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Geomagnetism, Earth rotation and the electrical conductivity of the lower mantle

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Abstract

A new approximation is derived for the electromagnetic torque acting on an electrically conducting mantle as a result of fluid flow in the core. The torque considered is that associated with the toroidal field induced in the mantle by advection of poloidal field (which is assumed to be frozen into an infinitely conducting liquid core) by the flow of liquid at the top of the core. The expression relies on the assumption that the mantle is an insulator apart from a 'thin' (with respect to the core radius) layer of finite conductance adjacent to the core–mantle boundary. This allows the toroidal field scalar in the mantle to be expressed as a first-order Taylor approximation. The time-dependent torque calculated at a sequence of epochs this century is compared with the torque which has previously been inferred from astronomical observations of the length of day. Although the initial results appear unpromising, a significant correlation exists when poorly determined components of the velocity field, which contribute substantially to variations in the calculated torque, are ignored. By regression analysis the conductance of the assumed thin layer is determined as 6.7×10^8 S and the lag of the electromagnetic torque behind the astronomical torque as 6 years, which is interpreted as the delay time for electromagnetic signals through the mantle. Finally, the implications of these results for the conductivity of the lower mantle are discussed. They imply that the bottom few hundred kilometres of the mantle probably have a conductivity of a few hundred to a few thousand siemens per metre; previously published lower-mantle conductivity models are examined in light of this conclusion. The close correlation of the variations in the calculated electromagnetic torque, which depend primarily on fluctuations in the core velocity field, with the length of day torque provides independent evidence that core flow is not steady on the decade time-scale.

1. Introduction

The observed period of the Earth's rotation, or 'length of day' (LOD) is known to vary on a wide range of time-scales. More precisely, what is actually observable are changes in the period of rotation of the crust and mantle. Most of these changes can be explained by exchange of angular

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momentum (AM) between the solid Earth and the hydrosphere and atmosphere, glacial rebound and a tidal, secular deceleration (see, e.g. Lambeck, 1980). Of the residual changes, by far the largest are the so-called 'decade fluctuations' — deviations of up to a few milliseconds from the mean length of day, lasting for 10 years or so. It is now generally accepted that such decade fluctuations must result from a transfer of angular momentum between the mantle and the core (see, e.g. Rochester, 1984; Hide, 1986). The questions of how the necessary exchange of angular momentum between liquid core and solid mantle occurs and whether or not the required changes in core angular momentum do occur are still unresolved and are the subject of much current research (see, e.g. Aldridge et al., 1990). The work presented here has tried to answer the first of these questions.

1.1. Angular momentum budget

Motions of the fluid adjacent to the core–mantle boundary (CMB) can be inferred from observations of changes in the geomagnetic field which arise primarily from such motions. Vestine and Kahle (1968) found that by assuming the exchange of AM is restricted to the upper 200 km of the core, reasonable agreement between the observed drift of the eccentric dipole and that inferred from the LOD fluctuations could be obtained. It was found that the observed eccentric dipole drift lags that inferred from LOD by about 5 years. Such a lag may be associated with a delay in the transmission of changes in the geomagnetic field through a partially electrically conducting mantle. Jault et al. (1988) recently derived a simple expression for the changes in the AM of the core using estimated surficial velocity fields computed under the assumption that the flow is tangentially geostrophic and the core is a perfect conductor. They found some agreement between the changes in the computed AM of the core and changes in the AM of the mantle inferred from observations of the LOD over the period 1969–1985 although the AM is a rather smooth function of time over this period. Jault et al. (1988) also claimed reasonable agreement before 1970 but did not show it. They found no lag between the geomagnetically and LOD inferred AM, however. The time-dependent core AM has also been calculated by Jackson et al., (1993), using geostrophic flows derived independently, for the whole of the twentieth century, and was found to be in qualitative agreement with the observed AM changes. The geomagnetically inferred changes lag the LOD inferred changes by some 5 and 25 years (Jackson's (1989) Fig. 6.14).

1.2. Core–mantle coupling

A number of candidate core–mantle coupling mechanisms have been proposed. These include 'topographic' coupling arising from pressure forces acting on departures from spherical (or more precisely ellipsoidal) symmetry on the CMB and 'viscous' coupling arising from viscous forces acting between core and mantle. The latter mechanism is usually considered to be too weak, in the case of the Earth, to account for the observations, and most investigations have concentrated on the former (e.g. Hide, 1989; R. Hide in Aldridge et al., 1990). Topographic coupling has excited much recent interest with the emergence of seismological evidence for significant CMB topography of up to 10 km (Morelli and Dziewonski, 1987) or even 20 km (Creager and Jordan, 1986) peak-to-trough amplitude. Although Jault and LeMouël (1989) considered that topographic coupling is a good candidate mechanism they also concluded that it may be difficult to compute, even if more precise knowledge of the CMB topography and flow at the top of the core were available. Using large-scale features of the flow and topography alone, Jault and LeMouël (1989) found the topographic torque to be two orders of magnitude too large to be compatible with LOD fluctuations using the topography model of Morelli and Dziewonski (1987), and also concluded that smaller-scale features would contribute significantly to the torque. However, small (no bigger than the likely uncertainties) modifications to the geometry of either the flow pattern or topography can bring the magnitudes into agreement. There is also significant uncertainty as to the true magnitude of the CMB topography: Neuberg and Wahr (1991) have, for example, recently argued that such topography may be limited to 2–3 km on lateral length scales up to 400 km. Furthermore, Gwinn et al. (1986) showed that the deviation from hydrostatic equilibrium can be represented by a P_2^0 harmonic with a peak-to-trough amplitude of 490 ± 110 m, which is an order of magnitude smaller than that of Morelli and Dziewonski (1987).

The other main candidate mechanism is 'electromagnetic' (EM) coupling. Any velocity disparity at the CMB gives rise to induced fields in the conducting mantle. Torques associated with these fields are believed to retard the rotation of the mantle (Rochester, 1960). Toroidal fields generated by convection in deeper, more rapidly rotating regions of the core are believed to 'leak' into the mantle, giving rise to torques which counterbalance the retarding torque (e.g. Stix and Roberts, 1984). Bullard et al. (1950) considered the role of electromagnetic coupling associated with the axial dipole but dismissed it as an insufficient mediator in the exchange of angular momentum between core and mantle. However, Rochester (1960) showed that the mantle torque is significantly increased by the inclusion of the effects of the non-dipole field and is of a sufficient order of magnitude to account for the decade fluctuations. Roden (1963) modified Rochester's (1960) uniformly conducting lower-mantle model to a three-layer model and concluded that the order of magnitude of the EM torque is sufficient to account for the irregular fluctuations in LOD — most efficiently if high conductivity is concentrated toward the CMB. Although knowledge of the geomagnetic field and motions of the core have now improved considerably, our understanding of lower-mantle conductivity is still poor, leaving considerable uncertainty in the calculated torque.

