Probabilistic Robotics Course

Registration on a Manifold

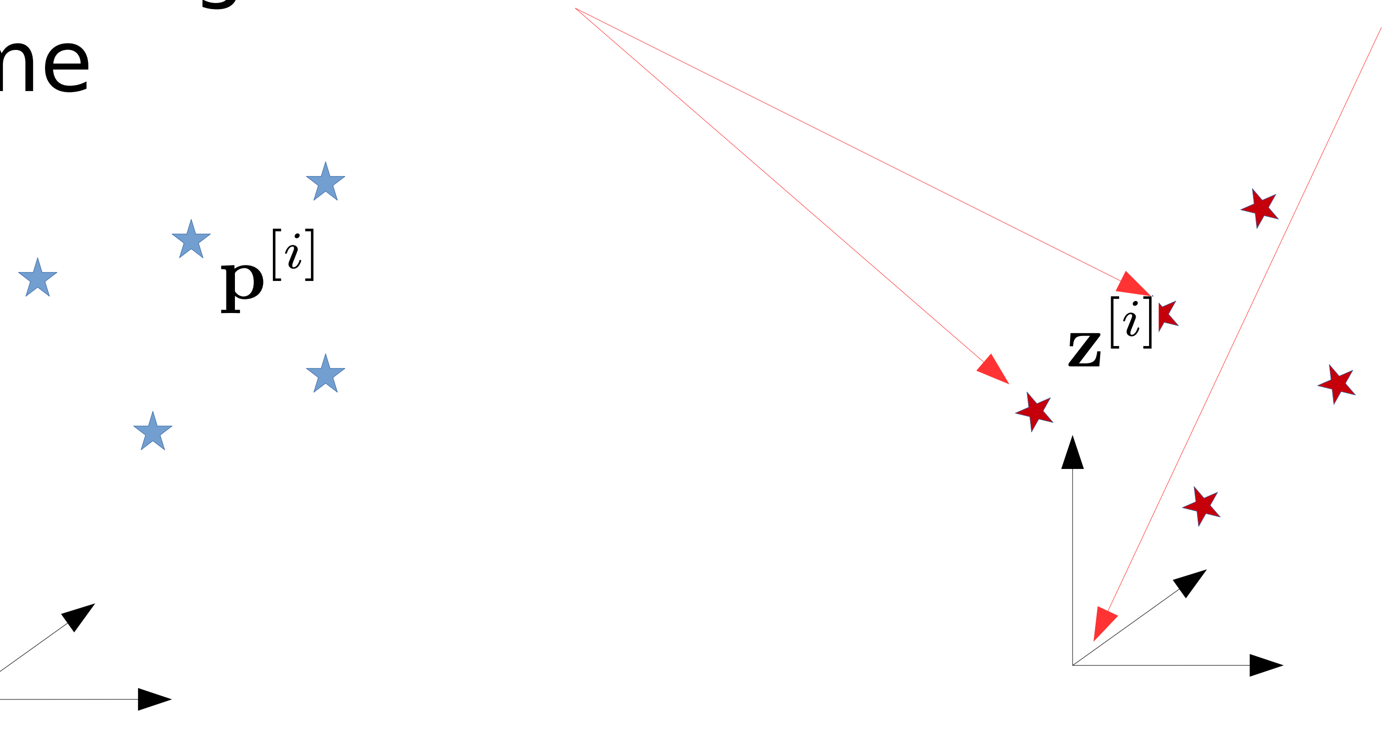
Giorgio Grisetti
{grisetti}@diag.uniroma1.it

Department of Computer, Control, and Management Engineering
Sapienza University of Rome

