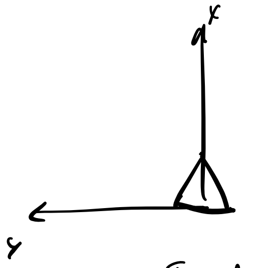
Kalman Filters Exercise

Velocity Model

Unicycle model



v

ω

ΔT

Integrate

(displacement) $\Delta\rho = v\,\Delta T$

$\Delta\theta = \omega\,\Delta T$

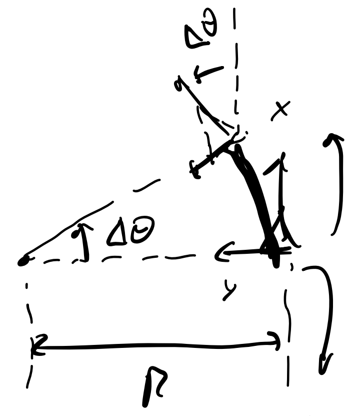
change in orientation

if I think constant v and ω,

moving always in a circle

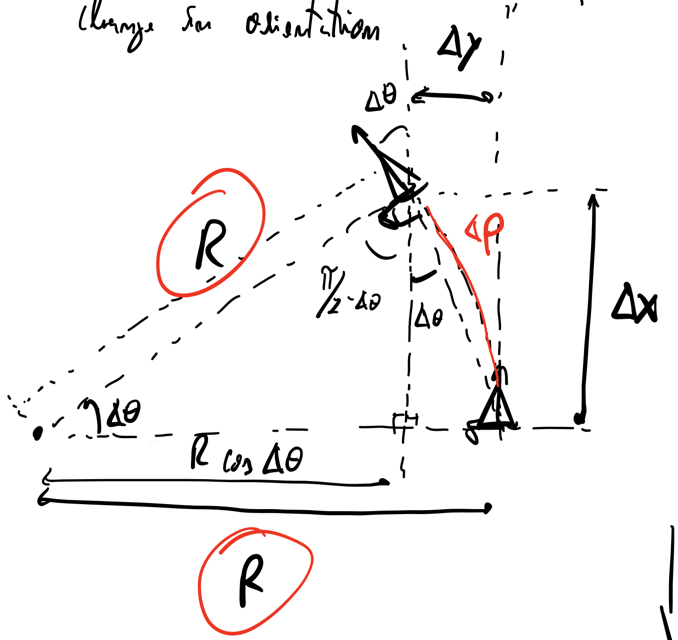
$v = \omega R$

Radians



$\Delta\rho$ (linear displacement = arc length)

$$\Delta\rho = R\,\Delta\theta \iff R = \frac{\Delta\rho}{\Delta\theta}$$



$$\begin{cases} \Delta x = R\sin(\Delta\theta) \\ \Delta y = R\,(1-\cos(\Delta\theta)) \end{cases}$$

→ Simple trigonometry

There is a singularity! $\left(\Delta\theta \to 0 \Downarrow R \to \infty\right)$

$$\Delta x = \frac{\Delta\rho}{\Delta\theta}\sin(\Delta\theta)$$

$$\Delta y = \frac{\Delta\rho}{\Delta\theta}(1-\cos(\Delta\theta))$$

finite limit for $\Delta\theta \to 0$

$$\begin{cases} \Delta x = \Delta\rho \cdot \boxed{\dfrac{\sin(\Delta\theta)}{\Delta\theta}} \\ \Delta y = \Delta\rho \cdot \boxed{\dfrac{1-\cos(\Delta\theta)}{\Delta\theta}} \end{cases}$$

→ Taylor Expansion

$$\frac{\text{sen}(\Delta\theta)}{\Delta\theta} \simeq S(\Delta\theta) = 1 - \frac{1}{6}\Delta\theta^2 + \frac{1}{120}\Delta\theta^4$$

$$\frac{1-\cos(\Delta\theta)}{\Delta\theta} \simeq C(\Delta\theta) = \frac{1}{2}\Delta\theta - \frac{1}{24}\Delta\theta^3 + \frac{1}{720}\theta^5$$

