Long-term changes in the Earth's climate: Milankovitch cycles as an exercise in classical mechanics

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Long-term changes in the Earth's climate: Milankovitch cycles as an exercise in classical mechanics

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Long-term changes in the tilt of the Earth's axis, relative to the plane of its orbit, are of great significance to long-term climate change, because they control the size of the arctic and Antarctic circles. These "Milankovitch cycles" have hitherto been calculated by classical perturbation methods or by direct numerical integration of Newton's equations of motion. This paper presents an approximate calculation from simple considerations of angular momentum using similar methods to those used to study the precession of a spinning top. It is an instructive exercise in classical mechanics and gives a simple explanation of the phenomenon in terms of angular momentum. It is shown that the main component of "Milankovitch cycles" has a period of 41,000 yr and is due to one of the modes of precession of the Earth-Venus system. The other mode of this system produces a component of period 29,500 yr, and a third component of period 54,000 yr results from the influence of the precession of the orbits of Jupiter and Saturn. These results agree closely with several of the numerical simulations in the literature and strongly suggest that some other results in the literature are incorrect.

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I. INTRODUCTION

Long-term changes in the Earth's climate are thought to be related to long-term changes in the Earth's orbital parameters.[1,2] The simplest of these is the 26,000-year precession of the spin axis of the Earth, observed in astronomy as the "precession of the equinoxes" and first analysed by Newton (Ref. 3, pp. 531–534). The qualitative explanation of this phenomenon is given in textbooks on classical mechanics, notably the one by Kibble and Berkshire.[4] Figure 1 is based on their Figs. 9.7 and 9.8 and shows the similarity of the spinning Earth to a spinning top. In the latter case, it is easy to see that the top is subject to a torque from its weight and the associated tip force, which causes it to precess about a vertical axis. In the former case, a torque arises from the action of the Sun's (and Moon's) gravity on the Earth's equatorial bulge: the side nearer the Sun is attracted more strongly than the side further from it, giving incremental forces of opposite signs, producing a torque, as shown in Fig. 1.

If the Earth's orbit around the sun were fixed and exactly circular, the climatic effect of the precession of the equinoxes would be nil. In fact it is small; the important effects climatically are the 41,000-year "Milankovitch cycles" in the size (as opposed to the orientation) of the tilt of the Earth's spin axis relative to the plane of the Earth's orbit. These produce corresponding cycles in the size of the arctic and Antarctic circles, which are widely believed[1,2] to be the explanation for the cycles of ice ages over the last million years. These ice-age cycles are revealed[5] in deposits of any material (e.g., seabed deposits from shelled creatures, limestone accumulations in caves, ancient ice in Antartica and Greenland) that has been laid down steadily over this period and that contains oxygen derived from the oceans. They all show regular variations in the small proportion of the stable heavy oxygen isotope ^{18}O; the period of these variations can be deduced from other evidence as approximately 100,000 yr. Seawater contains a small proportion of water containing ^{18}O. Such water evaporates less readily. Water vapour in the atmosphere, and thus, water in ice caps, is, thus, depleted in ^{18}O compared with sea water. When much of the Earth's water is trapped in ice, the concentration of ^{18}O in the oceans rises, and thus, the concentration in any material derived from seawater. (There are other interpretations of the data in the literature based on the temperature-dependence of the evaporation of water.)

Milankovitch cycles have hitherto been calculated by the classical 18th century perturbation methods (described in Chapters 6–7 of the textbook by Murray and Dermott[6]), or by direct numerical integration of Newton's equations of motion. An approximate calculation is also possible from simple considerations of angular momentum, as in Fig. 1, and is the subject of this paper. It is an instructive exercise in classical mechanics and gives a simple explanation of the phenomenon in terms of angular momentum. It also resolves a controversy, which has recently arisen in the literature over new computer simulations of Milankovitch cycles, which disagree with earlier ones (Ref. 7, Fig. 11).

The calculation begins with an analysis of the 18.6-year precession of the plane of the Moon's orbit around the Earth (i.e., lunar nodal precession, in astronomical parlance), where the geometry is simpler, and the results can be checked against observations. The agreement is quite close (3%), but not exact because the calculation assumes the Moon's orbit is circular. For our present purposes, the critical part of the analysis is the inclusion of the precession of the plane of the Earth's orbit around the Sun, which is driven by the gravitational effect of the other planets, as explained qualitatively by Kibble and Berkshire (p. 144). It is this precession of the Earth's orbit which causes Milankovitch cycles. Its effect on the precession of the Moon's orbit around the Earth is similar and can be checked against observations.

The calculation is then repeated for the case of the spin of the Earth about its axis. The Milankovitch cycles over the last million years are calculated and shown to contain components of periods 29,500 yr and 54,000 yr, as well as the fundamental component of 41,000 yr, all in agreement with the most authoritative computer simulations. Our analysis

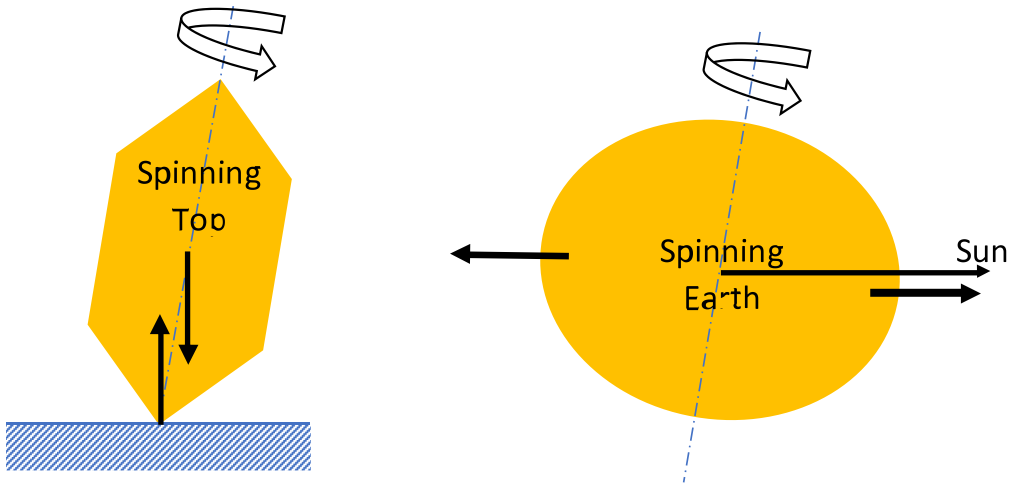

Fig. 1. Torque acting on a spinning top and on the Earth.

gives the physical explanation for these components as the two precession modes of the Venus-Earth system and the influence of the precession of the orbits of Jupiter and Saturn.