Courtesy of Li et. al

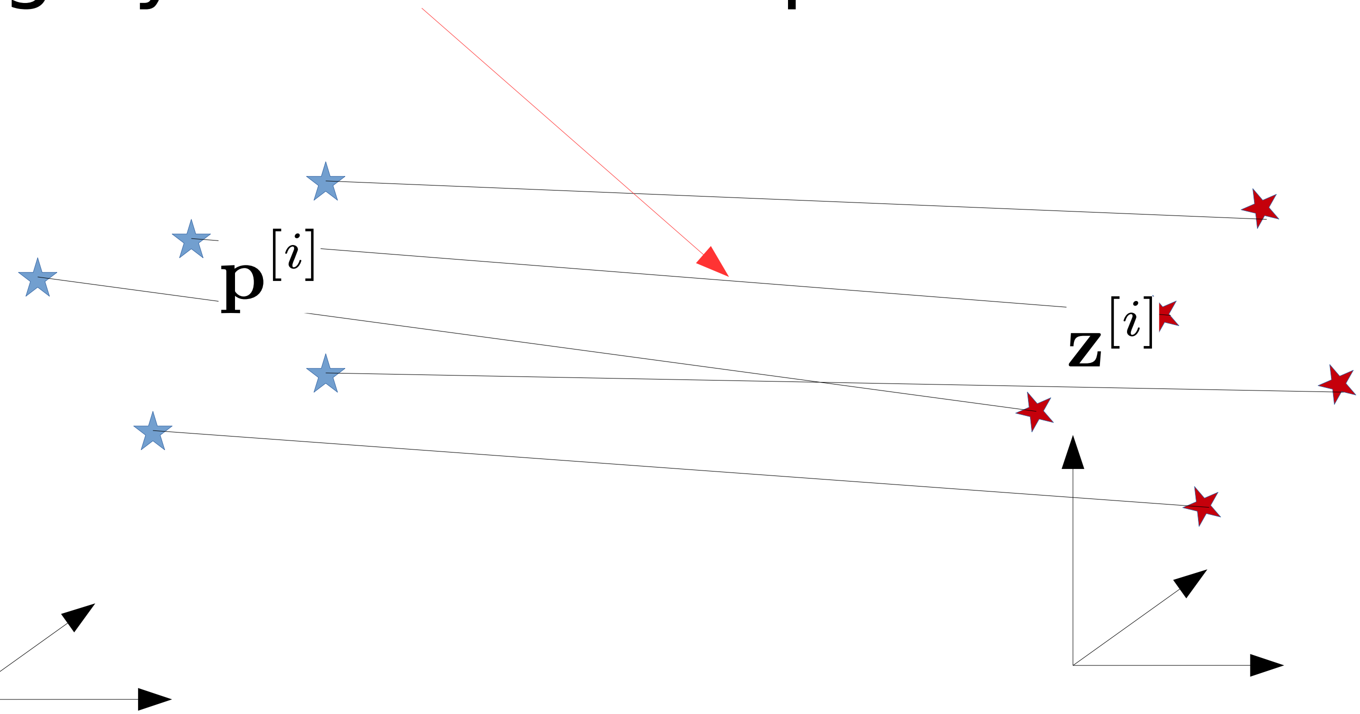
Example Registration in 3D

A set of generic measurements in the robot frame



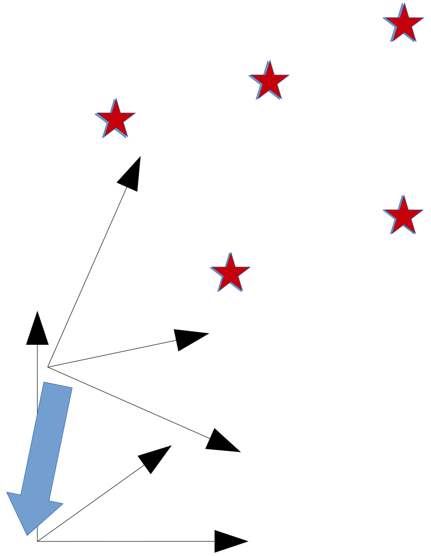
Example Registration in 3D

Roughly known correspondences



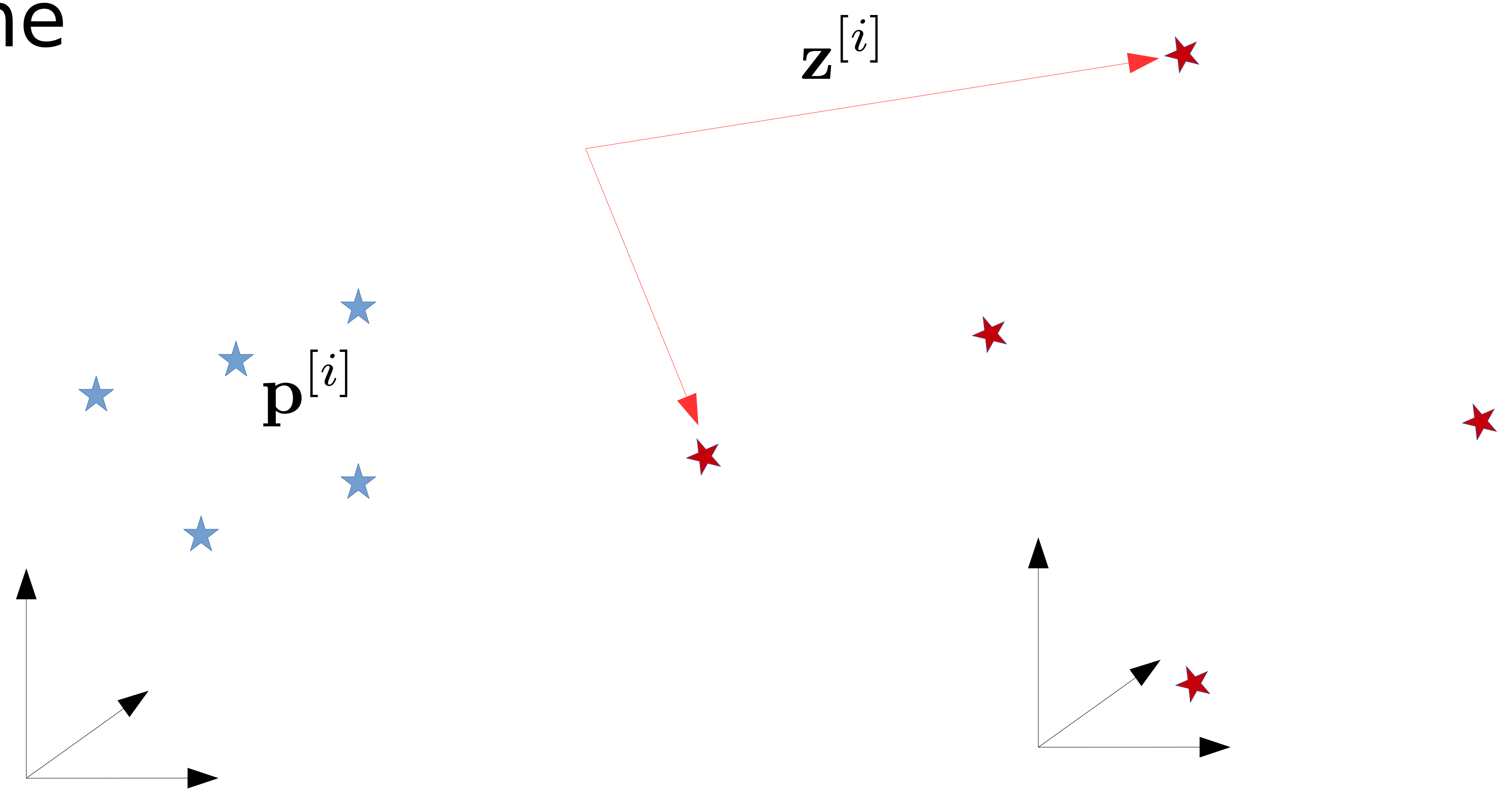
Example Registration in 3D

Such a transform will map the world in the robot frame



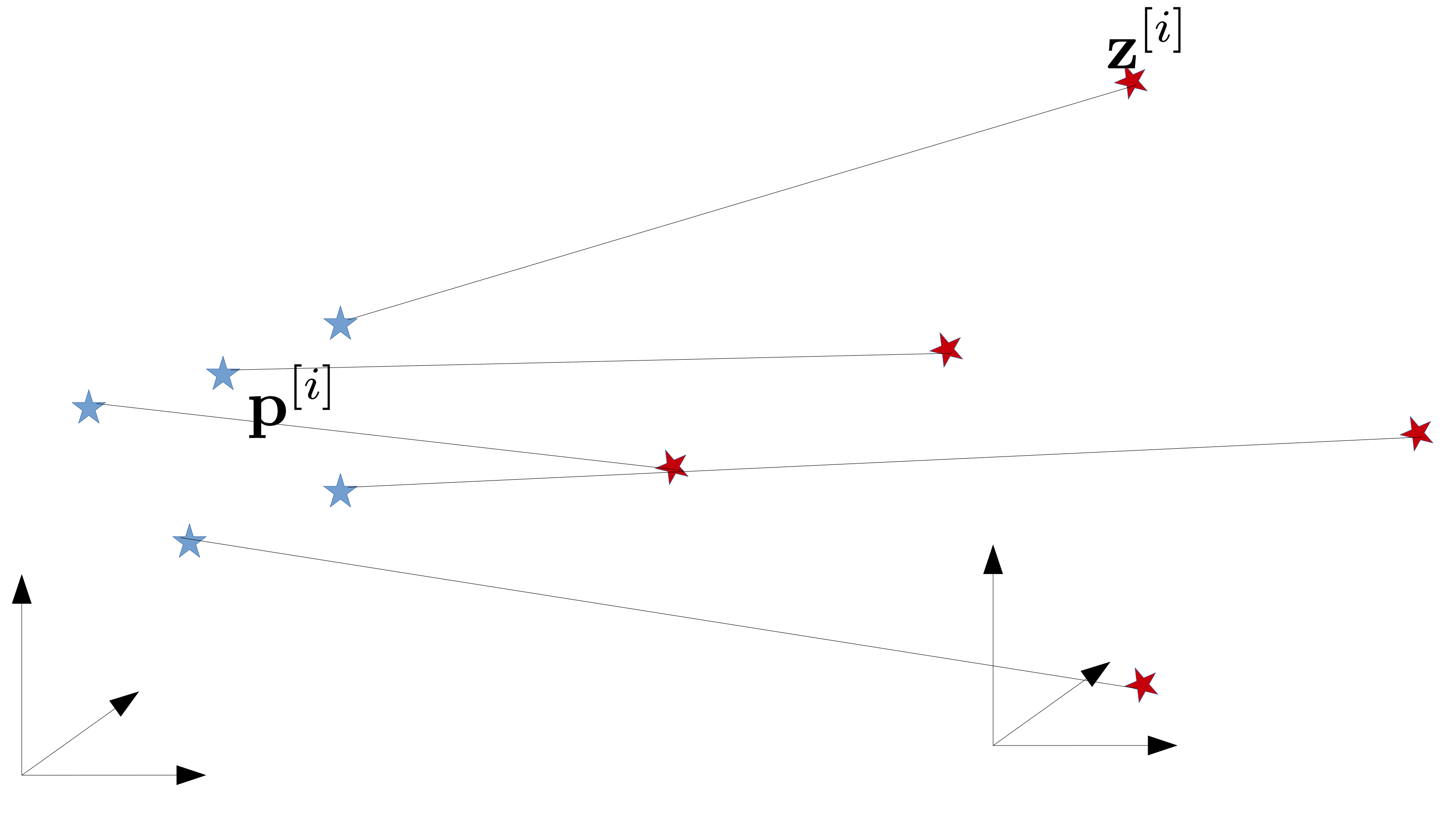
Example Similarity Registration in 3D

A set of generic measurements in the robot frame



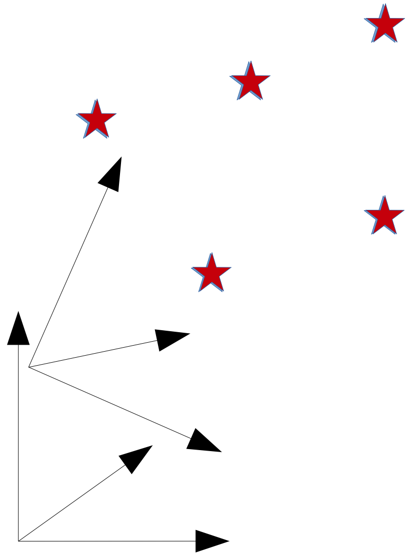
Example Similarity Registration in 3D

Roughly known correspondences



Example Registration in 3D

We want to find a transform (Sim3) that minimizes distance between corresponding points