$\longrightarrow$ very very close approximation

- no singularities
- polynomials more easy to compute

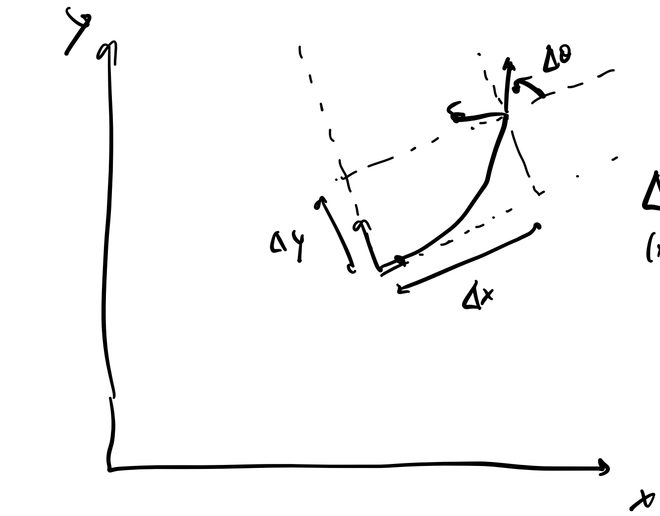
$$\Downarrow$$

$$\Delta x = \Delta\rho \cdot S(\Delta\theta)$$
$$\Delta y = \Delta\rho \cdot C(\Delta\theta)$$

$\xrightarrow{1^{st}\text{-order}}$

$$\boxed{\begin{aligned} \Delta x &\simeq \Delta\rho \\ \Delta y &\simeq \emptyset \end{aligned}}$$

but very bad approximation



ΔR — $R(\Delta\theta)$ (rotation)

$\Delta t \begin{cases} \Delta x \\ \Delta y \end{cases}$ (translation)

next state

$$R' = R(\theta)R(\Delta\theta) \equiv R(\theta + \Delta\theta)$$

$$t' = t + R\Delta t$$

NEW ORIENTATION

$$\boxed{\begin{aligned} x' &= x + \Delta\rho\cos(\theta) \\ y' &= y + \Delta\rho\,\text{sen}(\theta) \\ \theta' &= \theta + \Delta\theta \end{aligned}}$$

What if we assume 1^{st}-order approximation for $\omega = \emptyset$ and the real model for $\omega \neq \emptyset$?

$$\Delta x = \Delta\rho \cdot \frac{\text{sen}(\Delta\theta)}{\Delta\theta}$$

$$\Delta y = \Delta\rho \cdot \frac{1-\cos(\Delta\theta)}{\Delta}$$

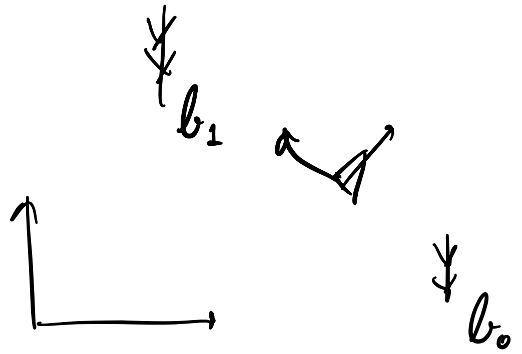
$$R = \begin{bmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{bmatrix}$$

$$R \cdot \begin{bmatrix} \Delta\rho \frac{\text{sen}(\Delta\theta)}{\Delta\theta} \\ \Delta\rho \cdot \frac{1-\cos(\Delta\theta)}{\Delta\theta} \end{bmatrix} = \begin{bmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{bmatrix} \cdot \begin{bmatrix} \Delta\rho \frac{\text{sen}(\Delta\theta)}{\Delta\theta} \\ \Delta\rho \cdot \frac{1-\cos(\Delta\theta)}{\Delta\theta} \end{bmatrix} = \begin{bmatrix} \Delta\rho \frac{\text{sen}(\Delta\theta)}{\Delta\theta} C_\theta - \Delta\rho \frac{\text{sen}(\Delta\theta)}{\Delta\theta} \cdot S_\theta \\ \Delta\rho \cdot \frac{1-\cos(\Delta\theta)}{\Delta\theta} \cdot S_\theta + \Delta\rho \cdot \frac{1-\cos(\Delta\theta)}{\Delta\theta} \cdot C_\theta \end{bmatrix}$$

