Multivariable Control

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António Paulo Moreira[1], Paulo G. Costa[2], Ricardo B. Sousa[3]

[1]Associated Professor of DEEC@FEUP

[2]Assistant Professor of DEEC@FEUP

[3]Ph.D. Student of PDEEC@FEUP

Outline

- Dynamics
- Multivariable Control
- SCARA Manipulator: Practical application of inverse dynamics

Dynamics

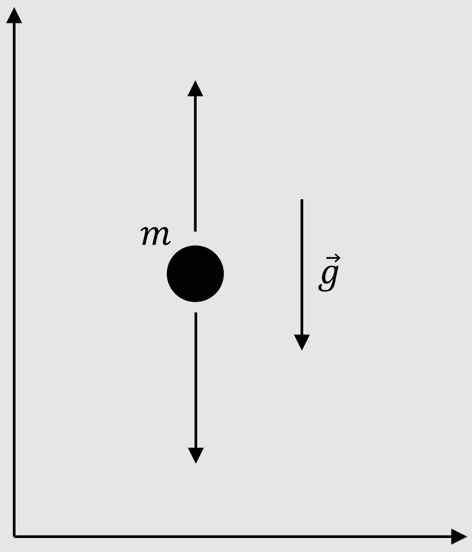
- Kinematic equations
  - Motion of the manipulator
  - No consideration for the forces and torques producing the motion

- Dynamics equations
  - Motion of the manipulator
  - Relationship between forces and motion

- **How do we consider the dynamics?** Euler-Lagrange equations
  - Evolution of a mechanical system subject to holonomic constraints
  - Based on the principle of virtual work
  - Requires the *Lagrangian* of the system: difference between kinetic and potential energy

One Dimensional System

- Equation of motion: $m\ddot{y} = f - mg$
  - Kinetic energy $K = \frac{1}{2}m\dot{y}^2$
  - $m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) = \frac{d}{dt}\frac{\partial K}{\partial \dot{y}}$
  - Potential energy $P = mgy$
  - $mg = \frac{\partial}{\partial y}(mgy) = \frac{\partial P}{\partial y}$

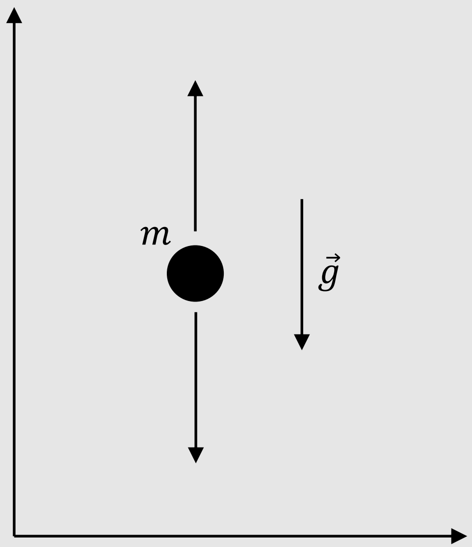


One Dimensional System

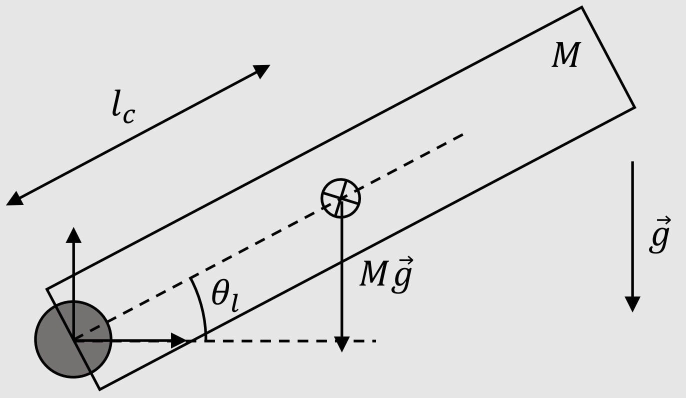
- Equation of motion: $m\ddot{y} = f - mg$

- Lagrangian $\mathcal{L} = K - P = \frac{1}{2}m\dot{y}^2 - mgy$
  - Where $\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial K}{\partial \dot{y}}$ and $\frac{\partial \mathcal{L}}{\partial y} = -\frac{\partial P}{\partial y}$

- Euler-Lagrange Equation: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = m\ddot{y} + mg = f$
  - **Goal:** describe the motion from the external force (equivalent to the Newton's 2nd law)
  - f: external force
  - Analysis in terms of kinetic and potential energy



Single Link Manipulator



- $\theta_m = r\theta_l$ (gear ratio r: 1)
- $K = \frac{1}{2}J_m\dot{\theta}_m^2 + \frac{1}{2}J_l\dot{\theta}_l^2 = \frac{1}{2}(r^2J_m + J_l)\dot{\theta}_l^2 = \frac{1}{2}J\dot{\theta}_l^2$
  - J_m : rotational inertia of the motor
  - J_l : rotational inertia of the link
- $P = Mgl_c \sin\theta_l$
- $\mathcal{L} = K - P = \frac{1}{2}J\dot{\theta}_l^2 - Mgl_c \sin\theta_l$
- $J\ddot{\theta}_l + Mgl_c \cos\theta_l = \tau_l$
  - τ_l: generalized force that represents the external forces and torques
  - $\tau_l = u - B\dot{\theta}_l$, where u is the motor torque relative to the link ($u = r\tau_m$)

General Case

- $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i$, where $i = 1, \dots, n$

- System described by the so-called *generalized coordinates*
  - n Denavit-Hartenberg joint variables (DH convention)
  - DH convention are equivalent to a set of generalized coordinates

- Require general expressions for kinetic (K) and potential (P) energy

General Expressions for Kinetic Energy K

- Concentrate the entire mass m of an object at its center of mass

- $K = K_{trans.} + K_{rot.} = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \mathfrak{I} \omega$
  - m: total mass of the object
  - v : linear velocity vector
  - $\mathfrak{I}$: symmetric 3×3 matrix called the Inertia Tensor (expressed in the inertial frame)
  - ω : angular velocity vector

