Probabilistic Robotics Course

Least Squares on Manifolds

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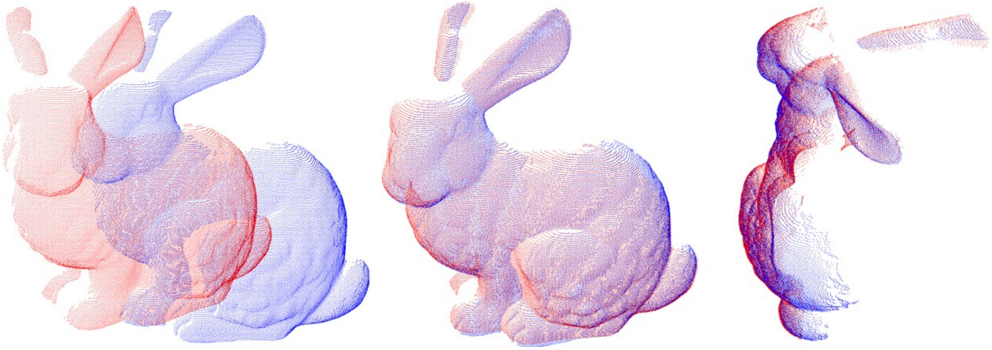
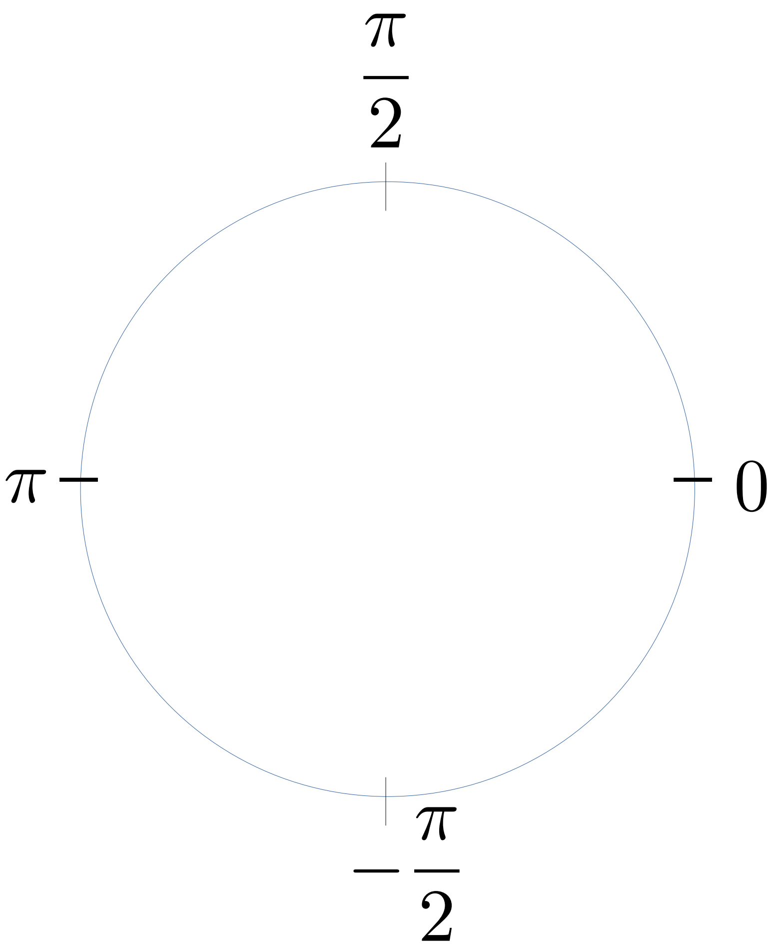