EXERCISE : BEACONS LOCALISATION WITH KALMAN differential robot $\left|\begin{matrix}\Delta\rho \\ \Delta\theta\end{matrix}\right.$ (control translational and rotational displacements) Moving in the plane

we know location of the two beacons (b_0, b_1)

robot senses | distance (robot, b_i) $\oplus$ beacons identity ("known association")



DOMAINS

- STATE $\longrightarrow$ at least $\{x, y, \theta\} \in SE(2) \Rightarrow X = (x, y, \theta)^T$
- CONTROL $\longrightarrow u = (\Delta\rho, \Delta\theta) \in \mathbb{R} \times SO(2) \rightarrow$ DO NOT LIVE IN AN EUCLIDEAN SPACE
- OBSERVATION $\rightarrow z = (z_1 \; z_2)^T \in \mathbb{R}^{2+}$ stacked distance between robot and 2 beacons; distances $\geq \emptyset$

State $X \in SE(2)$

Controls $U \in \underbrace{\mathbb{R}}^{\Delta\rho} \times \underbrace{SO(2)}^{\Delta\theta}$ rotation matric space ignoring the translation

Measurements $Z \in \mathbb{R}^{2+}$

TRANSITION FUNCTION - $X_t = f(x_{t-1}, u_{t-1})$

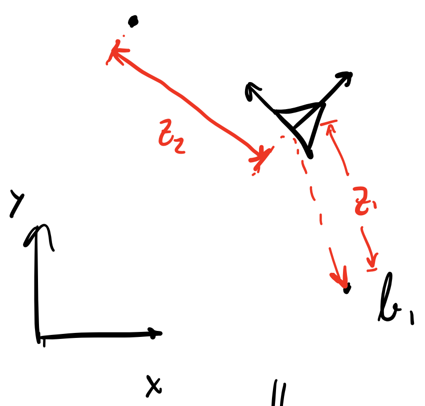
$$\begin{pmatrix} x_t \\ y_t \\ \theta_t \end{pmatrix} = \begin{pmatrix} x_{t-1} + \Delta\rho_{t-1} \cdot \cos(\theta_{t-1}) \\ y_{t-1} + \Delta\rho_{t-1} \cdot \sin(\theta_{t-1}) \\ \theta_{t-1} + \Delta\theta_{t-1} \end{pmatrix}$$

$\longrightarrow$ But we could use the complete model

$$A = \left.\frac{\partial f}{\partial x}\right|_{x_{t-1}} = \begin{pmatrix} 1 & \emptyset & -\Delta\rho_{t-1} \cdot \sin(\theta_{t-1}) \\ \emptyset & 1 & \Delta\rho_{t-1} \cdot \cos(\theta_{t-1}) \\ \emptyset & \emptyset & 1 \end{pmatrix} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{u_{t-1}} = \begin{pmatrix} \cos(\theta_{t-1}) & \emptyset \\ \sin(\theta_{t-1}) & \emptyset \\ \emptyset & 1 \end{pmatrix}$$

Observation Function – $z_t = h(x_t)$

b_2



$$\begin{pmatrix} z_{1,t} \\ z_{2,t} \end{pmatrix} = \begin{pmatrix} \sqrt{(x_t - x_1)^2 + (y_t - y_1)^2} \\ \sqrt{(x_t - x_2)^2 + (y_t - y_2)^2} \end{pmatrix}$$

$$\Downarrow$$

$$z_1 = \left\| \begin{pmatrix} x \\ y \end{pmatrix} - b_1 \right\|$$

$\Downarrow$

but do we always have the complete observation $(b_1 + b_2)$?

↳ only 1 observation:

$$C_i = \left.\frac{\partial h_i(x)}{\partial x}\right|_{x_t} = \frac{1}{\left\| \begin{pmatrix} x \\ y \end{pmatrix} - b_i \right\|} \left(x - x_i , \; y - y_i , \; \emptyset \right)$$

this makes sense, range lite does not give information about the orientation!

↳ complete model:

$$h(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} \qquad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Filter

↳ **Predict**: new control u_{t-1} comes in, need to update our current belief by integrating the control

$\mu_{t-1|t-1}$ → this is our current state estimation!