II. PRECESSION OF THE ORBIT OF THE MOON

The precession of the orbit of the Moon is explained qualitatively by Kibble and Berkshire (Ref. 4, p. 144) as being due to the gradient of the Sun's gravitational field. They define (p.130) a gravitational field as being the vector acceleration $\mathbf{g}(\mathbf{r})$ imparted by the field to a particle at position $\mathbf{r}$ but do not discuss the properties of its gradient. If $\mathbf{g}$ has components g_x, g_y, g_z in orthogonal directions x, y, z then a small change $\delta\mathbf{r}$ in position $\mathbf{r}$ with components $\delta r_x, \delta r_y, \delta r_z$ will produce a small change $\delta\mathbf{g}$ in $\mathbf{g}$ with components

$$\begin{bmatrix} \delta g_x \\ \delta g_y \\ \delta g_z \end{bmatrix} = \begin{bmatrix} \frac{\partial g_x}{\partial x} & \frac{\partial g_x}{\partial y} & \frac{\partial g_x}{\partial z} \\ \frac{\partial g_y}{\partial x} & \frac{\partial g_y}{\partial y} & \frac{\partial g_y}{\partial z} \\ \frac{\partial g_z}{\partial x} & \frac{\partial g_z}{\partial y} & \frac{\partial g_z}{\partial z} \end{bmatrix} \begin{bmatrix} \delta r_x \\ \delta r_y \\ \delta r_z \end{bmatrix}. \tag{1}$$

Following the tensor notation of Ref. 4 (p. 401), we will write this as

$$\delta\mathbf{g} = \mathsf{F}\boldsymbol{\delta r}, \tag{2}$$

where the gradient of the Sun's gravitational field is written as the tensor F. In their general discussion of tensors (Ref. 4, p. 401), Kibble and Berkshire distinguish the special case of a symmetric tensor, and F is such a case because (Ref. 4, eq. 6.3) $\mathbf{g}$ is the gradient of a potential Φ so that

$$\frac{\partial g_x}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial y} = \frac{\partial g_y}{\partial x}, \quad \text{etc.}$$

We conclude (Ref. 4, p. 404) that the orthogonal directions x, y, z can be chosen so that the matrix in Eq. (1) diagonalizes. Moreover, if these "principal directions" are chosen, then the sum of the diagonal elements of the matrix in Eq. (1) is

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{3}$$

by definition of a potential (Laplace's equation, Ref. 4, Eq. 6.49). We now make use of these properties of F.

We can take a frame of reference that is centered on the combined center of gravity C of the Earth and Moon but does not rotate relative to distant stars. In this frame, both the Moon and Sun appear to be in orbit; for the moment, we will assume the plane of the Sun's orbit to be fixed (i.e., we assume the normal $\mathbf{S}$ to the plane of the Sun's orbit is aligned with the normal $\mathbf{L}$ to the Laplace invariable plane, which is the plane normal to the (fixed) total angular momentum of the solar system[8]). We now assume both orbits to be circular (in fact their mean eccentricities are 0.055 and 0.0167, respectively[9,10]), a key approximation that we will discuss in detail at the end of this section. The radius of the Moon's orbit is only 0.00257 times that of the Sun, so for positions $\mathbf{r}$ relative to C that are within the Moon's orbit, we can take the Sun's gravitational field as

$$\mathbf{g}(C) + \mathsf{F}\mathbf{r}, \tag{4}$$

where $\mathbf{g}(C)$ is its value at C and F is its gradient there as just defined. If $\mathbf{r_c}$ and $\mathbf{r_c^*}$ are the positions of the centers of gravity of the Moon and Earth relative to C, and m and m^* are their masses, then the total gravitational force on them from the Sun is

$$m(\mathbf{g}(C) + \mathsf{F}\mathbf{r_c}) + m^*\left(\mathbf{g}(C) + \mathsf{F}\mathbf{r_c^*}\right) = (m + m^*)\mathbf{g}(C) + \mathsf{F}(m\mathbf{r_c} + m^*\mathbf{r_c^*}). \tag{5}$$

However, $m\mathbf{r_c} + m^*\mathbf{r_c^*} = 0$ since C is the center of gravity of m and m^* combined, so the second term on the RHS of Eq. (5) vanishes. The total gravitational force on m and m^* combined is, thus, the same as it would be if F were zero. Our frame is, thus, accelerating towards the Sun in the same way that it would if the Sun's gravitational field was everywhere equal to its value at C. In that case, its effects would be indistinguishable from an acceleration of the frame of reference (Ref. 4, p. 192). Thus, in our case (4), the effect of the uniform part of the Sun's gravitational field is cancelled out by the effect of the acceleration of our frame. We can, therefore, proceed as if our frame centered on C were fixed, provided we ignore the uniform part of the Sun's gravitational field, equal to its value at C. Specifically, we can consider the Moon on its own and equate the rate-of-change of its angular momentum about C, to the moment about C of the forces acting on it, excluding the forces from the first term in Eq. (4).

The gravitational force on the Moon from the Earth acts through C and so has zero moment about C. However, the second term in Eq. (4), i.e., the effect of gradient F of the Sun's gravitational field, produces a moment about C. This non-zero moment causes the precession of the Moon's orbit, as explained qualitatively by Kibble and Berkshire.

The period of the precession is much longer than the orbital period of the Earth and the Moon, so we are concerned with the long-term average value of the moment. Kibble and Berkshire suggest calculating the moment at each position of the Sun and Moon and taking the average (Ref. 4, p. 157). However, we can adopt the procedure introduced by Newton (as described in Ref. 3, p. 531) of smearing the masses of the Sun and Moon over their orbits seen in our frame. We are, therefore, concerned with the gravitational moment about C produced by one ring of mass M and radius R, corresponding to the Sun on a much smaller ring at its center, of mass m and radius r, corresponding to the Moon. Both rings are centered on C.

Figure 2 shows the Sun's ring, centered on C. For later use, we will define real and imaginary axes in the plane of the ring; these axes are fixed relative to distant stars. We will also define orthogonal axes x, y, and z. The z-axis is aligned with $\mathbf{S}$. The x and y axes are in the plane of the Sun's orbit but rotate relative to the distant stars in a way we now define, so that the angle φ increases (or decreases) steadily with time.