Courtesy of Li et. al

Non-Euclidean Spaces

In robotics we often encounter spaces that have a non-euclidean topology

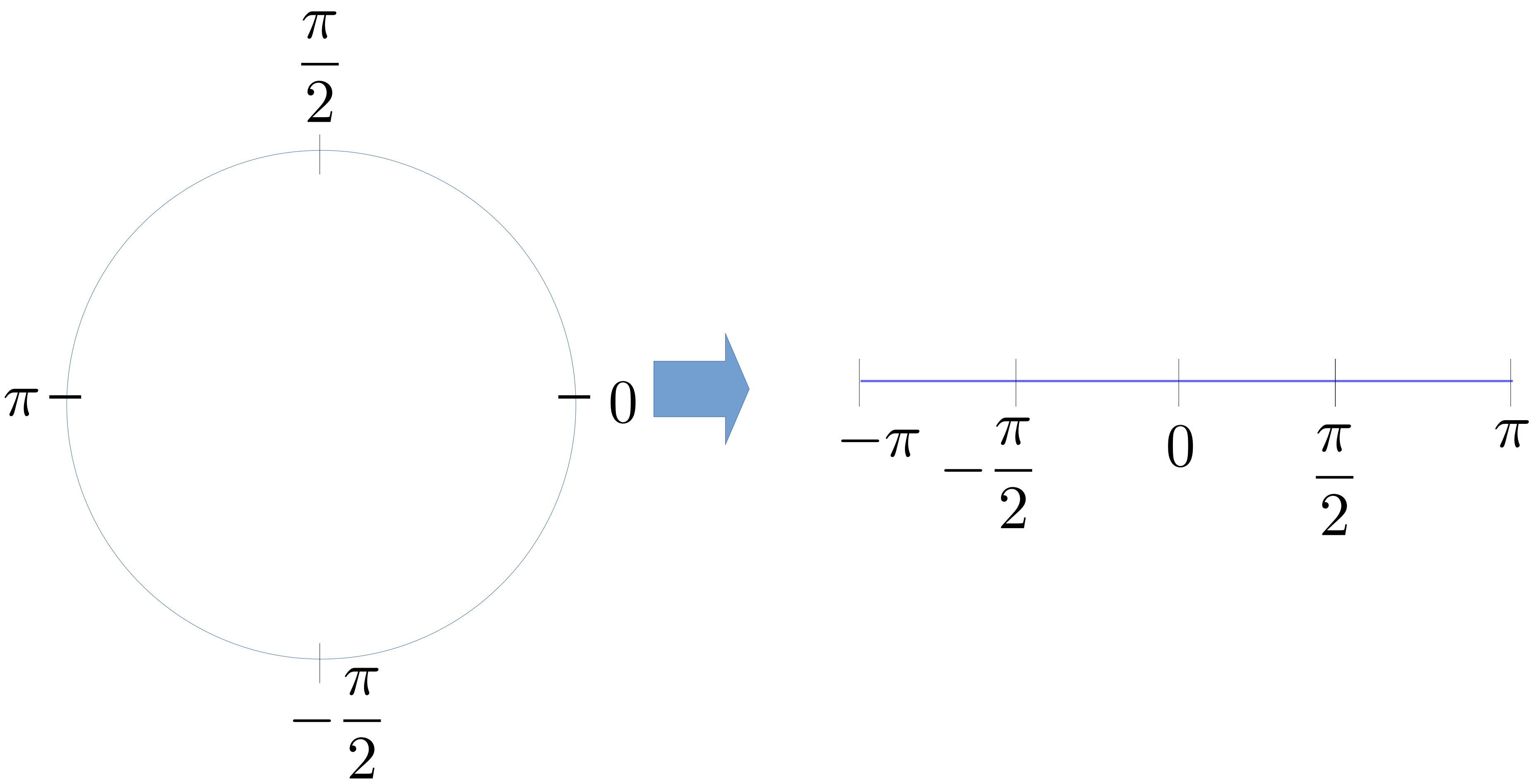
- E.g.: 2D angles



Non-Euclidean Spaces

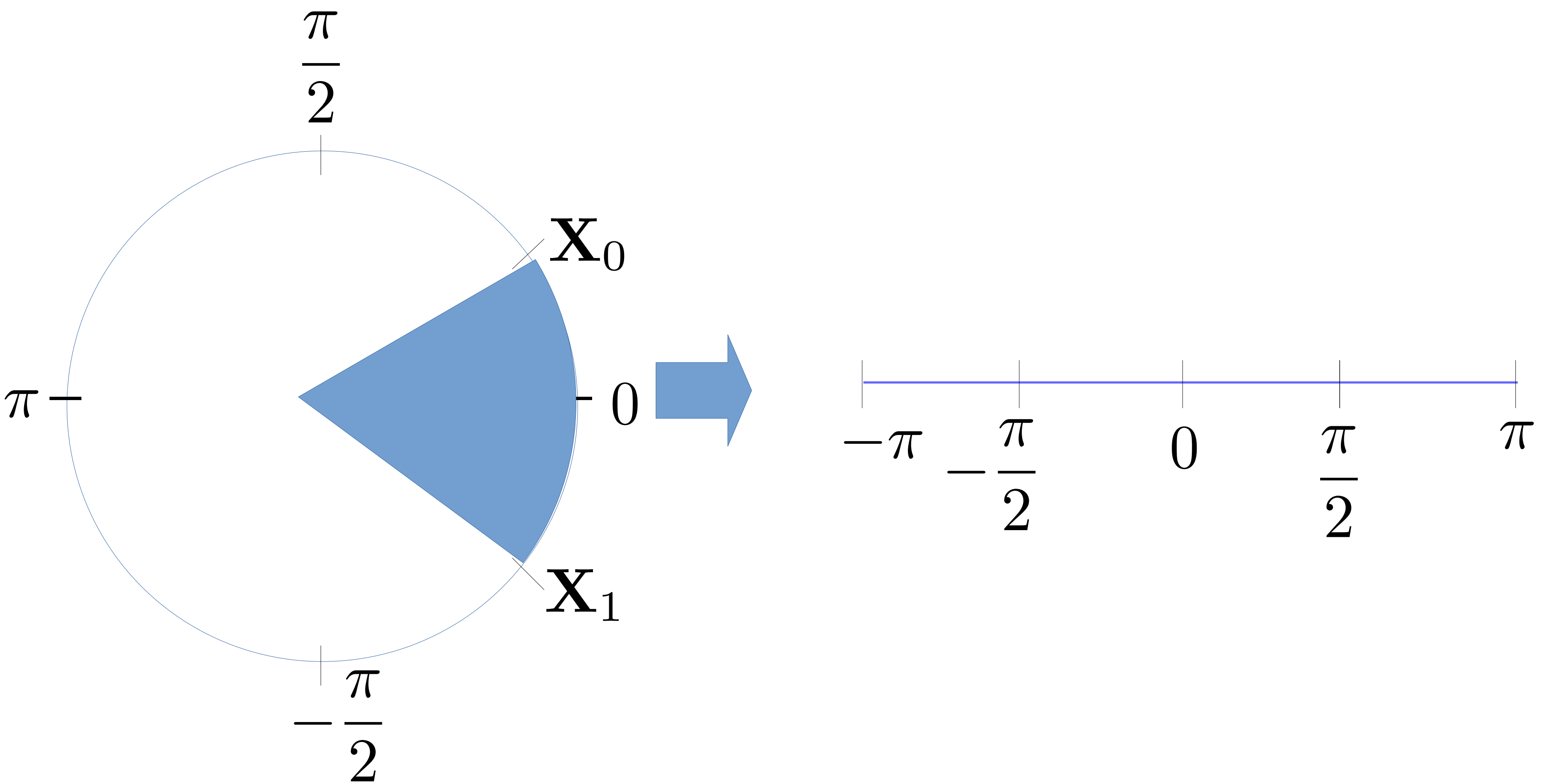
In such cases we commonly operate on a locally Euclidean parameterization