$\Downarrow$

$$\mu_{t|t-1} = f(\mu_{t-1|t-1}, u_{t-1})$$

$$A_t = \frac{\partial f(x,u)}{\partial u} \qquad B_t = \frac{\partial f(x,u)}{\partial u}$$

> linearize the model (state & control)

↑ usually constant, but may change over time

$$\Sigma_{t|t-1} = A_t \Sigma_{t-1|t-1} A_t^T + B_t \Sigma_u B_t^T$$

↳ predict the covariance (usually always greater than in previous step! ⇒ additive term $B_t \Sigma_u B_t^T$)

↳ **Update**: a new observation comes in, and we need to update the predicted belief by integrating the measurement

$\mu_z = h(\mu_{t|t-1})$ → predict the measurement based on our current state estimation

$C_t = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\mu_{t|t-1}}$ → linearize observation function

$$K_t = \Sigma_{t|t-1} C_t^T (\Sigma_z + C_t \Sigma_{t|t-1} C_t^T)^{-1}$$

$\mu_{t|t} = \mu_{t|t-1} + K_t (z_t - \mu_z)$ — mean

$\Sigma_{t|t} = (I - K_t C_t) \Sigma_{t|t-1}$ — covariance

update

Note:

→ We can test with zero noise in simulation to see if model and functions work

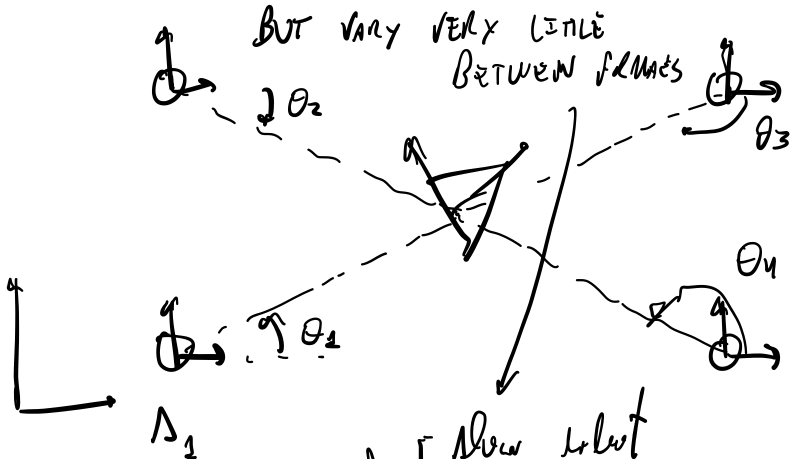
Exercise: Watch my Robot

differential robot

$\Delta\rho, \Delta\theta \rightarrow$ measurements taken

BUT VARY VERY LITTLE BETWEEN FRAMES

we want to track the robot position



Known locations!

slow robot / very fast filter

DOMAINS

STATE: $X = (x, y, \theta, \Delta\rho, \Delta\theta)^T$

$SE(2) \times \mathbb{R} \times SO(2)$

CONTROLS: no known control inputs

OBSERVATIONS: $z_i \in SO(2)$

$i = 1, \dots, N$

$\Downarrow$

$z \in SO(2)^N$

↳ TRANSITION:

$$\begin{pmatrix} x_t \\ y_t \\ \theta_t \\ \Delta\rho_t \\ \Delta\theta_t \end{pmatrix} = \begin{pmatrix} x_{t-1} + \Delta\rho_{t-1} \cdot \cos(\theta_{t-1}) \\ y_{t-1} + \Delta\rho_{t-1} \cdot \sin(\theta_{t-1}) \\ \theta_{t-1} + \Delta\theta_{t-1} \\ \Delta\rho_{t-1} \\ \Delta\theta_{t-1} \end{pmatrix}$$

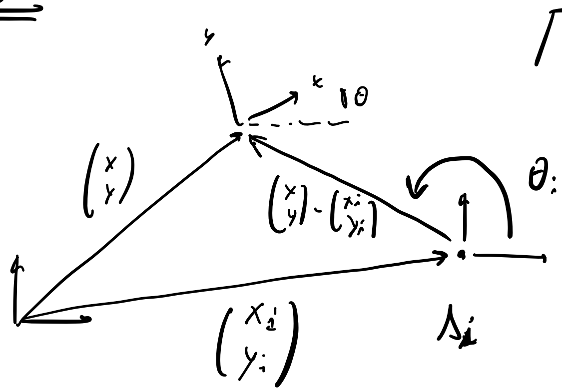
$\rightarrow$ we assume they remain constant...