Inertia Tensor

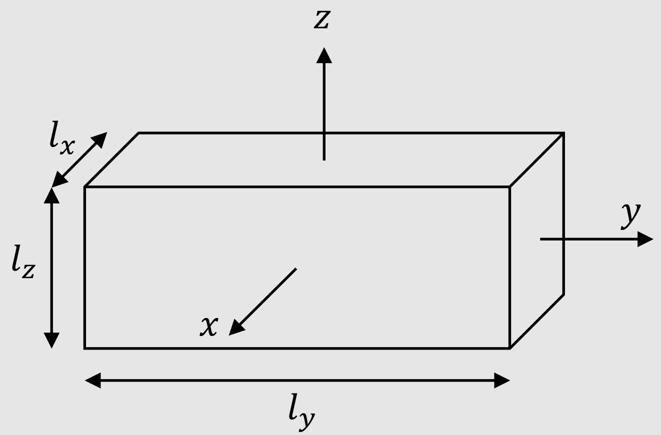
- $\mathfrak{I} = RIR^T$
  - R : orientation transformation between the body attached frame and the inertia frame
  - I : inertia tensor expressed in the body attached frame
- I is a constant matrix independent of the motion of the object
  - $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

  - $I_{xx} = \iiint (y^2 + z^2)\rho(x, y, z)\, dx\, dy\, dz$
  - $I_{yy} = \iiint (x^2 + z^2)\rho(x, y, z)\, dx\, dy\, dz$
  - $I_{zz} = \iiint (x^2 + y^2)\rho(x, y, z)\, dx\, dy\, dz$
  - $I_{xy} = I_{yx} = -\iiint xy\rho(x, y, z)\, dx\, dy\, dz$
  - $I_{xz} = I_{zx} = -\iiint xz\rho(x, y, z)\, dx\, dy\, dz$
  - $I_{yz} = I_{zy} = -\iiint yz\rho(x, y, z)\, dx\, dy\, dz$

  - Where $\rho(x, y, z)$ is the mass density of the object
  - **Note:** IF mass distribution is symmetric to the body frame, THEN inertia cross products are 0

Inertia Tensor: Uniform Rectangular Solid

- Uniform → $\rho(x, y, z) = \rho$ (constant)
- $I_{xx} = \int_{-l_z/2}^{l_z/2} \int_{-l_y/2}^{l_y/2} \int_{-l_x/2}^{l_x/2} (y^2 + z^2) \rho(x, y, z)\, dx\, dy\, dz = \rho \frac{l_x l_y l_z}{12} (l_y^2 + l_z^2)$
- $I_{yy} = \rho \frac{l_x l_y l_z}{12} (l_x^2 + l_z^2)$
- $I_{zz} = \rho \frac{l_x l_y l_z}{12} (l_x^2 + l_y^2)$
- Cross products of inertia are zero



- $I = \begin{bmatrix} \rho \frac{l_x l_y l_z}{12} (l_y^2 + l_z^2) & 0 & 0 \\ 0 & \rho \frac{l_x l_y l_z}{12} (l_x^2 + l_z^2) & 0 \\ 0 & 0 & \rho \frac{l_x l_y l_z}{12} (l_x^2 + l_y^2) \end{bmatrix}$

General Expressions for Kinetic Energy K

- n link robotic manipulator where the joint variables are the generalized coordinates (DH convention)
- m_i : mass of link i
- I_i : inertia matrix of link i
  - Evaluated around a coordinate frame $o_{c_i}x_{c_i}y_{c_i}z_{c_i}$ parallel to frame i but whose origin is at the link's center of mass

- $v_{c_i} = J_{v_{c_i}}(q)\dot{q}$: linear velocity vector at the center of mass of link i
- $\omega_{c_i} = J_{\omega_{c_i}}(q)\dot{q}$: angular velocity vector at the center of mass of link i

- $K = \frac{1}{2}\dot{q}^T \left(\sum_{i=1}^{n} m_i J_{v_{c_i}}(q)^T J_{v_{c_i}}(q) + J_{\omega_{c_i}}(q)^T R_{c_i}(q) I_i R_{c_i}(q)^T J_{\omega_{c_i}}(q)\right)\dot{q}$

General Expressions for Kinetic Energy K

- $K = \frac{1}{2}\dot{q}^T \left(\sum_{i=1}^{n} m_i J_{v_{c_i}}(q)^T J_{v_{c_i}}(q) + J_{\omega_{c_i}}(q)^T R_{c_i}(q) I_i R_{c_i}(q)^T J_{\omega_{c_i}}(q)\right)\dot{q}$

- In matrix form, $K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$
  - $D(q)$ is a symmetric positive definite matrix
  - $D(q)$ is in general **configuration dependent**

(see the final slides that present an example of how to compute $D(q)$ for the SCARA manipulator)

General Expressions for Potential Energy P

- Concentrate the entire mass m of an object at its center of mass

- $P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} -g^T o_{c_i} m_i$
  - g : gravitational vector in the inertial frame
  - o_{c_i} : coordinates of the center of mass of link i in the inertial frame
  - **Note:** the minus sign is required for the potential energy to be correctly computed

- IF robot contains elasticity (e.g., flexible joints), THEN potential energy P will include terms containing the energy stored in the elastic elements

- Potential energy P **depends on the configuration** of the robot **but not on its velocity**

(see the final slides that present an example of how to compute the potential energy P for the SCARA manipulator)

Equations of Motion

- Since $\mathcal{L} = K - P = \frac{1}{2}\sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j - P(q)$:
  - $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_j d_{kj}\ddot{q}_j + \sum_{i,j}\frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$
  - $\frac{\partial \mathcal{L}}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k}$

- Euler-Lagrange equation: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k$
  - $\sum_j d_{kj}\ddot{q}_j + \sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \sum_j d_{kj}\ddot{q}_j + \sum_{i,j} c_{ijk}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k$
  - Christoffel symbols of the 1st kind: $c_{ijk} = \left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\} = \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}$
    - Reduces the computation effort because $c_{ijk} = c_{jik}$ for a given k