- E.g. we map the angles in the interval [-pi:pi]



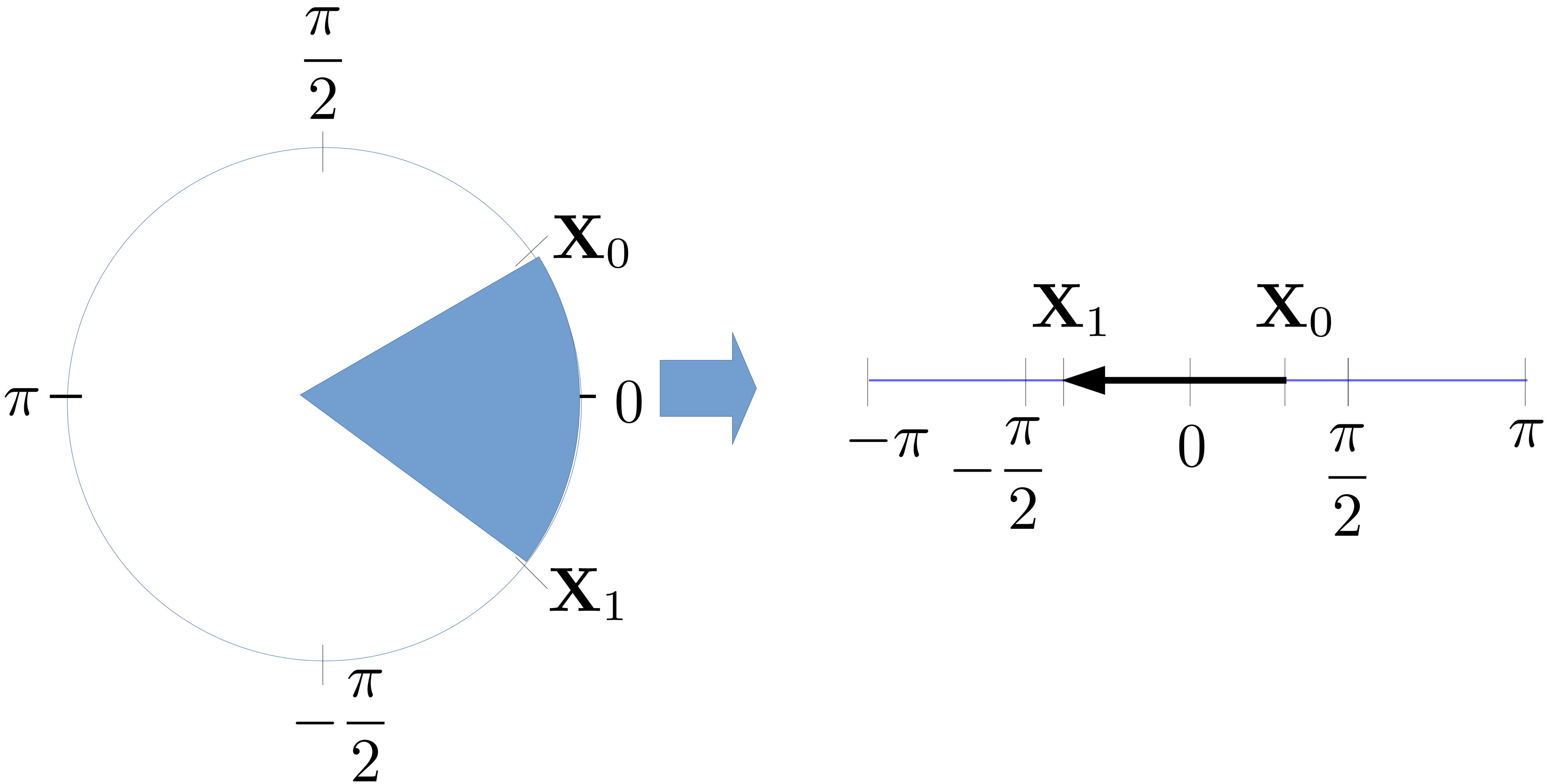
Non-Euclidean Spaces

We can then measure distances in the Euclidean mapping through a regular subtraction