Figure 3 shows the Moon's ring, viewed from the same angle as Fig. 2, and with the same axes x, y, z. The normal to the plane of its orbit is $\mathbf{l}$ (for lunar), and Fig. 3 also shows the plane containing $\mathbf{S}$ and $\mathbf{l}$ and the angle θ between $\mathbf{S}$ and $\mathbf{l}$. It is this plane that defines the direction of the x-axis. Because the orbit of the Moon is precessing, the x and y-axes rotate about the z-axis, as just described. In Fig. 3, we also define $\mathbf{l}'$ as the component of $\mathbf{l}$ in the x-direction, and α as the angle measured around the Moon's ring from the y-direction.

Because of the angle θ between the planes of the two rings, there is a moment on the Moon's ring from the gradient in the gravitational field from the Sun's ring. We will denote this gradient by $\bar{\mathsf{F}}$. The arguments of Eqs. (1)–(3) apply, so $\bar{\mathsf{F}}$ will be another symmetric tensor like F. By the symmetry of the ring, its principal axes are in the z-direction in Fig. 2 and in any two orthogonal directions in the plane of the Sun's ring, which we can, thus, take as the x and y directions.

The principal component of $\bar{\mathsf{F}}$ in the z-direction is easily calculated. If we move a distance δ from C in the z-direction, the gravitational forces from each part of the Sun's ring are no longer in its plane, but at an angle δ/R to it. Since the gravitational force from the Sun is GM/R^2 per unit mass, where G is the gravitational constant, we conclude that the principal component of $\bar{\mathsf{F}}$ in the z-direction is

$$-\frac{GM}{R^3}. \tag{6}$$

The other two principal components of $\bar{\mathsf{F}}$ are equal by symmetry and, thus, must be equal to

$$\frac{GM}{2R^3}, \tag{7}$$

because the sum of all three principal components is zero (3).

We now wish to find the moment on the Moon ring produced by $\bar{\mathsf{F}}$. By symmetry, it can have no components about the z and x axes; the moment thus acts entirely about the y-axis. The contributions to the moment from each of the principal components of $\bar{\mathsf{F}}$ need to be considered in turn. The simplest of the three to consider is the principal component of $\bar{\mathsf{F}}$ in the z-direction (6). We first consider the effect of this principal component on a single point on the Moon's ring of point mass δm. If we denote the angular position of δm on the ring as α, as defined above (see Fig. 3), then the z-coordinate of δm is

$$r \sin\alpha \sin\theta.$$

The principal component (6), thus, produces a force on δm of

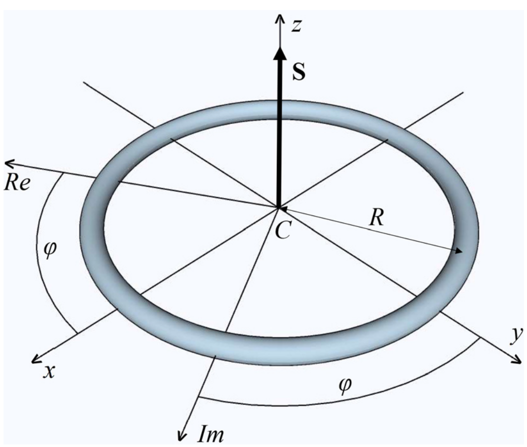

Fig. 2. The Sun in orbit around the combined centre of gravity C of the Earth and Moon with the Sun's mass smeared out into a ring, in the plane of the x, y and Re, Im axes. View from above the Earth's northern hemisphere. The Re, Im directions are fixed with respect to distant stars; the x, y directions are defined in Fig. 3.

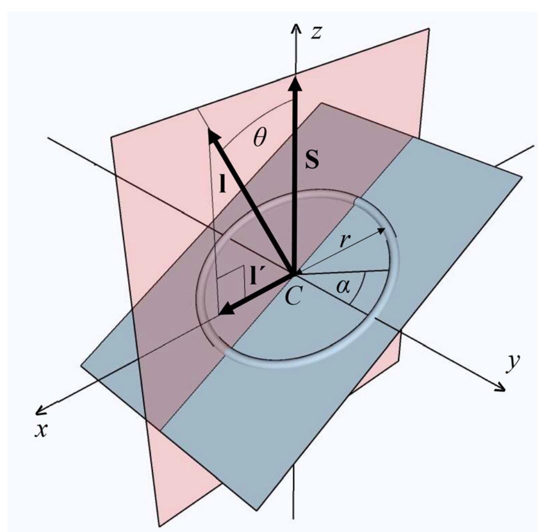

Fig. 3. The Moon in orbit around the combined centre of gravity C of the Earth and Moon, with the Moon's mass smeared out into a ring (same viewpoint and x, y, z axes as Fig. 2).

$$\delta m \frac{GM}{R^3} r \sin\alpha \sin\theta,$$

which acts in the sense of negative z since the sign of Eq. (6) is negative. Its lever arm about the y-axis is

$$r \sin\alpha \cos\theta = r \sin\alpha$$

since $\cos\theta \approx 1$ for our small θ (5.145°, see the end of Sec. III). Thus, it will produce a moment about the y-axis, in the sense to reduce θ, of

$$\delta m \frac{GM}{R^3} r^2 \sin^2\alpha \sin\theta.$$

The total moment on the Moon's ring will be the sum of the contributions from all the masses δm around the ring. Since the average value of $\sin^2\alpha$ around the ring is 0.5, we conclude that this total moment is

$$r^2 \sin\theta \frac{GMm}{2R^3} \tag{8}$$

acting about the y-axis in the sense to reduce θ.

The other two principal components (7) are equal, and it is convenient to consider them in combination. We then observe that in combination they will produce a force on a point mass δm, which is perpendicular to the z-axis, and proportional to the distance of δm from it (and acting away from the z-axis, since the sign of Eq. (7) is positive). Since $\cos\theta \approx 1$, the distance from the z-axis of all point masses on the Moon's ring is r, so from Eq. (7) the force on a single point mass δm is

$$\delta m \frac{GM}{2R^3} r. \tag{9}$$

The moment this force produces about the y-axis, in the sense to reduce θ, is due to its component in the x-direction (i.e., Eq. (9) times $\sin\alpha$), and its lever arm about the y-axis which is

$$r \sin\alpha \sin\theta$$

see Fig. 3. Thus, the moment about the y-axis, in the sense to reduce θ, is

$$\delta m \frac{GM}{2R^3} r^2 \sin^2\alpha \sin\theta.$$

The total moment on the Moon's ring will again be the sum of the contributions from all the masses δm around the ring. We again observe that the average value of $\sin^2\alpha$ around the ring is 0.5 and, thus, conclude that this total moment is

$$r^2 \sin\theta \frac{GMm}{4R^3} \tag{10}$$

acting about the y-axis in the sense to reduce θ. Thus overall, the gradient in the Sun's gravitational force produces an average moment on the Moon about C of Eqs. (8) + (10), i.e.,

$$3r^2 \sin\theta \frac{GMm}{4R^3} \tag{11}$$

acting about the y-axis in the sense to reduce θ, i.e., in the sense of $-\hat{\mathbf{y}}$.

