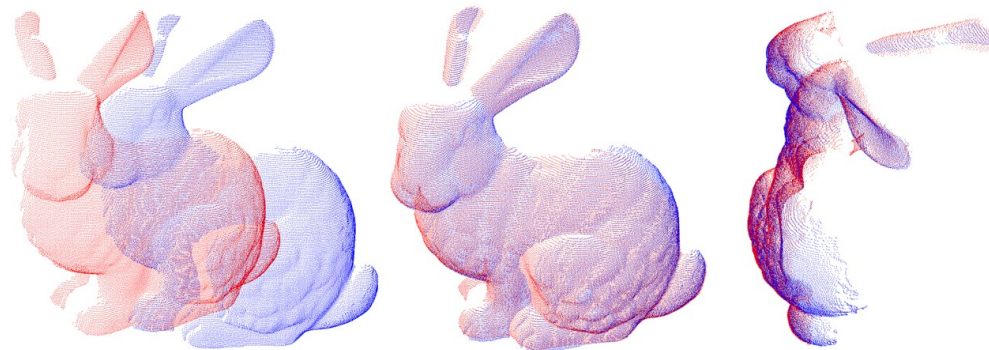


Probabilistic Robotics Course

Least Squares on Manifolds

Giorgio Grisetti
grisetti@diag.uniroma1.it

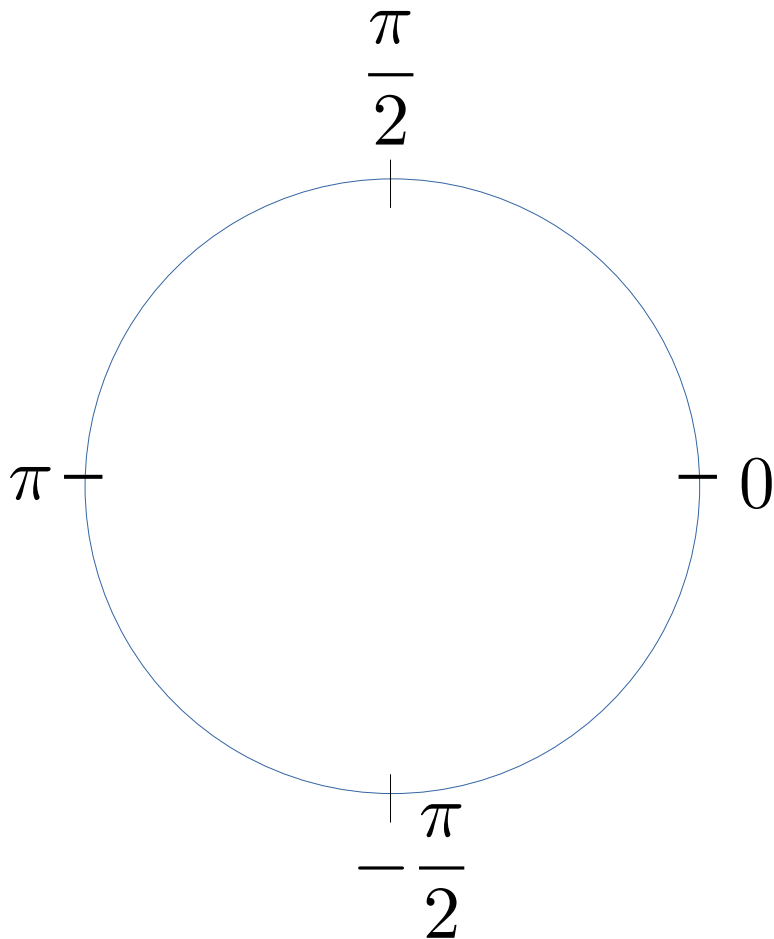
Department of Computer, Control and Management Engineering
Sapienza University of Rome



