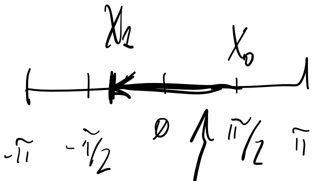
Linear Spaces on Manifolds → NON-EUCLIDEAN DOMAINS

example: ← vs

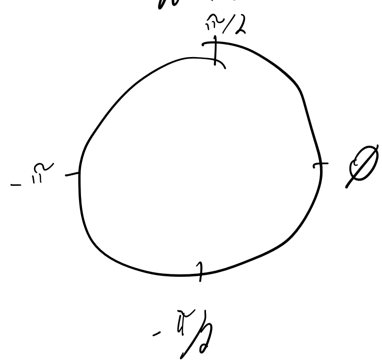
euclidean = map to $\mathbb{R}^n$ and can measure distance consistent to euclidean metric

Usually



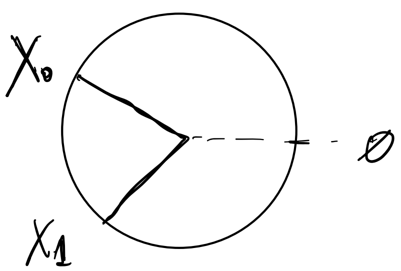
↳ distance

ANGLES



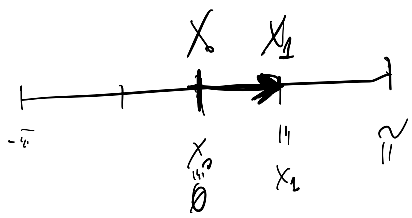
IS THIS CORRECT?

May lead to wrong result (which is not the minimum distance!) → specially in 3D, rotations are hell/tricky!

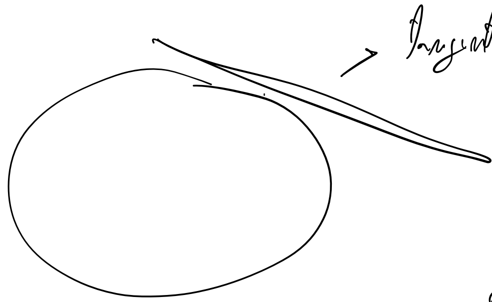


⇒ choosing this as origin of our mapping caused the invalidity of the angle difference!

Why not X_0 in the origin?

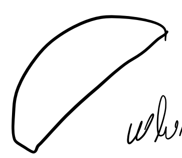


⇒ move the singularity far away!



→ tangent plane would be the best around X_0

or



where the most far away point is the mirrored one...

X_0 - start point on manifold → GOAL: around X_0

X_1 - end point manifold

Δx — difference

1. chart around x_0 → $x_0 \Rightarrow \emptyset$
2. location X_1 in relation to x_1 → but the chart-domain euclidean

$$X_1 \boxminus X_0 = \Delta x$$

≡ equivalent to the procedure (compute difference around x_0)

$\boxminus$: BOX MINUS

We can undo this process:

$$X_0 \boxplus \Delta x = X_1$$

$\boxplus$: BOX PLUS

≡

1. go to starting point
2. get a chart on starting point
3. move Δx and go back to manifold

≈ inverse process of the previous sequence

HOW TO EMBED IN THE LEAST SQUARE?

Euclidean

clear H and b:

$H \leftarrow \emptyset$; $b \leftarrow \emptyset$

for each measurement, update h and b:

$$e^{[i]} \leftarrow h^{[i]}(x^*) - z^{[i]}$$

$$J^{[i]} \leftarrow \left.\frac{\partial e^{[i]}(x)}{\partial x}\right|_{x=x^*}$$

evaluate at the current optimum

$$H \longleftarrow H + J^{[i]T}\,\Omega^{[i]}\,J^{[i]}$$

$$b \longleftarrow b + J^{[i]T}\,\Omega^{[i]}\,e^{[i]}$$

update the estimate w/ the perturbation:

$$\Delta x \longleftarrow \text{solve}(H\Delta x = -b)$$

→ should be euclidean

— straight forward?

$+ \Rightarrow \boxplus$

$- \Rightarrow \boxminus$

NOT QUITE EXACTLY...

$x^* \leftarrow x^* + \Delta x$ → should be euclidean

$$e^{[i]}(x) = h^{[i]}(x) \boxminus z_i \equiv$$

1. constant around z_i
2. find where $h^{[i]}$ is on the chart
3. calculate the Euclidean distance around z_i

euclidean distance

if spaces are Euclidean, $\boxminus \equiv -$ standard

$$e^{[i]}(x \boxplus \Delta x) = e^{[i]}(x) + \underbrace{\left.\frac{\partial e^{[i]}(x \boxplus \Delta x)}{\partial \Delta x}\right|_{\Delta x = \emptyset}}_{J^{[i]}} \cdot \Delta x$$

current optimum is x and $\left. x \boxplus \Delta x \right|_{\Delta x = \emptyset} = \emptyset$

allowing us to use the box-mechanism!

always lie on the manifold

linearization point → in 1 iteration, the origin of my chart does not move

ONLY 2 DIFFERENCES:

- Jacobian by differentiating $\left.\frac{\partial e(x^* \boxplus \Delta x)}{\partial \Delta x}\right|_{\Delta x = \emptyset}$
- $x^* \leftarrow x^* \boxplus \Delta x$

use of the box plus operator

$\Gamma\ x^* \longrightarrow$ does not need to be a vector → can be a data structure only Δx needs to be a vector!!

