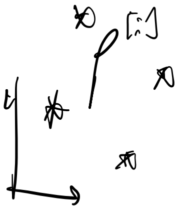
Registration in a Manifold

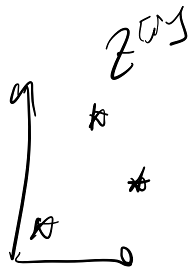
Standard example → registration in
(in the exam) a weird domain
(linear, transformation + scale, ...)

domain:
transformation + scaling factor
⇑



⇒ known map
(car position of satellites)
e.g.

roughly known correspondences....



→ observed points SCALED! (zoom factor)

if we knew scale, we would be able to register them

is also possible w/ linear relaxation

4D State ⇒ member of group of similarities

$$X = \begin{pmatrix} R & t \\ \emptyset & s \end{pmatrix} \in Sim3$$

⇒ allows to compose regular transformations

only 1 data structure

scaling factor (usually, here is 1)

$$\Delta x = (\underbrace{\Delta x \ \Delta y \ \Delta z}_{\Delta t} \ \underbrace{\Delta\alpha_x \ \Delta\alpha_y \ \Delta\alpha_z}_{\Delta R} \ ; \ \underbrace{\Delta s}_{\log s})^T$$

from gaussian formulation? .....

may be 3 elements
- $\mathbb{R}^4$ – quaternion
- $\mathbb{R}^3$ – translation
- $\mathbb{R}^+$ – scale

when $s = 1$, $\log 1 \Rightarrow \Delta s = \emptyset$

$$X \boxplus \Delta x = v2s(\Delta x) X$$

perturbation vector must be in a way that if it is $\emptyset$ and apply to one element of the manifold

$$X \boxplus \Delta x = X$$

when $s = \emptyset \Rightarrow \Delta s$ should be $\emptyset$ (identity matrix)

in our mapping will do $\log(s)$

(when identity is 1)

$\Downarrow$

small trick in manifold

$$v2s = \begin{pmatrix} R(\Delta\alpha) & \Delta t \\ \emptyset & \exp(\Delta\alpha) \end{pmatrix}$$

$\Downarrow$

vector 2 similarity

invert the operation

$\rightarrow$ if $\Delta X = \emptyset \Rightarrow v2s = I_{4\times 4}$

$\Downarrow$

as supposed to be

3D MEASUREMENTS $\rightarrow$ we just points $t \in \mathbb{R}^3$

$$h^{[i]}(X) = s\underbrace{\left[R p^{[i]} + t\right]}_{p'^{[i]}} \Rightarrow \text{WORLD} \Rightarrow \text{ROBOT FRAME}$$

$$h^{[i]}(X \boxplus \Delta x) = \exp(\Delta s) \cdot \left(R(\Delta\alpha) \cdot s \cdot \left[R p^{[i]} + t\right] + \Delta t\right)$$

prediction function
for the similarity domain

↳ in this case ⊟
not a problem
(measurements are euclidean)

$$e^{[:]}\left(X \boxplus \Delta x\right)_{3\times 7}$$

$$\left[\quad\right]_{x}$$

↑
other matrix?

4) Exercise

motion ⇒ in the exercise

simulation ⇒ is the demonstration that shows

damping ⟶ get the system to converge a bit bit slower
⊕
getting away to run w/ different conditions

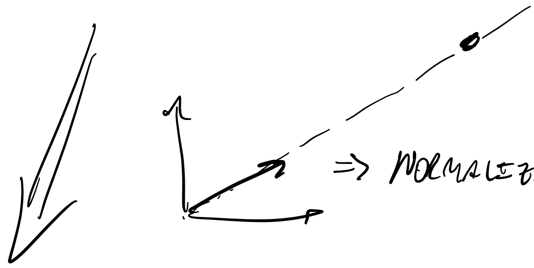
⇓

$\boxed{\text{Exercise}}$

Model: N points in 3D space

State : position ~~w.r.t.~~ w.r.t. robot

Observation => direction vectors !



~ bearing-only localization in SE(2)

generalization of camera problems

POINT LIES IN A SPHERE

$z = (x \ y \ z)^T \quad \| z \| = 1 \in S^2$

$$h^{[i]}(x) = \frac{R p^{[i]} + t}{\| R p^{[i]} + t \|}$$

similar to general 3D ICP with a normalization

do not use $\boxplus$ operator (for now / initially)

map S^2 in $\mathbb{R}^3$? ....