Non-Euclidean Spaces

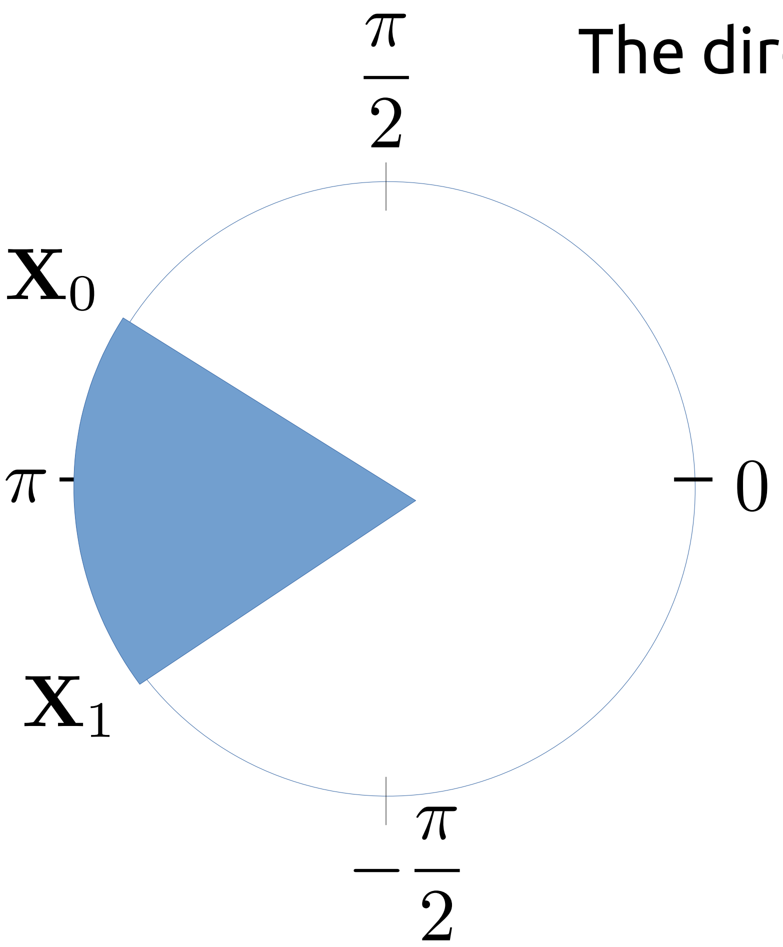
We can then measure distances in the Euclidean mapping through a regular subtraction

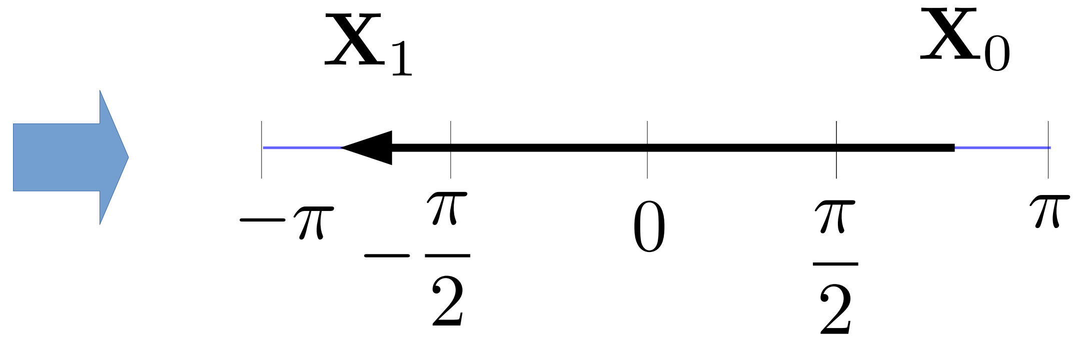


Non-Euclidean Spaces

We can then measure distances in the Euclidean mapping ~~through a regular subtraction~~

The direction of the distance is the opposite!!!

