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PAPER

Gravito-electromagnetic approach for the space-time of a plane gravitational wave

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Abstract

We build the Fermi frame associated to the world-line of a reference observer, arbitrary moving in a given space-time, and we show that local measurements can be described in terms of a gravito-electromagnetic analogy, where the gravito-electric and gravito-magnetic fields are related to the non inertial features of the observer's motion and to the curvature of space-time. We apply this formalism to the space-time of a plane gravitational wave and show that the interaction of the wave with antennas can be explained in terms of gravito-electromagnetic forces acting on test masses. Moreover, we show that, besides the known gravito-electric effects, on which present gravitational waves antennas are based, gravito-magnetic effects could in principle lead to other kinds of detectors.

1. Introduction

The observation of gravitational wave (GW) signals [1, 2] is just one of the latest successes of Einstein's theory of gravitation, General Relativity (GR): as a matter of fact, its predictions were verified with great accuracy during last century, even though challenges to the Einsteinian paradigm come from cosmological observations [3, 4]. Einstein's theory is based on the principle of general covariance, which requires physics laws to be expressed by tensorial equations in space-time; accordingly, physical measurements are meaningful only when the observer and the object of the observations are unambiguously identified [5]. The measurement process develops as follows: (i) observers possess their own space-time, in the vicinity of their world-lines; (ii) covariant physics laws are projected onto local space and time; (iii) predictions for the outcome of measurements in the local space-time of the observers are obtained. In practice, it is convenient for an observer to use a quasi-Cartesian coordinates system in his neighbourhood to describe the effects of gravitation; a continuous set of quasi-Cartesian coordinates associated to the observer's world-line defines the so called *Fermi coordinates* [6]. These coordinates have a concrete meaning, since they are the coordinates an observer would naturally use to make space and time measurements in the vicinity of his world-line. Basic studies on Fermi coordinates refer to geodesic [7] and accelerated [8] observers. The definition of these coordinates is relevant for measurements performed in arbitrary space-time: in particular, in this paper we will focus on the field of a gravitational-wave. Actually, in current literature, the interaction between gravitational waves and detectors is described in terms of a transverse and traceless tensor, which allows to introduce the so-called TT coordinates (see e.g. [9] and references therein for a thorough discussion on the various coordinates used to describe the interaction with gravitational waves). Here, following the approach described by Mashhoon [10], we are going to use the gravito-electromagnetic (GEM) analogy to describe, in Fermi coordinates, the effect of a gravitational-wave in the observer's frame.

It is a well known fact (see e.g. [11]) that Einstein equations, in weak-field approximation (small masses, low velocities), can be written in analogy with Maxwell equations for the electromagnetic field, where the mass density and current play the role of the charge density and current, respectively. Actually, these effects are very small, but there were many proposals in the past (see the review paper [11]) and also more recently to test them, such as the LAGEOS tests around the Earth [12], the Gravity Probe B mission [13], the LARES mission [14–17], the GINGER project [18–23], the LAGRANGE proposal [24] and other space-based tests [25]. Actually, it is possible to introduce a space-time curvature approach to gravito-electromagnetism [10] , which allows to express the curvature effects in analogy to classical electromagnetism, and this approach can be used in arbitrary curved space-time (see also [26, 27]). In particular, using this approach we will discuss some effects of gravitational waves on antennas constituted by systems of test masses; moreover, we will focus on the effect on spinning test masses that have been recently studied in [28].

The paper is organised as follows: in section 2 we briefly review the definition of Fermi coordinates, before introducing the gravito-electromagnetic formalism; then, in section 3 we recall the basic features of the space-time of gravitational plane wave; the interaction of the wave with antennas is explained in terms of gravito-magnetic forces in section 4. Eventually, conclusions are in section 5.

2. GEM effects in the Fermi frame

2.1. Fermi coordinates in arbitrary space-time

In this section we will briefly review the definition of a *Fermi frame*, that is a set of Fermi coordinates adapted to the world-line of an observer. In practice, one may think of a Fermi frame as the mathematical realisation of a laboratory frame in General Relativity.

Let us consider a congruence of observers in arbitrary motion in a gravitational field [29, 30]. In the background space-time describing the gravitational field, we choose a set of coordinates x^μ; accordingly, the world-line $x^\mu(\tau)$ of a reference observer as function of the proper time τ is determined by the following equation

$$\frac{\mathrm{D}x^\mu}{\mathrm{d}\tau} = a^\mu \leftrightarrow \ddot{x}^\mu + \Gamma^\mu_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = a^\mu, \tag{1}$$

where D stands for the covariant derivative along the world-line, a dot means derivative with respect to τ and a^μ is the four-acceleration. To define a set of Fermi coordinates for the reference observer, we proceed as follows[3]: in the tangent space along the world-line $x^\mu(\tau)$ we define the orthonormal tetrad of the observer $e^\mu_{(\alpha)}(\tau)$ such that $e^\mu_{(0)}(\tau)$ is the unit vector tangent to his world-line and $e^\mu_{(i)}(\tau)$ (for i = 1, 2, 3) are the spatial vectors orthogonal to each other and, also, orthogonal to $e^\mu_{(0)}(\tau)$. In summary, we have

$$e^\mu_{(\alpha)}(\tau)e^\mu_{\mu(\beta)}(\tau) = \eta_{(\alpha)(\beta)}. \tag{2}$$

where $\eta_{(\alpha)(\beta)}$ is the Minkowski tensor. The equation of motion of the tetrad is

$$\frac{\mathrm{D}e^\mu_{(\alpha)}}{\mathrm{d}\tau} = -\Omega^{\mu\nu}e_{\nu(\alpha)}, \tag{3}$$

where

$$\Omega^{\mu\nu} = a^\mu\dot{x}^\nu - a^\nu\dot{x}^\mu + \dot{x}_\alpha\Omega_\beta\epsilon^{\alpha\beta\mu\nu}. \tag{4}$$

In the above equation Ω^α is the four-rotation of the tetrad. In particular, we notice that for a geodesic ($a^\mu = 0$) and non rotating ($\Omega^\alpha = 0$) tetrad we have $\Omega^{\mu\nu} = 0$: consequently, in this case the tetrad is parallel transported. If $\Omega = 0$ and $a^\mu \neq 0$, the tetrad is Fermi-Walker transported; indeed, Fermi-Walker transport enables to define the natural *non rotating* moving frame for an accelerated observer [29].