Stix and Roberts (1984) appear to have been the first to attempt to evaluate a time-dependent EM torque (see also Paulus and Stix (1986)). Using main field and secular variation (SV) models (truncated at spherical harmonic degree and order five and six) as an approximation to the poloidal field at the CMB, the torque associated with poloidal and toroidal fields induced in the mantle by advection of the main field was computed for a series of epochs this century. Stix and Roberts (1984) concluded that the torque associated with the induced toroidal magnetic field exceeds that associated with the induced poloidal magnetic field by at least an order of magnitude. They also found the mean EM torque to be negative and two orders of magnitude larger than the mean LOD torque, whereas the fluctuations of the torque are of a comparable order of magnitude to those related to the LOD. Their results failed to show any close correspondence between the time-dependence of the computed and observed torques. This is likely to be due to the use of main field and SV models obtained by straightforward downward continuation to the CMB, and poorly determined models of CMB fluid flow. More recent work has shown greater agreement between the LOD and EM torques (Paulus, 1986; Paulus and Stix, 1989).

In the following we reconsider the viability of EM coupling between core and mantle. In Section 2 we derive an expression for the advective EM torque acting on the mantle assuming the presence of a thin layer of constant, uniform conductivity at its base. Time-dependent EM torques are then computed using recent models of the geomagnetic field at the CMB and flow at the top of the core. These results are presented in Section 3 and compared with the torque inferred from changes in Earth rotation. The possible implications of the results for the conductivity of the lower mantle are discussed in Section 4. Some concluding remarks are given in Section 5.

2. Calculation of the electromagnetic mantle torque

2.1. A thin conducting layer at the base of the mantle

The conductivity of the lower mantle is beyond the reach of induction studies based on externally induced field variations, with a generally accepted limit of around 1000 km depth (see, e.g. Parkinson and Hutton, (1989)), although Achache et al. (1981) used data in the period range 4 days to 11 years to obtain conductivity estimates down to 2000 km. Instead, estimates have relied primarily, though not exclusively, upon consideration of the propagation of the SV through the mantle, which will be considered further in Section 4. Estimates of lower-mantle conductivity vary by several orders of magnitude, although it is widely hypothesised that the lowermost regions may have a significant non-zero conductivity.

Proposed regions of enhanced conductivity near the CMB may correspond in part to the seismologically distinct D″ layer at the base of the mantle, which is of the order of 200–300 km thick. If this were the case, it is interesting to consider what effect such a layer would have on the EM interaction between the core and the mantle. Therefore, as a working model, it will be assumed that the mantle is an insulator except in a layer of constant conductivity σ_M, next to the CMB, of thickness δ. Conductivity is assumed to be effectively infinite in the core.

Let subscripts C and M relate to quantities in the core and in the conducting layer at the base of the mantle, respectively. The boundary condition for the electric field at the CMB, assumed spherical radius $r_C = 3485$ km, reduces to

$$-\boldsymbol{r}\times\left[(\nabla\times\boldsymbol{r}\psi)\times\boldsymbol{B}_C\right]=\boldsymbol{r}\times(\eta_M\nabla\times\boldsymbol{B}_M) \tag{1}$$

where ψ is the toroidal stream function for the flow at the top of the core, η_M is the diffusivity of the conducting layer and $\boldsymbol{r}$ is a radially outward vector of length r (see, e.g. Jacobs (1987)). Hence poloidal field in the core produces toroidal field in the mantle through rearrangement of field lines, from which the advective EM torque will be calculated.

Poloidal and toroidal decompositions of $\boldsymbol{B}_C$ and $\boldsymbol{B}_M$,

$$\boldsymbol{B}_M=\nabla\times(\nabla\times\boldsymbol{r}S_M)+\nabla\times\boldsymbol{r}T_M \tag{2}$$

$$\boldsymbol{B}_C=\nabla\times(\nabla\times\boldsymbol{r}S_C)+\nabla\times\boldsymbol{r}T_C \tag{3}$$

(where S and T are poloidal and toroidal scalars, respectively), can be substituted into (1), and $\boldsymbol{r}.\nabla\times$ applied to give

$$\begin{aligned}&\eta_M\boldsymbol{r}.\nabla\times\left[\boldsymbol{r}\times\nabla\times\nabla\times(\boldsymbol{r}T_M)\right]\\&\quad+\eta_M\boldsymbol{r}.\nabla\times\left[\boldsymbol{r}\times\nabla\times\nabla\times\nabla\times(\boldsymbol{r}S_M)\right]\\&\quad=-\boldsymbol{r}.\nabla\times\big(\boldsymbol{r}\times\{[\nabla\times(\boldsymbol{r}\psi)]\\&\quad\times[\nabla\times\nabla\times(\boldsymbol{r}S_C)]\}\\&\quad+\boldsymbol{r}\times\{[\nabla\times(\boldsymbol{r}\psi)]\times[\nabla\times(\boldsymbol{r}T_C)]\}\big)\end{aligned} \tag{4}$$

at $r = r_C$, where the magnetic diffusivity is related to conductivity by

$$\eta=\frac{1}{(\mu\sigma)}$$

μ being the permeability, taken as that of free space. As the toroidal field does not extend beyond the conducting region and is therefore not directly observable, it will be assumed that the only toroidal field present in the conducting layer is that generated by advection of the poloidal field by the toroidal flow at the top of the core. The EM torque arising from the interaction of poloidal flow with the poloidal field is likely to be much smaller (by about an order of magnitude (Stix and Roberts, 1984)) and so we choose to concentrate on the toroidal part of the flow.

