Test 3

Instructions:

- Exercises are grouped by topic and ordered by increasing difficulty within each topic.
- Submit your solutions in a well-organized Jupyter notebook.
- You are responsible for creating robust test cases for each question. Ensure your tests cover all possible cases, especially edge cases (e.g., with the largest possible N).
- Verify that all your solutions meet the specified time constraints. Unless otherwise stated, each question has a **1-second time limit**. Some of the later questions may have different limits, which will be indicated.
- Be sure that the inputs (which will be created by you) match the specifications, and your solutions meets the required time limit.
- Presentation matters: make sure your notebook is clean, well-structured, and that both solutions and test cases are clearly labeled.

Distinguishing Unitaries and States

Exercise 1. You are given an operation that implements a single-qubit unitary transformation: either the H gate or the X gate. The operation will have Adjoint and Controlled variants defined.

Your task is to perform necessary operations and measurements to figure out which unitary it was and to return 0 if it was the H gate or 1 if it was the X gate.

You are allowed to apply the given operation and its adjoint/controlled variants at most twice.

You have to implement an operation which takes a single-qubit operation as an input and returns an integer.

Exercise 2. You are given an operation that implements a single-qubit unitary transformation: either the identity (I gate) or one of the Pauli gates (X, Y or Z gate). The operation will have Adjoint and Controlled variants defined.

Your task is to perform necessary operations and measurements to figure out which unitary it was and to return

- 0 if it was the I gate,
- 1 if it was the X gate,
- 2 if it was the Y gate,
- 3 if it was the Z gate.

You are allowed to apply the given operation and its adjoint/controlled variants exactly once.

You have to implement an operation which takes a single-qubit operation as an input and returns an integer.

Exercise 3. You are given 3 qubits which are guaranteed to be in one of the two states:

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}(|100\rangle + \omega|010\rangle + \omega^2|001\rangle)$$
$$|\psi_1\rangle = \frac{1}{\sqrt{3}}(|100\rangle + \omega^2|010\rangle + \omega|001\rangle)$$

where $\omega = e^{2i\pi/3}$.

Your task is to perform necessary operations and measurements to figure out which state it was and to return 0 if it was $|psi_0\rangle$ state or 1 if it was $|\psi_1\rangle$ state. The state of the qubits after the operations does not matter.

You have to implement an operation which takes an array of 3 qubits as an input and returns an integer.

Preparing States

Exercise 4. You are given two qubits in state $|00\rangle$. Your task is to create the following state on them: $(|00\rangle + |01\rangle + |10\rangle)/\sqrt{3}$.

You have to implement an operation which takes an array of 2 qubits as an input and has no output. The "output" of your solution is the state in which it left the input qubits.

Exercise 5. You are given N qubits in the state $|0\cdots 0\rangle$, and an integer parity $\in \{0,1\}$. Your task is to prepare an equal superposition of all basis states that have the given parity of the number of 1s in their binary notation, i.e., the basis states that have an even number of 1s if parity=0 or the basis states that have an odd number of 1s if parity=1.

For example, for $N = 2$ the required state would be $(|00\rangle + |11\rangle)/\sqrt{2}$ if parity=0, and $(|01\rangle + |10\rangle)/\sqrt{2}$ if parity=1.

You are not allowed to use any gates except the Pauli gates (X, Y and Z), the Hadamard gate and the controlled versions of those (you are allowed to use multiple qubits as controls in the controlled versions of gates). However, you are allowed to use measurements.

You have to implement an operation which takes an array of N qubits and an integer as an input and has no output. The "output" of your solution is the state in which it left the input qubits.

Exercise 6. You are given two qubits which are guaranteed to be in one of the Bell states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Your task is to perform necessary operations and measurements to figure out which state it was and to return the index of that state (0 for Φ^+, 3 for Ψ^-, etc.). The state of the qubits after the operations does not matter.

You have to implement an operation which takes an array of two qubits as an input and returns an integer.

Quantum Oracles

Exercise 7. Implement a quantum oracle on N qubits which checks whether the bits in the input vector x alternate (i.e., implements the function $f(x) = 1$ if x does not have a pair of equal adjacent bits).

You have to implement an operation which takes the following inputs: an array of $N(1 \leq N \leq 7)$ qubits x in an arbitrary state (input register), a qubit y in an arbitrary state (output qubit), and performs a transformation $|x\rangle|y\rangle \to |x\rangle|y \oplus f(x)\rangle$.The operation doesn't have an output; its "output" is the state in which it leaves the qubits. Note that the input register x has to remain unchanged after applying the operation.

This question has a special time limit: 2 seconds.

Exercise 8. Implement a quantum oracle on N qubits which checks whether the bits in the input vector x form a periodic bit string (i.e., implements the function $f(x) = 1$ if x is periodic, and 0 otherwise).

A bit string of length N is considered periodic with period $P(1 \leq P \leq N - 1)$ if for all $i \in [0, N - P - 1] : x_i = x_i + P$. Note that P does not have to divide N evenly; for example, bit string "01010" is periodic with period P=2.

You have to implement an operation which takes the following inputs: an array of $N(1 \leq N \leq 7)$ qubits x in an arbitrary state (input register), a qubit y in an arbitrary state (output qubit), and performs a transformation $|x\rangle|y\rangle \to |x\rangle|y \oplus f(x)\rangle$.The operation doesn't have an output; its "output" is the state in which it leaves the qubits. Note that the input register **x** has to remain unchanged after applying the operation.

This exercise has a special time limit: 4 seconds.

Exercise 9. Implement a quantum oracle on N qubits which checks whether the bits in the input vector x form a palindrome bit string (i.e., implements the function $f(x) = 1$ if x is palindrome, and 0 otherwise).

You have to implement an operation which takes the following inputs: an array of $N(1 \leq N \leq 8)$ qubits x in an arbitrary state (input register), a qubit y in an arbitrary state (output qubit), and performs a transformation $|x\rangle|y\rangle \to |x\rangle|y \oplus f(x)\rangle$.The operation doesn't have an output; its "output" is the state in which it leaves the qubits. Note that the input register **x** has to remain unchanged after applying the operation.

This exercise has a special time limit: 2 seconds.