It is then possible to define a set of Fermi coordinates: the observer along the congruence measures time intervals according to his proper time, so the time coordinate is defined by $T = \tau$; to define the spatial coordinates, let us consider a point P along the world-line, corresponding to a given value of the proper time τ, and a space-like geodesic starting at P and defined by its unit tangent vector n^μ, whose components, with respect to the orthonormal tetrad are $n^{(i)} = n_{(i)} = n_\mu e^\mu_{(i)}(\tau)$, and $n^{(0)} = 0$. Let s be the distance parameter along this space-like geodesic: we define the Fermi coordinates X^μ of an arbitrary point Q along this curve as $(cT, X, Y, Z) = (c\tau, s\ n^{(1)}, s\ n^{(2)}, s\ n^{(3)})$, or in vector notation $(cT, \mathbf{X}) = (c\tau, s\mathbf{n})$. We point out that the explicit form of the transformation from the background coordinates system x^μ to Fermi coordinates X^μ can be found for instance in [30], or [28] where the case of a congruence of spinning test particles is considered. Accordingly, using Fermi coordinates, it is possible to parametrize the space-time in the vicinity of the reference world-line: in fact, since every point is reached by one space-like geodesic, it is possible to define its Fermi coordinates as above. Indeed, these coordinates are well defined in a neighbourhood of the world-line since, far away, it could be possible that space-like geodesics intersect, due to the space-time curvature.

In summary, Fermi coordinates are defined within a cylindrical space-time region of radius $\mathcal{R}$, in the vicinity of the reference world-line, where $\mathcal{R}$ is the space-time radius of curvature. These coordinates have a direct and operational meaning since, from the viewpoint of the reference observer, they measure proper times

[3] Greek indices refer to space-time coordinates, and assume the values 0,1,2,3, while Latin indices refer to spatial coordinates and assume the values 1,2,3, usually corresponding to the Cartesian coordinates x, y, z.

and distances away from the reference world-line. Here and henceforth, we will use Fermi coordinates only; as a consequence, for the the sake of clarity, we drop the brackets from tetrad indices. The observer's reference frame, equipped with Fermi coordinates, is our Fermi frame. Moreover, we will use bold-face symbols like $\mathbf{W}$ to refer to vectors in the Fermi frame.

In order to study the effect of the gravitational field in the Fermi frame, we need the expression of the space-time metric in the vicinity of the reference world-line. For geodesic world-lines the metric in Fermi coordinates is given by (see e.g. [7, 29])

$$ds^2 = -(1 + R_{0i0j}X^iX^j)c^2dT^2 - \frac{4}{3}R_{0jik}X^jX^kcdTdX^i + \left(\delta_{ij} - \frac{1}{3}R_{ikjl}X^kX^l\right)dX^idX^j. \quad (5)$$

The above expression is valid up to quadratic displacements $|X^i|$ from the reference world-line. Notice that $R_{\alpha\beta\gamma\delta}(T)$ is the projection of the Riemann curvature tensor on the orthonormal tetrad of the reference observer:

$$R_{\alpha\beta\gamma\delta}(T) = R_{\alpha\beta\gamma\delta}(\tau) = R_{\mu\nu\rho\sigma}e^{\mu}_{(\alpha)}(\tau)e^{\nu}_{(\beta)}(\tau)e^{\rho}_{(\gamma)}(\tau)e^{\sigma}_{(\delta)}(\tau), \quad (6)$$

and it is evaluated along the reference geodesic, where $T = \tau$ and $\mathbf{X} = 0$.

As an example of geodesic frame, we may think of satellites around the Earth or the Earth-centered Inertial (ECI) frame [31]. However, in actual experimental situations, the frame is in general non geodesic and, moreover, its axes are not Fermi-Walker transported: this is the case of terrestrial laboratories, which are fixed on the Earth surface, hence cannot be geodesic and, because of the daily motion, their axes are rotating and are not Fermi-Walker transported.

The space-time metric around the world-line of observers in accelerated motion with rotating tetrads is discussed in [30, 32, 33]. In particular, up to quadratic displacements $|X^i|$ from the reference world line, the metric turns out to be[4] [30]

$$\begin{aligned} ds^2 = &-\left[\left(1 + \frac{\mathbf{a}\cdot\mathbf{X}}{c^2}\right)^2 - \frac{1}{c^2}(\boldsymbol{\Omega}\wedge\mathbf{X})^2 + R_{0i0j}X^iX^j\right]c^2dT^2 + \left[\frac{1}{c}(\boldsymbol{\Omega}\wedge\mathbf{X})_i - \frac{4}{3}R_{0jik}X^jX^k\right]cdTdX^i \\ &+ \left(\delta_{ij} - \frac{1}{3}R_{ikjl}X^kX^l\right)dX^idX^j. \end{aligned} \quad (7)$$

In the above equation $\mathbf{X}$ is the position vector in the Fermi frame. The metric (7) encompasses both the gravitational effects, deriving from the curvature tensor, and the inertial effects, due to world-line acceleration $\mathbf{a}$ and the tetrad rotation $\boldsymbol{\Omega}$.

2.2. Gravito-electromagnetic approach

We suggest here that a deeper insight into the meaning of the various terms of the space-time metric (7) can be obtained by using an analogy with classical electromagnetism. Indeed, since in actual physical situations, such on the Earth and, more in general, in the Solar System, we deal with weak inertial and gravitational fields, the above metric (7) turns out to be a perturbation of the flat space-time Minkowski tensor. However, as a matter of fact, the application of general relativity is not limited to this scenario since, for instance, when we deal with black holes or study the large scale structure of the Universe, a more accurate description of the space-time metric is required.

We assume that acceleration $\mathbf{a}$ and rotation $\boldsymbol{\Omega}$ do not depend on time. In practice, neglecting the terms g_{ij} related to the spatial curvature, the above metric (7) can be written in the form [10]

$$ds^2 = -\left(1 - 2\frac{\Phi}{c^2}\right)c^2dT^2 - \frac{4}{c}(\mathbf{A}\cdot d\mathbf{X})dt + \delta_{ij}dX^idX^j. \quad (8)$$

in terms of the gravito-electromagntic (GEM) potentials $(\Phi, \mathbf{A})$: this allows to introduce the corresponding formalism. To begin with, we may separate the inertial contributions (those deriving from $\mathbf{a}$ and $\boldsymbol{\Omega}$) from the gravitational ones (those deriving from the curvature tensor $R_{\alpha\beta\gamma\delta}$). Accordingly, we define the *gravito-electric* (GE) $\Phi = \Phi(T, \mathbf{X})$ and *gravito-magnetic* (GM) $\mathbf{A} = \mathbf{A}(T, \mathbf{X})$ potentials:

$$\Phi(T, \mathbf{X}) = \Phi^I(\mathbf{X}) + \Phi^C(T, \mathbf{X}), \qquad \mathbf{A}(T, \mathbf{X}) = \mathbf{A}^I(\mathbf{X}) + \mathbf{A}^C(T, \mathbf{X}), \quad (9)$$

where in the GE potential $\Phi(T, \mathbf{X})$

$$\Phi^I(\mathbf{X}) = -\mathbf{a}\cdot\mathbf{X} - \frac{1}{2}\frac{(\mathbf{a}\cdot\mathbf{X})^2}{c^2} + \frac{1}{2}[|\boldsymbol{\Omega}|^2|\mathbf{X}|^2 - (\boldsymbol{\Omega}\cdot\mathbf{X})^2] \quad (10)$$