Equations of Motion

- In matrix form, $D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \phi(q) = \tau$
  - Where the (k,j)-th element of the matrix $C(q,\dot{q})$ is c_{kj}
  - $$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i = \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i$$
  - $\phi(q) = \frac{\partial P}{\partial q_k}$ is the influence of the gravity

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Multivariable Control

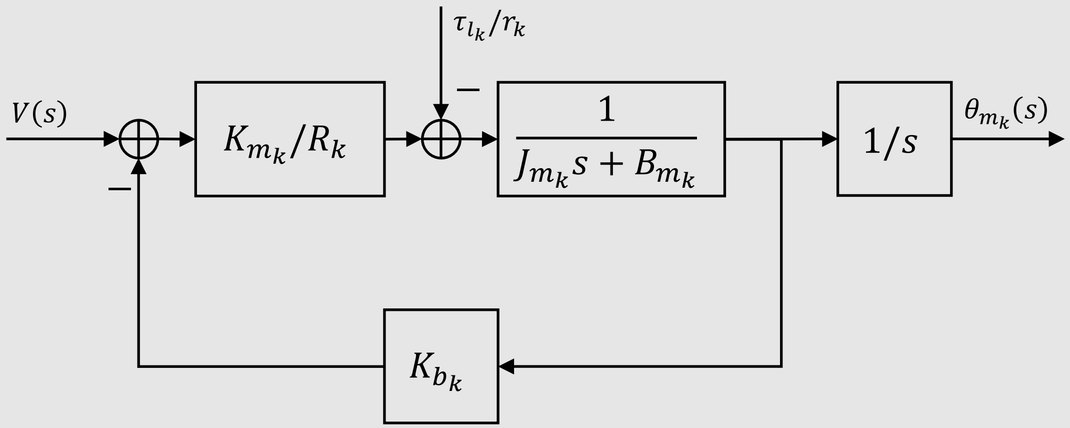
- **Previously** (independent joint control)**:**
  - Single-Input / Single-Output model
  - Coupling effects → treated as disturbances

- **Reality:** robot manipulator is a complex, non-linear, multivariable system!

- Multivariable Control
  - A more rigorous analysis of the performance of control systems
  - Design a robust and adaptive nonlinear control laws

Model of a Revolute Joint

- r_k : gear reduction ratio ($[r_k : 1]$)
- K_{m_k} : torque constant (N.m.A^{-1})
- K_{b_k} : speed constant (V.s.rad^{-1})
  - **Note:** $K_{m_k,S.I.} = K_{b_k,S.I.}$
- J_{m_k} : inertia of the motor (Kg.m^2)
- B_{m_k} : motor viscous constant (N.m.s)



- $J_{m_k}\ddot{\theta}_{m_k} + B_k\dot{\theta}_{m_k} = \frac{K_{m_k}V_k}{R_k} - \frac{\tau_{l_k}}{r_k}$, where $B_k = B_{m_k} + \frac{K_{b_k}K_{m_k}}{R_k}$ and $\theta_{m_k} = r_k q_k$

- $r_k^2 J_{m_k}\ddot{q}_k + r_k^2 B_k\dot{q}_k + \tau_k = \frac{r_k K_{m_k} V_k}{R_k} = u_k$

Equations of Motion

- Motor model:
  - $r_k^2 J_{m_k} \ddot{q}_k + r_k^2 B_k \dot{q}_k + \tau_k = \frac{r_k K_{m_k} V_k}{R_k} = u_k$
- Euler-Lagrange equation:
  - $\sum_j d_{kj} \ddot{q}_j + \sum_{i,j} \left\{ \frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i \dot{q}_j + \frac{\partial P}{\partial q_k} = \sum_j d_{kj} \ddot{q}_j + \sum_{i,j} c_{ijk} \dot{q}_i \dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k$

- $r_k^2 J_{m_k} \ddot{q}_k + \sum_j d_{kj} \ddot{q}_j + \sum_{i,j} c_{ijk} \dot{q}_i \dot{q}_j + r_k^2 B_k \dot{q}_k + \phi_k = u$
- $M(q)\ddot{q} + C(q, \dot{q})\dot{q} + B\dot{q} + \phi(q) = u$

  - $M(q) = D(q) + J$, where J is a diagonal matrix with elements $r_k^2 J_{m_k}$
  - B is a diagonal matrix with elements $B_{m_k} + \frac{K_{b_k} K_{m_k}}{R_k}$

Inverse Dynamics

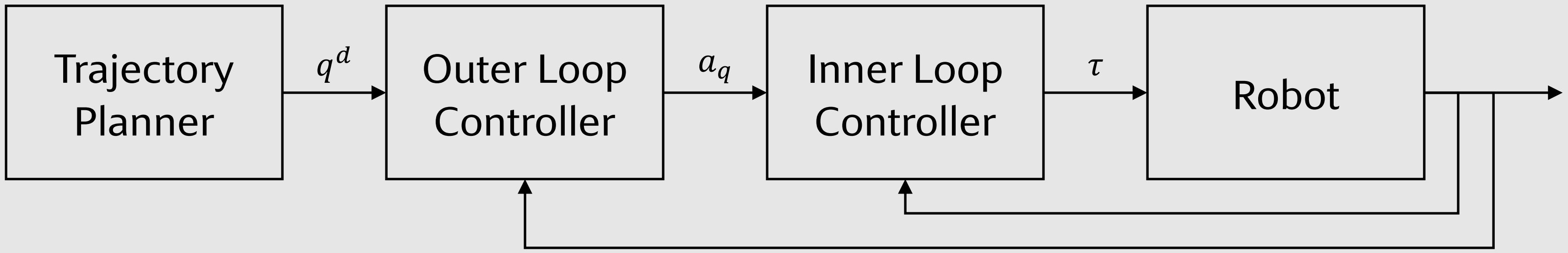
- **Goal:** nonlinear feedback control ($u = f(q, \dot{q}, t)$) with a linear closed loop system