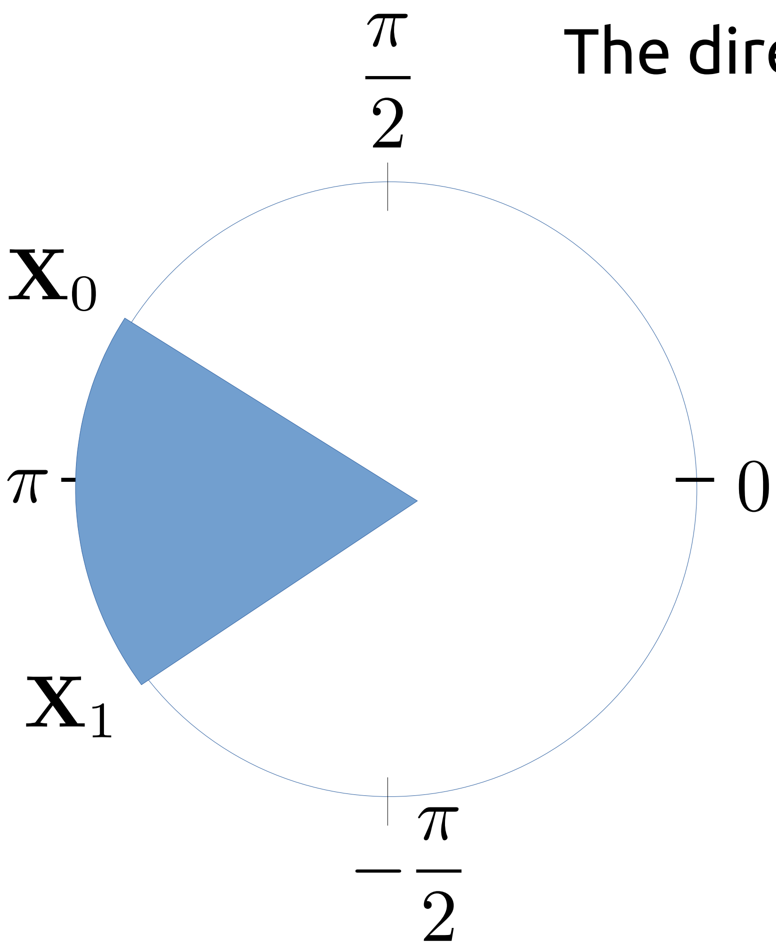


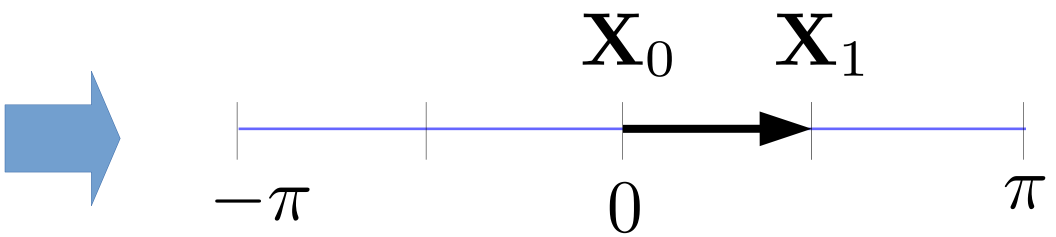


Non-Euclidean Spaces

Idea: when computing the distances, build the Euclidean mapping in the neighborhood of one of the points: the **chart around $\mathbf{X}_0$.**

The direction of the distance is the opposite!!!





Computing Differences

$\mathbf{x}_0$: start point, on manifold

$\mathbf{x}_1$: end point, on manifold

$\mathbf{\Delta x}$: difference, on chart

- Compute a chart around $\mathbf{x}_0$
- Compute the location of $\mathbf{x}_1$ on the chart
- Measure the difference between points in the chart
- Chart is Euclidean: $\mathbf{x}_0 = \mathbf{x}_1 \Rightarrow \mathbf{\Delta x} = 0$
- Use an operator $\mathbf{x}_1 \boxminus \mathbf{x}_0 = \mathbf{\Delta x}$

Hertzberg, Wagner, Frese and Schroeder 2011

Applying Differences

$\mathbf{X}_0$: start point, on manifold

$\mathbf{\Delta x}$: difference on chart

$\mathbf{X}_1$: end point, on manifold reachable from $\mathbf{X}_0$ by moving of $\mathbf{\Delta x}$ on the chart

- Compute a chart around $\mathbf{X}_0$
- Move of $\mathbf{\Delta x}$ on the chart and go back to the manifold
- Encapsulate the operation with an operator

$$\mathbf{X}_0 \boxplus \mathbf{\Delta x} = \mathbf{X}_1$$

Euclidean Least Squares

Clear **H** and **b**

$$\mathbf{H} \leftarrow 0 \qquad \mathbf{b} \leftarrow 0$$

For each measurement, update h and b

$$\begin{aligned}
\mathbf{e}^{[i]} &\leftarrow \mathbf{h}^{[i]}(\mathbf{x}^*) - \mathbf{z}^{[i]} \\
\mathbf{J}^{[i]} &\leftarrow \left.\frac{\partial \mathbf{e}^{[i]}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^*} \\
\mathbf{H} &\leftarrow \mathbf{H} + \mathbf{J}^{[i]T}\boldsymbol{\Omega}^{[i]}\mathbf{J}^{[i]} \\
\mathbf{b} &\leftarrow \mathbf{b} + \mathbf{J}^{[i]T}\boldsymbol{\Omega}^{[i]}\mathbf{e}^{[i]}
\end{aligned}$$

Update the estimate with the perturbation

$$\begin{aligned}
\boldsymbol{\Delta}\mathbf{x} &\leftarrow \mathrm{solve}(\mathbf{H}\boldsymbol{\Delta}\mathbf{x} = -\mathbf{b}) \\
\mathbf{x}^* &\leftarrow \mathbf{x}^* + \boldsymbol{\Delta}\mathbf{x}
\end{aligned}$$

Gauss in Non Euclidean Spaces

Beware of the + and – operators

- Error function

$$\mathbf{e}^{[i]}(\mathbf{x}) \quad = \quad \mathbf{h}^{[i]}(\mathbf{x}) \boxminus \mathbf{z}_i$$

- Taylor expansion

$$\mathbf{e}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x}) \quad = \quad \underbrace{\mathbf{e}^{[i]}(\mathbf{X})}_{\mathbf{e}^{[i]}} + \underbrace{\left.\frac{\partial \mathbf{e}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x})}{\partial \mathbf{\Delta x}}\right|_{\mathbf{\Delta x}=\mathbf{0}}}_{\mathbf{J}^{[i]}} \mathbf{\Delta x}$$

- Increments

$$\mathbf{X} \leftarrow \mathbf{X} \boxplus \mathbf{\Delta x}$$

Manifold Least Squares

Clear **H** and **b**

$$\mathbf{H} \leftarrow 0 \qquad \mathbf{b} \leftarrow 0$$

For each measurement

$$\begin{aligned}
\mathbf{e}^{[i]} &\leftarrow \mathbf{h}^{[i]}(\mathbf{X}^*) \boxminus \mathbf{Z}^{[i]} \\
\mathbf{J}^{[i]} &\leftarrow \left. \frac{\partial \mathbf{e}(\mathbf{X}^* \boxplus \boldsymbol{\Delta}\mathbf{x})}{\partial \boldsymbol{\Delta}\mathbf{x}} \right|_{\boldsymbol{\Delta}\mathbf{x}=\mathbf{0}} \\
\mathbf{H} &\mathrel{+}= \mathbf{J}^{[i]T} \boldsymbol{\Omega}^{[i]} \mathbf{J}^{[i]} \\
\mathbf{b} &\mathrel{+}= \mathbf{J}^{[i]T} \boldsymbol{\Omega}^{[i]} \mathbf{e}^{[i]}
\end{aligned}$$

