LEAST SQUARES

Maximum likelihood estimation $\quad x^* = \arg\max_x p(x \mid z) \quad \vdots \quad X_{\text{STATE}} \;\Big|\; \begin{array}{l}\text{ALL MEASUREMENT} \\ \text{ALL CONTROLS} \ldots\end{array}$

most likely value of this distribution → very complex in a probabilistic way

OBSERVATION MODEL

$$p(x \mid z) = \frac{\overbrace{p(z \mid x)}^{\text{OBSERVATION MODEL}}\; \overbrace{p(x)}^{=\text{PRIOR OVER POSSIBLE STATE}}}{p(z)}$$

if we do not know nothing, become uniform distribution (= constant)

$$\propto p(z \mid x)$$

$$= \prod p(z^{[i]} \mid x) \quad \rightarrow \text{assuming independent measurements}$$

Gaussian assumption → measurement affected by Gaussian Noise

$$\underbrace{p(z^{[i]} \mid x)}_{\text{likelihood}} = \mathcal{N}\big(z^{[i]}, \underbrace{h^{[i]}(x)}_{\text{prediction measurement}}, \Sigma^{[i]}\big)$$

$e^{[i]}(x)$ — error function

$$\propto \exp\Big(-\underbrace{\big(h^{[i]}(x) - z^{[i]}\big)}_{e^{[i]}}{}^T \underbrace{\Sigma^{[i]-1}}_{\Omega^{[i]} \,=\, \text{INFORMATION MATRIX}} \big(h^{[i]}(x) - z^{[i]}\big)\Big)$$

$$x^* = \arg\max_x \prod_i p(z^{[i]} | x)$$

OMEGA NORM

$$= \arg\max_x \prod_i \exp\left(-e^{[i]}(x)^T \Omega^{[i]} e^{[i]}(x)\right)$$

$$= \arg\min_x \sum_i e^{[i]}(x)^T \Omega^{[i]} e^{[i]}(x)$$

IS NOT A LINEAR FUNCTION!

iterative minimization

QUADRATIC FORM

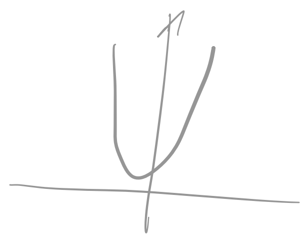
$\Updownarrow$

HIGHER DIMENSION OF 2ND GRADE POLINOMIAN

$$x \leftarrow x + \Delta x$$

$$F(x) = \sum_i e^{[i]}(x)^T \Omega^{[i]} e^{[i]}(x)$$

$$F(x+\Delta x) \cong \Delta x^T H \Delta x + 2b^T \Delta x + c$$



minimizing quadratic approximation!

PARABOLA (IF 1 DIMENSIONAL)

$$h \Delta x^2 + 2b\Delta + c$$

What will be the minimum?

DERIVATIVE $\rightarrow = \emptyset$

DERIVATIVE

$$\frac{\partial}{\partial \Delta x}\left(\Delta x^T H \Delta x + 2b^T \Delta x + c\right) =$$

(1 dim)

$$= H\Delta x + \Delta x^T H + 2b^T = \emptyset$$

$$2H\Delta x = -2b$$

$$\Delta x = -\frac{b}{H}$$

$$F(x+\Delta x) \cong \Delta x^T H \Delta x + 2b^T \Delta x + c$$ → HOW TO APPROXIMATE?

remember Taylor expansion

$$e^{[i]}(x^* + \Delta x) \cong \underbrace{e^{[i]}(x^*)}_{\mathbf{e}^{[i]}} + \underbrace{\left.\frac{\partial e^{[i]}(x)}{\partial x}\right|_{x=x^*}}_{\mathbf{J}^{[i]}} \cdot \Delta x$$