$$A = \frac{df}{dx} = \begin{bmatrix} 1 & \emptyset & -\Delta\rho_{t-1} \cdot \sin(\theta_{t-1}) & \emptyset & \emptyset \\ \emptyset & 1 & \Delta\rho_{t-1} \cdot \cos(\theta_{t-1}) & \emptyset & \emptyset \\ \emptyset & \emptyset & 1 & \emptyset & 1 \\ \emptyset & \emptyset & \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & 1 \end{bmatrix}$$

$B = \emptyset \rightarrow$ does not exist!

$$d\theta = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

↳ OBSERVATION:



$\theta_{A_i} = \operatorname{atan2}(y - y_{A_i}, x - x_{A_i}) \equiv h(x_t)$

$$\nabla \operatorname{atan2}(y, x) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

prediction of the measurement

$$C_{\Delta i} = \frac{1}{(x-x_i)^2 + (y-y_i)^2} \cdot \left(x - x_i, \; y - y_i, \; \emptyset, \; \emptyset, \; \emptyset \right)$$

this probably would be okay...

↳ what about knowing C_t, C_{t-1} → making "observable" Δp_{t-1} and $\Delta\theta_{t-1}$ ......

↳ Filter:

$$\mu_{t|t-1} = f(\mu_{t-1|t-1}, u_{t-1})$$

$$A_t = \frac{\partial f(x,u)}{\partial x}$$

$$\Sigma_{t|t-1} = A_t \Sigma_{t-1|t-1} A_t^T + \Sigma_{PROCESS}$$

small constant covariance

$$\begin{pmatrix} \emptyset & & & \\ & \emptyset & & \\ & & \emptyset & \\ & & & \sigma_{pp} \\ & & & & \sigma_{\theta\theta} \end{pmatrix}$$

position that the process

But this makes that will increase covariance of Δp and $\Delta\theta$ (WE HAVE NO INFORMATION ON THAT)

MY RESOLUTIONS....

Exercise: 2D GPS w/ global clock

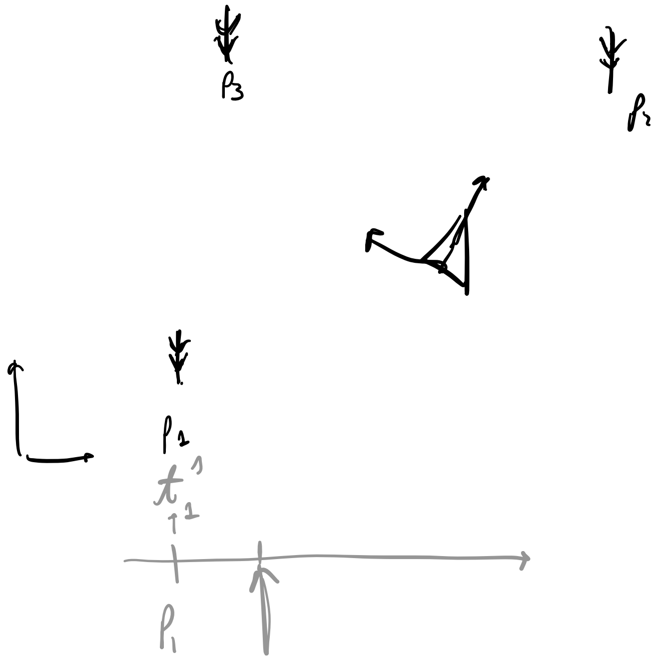
monocycle robot controlled in Δp and $\Delta\theta$ ⇒ we know action

robot equipped → antennas that receive the signal following set of beacons:

- unique ID
- its position on the plane p_i → I am going to assume that it's the robot position
- time stamp t_i^s (global clock)

(at transmission)

→ might not receive all beacons at time



Domain

State: $X \in SE(2)$
$X = (x, y, \theta)^T$

Controls: $U \in \mathbb{R} \times SO(2)$
$U = (\Delta\rho, \Delta\theta)^T$

Observation: time difference
$z \in \mathbb{R}^N$, $z = (z_1, \dots, z_N)^T$

t_1^a — arrival at robot ~ something related to signal propagation velocity!...

Transition → equal to first exercise!