Non-Euclidean Spaces

In robotics we often encounter spaces that have a non-euclidean topology

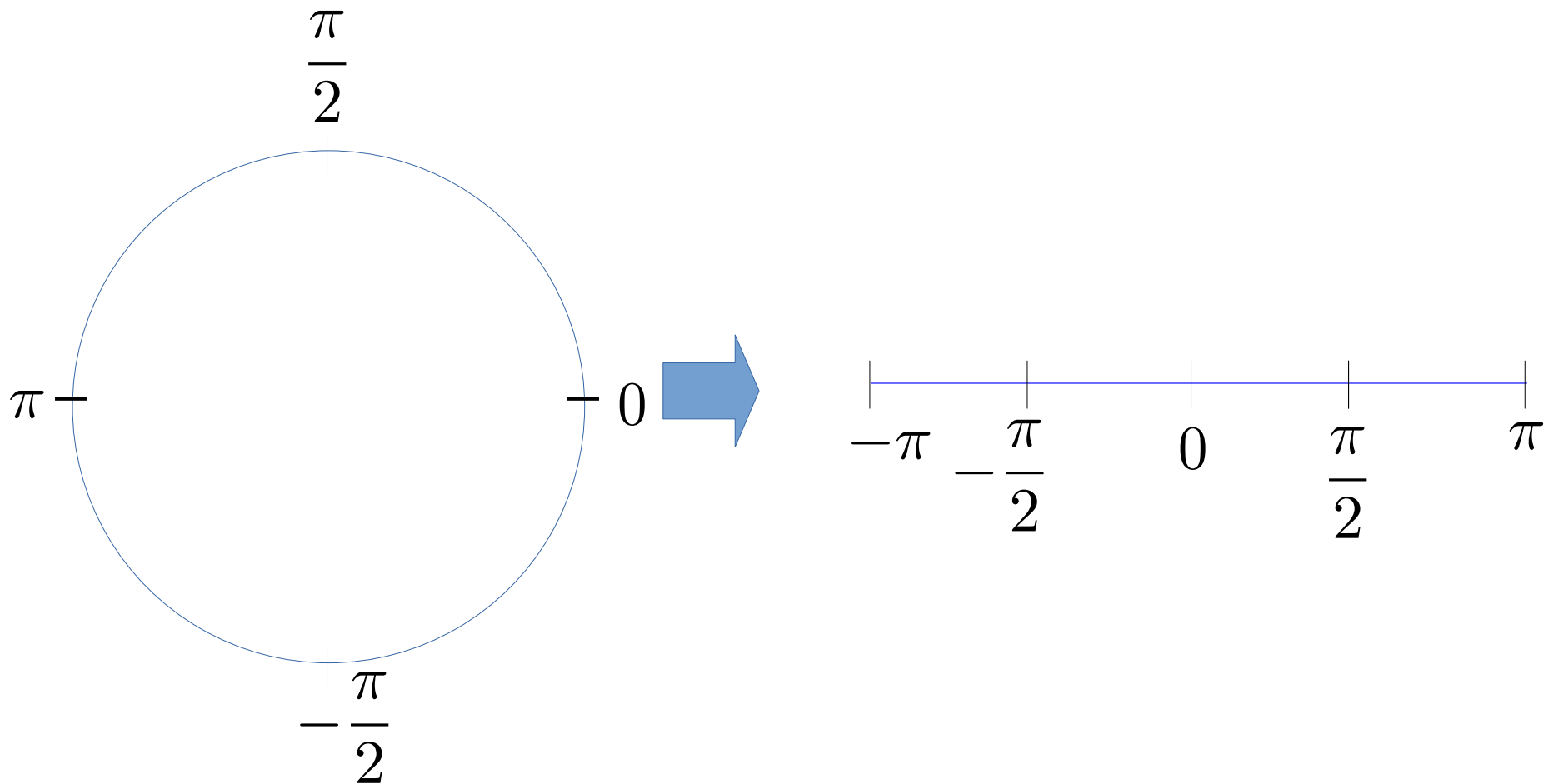
- E.g.: 2D angles



Non-Euclidean Spaces

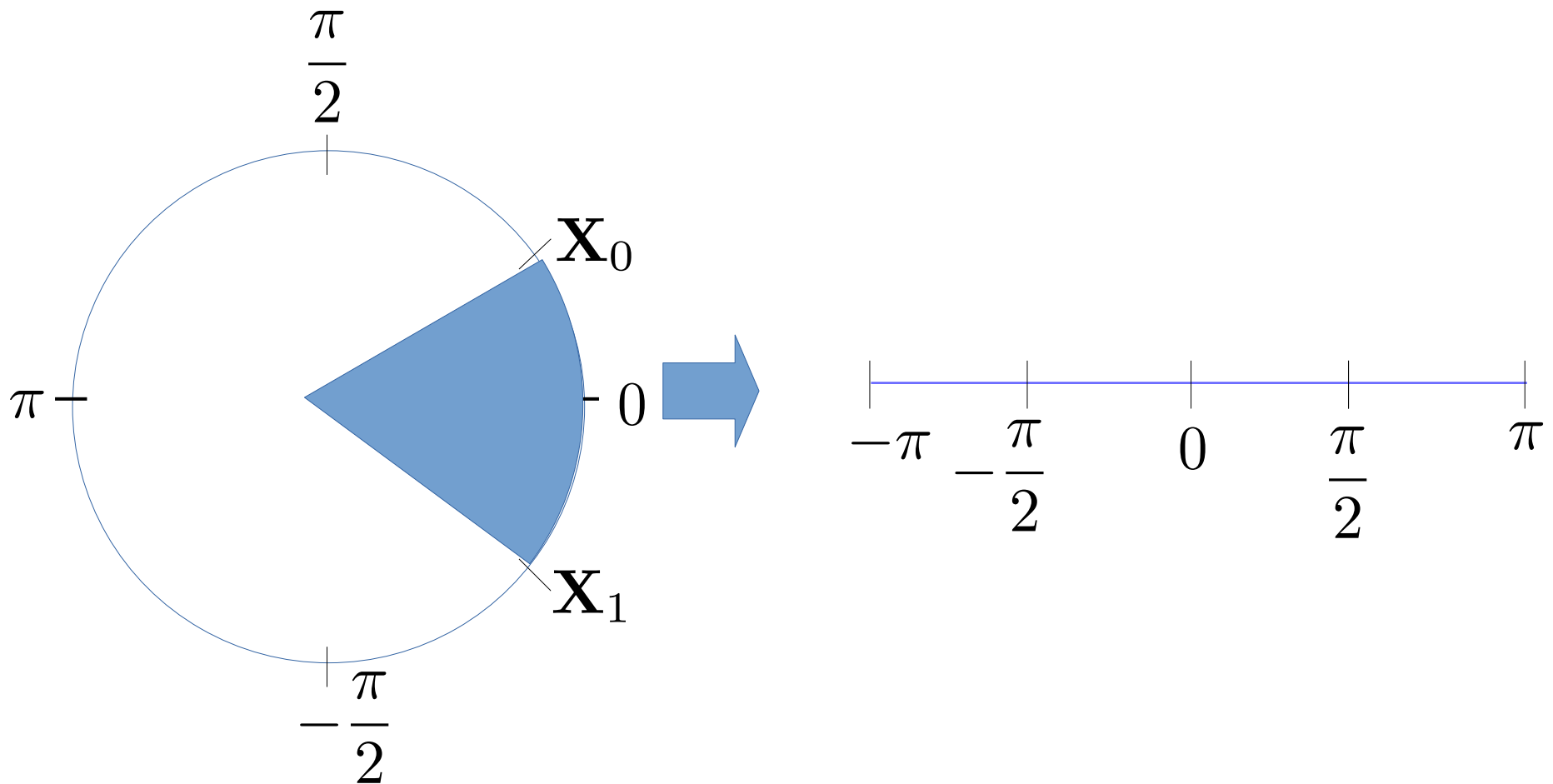
In such cases we commonly operate on a locally Euclidean parameterization