Because of the relationship

$$\begin{aligned}&\boldsymbol{r}\times\{[\nabla\times(\boldsymbol{r}\psi)]\times[\nabla\times(\boldsymbol{r}T_C)]\}\\&\quad=\boldsymbol{r}\times\{\boldsymbol{r}\nabla\psi.[\nabla\times(\boldsymbol{r}T_C)]-\nabla\psi\boldsymbol{r}.(\nabla T_c\times\boldsymbol{r})\}\\&\quad=0\end{aligned} \tag{5}$$

the second term in the right-hand side of (4) vanishes.

$$\eta_M\boldsymbol{r}.\nabla\times\left[\boldsymbol{r}\times\nabla\times\nabla\times(\boldsymbol{r}T_M)\right]=\eta_M\frac{\partial}{\partial r}(rL^2T_M) \tag{6}$$

$$\begin{aligned}&-\boldsymbol{r}.\nabla\times(\boldsymbol{r}\times\{[\nabla\times(\boldsymbol{r}\psi)]\times[\nabla\times\nabla\times(\boldsymbol{r}S_C)]\})\\&\quad-r^2\nabla_H.(\nabla_H\psi L^2S_C)\end{aligned} \tag{7}$$

and

$$\eta_M\boldsymbol{r}.\nabla\times\left[\boldsymbol{r}\times\nabla\times\nabla\times\nabla\times(\boldsymbol{r}S_M)\right]=0 \tag{8}$$

where ∇_H is the horizontal part of the gradient operator and L^2 is related to the horizontal part of the Laplacian by $L^2 = -r^2\nabla_H^2$. As $L^2S_M = L^2S_C$ (by continuity of the radial poloidal field across the core–mantle boundary), Eq. (4) can be reduced using (6), (7) and (8) to

$$\eta_M\frac{\partial}{\partial r}(rL^2T_M)=-r^2\nabla_H.(\nabla_H\psi L^2S_M) \tag{9}$$

at $r = r_C$. To determine the T_M in the thin conducting layer we consider the radial component of the induction equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(L^2T_M)+\eta_M\boldsymbol{r}.\nabla\times\nabla\times\nabla\times\boldsymbol{B}_M=0 \tag{10}$$

and note that since η_M is large we can neglect the first term in this expression giving

$$\begin{aligned} &\boldsymbol{r}.\nabla \times \nabla \times \nabla \times \boldsymbol{B}_M = 0 \\ &\quad \Rightarrow L^2 \nabla^2 T_M = 0 \end{aligned} \tag{11}$$

As T_M varies on a horizontal scale that is large compared with δ we can further approximate with

$$\nabla^2 T_M \approx \frac{\partial^2}{\partial r^2} T_M \tag{12}$$

Applying the boundary condition $T_M = 0$ at $r = r_C + \delta$, an approximate solution of (10) within the conducting layer is given by

$$T_M(r) = T_M(r_C)(r_C + \delta - r)\delta^{-1} \tag{13}$$

from which $\partial/\partial r\, T_M|_{r=r_C} = -T_M(r_C)\delta^{-1}$ follows. Since according to Eq. (9)

$$-\eta_M \frac{\partial}{\partial r} L^2 T_M \bigg|_{r=r_C} \approx r_C \nabla_H.(\nabla_H \psi L^2 S_M)|_{r=r_C} \equiv G \tag{14}$$

we can write solution (13) in the form

$$L^2 T_M = -(r - r_C - \delta)G/\eta_M \tag{15}$$

which is the required approximate relation between the toroidal and poloidal fields in the conducting layer.

2.2. Calculation of the torque integral

The torque Γ about the Earth's rotational axis (in the direction $\boldsymbol{k}$) can be expressed as

$$\mu\Gamma = \int\int\int_V (\hat{\boldsymbol{k}} \times \boldsymbol{r}).[(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}]\ \mathrm{d}V \tag{16}$$

where V is the volume of the conducting region of the mantle. Thus, in the case of the thin layer under consideration here, the axial torque arising from the induced toroidal field can be written as (see, e.g. Rochester (1960) and Gubbins and Roberts (1987, p. 62)).

$$\Gamma = \frac{\delta}{\eta_M \mu} \int\int_\Omega \left[\frac{\partial (L^{-2} G)}{\partial \theta} L^2 S_M \right] \bigg|_{r=r_C} \sin\theta\ \mathrm{d}\Omega \tag{17}$$

where $\int\int_\Omega$ denotes integration over the core–mantle boundary. This integral can be expressed in terms of spherical harmonic expansions of the magnetic field and fluid flow scalars in a summation involving Gaunt integrals. A more detailed explanation of this has been given by Stewart (1991).

2.2.1. Numerical implementation

Calculations were performed using the scalar spherical transform method, which is clearly related to the vector spherical transform method described by Lloyd and Gubbins (1990). The method relies on properties of the Gauss–Legendre quadrature method of numerical integration and discrete Fourier transformation: if $f(\theta, \phi)$ (where θ is colatitude and ϕ longitude) is a function defined on the surface of a sphere and $\{f_l^m\}$ the coefficients of its expansion in spherical harmonics, which terminates with degree $l = N$, there is an exact transformation between the grid of values $f_{ij} = f(\theta_i, \phi_j)$ and the spherical harmonic coefficients provided $\{\theta_i\}$ are the zeroes of a Legendre polynomial of degree $N_i \geq N$ and $\{\phi_j, j = 1 \ldots M\}$ are equally spaced with $M \geq 2N_1 + 1$. Given a computer program to transform between the spatial and spherical harmonic domains, it is easy to compute complicated functions on the sphere.

In this application, (17) represents the $l = 0$, $m = 0$ spherical harmonic term of the product $(\partial/\partial\theta)(L^{-2}G)L^2 S_M$.

We first calculate $G(\theta, \phi)$ by the following steps:

(1) rewrite (14) in the form

$$G(\theta, \phi) = \left[-\frac{1}{r} L^2 \psi L^2 S_M + \nabla_H \psi \cdot \nabla_H (L^2 S_M) \right]_{r=r_C} \tag{18}$$

(2) evaluate $L^2\psi$ from the spherical harmonic coefficients by multiplying each coefficient of degree l by $-l(l+1)$ and transform to the space domain;

(3) evaluate $L^2 S_M$ in the space domain in the same way;

(4) evaluate the θ and ϕ components of $\nabla_H \psi$ and $\nabla_H L^2 S_M$ in the space domain;

(5) multiply and sum the terms appearing in (18).

Second, we then obtain the contribution $\partial/\partial\theta\,(L^{-2}G)$ in (17) by the following steps:

(6) transform G to the spherical harmonic domain to give coefficients G_l^m;

(7) operate with L^{-2} by dividing coefficients of degree l by $-l(l+1)$ (the coefficient $G_{\gamma 0}^0$ is zero and need not be included);

(8) perform the sum with the new coefficients to give $\partial/\partial\theta(L^{-2}G)$ in the space domain.

Finally, we form the integrand in (17) by multiplying $\partial/\partial\theta(L^{-2}G)$ by $L^2 S_M$ in the space domain and transform the result to spherical harmonic coefficients. The $l=0$, $m=0$ coefficient is then proportional to the desired torque.

The method appears lengthy but is in fact easy to program (once the transform code is written) and reasonably fast. Furthermore, it eliminates the analysis required to express (17) in terms of Gaunt and Elsasser integrals, although it is still necessary to determine the maximum degree of expansions of products such as $\nabla_H \psi \cdot \nabla_H S_M$.

