MATH1510 Notes

Contents

1 Taylor's Theorem

$f(x) = f(c)+\frac{f'(c)}{1!}(x-c)^1+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{(n+1)}$,

i.e. $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{1!}(x-c)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{(n+1)}$

, where ξ is between c and x (either $c < x$ or $c > x$).

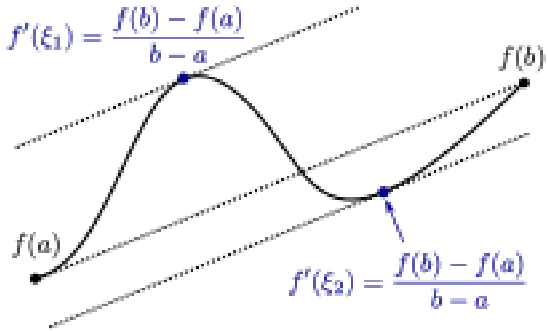
1.1 Simple case of Taylor's Theorem

Take $n = 1$, we have $f(x) = f(c) + f'(c)(x-c) + O((x-c)^2)$.

By dropping the error term, $f(x) = f(c) + f'(c)(x-c)$,
which is the equation of tangent line at $x = c3$.

1.2 Lagrange's Mean Value Theorem

$f(x) - f(c) = f'(\xi)(x-c)$
$\rightarrow \frac{f(x)-f(c)}{x-c} = f'(\xi)$ (Lagrange's Mean Value Theorem)



i.e. there is at last one point ξ, where slope of tangent line at ξ = slope of straight line joining $(x, f(x))$ and $(c, f(c))$.

Assumption 1: $f(t)$ is continuous in $[x, c]$. **(Closed interval!)**
Assumption 2: $f'(t)$ exists in (x, c). **(Open interval is enough.)**

> We require $f(t)$ be continuous in $[x, c]$ as we have to apply extreme value theorem.

1.3 Before Proving Lagrange's Mean Value Theorem

To prove Lagrange's Mean Value Theorem, we need to "deep" theorems about continuous functions. One of them is Intermediate Value Theorem (IVT).

1.3.1 Intermediate Value Theorem

Let $f(x)$ be continuous in $[a, b]$. Then for every v between $f(a)$ and $f(b)$, there is a value t in $[a, b]$ such that $f(t) = v$.

Specific application:

Let $f(x)$ be continuous in $[a, b]$ and $f(a) \cdot f(b) < 0$. Then there exists at least one $\xi \in [a, b]$ such that $f(\xi) = 0$.

Example:

Let $f(x) = x^3 + 3x - 2$. As $f(0) = -2$ and $f(1) = 2$, there is a root for $f(x) = 0$ in $[0, 1]$. (We can apply bisection method to approximate the root.)

1.3.2 Extreme Value Theorem

Let $f(x)$ be continuous in $[a, b]$. Then there exists global maximum M and global minimum m and $x_M, x_m \in [a, b]$,
such that $f(x_M) = M \geq f(x)$ and $f(x_m) = m \leq f(x)$ for all $x \in [a, b]$.

1.3.3 Rolle's Theorem

For any function $f(x)$ that is continuous in $[a, b]$, $f'(x)$ exists in (a, b), and $f(a) = f(b)$,
there exists $\xi \in [a, b]$ such that $f'(\xi) = 0$.

1.3.4 Cauchy's Mean Value Theorem

Let $f(x), g(x)$ be both continuous in $[a, b]$ and $f'(x), g'(x)$ both exist in (a, b). Also $g'(x) \neq 0$ in $[a, b]$. (i.e. $g'(x)$ is either increasing or decreasing.)

Then we have

$$\frac{g(b) - g(a)}{f(b) - f(a)} = \frac{f'(\xi)}{g'(\xi)}$$

for some $\xi \in [a, b]$.

1.4 Proof of Lagrange's Mean Value Theorem

First we prove Rolle's Theorem. Then we use it to prove Cauchy's Mean Value Theorem.

Lagrange's Mean Value Theorem is the application of Cauchy's Mean Value Theorem on $g(x) = x$.

1.4.1 Proving Cauchy's Mean Value Theorem from Rolle's Theorem

Produce a function $\Phi(x)$ satisfying $\Phi(a) = \Phi(b)$.

Rearranging Cauchy's Mean Value Theorem, we have

$$(f(b) - f(a))g'(\xi) - (g(b) - g(a))f'(\xi) = 0$$

This motivates us to let that

$$\Phi(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$$

We can check that $\Phi(a) = -f(a)g(b)+f(b)g(a)$ and $\Phi(b) = -f(a)g(b)+f(b)g(a)$. Hence $\Phi(a) = \Phi(b)$.