In vector notation, $\mathbf{S}$ is the unit normal to the plane of the Sun's orbit (i.e., in the z-direction at this stage), and we can define a unit vector $\mathbf{l}$ normal to the plane of the Moon's orbit, see Fig. 3. Since $\mathbf{l} \times \mathbf{S} = -\hat{\mathbf{y}} \sin\theta$, the moment (11) can then be written as

$$3r^2 \frac{GMm}{4R^3} \mathbf{l} \times \mathbf{S}. \tag{12}$$

Its effect is to make to $\mathbf{l}$ precess about $\mathbf{S}$ with constant angular velocity Ω (i.e., $\varphi = -\Omega t$ in Fig. 2), as described by Kibble and Berkshire (Ref. 4, Sec. 9.6). For later use, we will now re-work that argument in a different notation, first writing

$$\mathbf{l} = \mathbf{S} \cos\theta + \mathbf{l}', \tag{13}$$

where $\mathbf{l}'$ is the component of $\mathbf{l}$ perpendicular to $\mathbf{S}$ (i.e., in the x-direction, see Fig. 3). Thus, Eq. (12) becomes

$$3r^2 \frac{GMm}{4R^3} (\mathbf{S} \cos\theta + \mathbf{l}') \times \mathbf{S} = 3r^2 \frac{GMm}{4R^3} \mathbf{l}' \times \mathbf{S}.$$

We can equate this moment about C to the rate-of-change of the Moon's angular momentum about C. Ignoring the Moon's much smaller angular momentum about its own CG, the angular momentum of the Moon about C is in the direction of $\mathbf{l}$ (since the Moon rotates anti-clockwise in Fig. 3) and is

$$\frac{2\pi r^2 m}{T} \mathbf{l} = \frac{2\pi r^2 m}{T} (\mathbf{S} \cos\theta + \mathbf{l}'), \tag{14}$$

where T is the period of the Moon's orbit. Thus,

$$\frac{d}{dt}\left[\frac{2\pi r^2 m}{T} (\mathbf{S} \cos\theta + \mathbf{l}')\right] = 3r^2 \frac{GMm}{4R^3} \mathbf{l}' \times \mathbf{S}. \tag{15}$$

Since $\mathbf{l}' \times \mathbf{S}$ has no component in the direction of $\mathbf{S}$, we immediately see that $d/dt \cos\theta = 0$, so the inclination θ of the Moon's orbit is a constant. In addition,

$$\frac{d\mathbf{l}'}{dt} = \frac{3GMT}{8\pi R^3} \mathbf{l}' \times \mathbf{S}. \tag{16}$$

Although a little excessive in the present case, it will be convenient later to take real and imaginary axes in the x-y plane but not rotating relative to distant stars, and hence, not aligned with the x and y axes. In this way, we can denote the component of a vector in the x-y plane by a complex number, see Fig. 2. The vector $\mathbf{l}'$, in particular, we will denote by the complex number l', where $l' = \sin\theta(\cos\varphi + i \sin\varphi)$, see Figs. 2 and 3. The vector $\mathbf{l}' \times \mathbf{S}$ is also in the x-y plane, and in our complex notation, it is $-il'$, see Fig. 3. Thus, in our complex notation, Eq. (16) becomes

$$\frac{dl'}{dt} = -i \frac{3GMT}{8\pi R^3} l'. \tag{17}$$

The solution to this simple differential equation is

$$l' = a \exp\left(-i \frac{3GMT}{8\pi R^3} t\right),$$

where a is a complex constant and $|a| = \sin\theta$, see Fig. 3. Thus, $\mathbf{l}'$ rotates clockwise in Fig. 3 with angular frequency $\Omega = 3GMT/8\pi R^3$. The rotation of $\mathbf{l}'$ is the precession of $\mathbf{l}$ described in Ref. 4, where Ω is derived as Eq. (9.29), without having recourse to a complex notation. The precession period is

$$\frac{2\pi}{\Omega} = \frac{16\pi^2}{3}\frac{R^3}{GMT}.$$

Inserting the known values of R, G, M, and T in MKS units (Ref. 4, pp. xv, xvi), this comes to

$$\frac{16\pi^2}{3}\frac{(1.50\times 10^{11})^3}{6.67\times 10^{-11}\times 1.99\times 10^{30}\times 2.36\times 10^6} = 5.7\times 10^8\ \mathrm{s} = 18\ \mathrm{years}. \tag{18}$$

Agreement with the observed period of 18.6 yr[11] is reasonable (3%), but it also shows the limitations of our approximation that the orbits of the Sun and Moon are circular. The eccentricity of the Sun's orbit will affect the calculation above of the tensor $\bar{\mathsf{F}}$. In fact, its effect is small, and the symmetry of the tensor is preserved (Ref. 12, Appendix B). The eccentricity of the Moon's orbit is a much more serious matter. It affects the calculation above of the moment on the Moon produced by $\bar{\mathsf{F}}$. Additionally, there is an observed precession of the direction of the axes of the ellipse (i.e., apsidal precession, in astronomical parlance), whose period is 8.85 yr.[11] This is approximately 100 times the orbital period of the Moon, thus changing its angular momentum by approximately 1%. This angular momentum is ignored in our treatment above.

The accurate treatment of the nodal and apsidal precession of the Moon is a topic of considerable historical interest. There is a highly readable scientific account by Gutzwiller.[13] Lunar precession was the great scientific challenge of the 18th century: the problem was whether Newton's equations of motion, and his law of gravity, were exact or not. Newton had speculated that his law of gravity might not be exact (Ref. 13, p. 605, Sec. C), and also that Divine Intervention might be involved in the behaviour of the solar system (Ref. 3, pp. 586–589). The problem was solved by the (now classical) perturbation methods of Lagrange and Laplace, see Ref. 6, Chaps. 6 and 7, which successfully predicted both nodal and apsidal lunar precession from Newton's laws alone. Laplace was famously asked by his former pupil Napoleon whether he had found it necessary to invoke Divine Intervention in his great work *Traité de mécanique céleste*, to which he replied "Sire, I have no need of that hypothesis" (Je n'avais pas besoin de cette hypothèsis-là).

