Numeric examples. (To be moved to the Introduction Section.) In the numerical section, we develop several examples of robust pricing problems designed to show the improvement allowed by our result. We consider cost functions based on the the basket call option and a theoretical cross-product contract. Some of the examples use uniform or normal marginal distributions, while some use real-world marginal distributions implied by option prices of individual assets at two distinct future points in time.

In this section, we solve several versions of the problem of robust pricing of a financial contract involving the prices of a pair of assets at two distinct points in time. We apply the framework of [1], further developed in –EcksteinGuoLimObloj21– and analyze potential performance improvements allowed by our result about the geometry of optimal couplings. Our tests run accross different dimensions, cost functions and marginal shapes, including marginals derived from actual stock market option prices, as detailed below.

The model introduced in –EcksteinKupper2021– uses penalyzation and neural network optimization to solve the dual problem of VMOT. –EcksteinGuoLimObloj21– introduces the martingale constraint to the optimal transport problem. More precisely, the model approximates the potential functions $(\phi_i, \psi_i, h_i)_i$ of equation 2.6 using neural networks and use penalyzation to impose condition 2.7. The novelty of our method is in the improvement of θ, the sampling measure used for penalyzation: since we know that a solution (unique in the case of strict submodularity) is attained with monotone coupling of the X marginals $(\vec{\mu}_i)$, we determine θ^1, the X-side of θ, by those marginals. For the sake of comparison, we run tests with and without this improvement. For simplicity, we fix the penalyzation function as L_2 and the penalization parameter $\gamma = 1000$.

Example 1. (Basket option cost function, uniform marginals.) Let $d = 2$ and our cost function and marginals be

$$\begin{aligned} c(X_1, Y) &= (X_1 + 2X_2 - 1)^+ + (Y_1 + 2Y_2 - 1)^+ \\ X_1, X_2 &\sim U[-1, 1] \\ Y_1, Y_2 &\sim U[-2, 2] \end{aligned}$$

It is useful to calculate [bounds for p^+ and p^-? – check conditions for monotone coupling on Y] p^+ and p^- as references for the numerical output. For the maximum, consider $\hat{\pi}$ with positive monotone coupling on both $\hat{\pi}^1$ and $\hat{\pi}^2$. We calculate the maximum in this particular case as

$$p^+_{\hat{\pi}} = p^+_{X,\hat{\pi}} + p^+_{Y,\hat{\pi}}$$

where, using the formula on A1 below,

$$\begin{aligned} p^+_{X,\hat{\pi}} &= \frac{1}{3} \\ p^+_{Y,\hat{\pi}} &= \frac{25}{24} \end{aligned}$$

Thus, [a lower bound for p^+ is] $p^+ = \frac{1}{3} + \frac{25}{24} = \frac{11}{8} = 1.375$. For the minimum, let us consider $\tilde{\pi}$ with negative monotone coupling on both $\tilde{\pi}^1$ and $\tilde{\pi}^2$. Now we have, again from A1,

$$
\begin{aligned}
p^-_{X,\tilde{\pi}} &= 0 \\
p^-_{Y,\tilde{\pi}} &= \frac{1}{8}
\end{aligned}
$$

Thus, [an upper bound for p^- is] $p^- = \frac{1}{8} = 0.125$. Our first step in the computation process is to build two samples from the theoretical distributions μ_0 and θ in equation 2.3 of [1]. We repeat the target function to be optimized here, with adapted notation. We want to minimize or maximize

$$
\begin{aligned}
&\int \left(\sum_i f_i + \sum_i g_i \right) d\mu_0 + \int \beta_\gamma \left(c - \varphi\right) d\theta \\
&= \sum_i \left(\int f_i d\mu_i + \int g_i d\nu_i \right) + \int \beta_\gamma \left(c - \varphi\right) d\theta
\end{aligned}
$$

Notice that we are using the fact that $\int \sum_i h_i\left(x\right).\left(y_i - x_i\right) d\mu_0 = 0$ to eliminate this term from the first integral in the equation above. Given the separation in the first term of the RHS, μ_0 can be simply taken as the independent coupling of the marginals. As for θ, the only requirement besides the marginals is that a true solution π^* be absolutely convex with respect to θ. In its most general form, θ is composed of the independent coupling of the marginals. A more elaborate version comes from using our main result, when we set θ^1 to be monotonically coupled. We use both choices of θ and compare the outputs. Sample sizes are 100k in both cases. The graph below shows the convergence of the numeric value for the number of iterations in the horizontal axis. The shaded grey area covers ± 1 standard deviations of the numeric outputs.