- E.g. we map the angles in the interval $[-\pi:\pi]$



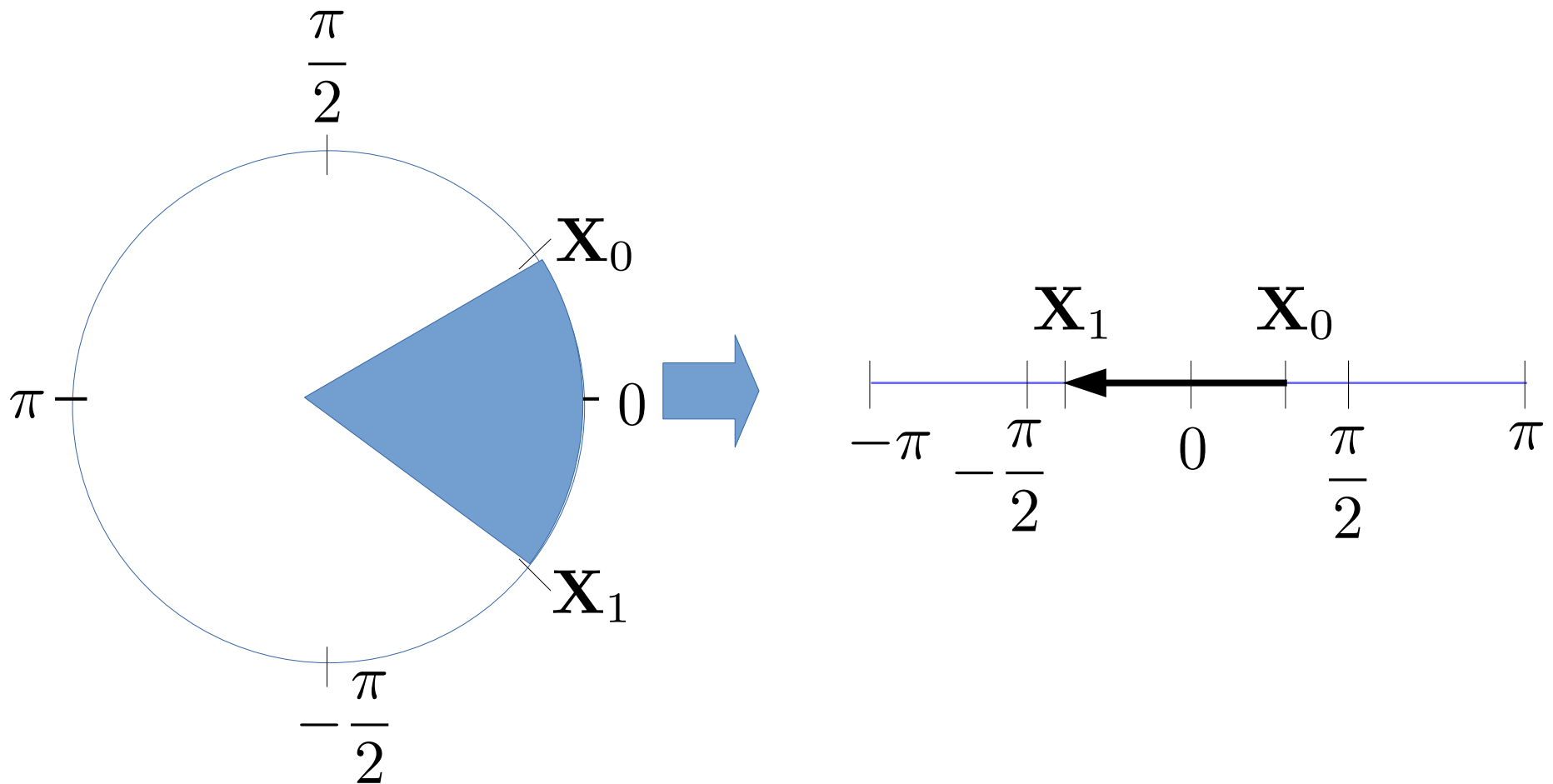
Non-Euclidean Spaces

We can then measure distances in the Euclidean mapping through a regular subtraction



