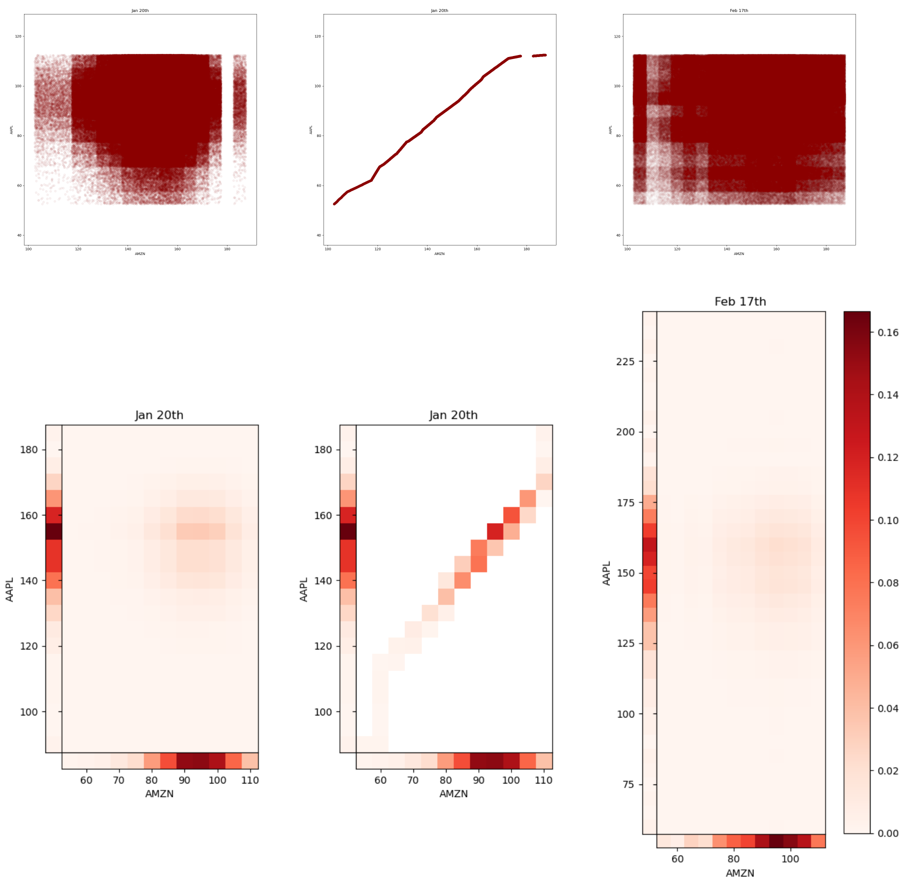
Empirical example. To test our method with real world data, we construct implicit marginal distributions for the prices of Apple and Amazon stocks at times Jan 20th and Feb 17th, 2023 based on call and put option prices as of Dec 16th, 2022 [note: update dates and prices before the last version of the paper] using the method described at [method used by Joshua]. We then solve the problem of an investor who wants to calculate upper and lower bounds for the fair price of a contract that pays off the covariance between the two stock prices on Feb 17th among all admissible joint distributions. [Note: motivation with portfolio mean-variance optimization to be provided in the section introduction.] As in [21], we analyze how the introduction of the information about an intermediate date (Jan 20th) affects the bounds. Formally, we define the payoff

$$f\left(X_1, X_2, Y_1, Y_2\right) = Y_1 Y_2$$

where X_1, X_2 are random variables denoting the prices of Amazon and Apple on Jan 20th and Y_1, Y_2 denote those prices on Feb 17th. $\mu_i\left(\nu_i\right)$ is the (market implied) distribution of $X_i\left(Y_i\right)$, and $\mathcal{M}\left(\vec{\mu}, \vec{\nu}\right)$ is the set of possible martingale joint distributions. We want to calculate

$$p^+ = \max_{\pi \in \mathcal{M}(\vec{\mu}, \vec{\nu})} \mathbb{E}_\pi f\left(x\right)$$
$$p^- = \min_{\pi \in \mathcal{M}(\vec{\mu}, \vec{\nu})} \mathbb{E}_\pi f\left(x\right)$$

We report the results of a few different approaches to the construction of our numeric sample. First notice that the structure of the implied marginal distributions is based on probability levels for a finite set of strike prices, typically multiples of \$5 or \$10 depending on the market practice. The probability density function is then calculated as a linear interpolation of those levels. Our first sampling method ("random sampling"?) consists in sampling random points directly from a coupling of these resulting PDF's. A second method ("strike sampling"?) is to take the set of all the possible combinations of strikes, coupling their associated probabilities in some sensible way. With that, we avoid the necessity of interpolation, which can lead to loss of information. [Discuss.] After deciding on that, as before we decide between a fully independent coupling or a monotone coupling on X. The resulting samples are illustrated on the graph below, where the first row presents the random sampling, the second row has the strike sampling; X points are shown in the first column with independent coupling and in the second coupling with monotone coupling, while the Y points are shown in the third column.


Jan 20th
Jan 20th
Feb 17th
Jan 20th
AAPL
AMZN
Jan 20th
AAPL
AMZN
Feb 17th
AAPL
AMZN
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00