SICP: State and Boxplus

State

$$\mathbf{X} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & s \end{pmatrix} \in Sim3$$

$$\mathbf{\Delta x} = (\underbrace{\Delta x \ \Delta y \ \Delta z}_{\mathbf{\Delta t}} \ \underbrace{\Delta \alpha_x \ \Delta \alpha_y \ \Delta \alpha_z}_{\mathbf{\Delta}\alpha}; \underbrace{\mathbf{\Delta} s}_{\log s})^T$$

$$\mathrm{v2s}(\mathbf{\Delta x}) = \begin{pmatrix} \mathbf{R}(\Delta\alpha) & \mathbf{\Delta t} \\ \mathbf{0} & \exp(\mathbf{\Delta} s) \end{pmatrix}$$

$$\mathbf{X} \boxplus \mathbf{\Delta x} = \mathrm{v2s}(\mathbf{\Delta x})\mathbf{X}$$

SICP: Operators

$$\mathbf{X} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & s \end{pmatrix} \in Sim3$$

$$\mathbf{X} \cdot \mathbf{p} = s\mathbf{R}\mathbf{p} + \mathbf{t}$$

$$\mathbf{X}_1 \cdot \mathbf{X}_2 = \begin{pmatrix} \mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_1\mathbf{t}_2 + s_2\mathbf{t}_1 \\ \mathbf{0} & s_1 s_2 \end{pmatrix}$$

SICP: Measurements

Measurements

$$
\begin{aligned}
\mathbf{z} &\in \Re^3 \\
\mathbf{h}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x}) &= (\mathbf{X} \boxplus \mathbf{\Delta x})\mathbf{p}^{[i]} \\
&= \exp(\mathbf{\Delta} s) \left(\mathbf{R}(\mathbf{\Delta}\alpha) \underbrace{s \, \mathbf{R}\mathbf{p}^{[i]} + \mathbf{t}}_{\mathbf{p}'^{[i]}} \right) + \mathbf{\Delta t}
\end{aligned}
$$

SICP: Error and Jacobian

Compelling fun with derivatives..

$$\mathbf{e}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x}) = \mathbf{h}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x}) - \mathbf{z}^{[i]}$$

$$\mathbf{J}^{[i]} = \left. \frac{\partial \mathbf{e}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x})}{\partial \mathbf{\Delta x}} \right|_{\mathbf{\Delta x}=0}$$

$$= \left. \left(\frac{\partial \mathbf{e}^{[i]}(\cdot)}{\partial \mathbf{\Delta t}} \; \frac{\partial \mathbf{e}^{[i]}(\cdot)}{\partial \mathbf{\Delta} \alpha} \; \frac{\partial \mathbf{e}^{[i]}(\cdot)}{\partial \mathbf{\Delta} s} \right) \right|_{\mathbf{\Delta x}=0}$$

$$= \left(\mathbf{I} \;\; - \left[\mathbf{p}'^{[i]} \right]_{\times} \mathbf{p}'^{[i]} \right)$$