Also $\Phi(x)$ is differentiable as it is a linear combination of $f(x)$ and $g(x)$.

Applying Rolle's Theorem to $\Phi(x)$, we have

$$\frac{d\Phi(x)}{dx}|_{x=\xi} = \frac{d}{dx}[(f(b) - f(a))g(x) - (g(b) - g(a))f(x)]$$

which is equal to $(f(b) - f(a))g'(\xi) - (g(b) - g(a))f'(\xi)$.

Hence

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

1.5 Discussion on continuity

$f(x)$ is continuous at c if $\lim_{t \to c} f(x) = f(c)$.

Similar to limit of a function, you can add, subtract, multiply and divide continuous functions.

$f(t) \oplus g(t)$ is continuous at $t = a$, where $\oplus$ is $+$, $-$, $\times$ or $\div$.
(When $\oplus$ is $\div$, $g(a)$ must not be 0.)

If $g(t)$ is continuous at $t = a$, and $f(u)$ is continuous at $t = g(a)$, then $f(g(t))$ is continuous at $t = a$.

> e.g. $\sqrt{1 + e^t}$ is a composite function, where $f(u) = \sqrt{u}$ and $g(t) = 1 + e^t$.

1.6 Application of Lagrange's Mean Value Theorem

Show $|\sin x - \sin y| \leq |x - y|$.

Proof:

Rewrite the question so that $\frac{|\sin x - \sin y|}{|x-y|} \leq 1$.

Let $f(t) = sin(t)$ be defined on $[x, y]$.

$$\frac{f(y) - f(x)}{y - x} = \frac{\sin y - \sin x}{y - x}$$

By Lagrange's Mean Value Theorem, this is equal to $f'(\xi) = \cos\xi$. As $\cos\xi \in [-1, 1]$, $|\cos\xi| \leq 1$, and this is proved.

2 Proof of Rolle's Theorem

2.1 Rolle's Theorem

For $f(x)$ continuous on $[a, b]$ and differentiable in (a, b), and $f(a) = f(b)$, there must exist at least one ξ in (a, b) that $f'(\xi) = 0$.

2.2 Fermat's Lemma

To prove Rolle's Theorem, we will use Fermat's Lemma.

> If $c \in (a, b)$ is a local maximum or minimum point and $f'(x)$ exists, then $f'(c) = 0$.
>
> A point $(c, f(c))$ is local maximum (minimum) if it is the maximum (minimum) in the interval $(c - h, c + h)$ for some $h > 0$.

Proof of Fermat's Lemma. Consider $\frac{f(c+h)-f(c)}{h}$, where c is a local maximum. Consider the left limit and right limit separately.

Note that the numerator is always ≤ 0 as c is a local maximum.

For the left limit, $h < 0$. Hence we have $\frac{f(c+h)-f(c)}{h} \geq 0$.
For the right limit, $h > 0$. Hence we have $\frac{f(c+h)-f(c)}{h} \leq 0$.

Then,

$$0 \leq \frac{f(c+h) - f(c)}{h} \leq 0$$

$$0 \leq \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \leq 0$$

$$0 \leq f'(x) \leq 0$$

$$f'(x) = 0$$

2.3 Proof of Rolle's Theorem

Now that we have proved Fermat's Lemma, we can prove Rolle's Theorem with the help of Extreme Value Theorem.

Extreme Value Theorem states that for $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$, there must be a global minimum x_m and global maximum x_M, such that $f(x_m) \leq f(x) \leq f(x_M)$ for all x in $[a, b]$.

Note on $[a, b]$. The interval is $[a, b]$ instead of (a, b). Consider $\tan x$, which is continuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$ but not on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
It has neither global minimum nor global maximum. (The range of $\tan x$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is $(-\infty, \infty)$.)

Then there are two cases for $f(x)$.

Case 1: at least one of x_m or x_M lies in (a, b). By Fermat's Lemma, the minimum or maximum that lies in (a, b) satisfies $f'(x) = 0$.

Case 2: both of x_m and x_M lies on the boundary, i.e. $\{x_m, x_M\} = \{a, b\}$. As $f(x_m) \leq f(x) \leq f(x_M)$, and $f(a) = f(b)$, we have

$$f(a) = f(b) = f(x_m) = f(x_M) = f(x) \text{ for all } x$$

Which means that for all $x \in (a, b)$, $f'(x) = 0$.

