MATH1510 Notes

Contents

1 Taylor's Theorem

$f(x) = f(c) + \frac{f'(c)}{1!}(x-c)^1 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{(n+1)}$,

i.e. $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{1!}(x-c)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{(n+1)}$

, where ξ is between c and x (either $c < x$ or $c > x$).

1.1 Simple case of Taylor's Theorem

Take $n = 1$, we have $f(x) = f(c) + f'(c)(x-c) + O((x-c)^2)$.

By dropping the error term, $f(x) = f(c) + f'(c)(x-c)$,
which is the equation of tangent line at $x = c$.

1.2 Lagrange's Mean Value Theorem

$f(x) - f(c) = f'(\xi)(x-c)$
$\rightarrow \frac{f(x)-f(c)}{x-c} = f'(\xi)$ (Lagrange's Mean Value Theorem)

MVT.png

i.e. there is at last one point ξ, where slope of tangent line at ξ = slope of straight line joining $(x, f(x))$ and $(c, f(c))$.

Assumption 1: $f(t)$ is continuous in $[x, c]$. **(Closed interval!)**

Assumption 2: $f'(t)$ exists in (x, c). **(Open interval is enough.)**

> We require $f(t)$ be continuous in $[x, c]$ as we have to apply extreme value theorem.

1.3 Before Proving Lagrange's Mean Value Theorem

To prove Lagrange's Mean Value Theorem, we need to "deep" theorems about continuous functions. One of them is Intermediate Value Theorem (IVT).

1.3.1 Intermediate Value Theorem

Let $f(x)$ be continuous in $[a, b]$. Then for every v between $f(a)$ and $f(b)$, there is a value t in $[a, b]$ such that $f(t) = v$.

Specific application:

Let $f(x)$ be continuous in $[a, b]$ and $f(a) \cdot f(b) < 0$. Then there exists at least one $\xi \in [a, b]$ such that $f(\xi) = 0$.

Example:

Let $f(x) = x^3 + 3x - 2$. As $f(0) = -2$ and $f(1) = 2$, there is a root for $f(x) = 0$ in $[0, 1]$. (We can apply bisection method to approximate the root.)

1.3.2 Extreme Value Theorem

Let $f(x)$ be continuous in $[a, b]$. Then there exists global maximum M and global minimum m and $x_M, x_m \in [a, b]$,
such that $f(x_M) = M \geq f(x)$ and $f(x_m) = m \leq f(x)$ for all $x \in [a, b]$.

1.3.3 Rolle's Theorem

For any function $f(x)$ that is continuous in $[a, b]$, $f'(x)$ exists in (a, b), and $f(a) = f(b)$,
there exists $\xi \in [a, b]$ such that $f'(\xi) = 0$.

1.3.4 Cauchy's Mean Value Theorem

Let $f(x), g(x)$ be both continuous in $[a, b]$ and $f'(x), g'(x)$ both exist in (a, b). Also $g'(x) \neq 0$ in $[a, b]$. (i.e. $g'(x)$ is either increasing or decreasing.)

Then we have

$$\frac{g(b) - g(a)}{f(b) - f(a)} = \frac{f'(\xi)}{g'(\xi)}$$

for some $\xi \in [a, b]$.

1.4 Proof of Lagrange's Mean Value Theorem

First we prove Rolle's Theorem. Then we use it to prove Cauchy's Mean Value Theorem.

Lagrange's Mean Value Theorem is the application of Cauchy's Mean Value Theorem on $g(x) = x$.

1.4.1 Proving Cauchy's Mean Value Theorem from Rolle's Theorem

Produce a function $\Phi(x)$ satisfying $\Phi(a) = \Phi(b)$.

Rearranging Cauchy's Mean Value Theorem, we have

$$(f(b) - f(a))g'(\xi) - (g(b) - g(a))f'(\xi) = 0$$

This motivates us to let that

$$\Phi(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$$

We can check that $\Phi(a) = -f(a)g(b) + f(b)g(a)$ and $\Phi(b) = -f(a)g(b) + f(b)g(a)$. Hence $\Phi(a) = \Phi(b)$.

Also $\Phi(x)$ is differentiable as it is a linear combination of $f(x)$ and $g(x)$.

Applying Rolle's Theorem to $\Phi(x)$, we have

$$\frac{d\Phi(x)}{dx}|_{x=\xi} = \frac{d}{dx}[(f(b)-f(a))g(x)-(g(b)-g(a))f(x)]$$

which is equal to $(f(b)-f(a))g'(\xi)-(g(b)-g(a))f'(\xi)$.

Hence

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

1.5 Discussion on continuity

$f(x)$ is continuous at c if $\lim_{t\to c} f(x) = f(c)$.

Similar to limit of a function, you can add, subtract, multiply and divide continuous functions.

$f(t) \oplus g(t)$ is continuous at $t = a$, where $\oplus$ is $+$, $-$, $\times$ or $\div$.
(When $\oplus$ is $\div$, $g(a)$ must not be 0.)

If $g(t)$ is continuous at $t = a$, and $f(u)$ is continuous at $t = g(a)$,
then $f(g(t))$ is continuous at $t = a$.

> e.g. $\sqrt{1+e^t}$ is a composite function, where $f(u) = \sqrt{u}$ and $g(t) = 1+e^t$.

1.6 Application of Lagrange's Mean Value Theorem

Show $|\sin x - \sin y| \leq |x-y|$.

Proof:

Rewrite the question so that $\frac{|\sin x - \sin y|}{|x-y|} \leq 1$.