[graph: convergence max]

[graph: convergence min]

Results after 200 iterations.

Maximization.

Coupling	Independent	Positive θ^1
Target	1.375	
Dual approx.value	1.2807	1.3170
Standard deviation	0.0254	0.0288
Penalty	0.0174	0.0055

Minimization. [to be updated]

Coupling	Independent	Negative θ^1
Target	0.125	
Dual approx.value	0.1433	0.1311
Standard deviation	0.0230	0.0251
Penalty	0.0082	0.0075

Example 2. Covariance cost function, normal marginals. The general form for our cost function and marginals is

$$
\begin{aligned}
c(X,Y) &= \sum_i \sum_{j>i} a_{ij} X_i X_j + b_{ij} Y_i Y_j \\
X_i &\sim N\left(0, \sigma_i^2\right) \\
Y_i &\sim N\left(0, \rho_i^2\right)
\end{aligned}
$$

We start with $d = 2$ and

$$
\begin{aligned}
c(X,Y) &= 2X_1X_2 + 3Y_1Y_2 \\
(\sigma_1, \sigma_2) &= (2,1) \\
(\rho_1, \rho_2) &= (3,4)
\end{aligned}
$$

Proposition 3.7 gives us the exact solution as a reference. We have $\lambda_1 = \sqrt{5}, \lambda_2 = \sqrt{15}$ and

$$p^+ = 10 + 15\sqrt{3} \approx 35.981$$

To avoid noisy behavior in the potential functions, we sample from normal distributions clipped at $\pm 4\sigma$. Sample sizes are 100k. To provide a reference for the convergence process, we also use the fact that an optimal θ^* is be given by [reference]. The value convergence using the three distributions – with independent and positively monotone θ^1 and with full optimal coupling – is shown below.

[graph: convergence]

The convergence graph shows similar pattern between independent and monotone coupling on θ^1. [comment]

We construct candidate distributions $\hat{\pi}_{\text{independent}}$ and $\hat{\pi}_{\text{monotone}}$ derived from the numerically constructed potential functions according to [1], equation 2.6.

[graphs: $\hat{\pi}$ for both cases]

Results after 2000 iterations.

Cupling	Independent	Positive θ^1	Fully optimal
Target	35.98		
Dual approx.value			
Standard deviation			
Penalty			

Higher dimensions. To compare performance in higher dimensions, we also run examples with $d = 3$ and $d = 5$. Our cost is simplified as

$$c(x,y) = \sum_i \sum_{j>i} b_{ij} y_i y_j$$

For $d = 3$, the cost, marginals and target maximum values are given by

$$c(x, y) = y_1 y_2 + 2y_1 y_3 + \frac{1}{2} y_2 y_3$$
$$(\mu_1, \mu_2, \mu_3) = (1, 2, 3)$$
$$(\rho_1, \rho_2, \rho_3) = (2, 3, 8)$$
$$p^+ \approx 48.855$$

Below is a graph of the convergence
[graph: convergence]
[comment]
Results after 2000 iterations.

Cupling	Independent	Positive θ^1
Target	48.855	
Dual approx.value		
Standard deviation		
Penalty		

For the $d = 5$, our setting is

$$c(x, y) = \sum_i \sum_{j>i} y_i y_j$$
$$(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) = (1, 2, 2, 3, 3)$$
$$(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) = (2, 3, 4, 5, 6)$$
$$p^+ \approx 153.751$$

Below is a graph of the convergence
[graph: convergence]
[comment]
Results after 2000 iterations.

Cupling	Independent	Positive θ^1
Target	153.751	
Dual approx.value		
Standard deviation		
Penalty		

Example 4. Covariance cost function with real stock market marginals. To be developed. Include graphs of dual functions and discussion.