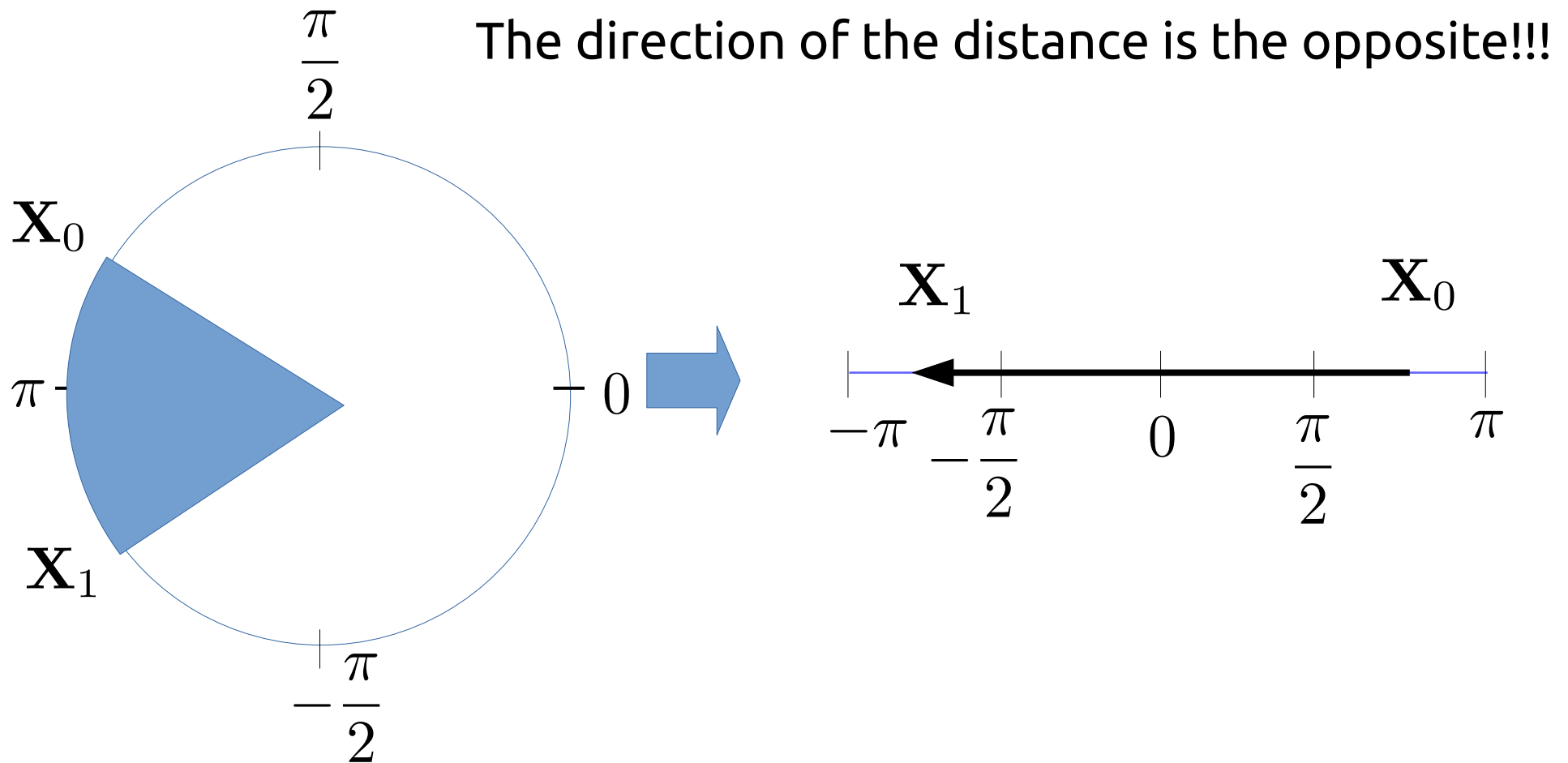
Non-Euclidean Spaces

We can then measure distances in the Euclidean mapping through a regular subtraction



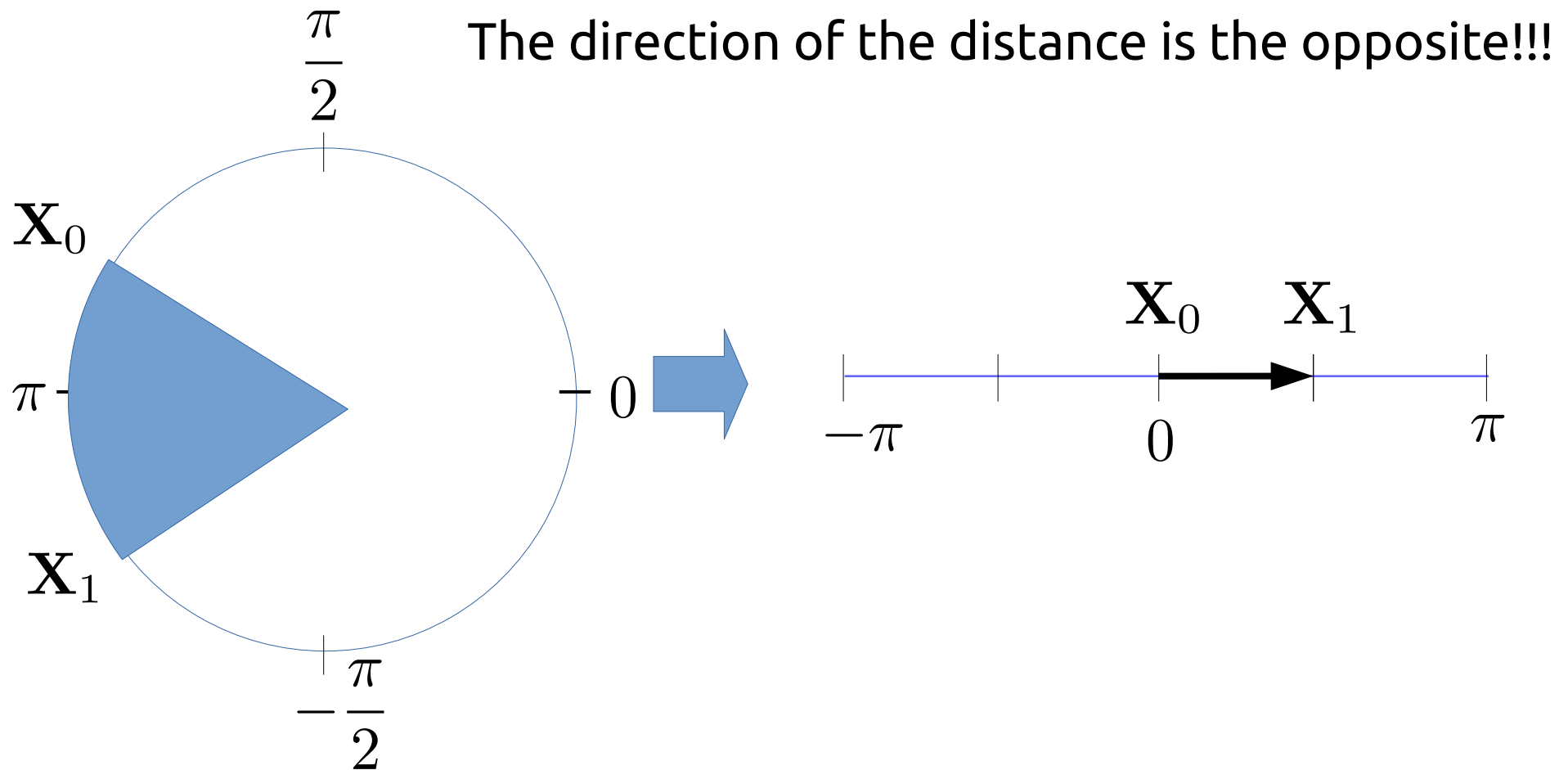
Non-Euclidean Spaces

We can then measure distances in the Euclidean mapping ~~through a regular subtraction~~



Non-Euclidean Spaces

Idea: when computing the distances, build the Euclidean mapping in the neighborhood of one of the points: the **chart around X_0** .