Let $f(t) = sin(t)$ be defined on $[x, y]$.

$$\frac{f(y)-f(x)}{y-x} = \frac{\sin y - \sin x}{y-x}$$

By Lagrange's Mean Value Theorem, this is equal to $f'(\xi) = \cos\xi$.
As $\cos\xi \in [-1,1]$, $|\cos\xi| \leq 1$, and this is proved.

2 Proof of Rolle's Theorem

2.1 Rolle's Theorem

For $f(x)$ continuous on $[a,b]$ and differentiable in (a,b), and $f(a) = f(b)$, there must exist at least one ξ in (a,b) that $f'(\xi) = 0$.

2.2 Fermat's Lemma

To prove Rolle's Theorem, we will use Fermat's Lemma.

> If $c \in (a,b)$ is a local maximum or minimum point and $f'(x)$ exists, then $f'(c) = 0$.
>
> A point $(c, f(c))$ is local maximum (minimum) if it is the maximum (minimum) in the interval $(c-h, c+h)$ for some $h > 0$.

Proof of Fermat's Lemma. Consider $\frac{f(c+h)-f(c)}{h}$, where c is a local maximum. Consider the left limit and right limit separately.

Note that the numerator is always ≤ 0 as c is a local maximum.

For the left limit, $h < 0$. Hence we have $\frac{f(c+h)-f(c)}{h} \geq 0$.
For the right limit, $h > 0$. Hence we have $\frac{f(c+h)-f(c)}{h} \leq 0$.

Then,

$$0 \leq \frac{f(c+h)-f(c)}{h} \leq 0$$

$$0 \leq \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} \leq 0$$

$$0 \leq f'(x) \leq 0$$
$$f'(x) = 0$$

2.3 Proof of Rolle's Theorem

Now that we have proved Fermat's Lemma, we can prove Rolle's Theorem with the help of Extreme Value Theorem.

> Extreme Value Theorem states that for $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$, there must be a global minimum x_m and global maximum x_M, such that $f(x_m) \leq f(x) \leq f(x_M)$ for all x in $[a, b]$.
>
> **Note on $[a, b]$.** The interval is $[a, b]$ instead of (a, b). Consider $\tan x$, which is continuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$ but not on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
> It has neither global minimum nor global maximum. (The range of $\tan x$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is $(-\infty, \infty)$.)

Then there are two cases for $f(x)$.

Case 1: at least one of x_m or x_M lies in (a, b). By Fermat's Lemma, the minimum or maximum that lies in (a, b) satisfies $f'(x) = 0$.

Case 2: both of x_m and x_M lies on the boundary, i.e. $\{x_m, x_M\} = \{a, b\}$. As $f(x_m) \leq f(x) \leq f(x_M)$, and $f(a) = f(b)$, we have

$$f(a) = f(b) = f(x_m) = f(x_M) = f(x) \text{ for all } x$$

Which means that for all $x \in (a, b)$, $f'(x) = 0$.

Therefore we have proved Rolle's Theorem for both cases.

3 Some Applications of Lagrange's Mean Value Theorem

3.1 Strictly Increasing Functions

Assumption: $f'(x) > 0$ for all x in $[a, b]$.

Conclusion: $f(x)$ is strictly increasing in (a, b).

Proof: Let $x_1 < x_2$, where x_1 and x_2 are two real numbers.

Applying the Lagrange's Mean Value Theorem, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi)$$

where $x_1 < \xi < x_2$.

However, we have $f'(\xi) > 0$ for all ξ. Also we have $x_2 - x_1 > 0$. Therefore we conclude that $f(x_2) - f(x_1) > 0$ and hence $f(x_2) > f(x_1)$.

3.2 Linear Function

Assumption: $f'(x)$ exists for any real number x and $f'(x)$ is a constant.

Conclusion: $f(x) = ax + b$ for some real numbers a and b.

Proof: (Tips: using Lagrange's Mean Value Theorem)

3.3 L'Hopital Rule

Assumption: $f(x)$ and $g(x)$ are continuous functions on an interval containing $x = a$, with $f(a) = g(a) = 0$.
Also f and g are differentiable, and that f' and g' are continuous.
Also $g'(a) \neq 0$.

Proof:

Consider $\frac{f(x)-0}{g(x)-0}$. This is same as $\frac{f(x)-f(a)}{g(x)-g(a)}$. $(f(a) = g(a) = 0)$

We can then apply Cauchy's Mean Value Theorem. This means that $\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$, where ξ is between x and a.

Then we take the limit as $x \to a$. We have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(\xi)}{g'(\xi)} = \frac{f'(a)}{g'(a)}$$

Note: the original limit must be in the form of $\pm 3.\frac{\infty}{\infty}$ or $\frac{0}{0}$

Remarks:

1. We can use L'Hopital rule to help us find limits like $\lim_{x \to 0^+} x^x$.
2. Let $f_n(x) = x^{x^{\cdots^x}}$, where there are n xs. (e.g. $f_2(x) = x^x$)
   We cannot easily find the minimum or maximum point of $f_n(x)$ for $n \geq 3$.

3.4 Taylor's Theorem

Recall that when $n = 0$, Taylor's Theorem is

$$f(x) = f(a) + \frac{f^{(0+1)}(\xi)}{(0+1)!}(x-a)^{(0+1)}$$

which is

$$f(x) = f(a) + f'(\xi)(x-a)$$

, a rearrangement of Lagrange's Mean Value Theorem

$$\frac{f(x) - f(a)}{x - a} = f'(\xi)$$

For $n = 1$, we have

$$f(x) = f(a) + f'(a) \cdot (x-a) + \frac{f''(\xi)}{2!}(x-a)^2$$

, where the last term is the error term.

