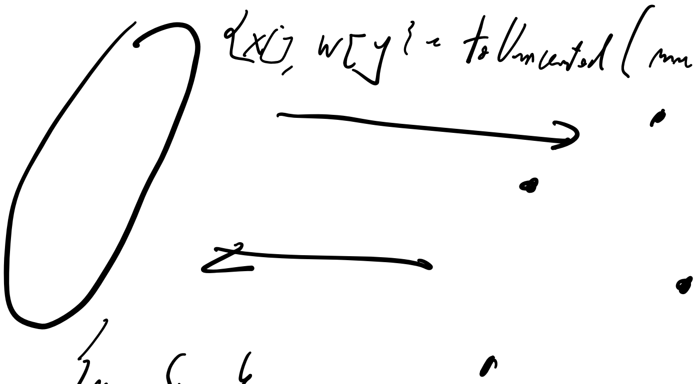
UNSCENTED TRANSFORM

— alternative way of parameterize the Gaussian distribution

Gaussian → may be determined experimentally → Unscented
- control points = SAMPLES selected → deterministically
- Weights (based on parameters of Gauss.)
  - ↳ the heavier a control point would be, the more the mean would lean towards the point

$\{X[\,], W[\,]\} = \text{toUnscented}(mu, Sigma)$



$\{mu, Sigma\} = \text{fromUnscented}(X[\,], W[\,])$

↳ sample w/ weight = sigma points

SIGMA POINT $\mathcal{X}^{(i)}$ =
- position — $x^{(i)} \in \Omega$
- weight for the mean — $W_m^{(i)} \in \mathbb{R}^+$
- weight for the covariance — $W_c^{(i)} \in \mathbb{R}^+$

2 scalars to reconstruct mean and covariance, respectively

$$\mu = \sum W_m^{(i)} x^{(i)}$$

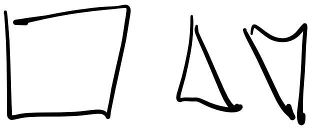
$$\Sigma = \sum \underbrace{W_c^{(i)}}_{\text{scale}} \oplus \underbrace{(x^{(i)} - \mu)(x^{(i)} - \mu)^T}_{\text{outer product}}$$

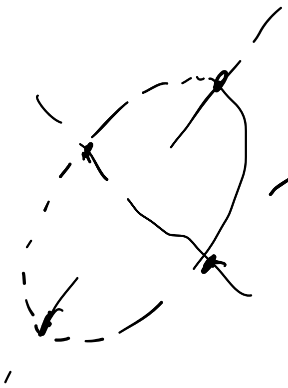
≈ similar experimental computation of the Gaussian parameters

Cholesky decomposition : CHOL (MATLAB)

$$A = L L^T \qquad L = \sqrt{A}$$

↓ ABUSE OF NOTATION !





putting them along the axes of the ellipse would require the computation of eigenvalues (very costly for high dimensionality)

$\Downarrow$

Cholesky Decomposition

$K \geq 0$
$\alpha \in [0, 1]$
$\beta = 2$

$\longrightarrow$ HOW TO SELECT?

$K = 0$
$\alpha = 10^{-3}$
$\beta = 2$

$\longrightarrow$ The important thing is that going forward and backward we have the same Gaussian dist.

$\downarrow$

3 parameters that determine how to select the sigma points given that I selected a set of directions

$n \equiv$ dimension

$L = \sqrt{(n + \lambda) A}$ $\longrightarrow$ L is the Cholesky decomposition of $(n-\lambda) A$

$\Downarrow$

Lower triangular matrix OF VECTORS:

$X^{(0)} = \mu$

$X^{(i)} = \mu + [L]_i \quad , i = 1, \dots, n$

$X^{(i)} = \mu - [L]_{n-i} \quad , i = n+1 .. 2n$

inside order create 2 sigma points

$\oplus$ order does not matter

(mean operation destroys the order)

$\Downarrow$

⇓

Goal: simpler operations when using the Gaussian

Jacobian may be error prone...

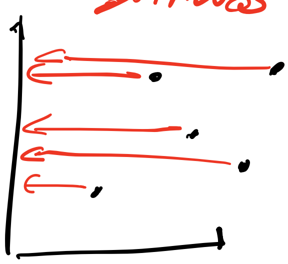
Unscented transform allows that!

Unscented Marginalisation

(Marginalization)

$X = \begin{pmatrix} X_a \\ X_b \end{pmatrix}$ be a random variable $X \sim UT(x; x^{(i)}, w_m^{(i)}, w_c^{(i)})$

Marginal $p(x_a) = \int_{x_b} p(x_a, x_b)\, dx_b$, $X_a \sim UT(X_a; x_a^{(i)}, w_m^{(i)}, w_c^{(i)})$



$\Longrightarrow$ KEEP SIGMA POINTS $\oplus$ SUPPRESS THE MARGINALIZED DIMENSION!

(projected in the part that makes the marginalization)

but we end up with more sigma points than what we need (reduced dimension)

$\downarrow$

$\vdots \Rightarrow \mu, \Sigma \Rightarrow \vdots$

to minimize # sigma pts

$x_a \sim UT(x_a; x_a^{(i)}, w_m^{(i)}, w_c^{(i)})$ (TRANSFORMATION FUNCTION)

$x_b = f(x_a)$ $\Longrightarrow$ x_b also becomes a random variable

(do for all sigma points)

$x_b \sim UT(x_b; x_b^{(i)}, w_m^{(i)}, w_c^{(i)})$ where $\mathcal{X}_b^{(i)} = f(\mathcal{X}_a^{(i)})$

weights are preserved !

may be NON-LINEAR (but not guaranteed if is still Gaussian)

if AFFINE, represent Gaussian ! (if A Gaussian)

inner noise of the conditional distributions

(UNSCENTED CHAIN RULE)

$x_a \sim \mathcal{U}(x_a; x_a^{(i)}, w_m^{(i)}, w_c^{(i)})$

$p(x_b \mid x_a) = \mathcal{N}(x_b; f(x_a), \Sigma_{b|a})$

$p(x_a, x_b) = \mathcal{N}(x_{a,b}; \mu_{a,b}, \Sigma_{a,b})$

$$\mu_{a,b} = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix} = \begin{pmatrix} \mu_a \\ f(\mu_a) \end{pmatrix}$$

$$\Sigma_{a,b} = \begin{pmatrix} \Sigma_a & \Sigma_{a,b} \\ \Sigma_{b,a} & \Sigma_{b|a} + \Sigma_{b,b} \end{pmatrix}$$

cross-condition coefficients

covariance due to projection x_a through $f(x_a)$

$\Sigma_{a,b} =$

as tall as x_a
as wide as x_b

$$\begin{pmatrix} \mathcal{X}_a \\ f(\mathcal{X}_a) \end{pmatrix}$$

new sigma points

how to recover covariance ?

OUTER MOMENTS

$\sum_i w_c^{(i)} (\dots)$

Conditioning → with sigma points we do not have continuous ?

unscented transform conditioning go back to Gaussian

$$\Sigma_{z,z} = \Sigma_{z|x} + \sum_i w_c^{(i)} \left(z_t^{(i)} - \mu_z \right) \left(z_t^{(i)} - \mu_z \right)^T$$