Methodology

- State space - x → what truly is? (not how we represent it)
  - transformation 3D ⇒ SE(3)
  - line fitting / plane ...
    - is not euclidean: [normal + distance of the plane to origin (4 "coordinates", but normal constrained to unit sphere) / vector to origin to minimum plane ... ✗]
    - ⇒ euclidean parametrization for perturbation ⇓ SHOULD BE MINIMAL
    - e.g., minimal motions ... (azimuth + angle + move along normal)
- measurement space
  - · box minus operator → how to determine set of motion that ... 6 ...
- $h(x)$
- $e(x)$

ICP in 2D

① we know the correspondence

⇒ GOAL: find transformation that minimizes the distance between points

lines $\longleftrightarrow$ lines
(REGISTRATION)

↳ edge detection on the image
↓ line extraction
↳ we would want to register lines
(projective line registration)

1° - specify state space : $X \in SE(2)$, $\begin{bmatrix} R & t \\ \emptyset & 1 \end{bmatrix}$ → We can represent it in any form! (any convenient way)

↳ perturbation vector $\Delta x = (\Delta x \ \Delta y \ \Delta\theta)^T$
if $\emptyset$, would remain the same!

$X \boxplus \Delta x = v2t(\Delta x) \cdot X$
left multiplication ⟹ but could be right multiplication (by inverting $v2t(\Delta x)$)

$= \Delta X \cdot X$ → will remain in the manifold $SE(2)$!

2° - define the measurements
$z \in \mathbb{R}^2$ → are euclidean → does not need to be defined the box $\boxplus$ operator

$l^{[i]}(x) = R p^{[i]} + t$
$= X p^{[i]}$

$l^{[i]}(X \boxplus \Delta x) = (X \boxplus \Delta x)$
$= v2t(\Delta x)\, X p^{[i]}$

being a transformation matrix $\tilde{p}^{[i]}$

$$= R(\Delta\theta)\,\tilde{p}^{[i]} + \Delta t$$

$$\left.\frac{\partial h^{[i]}(x \boxplus \Delta x)}{\partial \Delta x}\right|_{\Delta x = 0} = \begin{pmatrix} \dfrac{\partial h(\cdot)}{\partial \Delta t} & \dfrac{\partial h(\cdot)}{\partial \Delta \theta} \end{pmatrix}$$

$[2 \times 2]$ $\quad$ $[2 \times 1]$

$$\left.\frac{\partial h}{\partial \Delta t}\right|_{\Delta x = 0} = I$$

$$\left.\frac{\partial h}{\partial \Delta\theta}\right|_{\Delta x = 0} = R'(0)\,\tilde{p}^{i}$$

$$= \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\text{skew symmetric matrix}} \tilde{p}^{[i]} = \begin{pmatrix} -\tilde{p}^{[i]}.y \\ \tilde{p}^{[i]}.x \end{pmatrix}$$

octave : $X = v2t(dx) * X;$

$\Downarrow$

single difference on the top 2d!

UNCERTAINTY – new manifold formulation returns a probability distribution over THE PERTURBATION VECTOR !

$H \rightarrow$ information matrix of the probability around the optimum

How to obtain the cov mat in the domain the manifold?

→ Apply 1st covariance propagation

$$X = X^* \boxplus \Delta x$$

function of random variable

random variable $= g\,x^*(\Delta x)$

random variable

has gaussian distribution

1st order propagation ≡ LINEARIZATION

$$J_x = \left.\frac{\partial X^* \boxplus \Delta x}{\partial \Delta x}\right|_{\Delta x = 0}$$

$$X \sim \mathcal{N}(X;\ X^*,\ J_x H^{-1} J_x^T)$$

project $\Delta x^{(i)}$ sigma points through $X^* \boxplus \Delta x^{(i)} = x^{(i)}$

Measurement uncertainty

$$e^{[i]} = h^{[i]} \boxminus z^{[i]}$$

→ remap $h^{[i]}$ and $z^{[i]}$ to euclidean mediums compatible with info matrix

→ assumed to be in the manifold

NOT CONFORMANT

$$H += J^{[i]} \Omega^{[i]} J^{[i]}$$

$$e = h \boxminus z$$

Can describe a gaussian distribution $\Rightarrow e^{[i]}(x) \sim \mathcal{N}\left(h^{[i]}(x) \boxminus z^{[i]}, \Omega_x\right)$

1^{st}-ORDER PROPAGATION

$$J_e^{z} \rightarrow \frac{\partial h \boxminus z}{\partial z}$$

$$\Omega_x^{[i]} = \left(J_e \Omega^{-1} J_e\right)^{-1}$$

or

Unscented Transformation

IF measurement space is not euclidean?

e ~~e~~ h ∃ z

worn case that
can happen

ALTERNATIVE

→ instead of densifying the error in the parameters space, densify the perturbation around the unconditionate

only computed 1! / once
⊕
always attached to the chart

From now on,

OPERATE ONLY ON THE MANIFOLDS!

- easier computations
- a lot of things become constant

→ at each iteration compute an EUCLIDEAN approximation of our problem

⇓

to put faster always singularities