Compute and apply the perturbation

$$\begin{aligned}
\boldsymbol{\Delta}\mathbf{x} &\leftarrow \mathrm{solve}(\mathbf{H}\boldsymbol{\Delta}\mathbf{x} = -\mathbf{b}) \\
\mathbf{X}^* &\leftarrow \mathbf{X}^* \boxplus \boldsymbol{\Delta}\mathbf{x}
\end{aligned}$$

Methodology

State space **X**

- Qualify the Domain
- Define an Euclidean parameterization for the perturbation
- Define boxplus operator

Measurement space(s) **Z**

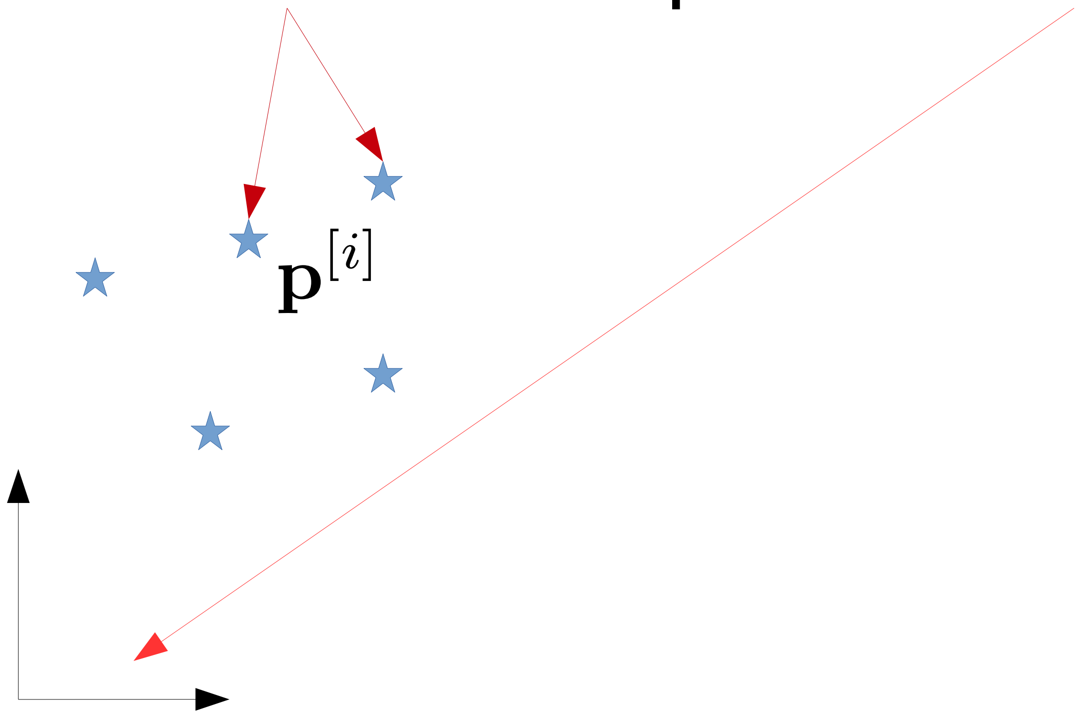
- Qualify the Domain
- Define an Euclidean parameterization for the perturbation
- Define boxminus operator