- Outer loop: $\ddot{q} = a_q$
  - Output of the loop : $a_q = -K_0 q - K_1 \dot{q} + r$
  - Reference input : $r = \ddot{q}^d + K_0 q^d + K_1 \dot{q}^d$
  - Gain matrices K_0, K_1 : (possible values)
    - $K_0 = \mathrm{diag}\{\omega_1^2, \dots, \omega_n^2\}$
    - $K_1 = \mathrm{diag}\{2\omega_1, \dots, 2\omega_n\}$

- Inner loop: $u = M(q)a_q + C(q, \dot{q})\dot{q} + B\dot{q} + \phi(q)$

(see the final slides that present a practical application of inverse dynamics with the SCARA manipulator)

Inverse Dynamics



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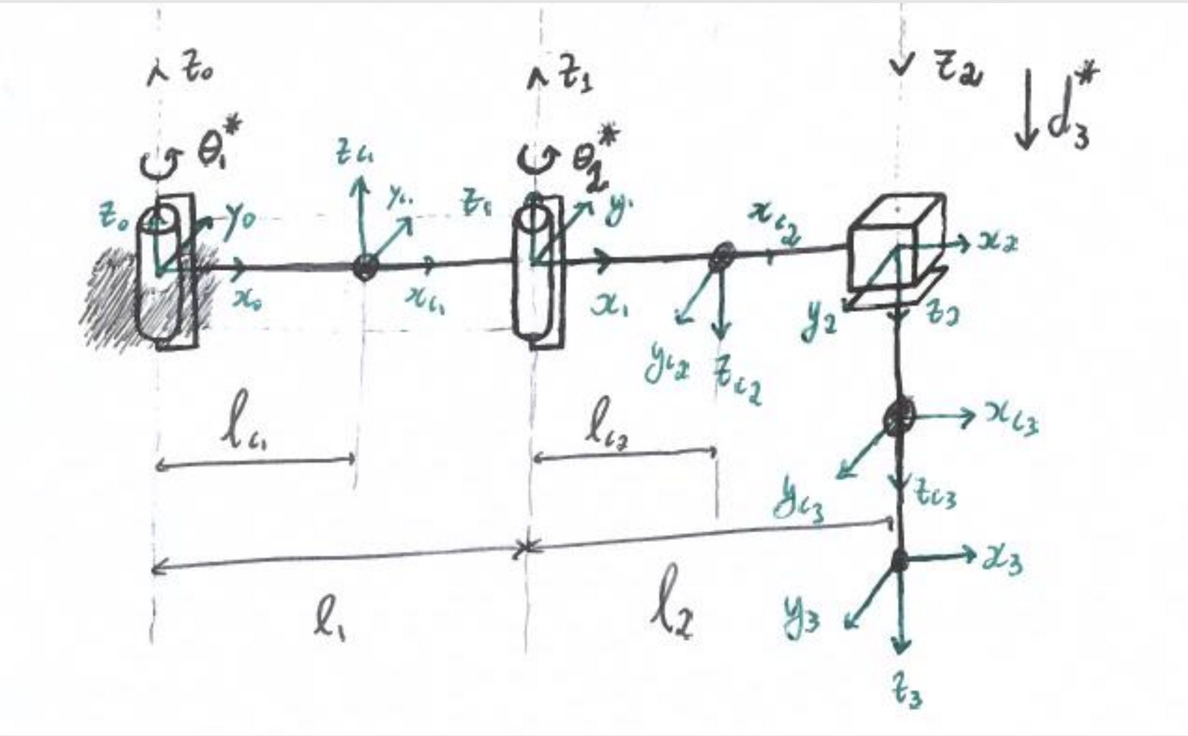
SCARA Manipulator

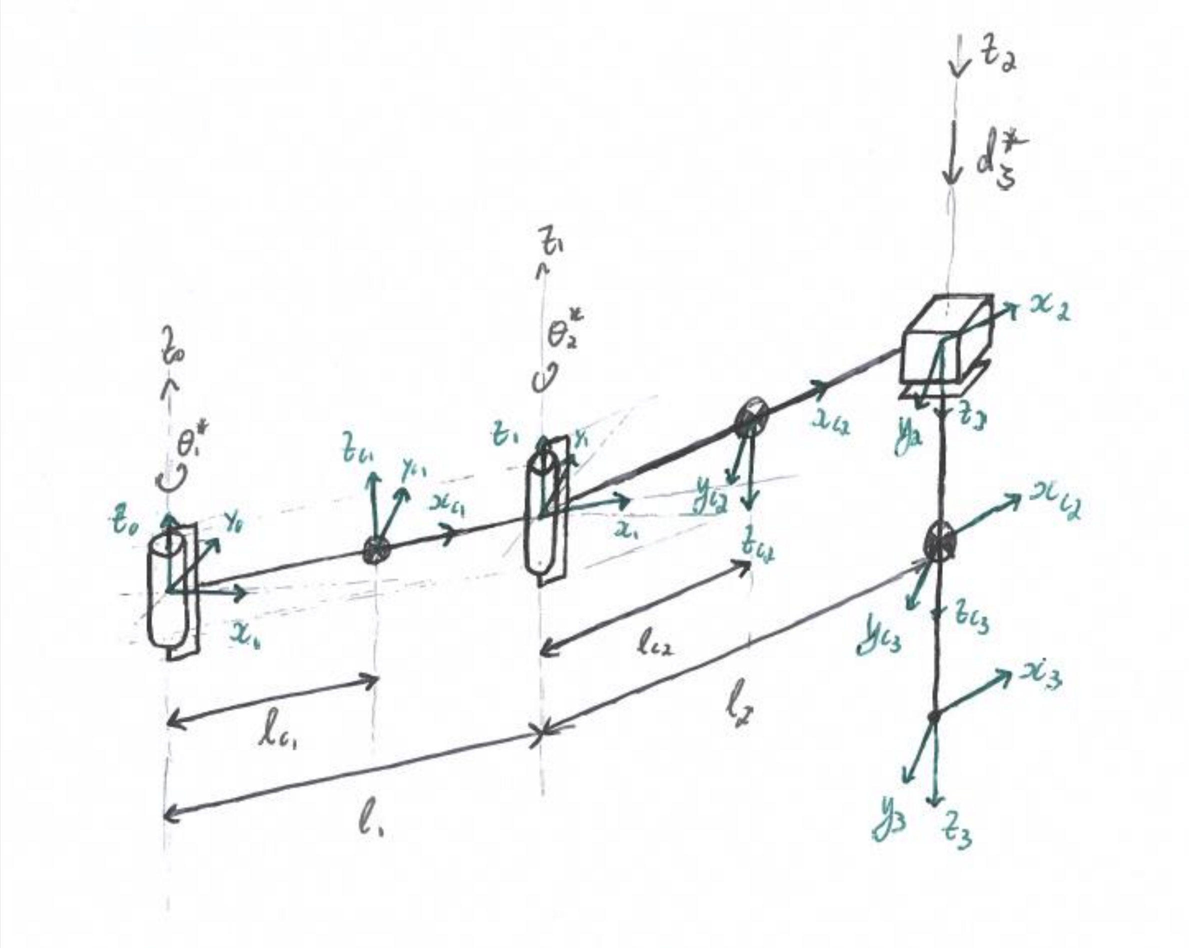
- SCARA: Selective Compliant Articulated Robot for Assembly
- Popular manipulator tailored for assembly operations
- RRP configuration