Therefore we have proved Rolle's Theorem for both cases.

3 Some Applications of Lagrange's Mean Value Theorem

3.1 Strictly Increasing Functions

Assumption: $f'(x) > 0$ for all x in $[a, b]$.

Conclusion: $f(x)$ is strictly increasing in (a, b).

Proof: Let $x_1 < x_2$, where x_1 and x_2 are two real numbers.

Applying the Lagrange's Mean Value Theorem, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi)$$

where $x_1 < \xi < x_2$.

However, we have $f'(\xi) > 0$ for all ξ. Also we have $x_2 - x_1 > 0$. Therefore we conclude that $f(x_2) - f(x_1) > 0$ and hence $f(x_2) > f(x_1)$.

3.2 Linear Function

Assumption: $f'(x)$ exists for any real number x and $f'(x)$ is a constant.

Conclusion: $f(x) = ax + b$ for some real numbers a and b.

Proof: (Tips: using Lagrange's Mean Value Theorem)

3.3 L'Hopital Rule

Assumption: $f(x)$ and $g(x)$ are continuous functions on an interval containing $x = a$, with $f(a) = g(a) = 0$.
Also f and g are differentiable, and that f' and g' are continuous.
Also $g'(a) \neq 0$.

Proof:

Consider $\frac{f(x)-0}{g(x)-0}$. This is same as $\frac{f(x)-f(a)}{g(x)-g(a)}$. ($f(a) = g(a) = 0$)

We can then apply Cauchy's Mean Value Theorem. This means that $\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$, where ξ is between x and a.

Then we take the limit as $x \to a$. We have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(\xi)}{g'(\xi)} = \frac{f'(a)}{g'(a)}$$

Note: the original limit must be in the form of $\pm 3. \frac{\infty}{\infty}$ or $\frac{0}{0}$

Remarks:

1. We can use L'Hopital rule to help us find limits like $\lim_{x \to 0^+} x^x$.
2. Let $f_n(x) = x^{x^{\cdots^{x}}}$, where there are n xs. (e.g. $f_2(x) = x^x$) We cannot easily find the minimum or maximum point of $f_n(x)$ for $n \geq 3$.

3.4 Taylor's Theorem

Recall that when $n = 0$, Taylor's Theorem is

$$f(x) = f(a) + \frac{f^{(0+1)}(\xi)}{(0+1)!}(x-a)^{(0+1)}$$

which is

$$f(x) = f(a) + f'(\xi)(x-a)$$

, a rearrangement of Lagrange's Mean Value Theorem

$$\frac{f(x)-f(a)}{x-a}=f'(\xi)$$

For $n=1$, we have

$$f(x)=f(a)+f'(a)\cdot(x-a)+\frac{f''(\xi)}{2!}(x-a)^2$$

, where the last term is the error term.

We intend to prove Taylor Theorem for the case of $n=1$. We can rearrange the theorem to become

$$f(x)-f(a)-f'(a)(x-a)=\frac{f''(\xi)}{2!}(x-a)^2$$

We can reconsider the case $n=0$, where we have $f(x)-f(a)=f'(\xi)(x-a)$. Similarly, we can try to rearrange the terms and obtain

$$\frac{f(x)-f(a)-f'(a)(x-a)}{(x-a)^2}=\frac{f''(\xi)}{2!}$$

Let $A(x)=f(x)-f(a)-f'(a)(x-a)$ and $B(x)=(x-a)^2$. We notice that $A(a)=0$ and $B(a)=0$. Hence the left hand side is equivalent to $\frac{A(x)-A(a)}{B(x)-B(a)}$. Also

$$\frac{d}{dx}(f(x)-f(a)-f'(a)(x-a))|_{x=\xi}=f'(\xi)-f'(a)$$

$$\frac{d}{dx}(x-a)^2|_{x=\xi}=2(x-a)|_{x=\xi}=2(\xi-a)$$

Hence

$$\frac{f(x)-f(a)-f'(a)(x-a)}{(x-a)^2} = \frac{1}{2}(\frac{f'(\xi)-f'(a)}{\xi-a}) = \frac{1}{2}\frac{df'(\xi)}{d\xi}|_{x=\alpha} = \frac{f''(\alpha)}{2}$$

4 Miscellaneous

4.1 Continuity of Composite Functions

Let $f(x), g(y)$ be functions. They are continuous at $x = a$ if

1. $f(x)$ is continuous at $x = a$; and
2. $g(y)$ is continuous at $y = f(a)$.