Identify the prediction functions **h(X)**

Define the error functions **e(X)**

Example ICP Optimization 2D

Given a set of 2D points in the world frame

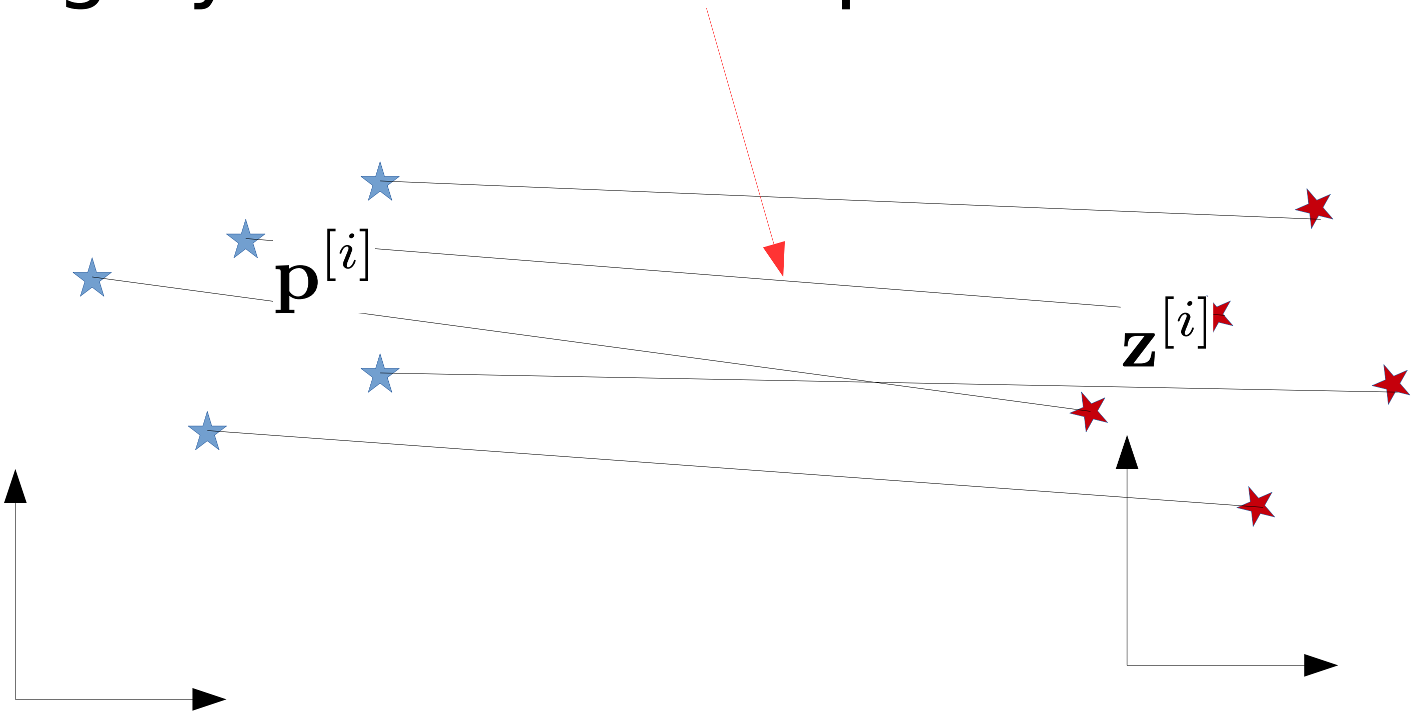


Example ICP Optimization 2D

A set of 2D measurements in the robot frame

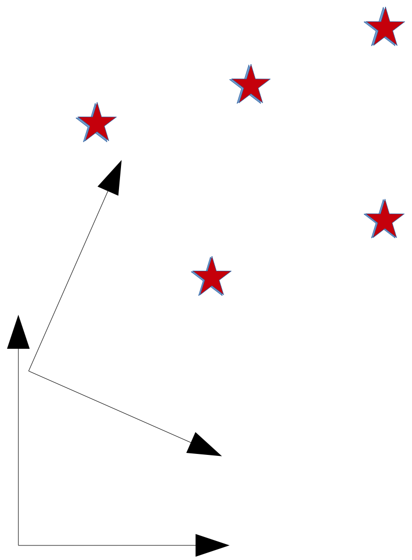
Example ICP Optimization 2D

Roughly known correspondences



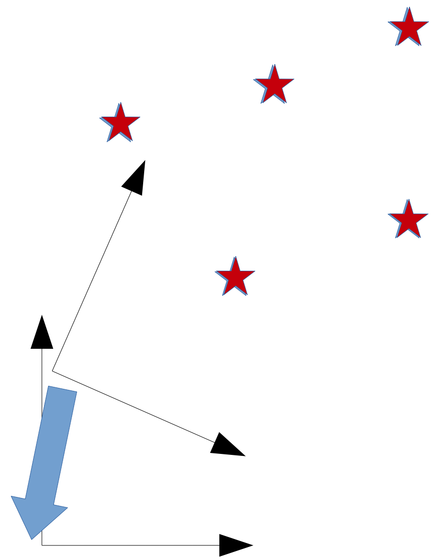
Example ICP Optimization 2D

We want to find a transform that minimizes distance between corresponding points



Example ICP Optimization 2D

Such a transform will be the pose of world w.r.t. robot



Note: we can also estimate robot w.r.t world, but it leads to longer calculations

ICP: State Space

State

$$\mathbf{X} \in SE(2), \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

Manifold representation as homogeneous transformation

$$\mathbf{\Delta x} = (\underbrace{\Delta x \ \Delta y}_{\mathbf{\Delta t}} \ \Delta\theta)^T$$

Euclidean parameterization for the chart

$$\mathbf{\Delta X} = v2t(\mathbf{\Delta x}) = \begin{bmatrix} \mathbf{\Delta R} & \mathbf{\Delta t} \\ \mathbf{0} & 1 \end{bmatrix}$$

Convenient function that converts a perturbation into a matrix

$$\begin{aligned} \mathbf{X} \boxplus \mathbf{\Delta x} &= v2t(\mathbf{\Delta x})\mathbf{X} \\ &= \mathbf{\Delta X} \cdot \mathbf{X} \end{aligned}$$

Definition of the boxplus operator

ICP: Measurements

$$
\begin{aligned}
\mathbf{z} &\in \Re^2 \\
\mathbf{h}^{[i]}(\mathbf{X}) &= \mathbf{R}\mathbf{p}^{[i]} + \mathbf{t} \\
&= \mathbf{X}\mathbf{p}^{[i]} \\
\mathbf{h}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x}) &= (\mathbf{X} \boxplus \mathbf{\Delta x})\mathbf{p}^{[i]} \\
&= v2t(\mathbf{\Delta x})\,\underbrace{\mathbf{X}\mathbf{p}^{[i]}}_{\tilde{\mathbf{p}}^{[i]}} \\
&= \mathbf{R}(\Delta\theta)\tilde{\mathbf{p}}^{[i]} + \mathbf{\Delta t}
\end{aligned}
$$

ICP: Jacobian

$$\mathbf{h}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x}) = \mathbf{R}(\mathbf{\Delta}\theta)\tilde{\mathbf{p}}^{[i]} + \mathbf{\Delta t}$$

$$\left.\frac{\partial \mathbf{h}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x})}{\partial \mathbf{\Delta x}}\right|_{\mathbf{\Delta x}=0} = \left.\left(\frac{\partial \mathbf{h}^{[i]}(\cdot)}{\partial \mathbf{\Delta t}} \;\; \frac{\partial \mathbf{h}^{[i]}(\cdot)}{\partial \Delta \theta} \right)\right|_{\mathbf{\Delta x}=0}$$

$$\left.\frac{\partial \mathbf{h}^{[i]}(\cdot)}{\partial \mathbf{\Delta t}}\right|_{\mathbf{\Delta x}=0} = \mathbf{I}$$

$$\left.\frac{\partial \mathbf{h}^{[i]}(\cdot)}{\partial \Delta \theta}\right|_{\mathbf{\Delta x}=0} = \mathbf{R}'(0)\tilde{\mathbf{p}}^{[i]}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{\mathbf{p}}^{[i]} = \begin{pmatrix} -\tilde{p}^{[i]}.y \\ \tilde{p}^{[i]}.x \end{pmatrix}$$

