Novel Principles for Grand Unification Theories

Written by

Sophie Lund Wagner – sowag21@student.sdu.dk

Under the guidance of

Francesco Sannino – sannino@cp3.sdu.dk

Department of Physics, Chemistry and Pharmacy
University of Southern Denmark
Odense, Denmark

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Abstract

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Chapter 1

Mathematical theory

I will write some notes here [1]

1.1 Group theory

1.1.1 Groups

Groups are used as a fundamental mathematical language to describe, understand and predict symmetries and physical phenomena within physics particular in quantum field theory. A group G is a set of elements $\{g\}$ with an operation that assigns to every pair of elements a third element within that same set. The operation must be

1. Closure: If $f, g \in G$ then $h = fg \in G$.
2. Associativity: For all $g,\ h,\ k, \in G$, $(gh)k = g(hk)$.
3. Identity element: There exist an element, $e \in G$ such that for all $f \in G$ we have $ef = fe = f$.
4. Inverse element: For every $f \in G$ there exist an inverse element such that $gg^{-1} = g^{-1}g = e$.

Groups can be interpreted more visual as a a multiplication table, as seen in table **??** (for discrete group elements). An example of this for the group Z_3 see table **??**.

A group can be both finite and infinite. Z_3 is a finite group because it has a finite number of group elements, where an infinite group will have an infinite number of group elements. The number of elements in a finite group is called the order of the group. Thus, Z_3 is of order 3. Groups can also be categorized by being abelian or non-abelian. The operation of an abelian group has an extra axiom, restricting the multiplication law

1. Communicative: For all $g, h \in G$, $gh = hg$.

From the multiplication table for Z_3 **??** it can be seen that the group is abelian as well.

1.1.2 Representation

A representation of a group G is a mapping, D of the elements of the group onto a set of linear operators acting on a vector space V. This makes the abstract group easier to realize, hence representations are matrices (for finite groups). This is useful because without this a mathematical abstract object like a group, can't directly be understood but with a representation of the group, we can realize it using one of the things we understand well: matrices. The group representations are useful because they live in linear space. This also means that we can always change the representation by peforming a linear transformation. The mapping D need to uphold two conditions

- The representation of the identity element $e \in G$ must yield the identity operator in the respective vector space: $D(e) = 1$.
- The groups multiplication law is mirrored by the multiplication of their corresponding linear operators on the vector space: $D(g_1)D(g_2) = D(g_1g_2)$.

An example of this is the 3D rotation group: $SO(3)$. The group has a representation that consist of 3×3 matrices. The dimension of a representation is defined by the dimensions of the space that it acts upon. In this example the dimensions is thus 3. Each group has multiple different representations and they don't have to have the same dimensions. A representation is regular if the dimensions of the representation is the order of the group.

Representations can also be described by being reducible, completely reduicble and irreducible. To understand these definitions, we need to first understand invariant subspaces. An invariant subspace is when all mappings $D(g)$ on any vector w in the subspace $W \subset V$ maps the vector back into the subspace W:

$$D(g)w \in W \quad \forall g \in G, w \in W \ . \tag{1.1}$$

A representaiton is reducible if it has a non-trivial invariant subspace. Non-trivial meaning that it has to be an invariant subspace which is not just $\{0\}$ or the entire set of V. An irreducible representation (irrep) is when the repersentation has no non-trivial invariant subspaces. Thus you cannot "break" the repersentation further down. A completly reducible representation is on block diagonal form and can thus be written as a direct sum of irreducible representations

$$D_1 \oplus D_2 \oplus \cdots \ . \tag{1.2}$$

An important property of finite groups is that their representations always is completely reducible [2].