Computing Differences

\mathbf{X}_0 : start point, on manifold

\mathbf{X}_1 : end point, on manifold

$\Delta \mathbf{x}$: difference, on chart

- Compute a chart around \mathbf{X}_0
- Compute the location of \mathbf{X}_1 on the chart
- Measure the difference between points in the chart
- Chart is Euclidean: $\mathbf{X}_0 = \mathbf{X}_1 \Rightarrow \Delta \mathbf{x} = 0$
- Use an operator $\mathbf{X}_1 \boxminus \mathbf{X}_0 = \Delta \mathbf{x}$

Applying Differences

\mathbf{X}_0 : start point, on manifold

$\Delta \mathbf{x}$: difference on chart

\mathbf{X}_1 : end point, on manifold reachable from \mathbf{X}_0
by moving of $\Delta \mathbf{x}$ on the chart

- Compute a chart around \mathbf{X}_0
- Move of $\Delta \mathbf{x}$ on the chart and go back to the manifold
- Encapsulate the operation with an operator

$$\mathbf{X}_0 \boxplus \Delta \mathbf{x} = \mathbf{X}_1$$

Euclidean Least Squares

Clear **H** and **b**

$$\mathbf{H} \leftarrow 0 \quad \mathbf{b} \leftarrow 0$$

For each measurement, update h and b

$$\mathbf{e}^{[i]} \leftarrow \mathbf{h}^{[i]}(\mathbf{x}^*) - \mathbf{z}^{[i]}$$

$$\mathbf{J}^{[i]} \leftarrow \left. \frac{\partial \mathbf{e}^{[i]}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^*}$$

$$\mathbf{H} \leftarrow \mathbf{H} + \mathbf{J}^{[i]T} \mathbf{\Omega}^{[i]} \mathbf{J}^{[i]}$$

$$\mathbf{b} \leftarrow \mathbf{b} + \mathbf{J}^{[i]T} \mathbf{\Omega}^{[i]} \mathbf{e}^{[i]}$$

Update the estimate with the perturbation

$$\Delta \mathbf{x} \leftarrow \text{solve}(\mathbf{H} \Delta \mathbf{x} = -\mathbf{b})$$

$$\mathbf{x}^* \leftarrow \mathbf{x}^* + \Delta \mathbf{x}$$

Gauss in Non Euclidean Spaces

Beware of the + and - operators

- Error function

$$\mathbf{e}^{[i]}(\mathbf{x}) = \mathbf{h}^{[i]}(\mathbf{x}) \boxminus \mathbf{z}_i$$

- Taylor expansion

$$\mathbf{e}^{[i]}(\mathbf{X} \boxplus \Delta \mathbf{x}) = \underbrace{\mathbf{e}^{[i]}(\mathbf{X})}_{\mathbf{e}^{[i]}} + \underbrace{\frac{\partial \mathbf{e}^{[i]}(\mathbf{X} \boxplus \Delta \mathbf{x})}{\partial \Delta \mathbf{x}} \bigg|_{\Delta \mathbf{x}=0}}_{\mathbf{J}^{[i]}} \Delta \mathbf{x}$$

- Increments

$$\mathbf{X} \leftarrow \mathbf{X} \boxplus \Delta \mathbf{x}$$

Manifold Least Squares

Clear **H** and **b**

$$\mathbf{H} \leftarrow 0 \quad \mathbf{b} \leftarrow 0$$

For each measurement

$$\mathbf{e}^{[i]} \leftarrow \mathbf{h}^{[i]}(\mathbf{X}^*) \boxminus \mathbf{Z}^{[i]}$$

$$\mathbf{J}^{[i]} \leftarrow \left. \frac{\partial \mathbf{e}(\mathbf{X}^* \boxplus \Delta \mathbf{x})}{\partial \Delta \mathbf{x}} \right|_{\Delta \mathbf{x} = 0}$$

$$\mathbf{H} \quad + = \quad \mathbf{J}^{[i]T} \boldsymbol{\Omega}^{[i]} \mathbf{J}^{[i]}$$

$$\mathbf{b} \quad + = \quad \mathbf{J}^{[i]T} \boldsymbol{\Omega}^{[i]} \mathbf{e}^{[i]}$$

Compute and apply the perturbation

$$\Delta \mathbf{x} \leftarrow \text{solve}(\mathbf{H} \Delta \mathbf{x} = -\mathbf{b})$$

$$\mathbf{X}^* \leftarrow \mathbf{X}^* \boxplus \Delta \mathbf{x}$$

Methodology

State space \mathbf{X}

- Qualify the Domain
- Define an Euclidean parameterization for the perturbation
- Define boxplus operator

Measurement space(s) \mathbf{Z}

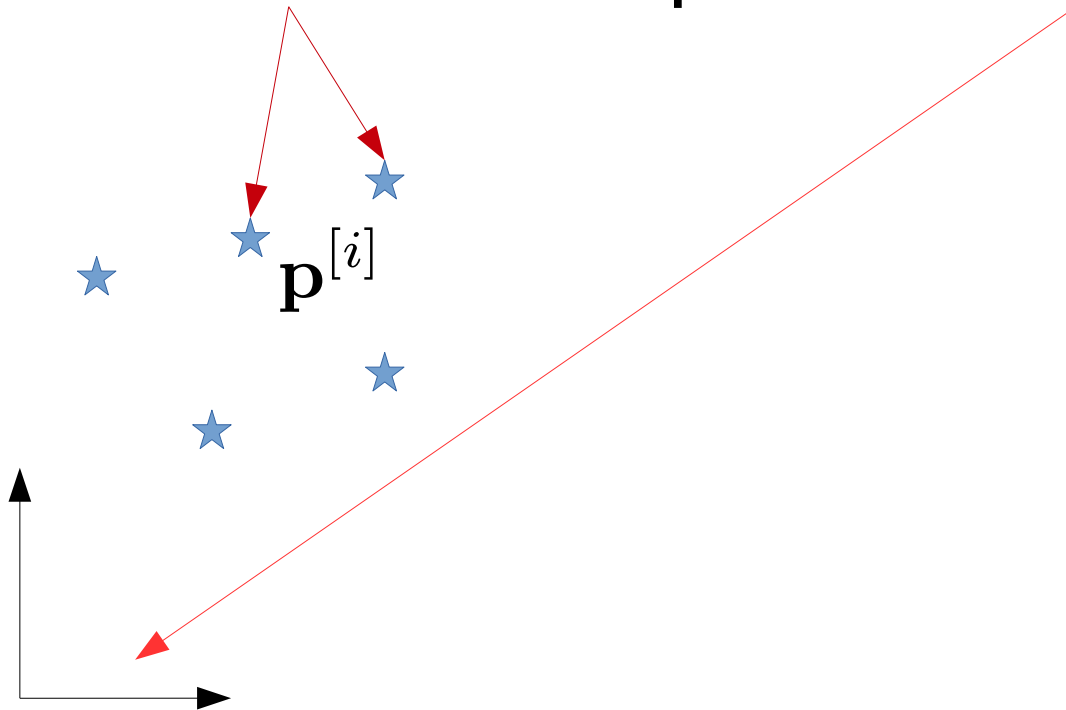
- Qualify the Domain
- Define an Euclidean parameterization for the perturbation
- Define boxminus operator