Epson SCARA E2L653S

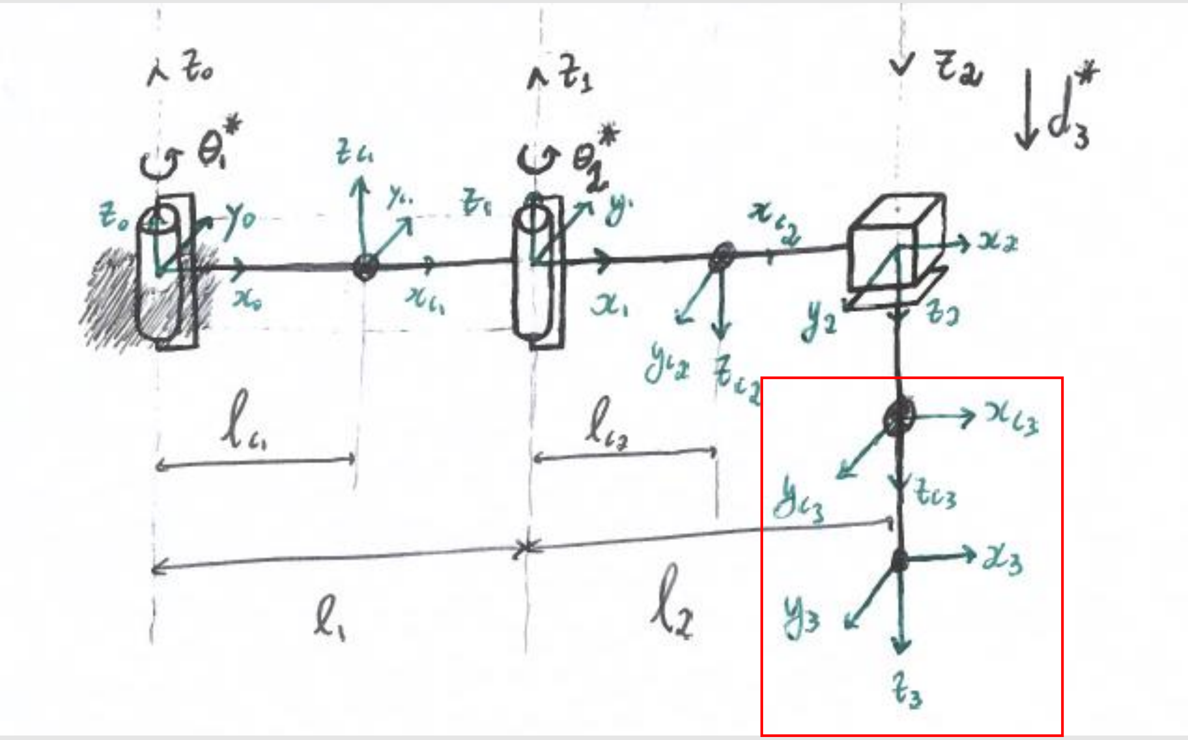
ABB IRB 910SC

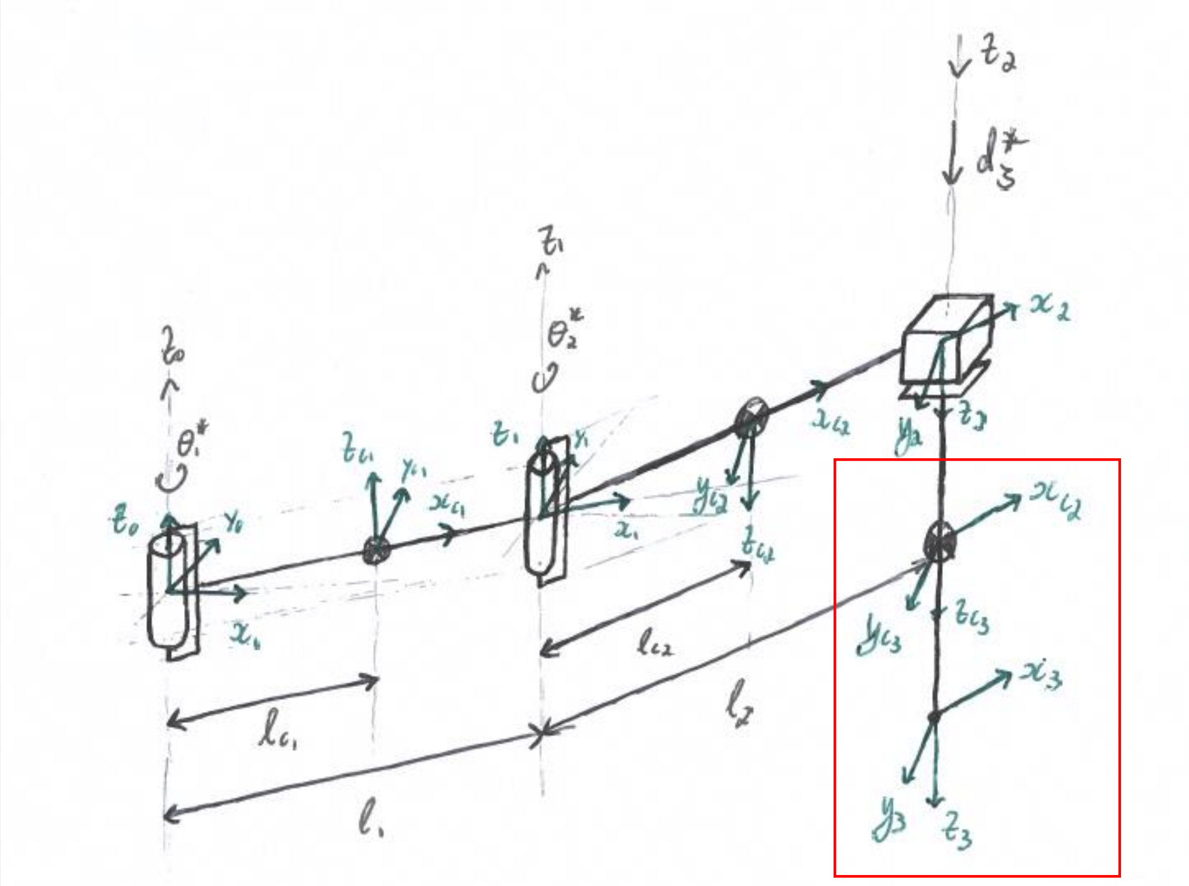
Forward Kinematics (DH convention)