[Comment: Need to write about unitary repersentations]

1.1.3 Subgroups

A subgroup H is a group with a set of elements which all is a part of a group G. Then H will be a subgroup of G. There are two trivial subgroups of G, namely the identity and the group itself. Subgroups thus capture a smaller symmetry structure inside a larger symmetry. Using subspaces we can define cosets. A coset can both be a right- or a left-coset. A right-coset of a subgroup H in a group G is defined to be all the elements on the form Hg, with $g \in G$. This means that all elements of the subgroup multiplied with a group element to the right is a right-coset. The set which holds all right-cosets of the subgroup is denoted as G/H. This is called a coset-space. From the definition of cosets normal subgroups can be defined as having their righ-coset and left-coset being equal

$$gH = Hg \ . \tag{1.3}$$

1.1.4 Schur's Lemma

We won't prove Schur's Lemma here, mearly state it. Take two irreducible representations D_1 and D_2 of the group G. Now consider a linear map that take one representation to the other

$$T: \ V \to W \ , \tag{1.4}$$

while respecting the group rules. This is also called a homomorphisme of representations.

1.1.5 Characters

A character is the trace of a representation

$$\chi(g) = \text{tr}\, D(g) \ , \tag{1.5}$$

The characters of a group representation are useful for two main reasons. First, using the cyclic property of the trace tr $AB =$ tr BA, means that under a similarity transformation[I] the charchters will not differ. This also means that characters are constant on conjugate classes[II]. Second, character tables are useful to describe the representation quite well. Let us look at an example of S_3. This group is a permutation group of the elements $1, 2, 3$. This means that it has six elements

$$\{e, (12), (13), (23), (123), (132)\} \ , \tag{1.7}$$

and the conjugate classes are easy to find. They preserve the shape of the permutations, so the three groups are

$$C_1 = \{e\}, \ C_2 = \{(12), (13), (23)\}, \ C_3 = \{(123), (132)\} \ . \tag{1.8}$$

The trivial representation D_0 is defined as $D_0(g) = 1 \ \forall \ g$. Thus, this representation has characters $\chi_0(g) = 1$. We know from the number of conjugate classes that there exist three irreducible representations. We also know that the dimensions of the representations depend of the number of elements in the group

$$\sum_a n_a^2 = N \ , \tag{1.9}$$

[I]A similarity transformation of a matrix M is given by

$$M' = PMP^{-1} \ , \tag{1.6}$$

where the matrices M and M' is said to be *similar*. They have the same trace, determinant and eigenvalues but in different bases. [**missing source**]

[II]For $g_1 \in G$ the conjugate class is $\{g^{-1} g_1 g g \in G$, so for a matrix, the conjugate will be a similar matrix.

	e	(12), (13), (23)	(123), (132)
0	1	1	1
1	1	-1	1
2	2	0	-1

in our case we must thus have the dimensions of the two remaining representations to be one and two. The one second one dimensional representations is almost the same as the first, only the sign differs. Even permutations will have $+1$ and odd -1. Therefore, the second conjugate class is odd (only one permutation) and the third one is even (two permutations). Now the two-dimensional trivial representation must have a character with the value 2. To find the remaining characters, the orthogonality:

$$\frac{1}{N}\sum_g \chi_{D_a}(g)^* \chi_{D_b}(g) = \delta_{ab} \tag{1.10}$$

now, using these two equations can be made, and the result gives us the character table for the group.

1.1.6 Young Tableaux

1.1.7 Lie groups

Non-abelian group representations

Lie groups are continuously generated groups, where continuously indicates that using infinitesimal elements of the group a general element can be reached. This also requires, that the group elements are arbitrarily close to the identity. An infinitesimal element can be written

$$g(\alpha) = 1 + i\alpha^a T^a + \mathcal{O}^a \ , \tag{1.11}$$

where T^a is the generators of the group and α^a is the infinitesimal group parameters. The generators span the space of the infinitesimal group transformations and they obey the commutation relation

$$[T^a, T^b] = i f^{abc} T^c \tag{1.12}$$

which is a linear combination of generators. The vector space spanned by the generators are called the Lie algebra [1]. The structure constants f^{abc} obey the Jacobi identity