III. EFFECT OF THE PRECESSION OF THE ORBIT OF THE EARTH AROUND THE SUN

We assumed in Sec. II that the plane of the Earth's orbit around the Sun (and thus the plane of the Sun's orbit as seen in our frame centered on C) does not change with time. In fact it is the Laplace invariable plane, whose normal unit vector $\mathbf{L}$ is in the direction of the total angular momentum of the solar system, which does not vary with time. The normal to the plane of the Earth's orbit around the Sun (and thus the normal $\mathbf{S}$ to the plane of the Sun's orbit seen in our frame) is inclined at an angle ξ to $\mathbf{L}$ and has a slow precession around it, which is caused by the influence of the other planets, as explained qualitatively by Kibble and Berkshire (Ref. 4, p. 144). This precession can be analysed[12] by the methods of Sec. II, taking into account the most important planetary influences, which are from Venus, Jupiter, and Saturn.

We will now take the z-direction in Fig. 2 as aligned with $\mathbf{L}$, so that $\mathbf{S}$ has a component $\mathbf{S}'$ in the x-y plane (now the Laplace invariable plane). Because of the importance of the Moon to the precession of the Earth's spin axis (see Sec. IV), we need to find the effect of this precession of $\mathbf{S}$, on the precession of $\mathbf{l}$. Instead of Eq. (13), we now have

$$\mathbf{l} = \mathbf{L}\cos\theta + \mathbf{l}'. \tag{19}$$

We also now have

$$\mathbf{S} = \mathbf{L}\cos\xi + \mathbf{S}'.$$

The analysis now follows the development in Sec. II, except that $\mathbf{l}\times\mathbf{S}$ in Eq. (12) is now

$$[\mathbf{L}\cos\theta + \mathbf{l}']\times[\mathbf{L}\cos\xi + \mathbf{S}'] = \mathbf{l}'\times\mathbf{L}\cos\xi + (\mathbf{L}\cos\theta)\times\mathbf{S}' + \mathbf{l}'\times\mathbf{S}'. \tag{20}$$

Since $|\mathbf{l}'|$ and $|\mathbf{S}'|$ are both $\ll 1$, we can proceed with a first order analysis, in which we ignore their squares and higher powers, and also their product. Thus, we can ignore the third term on the RHS of Eq. (20) and also write $\cos\theta = \cos\xi = 1$, so that Eq. (15) becomes

$$\frac{d}{dt}\left[\frac{2\pi r^2 m}{T}(\mathbf{L}+\mathbf{l}')\right] = \frac{3r^2 GMm}{4R^3}(\mathbf{l}'\times\mathbf{L} + \mathbf{L}\times\mathbf{S}'). \tag{21}$$

Since $d\mathbf{L}/dt = 0$, we have

$$\frac{d\mathbf{l}'}{dt} = \frac{3GMT}{8\pi R^3}(\mathbf{l}'\times\mathbf{L} + \mathbf{L}\times\mathbf{S}'). \tag{22}$$

This equation can be compared with Eq. (16), where $\mathbf{S}$ was assumed fixed and is now replaced by the fixed $\mathbf{L}$, and additionally we have the term $\mathbf{L}\times\mathbf{S}'$ on the RHS, which is evidently the effect of the precession of $\mathbf{S}$ about $\mathbf{L}$. We can again replace $3GMT/8\pi R^3$ with the angular frequency Ω, and now not only denote $\mathbf{l}'$ by the complex number l', but also denote $\mathbf{S}'$ by the complex number S', so that $\mathbf{l}'\times\mathbf{L}$ and $\mathbf{L}\times\mathbf{S}'$ are denoted by $-il'$ and iS', respectively. From the analysis cited earlier, we have (Ref. 12, Eq. II.23 and Table II)

$$S' = b_1 e^{-i\Omega_1 t} + b_2 e^{-i\Omega_2 t} + b_s e^{-i\Omega_3 t}, \tag{23}$$

where

(i) $b_1 = 0.0262e^{i4.456}$ is the complex amplitude of the first orbital precession mode of the Venus-Earth system. It has angular frequency $\Omega_1 = 8.75$ rad/10,000 yr and thus period of $2\pi/8.75 = 72{,}000$ yr, rounding to the accuracy of the calculation.

(ii) $b_2 = 0.0121e^{i0.0631}$ is the complex amplitude of the second orbital precession mode of the Venus-Earth system. It has angular frequency $\Omega_2 = 2.87$ rad/10,000 yr and thus period of $2\pi/2.87 = 220{,}000$ yr, rounding to the accuracy of the calculation.

(iii) $b_3 = 0.00252e^{i5.357}$ is the complex amplitude of the precession of the Earth's orbit caused by the orbital precession of Jupiter and Saturn. It has angular frequency $\Omega_3 = 12.57$ rad/10,000 yr and thus period of $2\pi/12.57 = 50{,}000$ yr, rounding to the accuracy of the calculation.

Thus, in our complex notation, Eq. (22) becomes

$$\frac{dl'}{dt} = -i\Omega(l' - S') \\ = -i\Omega(l' - b_1 e^{-i\Omega_1 t} - b_2 e^{-i\Omega_2 t} - b_s e^{-i\Omega_3 t}). \tag{24}$$

This simple differential equation is readily solved as

$$l' = ae^{-i\Omega t} + \frac{b_1}{1 - \Omega_1/\Omega} e^{-i\Omega_1 t} + \frac{b_2}{1 - \Omega_2/\Omega} e^{-i\Omega_2 t} \\ + \frac{b_3}{1 - \Omega_3/\Omega} e^{-i\Omega_3 t}, \tag{25}$$

where a is again a complex constant. Our interest in Sec. IV will be in the precession of the axis **l** of the orbit of the moon relative to the axis **S** of the orbit of the Sun, which in our complex notation is

$$l' - S' = ae^{-i\Omega t} + \frac{b_1}{\Omega/\Omega_1 - 1} e^{-i\Omega_1 t} \\ + \frac{b_2}{\Omega/\Omega_2 - 1} e^{-i\Omega_2 t} + \frac{b_3}{\Omega/\Omega_3 - 1} e^{-i\Omega_3 t}. \tag{26}$$

We now observe that $\Omega_1, \Omega_2, \Omega_3 \ll \Omega$ so the last three terms on the RHS of Eq. (26) are very small compared with the first. The axis of Moon's orbit therefore effectively precesses around **S**, although **S** is itself precessing, much more slowly. This is as observed: the inclination of the Moon's orbit to that of the Sun is a constant 5.145° throughout the 18-year precession cycle.[9] If the precession were not around **S**, the angle would vary.

