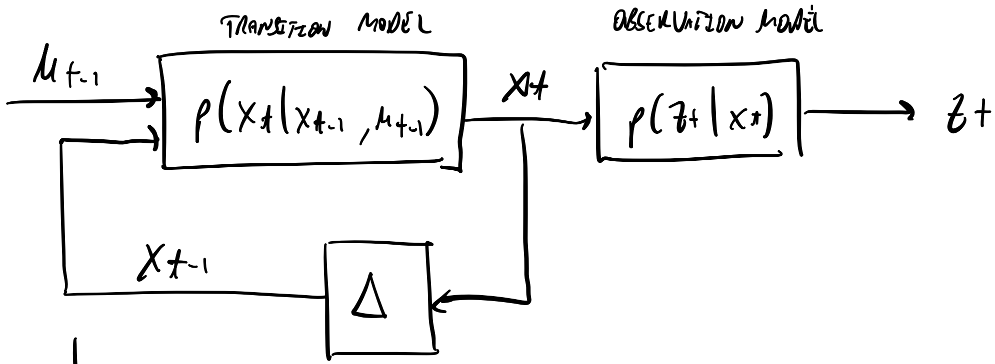
Kalman Filters

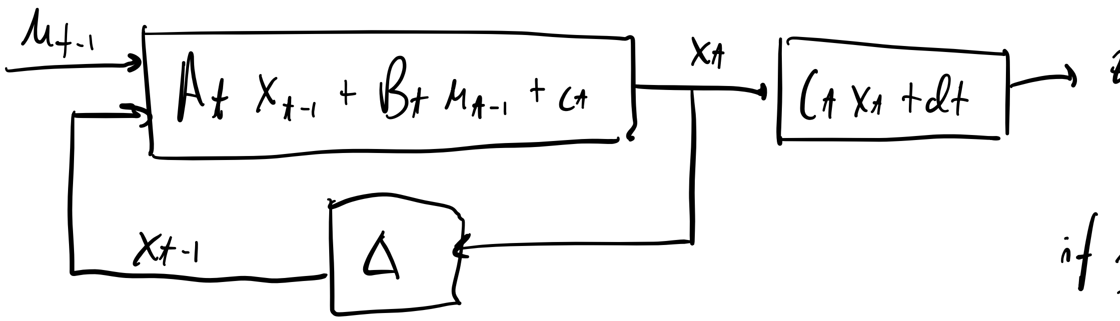
Note: everything on Gaussian manipulations is **indeed** Kalman derivation!!

Stochastic System Models



↓

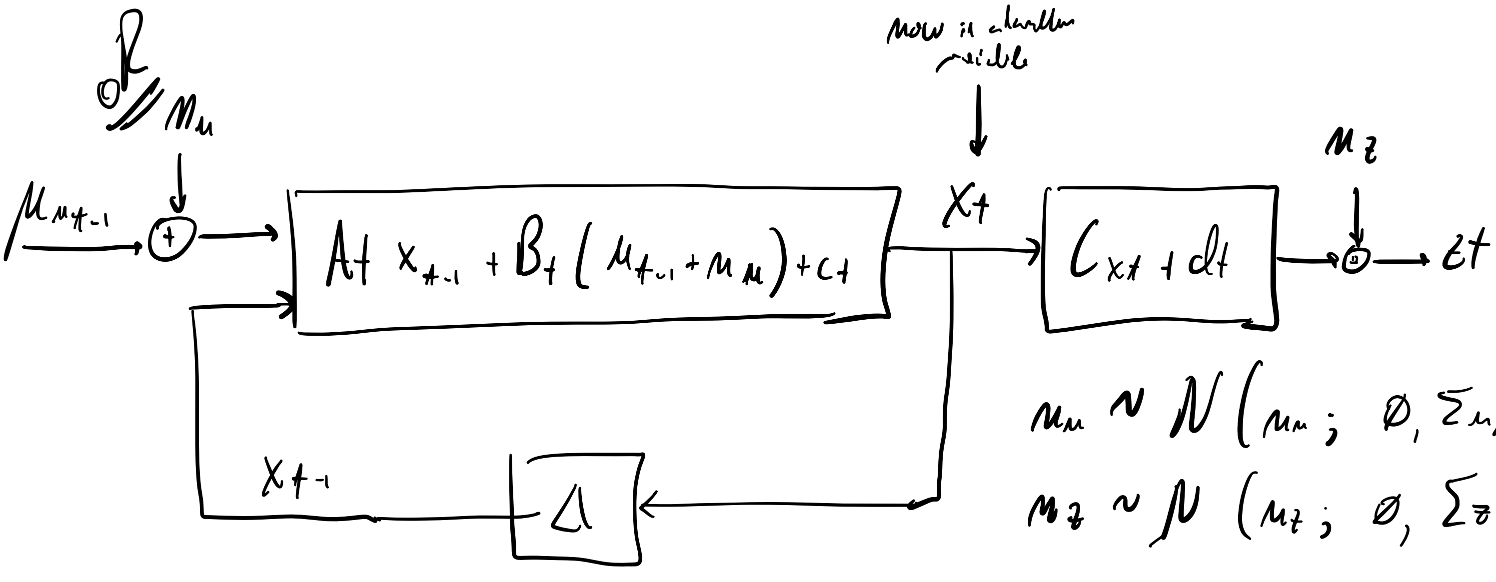
Linear Systems = transition + observation are affine transforms!