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0 \ . \tag{1.13}$$

[Comment: der skal skrives noget mere forklarende her, kap 70 og 15.4 i peskin skal bare ind her] Two useful characterstics of representations are the quadratic Casimir operator $C_2(R)$ and the Dynkin index $T(R)$ [3]. The Dynkin index is defined

$$\operatorname{tr} T^a_R T^b_R = T(R)\delta^{ab} \ . \tag{1.14}$$

[Comment: mere mat]

$$\operatorname{tr} T^a_R T^b_R = \frac{1}{2}\delta^{ab} \ . \tag{1.15}$$

[Comment: mere mat] If we now consider anti-commutator relations we have another invariant symbol

$$A(R)d^{abc} \equiv \frac{1}{2}\,\mathrm{tr}\,T^a_R\{T^b_R, T^c_R\}\ , \tag{1.16}$$

with $A(R)$ being the anomaly coefficient [3]. Using the cyclic property of the trace and that $(T^a_R)^i{}_j = -(T^a_{\bar{R}})_j{}^i$, we have

$$A(R) = -A(\bar{R})\ . \tag{1.17}$$

For real or pseudoreal representation the anomaly coefficient is thus $A(R) = 0$.

SU(2)

1.1.8 Roots

1.1.9 Dynkin Diagrams

Chapter 2

Physics theory

2.1 Effective potential

The effective potential is a function which to lowest order in pertubation theory is equal to the classical potential energy and to higher orders would include qunatum corrections [1]. These corrections are generally not finite and owuld require renormalisation. The effective potential can be used for studying spontaneous symmetry breaking, which is based on minimizing the classical potential $V(\phi)$. However, the vacuum at a qunatum level isn't static, but rather fluctuating. Thus, using the effective potetial these qunatum fluctuation is taken intro account [**zee**]. To define the effective potential, we first need to define the effective action. To do this, we consider a scalar field theory ϕ with energy as the connected generating functional $E[J]$

$$Z[J] = e^{-iE[J]} = \int \mathcal{D}\phi\, \phi \exp \left[i \int d^4x (\mathcal{L}[\phi] + J\phi) \right] , \tag{2.1}$$

where both the external source J and the field ϕ depends on the spacetime coordinate x. The equation is in the functional integral formalism. If we translate it to the cannonical formalism we recognize that the right hand side is simply an amplitud on the form

$$\langle \Omega | \exp \left[-iHT \right] | \Omega \rangle , \tag{2.2}$$

where $|\Omega\rangle$ is the vacuum state of the interacting Hamiltonian H, and T is the relevant time interval. Thus, it is simply a vacuum to vacuum amplitude, in the presence of J. So $E[J]$ must represent the vacuum energy [1]. To define an effective action, we first need to compute the expectation value of the field. This holds information on the state of the vacuum and the systems response to external perturbations

$$\phi_c(x) \equiv \frac{\delta E[J]}{\delta J(x)} , \tag{2.3}$$

the expectation value $\phi_c(x)$ in the canonical formalism $\langle \Omega | \phi(x) | \Omega \rangle_J$ is a weighted average of the fluctuations of the vacuum configuration and is sometimes called the classical field [1]. Now evaluating the functional derivative

$$\phi_c(x) = -\frac{1}{Z[J]} \int \mathcal{D}\phi\, \phi(x) \exp \left[i \int d^4x (\mathcal{L}[\phi] + J\phi) \right] = \langle \Omega | \phi(x) | \Omega \rangle_J . \tag{2.4}$$

Now we can perform a Legendre transformation. This is done analogous to statistical mechanics, where the Legendre transformation relates free energy to the energy: $F = E - TS$, where the free energy is a function of temperature T and the energy is a function of entropy S. In this case we have a function E of J and we transform it to the effective action Γ which is a function of ϕ_c[I]

$$\Gamma[\phi_c] = -E[J] - \int d^4y\, J(y)\phi_c(y)\ . \tag{2.5}$$

To better understand the role of the effective action, we take the functional derivative of it with respect to the classical field. This is analogue to the classic action $S[\phi] = \int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$ where varying the action with respect to the field $\phi(x) \to \phi(x) + \delta\phi(x)$ gives the equations of motion

$$\frac{\delta S[\phi]}{\delta\phi} = 0\ . \tag{2.6}$$

Now let us do the same for the quantum corrected action

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -\frac{\delta E[J]}{\delta\phi_c(x)} - \int d^4y\, J(y)\frac{\delta\phi_c(y)}{\delta\phi_c(x)} - \int d^4y\, \frac{\delta J(y)}{\delta\phi_c(x)}\phi_c(y)\ , \tag{2.7}$$

using the definition of the functional derivative **??** and that $E[J]$ depends on the source, we can rewrite the functional derivative in the first term using the chain rule [II]