[4] We neglect here higher order terms, proportional to the product of curvature tensor and the tetrad rotation $\boldsymbol{\Omega}$.

is the *inertial* contribution, while

$$\Phi^C(T, \mathbf{X}) = -\frac{1}{2}R_{0i0j}(T)X^iX^j \tag{11}$$

is the *curvature* contribution; similarly, for the GM potential $\mathbf{A}(T, \mathbf{X})$ we have the *inertial* contribution

$$A_i^I(\mathbf{X}) = -\left(\frac{\boldsymbol{\Omega} c}{2} \wedge \mathbf{X}\right)_i, \tag{12}$$

and the *curvature* contribution:

$$A_i^C(T, \mathbf{X}) = \frac{1}{3}R_{0jik}(T)X^jX^k. \tag{13}$$

The gravito-electric and gravito-magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are defined in terms of the potentials by

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial}{\partial T}\left(\frac{1}{2}\mathbf{A}\right), \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{14}$$

In particular, using the definitions (14) we obtain for the GM field (up to linear order in $|X^i|$)

$$\mathbf{E}^I = \mathbf{a}\left(1 + \frac{\mathbf{a} \cdot \mathbf{X}}{c^2}\right) + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{X}), \quad E_i^C(T, \mathbf{X}) = c^2R_{0i0j}(T)X^j. \tag{15}$$

As for the GM field, we get

$$\mathbf{B}^I = -\boldsymbol{\Omega} c, \quad B_i^C(T, \mathbf{R}) = -\frac{c^2}{2}\epsilon_{ijk}R^{jk}{}_{0l}(T)X^l. \tag{16}$$

Starting from the above definitions the gravito-electricmagnetic fields are written in the form

$$\mathbf{E} = \mathbf{E}^I + \mathbf{E}^C, \quad \mathbf{B} = \mathbf{B}^I + \mathbf{B}^C. \tag{17}$$

It is possible to show (see e.g. [10]) that the curvature part of the GEM fields defined before can be combined in the GEM Faraday tensor

$$F_{\alpha\beta} = -c^2R_{\alpha\beta0i}X^i \tag{18}$$

where $F_{0i} = -E_i^C$ and $F_{ij} = \epsilon_{ijk}B_k^C$. Hence, Maxwell's equations $F_{[\alpha\beta,\gamma]} = 0$ and $F^{\alpha\beta}{}_{,\beta} = \frac{4\pi}{c}J^\alpha$ are satisfied to linear order in $|X^i|$ with

$$\frac{4\pi}{c^3}J_\alpha(T, \mathbf{0}) = -R_{0\alpha} = -\frac{8\pi G}{c^4}\left(T_{0\alpha} - \frac{1}{2}\eta_{0\alpha}T^\beta{}_\beta\right), \tag{19}$$

along the reference world-line which, indeed, justifies the name of GEM fields. Consequently, it is possibile to define the stress-energy tensor, in analogy with electromagnetism:

$$T^{\alpha\beta} = \frac{1}{4\pi G}\left(F^\alpha{}_\gamma F^{\beta\gamma} - \frac{1}{4}g^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}\right), \tag{20}$$

or, in terms of the components of the curvature tensor

$$T^{\alpha\beta} = \frac{1}{4\pi G}\left(R^\alpha{}_{\gamma0i}R^{\beta\gamma}{}_{0j} - \frac{1}{4}\eta^{\alpha\beta}R_{\gamma\delta0i}R^{\gamma\delta}{}_{0j}\right)X^iX^j. \tag{21}$$

Notice that along the reference world-line both the Faraday tensor and the stress-energy tensor vanish: this is ultimately a consequence of Einstein's principle of equivalence. Indeed, as discussed in [10], a coordinate-independent measure of the stress-energy content of the gravitational field can be obtained by an averaging process in the vicinity of the reference world-line.

The analogy with electromagnetic fields can be exploited also for for describing the motion of free test masses. More precisely, this is the motion of free test masses *relative* to a reference mass, at rest at origin of the Fermi frame. This motion is determined by the geodesics of the space-time metric (7). In particular the motion of free test particles in the Fermi frame can be written in the form of a Lorentz-like force equation [10]

$$m\frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}T^2} = q_E\mathbf{E} + q_B\frac{\mathbf{V}}{c} \times \mathbf{B}, \tag{22}$$

up to linear order in the particle velocity $\mathbf{V} = \frac{\mathrm{dX}}{\mathrm{d}T}$ (which is indeed the *relative velocity* with respect to the reference mass). In the Lorentz-like force equation, $q_E = -m$ is the GE charge, and $q_B = -2m$ is the GM one (the minus sign takes into account that the gravitational force is always attractive). As a consequence, the Lorentz-like force equation becomes

$$m\frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}T^2} = -m\mathbf{E} - 2m\frac{\mathbf{V}}{c} \times \mathbf{B}. \tag{23}$$

In particular, if we consider the inertial fields $\mathbf{E}^I$, $\mathbf{B}^I$ only, the equation of motion of the test particles, equation (23) becomes

$$\frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}T^2} = -\mathbf{a}\left(1 + \frac{\mathbf{a}\cdot\mathbf{X}}{c^2}\right) - \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{X}) - 2\boldsymbol{\Omega} \wedge \mathbf{V}. \tag{24}$$

The above expression is the same used in classical physics for describing the motion of free particles in a non inertial frame, except for the factor $\left(1 + \frac{\mathbf{a}\cdot\mathbf{X}}{c^2}\right)$, which is due to a redshift effect and, hence, is a purely relativistic term.

There are other analogies that can be borrowed from classical electrodynamics for the GEM fields. A charged spinning test particle has a magnetic moment $\boldsymbol{\mu} = \frac{q}{2mc}\mathbf{S}$, where m, q, $\mathbf{S}$ are its mass, charge and spin, respectively. The magnetic dipole in an external magnetic field $\mathbf{B}$ undergoes a torque $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$. Taking into account the GEM analogy, we may say that a test spinning particle with mass m and spin $\mathbf{S}$ has a gravito-magnetic charge $q_B = -2m$ and, as a consequence, it possesses a *gravito-magnetic dipole moment* $\boldsymbol{\mu}_g = -\frac{\mathbf{S}}{c}$ [10]. Hence, in an external gravitomagnetic field $\mathbf{B}$, its evolution equation is

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}T} = \boldsymbol{\mu}_g \times \mathbf{B} = -\frac{1}{c}\mathbf{S} \times \mathbf{B} = \frac{1}{c}\mathbf{B} \times \mathbf{S}. \tag{25}$$

Furthermore (see e.g. [34]) due to the coupling with the gravito-magnetic field, the spinning particle undergoes a force[5]

$$\mathbf{F}_S = \boldsymbol{\nabla}(\boldsymbol{\mu}_g \cdot \mathbf{B}) = -\frac{1}{c}\boldsymbol{\nabla}(\mathbf{S} \cdot \mathbf{B}) = -\frac{1}{c}(\mathbf{S} \cdot \boldsymbol{\nabla})\mathbf{B}. \tag{26}$$

Notice that this force depends on the inhomogeneity of the gravitational field: in other words, a uniform field does not produce such a force.