SICP: Code

```
#from 7d vector to similiarity matrix Sim(3)
function S=v2s(v)
  S=eye(4);
  S=v2t(v(1:6));
  S(4,4) = exp(v(7));
endfunction

function [e,J]=errorAndJacobianManifold(X,p,z)
  t=X(1:3, 4);
  R=X(1:3, 1:3);
  s=X(4,4);
  z_hat=s*(R*p+t);
  e=z_hat-z;
  J=zeros(3,7);
  J(1:3,1:3)=eye(3);
  J(1:3,4:6)=skew(-z_hat);
  J(1:3,7)=z_hat;
endfunction
```

SICP: Code

```
function [X, chi_stats, num_inliers]= doICP(x_guess, P, Z, num_iterations, damping, kernel_threshold)
  X=v2s(x_guess);
  chi_stats=zeros(1,num_iterations);
  num_inliers=zeros(1,num_iterations);
  for (iteration=1:num_iterations)
    H=zeros(7,7);
    b=zeros(7,1);
    chi_stats(iteration)=0;
    for (i=1:size(P,2))
      [e,J] = errorAndJacobian(X, P(:,i), Z(:,i));
      chi=e'*e;
      if (chi>kernel_threshold)
        e*=sqrt(kernel_threshold/chi);
        chi=kernel_threshold;
      else
        num_inliers(iteration)++;
      endif;
      chi_stats(iteration)+=chi;
      H+=J'*J;
      b+=J'*e;
    endfor
    H+=eye(7)*damping;
    dx=-H\b;
    X=v2s(dx)*X;
  endfor
endfunction
```

Testing

Run the program (23b, icp with similarities)

Exercise

Model:

- N points in the 3D space

Measurements:

- direction vectors of these points expressed in the reference frame of the sensor

$$\mathbf{z} \in S^2$$

$$\mathbf{h}^{[i]}(\mathbf{X}) = \frac{\mathbf{R}\mathbf{p}^{[i]} + \mathbf{t}}{\|\mathbf{R}\mathbf{p}^{[i]} + \mathbf{t}\|}$$

- Hint: map S2 in R3 (overparameterize)

$$\mathbf{z} = (\mathrm{x} \;\; \mathrm{y} \;\; \mathrm{z})^T \; s.t. ||\,\mathbf{z}\,|| = 1$$

State

State

$$
\begin{aligned}
\mathbf{X} &= [\mathbf{R}|\mathbf{t}] \in SE(3) \\
\boldsymbol{\Delta}\mathbf{x} &= (\underbrace{\Delta x \; \Delta y \; \Delta z}_{\boldsymbol{\Delta}\mathbf{t}} \; \underbrace{\Delta\alpha_x \; \Delta\alpha_y \; \Delta\alpha_z}_{\boldsymbol{\Delta}\alpha})^T \\
\mathbf{X} \boxplus \boldsymbol{\Delta}\mathbf{x} &= \mathrm{v2t}(\boldsymbol{\Delta}\mathbf{x})\mathbf{X} \\
&= [\mathbf{R}(\boldsymbol{\Delta}\alpha)\mathbf{R}|\mathbf{R}(\boldsymbol{\Delta}\alpha)\mathbf{t} + \boldsymbol{\Delta}\mathbf{t}]
\end{aligned}
$$

Measurements

The domain is a manifold

$$\begin{aligned}\mathbf{z} &= (x\,y\,z)^T \in S^2 \\ \mathbf{z}_1 \boxminus \mathbf{z}_2 &= \mathbf{\Delta z} = (\theta\ ,\ \varphi\ ) : \text{relative angle on azimuth and elevation}\end{aligned}$$