if x, z have random noise....

⇓ if we have gaussian noise

$$p(x_t | x_{t-1}, u_{t-1}) = \mathcal{N}(x_t;\ A x_{t-1} + B u_{t-1} + c,\ \Sigma_x)$$
$$p(z_t | x_t) = \mathcal{N}(z_t;\ C x_t + d_t,\ \Sigma_z)$$



$$n_u \sim \mathcal{N}(n_u;\ \emptyset, \Sigma_u)$$
$$n_z \sim \mathcal{N}(n_z;\ \emptyset, \Sigma_z)$$

$u_{t-1} \sim \mathcal{N}(u_{t-1} ; \mu_{u,t-1} , \Sigma_{u,t})$

describes the white noise $(\Sigma_{u,t} \equiv \Sigma_u$ of $u_u)$

FILTERING

Kalman Filter $\equiv$ Bayes Filter
- system linear
- noise gaussian

Considerations
- initial belief is Gaussian
- noise is Gaussian
- all function are affine

↓

Gaussian distributions are closed under
- affine transformation
- chain rule
- marginalization
- conditioning

⇒ belief remains Gaussian !

↳ PREDICT

(JOINT) RANDOM VARIABLE x — AFFINE TRANSFORMATION

$$X_t = \underbrace{(A_t \;\; B_t)}_{A} \begin{pmatrix} X_{t-1} \\ u_{t-1} \end{pmatrix} + c_t \Rightarrow X_{t|t-1}$$

2 variables are independent ↓ non-correlation factor is $\emptyset$

$$\underbrace{\begin{pmatrix} X_{t-1|t-1} \\ u_{t-1} \end{pmatrix}}_{\text{JOINT DISTRIBUTION}} \sim \mathcal{N}\left[\begin{pmatrix} \mu_{t-1|t-1} \\ \mu_{u,t-1} \end{pmatrix} ; \begin{pmatrix} \Sigma_{t-1|t-1} & \emptyset \\ \emptyset & \Sigma_{u,t-1} \end{pmatrix} \right]$$

1° 2 distributions as input → transition model ⇒ next state (predict)

2° apply affine transform theorem

$p(x_{t|t-1})$ → is still Gaussian

$$p(x_{t|t-1}) \sim \mathcal{N}(x_{t|t-1} ; \mu_{t|t-1} , \Sigma_{t|t-1})$$

$$\begin{cases} \mu_{t|t-1} = A_t \mu_{t-1|t-1} + B_t \mu_{u,t-1} + c_t \\ \Sigma_{t|t-1} = A_t \Sigma_{t-1|t-1} A_t^T + B_t \Sigma_u B_t^T \end{cases}$$

(PREDICT)

$$\mu_{t|t-1} = \begin{pmatrix} A_t & B_t \end{pmatrix} \begin{pmatrix} \mu_{t-1|t-1} \\ \mu_{u,t-1} \end{pmatrix} + C_t = A_t \cdot \mu_{t-1|t-1} + B_t \cdot \mu_{u,t-1} + C_t$$

$$\Sigma_{t|t-1} = \begin{pmatrix} A_t & B_t \end{pmatrix} \begin{pmatrix} \Sigma_{t-1|t-1} & \emptyset \\ \emptyset & \Sigma_{u,t-1} \end{pmatrix} \begin{pmatrix} A_t^T \\ B_t^T \end{pmatrix} =$$

$$= \begin{pmatrix} A_t & B_t \end{pmatrix} \begin{pmatrix} \Sigma_{t-1|t-1} A_t^T & \emptyset \\ \emptyset & \Sigma_{u,t-1} B_t^T \end{pmatrix} =$$

$$= A_t \Sigma_{t-1|t-1} A_t^T + B_t \Sigma_{u,t-1} B_t^T$$

⇒ **UPDATE** : using the information of observation to refine our state estimation