Before going on and studying the effects of the field of a GW in the Fermi frame in terms of GEM fields, let us briefly comment on the operational meaning of the equation of motion (23). From the viewpoint of the reference observer, this equation describes the evolution of a test mass; in other words how its spatial coordinates X, Y, Z (which, by construction, measure proper distances away from the reference world-line) change due to the action of inertial and gravitational fields. This action can be simply described in the Fermi frame in terms of Newtonian GEM forces; of course, if other forces are present (such as mechanical or electromagnetic ones) they should be added to the equation of motion.

In what follows we are interested in the description of the effects of a GW in the Fermi frame in terms of the GEM fields; hence, we will not consider inertial effects and, moreover, we will focus on the Riemann curvature of the wave neglecting the contributions due to local gravitational fields (such as the one of the Earth), whose effects could be in any case linearly added, since in actual experimental situations (at least in the Solar System) they are small perturbations of flat space-time.

3. Gravito-electromagnetism in the gravitational wave space-time

We start by briefly recalling the basic features of the space-time describing the field of a plane GW. To this end, we remember that plane GWs are solutions of the vacuum linearised Einstein's equations

$$\Box\bar{h}_{\mu\nu} = 0. \tag{27}$$

In order to obtain the above equations, we start from Einstein's field equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{28}$$

and suppose that the space-time metric $g_{\mu\nu}$ is in the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$ is a small perturbation of the Minkowski tensor $\eta_{\mu\nu}$ of flat space-time. Setting $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, with $h = h^{\mu}_{\mu}$, Einstein's field equations (28) in the Lorentz gauge $\partial_{\mu}\bar{h}_{\mu\nu} = 0$ turn out to be

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}. \tag{29}$$

Consequently, in vacuum we obtain equations (27). Exploiting the gauge freedom and using the so-called transverse—traceless coordinates (TT) $x^{\mu} = (ct, x, y, z)$ (see e.g. [9]), if we suppose that the plane GW

[5] Indeed, this the projection of the spin-curvature force $F^{\alpha} = -\frac{c}{2}R^{\alpha}{}_{\beta\mu\nu}u^{\beta}S^{\mu\nu}$ onto the Fermi frame, where u^{β} is the four velocity and $S^{\mu\nu}$ is the spin tensor.

propagates along the x axis, a solution of (27) can be written in the form

$$\bar{h}_{\mu\nu} = -(h^{+}e^{+}_{\mu\nu} + h^{\times}e^{\times}_{\mu\nu}) \tag{30}$$

where

$$h^{+} = A^{+}\sin(\omega t - kx), \quad h^{\times} = A^{\times}\cos(\omega t - kx) \tag{31}$$

and

$$e^{+}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad e^{\times}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{32}$$

are the polarisation tensors of the wave. In the above definitions, A^{+}, $A^{\times}$ are the amplitude of the wave in the two polarisation states, while ω is its frequency and k the wave number, so that the wave four-vector is $k^{\mu} = \left(\frac{\omega}{c}, k, 0, 0\right)$.

Accordingly, in TT coordinates the gravitational field of the wave is described by the line element

$$ds^2 = -c^2dt^2 + dx^2 + (1 - h^{+})dy^2 + (1 + h^{+})dz^2 - 2h_{\times}dydz. \tag{33}$$

We work in linear approximation in the perturbation $h_{\mu\nu}$ and, consequently, in the amplitude A^{+}, $A^{\times}$ of the wave. In this approximation, the Riemann tensor is given by the following expression (see e.g [29]):

$$R_{ij0l}(x) = \frac{1}{2}(\partial_0\partial_j h_{il} - \partial_0\partial_i h_{jl} + \partial_l\partial_i h_{j0} - \partial_l\partial_j h_{i0}). \tag{34}$$

In particular, since $h_{i0} = 0$ in the TT metric, the above expression of the Riemann tensor simplifies to

$$R_{ij0l}(x) = \frac{1}{2}(\partial_0\partial_j h_{il} - \partial_0\partial_i h_{jl}). \tag{35}$$

Actually, as we have seen before, the effects of the curvature tensor in the observer frame can be described in terms of the GEM fields $\mathbf{E}^{C}$, $\mathbf{B}^{C}$. Ssince we consider a geodesic and non rotating frame, from equation (17) we simply have $\mathbf{E} = \mathbf{E}^{C}$ and $\mathbf{B} = \mathbf{B}^{C}$, so that the GEM fields are due to the curvature tensor only. In order to evaluate these fields, we should in principle calculate the expression of the Riemann tensor in Fermi coordinates. However, in weak field approximation—that is to say up to linear order in $h_{\mu\nu}$ - the Riemann tensor is invariant with respect to coordinate transformations, hence it has the same expression in terms of the new coordinates. Consequently, we may use the TT values for the perturbations $h_{\mu\nu}$ given by equation (33), and express them in Fermi coordinates. Notice that, according to what we have discussed in section 2.1, in order to define the gravito-electrogmagnetic fields, we need the projection of the Riemann tensor along the observer's tetrad (6). Since we are dealing with GWs, we have also to consider that the extension of our frame should not be comparable with the wavelength, otherwise the spatial variation of the wave field should be taken into account.

Using the set of unit vectors $\mathbf{u}_X$, $\mathbf{u}_Y$, $\mathbf{u}_Z$ in the Fermi frame, the field gravito-electric field (15) is $\mathbf{E} = E_X\mathbf{u}_X + E_Y\mathbf{u}_Y + E_Z\mathbf{u}_Z$, where

$$E_X = 0, \quad E_Y = -\frac{\omega^2}{2}[A^{+}\sin(\omega T)Y + A^{\times}\cos(\omega T)Z], \quad E_Z = -\frac{\omega^2}{2}[A^{\times}\cos(\omega T)Y - A^{+}\sin(\omega T)Z]. \tag{36}$$

The gravito-magnetic field (16) is $\mathbf{B} = B_X\mathbf{u}_X + B_Y\mathbf{u}_Y + B_Z\mathbf{u}_Z$, where

$$B_X = 0, \quad B_Y = -\frac{\omega^2}{2}[-A^{\times}\cos(\omega T)Y + A^{+}\sin(\omega T)Z], \quad B_Z = -\frac{\omega^2}{2}[A^{+}\sin(\omega T)Y + A^{\times}\cos(\omega T)Z]. \tag{37}$$

Notice that both fields are perpendicular to the propagation direction: the GW, like an electromagnetic one, is transverse. Moreover, it is easy to check that $\mathbf{E}\cdot\mathbf{B} = 0$, in other words the two field are perpendicular (see also figures 1 and 2) and $|\mathbf{E}|^2 - |\mathbf{B}|^2 = 0$.