Forward Kinematics (DH convention)





We will consider that the center of mass of link 3 is align with the end-effector's coordinate frame

Forward Kinematics (DH convention)

- $H_1^0 = \begin{bmatrix} c_{q_1} & -s_{q_1} & 0 & l_1 c_{q_1} \\ s_{q_1} & c_{q_1} & 0 & l_1 s_{q_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Link i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	q_1	0 m	l_1	0º
2	q_2	0 m	l_2	180º
3	0º	q_3	0 m	0º

- $H_2^0 = \begin{bmatrix} c_{q_1+q_2} & s_{q_1+q_2} & 0 & l_1 c_{q_1} + l_2 c_{q_1+q_2} \\ s_{q_1+q_2} & -c_{q_1+q_2} & 0 & l_1 s_{q_1} + l_2 s_{q_1+q_2} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- $H_3^0 = \begin{bmatrix} c_{q_1+q_2} & s_{q_1+q_2} & 0 & l_1 c_{q_1} + l_2 c_{q_1+q_2} \\ s_{q_1+q_2} & -c_{q_1+q_2} & 0 & l_1 s_{q_1} + l_2 s_{q_1+q_2} \\ 0 & 0 & -1 & -q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Forward Kinematics (DH convention)

- $H_1^0 = \begin{bmatrix} c_{q_1} & -s_{q_1} & 0 & l_1 c_{q_1} \\ s_{q_1} & c_{q_1} & 0 & l_1 s_{q_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

...

- $H_2^0 = \begin{array}{c} \begin{array}{cccc} x_2^0 & y_2^0 & z_2^0 & o_2^0 \end{array} \\ \begin{bmatrix} c_{q_1+q_2} & s_{q_1+q_2} & 0 & l_1 c_{q_1} + l_2 c_{q_1+q_2} \\ s_{q_1+q_2} & -c_{q_1+q_2} & 0 & l_1 s_{q_1} + l_2 s_{q_1+q_2} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$

...

- $H_3^0 = \begin{bmatrix} c_{q_1+q_2} & s_{q_1+q_2} & 0 & l_1 c_{q_1} + l_2 c_{q_1+q_2} \\ s_{q_1+q_2} & -c_{q_1+q_2} & 0 & l_1 s_{q_1} + l_2 s_{q_1+q_2} \\ 0 & 0 & -1 & -q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Link i	$\boldsymbol{\theta_i}$	$\boldsymbol{d_i}$	$\boldsymbol{a_i}$	$\boldsymbol{\alpha_i}$
1	q_1	0 m	l_1	0º
2	q_2	0 m	l_2	180º
3	0º	q_3	0 m	0º

Inverse Kinematics

- $c_2 = \frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}$

- $q_2 = \mathrm{atan2}\left(-\sqrt{1-c_2^2}, c_2\right)$
- $q_1 = \mathrm{atan2}(y,x) - \mathrm{atan2}\left(l_2 \mathrm{s}_{q_2}, l_1 + l_2 \mathrm{c}_{q_2}\right)$
- $q_3 = -z$

- **Note:** the inverse kinematics are still require for the inverse dynamics (compute q^d)

Velocity Kinematics

- $J_3^0 = \begin{bmatrix} -l_1 s_{q_1} - l_2 s_{q_1+q_2} & -l_2 s_{q_1+q_2} & 0 \\ l_1 c_{q_1} + l_2 c_{q_1+q_2} & l_2 c_{q_1+q_2} & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

- Matrix J_3^0 defines the velocity kinematics of the end-effector relative to the coordinate frame 0
- However, we can define the velocity kinematics relative to any point of the robot manipulator

- **Note:** the inverse kinematics require the definition of the velocity kinematics of the links' centers of mass

Velocity Kinematics

- Let us define the forward kinematics of each link's center of mass o_{c_i}:

  - $H_{c_1}^0 = \begin{bmatrix} c_{q_1} & -s_{q_1} & 0 & l_{c_1}c_{q_1} \\ s_{q_1} & c_{q_1} & 0 & l_{c_1}s_{q_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\rightarrow J_{v_{c_1}} = \begin{bmatrix} -l_{c_1}s_{q_1} & 0 & 0 \\ l_{c_1}c_{q_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $J_{\omega_{c_1}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