NORM / EXERCISE

CHAIN RULE CAN BE USED FOR NORMALIZATION FUNCTION

Euclidean Least Squares

$H \leftarrow \emptyset$, $b \leftarrow \emptyset$

For each measurement, update H and b

$$e^{[i]} \leftarrow h^{[i]}(x^*) - z^{[i]}$$

$$J^{[i]} \leftarrow \left.\frac{\partial e^{[i]}(x)}{\partial x}\right|_{x=x^*}$$

$$H \leftarrow H + J^{[i]^T} \Omega^{[i]} J^{[i]}$$

← 1st order propagation / linearization

$$b \leftarrow b + J^{[i]^T} \Omega^{[i]} e^{[i]}$$

UPDATE THE ESTIMATE WITH THE PERTURBATION

$$\Delta x \leftarrow \text{solve}\left(H \Delta x = -b\right)$$

$$x^* \leftarrow x^* + \Delta x$$

Manifold Least Squares

$H \leftarrow \emptyset$, $b \leftarrow \emptyset$

For each measurement

$$e^{[i]} \leftarrow h^{[i]}(x^*) \boxminus z^{[i]}$$

$$J^{[i]} \leftarrow \left.\frac{\partial e(x^* \boxplus \Delta x)}{\partial \Delta x}\right|_{\Delta x=\emptyset}$$

$$H \mathrel{+}= J^{[i]^T} \Omega^{[i]} J^{[i]}$$

$$b \mathrel{+}= J^{[i]^T} \Omega^{[i]} e^{[i]}$$

$$\Delta x \leftarrow \text{solve}\left(H \Delta x = -b\right)$$

$$x^* \leftarrow x^* \boxplus \Delta x$$

⇓

Methodology:

1. State domain X
   - qualify domain
   - define euclidean parametrization for the perturbation
   - define boxplus $\boxplus$ operator
2. Measurement space Z
   - qualify domain
   - define euclidean parametrization for the perturbation
   - define boxminus $\boxminus$ operator
3. Identify prediction function $h(x)$
4. Define error function $e(x)$

State:

$X = [R|t] \in SE(3)$ — transformation of the world w.r.t. robot coordinate frame

3D ✓

$$\Delta x = (\underbrace{\Delta x \ \Delta y \ \Delta z}_{\Delta t} \ \underbrace{\Delta\alpha_x \ \Delta\alpha_y \ \Delta\alpha_z}_{\Delta\alpha})^T \in \mathbb{R}^6$$

$$X \boxplus \Delta x = v2t(\Delta x) \cdot X =$$

$$= \begin{bmatrix} R(\Delta\alpha) & \Delta t \\ \emptyset & 1 \end{bmatrix} \cdot \begin{bmatrix} R & t \\ \emptyset & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} R(\Delta\alpha) \cdot R & R(\Delta\alpha) t + \Delta t \\ \emptyset & 1 \end{bmatrix} = \left[R(\Delta\alpha) \cdot R \mid R(\Delta\alpha) \cdot t + \Delta t \right]$$

Measurements:

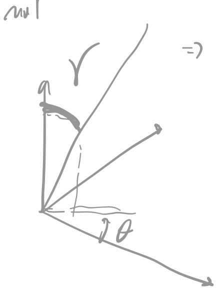
$z = (x \ y \ z)^T \in \cancel{\mathbb{R}^3}$

This way facilitates in terms of it being euclidean! → why not

$$h(x^*) = \frac{R p^{[i]} + t}{\| R p^{[i]} + t \|}$$

⇓

$\cancel{\boxminus \equiv -}$



⇒ indeed, we only observe a direction!

↳ measurement $z = (x, y, z)^T \in S^2$ → DOMAIN IS A MANIFOLD ok

$$h(x^*) = \text{norm}\left(h_{3D,ICP}(x^*)\right)$$

$$h_{3D,ICP}(x^*) = R \cdot p^{[i]} + t$$

$$\text{norm}(y) = \frac{y}{\|y\|} = \frac{y}{\sqrt{y_x^2 + y_y^2 + y_z^2}}$$

(Sphere)

$\Downarrow$

$$z = (x\ y\ z)^T \in S^2$$

$$z_1 \boxminus z_2 = \Delta z = (\theta, \varphi)$$

$\parallel$

relative angle on azimuth and elevation

$$h_{3D,ICP}(x^* \boxplus \Delta x) = R(\Delta\alpha) \cdot R\, p^{[i]} + R(\Delta\alpha)\, t + \Delta t \quad \Big| \quad = R(\Delta\alpha)\, p^{[i]} + \cancel{\Delta t}$$

$x^* = \emptyset$? how to do with $x \boxplus \Delta x$ as x were the origin? ....