1. Chain rule

$$p(z_t | x_t) = \mathcal{N}(z_t ; C_t x_t + d_t , \Sigma_{z,t})$$

$$p(x_t) = p(x_{t|t-1})$$

$$= \mathcal{N}(x_{t|t-1} ; \mu_{t|t-1} , \Sigma_{t|t-1})$$

joint dist. describes the possible set of states I could be in and the possible set of measurements I would obtain

Compute the joint:

$$p(z_t, x_t) = \mathcal{N}\left[\begin{pmatrix} \mu_{t|t-1} \\ \mu_z \end{pmatrix} ; \begin{pmatrix} \overbrace{\Sigma_{t|t-1}}^{\Sigma_{aa}} & \overbrace{\Sigma_{t|t-1} C_t^T}^{\Sigma_{ab}} \\ \underbrace{C_t \Sigma_{t|t-1}}_{\Sigma_{ba}} & \underbrace{\Sigma_z + C_t \Sigma_{t|t-1} C_t^T}_{\Sigma_{bb}} \end{pmatrix} \right]$$

$$\mu_z = C_t \mu_{t|t-1} + d_t$$

2° — conditioning on the measurement itself

$$p(x_{t|t} \mid z_t) = \mathcal{N}(x_{t|t} ; \mu_{t|t} , \Sigma_{t|t})$$

$$\mu_{t|t} = \mu_{t|t-1} + \underbrace{\Sigma_{t|t-1} C_t^T \left[\Sigma_z + C_t \Sigma_{t|t-1} C_t^T\right]^{-1}}_{K_t} (z_t - \mu_z)$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \underbrace{\Sigma_{t|t-1} C_t^T \left[\Sigma_z + C_t \Sigma_{t|t-1} C_t^T\right]^{-1}}_{K_t} C_t \Sigma_{t|t-1}$$

PREDICT

IF ALL ASSUMPTIONS ARE EXPECTED

↳ we have a perfect estimation!

K_t → KALMAN GAIN

(explains how much difference between measurement and ~~expected~~ expected value influences current state)

NON-LINEAR SYSTEM

KF → EKF [to deal w/ non-linear system; NOT OPTIMAL

⇓

Linearize a system : linearization point (= most likely position = mean)

$$f(x,u) \simeq f(x_0, u_0) + \underbrace{\frac{\partial f(x_0,u)}{\partial x}}_{A} (x - x_0) + \underbrace{\frac{\partial f(x,u)}{\partial x}}_{B} (u - u_0)$$

$$= Ax + Bu + \underbrace{f(x_0, u_0) - Ax_0 - Bu_0}_{c}$$

Jacobian (symbolic)

$$h(x) \approx h(x_0) + \underbrace{\frac{\partial h(x)}{\partial x}}_{C} (x - x_0)$$

$$= Cx + \underbrace{h(x_0) - Cx_0}_{d}$$

<u>PREDICT</u>: incorporate new control

$$\mu_{t|t-1} = f(\mu_{t-1|t-1}, \mu_{u|t-1})$$

$$A_t = \left. \frac{\partial f(x, u)}{\partial x} \right|_{x = \mu_{t-1|t-1}}$$

$$B_t = \left. \frac{\partial f(x, u)}{\partial u} \right|_{u = \mu_{u,t-1}}$$

$$\Sigma_{t|t-1} = A_t \Sigma_{t-1|t-1} A_t^T + B_t \Sigma_u B_t^T$$

<u>UPDATE</u>: incorporate new measurement

$$\mu_z = h(\mu_{t|t-1})$$

$$C_t = \left. \frac{\partial h(x)}{\partial x} \right|_{x = \mu_{t|t-1}}$$

$$K_t = \Sigma_{t|t-1} C_t^T (\Sigma_z + C_t \Sigma_{t|t-1} C_t^T)^{-1}$$

$$\mu_{t|t} = \mu_{t|t-1} + K_t (z_t - \mu_z)$$

$$\Sigma_{t|t} = (I - K_t C_t) \Sigma_{t|t-1}$$