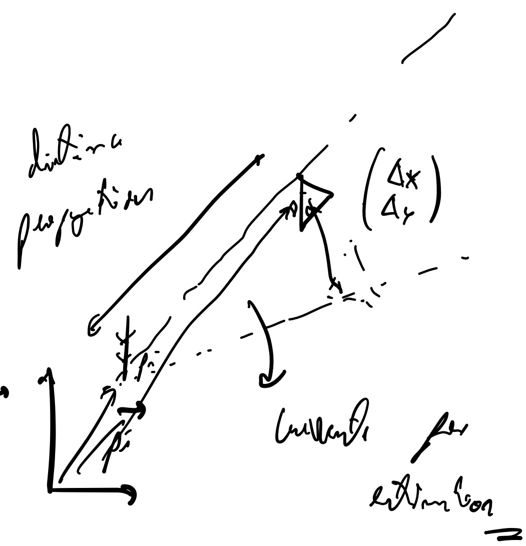
Observation

$$\Delta t_{\text{ime-signal}} = \frac{\Delta \text{ distance propagation}}{v \text{ propagation}}$$

$$t_1^a - t_1^s = \frac{|x - p_i|}{c}$$

$\begin{pmatrix} x \\ y \end{pmatrix}$ = x

c → speed of light → the text of the exercise does not say if signal is | light | RF | ....



$$t_i^a = \frac{|x - p_i|}{c} + t_i^s$$

$$z_i = h_i(x) = t_i^a - t_i^s = \frac{|x - p_i|}{c}$$

$$\triangleright \; C_i = \frac{1}{c\,|x - p_i|}\left(x - x_i,\; y - y_i,\; 0\right)$$

Exercise: 2D GPS w/ global clock

unicycle robot, we have access to $(\Delta\rho, \Delta\theta)$

at beacons:
- ID
- position on plane, p_i (of the beacon)
- $\hat{t}_i$, when beacon transmits the sign

↓

global clock shared by all beacons!

$$t_{robot} = t_{global} + \tau$$

↓

small slowly changing bias

Domain

State: $X = (x, y, \theta)$ only? what about the bias?

$\Downarrow$

$$X = (x, y, \theta, \tau)^T \in SE(2) \times \mathbb{R}$$

may be positive, negative or even $\emptyset$.....

Controls: $U = (\Delta\rho, \Delta\theta)^T \in \mathbb{R}^+ \times SO(2)$

Observation: time difference between t_i^{s} (global) and t_i^{a} (robot)

↓

must be referenced to the same time clock

Transition

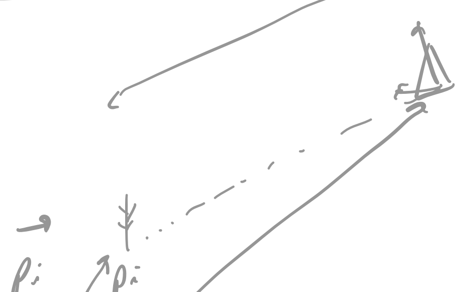
$$\begin{pmatrix} x_t \\ y_t \\ \theta_t \\ \tau_t \end{pmatrix} = \begin{pmatrix} x_{t-1} + \Delta\rho_t \cdot \cos(\theta_t) \\ y_{t-1} + \Delta\rho_t \cdot \sin(\theta_t) \\ \theta_{t-1} + \Delta\theta_t \\ \tau_{t-1} \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & \emptyset & -\Delta\rho_t \cdot \sin(\theta_t) & \emptyset \\ \emptyset & 1 & \Delta\rho_t \cdot \cos(\theta_t) & \emptyset \\ \emptyset & \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & \emptyset & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} \cos(\theta_t) & \emptyset \\ \sin(\theta_t) & \emptyset \\ \emptyset & 1 \\ \emptyset & \emptyset \end{pmatrix}$$

Observation

propagation distance



translation part

$$t_i^a - t_i^s = \frac{|t - p_i|}{c} \quad \text{(global)}$$

$\Downarrow$

$$z_i = h_i(X) = t_i^{a,robot} - t_i^{s,global} =$$

distance $= |t - p_i|$

$$= \frac{|t - p_i|}{c} - \tau$$

the dsc is varying positive? ....

probably does not matter,
but must be consistent!

Filter → we can add
small items to the covariance element $\sigma_{\tau\tau}$
to inject noise in the time skew!
(periodically)
τ is only estimated, cannot be controlled