At a fixed time T, the components of the gravito-electric and gravito-magnetic fields are plotted in figures 1 and 2 respectively. We see that, for both fields, the $A^{\times}$ components are obtained from the A^{+} with a rotation of $\pi/4$.

On the basis of the GEM analysis, and taking into account the definition of the stress-energy tensor (20), we may define the Poynting vector

$$\mathbf{P} = \frac{c}{4\pi G}\mathbf{E}\times\mathbf{B}. \tag{38}$$

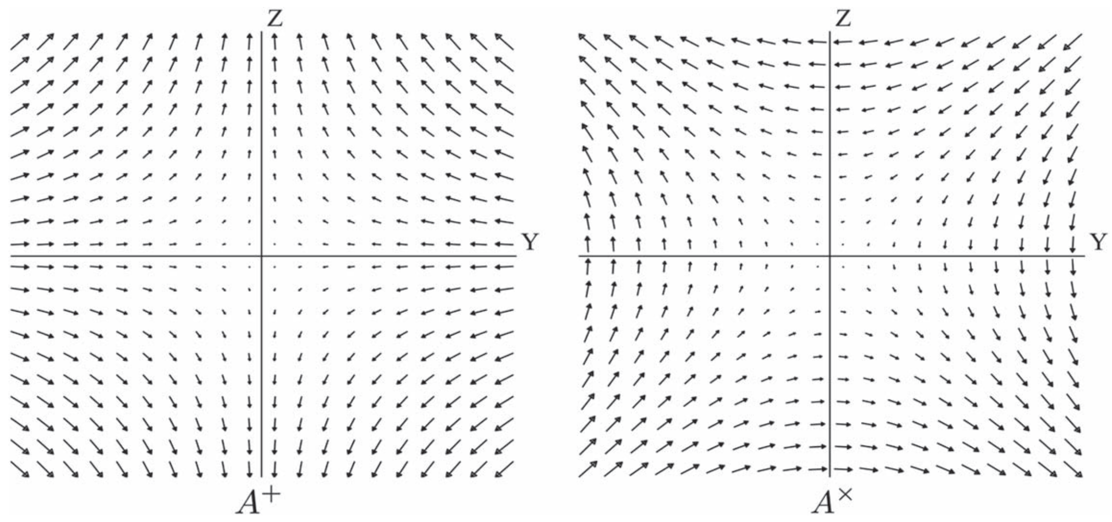

Figure 1. The gravito-electric field **E**: on the left, we suppose that the GW has just the A^+ polarisation; on the right, we suppose that the GW has just the $A^\times$ polarisation. Notice that the two polarisations differ by a rotation of $\pi/4$.

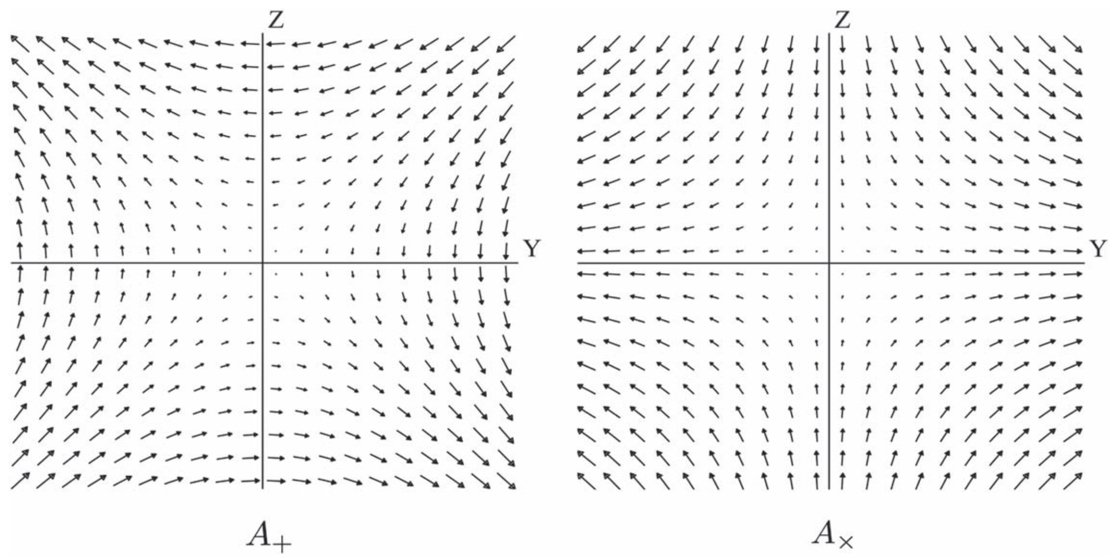

Figure 2. The gravito-magnetic field **B**: on the left, we suppose that the GW has just the A^+ polarisation; on the right, we suppose that the GW has just the $A^\times$ polarisation. Notice that the two polarisations differ by a rotation of $\pi/4$.

In particular, for a circularly polarised wave, since $A^+ = A^\times = A$, we obtain

$$\mathbf{P} = \frac{\omega^4 A^2 c}{16\pi G}(Y^2 + Z^2)\mathbf{u}_X. \tag{39}$$

The above expression describes the energy per unit time and unit of surface, which the wave transports along its direction of propagation. If we consider a circular detector of radius L, orthogonal to the propagation direction, the energy per unit time is then

$$\frac{\mathrm{d}E}{\mathrm{d}T} = \frac{c}{32G}\omega^4 A^2 L^4. \tag{40}$$

4. Detection of gravitational waves and gravito-electromagnetic fields

According to Press and Thorne [35], gravitational waves can be thought of as a 'field of (relative) gravitational forces propagating with the speed of light'. This definition gets an operational meaning in the Fermi frame, where physical quantities, such as displacements, are relative to the reference world-line; in our GEM approach, this is emphasised by the fact that the gravito-electric and gravito-magnetic fields are position depending, so that they act differently on test masses located at different locations in the frame, thus producing tidal effects. Then, the passage of a GW provokes a space-time deformation which, in the Fermi frame, can be described in terms of forces due to the GEM fields. Accordingly, we may thought of a GW as a field of stresses: a gravitational waves antenna (GW-antenna) or detector is a physical system on which the wave acts producing displacements and motion, relative to the reference world-line. Stresses produce strains, which can be evidenced by suitable devices, acting as sensors.