A1. Portfolio option price – direct calculation. Let

$$f(x_1, x_2) = (a_1 x_1 + a_2 x_2 - k)^+$$
$$X_i \sim \mu_i \equiv U[-m, m], i = 1, 2$$

with $a_i \geq 0; m > 0$. We are interested in the maximum and minimum of the expected value of f over all possible couplings of μ_1 and μ_2. Denote

$$p^+ = \max_{\pi \in \Pi(\mu_1, \mu_2)} \mathbb{E}_\pi f(x)$$
$$p^- = \min_{\pi \in \Pi(\mu_1, \mu_2)} \mathbb{E}_\pi f(x)$$

By [REF], the maximum is attained at the positive-diagonal monotone coupling of μ_1, μ_2, named π^+. Since $x_1 = x_2$ in the support of π^+, we have

$$\begin{aligned}
p^+ &= \int f(x) \mathbb{P}(x)\, d\pi^+ \\
&= \frac{1}{2m} \int_{-m}^{m} f((t,t))\, dt \\
&= \frac{1}{2m} \int_{-m}^{\frac{k}{a_1+a_2}} f((t,t))\, dt + \frac{1}{2m} \int_{\frac{k}{a_1+a_2}}^{m} f((t,t))\, dt
\end{aligned}$$

Notice that the first integral is zero, and so is the second one if $m \leq \frac{k}{a_1+a_2}$. If $m \geq \frac{k}{a_1+a_2}$ then we have

$$\begin{aligned}
p^+ &= \frac{1}{2m} \int_{\frac{k}{a_1+a_2}}^{m} f((t,t))\, dt \\
&= \frac{1}{2m} \int_{\frac{k}{a_1+a_2}}^{m} \left[(a_1+a_2)t - k\right] dt \\
&= \frac{a_1+a_2}{4m} \left[m^2 - \left(\frac{k}{a_1+a_2}\right)^2\right] - k\left(m - \frac{k}{a_1+a_2}\right) \\
&= \frac{a_1+a_2}{4m} \left[m^2 - 2m\frac{k}{a_1+a_2} + \left(\frac{k}{a_1+a_2}\right)^2\right] \\
&= \frac{a_1+a_2}{4m} \left(m - \frac{k}{a_1+a_2}\right)^2
\end{aligned}$$

Similarly, for the minimum we consider the negative-diagonal monotone coupling π^- in whose support $x_2 = -x_1$. Thus we have

$$\begin{aligned}
p^- &= \int f(x) \mathbb{P}(x)\, d\pi^- \\
&= \frac{1}{2m} \int_{-m}^{m} f((t,-t))\, dt \\
&= \frac{1}{2m} \int_{-m}^{\frac{k}{a_1-a_2}} f((t,-t))\, dt + \frac{1}{2m} \int_{\frac{k}{a_1-a_2}}^{m} f((t,-t))\, dt
\end{aligned}$$

Now the second integral is zero and so is the first if $-m \geq \frac{k}{a_1-a_2}$, or $m \leq \frac{k}{a_2-a_1}$. If $m \geq \frac{k}{a_2-a_1}$ then

$$\begin{aligned}
p^- &= \frac{1}{2m}\int_{-m}^{\frac{k}{a_1-a_2}} f\left((t,-t)\right) dt \\
&= \frac{1}{2m}\int_{-m}^{\frac{k}{a_1-a_2}} \left[(a_1-a_2)\, t - k\right] dt \\
&= \frac{a_1-a_2}{4m}\left[\left(\frac{k}{a_1-a_2}\right)^2 - m^2\right] - k\left(\frac{k}{a_1-a_2} + m\right) \\
&= \frac{a_1-a_2}{4m}\left[-\left(\frac{k}{a_1-a_2}\right)^2 - m^2 - 2km\right] \\
&= \frac{a_2-a_1}{4m}\left(m - \frac{k}{a_2-a_1}\right)^2
\end{aligned}$$

References

[1] S. Eckstein and M. Kupper. Computation of optimal transport and related hedging problems via penalization and neural networks. *Applied Mathematics and Optimization*, April 2021.