# Mathematical modeling exercises

Renan Vicente Pinto

July 8, 2025

## 1 Optimal dimensions of a can

**Determine the dimensions (height and radius) of a cylindrical can that will either maximize its volume or minimize its cost while satisfying material constraints.**

*Solution*: Let $h$ and $r$ be the height and the radius of the can, respectively. We will consider the area of the surface of the can as a constant $A$. We write the following model to maximize its volume:

$$\begin{array}{ll} \max & \pi\, r^2\, h \\ \text{s. t.} & 2\,\pi\, r^2 + 2\,\pi\, r\, h \;=\; A \\ & r, h \;\geq\; 0\,. \end{array}$$

Let $c$ be the cost per unit of area. To minimize its cost, we consider its volume as a constant $V$:

$$\begin{array}{ll} \min & c\,(2\,\pi\, r^2 + 2\,\pi\, r\, h) \\ \text{s. t.} & \pi\, r^2\, h \;=\; V \\ & r, h \;\geq\; 0\,. \end{array}$$

## 2 Facility location problem

**Determine the optimal location for two new warehouses to minimize transportation costs to a set of destinations. The transportation cost is proportional to the distance and the quantity to be transported.**

*Solution*: Let $x_i$, $i = 1, \ldots, 2$, be the locations of the warehouses. Let $p_j$, $j = 1, \ldots, n$, be the locations of the set of $n$ destinations. Let $d_j$ be the demand of destination $j$ and let $q_{ij}$ be the quantity to be transported from warehouse $i$ to destination $j$. Finally, let $c$ be the unit cost per unit of distance. We write the following model:

$$
\begin{array}{lll}
\min & c \sum_{i=1}^{2} \sum_{j=1}^{n} \|x_i - x_j\| \, q_{ij} & \\
\text{s. t.} & \sum_{i=1}^{2} q_{ij} \;\geq\; d_j\,, & \forall\, j = 1, \ldots, n \\
& q_{ij} \;\geq\; 0\,, & \forall\, i = 1, 2\,,\ \forall\, j = 1, \ldots, n.
\end{array}
$$

# 3 Network flow optimization

**Given a directed graph $G = (V, A)$, where each arc has an associated capacity value, determine the maximum flow that can be sent from a source vertex $s$ to a sink vertex $t$.**

*Solution*: Let $c_{ij}$ be the capacity of arc $(i, j) \in A$. The maximum flow from vertex $s$ to vertex $t$ is the solution of the following problem, considering an artificial arc $(t, s)$:

$$
\begin{array}{lll}
\max & f_{ts} & \\
\text{s. t.} & \sum_{(i,j) \in A} f_{ij} \;=\; \sum_{(j,i) \in A} f_{ji}\,, & \forall\, i \in V \\
& 0 \;\leq\; f_{ij} \;\leq\; c_{ij}\,, & \forall\, (i, j) \in A\,.
\end{array}
$$

# 4 Kepler's problem: volume of a parallelepiped inscribed in an ellipsoid

**Maximize the volume of a parallelepiped that can be inscribed within a given ellipsoid.**

*Solution*: As the volume is invariant by translations and rotations, we consider the ellipsoid to be defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \;=\; 1\,.$$

The maximum volume parallelepiped can be found by:

$$\begin{array}{ll} \max & (2x)(2y)(2z) \\ \text{s. t.} & \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \;=\; 1\,. \end{array}$$

Due to the problem symmetry, we could consider $x, y, z \geq 0$.

Note: This formulation assumes that the maximum-volume parallelepiped has edges parallel to the axes. Is this always true?