We intend to prove Taylor Theorem for the case of $n = 1$. We can rearrange the theorem to become

$$f(x) - f(a) - f'(a)(x-a) = \frac{f''(\xi)}{2!}(x-a)^2$$

We can reconsider the case $n = 0$, where we have $f(x) - f(a) = f'(\xi)(x-a)$. Similarly, we can try to rearrange the terms and obtain

$$\frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)^2} = \frac{f''(\xi)}{2!}$$

Let $A(x) = f(x) - f(a) - f'(a)(x-a)$ and $B(x) = (x-a)^2$. We notice that $A(a) = 0$ and $B(a) = 0$. Hence the left hand side is equivalent to $\frac{A(x)-A(a)}{B(x)-B(a)}$. Also

$$\frac{d}{dx}(f(x) - f(a) - f'(a)(x-a))|_{x=\xi} = f'(\xi) - f'(a)$$
$$\frac{d}{dx}(x-a)^2|_{x=\xi} = 2(x-a)|_{x=\xi} = 2(\xi - a)$$

Hence

$$\frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)^2} = \frac{1}{2}(\frac{f'(\xi) - f'(a)}{\xi - a}) = \frac{1}{2}\frac{df'(\xi)}{d\xi}|_{x=\alpha} = \frac{f''(\alpha)}{2}$$

4 Taylor Theorem Continued

4.1 $n = 3$ case of Taylor's Theorem

Consider $f(x) = f(c) + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(\xi)}{3!}(x-c)^3$.

By rearranging the terms, we have

$$\frac{f(x) - f(c) - \frac{f'(c)}{1!}(x-c) - \frac{f''(c)}{2!}(x-c)^2}{(x-c)^3} = \frac{f^3(\xi)}{3!}$$

Let $F(x) = \text{numerator} = f(x) - f(c) - \frac{f'(c)}{1!}(x-c) - \frac{f''(c)}{2!}(x-c)^2$, and $G(x) = (x-c)^3$. Hence we have $F(c) = 0$ and $G(c) = 0$. Hence we have

$$\frac{F(x) - F(c)}{G(x) - G(c)} = \frac{f^3(\xi)}{3!}$$

$F'(\xi) = f'(\xi) - f'(c) - 2 \cdot \frac{f''(c)}{2!}(\xi - c) = f'(\xi) - f'(c) - f''(c)(\xi - c)$
$G'(\xi) = 3(\xi - c)^2$

Hence

$$\frac{f(x) - f(c) - f'(c)(x-c) - \frac{f''(c)}{2}(x-c)^2}{(x-c)^3} = \frac{f'(\xi) - f'(c) - f''(c)(\xi - c)}{3(\xi - c)^2}$$

We can apply Cauchy's Mean Value Theorem once more and obtain

$$\frac{f'(\xi) - f'(c) - f''(c)(\xi - c)}{3(\xi - c)^2} = \frac{f''(\eta) - f''(c)}{6(\eta - c)}$$

Then we can apply Lagrange's Mean Value Theorem and obtain

$$\frac{f''(\eta) - f''(c)}{6(\eta - c)} = \frac{1}{6} f^{(3)}(\delta) = \frac{f^{(3)}(\delta)}{3!}$$

for some δ between x and c.

4.2 Some terminologies related to Taylor's Theorem

Terms:

1. Taylor's Theorem
2. Taylor Polynomial
3. Taylor Series
4. MacLaurin Series
5. Center

Taylor's Theorem:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$
$$+ \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{(n+1)}$$

Assumptions: f has $n+1$ derivatives for an open interval containing c.

Remark: the point c is special. If we substitute $x = c$ on the right hand side, all terms except $f(c)$ vanish. Hence $f(x) = f(c)$ and it is named **center**.

The polynomial is a good approximation of $f(x)$ near the center c. It's called **Taylor polynomial of degree** n. **It does not contain the error term.**

MacLaurin polynomial: Taylor polynomial centered at $x = 0$

MacLaurin series: Taylor series centered at $x = 0$

4.3 Examples

4.3.1 Exponential Function

Find Taylor Polynomial of degree 1, 2 and 3 centered at $x = 0$, for the function $f(x) = e^x$.

$TP(1) = f(0) + \frac{f'(0)}{1!}(x-0)^1 = e^0 + \frac{e^0}{1!}x = 1 + x$
$TP(2) = f(0) + \frac{f'(0)}{1!}(x-0)^1 + \frac{f'(0)}{2!}(x-0)^2 = e^0 + \frac{e^0}{1!}x + \frac{e^0}{2!}x^2 = 1 + x + \frac{x^2}{2!}$
$TP(3) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

4.4 Taylor Series

Question: if $f(x)$ has all derivatives in an open interval containing the center C, is it true that

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!}(x-c)^i$$

(without error term)? No! (Not always)

If a function can be written as a Taylor polynomial with no error term, then the function is a Taylor series.

Examples:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

Examples of not Taylor Series:

All 3 converge for only $-1 < x < 1$:

$$\ln(1+x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \dots$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$
$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots$$

5 Section of Taylor's Theorem

5.1 Estimation of Taylor's Theorem

For all practical purposes, we're only interested in "truncated Taylor series" or Taylor polynomial of degree (order) n.

$$TP_n(x,c) = f(c) + \sum_{i=1}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i$$

5.2 Limitation of Taylor Polynomial

Some functions cannot be represented by Taylor Polynomial. For example,

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$$

$f^{(n)}$ for all $n > 0$ exists, and $f^{(n)}(0) = 0$ for all $n \geq 1$.

