Gaussian Exercise

(note: no solution, may be wrong...)

Let $X_{t-1} \sim \mathcal{N}(\mu_x, \Sigma_x)$

$\quad\ \ u_{t-1} \sim \mathcal{N}(\mu_u, \Sigma_u)$

① Compute the parameters of the joint distribution $p(X_{t-1}, u_{t-1})$, assuming X and u are independent

$$\mu_{x,u} = \begin{pmatrix} \mu_x \\ \mu_u \end{pmatrix} \qquad \Sigma_{x,u} = \begin{pmatrix} \Sigma_x & \emptyset \\ \emptyset & \Sigma_u \end{pmatrix}$$

If independent, it means no affine transformation relates X with u
("ellipse is not rotated!")

② Consider the following affine transformation

$$X_t = \begin{pmatrix} A_t & B_t \end{pmatrix} \begin{pmatrix} X_{t-1} \\ u_{t-1} \end{pmatrix} + c_t \quad \rightsquigarrow \quad X_t = A\, X_{t-1} + B\, u_{t-1} + c_t$$

compute the parameters of $p(X_t)$, $\mu_{t|t-1}$ and $\Sigma_{t|t-1}$

$$X = \begin{pmatrix} X_{t-1} \\ \cancel{\mu}\, u_{t-1} \end{pmatrix} \longrightarrow$$

partitioning of a random vector X

①

$$\mu = \begin{pmatrix} \mu_{x_{t-1}} \\ \mu_{u_{t-1}} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$$

Σ_{ab}^{T} ///

...

Theorem 1: Marginalisation

$$p(x_a) = \mathcal{N}(x_a;\ \mu_a,\ \Sigma_{aa}) \longrightarrow$$

see - additional - details.pdf

Theorem 2: Conditioning

$$p(x_a | x_b) = \mathcal{N}(x_a;\ \mu_{a|b},\ \Sigma_{a|b})$$

$$\begin{cases} \mu_{a|b} = \mu_a + \Sigma_{ab} \Sigma_{bb}^{-1} (x_b - \mu_b) \\ \Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba} \end{cases}$$

OR //

$$\begin{cases} \mu_{a|b} = \mu_{a|b} - \Omega_{aa}^{-1} \Omega_{ab} (x_b - \mu_b) \\ \Sigma_{a|b} = \Omega_{aa}^{-1} \end{cases}$$

$$X_t = \begin{pmatrix} A_t & B_t \end{pmatrix} \begin{pmatrix} X_{t-1} \\ u_{t-1} \end{pmatrix} + C_t = A_t \cdot X_{t-1} + B_t \cdot u_{t-1} + C_t$$

↳ CONSTANT VALUE

$$\begin{cases} p(X_{t-1}) = \mathcal{N}(X_{t-1}; \mu_{t-1}, \Sigma_{t-1}) \\ p(u_{t-1}) = \mathcal{N}(u_{t-1}; \mu_{u_{t-1}}, \Sigma_{u_{t-1}}) \\ p(X_t) = \mathcal{N}(X_t; \mu_t, \Sigma_t) \end{cases}$$

$$p(X_{t|t-1}) = p(X_t \mid X_{t-1}, u_{t-1})$$

$$\mu_{t|t-1} = \mu_t + \Sigma_{t,t-1} \Sigma_{t-1,t-1}^{-1} ($$

this seem like the derivation of Kalman filter!