IV. PRECESSION OF THE SPIN AXIS OF THE EARTH, RELATIVE TO THE PLANE OF THE EARTH'S ORBIT

We can now apply the methods of Secs. II and III to the central topic of this paper, which is the precession of the spin axis of the Earth, relative to the plane of the Earth's orbit. Kibble and Berkshire[4] explain (pp. 214 and 215) that it is caused by the gradients in the gravitational field of the Sun and Moon, acting on the slightly ellipsoidal shape of the Earth (Ref. 4, pp. 140–143). They discuss its importance in astronomy, where it is observed as the 26,000-year-period "precession of the equinoxes," first noticed by the ancient Greeks and analyzed quantitatively by Newton (Ref. 3, pp. 531–534). We cannot simply assume that the axis of this precession is fixed, or that it is normal to the plane of the Earth's orbit. We saw in the last section that it will be somewhere between these two extremes, depending on the ratio of the precession periods.

Our analysis follows Sec. II closely with the frame of reference now centered on the center of gravity of the Earth, and the Earth subject to the gradients in the gravitational fields of the Sun and Moon (the influence of the other planets being negligible[12]) with the effect of the uniform parts of these fields being cancelled out by the acceleration of the frame. Again the mass of the Sun can be smeared out into a ring, and the Moon too, averaging over the precession of its orbital plane to give a single ring (or not—including its precession allows us to calculate the associated tiny wobble in the precession of the Earth's axis, described on p. 215 of Ref. 4, which gives a useful cross-check on the argument (19)–(24), see Ref. 12, Appendix D). It is in the same plane as the Sun's ring, because the precession axis of the Moon's orbit is the same as the axis of the Sun's orbit, to a very close approximation, as shown at the end of the last section. The gradients of the gravitational fields of the Sun and Moon are, thus, tensors with the same principal axes, so we can simply add their components (6) and (7), those from the Moon being approximately twice those from the Sun (Ref. 4, p. 215).

The next stage of the calculation in Sec. II is to consider the moment produced by this tensor on a ring which represented the Moon in that section but is now one of many rings at various latitudes, forming the out-of-spherical part of the Earth. The moment on each needs to be calculated, and the results added. Figure 3 now illustrates a ring at the equator; the other rings will be displaced up or down towards the poles. As the Earth tilts its spin axis at an angle ψ to **S**, this will introduce from Eq. (7) an additional force in the x-y plane on each ring proportional to its latitude and $\sin\psi$. The rings at the same latitude in the northern and southern hemisphere can be paired, so that these forces combine to produce a moment. Since it is also proportional to $\sin\psi$, the combined effect of all the rings is simply another version of Eq. (11) with some other constant rather than 3/4.

Similarly the angular momentum (14) will have some other constant representing the Earth's moment of inertia about its spin axis, and thus, the rest of the argument in Sec. II will be the same, except for different constants. We need not trouble here to calculate them, nor correct them for ψ not being small (≈0.4 radians), since the resulting precession period of $2\pi/\Omega_4 = 26{,}000$ yr given on p. 215 in Ref. 4 is not at issue—it is the changes in the magnitude ψ of the precession, rather than its period, which are the subject of this paper.

An important observation is that to change the magnitude of θ in Fig. 3, we require a moment component which is not perpendicular to the angular momentum, i.e., a moment in the x-z plane. However, we are assuming the rings in Figs. 2 and 3 are circular, so neither an x-component or a z-component is possible, by symmetry. In fact their eccentricity is immaterial, because the symmetry of their gradient tensors is the same as for circular rings (Ref. 12, Appendix B), as we noted at the end of Sec. II.

All this assumes as in Sec. II that the plane of the Sun's orbit, with its normal **S**, is fixed. Thus, we conclude that the changes in the inclination ψ of the Earth's spin axis to **S** are the result of **S** not being fixed but precessing in the way described in the last section. The effect of the precession of **S** on the precession of the orbit of the Moon was calculated there, taking advantage of the fact that the inclination θ of the axis **l** of the Moon's orbit to **L** is small. This time, the equivalent inclination χ of the unit vector **p** in the Earth's spin axis to **L** is about 0.4 radians, so such an approach is questionable. The analysis below can accordingly be repeated, see Ref. 12, Appendix E, without assuming that χ is small, merely that its variations $\delta\chi$ from its average value

χ_0 is small. The conclusion is that the analysis in the last section stands but with additional correction factors.

The new version of Eq. (26) for the precession of the Earth's spin axis **p** relative to the axis **S** of the Sun's orbit is

$$p' - S' = de^{-i\Omega_4 t} + 0.772\frac{b_1 e^{-i\Omega_1 t}}{\Omega_4/\Omega_1 - 1} + 0.304\frac{b_2 e^{-i\Omega_2 t}}{\Omega_4/\Omega_2 - 1} + 0.842\frac{b_3 e^{-i\Omega_3 t}}{\Omega_4/\Omega_3 - 1}, \tag{27}$$

where p' is our complex notation for the component $\mathbf{p}'$ of $\mathbf{p}$, which is perpendicular to $\mathbf{L}$, and d is a complex amplitude. The additional factors 0.772, 0.304, and 0.842 are from Ref. 12, Appendix E, as just described.

Evaluating the fractions numerically using the values given in (i)–(iii) in the last section and the value

$$\Omega_4 = 2\pi/0.26 = 24.17\ \text{rad}/100{,}000\,\text{yr}$$

given above, we obtain

$$p' - S' = 0.3970e^{i(3.109-24.17t)} + 0.0115e^{i(4.456-8.75t)} + 0.000496e^{i(0.0631-2.87t)} + 0.00230e^{i(5.357-12.57t)}, \tag{28}$$

where we have also evaluated d numerically, from the fact that $t=0$ is the year 2000, when the direction of the Sun at the March equinox is zero celestial longitude and our imaginary axis, so $\arg(p' - S') = \pi$ at $t=0$. Also $\psi = 0.4091$ radians from Ref. 10, so $|p' - S'| = \sin 0.4091 = 0.3978$.