Consider the Taylor Polynomial

$$f(c) + \sum_{i=1}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i$$

which is equal to 0.

We can also consider the similar function

$$g(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

which has similar properties.

5.3 Analytic Functions

If

$$f(x) = f(c) + \sum_{i=1}^{\infty} \frac{f^{(i)}(c)}{i!}(x-c)^i$$

then we say $f(x)$ is analytic at $x = c$.

For example, $\sin(x), \cos(x), \ln(1+x), e^x$ are all analytic functions.

5.4 Estimation with Taylor Series

Suppse we want to calculate the value of e. How to do it if the allowed error is 0.01?

Solution: use truncated Taylor Series entered at some easy-to-calculate point c. How to truncate, or how to choose n?

Assume for simplicity that $e < 3$. Choose the function $f(x) = e^x$ and $c = 0$. Then we have

$$e = f(1) = f(0) + \frac{f'(0)}{1!}(x-0)^1 + \cdots + \frac{f^{(n)}(0)}{n!}(x-0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-0)^{(n+1)}$$

which is equal to

$$1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!} + \frac{e^\xi \cdot x^{(n+1)}}{(n+1)!}$$

and we have $0 < \xi < x$, i.e. $0 < \xi < 1$.

Hence we need to control the error term, so that it is ≤ 0.01, i.e. $|\frac{e^\xi}{(n+1)!}| \leq 0.01$

Then we need to find an "n" satisfying this inequality. We cannot solve the equation easily, so we solve $|\frac{3}{(n+1)!}| \leq 0.01$ instead.

Hence, we have $n = 5$.

$$e^1 = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{5!}$$

approximates e^1 with error < 0.01.

5.5 Another Problem Involving Taylor Series

Find n such that Taylor polynomial of degree n of $\sin x$ estimates $\sin 1$ with error < 0.01.

Solution: Let $f(x) = \sin x$ and choose $c = 0$. We have

$$f^{(n)}(0) = \begin{cases} 0 & , n \equiv 0 \pmod 4 \\ 1 & , n \equiv 1 \pmod 4 \\ 0 & , n \equiv 2 \pmod 4 \\ -1 & , n \equiv 3 \pmod 4 \end{cases}$$

Then we can solve the error term < 0.01, i.e.

$$\frac{f^{(n+1)}(0)}{(n+1)!} < 0.01$$

We have $-1 \leq f^{(n+1)}(0) \leq 1$. Hence we can solve the looser condition $|\frac{1}{(n+1)!}| < 0.01$ instead.

5.6 Two Interesting Problems

1. $\ln(1+x)$: use this function to find $\ln(1.5)$ with error error < 0.01.
2. $\sqrt{1+x}$: use this function to find $\sqrt{1.5}$ with error < 0.01. (Note: use Taylor's Theorem, not Newton's generalized binomial theorem, $-1 < x < 1$)

Problem 1: let $f(x) = \ln(1+x)$, $c = 0$.

$$f(0) = \ln 1 = 0$$
$$f'(0) = 1$$
$$f''(0) = -1$$
$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$
$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$$

The domain of the Taylor series (radius of convergence) is $(-1, +1)$.

5.7 An Example about Integration

Question: find $\int_0^1 e^{x^2} dx$ with error < 0.01.

Note: $\int e^{x^2}$ and $\int e^{-x^2}$ has no closed form solution.

Solution: to approximate $\int_0^1 e^{x^2} dx$, we can use Taylor's Theorem (and control the error term):

Wrong idea: use Taylor series and let $f(x) = e^{x^2}$. The expansion of the Taylor polynomial is extremely complicated.

Correct idea (not necessarily the most accurate):

$$\int_0^1 e^{x^2} dx = \int_0^1 (\sum_{k=0}^{n} \frac{x^{2k}}{k!}) dx + \int_0^1 \frac{e^{\xi}}{(n+1)!} x^{2n+2} dx$$

We want the error term to be < 0.01, i.e. $\int_0^1 \frac{e^{\xi}}{(n+1)!} x^{2n+2} dx < 0.01$

We have $\xi \in [0,1] \implies e^{\xi} \in [1, e]$ and $x \in [0,1] \implies x^{2n+2} \in [0,1]$. Hence the error term $\int_0^1 \frac{e^{\xi}}{(n+1)!} x^{2n+2} dx < \int_0^1 \frac{e}{(n+1)!} dx < \frac{e}{(n+1)!}$.

5.8 Summary

Example of Taylor Series / Theorem: $\sin 1, \ln(1+x), \sqrt{1+x}, \int_0^1 e^{x^2} dx$

Mentioned the domain of the Taylor Polynomial may not be the same as the left hand side

$+, -, \times, \div, \circ$(composition) of taylor series

5.9 Power series of $\frac{\sin x}{1-x}$

Question: We want to find the power series of $\frac{\sin x}{1-x}$.

Notes:

$$\frac{\sin x}{1-x} = (x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^{2n+1} \cdot \frac{x^{2n+1}}{(2n+1)!} + \dots)$$
$$\cdot (1 + x + x^2 + \dots + x^n + \dots)$$

By collecting terms, we have

$$\begin{aligned}\sin x &= x + x^2 + \frac{5}{6}x^3 + \dots \\ &= (\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}) \cdot (\sum_{k=0}^{\infty} x^k) \\ &= \sum_{k=0}^{\infty} (\sum_{l=0}^{k} (\dots)) x^k\end{aligned}$$

6 Curve Sketching and Related Concepts

6.1 Concavity

Concave upwards ('Convex'): $f''(x) > 0$

Concave downwards ('Concave'): $f''(x) < 0$

6.2 Local maximum / minimum

To 'sketch' the curve (more officially 'graph') of a function f, we need, among other things, to find 'local maximum / minimum points'.