$\mathbf{e}^{[i]}$ = coefficient

$\mathbf{J}^{[i]}$ = Jacobian of e error function

evaluate at current linearization point

$$F(x^* + \Delta x) = \sum_i e^{[i]}(x^* + \Delta x)^T \Omega^{[i]} e^{[i]}(x^* + \Delta x) \cong$$

$$\cong \sum_i \left(e^{[i]} + J^{[i]} \cdot \Delta x\right)^T \cdot \Omega^{[i]} \cdot \left(e^{[i]} + J^{[i]} \cdot \Delta x\right) =$$

$$= \sum_i \left[e^{[i]T} \Omega^{[i]} e^{[i]} + e^{[i]T} \Omega^{[i]} J^{[i]} \cdot \Delta x + \Delta x^T J^{[i]T} \cdot \Omega^{[i]} \cdot e^{[i]} + \left(J^{[i]} \Delta x\right)^T \Omega^{[i]} J^{[i]} \Delta x \right]$$

$$= \sum_i \left[\Delta x^T \underbrace{J^{[i]T} \cdot \Omega^{[i]} \cdot J^{[i]}}_{H^{[i]}} \Delta x + 2 \underbrace{J^{[i]T} \Omega^{[i]} e^{[i]}}_{b^{[i]T}} \cdot \Delta x + \underbrace{e^{[i]T} \cdot \Omega^{[i]} \cdot e^{[i]}}_{c^{[i]}} \right]$$

$$v^T \Omega u \overset{\text{vs}}{=} u^T \Omega v$$

$$\Downarrow$$

$$\Delta x^* \cong \operatorname*{argmin}_{\Delta x} \left(\Delta x^T H \Delta x + 2b^T \Delta x + c \right)$$

$$\emptyset = \frac{\partial \left[\Delta x^T H \Delta x + 2b^T \Delta x + c\right]}{\partial \Delta x}$$

$$-b = H \Delta x$$

use as accumulators

$$H = \sum_i H^{[i]}$$

...

dim $(H) = ?$

dim $(b) = ?$

For each measurement:

$e^{[i]}$

$J^{[i]}$

H

b

$\sim$

END:

update estimate with perturbation

$\Delta x \leftarrow$ solve $(H \Delta x = -b)$

$x^* \leftarrow x^* + \Delta x$

if $y = mx + b$

w/ $e^{[i]}$ as affine $\Rightarrow$ 1 iteration to find minimum

$\left(\frac{\partial y}{\partial x} = m \right.$ ~ coefficients denote constant $\left. \right)$

Gaussian - Newton → assumes linear parametrization

⇓

we need to find local Euclidean parametrization

in SLAM

- Calibration
- Registration (ICP, Point)
- Global Optimization (Pose SLAM, Bundle)

assuming

Known Data Association!

if initial guess is available → need a strategy to improve initial pose estimation

$$\boxed{\text{Iterative Closest Point (ICP) 2D}}$$

EXAMPLE

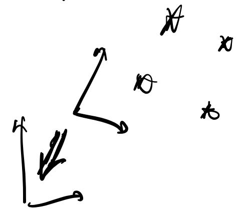
w/ known correspondences

Real ICP ⇒ we do not know the correspondences

⇓

they are computed at each iteration

Goal: find transformation



STATE

$X \in SE(2)$

$$X = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \begin{matrix} \} \text{ translation} \\ \\ \text{rotation} \end{matrix}$$

measurements

$z \in \mathbb{R}^2 = \begin{bmatrix} x \\ y \end{bmatrix}$

$$h^{[i]}(X) = R(\theta)\, p^{[i]} + t$$

$$e^{[i]} = \left(h^{[i]} - z^{[i]} \right)$$

does not depend on the state (IN THIS CASE!)

$$\frac{\partial h^{[i]}(x)}{\partial x} = \begin{pmatrix} \frac{\partial h^{[i]}(x)}{\partial t} & \frac{\partial h^{[i]}(x)}{\partial \theta} \end{pmatrix} \rightarrow \text{measurement Jacobian}$$

$\|$

$R^{1}(\theta)\, p^{[i]}$

Note the

$$C = \sum_i c^{[i]} = \sum e^{[i]^T} \Omega\, e^{[i]} \leftarrow \underline{\underline{chi}}$$

(CUMULATIVE ERROR)

how far we are from the optimum

Proposal: ICP with only observable the bearing of measurements!

FUNCTION z_new = extract Azimuth (P)

FOR i = 1 : length (P)

z_new (i) = atan2(P(2,:), P(1,:));

ENDFOR

ENDFUNCTION

error → mono dimensional

(it's only an angle distance)

FUNCTION ErrorJacobian (x, y, z) (bearing Only)

p_pred

z_hat = atan2(p_pred(2), p_pred(1))

e = z_hat − z → must be normalized!   (e = atan2(sin(e), cos(e)))

:

Jatan2 = [

]

END FUNCTION