We can factorize Eq. (28) to

$$p' - S' = e^{i(3.109-24.17t)}\big(0.3970 + 0.0115e^{i(1.347+15.42t)} + 0.000496e^{i(3.237+21.30t)} + 0.00230e^{i(2.248+11.60t)}\big). \tag{29}$$

The inclination ψ of the Earth's spin axis to **S** is $\sin^{-1}|p' - S'|$, thus

$$\psi = \sin^{-1}\left|0.3970 + 0.0115e^{i(1.347+15.42t)} + 0.000496e^{i(3.237+21.30t)} + 0.00230e^{i(2.248+11.60t)}\right|. \tag{30}$$

The variation in ψ over the last 300,000 yr is shown in Fig. 4.

From the form of Eq. (30), we observe that the main frequency-component of the variations in ψ has a period of $100{,}000 \times (2\pi/15.42) = 41{,}000$ yr and is caused by the first orbital precession mode of the Venus-Earth system, see (i)–(iii) in Sec. III. There are lesser frequency-components of periods $100{,}000 \times (2\pi/21.30) = 29{,}500$ yr and $100{,}000 \times (2\pi/11.60) = 54{,}000$ yr, caused, respectively, by the second orbital mode of precession of the Venus-Earth system, and the precession of the orbits of Jupiter and Saturn. To be more

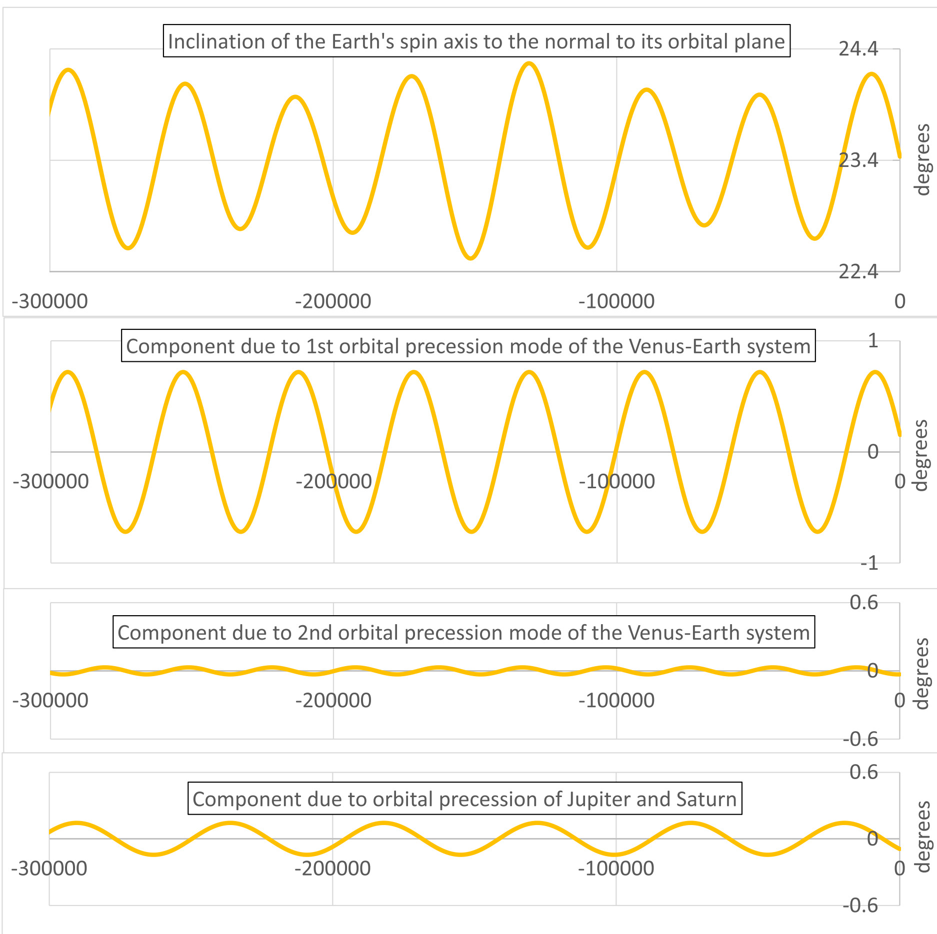

Fig. 4. Variation in the inclination, in degrees, of the Earth's spin axis (relative to the normal to the plane of its orbit) over the past 300,000 yr (AD 2000 = 0), from Eq. (30). Also components from different causes, from Eq. (33).

precise, if we put $\psi = \psi_0 + \delta\psi$, where $\sin\psi_0 = 0.3970$, and take $\cos(\delta\psi) = 1$ and $\sin(\delta\psi) = \delta\psi$, then

$$\begin{aligned}\sin\psi &= \sin\psi_0 + \delta\psi\cos\psi_0 \\ &= |0.3970 + 0.0115e^{i(1.347+15.42t)} \\ &\quad + 0.000496e^{i(3.237+21.30t)} \\ &\quad + 0.00230e^{i(2.248+11.60t)}|, \end{aligned} \tag{31}$$

i.e.,

$$\begin{aligned}&0.3970 + \delta\psi\cos\psi_0 \\ &\quad \approx 0.3970 + 0.0115\cos(1.347 + 15.42t) \\ &\qquad + 0.000496\cos(3.237 + 21.30t) \\ &\qquad + 0.00230\cos(2.248 + 11.60t). \end{aligned} \tag{32}$$

So that we can obtain the approximate frequency components of ψ as

$$\begin{aligned}\psi \approx \psi_0 + \frac{1}{\cos\psi_0}&[0.0115\cos(1.347 + 15.42t) \\ &+ 0.000496\cos(3.237 + 21.30t) \\ &+ 0.00230\cos(2.248 + 11.60t)]. \end{aligned} \tag{33}$$

These have been added to Fig. 4. Note that the relative importance of the second orbital precession mode of the Venus-Earth system (third plot) and the orbital precession of Jupiter and Saturn (fourth plot) is reversed, compared with their contributions to the inclination of the Sun's orbit to the Laplace invariable plane, see (i)–(iii) in Sec. III. This is because the different ratios of their periods to the period of the precession of the Earth's spin axis and because of the different correction factors to allow for χ not being small, see Eq. (27).

Figure 4 can be compared with the computations of Ref. 14 at their Fig. 7(b), which is equivalent to the top plot in our Fig. 4 and are shown as the top plot in our Fig. 5. The results are similar, with the same 41,000 yr period, the same amplitude of about $\pm 1^\circ$, and the same phasing. Muller and MacDonald[2] have carried out further analysis of the results of Ref. 14. They find (Ref. 2, p. 37) that the additional frequency components of the results of Quinn *et al*. have periods of 29,000 yr and 53,000 yr. These are in strikingly close agreement with our figures above. However, there are detailed differences, for example, the most recent cycle is from 22.7° to 24.2° in Fig. 4, but from 0.388 to 0.423 radians, i.e., from 22.2° to 24.2° in Fig. 7(b) of Ref. 14 (see the top plot in Fig. 5). The most significant difference, however, is that the amplitude of the oscillations in Fig. 7(b) of Ref. 14 decays markedly further back in time to 1×10^6 yr ago.

