding expression for $f(x-h)$ and $f(x+h)$, which yields

$$f(x+h)+f(x-h)=2f(x)+h^2f''(x)$$

and rearranging yields

$$f''(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2}.$$

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**Convex functions:**

**Definition** A function $f$ is convex if for any $\boldsymbol{x}^1,\ \boldsymbol{x}^2 \in C$ and $0 \le \lambda \le 1$

$$f(\lambda\boldsymbol{x}^1+(1-\lambda)\boldsymbol{x}^2) \le \lambda f(\boldsymbol{x}^1)+(1-\lambda)f(\boldsymbol{x}^2).$$

Hessian $\nabla f^2(\boldsymbol{x})$ is PD $\Longrightarrow$ strictly convex function.

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For a convex function of one variable it holds that $f''(x) \ge 0$.

From the finite differences approximation of the second derivative and $h>0$, expression

$$f''(x)=\frac{f(x+h)-2f(x)+f(x-h)}{h^2} \ge 0$$

can be re-arranged as

$$f(x) \le \frac{f(x+h)+f(x-h)}{2},$$

which is equivalent to the definition of a convex function.