2.3. Field and flow at the core–mantle boundary

Single epoch 'snap-shots' of the geomagnetic field at the Earth's surface were taken from an unpublished time-dependent model of Bloxham and Jackson for the period 1840–1980 (to be referred to here as BJ1). This was derived by a very similar method to that of Bloxham and Jackson (1992) (A. Jackson, personal communication, 1991) with a temporal representation in B-splines. This gives a much improved temporal fit over earlier time-dependent models (Bloxham, 1987; Bloxham and Jackson, 1989) and eliminates end effects (A. Jackson, personal communication, 1991). The inversion method used also ensures spatial convergence of the field at the CMB, which represents a significant improvement over more traditional methods, for example those involving an arbitrary truncation of the spherical harmonic expansion. Previous attempts to calculate the time-dependent electromagnetic torque (Stix and Roberts, 1984; Paulus and Stix, 1986), have used field models which are inappropriate for downward continuation.

For the evaluation of the torque, estimates of the toroidal part of the flow at the top of the core are required. The problem of inferring the flow at the top of the core from the secular variation has been the focus of much research in recent years. In such work it is usually assumed that the conductivity of the core is sufficiently high that diffusion can be neglected and field lines are effectively frozen into the liquid and act as a tracer; this is commonly known as the frozen-flux hypothesis (Roberts and Scott, 1965). As previously mentioned, diffusion in the core has also been ignored in the torque calculation of Section 2.1. The inherent non-uniqueness of the problem (Backus, 1968) can be at least partially surmounted through the imposition of prior physical assumptions concerning the flow, such as steady flow, tangentially geostrophic flow and toroidal flow (LeMouël, 1984; Voorhies and Backus, 1985; Voorhies, 1986; Lloyd and Gubbins, 1990; Whaler, 1991; Jackson and Bloxham, 1991). Of course, any torque calculated from such flow models depends both on the constrained and unconstrained parts of the flow. A suite of flow models derived under the assumption of tangential geostrophy was used here (Jackson and Bloxham, 1990). Although the models include toroidal, poloidal and shear coefficients, only the toroidal coefficients were used, in line with our earlier decision. The flows were of maximum degree and order 14 and were available for epochs 1900–1980 at 5 year intervals.

2.4. Astronomical observations of Earth rotation

From astronomically inferred variations in the period of the Earth's rotation, Morrison (1979) found that 10^{18} N m can be considered as an approximate upper bound on the torque acting upon the mantle associated with the decade fluctuations. A thorough analysis of occultation data was performed, resulting in a time-series of values of ΔT, the deviation of the instantaneous period of rotation from its mean or excess length of day, for the period 1861–1978. This can then be scaled to give $\Delta\theta$, the sidereal displacement

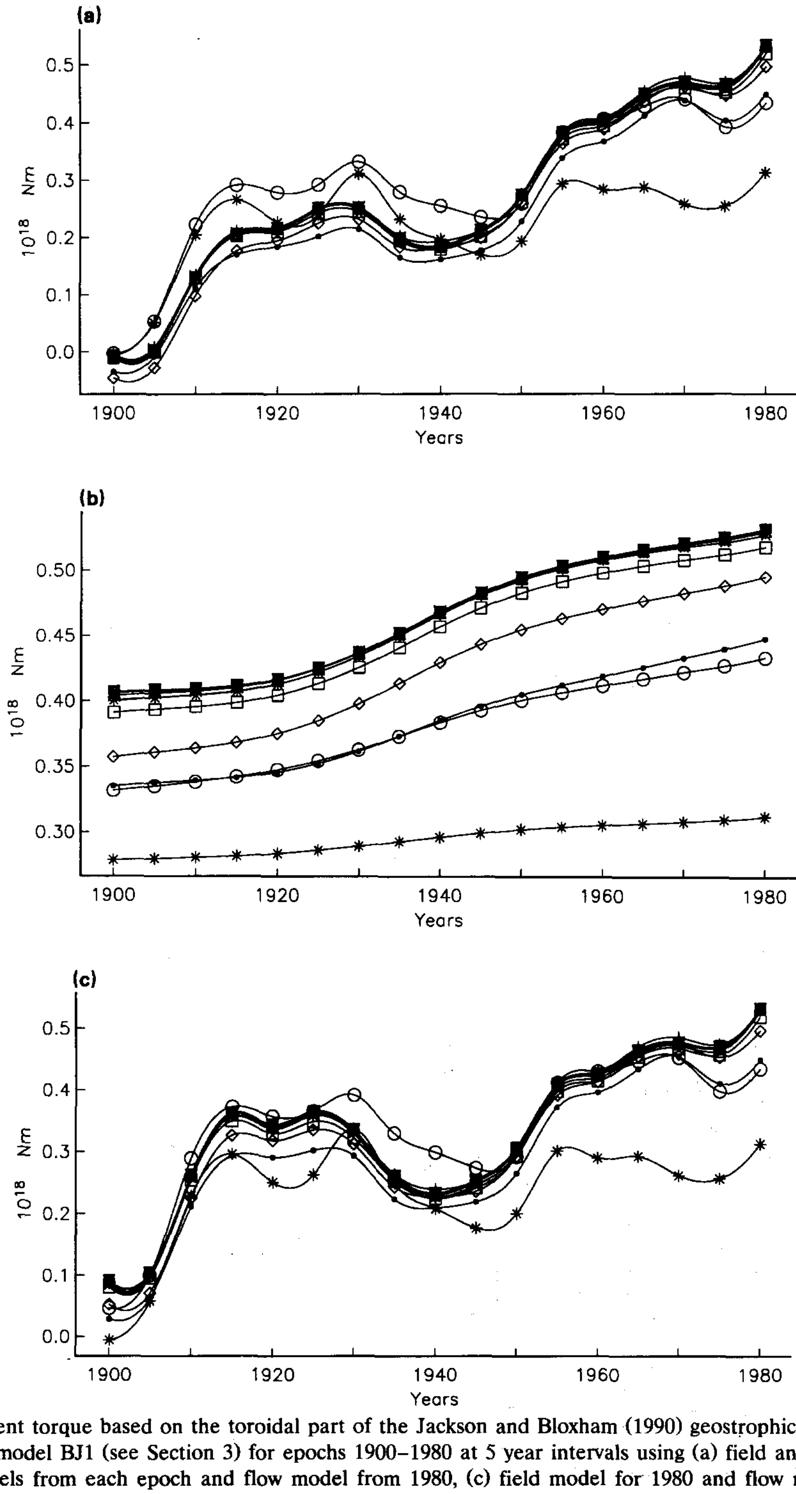

Fig. 1. Time-dependent torque based on the toroidal part of the Jackson and Bloxham (1990) geostrophic flows and field models derived from spline model BJ1 (see Section 3) for epochs 1900–1980 at 5 year intervals using (a) field and flow models for each epoch, (b) field models from each epoch and flow model from 1980, (c) field model for 1980 and flow models for each epoch. Curves are interpolated using splines. In each plot, different symbols are for increasing maximum degree of the flows included in the torque calculation. (Maximum degrees 1–4 are *, ○, ● and ◇ respectively). (Torque units are 10^{18} N m).