In this section we are going to see how the effects of GWs on antennas can be described in terms of GEM fields. A thorough analysis of different kinds of GW receivers can be found in the already cited review paper [35] which, even though is not updated with the recent technological developments, gives a very accurate description of the basic principles of GWs detection. Indeed, all detection methods described in the aforementioned review paper are essential based on the effect of the gravito-electric field; however, we have shown that in the GEM approach also the gravito-magnetic field is present, so we are going to suggest some detection processes based on the action of the gravito-magnetic field.

4.1. Gravito-electric effects

4.1.1. Free-mass or almost free-mass GW-antennas

The simplest GW-antenna is made of two free masses, which are at rest before the passage of the wave; in particular, we suppose that one of them is at the origin of the Fermi frame, so we are interested in the motion of the other mass, due to the passage of the wave.

Before going on, it is useful to stress the limits of our approximation: we work at first order in the wave amplitude, so we have to deal with equations in a self-consistent way. GW-antennas are physical systems made of test masses, which may possess also spin. If we suppose that $\mathbf{V}^0$ is the velocity of a test mass before the passage of the wave, the latter provokes a change $\mathbf{V}(T) = \mathbf{V}^0 + \delta\mathbf{V}(T)$, where the variation $\delta\mathbf{V}(T)$ is of the order of the wave amplitude A: $\delta\mathbf{V}(T) = O(A)$. As a consequence, since we work in linear approximation, in the equation of motion (23) we can neglect the contribution of the gravito-magnetic field if the test masses are at rest before the passage of the wave; things are different if we consider masses in motion before the passage of the wave, or if we consider spinning test masses (see below).

Accordingly, the test mass is acted upon by the gravito-electric field only, and its equation of motion is

$$\frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}T^2} = -\mathbf{E} \tag{41}$$

Let us suppose that the polarisation of the wave is such that $A^\times = 0$ (as we have seen before, the effect of the $A^\times$ polarisation is qualitatively the same); according to equations (36) the gravito-electric field is given by

$$E_X = 0, \quad E_Y = -\frac{\omega^2}{2}[A^+ \sin(\omega T) Y], \quad E_Z = \frac{\omega^2}{2}[A^+ \sin(\omega T) Z]. \tag{42}$$

Let the location of the mass before the passage of the wave be $\mathbf{X}_0 = (0, L, 0)$, so that the physical distance between the two masses is L. Then, the the solution of equation (41) up to linear order in the wave amplitude, is

$$X(T) = 0, \quad Y(T) = L\left[1 - \frac{A^+}{2}\sin(\omega T)\right], \quad Z(T) = 0. \tag{43}$$

The distance between the two masses changes with time. Indeed, the possibility of having two *free masses* is quite complicated unless they are located in space: this is the case of the proposed space mission LISA (Laser Interferometer Space Antenna) [36], which is based on a constellation of three satellites, orbiting around the Sun. The variation of the distance between the satellites will be measured using laser interferometry. In Earth-based experiments, such as LIGO[37]and VIRGO[38], the masses are held by suspensions and allowed to oscillate at the passage of the wave; also in this case, the distance variations are measured by interferometric techniques.

4.1.2. Resonant GW-antennas

Resonant GW-antennas are important from an historical viewpoint, since they were the first detectors developed for the search of GWs and, also, because of the works of Weber [39, 40] who, since 1969 observed coincident excitations of two resonant antennas. The Weber experiments and the problems raised by the interpretation of its results in terms of characteristic of the GWs and their sources, are discussed in [35]. The basic principle is the mechanical resonator: when a GW reaches a mechanical resonator, the normal modes are excited. A toy model is a damped spring-mass system which, in our approach can be described considering the elastic and damping forces, besides the gravitational force $-m\mathbf{E}$. As a consequence, we have the equation of motion

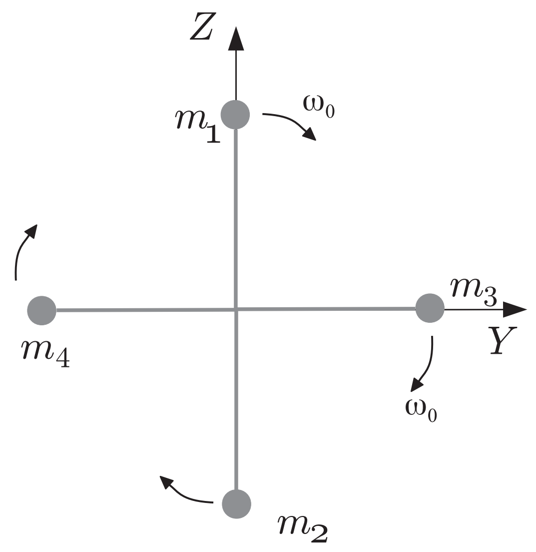

Figure 3. Two identical dumbbells are made by test masses $m_1 = m_2 = m_3 = m_4 = m$ at fixed distance R, and rotated by $\pi/2$ with respect each other. They independently rotate with frequency ω_0, before the passage of the wave.

$$m\frac{d^2Y}{dT^2} + m\gamma\frac{dY}{dT} + m\omega_0^2(Y - Y_0) = m\frac{\omega^2}{2}A^+ \sin(\omega T) Y_{0.} \tag{44}$$

In the above equation, γ is the damping constant, ω_0 is the proper frequency of the system, and we suppose that the rest length is Y_0. The solution of the above equation is

$$Y(T) = Y_0 + \frac{A^+\omega^2 Y_0}{2}\frac{\sin(\omega T - \varphi)}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}}, \tag{45}$$

where $\varphi = \arctan \gamma/(\omega_0^2 - \omega^2)$. The amplitude of the induced oscillation reaches it maximum $\frac{A^+\omega_0 Y_0}{2\gamma}$ in resonance condition, i.e. when $\omega = \omega_0$

Also the so-called heterodyne antenna [41, 42] is based on the resonance principle. In this case the GW-antenna is a quadupolar system, made of two dumbbells crossed at an angle of $\pi/2$, with length R. They independently rotate in the plane orthogonal to the propagation direction with the same frequency ω_0. Let us suppose that at $T = 0$ the configuration of the dumbbells is that of figure 3, i.e. the four masses $m_1 = m_2 = m_3 = m_4 = m$ are along the axes Y and Z. The coordinates of the mass m_1, whose position at $T = 0$ is $\mathbf{X}_0 = (0, 0, R)$, are

$$X_1 = 0, \quad Y_1 = R \sin \omega_0 T, \quad Z_1 = R \cos \omega_0 T. \tag{46}$$