Also shown in Fig. 7(b) of Ref. 14 are the results of the computations of Berger[15,16] based on the work of Bretagnon[17] (again see the top plot in Fig. 5). This work appears to be a higher-order extension of Lagrange's perturbation method and gives results as a series of frequency-components. Up to 100,000 yr ago, they agree closely with the results of Quinn *et al*., but further back in time they start to differ markedly, with Berger's results, like ours, not showing the long-term reduction in the magnitude of the variations in ψ, seen in Fig. 7(b) of Quinn *et al.*[14] Berger's results

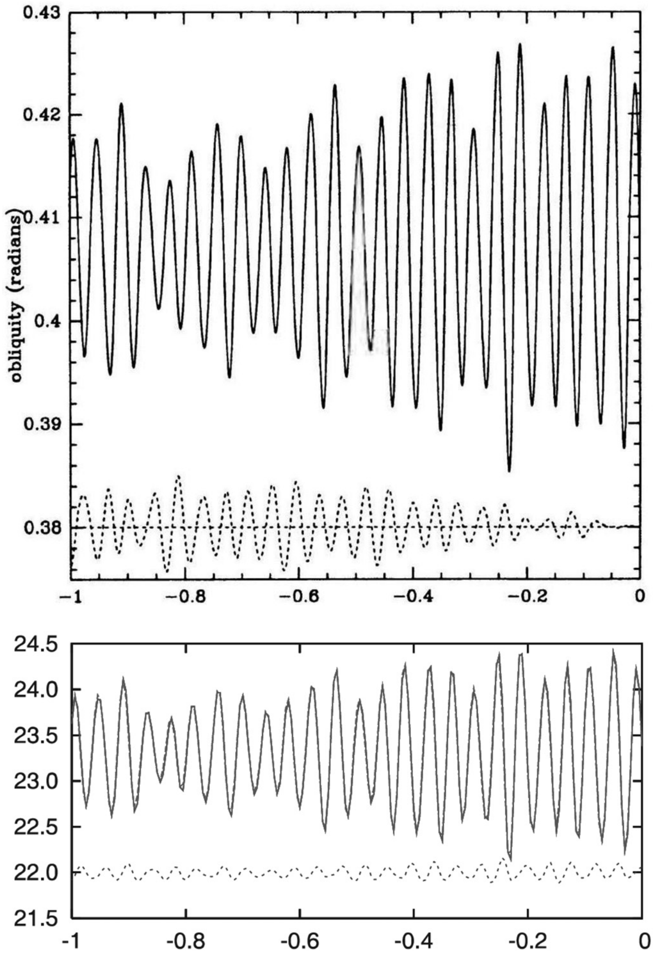

Fig. 5. Variation in the Earth's obliquity (i.e., inclination of its spin axis to the normal to its orbit) over the last million years, reproduced from Fig. 7(b) of Quinn *et al*. (Ref. 14) (top) and Fig. 15 of Laskar *et al*. (Ref. 17) (bottom). Also shown (offset dashed lines) are the differences from Berger's results (Ref. 15) (top) and the differences between full and truncated solutions (bottom).

do not appear to show any systematic differences from Fig. 4: Although the most recent cycle is larger than that shown in Fig. 2b of Berger,[15] by the fourth cycle the position is reversed.

Overall, the broad agreement among all three sets of results very strongly suggests that all three are essentially correct, since they were obtained by three quite different methods. They also broadly agree with two of the results (from Refs. 18 and 19) shown in Fig. 11 in a recent paper by Smulsky.[7] This agreement very strongly suggests that Smulsky's own results in his Fig. 11 are incorrect. Further light is shed by Fig. 15 of Ref. 18, which gives the variation in the Earth's obliquity over the last million years, in exactly the same format as Fig. 7(b) of Ref. 14. The agreement is extremely impressive, see Fig. 5, given the completely different methods employed. This strongly suggests that the long-term reduction in the magnitude of the variations in ψ, seen in both, is correct. It is the very long term variations in orbital elements (e.g., over 500×10^6 yr in Ref. 18, much longer than the 2×10^6 yr relevant to the recent ice ages), which are of astronomical interest, because they reveal behaviour which is ultimately chaotic. This is a recent discovery of fundamental importance in solar system dynamics, see Ref. 6, Sec. 9.3.

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts of interest.

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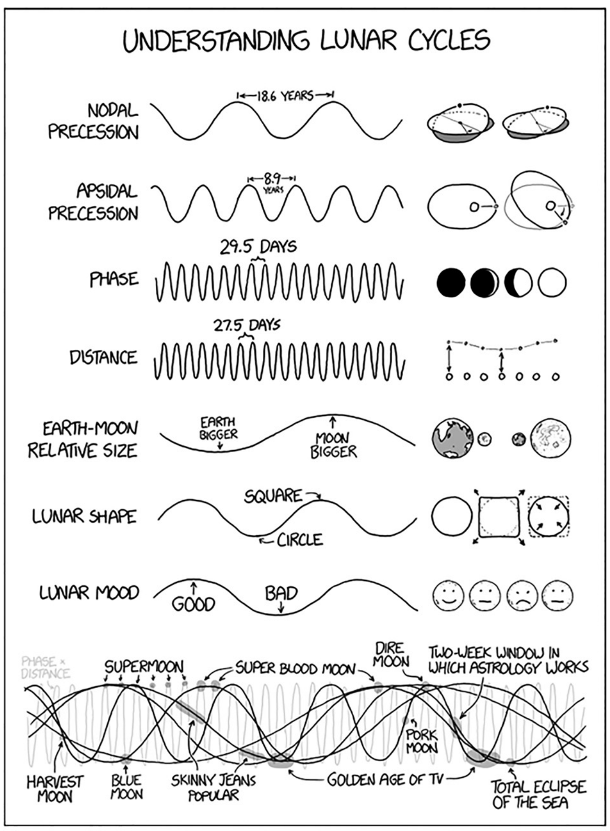

The Antikythera mechanism had a whole set of gears specifically to track the cyclic popularity of skinny jeans and low-rise waists. (Source: https://xkcd.com/2172/)