angle of the Greenwich meridian. The associated torque can then be calculated from the second time derivative of this quantity. Morrison evaluated this as the second derivative of a quadratic fitted to his data over an 11 year window yielding a series of torques for the period 1866–1973. This

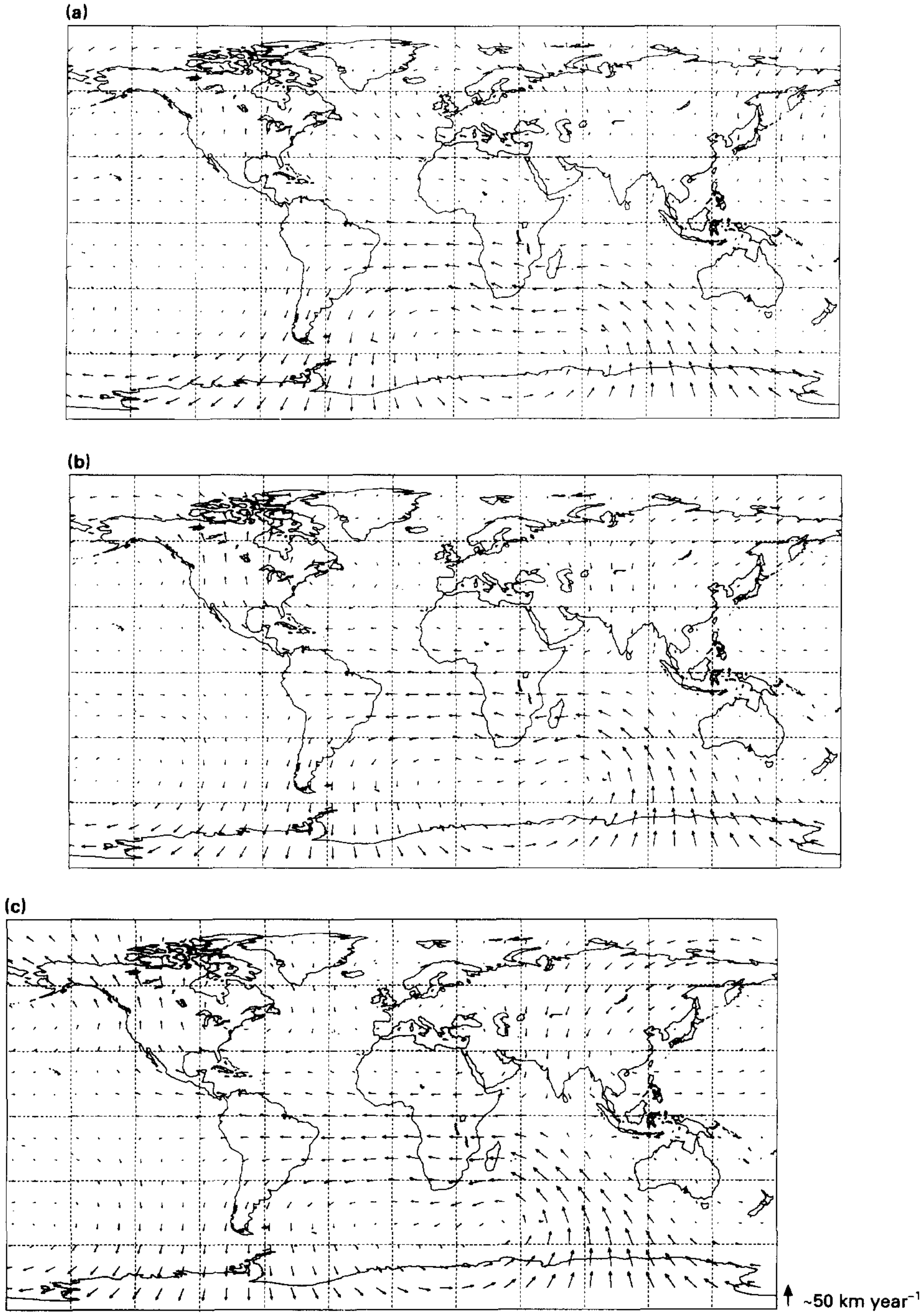

Fig. 2. Toroidal part of the geostrophic flows of Jackson and Bloxham (1990) for epochs (a) 1950, (b) 1965, (c) 1980. Grid spacing is 30°. (Note the marked increase in flow beneath Siberia, Central Asia and Arctic Canada from 1950 to 1980.)

has been repeated here though using the estimated data uncertainties to perform a weighted quadratic fit. This modification was found to make almost no difference to the resulting curves although it allowed standard deviations based on the weighted misfit to be derived. Morrison's data were first supplemented by post-1978 data from McCarthy and Babcock (1986), whose results for the excess length of day essentially reproduce those of Morrison (1979) and Stephenson and Morrison (1984) before 1978. This yielded a series of torque assessments for the years 1866–1979. For consistency in the weighted fit, the uncertainties of Morrison (1979) are used rather