ICP: Octave Program

```
function [e,J]=errorAndJacobianManifold(X,p,z)
 t=X(1:2,3);
 R=X(1:2,1:2);
 z_hat=R*p+t;
 e=z_hat-z;

 J=zeros(2,3);
 J(1:2,1:2)=eye(2);
 J(1:2,3)=[-z_hat(2),
   z_hat(1)]';
endfunction;
```

ICP: Octave Program

```
function [chi,X]=icp2dManifold(X,P,Z)
  chi=0;           %cumulative chi2
  H=zeros(3,3); %accumulators for H and b
  b=zeros(3,1);
  for(i=1:size(P,2))
     p=P(:,i); z=Z(:,i);
     [e,J]=errorAndJacobianManifold(X,p,z);
       H+=J'*J;                %assemble H and B
     b+=J'*e;
     chi+=e'*e;                %update cumulative error
  endfor
  dx=-H\b;                     %solve the linear system
  X=v2t(dx)*X;                      %apply perturbation
endfunction
```

Uncertainty of the Solution

Optimizing on a Manifold generates a H matrix that is computed **on the chart**

$\mathbf{H}^{-1}$ represents a covariance of the solution around the origin of the chart

We can write that around the optimum

$$\mathbf{\Delta x} \sim \mathcal{N}(\mathbf{\Delta x}; \mathbf{0}, \mathbf{H}^{-1})$$

The Gaussian approximation of the distribution of solution around the optimum, is related to the chart through boxplus

$$\begin{aligned} \mathbf{X} &= \mathbf{X}^* \boxplus \mathbf{\Delta x} \\ &= g_{\mathbf{X}^*}(\mathbf{\Delta x}) \end{aligned}$$

Uncertainty (cont)

Using our manipulation skills on the Gaussian distribution, we can compute the approximation of a Gaussian in X, by either

- Linearization

$$\mathbf{J_X} = \left. \frac{\partial \mathbf{X}^* \boxplus \mathbf{\Delta x}}{\partial \mathbf{\Delta x}} \right|_{\mathbf{\Delta x}=\mathbf{0}}$$

$$\mathbf{X} \sim \mathcal{N}(\mathbf{X}; \mathbf{X}^*, \mathbf{J_X} \mathbf{H}^{-1} \mathbf{J_X}^T)$$

- Unscented Transform

$$\mathbf{X}^{(i)} = \mathbf{X}^* \boxplus \mathbf{\Delta x}^{(i)}$$

Where $\mathbf{\Delta x}^{(i)}$ are sigma points extracted from the Gaussian distribution in the chart.

Measurement Uncertainty

We can express the uncertainty in the measurements on a chart centered on the measurement mean

chart perturbation

$$\mathbf{\Delta z}^{[i]} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}^{[i]}_{\mathbf{\Delta z}})$$

Random variable obtained by composing

$$\mathbf{Z}^{[i]} = \mu^{[i]}_{\mathbf{Z}} \boxplus \mathbf{\Delta z}^{[i]}$$

measurement mean

perturbation

$$\mathbf{e}^{[i]}(\mathbf{X}) \sim \mathcal{N}(\underbrace{\mathbf{h}^{[i]}(\mathbf{X}) \boxminus \mu^{[i]}_{\mathbf{Z}}}, \mathbf{\Sigma}^{[i]}_{\mathbf{\Delta z}})$$

This is the difference between prediction and mean **on the chart**

Manifold Least Squares (Omega on Chart)

Clear **H** and **b**

$$\mathbf{H} \leftarrow 0 \qquad \mathbf{b} \leftarrow 0$$

For each measurement

$$\begin{aligned}
\mathbf{e}^{[i]} &\leftarrow \mathbf{h}^{[i]}(\mathbf{X}^*) \boxminus \mu_{\mathbf{Z}}^{[i]} \\
\mathbf{J}^{[i]} &\leftarrow \left.\frac{\partial \mathbf{e}(\mathbf{X}^* \boxplus \mathbf{\Delta x})}{\partial \mathbf{\Delta x}}\right|_{\mathbf{\Delta x}=\mathbf{0}} \\
\mathbf{H} &\mathrel{+}= \mathbf{J}^{T\,[i]} \mathbf{\Omega}^{[i]} \mathbf{J}^{[i]} \\
\mathbf{b} &\mathrel{+}= \mathbf{J}^{T\,[i]} \mathbf{\Omega}^{[i]} \mathbf{e}^{[i]}
\end{aligned}$$

Compute and apply the perturbation

$$\begin{aligned}
\mathbf{\Delta x} &\leftarrow \mathrm{solve}(\mathbf{H}\mathbf{\Delta x} = -\mathbf{b}) \\
\mathbf{X} &\leftarrow \mathbf{X} \boxplus \mathbf{\Delta x}
\end{aligned}$$

Conclusions

- Least squares on smooth manifolds offers a more robust formulation of non-linear least squares on non-Euclidean spaces
- Key idea: linearize the problem with respect to the current optimum, around the perturbations
- Using the boxplus and boxminus operators to encapsulate operations on the manifolds

  Download the octave code and compare plain vs manifold ICP
- Beware of the Uncertainties