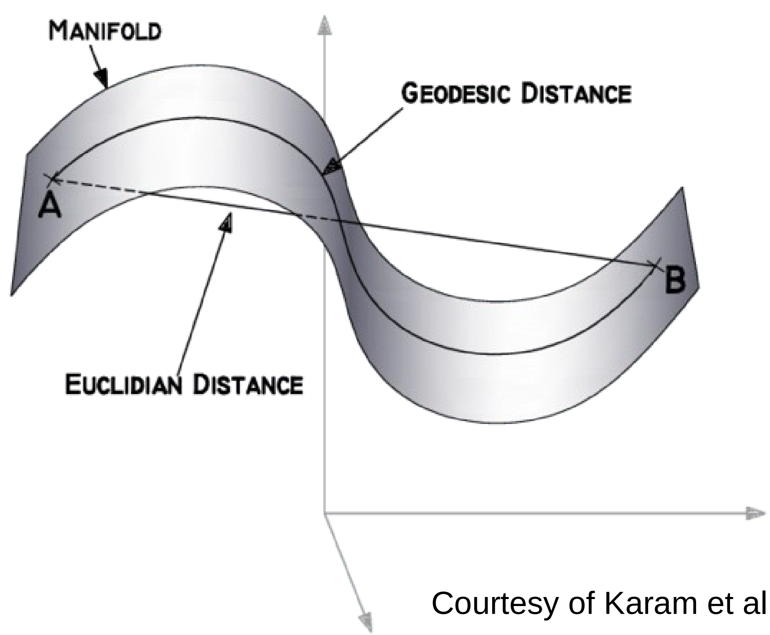
Complicated:

1. construct a rotation matrix R(z) = R(θ)R(φ)
2. compute such a rotation matrix for both z1 and z2
3. compute the rotation difference as (R2)^T*R1
5. extract azimuth and elevation from rotation difference (atan2(-R(1,2),R(2,2)), atan2(-R(3,1),R(3,3))

Measurement (simplified)

We can simplify the problem (loosing the benefits of operating on a geodesic distance), by treating Z as if it were euclidean.

This replaces the distance along the geodesic with the chordal distance.



Courtesy of Karam et al.

Error Function

$$
\begin{aligned}
\mathbf{e}^{[i]}(\mathbf{X}) &= \mathbf{h}^{[i]}(\mathbf{X}) - \mathbf{Z}^{[i]} \\
\mathbf{h}^{[i]}(\mathbf{X}) &= \operatorname{normalize}(\underbrace{\mathbf{R}\mathbf{p}^{[i]} + \mathbf{t}}_{\mathbf{p}'^{[i]}}) \\
\mathbf{h}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x}) &= \operatorname{normalize}(\underbrace{\mathbf{\Delta R}\mathbf{p}'^{[i]} + \mathbf{\Delta t}}_{\mathbf{h}_{\mathrm{icp}}}) \\
\operatorname{normalize}(\mathbf{v}) &= \frac{\mathbf{v}}{\sqrt{\mathbf{v}^T\mathbf{v}}} \\
\frac{\partial \operatorname{normalize}(\mathbf{v})}{\partial \mathbf{v}} &= \frac{\mathbf{I}\|\mathbf{v}\| - \mathbf{v}\mathbf{v}^T/\|\mathbf{v}\|}{\|\mathbf{v}\|^2} = \frac{1}{\|\mathbf{v}\|}\mathbf{I} - \frac{1}{\|\mathbf{v}\|^3}\mathbf{v}\mathbf{v}^T
\end{aligned}
$$

Jacobian

Chain rule

$$\left.\frac{\partial \mathbf{e}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x})}{\partial \mathbf{\Delta x}}\right|_{\mathbf{\Delta x}=\mathbf{0}} = \left.\frac{\partial \mathrm{normalize}(\mathbf{p})}{\partial \mathbf{p}}\right|_{\mathbf{p}=\mathbf{p}'^{[i]}} \left.\frac{\partial \mathbf{h}_{\mathrm{icp}}^{[i]}(\mathbf{X} \boxplus \mathbf{\Delta x})}{\partial \mathbf{\Delta x}}\right|_{\mathbf{\Delta x}=\mathbf{0}}$$

$$= \frac{1}{\|\mathbf{p}'^{[i]}\|^3} \left(\mathbf{I}\|\mathbf{p}'^{[i]}\|^2 - \mathbf{p}'^{[i]}\mathbf{p}'^{[i]T}\right) \left(\mathbf{I} \ \lfloor -\mathbf{p}'^{[i]} \rfloor_\times \ \right)$$