We suppose that the wave is circularly polarised, so that $A^+ = A^\times = A$, and taking into account the expression of the gravito-electric field (36), the gravito-electirc force acting on the mass is $\mathbf{F}_1^E = -m\mathbf{E}$

$$F_{1,X}^E = 0, \quad F_{1,Y}^E = \frac{m\omega^2 AR}{2}\cos(\omega - \omega_0)T, \quad F_{1,Z}^E = -\frac{m\omega^2 AR}{2}\sin(\omega - \omega_0)T. \tag{47}$$

We see that if $\omega_0 = \omega/2$, the above expression becomes

$$F_{1,X}^E = 0, \quad F_{1,Y}^E = \frac{m\omega^2 AR}{2}\cos\frac{\omega}{2}T, \quad F_{1,Z}^E = -\frac{m\omega^2 AR}{2}\sin\frac{\omega}{2}T \tag{48}$$

This force has a constant magnitude, and it is always orthogonal to the dumbbell: the mass experiences a force of constant magnitude $|\mathbf{F}_1^E| = \frac{m\omega^2 AR}{2}$. It easy to check that the other mass m_2 undergoes an equal force, directed in opposite direction. Hence, due to the action of the GW, a constant torque $\boldsymbol{\tau}_{12} = -m\omega^2 AR^2\mathbf{u}_X$ acts on the dumbbell, with the effect of *accelerating* its rotation. Now, if we use the same approach with the other dumbbell, we see that it is acted upon by a constant torque $\boldsymbol{\tau}_{34} = m\omega^2 AR^2\mathbf{u}_X$, with the effect of *decelerating* its rotation. In summary, with this choice of the rotation frequency, one dumbbell is accelerated and the other is decelerated, so that the masses come closer: the angular separation θ between the two dumbbells evolves with time with the law $\Delta(\theta)(T) = \frac{\pi}{2} - \delta\theta(T) = \frac{\pi}{2} - \frac{1}{2}\omega^2 AT^2$, which is independent of the length R. We obtain the following estimate:

$$\delta\theta(T) = 5 \times 10^{-8}\left(\frac{\omega}{10^3\mathrm{Hz}}\right)^2\left(\frac{A}{10^{-21}}\right)\left(\frac{T}{10^4\mathrm{s}}\right)^2 \tag{49}$$

4.2. Gravito-magnetic effects

4.2.1. Gravito-magnetic resonance in hetorodyne antenna

Besides the effect described above and discussed in [41, 42] as possible candidate for a GW-antenna, if we consider the same double-dumbbell device in figure 3, there is an additional effect, due to the action of the gravito-magnetic field on the rotating masses.

Starting from the expression of the gravito-magnetic field (37), the force acting on a mass moving with speed $\mathbf{V}$, is $\mathbf{F}^B = -2m\frac{\mathbf{V}}{c} \times \mathbf{B}$. Since we are working at linear order in the wave amplitude, we use in this expression the velocity of the system before the passage of the wave.

Let the rotation frequency be $\omega/2$: the gravito-magnetic field acting on the the mass m_1 is

$$B_X = 0, \quad B_Y = -\frac{\omega^2 AR}{2}\sin\frac{\omega}{2}T, \quad B_Z = -\frac{\omega^2 AR}{2}\cos\frac{\omega}{2}T \tag{50}$$

We see that gravito-magnetic field has constant magnitude, and it is always directed toward the center; the mass m_1 undergoes the force $\mathbf{F}_1^B = \frac{m\omega^3 AR^2}{2c}\mathbf{u}_X$. The other mass m_2 undergoes to the same force, so that the total force acting on the first dumbbell is $\mathbf{F}_{12}^B = \frac{m\omega^3 AR^2}{c}\mathbf{u}_X$. If we consider the other dumbbell, using the same approach we see that it undergoes a total force $\mathbf{F}_{34}^B = -\frac{m\omega^3 AR^2}{c}\mathbf{u}_X$. The first dumbbell moves in the direction of propagation of the wave, while he other one moves in the opposite direction: accordingly, their distance d changes with time according to $d(T) = \frac{\omega^3 AR^2}{c}T^2$. We obtain the following estimate:

$$d(T) = 3 \times 10^{-17}\mathrm{m}\left(\frac{\omega}{10^3\mathrm{Hz}}\right)^3\left(\frac{A}{10^{-21}}\right)\left(\frac{R}{1\mathrm{m}}\right)^2\left(\frac{T}{10^4\mathrm{s}}\right)^2 \tag{51}$$

4.2.2. Gravito-magnetic spin precession

As we have shown in section 2, the gravito-magnetic field acts on spinning test masses too; up to now, we have just considered the effect of the gravito-electromangetic fields on test masses, described in terms of Lorentz-force. However, a spinning test particle undergoes a torque $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$ in a gravito-magnetic field, so that its evolution equation is

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}T} = \frac{1}{c}\mathbf{B} \times \mathbf{S} \tag{52}$$

We explicitly obtain the following equations

$$\frac{\mathrm{d}S_X}{\mathrm{d}T} = \frac{\omega^2}{2c}[A^+\sin(\omega T)(Y^0S_Y^0 - Z^0S_Z^0) + A^\times\cos(\omega T)(Y^0S_Z^0 + Z^0S_Y^0)], \tag{53}$$

$$\frac{\mathrm{d}S_Y}{\mathrm{d}T} = -\frac{\omega^2}{2c}[A^+\sin(\omega T)Y^0S_X^0 + A^\times\cos(\omega T)Z^0S_X^0], \tag{54}$$

$$\frac{\mathrm{d}S_Z}{\mathrm{d}T} = -\frac{\omega^2}{2c}[A^\times\cos(\omega T)Y^0S_X^0 - A^+\sin(\omega T)Z^0S_X^0] \tag{55}$$

In order to have a simple interpretation of the spin evolution, let us suppose that $\mathbf{S}^0 = (S^0, 0, 0)$; moreover, we suppose that the wave is circularly polarised, so that $A^+ = A^\times = A$, and the location of the spinning particle is $\mathbf{X}^0 = (0, 0, Z^0)$. Since at the location $(0, 0, Z^0)$ the GM field (37) becomes

$$B_X = 0, \quad B_Y = -\frac{\omega^2}{2}A\sin(\omega T)Z^0, \quad B_Z = -\frac{\omega^2}{2}A\cos(\omega T)Z^0, \tag{56}$$

from the equations (53)–(55) we obtain the following solutions

$$S_X(T) = S^0, \tag{57}$$

$$S_Y(T) = -\frac{\omega}{2c}A\sin(\omega T)Z^0S^0, \tag{58}$$

$$S_Z(T) = -\frac{\omega}{2c}A\cos(\omega T)Z^0S^0. \tag{59}$$

In this configuration the spin vector rotates around its initial direction: we see that the S_X component is unchanged, while the spin gets a component in the plane orthogonal to the wave $\mathbf{S}_\perp = S_Y\mathbf{u}_Y + S_Z\mathbf{u}_Z$, which rotates with the wave frequency. We may evaluate the effect of the GW by considering the ratio:

$$\frac{|\mathbf{S}_\perp|}{S^0} = \frac{\omega}{2c} A Z^0 = \pi A \frac{Z^0}{\lambda}. \tag{60}$$

We see that the precession relative to the reference spin at the origin of the frame depends on the distance, so it can be increased by using a large separation Z^0, which needs to be in any case smaller than wavelength λ. In any case, if we set $\frac{Z^0}{\lambda} = 10^{-1}$, we obtain a relative spin variation $\frac{|\mathbf{S}_\perp|}{S^0} \simeq 10^{-22}$, with $A \simeq 10^{-21}$.