# 5 Hero of Alexandria: Least Distance Principle

**Consider a reflecting line $l\,:\,ax + by = c$ in the plane, and two points $P_i = (x_i, y_i)$, for $i \in \{A, B\}$, on the same side of the line, say $a\,x_i + b\,y_i \leq c$, for $i \in \{A, B\}$. A ray of light travels from $P_A$ to the line $l$ where it is reflected at a point $P$ and finally arrives at point $P_B$. Find the point $P$, considering that the light travels along the shortest path possible.**

*Solution*:

$$\begin{array}{ll} \min & \|P_A - P\| + \|P_B - P\| \\ \text{s. t.} & P \in l \end{array}$$

or

$$\begin{array}{ll} \min & \sqrt{(x_A - x)^2 + (y_A - y)^2} + \sqrt{(x_B - x)^2 + (y_B - y)^2} \\ \text{s. t.} & a\,x + b\,y = c\,. \end{array}$$

# 6 Hanging Bar

**Consider a bar $AB$ of length $\ell$. We assume the mass is uniformly distributed along the bar. The bar is attached to a string at a point $C$ located at a distance $a$ from end $A$. The other end of the string is fixed to the ceiling. The system is left free. Compute the equilibrium position of the bar (the position of minimum potential energy).**

*Solution*: Let $m$ be the mass of the bar, let $s$ be length of the string and let $g$ be the gravitational constant. We have:

$$\begin{array}{ll} \min & m\,g\,\frac{1}{2}\,(y_A + y_B) \\ \text{s. t.} & \|A - B\| \;=\; \ell \\ & C = A + \dfrac{a}{\ell}\,(B - A) \\ & \|C\| \;\leq\; s \end{array}$$

or, taking the squares of the expressions involving norms,

$$\begin{array}{ll} \min & m\,g\,\frac{1}{2}\,(y_A + y_B) \\ \text{s. t.} & (x_A - x_B)^2 + (y_A - y_B)^2 \;=\; \ell^2 \\ & x_C = x_A + \dfrac{a}{\ell}\,(x_B - x_A) \\ & y_C = y_A + \dfrac{a}{\ell}\,(y_B - y_A) \\ & x_C^2 + y_C^2 \;\leq\; s^2 \end{array}$$

# 7 Hanging Triangle

**Consider a triangle $ABC$ formed by solid bars $AB$, $BC$ and $AC$, of lengths $l_k$ , for $k \in \{AB, BC, AC\}$. We assume that each bar's mass is uniformly distributed according to its line density $\rho_k$ , for $k \in \{AB, BC, AC\}$. The triangle is suspended from the ceiling at the vertex $A$ and is left free to the action of its weight. Determine the equilibrium position of the triangle.**

$$\begin{array}{ll} \min & \rho_{AB}\,\ell_{AB}\,\frac{1}{2}\,(y_A + y_B) + \rho_{AC}\,\ell_{AC}\,\frac{1}{2}\,(y_A + y_C) + \rho_{BC}\,\ell_{BC}\,\frac{1}{2}\,(y_B + y_C) \\ \text{s. t.} & \|A - B\| \;=\; \ell_{AB} \\ & \|A - C\| \;=\; \ell_{AC} \\ & \|B - C\| \;=\; \ell_{BC} \\ & \|A\| \;\leq\; s \end{array}$$

# 8 Electric Static Equilibrium

**Given a metallic ring of radius $R > 0$ and a set of electrical charges $q_i$, for $i = 1, \ldots, N$, that are left free on the ring at positions $r_i$, for $i = 1, \ldots, N$, find the set of equilibrium position of the charges. The potential energy associated with two charges $q_i$ and $q_j$ is given by $U(q_i, q_j) = \dfrac{1}{4\pi\epsilon_0} \dfrac{q_i\, q_j}{\|r_i - r_j\|}$.**

$$
\begin{array}{lll}
\min & \dfrac{1}{4\pi\epsilon_0} \displaystyle\sum_{i=1}^{N} \sum_{j=1}^{N} \dfrac{q_i\, q_j}{\|r_i - r_j\|} & \\
\text{s. t.} & \|r_i\| = R\,, & \forall\, i = 1, \ldots, N\,.
\end{array}
$$