  - $H_{c_2}^0 = \begin{bmatrix} c_{q_1+q_2} & s_{q_1+q_2} & 0 & l_1c_{q_1} + l_{c_2}c_{q_1+q_2} \\ s_{q_1+q_2} & -c_{q_1+q_2} & 0 & l_1s_{q_1} + l_{c_2}s_{q_1+q_2} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\rightarrow J_{v_{c_2}} = \begin{bmatrix} -l_1s_{q_1} - l_{c_2}s_{q_1+q_2} & -l_{c_2}s_{q_1+q_2} & 0 \\ l_1c_{q_1} + l_{c_2}c_{q_1+q_2} & l_{c_2}c_{q_1+q_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $J_{\omega_{c_2}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

  - $H_{c_3}^0 = \begin{bmatrix} c_{q_1+q_2} & s_{q_1+q_2} & 0 & l_1c_{q_1} + l_2c_{q_1+q_2} \\ s_{q_1+q_2} & -c_{q_1+q_2} & 0 & l_1s_{q_1} + l_2s_{q_1+q_2} \\ 0 & 0 & -1 & -q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\rightarrow J_{v_{c_3}} = \begin{bmatrix} -l_1s_{q_1} - l_2s_{q_1+q_2} & -l_2s_{q_1+q_2} & 0 \\ l_1c_{q_1} + l_2c_{q_1+q_2} & l_2c_{q_1+q_2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $J_{\omega_{c_3}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

  - **Note:** l_{c_i} is the distance from $o_{c_{i-1}}$ to the link i's center of mass (see the initial figure that defines all the coordinate frames)

Velocity Kinematics

- Let us define the forward kinematics of each link's center of mass o_{c_i}:
  - $H_{c_1}^0 = \left[\begin{array}{cc} \overset{R_{c_1}}{\boxed{\begin{matrix} c_{q_1} & -s_{q_1} & 0 \\ s_{q_1} & c_{q_1} & 0 \\ 0 & 0 & 1 \end{matrix}}} & \begin{matrix} l_{c_1}c_{q_1} \\ l_{c_1}s_{q_1} \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{array}\right] \rightarrow J_{v_{c_1}} = \begin{bmatrix} -l_{c_1}s_{q_1} & 0 & 0 \\ l_{c_1}c_{q_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad J_{\omega_{c_1}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
  - $H_{c_2}^0 = \left[\begin{array}{cc} \overset{R_{c_2}}{\boxed{\begin{matrix} c_{q_1+q_2} & s_{q_1+q_2} & 0 \\ s_{q_1+q_2} & -c_{q_1+q_2} & 0 \\ 0 & 0 & -1 \end{matrix}}} & \begin{matrix} l_1c_{q_1} + l_{c_2}c_{q_1+q_2} \\ l_1s_{q_1} + l_{c_2}s_{q_1+q_2} \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{array}\right] \rightarrow J_{v_{c_2}} = \begin{bmatrix} -l_1s_{q_1} - l_{c_2}s_{q_1+q_2} & -l_{c_2}s_{q_1+q_2} & 0 \\ l_1c_{q_1} + l_{c_2}c_{q_1+q_2} & l_{c_2}c_{q_1+q_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad J_{\omega_{c_2}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$
  - $H_{c_3}^0 = \left[\begin{array}{cc} \overset{R_{c_3}}{\boxed{\begin{matrix} c_{q_1+q_2} & s_{q_1+q_2} & 0 \\ s_{q_1+q_2} & -c_{q_1+q_2} & 0 \\ 0 & 0 & -1 \end{matrix}}} & \begin{matrix} l_1c_{q_1} + l_2c_{q_1+q_2} \\ l_1s_{q_1} + l_2s_{q_1+q_2} \\ -q_3 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{array}\right] \rightarrow J_{v_{c_3}} = \begin{bmatrix} -l_1s_{q_1} - l_2s_{q_1+q_2} & -l_2s_{q_1+q_2} & 0 \\ l_1c_{q_1} + l_2c_{q_1+q_2} & l_2c_{q_1+q_2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad J_{\omega_{c_3}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

- **Note:** l_{c_i} is the distance from $o_{c_{i-1}}$ to the link i's center of mass (see the initial figure that defines all the coordinate frames)

Inertia Matrix $D(q)$

- For this SCARA manipulator, $D(q)$ is a 3×3 matrix

- $D(q) = \sum_{i=1}^{n} m_i J_{v_{c_i}}(q)^T J_{v_{c_i}}(q) + J_{\omega_{c_i}}(q)^T R_{c_i}(q) I_i R_{c_i}(q)^T J_{\omega_{c_i}}(q)$

- Also, let's assume that all three links are a uniform rectangular solid:
  - $\rho_i(x, y, z) = \rho_i = \frac{m_i}{V_i} = \frac{m_i}{l_{x_i} l_{y_i} l_{z_i}}$
  - $I_i = \begin{bmatrix} \frac{m_i}{12}(l_{y_i}^2 + l_{z_i}^2) & 0 & 0 \\ 0 & \frac{m_i}{12}(l_{x_i}^2 + l_{z_i}^2) & 0 \\ 0 & 0 & \frac{m_i}{12}(l_{x_i}^2 + l_{y_i}^2) \end{bmatrix}$
  - **Note:** the l_{x_i}, l_{y_i}, and l_{z_i} are relative to the coordinate frame $o_{c_i} x_{c_i} y_{c_i} z_{c_i}$

Matrix $C(q,\dot{q})$

- $C(q,\dot{q})$, where $c_{kj} = \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i$

- $C(q,\dot{q}) = \begin{bmatrix} c_{211}\dot{q}_2 & c_{121}\dot{q}_1 + c_{221}\dot{q}_2 & 0 \\ c_{112}\dot{q}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

  - $c_{211} = c_{121} = c_{221} = m_2 \mathrm{c}_{q_1+q_2} l_{c_2}\left(l_1 \mathrm{s}_{q_1} + l_{c_2}\mathrm{s}_{q_1+q_2}\right) + m_3 \mathrm{c}_{q_1+q_2} l_2\left(l_1 \mathrm{s}_{q_1} + l_2 \mathrm{s}_{q_1+q_2}\right) -$
    $- m_2 \mathrm{s}_{q_1+q_2} l_{c_2}\left(l_1 \mathrm{c}_{q_1} + l_{c_2}\mathrm{c}_{q_1+q_2}\right) - m_3 \mathrm{s}_{q_1+q_2} l_2\left(l_1 \mathrm{c}_{q_1} + l_2 \mathrm{c}_{q_1+q_2}\right)$