Identify the prediction functions $\mathbf{h}(\mathbf{X})$

Define the error functions $\mathbf{e}(\mathbf{X})$

Example ICP Optimization 2D

Given a set of 2D points in the world frame



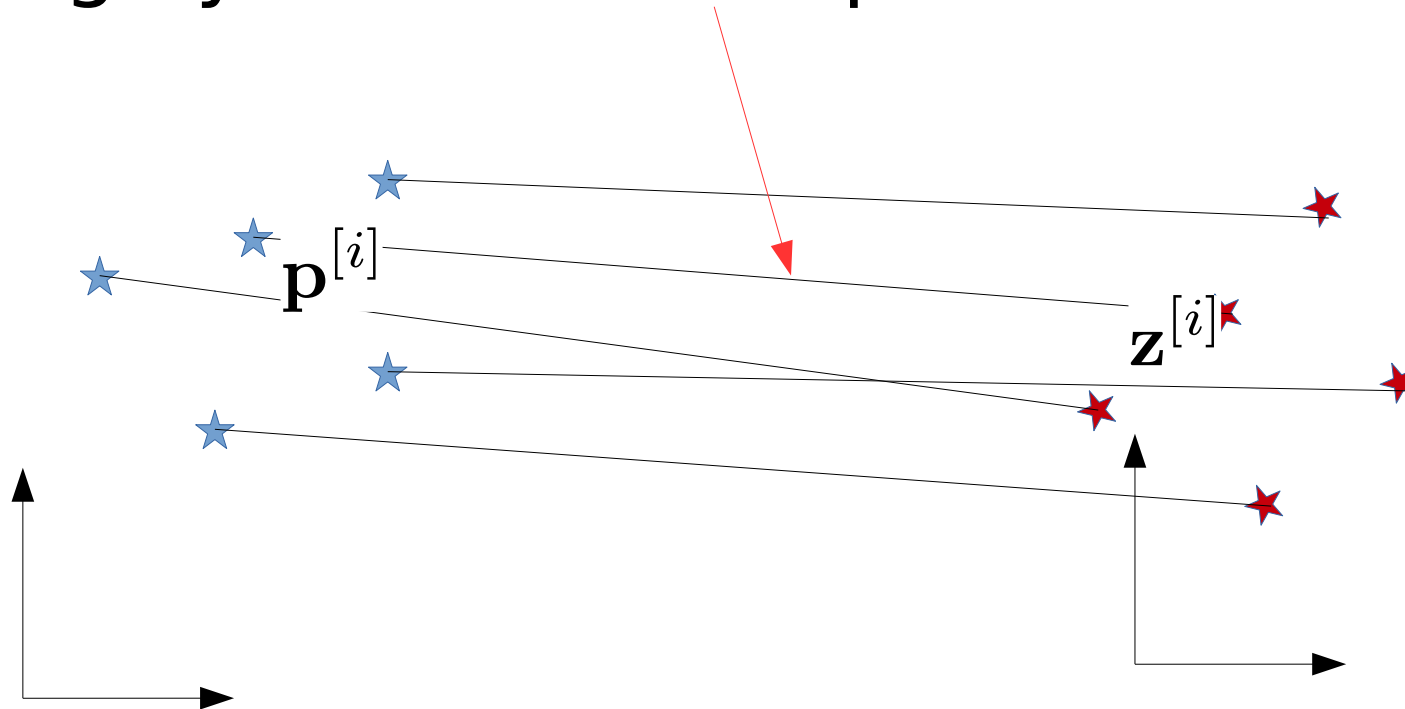
Example ICP Optimization 2D

A set of 2D measurements in the robot frame



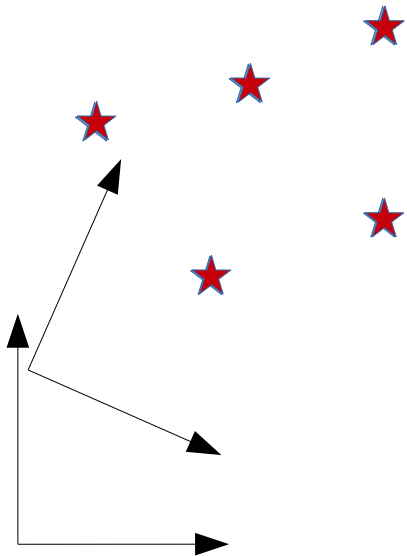
Example ICP Optimization 2D

Roughly known correspondences



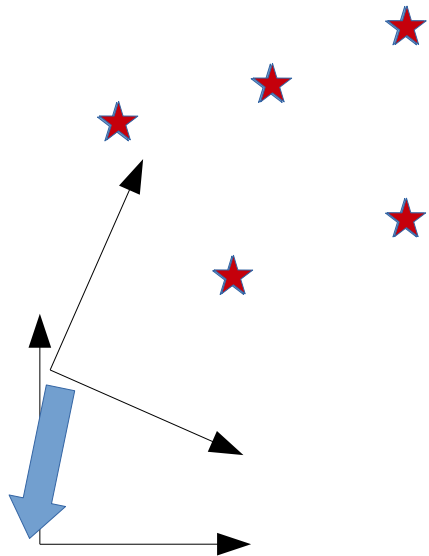
Example ICP Optimization 2D

We want to find a transform that minimizes distance between corresponding points



Example ICP Optimization 2D

Such a transform will be the pose of world w.r.t. robot



Note: we can also estimate robot w.r.t world, but it leads to longer calculations