To find them, we can use **first derivative test** or **second derivative test**.

6.2.1 First Derivative Test

A point p is called the local minimum (maximum) of $f(x)$ if

1. $f'(p-\epsilon) < 0$ ($f'(p-\epsilon) > 0$) for small positive ϵ
2. $f'(p+\epsilon) > 0$ ($f'(p+\epsilon) < 0$) for small positive ϵ
3. f is continuous at $x = p$

This can be proved by integration or Mean Value Theorem.

6.2.2 Second Derivative Test

A point p is called the local minimum (maximum) of $f(x)$ if

1. **Assumption:** $f''(x)$ is continuous near p
2. $f'(p) = 0$
3. $f''(p) < 0$ ($f''(p) > 0$)

This implies that $f'(p)$ is strictly decreasing (strictly increasing), and hence by the First Derivative Test, p is the local maximum (local minimum) point.

6.3 Inflection Point

The point at which $f''(x)$ changes sign, or the concavity of $f(x)$ changes.

7 Integration Theory

There are two main concepts in integration:

Indefinite Integration (not about area)

Definite Integration (about area)

7.1 Indefinite Integration

What is $\int f(x)dx$? How is it related to derivatives?

If we let $F(x) = \int f(x)dx$, then we know that

$$\frac{d}{dx}F(x) = f(x)$$

This equation is also an example of a **first order ordinary differential equation**, which is an equation of the form

$$\frac{dF(x)}{dx} = f(x)$$

($f(x)$ is given, and $F(x)$ is the unknown.)

Example:

$$\frac{dF(x)}{dx} = x \implies F(x) = \frac{1}{2}x^2 + C$$

That C is a constant can be proved with Mean Value Theorem.

7.2 Phase Plane

Reference: Phase Plane Plotter

The following set of equation is a pendulum motion:

$$\begin{aligned} x' &= y \\ y' &= \sin x \end{aligned}$$

For our context, we want to solve a general differential equation like

$$\frac{dy}{dx} = x$$

We can let $t = x$, then

$$\frac{dx}{dt} = \frac{dx}{dx} = 1$$

$$\frac{dy}{dt} = \frac{dy}{dx} = x$$

If we try to solve the pendulum equation, we have

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin x}{y}$$
$$\frac{dy}{dx} = \frac{\sin x}{y}$$

which leads us to nothing. We can try

$$\frac{dy}{dt} = \frac{d(\frac{dx}{dt})}{dt} = \sin x$$

which is

$$\frac{d^2x}{dt^2} = \sin x$$

the basic pendulum equation.

Interestingly, we can see the phase plane diagram and notice that only for small angles the pendulum have a period.

Also, only for very simple functions (generally those mentioned in DSE syllabus), obtained from integration, the phase diagram has no intersection points.

7.3 More on Indefinite Integral

Indefinite integral $\int f(x)dx$ are solutions of the differential equation

$$\frac{dF(x)}{dx} = f(x)$$

and the solution curves of $y = F(x)$ do not intersect.

7.3.1 Prove for nonexistence of intersection

Claim: let $y = F_1(x), y = F_2(x)$ be two solutions of the differential equation $\frac{dF(x)}{dx} = f(x)$. Then they differ only by a constant.

Proof: consider the difference $F_2(x) - F_1(x)$. Then we have

$$\frac{d}{dx}(F_2(x) - F_1(x)) = f(x) - f(x) = 0$$

As $\frac{d}{dx}(F_1(x) - F_2(x)) = 0$ in some interval $[a, b]$, we have $F_1(x) - F_2(x) \equiv C$.

Proof: Let $G(x) = F_1(x) - F_2(x)$. We have

$$\frac{G(b) - G(a)}{b - a} = G'(\xi)$$

for some ξ and all a, b.

However, $G'(\xi) = 0$ for any ξ.

Then we have

$$\frac{G(b) - G(a)}{b - a} = 0 \implies G(b) - G(a) = 0 \implies G(a) = G(b)$$

for all a, b.

Therefore, $G(x) \equiv C$.

Proof by contradiction: suppose $G(x) \not\equiv C$. Then there exists two points x_1, x_2 in the domain such that

$$G(x_1) \neq G(x_2)$$

By Lagrange Mean Value Theorem, there exists at least one ξ such that

$$0 \neq \frac{G(x_1) - G(x_2)}{x_1 - x_2} = G'(\xi)$$

i.e. there exists at least one ξ such that $G'(\xi) \neq 0$. However $G'(\xi) = 0$ as proved, and there is a contradiction.

Hence $G(x) \equiv C$.

7.3.2 Existence of Solutions

Next, we study whether the equation $\frac{dF(x)}{dx} = f(x)$ has solution, where x is in $[a, b]$.

Assumptions needed:

1. $f(x)$ is a continuous function.
2. $[a, b]$ is a closed interval.

With these assumptions, we also know that $F(x)$ is differentiable **in** (a, b). ($f(x)$, the derivative, is continuous), i.e. $F(x)$ is smoother than $f(x)$.

For example, what is $\int |x|dx$?

Answer: let $F(x)$ be the solution.

$F(x) = \frac{x^2}{2} + C$ for $x \geq 0$ and $F(x) = -\frac{x^2}{2} + C$ for $x < 0$.

$$F(x) = \int |x|dx + C = \begin{cases} \frac{x^2}{2} + C_1 & (x \geq 0) \\ -\frac{x^2}{2} + C_2 & (x < 0) \end{cases}$$

Also this function is continuous for each x. So we have $C_1 = C_2 = C$ for some C.