1st alternative (complicated):

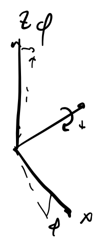
- construct rotation matrix $R(z) = R(\theta)\, R(\varphi)$ ——— $= R_z(\theta)\, R_y(\varphi)$? $= \begin{bmatrix} c_\theta c_\varphi & -s_\theta & c_\theta s_\varphi \\ s_\theta c_\varphi & c_\theta & s_\theta s_\varphi \\ -s_\varphi & \emptyset & c_\varphi \end{bmatrix}$ ✓

$$R_z(\theta) = \begin{bmatrix} c_\theta & -s_\theta & \emptyset \\ s_\theta & c_\theta & \emptyset \\ \emptyset & \emptyset & 1 \end{bmatrix} \qquad R_y(\varphi) = \begin{bmatrix} c_\varphi & \emptyset & s_\varphi \\ \emptyset & 1 & \emptyset \\ -s_\varphi & \emptyset & c_\varphi \end{bmatrix}$$

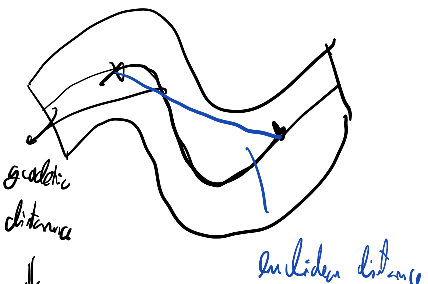
- compute rotation matrix for both z_1 and z_2
- compute the rotation difference as $R_2^T R_1$
- extract azimuth and elevation from rotation difference

$$\theta = -\text{atan2}(r_{12}, r_{22}) \quad ✓$$

$$\varphi = -\text{atan2}(r_{31}, r_{33})$$



2nd alternative (simplified): treat z as euclidean



$\Downarrow$

has some benefits!

$$\text{norm}(v) = \frac{v}{\sqrt{v^T v}}$$

$$\partial\, x^2 + y^2 + z^2 = (2x\ \ 2y\ \ 2z) \;\; \parallel \;\; 2v^T$$

$$\frac{\partial \sqrt{v^T v}}{\partial v} = (v^T v)' \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{v^T v}} = \frac{2 \cdot v^T}{2\|v\|} = \frac{v^T}{\|v\|}$$

$$\frac{\partial\, \text{norm}(v)}{\partial v} = \frac{v' \|v\| - v \|v\|'}{\|v\|^2} = \frac{I \cdot \|v\| - v\, v^T / \|v\|}{\|v\|^2}$$

$$= \frac{I}{\|v\|} - \frac{v\, v^T}{\|v\|^3}$$

$\Downarrow$

$$e^{[i]}(x) = h^{[i]}(x) - z^{[i]}$$

$$h^{[i]}(x) = \text{normalize}\left(R\, p^{[i]} + t\right)$$

$p'^{[i]}$ = point w.r.t robot coordinate frame (≈ estimated coordinate)

$\Downarrow$

$$h^{[i]}(X \boxplus \Delta x) = \text{normalize}\big(\underbrace{R(\Delta\alpha)\, p'^{[i]} + \Delta t}_{h_{ICP}}\big)$$

chain rule

$$\frac{\partial e(x \boxplus \Delta x)}{\partial \Delta x}\bigg|_{\Delta x = \emptyset} = \frac{\partial\, \text{normalize}(p)}{\partial p}\bigg|_{p = p'^{[i]}} \frac{\partial h_{ICP}(x \boxplus x)}{\partial \Delta x}\bigg|_{\Delta x = \emptyset}$$

$$= \left(\frac{1}{\|v\|} \cdot I - \frac{1}{\|v\|^3} \cdot v v^T\right) \cdot \left[\, I \qquad - \left[p'^{[i]}\right]_x \right]$$

$v = p'^{[i]}$

$$\left[p'^{[i]}\right]_x = \begin{bmatrix} \emptyset & p_z' & -p_y' \\ -p_z' & \emptyset & p_x' \\ p_y' & -p_x' & \emptyset \end{bmatrix} = -\text{skew}\left(p'^{[i]}\right)$$

remember properties of deriving rotation matrices !!!

???

$$\frac{\partial\left(R(\Delta\alpha)\, p' + \Delta t\right)}{\partial \Delta\alpha} = \frac{\partial R}{\partial \Delta\alpha} \cdot p' + \Delta t$$

$R = R_x R_y R_z$

$$\frac{\partial}{\partial \Delta x} = R'_{\Delta\alpha_x}\Big|_{\Delta\alpha_x = \emptyset} \cdot \underbrace{R_{\Delta\alpha_y}\Big|_{\Delta\alpha_y = \emptyset}}_{I_{3\times3}} \underbrace{R_{\Delta\alpha_z} \ldots}_{I_{3\times3}} \cdot p'$$

$$= \left(\emptyset \quad -p_z' \quad p_y'\right)$$

$$\frac{\partial}{\partial \Delta y} = \left(p_z' \quad \emptyset \quad -p_x'\right)$$

$$\frac{\partial}{\partial \Delta z} = \left(-p_y' \quad p_x' \quad \emptyset\right)$$