ICP: State Space

State

$$\mathbf{X} \in SE(2), \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

Manifold representation
as homogeneous
transformation

$$\Delta \mathbf{x} = \underbrace{(\Delta x \ \Delta y \ \Delta \theta)}_{\Delta \mathbf{t}}^T$$

Euclidean
parameterization for the
chart

$$\Delta \mathbf{X} = v2t(\Delta \mathbf{x}) = \begin{bmatrix} \Delta \mathbf{R} & \Delta \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

Convenient
function that
converts a
perturbation
into a matrix

$$\begin{aligned} \mathbf{X} \boxplus \Delta \mathbf{x} &= v2t(\Delta \mathbf{x}) \mathbf{X} \\ &= \Delta \mathbf{X} \cdot \mathbf{X} \end{aligned}$$

Definition of the
boxplus
operator

ICP: Measurements

$$\begin{aligned}\mathbf{z} &\in \mathbb{R}^2 \\ \mathbf{h}^{[i]}(\mathbf{X}) &= \mathbf{R}\mathbf{p}^{[i]} + \mathbf{t} \\ &= \mathbf{X}\mathbf{p}^{[i]} \\ \mathbf{h}^{[i]}(\mathbf{X} \boxplus \Delta\mathbf{x}) &= (\mathbf{X} \boxplus \Delta\mathbf{x})\mathbf{p}^{[i]} \\ &= v2t(\Delta\mathbf{x}) \underbrace{\mathbf{X}\mathbf{p}^{[i]}}_{\tilde{\mathbf{p}}^{[i]}} \\ &= \mathbf{R}(\Delta\theta)\tilde{\mathbf{p}}^{[i]} + \Delta\mathbf{t}\end{aligned}$$

ICP: Jacobian

$$\mathbf{h}^{[i]}(\mathbf{X} \boxplus \Delta \mathbf{x}) = \mathbf{R}(\Delta \theta) \tilde{\mathbf{p}}^{[i]} +$$

$$\left. \frac{\partial \mathbf{h}^{[i]}(\mathbf{X} \boxplus \Delta \mathbf{x})}{\partial \Delta \mathbf{x}} \right|_{\Delta \mathbf{x}=0} = \left(\left. \frac{\partial \mathbf{h}^{[i]}(\cdot)}{\partial} \right|_{\Delta \mathbf{x}=0} \quad \left. \frac{\partial \mathbf{h}^{[i]}(\cdot)}{\partial \Delta \theta} \right|_{\Delta \mathbf{x}=0} \right)$$

$$\left. \frac{\partial \mathbf{h}^{[i]}(\cdot)}{\partial} \right|_{\Delta \mathbf{x}=0} = \mathbf{I}$$

$$\left. \frac{\partial \mathbf{h}^{[i]}(\cdot)}{\partial \Delta \theta} \right|_{\Delta \mathbf{x}=0} = \mathbf{R}'(0) \tilde{\mathbf{p}}^{[i]}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{\mathbf{p}}^{[i]} = \begin{pmatrix} -\tilde{p}^{[i]}.y \\ \tilde{p}^{[i]}.x \end{pmatrix}$$

ICP: Octave Program

```
function [e,J]=errorAndJacobianManifold(X,p,z)
    t=X(1:2,3);
    R=X(1:2,1:2);
    z_hat=R*p+t;
    e=z_hat-z;

    J=zeros(2,3);
    J(1:2,1:2)=eye(2);
    J(1:2,3)=[-z_hat(2),
               z_hat(1)]';
endfunction;
```

ICP: Octave Program

```
function [chi,X]=icp2dManifold(X,P,Z)
    chi=0;           %cumulative chi2
    H=zeros(3,3); %accumulators for H and b
    b=zeros(3,1);
    for(i=1:size(P,2))
        p=P(:,i); z=Z(:,i);
        [e,J]=errorAndJacobianManifold(X,p,z);
        H+=J'*J;      %assemble H and B
        b+=J'*e;
        chi+=e'*e;    %update cumulative error
    endfor
    dx=-H\b;          %solve the linear system
    X=v2t(dx)*X;      %apply perturbation
endfunction
```

Uncertainty of the Solution

Optimizing on a Manifold generates a \mathbf{H} matrix that is computed **on the chart**

\mathbf{H}^{-1} represents a covariance of the solution around the origin of the chart

We can write that around the optimum

$$\Delta \mathbf{x} \sim \mathcal{N}(\Delta \mathbf{x}; \mathbf{0}, \mathbf{H}^{-1})$$

The Gaussian approximation of the distribution of solution around the optimum, is related to the chart through boxplus

$$\begin{aligned} \mathbf{X} &= \mathbf{X}^* \boxplus \Delta \mathbf{x} \\ &= g_{\mathbf{X}^*}(\Delta \mathbf{x}) \end{aligned}$$

Uncertainty (cont)

Using our manipulation skills on the Gaussian distribution, we can compute the approximation of a Gaussian in \mathbf{X} , by either

- Linearization

$$\mathbf{J}_{\mathbf{X}} = \left. \frac{\partial \mathbf{X}^* \boxplus \Delta_{\mathbf{X}}}{\partial \Delta_{\mathbf{X}}} \right|_{\Delta_{\mathbf{X}}=0}$$
$$\mathbf{X} \sim \mathcal{N}(\mathbf{X}; \mathbf{X}^*, \mathbf{J}_{\mathbf{X}} \mathbf{H}^{-1} \mathbf{J}_{\mathbf{X}}^T)$$