7.4 Some Facts about Integration

1. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
2. $\int kf(x)dx = k \int f(x)dx$

The first two properties are called "linearity". (Related to the existence of only first derivatives in differential equation.)

7.5 Analog of Product, Quotient and Chain Rule

Question: for differentiation, we have product, quotient, chain rules. How about integration?

Answer:

Chain rule $\approx$ method of substitution.

For example, evaluate $\int \cos 2x dx$.
Let $u = 2x$. Then $dx = \frac{1}{2}du$.
$\int \cos 2x dx = \int \cos u \frac{du}{2} = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$.

Product rule $\approx$ integration by part.

Let $u(x), v(x)$ be two differentiable functions. Then

$$\begin{aligned}
&d(u(x)v(x)) \\
&= (v(x)\frac{du(x)}{dx} + u(x)\frac{dv(x)}{dx}) \cdot dx \\
&= v(x)du(x) + u(x)dv(x)
\end{aligned}$$

$$d(uv) = vdu + udv$$

$$\int d(uv) = \int vdu + \int udv$$

$$\int udv = -\int vdu + \int 1d(uv) = -\int vdu + uv$$

7.6 Examples of Integration

We have some commonly used results like

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$\int \cos x dx = \sin x + C$

$\int \sin x dx = -\cos x + C$

$\int e^x dx = e^x + C$

$\int x^{-1} dx = \ln|x| + C$

The constant is important: let's consider $\int x^{-1}dx$.

$$\begin{aligned}\int x^{-1}dx &= -\int x(-x^{-2})dx + 1(u = x^{-}1, v = x, du = -x^{-2}dx)\\ &= \int x^{-1}dx + 1\end{aligned}$$

Where is the mistake?

Some problems to solve:

1. $I_n = \int x^n e^x dx$ (easy)
2. $\int \cos x e^x dx$ (difficult)

By using **reduction formula**, we have

$$\begin{aligned}I_n &= \int x^n d(e^x)\\ &= -\int e^x d(x^n) + x^n e^x\\ &= -n\int x^{n-1}e^x + x^n e^x\\ &= -nI_{n-1} + x^n e^x\end{aligned}$$

$$\begin{aligned}\int e^x \cos x dx &= e^x \cos x + \int e^x \sin x dx\\ &= e^x \cos x + e^x \sin x - \int e^x \cos x dx\end{aligned}$$

Hence

$$\int e^x \cos x dx = \frac{1}{2}e^x(\sin x + \cos x) + C$$

Problem to think:
Given $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$. Find reduction formula of I_n and show $\lim_{n\to\infty} I_n = 0$.

7.7 Malthus Population Model

We have a population model based on the differential equation

$$\frac{dP}{dt} = kP$$
$$\int \frac{dP}{P} = \int kdt$$
$$P = Ce^{kt}$$

But the model is too idealistic so we modify the differential equation to become

$$\frac{dP}{dt} = kP(1 - \frac{P}{K})$$

where K is the maximum population. This equation is also separable.

We can prove that P converges to K.

This can also be generalized to a predator-prey model, for example

$$\frac{dx}{dt} = f_x(x, y) = k_1 x(x - y)$$
$$\frac{dy}{dt} = f_y(x, y) = k_2 x(x + y)$$

7.8 Partial Fraction

To solve the equation $\frac{dP}{dt} = kP(\frac{K-P}{K})$ or similar differential equations, we solve $\frac{dP}{P(K-P)} = \int kdt$ instead using partial fraction method.

For simplicity let $K = 1$.

Our target is to resolve the left hand side to some integral similar to $\int dPP$.

We recall that it is possible to decompose a fraction like $\frac{1}{6}$ by doing the following:

$$\frac{1}{6} = \frac{1}{2 \cdot 3} = \frac{1}{2} - \frac{1}{3}$$

and the ‘dictionary‘ integer $\longleftrightarrow$ polynomial and rational number $\longleftrightarrow$ rational function.

For exmaple, let us try to decompose $\frac{1}{x^2+3x+2}$. As $\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)}$, we have

$$\frac{1}{x^2 + 3x + 2} = \frac{A}{x+1} + \frac{B}{x+2}$$

where A, B are rational numbers. We can then solve the identity

$$1 \equiv A(x+2) + B(x+1)$$

to find the coefficients, and obtain the final equation

$$\frac{1}{x^2 + 3x + 2} = \frac{1}{x+1} - \frac{1}{x+2}$$

Integrating both sides, we have

$$\int \frac{1}{x^2 + 3x + 2} dx = \ln|x+1| - \ln|x+2|$$

Problem: how do we integrate functions like $\frac{x+1}{x^2+x+1}$?

7.9 Integration Problem Examples

$$\begin{aligned}\int 3x^2 \arctan(x^3) dx &= \int \arctan(x^3) d(x^3) \\ &= x^3 \arctan(x^3) - \int x^3 \cdot \frac{3x^2}{1+x^3} dx\end{aligned}$$

7.10 Half Angle t-Substitution

How do we evaluate the following integral?

$$\int \frac{1}{1 - \sin x + \cos x} dx$$

Or, actually, any rational functions of $\sin x$ and $\cos x$ like $\frac{\sin x - 3\cos x}{2\sin x + 5\cos x + 1}$?