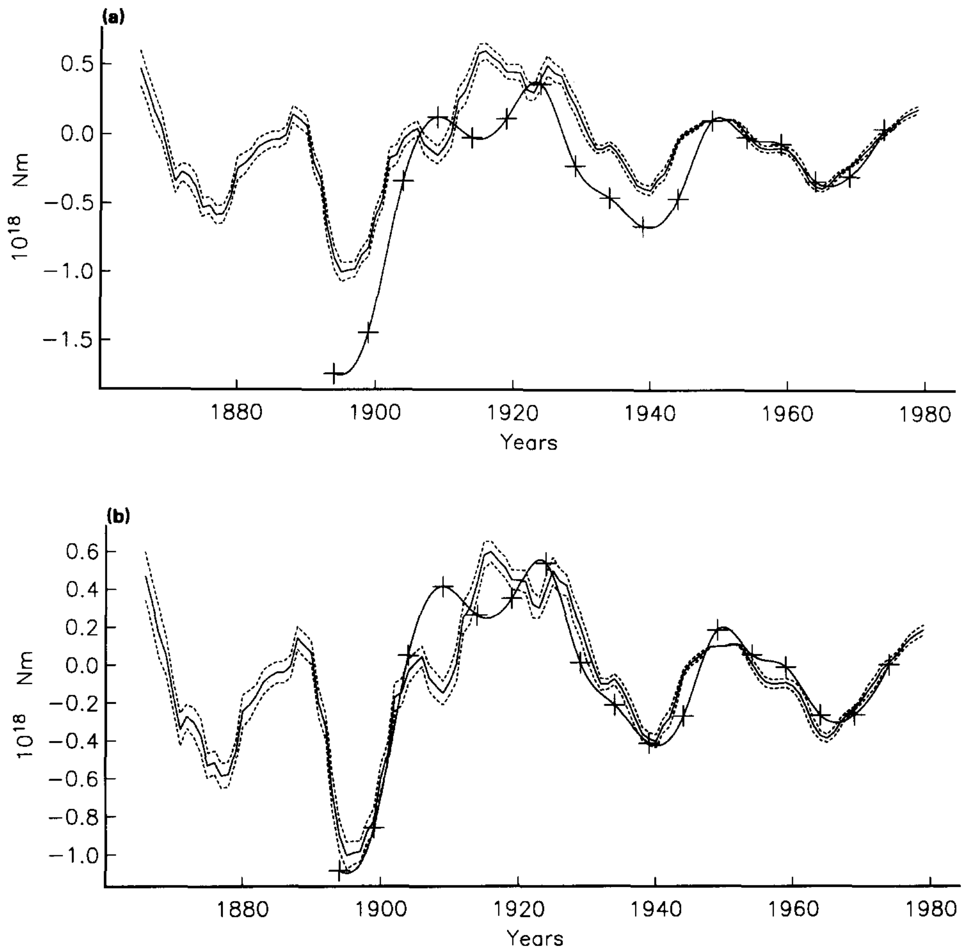

Fig. 3. Time-dependent electromagnetic torque based on zonal toroidal part of geostrophic flows all taken to spherical harmonic degree and order 14. Electromagnetic torque has been scaled, linearly detrended, offset and phase shifted by −6 years. Its values are marked by + and interpolated with cubic-splines. Comparison is with the torque required to explain the decade fluctuations in the LOD (continuous line with one standard error envelope (dashed lines)) as derived from the data of Morrison (1979) (1861–1978) and McCarthy and Babcock (1986) (1979–1984). Detrending, etc. computed for the time periods (a) 1939–1974 and (b) 1894–1974. (Torque units are 10^{18} N m.)

than those of McCarthy and Babcock. The resulting LOD torque with a one standard deviation error envelope can be seen in Fig. 3.

3. Results

From (17) it is clear that the torque depends linearly on the conductance $\delta\sigma_M$. Calculations were carried out using a layer of conductance of 10^8 S which could, for example, be a layer of thickness $\delta = 100$ km with a conductivity of $\sigma_M = 10^3$ S m^{-1}. The calculated torque is presented in Fig. 1(a). The torque shows significant fluctuations on the decade time-scale but is dominated by a long-term increase. This figure also shows that the torque converges rapidly (by about degree three or four) as flow coefficients of higher degree are included. Comparison of Figs. 1(b) and 1(c) shows that fluctuations in the torque arise primarily from changes in the flow. Therefore the reliability of the torque calculation depends heavily on the reliability of the flow models used.

The visual agreement between the EM torque of Fig. 1(a) and the LOD torque is poor even allowing for the significant linear trend in the former. As the fluctuations in the torque arise primarily from the time-dependence of the flow, this disagreement may arise from ambiguity in the flow models. To test this, let us consider the significant increase in the computed torque over the period 1950–1980 (Fig. 1(a)). Now we consider the plots of the toroidal part of the geostrophic flows for epochs 1950, 1965 and 1980 (Fig. 2). The main feature that develops in these flows over the period 1950–1980 is the increase in flow beneath Siberia, Central Asia and the Arctic Sea north of Alaska, and so it is likely that this part of the flow model is responsible for the changes in the predicted EM torque over this period. However, these areas correspond to regions in which the geostrophic constraint does not completely resolve the non-uniqueness of the velocities (see, e.g. Backus and LeMouël, 1986; Jackson and Bloxham, 1991). Such regions account for some 41% of the core surface (Bloxham and Jackson, 1991).

One might hope to improve the torque estimates by eliminating the contribution from potentially spurious velocity components in the regions of non-uniqueness. The torque calculation requires an integration involving the velocity field over the entire core surface, however, and so it is not possible to eliminate these regions from the calculation precisely. As the features of the flow identified above are primarily non-zonal, one might hope to improve the calculation by using only the zonal part of the flow. This is justified because the zonal components of the flow, being constant along lines of equal latitude, are determined uniquely provided no region of non-uniqueness encloses a complete circle of latitude (a proviso which is satisfied). Therefore the torques were recomputed, again for a layer of conductance 10^8 S using only the zonal toroidal coefficients of the geostrophic flows.

The relationship between the calculated EM torque and the LOD torque was investigated using regression analysis. A simultaneous fit to the EM torques by a linear trend plus a scalar multiple of the LOD torques was undertaken for the periods 1861–1979 (the full period for which LOD data are available) and 1945–1979 (for which higher-quality magnetic data are available). This was done allowing lags between the EM torque and the LOD torque ranging from -30 to $+20$ years. It may be useful to recall that EM torque estimates were available based on geomagnetic data for the period 1900–1980 at 5 year intervals. The exact period over which each regression could be calculated depended, therefore, on the overlap between the data for which LOD torque estimates were available (1866–1979) and the periods for which EM torque estimates were used (1900–1980 or 1945–1980).

The resulting correlation coefficients are plotted as a function of lag in Fig. 4. This shows clearly that the optimal correlation occurs for a lag of -6 years, i.e. when torque associated with flows based on geomagnetic observations for 1980 are assigned to the year 1974. The strength of the correlation is much less for the EM torques based on the full geostrophic flow, however. The regression statistics are given in Table 1 both for the EM torque based on zonal geostrophic flow and

Table 1
Results of a simultaneous least-squares fit of the EM torques to the astronomically derived torques plus a linear trend, for (n) epochs (a, c) 1945,...,1980 and (b, d) 1990,...,1980 with the EM torques phase shifted by -6 years

	n	Scale factor	Offset (10^{18} N m)	Trend (10^{18} N m year^{-1})
(a)	6	7.88 ± 1.22	1.36 ± 0.28	0.014 ± 0.004
(b)	17	6.67 ± 0.88	0.76 ± 0.14	0.017 ± 0.003
(c)	6	43.44 ± 97.10	3.19 ± 7.66	0.244 ± 0.547
(d)	17	12.57 ± 5.07	0.64 ± 0.39	0.072 ± 0.029

When the EM torque for a layer of conductance 10^8 s m^{-1} is multiplied by the scale factor, and the trend and offset removed, the result fits the LOD torque with the indicated misfit over the n epochs. (See Fig. 3.) (a) and (b) are based on zonal flows, (c) and (d) are based on the full flows.

the EM torque based on the full geostrophic flow. In both cases the optimal fit was found for a 6 year lag of the EM torque after the LOD torque. However, the large uncertainties in the results based on the full flow emphasise the poorer correlation indicated by Fig. 4. For the regression over the period 1939–1974 the fit of the EM torques based on zonal flows lies mostly within the one standard error limits of the LOD data but for both zonal cases the fit is reasonable for the whole 1894–1974 period.