  - $c_{112} = m_2 \mathrm{s}_{q_1+q_2} l_{c_2}\left(l_1 \mathrm{c}_{q_1} + l_{c_2}\mathrm{c}_{q_1+q_2}\right) + m_3 \mathrm{s}_{q_1+q_2} l_2\left(l_1 \mathrm{c}_{q_1} + l_2 \mathrm{c}_{q_1+q_2}\right) -$
    $- m_2 \mathrm{c}_{q_1+q_2} l_{c_2}\left(l_1 \mathrm{s}_{q_1} + l_{c_2}\mathrm{s}_{q_1+q_2}\right) - m_3 \mathrm{c}_{q_1+q_2} l_2\left(l_1 \mathrm{s}_{q_1} + l_2 \mathrm{s}_{q_1+q_2}\right) =$
    $= -c_{211}$

Gravity Influence $\phi(q)$

- Note that $\phi(q)$ depends on the direction of the gravity vector $\vec{g}$ in the inertia frame

- Horizontal SCARA
  - $\phi(q) = \begin{bmatrix} 0 \\ 0 \\ -m_3 g \end{bmatrix}$

- Vertical SCARA
  - $\phi(q) = \begin{bmatrix} m_1 g l_{c_1} \mathrm{c}_{q_1} + m_2 g\left(l_1 \mathrm{c}_{q_1} + l_{c_2} \mathrm{c}_{q_1+q_2}\right) + m_3 g\left(l_1 \mathrm{c}_{q_1} + l_2 \mathrm{c}_{q_1+q_2}\right) \\ m_2 g l_{c_2} \mathrm{c}_{q_1+q_2} + m_3 g l_2 \mathrm{c}_{q_1+q_2} \\ 0 \end{bmatrix}$

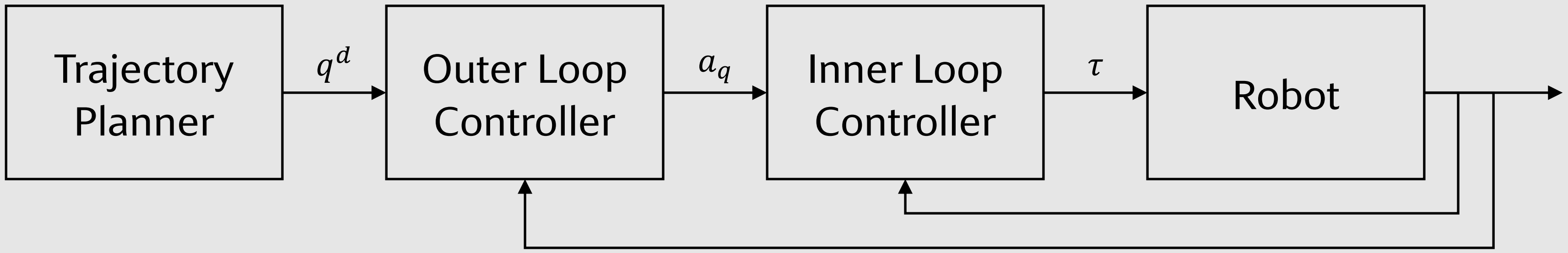
Inverse Dynamics

- Outer loop: $\ddot{q} = a_q$
  - Input of the system : $a_q = -K_0 q - K_1 \dot{q} + r$
  - Reference input : $r = \ddot{q}^d + K_0 q^d + K_1 \dot{q}^d$
  - Gain matrices K_0, K_1 : (possible values)
    - $K_0 = \text{diag}\{\omega_1^2, \dots, \omega_n^2\}$
    - $K_1 = \text{diag}\{2\omega_1, \dots, 2\omega_n\}$

- Inner loop: $u = M(q)a_q + C(q, \dot{q})\dot{q} + B\dot{q} + \phi(q)$

(see the final slides that present a practical application of inverse dynamics with the SCARA manipulator)

Inverse Dynamics



Next Steps: Task Space

- $X \in \mathbb{R}^6$: end-effector pose using minimal representation of $SO(3)$
- $\dot{X} = J(q)\dot{q}$
- $\ddot{X} = J(q)\ddot{q} + \dot{J}(q)\dot{q} = a_X$
  - Where $J = J_a$ (analytical Jacobian; see section 4.8 of the book for further details)

- **Modification:** outer loop (the inner loop remains unchanged)
  - $a_X = \ddot{X}^d + K_P(X^d - X) + K_D(\dot{X}^d - \dot{X})$
  - $a_q = \ddot{q} = J^{-1}(q) \cdot \{a_X - \dot{J}(q)\dot{q}\}$

  - **Note:** the task space does not require computing the desired trajectory X^d in the joint space

Final Observations

- Did you thought of doing all the computations for inverse dynamics by hand?

- **No!** For example, use math solvers (MATLAB, Symbolab, etc.) to compute all the matrices required to implement inverse dynamics.

- **Note:** if you use symbolic variables in MATLAB with transposed matrices, it will give you complex numbers:
  - MATLAB assumes that the symbolic variable could be a real or a complex number
  - So, transposing a matrix, will not only interchange but also switch the imaginary part of each number (conjugate)
  - Implement a specific transpose function to only interchange the matrix's elements (we know that all symbolic variables in the scope of inverse dynamics – l_i, l_{c_i}, q_i, etc. – are real numbers)

References

- Spong, M. W., Hutchinson, S., and Vidyasagar, M. 2005. *Robot Modeling and Control*. 1st edition. John Wiley & Sons.
  - Chapter 6: Dynamics
  - Chapter 7: Independent Joint Control
  - Chapter 8: Multivariable Control

Multivariable Control

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António Paulo Moreira[1], Paulo G. Costa[2], Ricardo B. Sousa[3]

[1]Associated Professor of DEEC@FEUP

[2]Assistant Professor of DEEC@FEUP

[3]Ph.D. Student of PDEEC@FEUP