We can use the technique t-substitution, but letting $t = \tan\frac{x}{2}$. Then we can express $\sin x, \cos x$ and dx in terms of t.

$$\begin{aligned}
\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{3} \\
&= 2 \cdot \frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} \\
&= \frac{2t}{1+t^2}
\end{aligned}$$

$$\begin{aligned}
\cos x &= \cos^2 x - \sin^2 x \\
&= (\frac{1}{\sqrt{1+t^2}})^2 - (\frac{t}{\sqrt{1+t^2}})^2 \\
&= \frac{1-t^2}{1+t^2}
\end{aligned}$$

$$\begin{aligned}
dt &= d(\tan\frac{x}{2}) \\
&= \frac{1}{2}\sec^2\frac{x}{2} dx \\
&= \frac{1}{2}(1+t^2) dx
\end{aligned}$$

$$dx = \frac{2dt}{1+t^2}$$

Therefore

$$\begin{cases} \sin x = \frac{2t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} \\ dx = \frac{2dt}{1+t^2} \end{cases}$$

7.11 Trigonometric Substitution Revisited

Let us consider integrals of the form $\int_{x=-R}^{x=R} \sqrt{R^2 - x^2} dx$. Let us compute the integral, taking into the account the domain of the integrands (i.e. $\sqrt{R^2 - x^2}$).

Note that the integrand must be continuous and the domain must be closed.

Let $x = R\sin\theta$.

What is the domain of θ?

$$x = -R \implies R = R\sin\theta \implies \theta = -\frac{\pi}{2}$$
$$x = R \implies R = R\sin\theta \implies \theta = \frac{\pi}{2}$$

In the two equations above, there are many solutions. The 'branch' (domain, term from complex function theory) $[-\frac{pi}{2}, \frac{pi}{2}]$ was chosen.

Then, $dx = R\sin\theta d\theta$.

Finally,

$$\begin{aligned} \int_{-R}^{R} \sqrt{R^2 - x^2} dx &= \int_{\theta=-\frac{pi}{2}}^{\theta=\frac{\pi}{2}} \sqrt{R^2 - R^2\sin^2\theta} \cdot R\cos\theta d\theta \\ &= \int_{\theta=-\frac{pi}{2}}^{\theta=\frac{\pi}{2}} |R\cos\theta| \cdot R\cos\theta d\theta \\ &= \int R^2\cos^2\theta \text{ (for the branch } [-\frac{\pi}{2}, \frac{\pi}{2}],\ \cos\theta \geq 0) \end{aligned}$$

Also, for the branch chosen, we have $\frac{d \sin \theta}{d\theta} \geq 0$ and $\cos \theta \geq 0$.

> Interestingly, the proof above is actually a circular proof. When we compute dx, we used the fact that $\frac{d \sin \theta}{d\theta} = \cos \theta$, which relies on the fact that $\lim_{h \to 0} \frac{\sin h}{h} = 1$, which is derived from the area of circle $A = \frac{1}{2} r^2 \theta$.

7.12 Integration by Rationalization

Problem: Evaluate $\int \frac{\sqrt{x+1}}{x} dx$.

Solution: Let $u = \sqrt{x+1}$.

Then $u^2 = x + 1 \implies 2udu = dx$.

$$\begin{aligned} \int \frac{\sqrt{x+1}}{x} &= \int \frac{u}{u^2 - 1} \cdot 2udu \\ &= \int \frac{2}{u^2 - 1} du \end{aligned}$$

Which is a rational function.

This can also be used to solve integration problems like $\int \sqrt{\frac{ax+b}{cx+d}} dx$.

8 Riemann Sum

8.1 Introduction

Goal: To compute the 'area' (signed area) 'under' the curve $y = f(x)$, for $a \leq x \leq b$.

Idea: To copy the idea of Archimedes of calculating area of circle. To do this, put 'suitable' rectangles under the curve to approximate the area, and increase the number of rectangles so the sum of area of rectangles gets close to the area under the curve.

There are several ways to choose the rectangles. For example, the **lower sum** can be used.

8.2 An Example of Riemann Sum

Consider the curve $y = x$, and $-1 \leq x \leq 1$. Let $n = 4$. Then

$$x_0 = -1, x_1 = -0.5, x_2 = 0, x_3 = 0.5, x_4 = 1$$

is one method of choosing x.

Using **lower sum**, we have

$$\begin{aligned}\textbf{Area} &= (-1) \cdot 0.5 + (-0.5) \cdot 0.5 + 0 \cdot 0.5 + 0.5 \cdot 0.5 \\ &= -0.5\end{aligned}$$

Using **midpoint sum**, we have

$$\begin{aligned}\textbf{Area} &= (-0.75) \cdot 0.5 + (-0.25) \cdot 0.5 + 0.25 \cdot 0.5 + 0.75 \cdot 0.5 \\ &= 0\end{aligned}$$

The area under the curve can be approximated as follows:

$$\int_{x=a}^{x=b} f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

where $\xi_i \in [x_{i-1}, x_i]$. Then it can be proved that any ξ_i can be chosen without affecting the limit.

8.3 Condition for Integrable Functions

1. The subdivision of $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ must satisfy the condition that $|\Delta x_i| \to 0$ as $n \to \infty$.

Why? In order to avoid the situation like ($a = 0, b = 1$) $x_0 = 0, x_i = \frac{1}{2^i}$ for $i > 0$.

2. $f(x)$ needs to be a continuous function on $[a, b]$.

If condition 1 and 2 are satisfied, then it can be proved (using epsilon-delta definition) that $\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$ exists (and is finite), and is defined as the area under the curve.

This limit is denoted (given the notation) $\int_a^b f(x)dx$.