The agreement between the EM and LOD torques in Fig. 3 suggests that EM coupling may be an important mechanism by which angular momentum is transferred between core and mantle. The scale factor required to make the decade fluctuations in the torques agree implies a layer at the base of the mantle of conductance $(6.67 \pm 0.88) \times 10^8$ S m^{-1} when the regression is calculated over the period 1894–1974 for a lag of -6 years.

The offset and trend need some explanation. Stix and Roberts (1984) found their EM torque to have a mean value of -1.5×10^{18} N m and suggested that in reality this may be balanced by the couple owing to 'leakage' of toroidal flux from the core into the mantle, which has not been accounted for in the present work. This argument rests on the supposition that the toroidal field in the core is generated by motions of the deeper core with a longer characteristic time-scale than the free stream at the top of the core. The presence of the significant trend amounting to approximately 0.7×10^{18} N m in 50 years is a

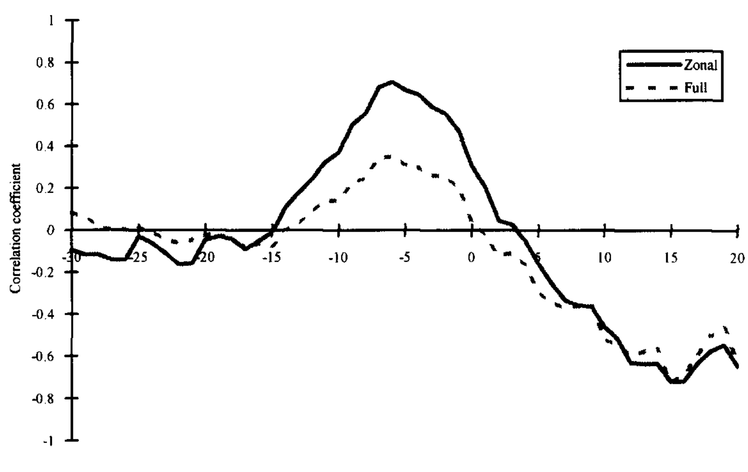

Fig. 4. Coefficients of correlation between the astronomically derived LOD torque and the best-fitting electromagnetic torque (with linear trend and offset removed as discussed in Section 3). Correlation is shown vs. lag between electromagnetic and astronomical torques. Curves are presented for electromagnetic torque based on zonal part of flows and full flows.

little more surprising. This may well result from the rather bold assumption used earlier that the poloidal field at the Earth's surface may be downward continued to the base of the mantle assuming the mantle to be insulating. This crude approximation could be refined and would be worth future investigation.

The calculation performed here differs from that of Love and Bloxham (1994), who solved an inverse problem for the toroidal field in the mantle required to give an EM (advective) torque consistent with the LOD torque. They concluded that EM coupling is not able to explain the LOD torque as the inferred toroidal field was too strong, on the grounds that it would produce electric fields at the Earth's surface which are stronger than measured values and generates ohmic heating that exceeds, or contributes an unacceptably large fraction of, the Earth's surface heat flow. The root mean square (r.m.s.) toroidal field strength implied by the best-fitting EM torque in our calculation is 3.5 mT, about an order of magnitude larger than the r.m.s. poloidal field strength at the CMB. This is typical of the toroidal field strength in the core obtained by balancing the Lorentz and Coriolis forces. The main differences between the assumptions made in our forward calculation and the inversion undertaken by Love and Bloxham (1994) are that the latter included non-zonal flow coefficients, which we found severely degraded the correlation between EM and LOD torques. Presumably they were attempting to find a toroidal field to reproduce the offset between the two torques, and possibly a trend, if their flow model also produced an EM torque increasing with time.

4. Lower-mantle conductivity

A number of published estimates of lower-mantle conductivity are summarised in Table 2. Seven of these adopt the 'power-law' profile first introduced by McDonald (1957). This takes the form

$$\sigma = \sigma_0\left(\frac{r_C}{r}\right)^{\alpha} \tag{19}$$

valid for a specific range of depths, where r is the radius, r_C is the radius of the CMB (assumed spherical) and σ_0 is the conductivity of the mantle at the CMB. Although there is a variation of several orders of magnitude in the estimates of the conductivity at the base of the mantle, what most of the models in Table 2 have in common is a rapid drop-off of conductivity with distance from the CMB and hence a concentration of conductance towards the CMB. Thus many of

Table 2
Summary of various lower-mantle electrical conductivity distributions considered

Authors	r_{min} (km)	r_{max} (km)	σ_0 (s m^{-1})	α	Method
Power-law profile					
Alldredge (1977)	3485	5371	100000	25.0	15/25 year waves
Backus (1983)	3485	5731	3000	11.3	1969–1970 jerk
Courtillot and LeMouël (1984)	3485	5731	1000	19.0	1969–1970 jerk
Kolomiytseva (1972)	3485	6371	710000	30.0	60 year wave
McDonald (1957)	3485	5671	223	5.1	SV spectrum
Paulus and Stix (1989)	3485	5485	600	0.5	EM CM coupling
Stix and Roberts (1984)	3485	5485	3000	30.0	EM CM coupling
Slab / layered profile					
Achache et al. (1980)	3485	5485	100	–	1969–1970 jerk
Currie (1968)	3485	5485	200	–	MHD turbulence
Ducruix et al. (1980)	3485	3871	100	–	1969–1970 jerk
	3871	4871	30	–	
	4871	5871	1	–	

SV, Secular variation; EM, electromagnetic; CM, core–mantle; MHD, magnetohydrodynamic.