Possibile candidates for measuring such a small effect could be the optical magnetometers: in particular GNOME (Global Network of Optical Magnetometers for Exotic Physics) [43] is based on synchronous measurements of optical-magnetometer signals from several devices operating in magnetically shielded environments in distant locations: by synchronously detecting and correlating magnetometer signals, transient events of global character may be identified. These devices are accurate magnetic field sensors, and their measurements can be related to spin dynamics. Another possibility is the use of a ferromagnetic sample: in fact, since a GW induces spin precessions, the magnetization due to electron spins in ferromagnetic samples will change after the passage of the GW. On the other hand, if we consider mechanical gyroscopes, in terms of angles, the gravitational wave induces a precession angle of 10^{-22} rad; by comparison, remember that in the Gravity Probe B mission [13] the (1-year) Lense-Thirring effect was 10^{-9} rad, hence it seems very difficult to use these devices to measure the effect of the gravitational waves.

4.2.3. Gravito-magnetic effect on spinning particle motion

If the gravito-magnetic field is not homogeneous, the spinning particle is acted upon by the force

$$F_S = -\frac{1}{c}(\mathbf{S}\cdot\boldsymbol{\nabla})\mathbf{B} \tag{61}$$

According to equation (61), the behaviour of spinning and spinless particles at rest before the passage of the wave is different. These features have been investigated for instance in [44, 45] and also in [28]: indeed, GWs carry angular momentum and, hence, spinning particles interact with GWs differently from non spinning ones. According to our approach, up to linear order in the wave amplitude, while spinless particles are only acted upon by the gravito-electric force $\mathbf{F}^E = q_E\mathbf{E} = -m\mathbf{E}$, since the gravito-magnetic contribution in the Lorentz force (23) is negligible, spinning particles, on the contrary, are subjected to the total force $\mathbf{F}^E + \mathbf{F}^S = -m\mathbf{E} - \frac{1}{c}(\mathbf{S}\cdot\boldsymbol{\nabla})\mathbf{B}$, so that their equations of motion turns out to be

$$\frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}T^2} = -\mathbf{E} - \frac{1}{c}(\mathbf{S}\cdot\boldsymbol{\nabla})\mathbf{B}. \tag{62}$$

Since we work up to linear order in A, the GEM fields are evaluated at the initial position which is supposed to be $\mathbf{X}^0 = (0, Y_0, Z_0)$; for the same reason, we consider for the spin vector its initial value $\mathbf{S}^0$. We suppose the the wave is circularly polarised, so that $A^+ = A^\times = A$.

From equations (36) and (37) we obtain the following expressions for the gravito-electric force $\mathbf{F}^E$ in the plane of the wave

$$F_Y^E = m\frac{\omega^2 A}{2}[\sin(\omega T)Y_0 + \cos(\omega T)Z_0], \tag{63}$$

$$F_Z^E = m\frac{\omega^2 A}{2}[\cos(\omega T)Y_0 - \sin(\omega T)Z_0]. \tag{64}$$

and the spin force $\mathbf{F}^S$

$$F_Y^S = \frac{\omega^2 A}{2}\left[\sin(\omega T)\frac{S_Z^0}{c} - \cos(\omega T)\frac{S_Y^0}{c}\right], \tag{65}$$

$$F_Z^S = \frac{\omega^2 A}{2}\left[\cos(\omega T)\frac{S_Z^0}{c} + \sin(\omega T)\frac{S_Y^0}{c}\right]. \tag{66}$$

Then, the equations of motion (62) explicitly read

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}T^2} = \frac{\omega^2 A}{2}\left[\sin(\omega T)\left(Y_0 + \frac{S_Z^0}{mc}\right) + \cos(\omega T)\left(Z_0 - \frac{S_Y^0}{mc}\right)\right], \tag{67}$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}T^2} = \frac{\omega^2 A}{2}\left[\cos(\omega T)\left(Y_0 + \frac{S_Z^0}{mc}\right) - \sin(\omega T)\left(Z_0 - \frac{S_Y^0}{mc}\right)\right]. \tag{68}$$

The wave-driven solutions are

$$Y(T) = -\frac{A}{2}\left[\left(Y_0 + \frac{S_Z^0}{mc}\right)\sin(\omega T) + \left(Z_0 - \frac{S_Y^0}{mc}\right)\cos(\omega T)\right], \tag{69}$$

$$Z(T) = -\frac{A}{2}\left[\left(Y_0 + \frac{S_Z^0}{mc}\right)\cos(\omega T) - \left(Z_0 - \frac{S_Y^0}{mc}\right)\sin(\omega T)\right]. \tag{70}$$

So we see the difference between spinning particles and spinless ones. The terms in the form $\frac{S}{mc}$ are, for electrons, of the order of 10^{-12} m.

5. Conclusions

In this paper, we have considered the construction of the Fermi frame for observers arbitrary moving in space-time and we have shown that it is possible to describe the effects of both inertial and gravitational fields in terms of a gravito-electromagnetic analogy. Consequently, the motion of test masses in the Fermi frame can be explained in terms of gravito-electromagnetic forces, in analogy to what happens in classical electrodynamics. Then, we have applied this formalism to the case of the space-time of a plane gravitational wave; we have shown that the passage of a gravitational wave provokes a space-time deformation which, in the Fermi frame, can be described in terms of forces due to the gravito-electric and gravito-magnetic fields. Current gravitational waves antennas are designed to measure gravito-electric effects only, and we have shown how they can be described in our formalism. In addition, we have emphasised that the gravito-magnetic part of the wave field acts on moving or spinning test masses; as for the gravito-magnetic spin precession, we have preliminarily suggested an experimental setup based on the use of optical magnetometers or ferromagnetic samples. More in general, we have given some rough and preliminary estimates for these new gravito-magnetic effects due to the presence of a gravitational wave. The effects are very small, as expected since we are dealing with gravitational waves: consequently, the actual possibility of projecting and performing dedicated experiments deserves careful analysis, which is beyond the scopes of this paper. However, we believe that our formalism could be useful to investigate new effects, connected with the passage of a gravitational-wave.

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