8.4 Rough Idea of Proof

Bound (control, find the upper and lower limit) $f(\xi_i)$ from above and below.

find m_i, M_i such that $m_i \leq f(\xi_i) \leq M_i$. m_i and M_i are the **absolute minimum** and **absolute maximum** of $f(x)$ in $[x_{i-1}, x_i]$ respectively.

> As we are consider the **closed** interval $[x_{i-1}, x_i]$, the absolute minimum and maximum must exist.

Therefore $m_i\Delta x_i \leq f(\xi_i)\Delta x_i \leq M_i\Delta x_i$.

> We wish that as $n \to \infty$, $\lim \sum_{i=1}^{n} m_i\Delta x_i = \lim \sum_{i=1}^{n} M_i\Delta x_i$.
>
> Then by sandwich theorem,
>
> $$\lim_{n\to\infty} \sum_{i=1}^{n} m_i\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} M_i\Delta x_i$$
>
> and it doesn't matter how we choose ξ_i as long as $\xi_i \in [x_{i-1}, x_i]$.

$\lim_{n\to\infty} \sum_{i=1}^{n} m_i\Delta x_i$ is named the **lower sum** and $\lim_{n\to\infty} \sum_{i=1}^{n} M_i\Delta x_i$ is denoted the **upper sum**. They are also called the **Darboux sums**.

Instead of finding out the upper sum / the lower sum, we estimate $\sum_{i=1}^{n}(M_i - m_i)\Delta x_i$.

$M_i - m_i$ measures the 'change' of $f(x)$ when $x_{i-1} \leq x \leq x_i$.

If f is continuous, it can be proved that $M_i - m_i$ can be made arbitrary small.

Applications: this can be used to find the normal distribution table.

8.5 Another Example of Riemann Sum

Consider the function $y = f(x) = x$.

Step 1: let $x_0 = 0, x_1 = \frac{1}{n}, \ldots, x_i = \frac{i}{n}, x_n = 1$.
This way of partitioning $[a, b]$ is called **equipartition**.

Step 2: $\Delta x_i = x_i - x_{i-1} = \frac{i}{n} - \frac{i-1}{n} = \frac{1}{n}$.

Step 3: We will choose the upper sum for this problem.

$$\begin{aligned}
\text{Upper sum} &= U_n \\
&= \sum_{i=1}^{n} \frac{1}{n} \cdot in \\
&= \frac{1}{n^2} \sum_{i=1}^{n} i \\
&= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} \\
&= \frac{n+1}{2n} \\
&= \frac{1}{2}(1 + \frac{1}{n})
\end{aligned}$$

Step 4:

$$\lim_{n \to \infty} U_n = \frac{1}{2} = \int_0^1 x dx = \left[\frac{x^2}{2} \right]_0^1$$

Remark: if $f(x) = x^\alpha$ for $\alpha > 1$, then equipartition does not work well.

Note: Mathematician *Otto Toeplitz*'s approach of calculus.

9 Fundamental Theorem of Calculus

9.1 An Example

Consider the problem

$$\int_0^1 x dx$$

First, we find $\int x dx = \frac{x^2}{2} + C$.
Then, let $F(x) := \frac{x^2}{2} + C$.
Finally, compute $F(1) - F(0) = \int_0^1 x dx$.

How do we prove that this is correct?

Consider $f(x) : [a, b] \to \mathbb{R}$, and let $x \in (a, b)$. Then we notice that the area of the curve $y = f(x)$ between x and $x + \Delta x \approx f(x) \cdot \Delta x$.

But the area of the small rectangle is also equal to $\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt$, which is equal to $\int_x^{x+\Delta x} f(t)dt$.

Or, by algebra,

$$\begin{aligned}
&\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt \\
=&\int_a^{x+\Delta x} f(t)dt + \int_x^a f(t)dt \\
=&\int_x^{x+\Delta x} f(t)dt
\end{aligned}$$

where we have used

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$ (from area of rectangle)
2. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

But then because $\int_a^{x+\Delta x} f(t)dt = G(x+\Delta x)$, and $\int_a^x f(t)dt = G(x)$, where $G(x) := \int_a^x f(t)dt$,
so $\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt = G(x+\Delta x) - G(x)$.

Dividing Δx from both sides, we have

$$\frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt \approx \frac{f(x)\Delta x}{\Delta x} = f(x)$$

Also

$$\lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x} \approx f(x)$$

We can use the **Integral Mean Value Theorem** to prove the two statements above more formally.

> The **Integral Mean Value Theorem** states that if f is continuous in $[a,b]$, then there exists ξ in $[a,b]$ such that
>
> $$\int_a^b f(x)dx = f(\xi)(b-a)$$
>
> or, equivalently,
>
> $$f(\xi) = \frac{1}{b-a}\int_a^b f(x)dx$$

Using this theorem, we have

$$\int_x^{x+\Delta x} f(t)dt = f(\xi)\cdot \Delta x$$

for some $\xi \in [x, x+\Delta x]$. Then,

$$\lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x} = \lim_{\Delta x \to 0} f(\xi)$$

As $x \leq \xi \leq x + \Delta x$, by sandwich theorem, $\xi \to x$.

Remark: $\int_a^b x dx$

Step 1: $\int x dx = \frac{x^2}{2} + C$, i.e. find $G(x)$ such that $G'(x) = x$.

Step 2: If we can find the function $G(x)$, then $\int_a^b x dx = G(b) - G(a)$.

10 Miscellaneous

10.1 Continuity of Composite Functions

Let $f(x), g(y)$ be functions. They are continuous at $x = a$ if

1. $f(x)$ is continuous at $x = a$; and
2. $g(y)$ is continuous at $y = f(a)$.