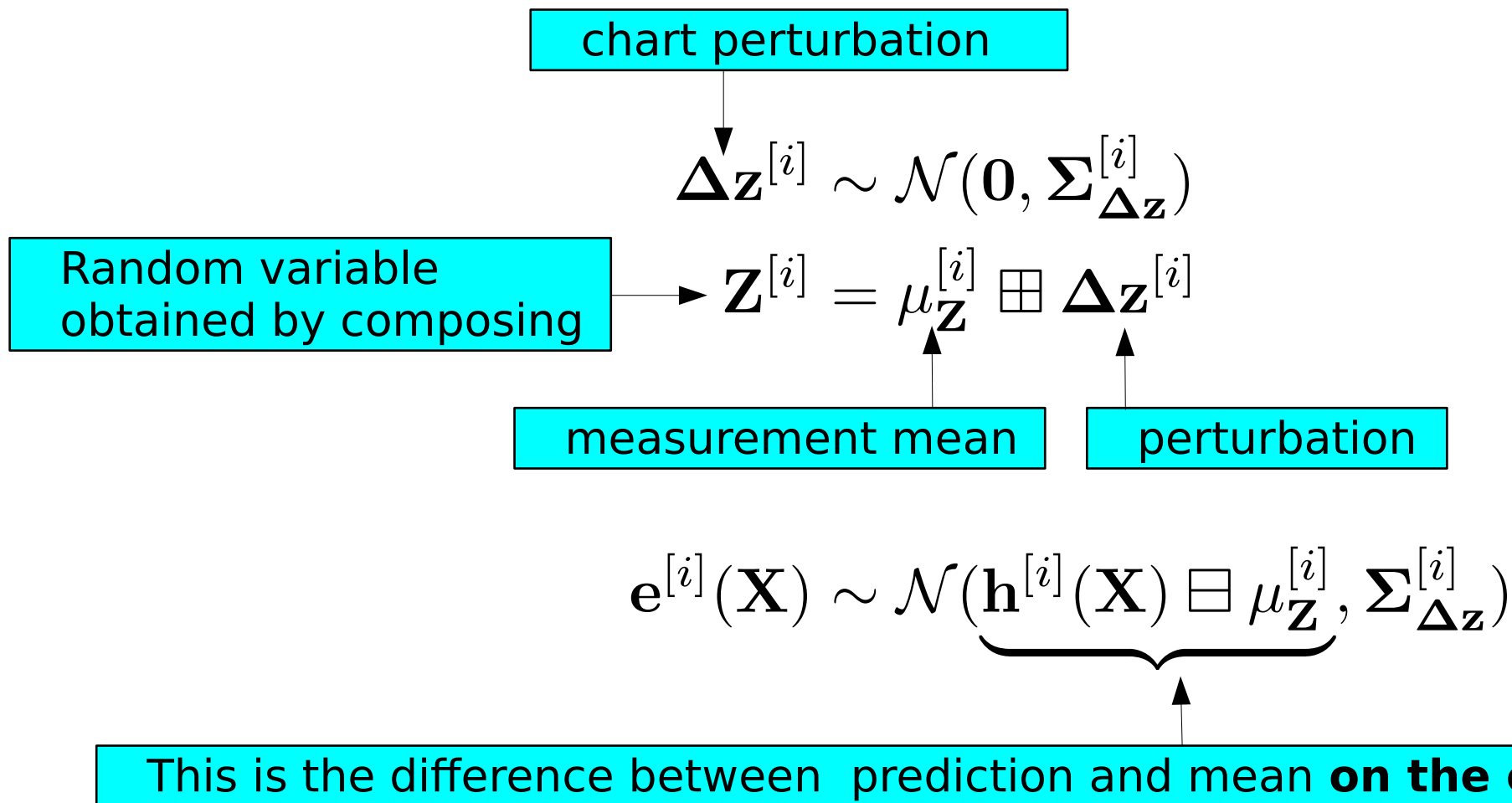
- Unscented Transform

$$\mathbf{X}^{(i)} = \mathbf{X}^* \boxplus \Delta_{\mathbf{X}}^{(i)}$$

Where $\Delta_{\mathbf{X}}^{(i)}$ are sigma points extracted from the Gaussian distribution in the chart.

Measurement Uncertainty

We can express the uncertainty in the measurements on a chart centered on the measurement mean



Manifold Least Squares (Omega on Chart)

Clear **H** and **b**

$$\mathbf{H} \leftarrow 0 \quad \mathbf{b} \leftarrow 0$$

For each measurement

$$\begin{aligned} \mathbf{e}^{[i]} &\leftarrow \mathbf{h}^{[i]}(\mathbf{X}^*) \boxminus \mu_{\mathbf{z}}^{[i]} \\ \mathbf{J}^{[i]} &\leftarrow \left. \frac{\partial \mathbf{e}(\mathbf{X}^* \boxplus \Delta \mathbf{x})}{\partial \Delta \mathbf{x}} \right|_{\Delta \mathbf{x} = 0} \\ \mathbf{H} &+ = \mathbf{J}^{T[i]} \boldsymbol{\Omega}^{[i]} \mathbf{J}^{[i]} \\ \mathbf{b} &+ = \mathbf{J}^{T[i]} \boldsymbol{\Omega}^{[i]} \mathbf{e}^{[i]} \end{aligned}$$

Compute and apply the perturbation

$$\begin{aligned} \Delta \mathbf{x} &\leftarrow \text{solve}(\mathbf{H} \Delta \mathbf{x} = -\mathbf{b}) \\ \mathbf{X} &\leftarrow \mathbf{X} \boxplus \Delta \mathbf{x} \end{aligned}$$

Conclusions

- Least squares on smooth manifolds offers a more robust formulation of non-linear least squares on non-Euclidean spaces
 - Key idea: linearize the problem with respect to the current optimum, around the perturbations
 - Using the boxplus and boxminus operators to encapsulate operations on the manifolds
- Download the octave code and compare plain vs manifold ICP
- Beware of the Uncertainties