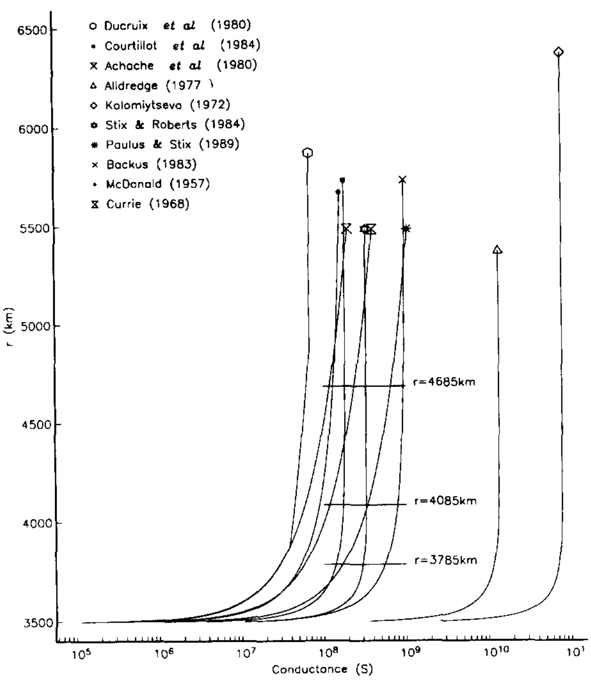

Fig. 5. Integrated conductivity (conductance) for the radius intervals $[r_C, r]$ ($r_C = 3485$ km) from ten published profiles of lower-mantle electrical conductivity (summarised in Table 2 and described in Section 4). Horizontal bars mark the conductance range 10^8–10^9 S. Curves are terminated at their upper radius of validity by their key symbol.

these models could be reasonably well approximated by a thin layer of constant conductance as used to derive the electromagnetic torques in Sections 2 and 3.

If it is assumed that electromagnetic coupling is the sole agent of angular momentum transfer between core and mantle, then the regression analysis of Section 3 suggests that the conductance of such a layer should be approximately 10^8–10^9 S. The horizontal bars in Fig. 5 mark this conductance range at three radii: 3785 km, 4085 km and 4685 km, corresponding to layers of thickness 300 km, 600 km and 1200 km, respectively. This figure can be used to identify immediately which of the published conductivity profiles summarised in Table 2 are compatible with the mantle torque.

The model of Ducruix et al. (1980) is at least half an order of magnitude too small to explain the LOD fluctuations, whereas the models of Alldredge (1977) and Kolomiytseva (1972) have conductances within 500 km of the CMB that are two to three orders of magnitude too large to be consistent with the required torques. This leaves seven models that could conceivably be approximated by thin layer models with appropriate conductance. The concentration of conductance in the three layers for these seven is summarised in Table 3. Those of Achache et al. (1980) and McDonald (1957), however, are only just adequate (Fig. 5) and only then if the layer thickness is 1200 km, when the thin layer approximation for the induced toroidal field in Section 2 would probably be invalid. For a layer thickness of 300 km or less, the model of Currie (1968) is also excluded, which leaves four candidate models as indicated in Table 3. The Paulus and Stix (1989)

Table 3
Conductivity models from Table 2 for which the lower δ kilometres have a sufficient conductance to produce EM torques to account for the decade fluctuations in the LOD

Models	$\delta = 300$ (%)	$\delta = 600$ (%)	$\delta = 1200$ (%)	Total conductance (10^{18} s)
Backus (1983) [a]	56	80	96	9.8
Courtillot and LeMouël (1984) [a]	76	94	99	1.8
McDonald (1957)	32	55	81	1.6
Paulus and Stix (1989) [a]	16	32	62	10.6
Stix and Roberts (1984) [a]	90	99	100	3.3
Achache et al. (1980)	14	30	60	2.0
Currie (1968)	14	30	60	4.0

Figures given are the percentage of the conductance of the whole model which is contained in the layer.
[a] Candidate models (see text).

profile is particularly flat and would require the full 1200 km thickness to account for the torque (see Table 3). The models of Stix and Roberts (1984) and Courtillot and LeMouël (1984) have most of the conductivity concentrated in the first 300 km but the conductivities would need to be increased by factors of about three and five, respectively. Backus' (1983) model has the most appropriate conductance of all the models and the equivalent layer would need to be about 600 km thick. It is not surprising, looking back at Table 2, to find that these last three models are all qualitatively very similar: a conductivity of a few thousand Siemens per metre at the base of the mantle with a rapid drop-off away from the CMB.

4.1. Propagation time and layer thickness

For a mantle of finite, non-zero conductance there is a delay time associated with the propagation of geomagnetic variations from the core to the surface. In Section 3 it was found that the optimal lag between the astronomically and geomagnetically inferred torques is 6 years. Vestine and Kahle (1968) found a lag of 5 years based on computations of angular momentum derived from observations of the eccentric dipole.

Backus (1983) showed that the geomagnetic impulse propagation time is dependent on the spatial harmonic degree of the impulse field. Considering the first six harmonic degrees only and conductance 6.7×10^8 S, this would imply an impulse delay time of 3–4 years for a 300 km thick layer or 5–7.5 years for a 600 km thick layer. The corresponding smoothing time (Backus, 1983) is, however, not consistent with the apparent rapidity of the 1970 and 1978 geomagnetic impulses unless significant spatial mode mixing is present (Stewart, 1991).

For a periodic signal, a 400 km thick layer of material of constant conductivity with total conductance of 6.7×10^8 S would have a delay time of 6.1 years. This is in broad agreement with the delay found here and those of Vestine and Kahle (1968) and Backus (1983). Stewart (1991) has revisited the question of the location of the 'electromagnetic core–mantle boundary' (Hide, 1978) using time-dependent field models such as those used here and concluded that there may be a sharp transition in conductivity at about 300 km above the core–mantle boundary.

5. Conclusions

A clear correlation over an 80 year period has been found between the calculated time-varying electromagnetic torque acting on the Earth's mantle and the torque generally believed to act on the mantle as inferred from astronomically based observations of the Earth's rotation. The torque calculation rests on a number of assumptions and approximations, the foremost of which is that the mantle is an electrical insulator except for a thin layer of finite conductance adjacent to the core–mantle boundary. The conductance inferred from regression analysis is $(6.7 \pm 0.9)10^8$ S. The optimal correlation is found for a lag of the electromagnetic torque after the astronomical torque of 6 years.

Only the zonal portion of flows derived under the tangential geostrophy constraint has been used in the calculation of the torque. When the full flow models are used the correlation breaks down. We believe this is due to flow in areas of ambiguity where tangential geostrophy does not constrain the flow calculation. Model flow in these areas is primarily non-zonal and possibly spurious.

The results presented provide evidence that electromagnetic forces are the dominant mediator of angular momentum transfer between the liquid outer core and the solid mantle on the decade time-scale. This also provides independent evidence that the surficial flow at the top of the core is non-steady on the decade time-scale.

A review of previously published conductivity models in light of the results obtained here suggests that several estimates of lower-mantle conductivity are too high, in particular those of Kolomiytseva (1972) and Alldredge (1977). It seems more likely that conductivity is restricted to a thin layer at the base of the mantle, the conductance of which is between 10^8 and 10^9 S. The apparent delay in propagation of geomagnetic variations through the mantle tends to favour a 300 km thick layer with a conductivity of between 330 and 3300 S m^{-1}.

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