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -\int d^4y\, \frac{\delta J(y)}{\delta\phi_c(x)}\frac{\delta E[J]}{\delta J(y)} - J(x) - \int d^4y\, \frac{\delta J(y)}{\delta\phi_c(x)}\phi_c(y)\ , \tag{2.8}$$

using the definition of the classical field in Equation 2.5, we obtain

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -\int d^4y\, \frac{\delta J(y)}{\delta\phi_c(x)}(-\phi_x(y)) - J(x) - \int d^4y\, \frac{\delta J(y)}{\delta\phi_c(x)}\phi_c(y) = -J(x)\ . \tag{2.9}$$

Thus, with no external source J, the effective action has the same form as the classic action, when computing the equations of motion. The equation can therefore be interpreted as the quantum equation of motion. Using that the effective action, actually is an action we can expand the functional derivative $\Gamma[\phi_c]$

$$\Gamma[\phi_c] = \int \mathrm{d}^4x\, \mathcal{L}_{\text{eff}}(\phi_c, \partial_\mu\phi_c) = \int \mathrm{d}^4x\, \left[-V_{\text{eff}}(\phi_c) + Z(\phi_c)(\partial_\mu\phi_c)^2 + \ddots\right]\ , \tag{2.10}$$

where the $\mathcal{L}_{\text{eff}}$ is the effective lagrangian, $V_{\text{eff}}(\phi_c)$ is the effective potential, and the dots indicate higher powers of derivatives. Using this expansion of the effective action, the functional derivative can aslo be written as

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -\frac{\delta V_{\text{eff}}(\phi_c)}{\delta\phi_c(x)} frac\delta\delta\phi_c(x) \int \mathrm{d}^4x\, \left[Z(\phi_c)(\partial_\mu\phi_c)^2 + \ddots\right] = -\frac{\delta V_{\text{eff}}(\phi_c)}{\delta\phi_c(x)}\ , \tag{2.11}$$

[I] The difference in parenteses used () and [] refers to the function of or the functional of.

[II] The chain rule for functional derivatives is a generalization of the standard differentiation chain rule $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial g} frac\partial g\partial x$ for a composotite function $f(g(x))$. For a functional derivative the difference is that $g(x) \to g_i(\vec{x})$. This means that f is a functional of g which is a function of x, this is why we consider the continium limit and we have

$$\frac{\delta f[g]}{\delta g(x)} = \int \frac{\delta f[g]}{\delta g(x)}\frac{\delta g(x)}{\delta x(y)}$$

[**<empty citation>**] ***[Comment: rewrite this is not super correct]*** for $\phi_c(x)$ constant in x, and thereby J independent of x. Thus, we can write the effective potential as

$$\frac{\delta V_{\text{eff}}(\phi_c)}{\delta\phi_c(x)} = J(x) \ . \tag{2.12}$$

For the case of no source, the condition that the effective action has an extrenum is converted into a condition on the effective potential to have a minimum ***[Comment: is minimimum correct here? it is not just an extrenum]***

$$\frac{\delta V_{\text{eff}}(\phi_c)}{\delta\phi_c(x)} = 0 \ . \tag{2.13}$$

Consider Equation 2.7 with no source term the vacuum expectation value is determined by minimizing V_{eff} [**zee**].

2.2 The renormalisation group

2.2.1 The Callan-Symanzik Equation

2.2.2 The beta function and anomalous dimensions

An example: Non-Abelian Lagrangian

2.2.3 Fixed point analysis

Banks-Zacks fixed point

So the result is

$$\beta_{g_{1-loop}} = -\alpha_g^2 b_0 = \frac{-g^3}{16\pi^2}\left[\frac{11}{3}C_2(H) - \frac{4}{3}n_f C(R)\right] \ . \tag{2.14}$$

2.3 Chiral theories

We want to consider chiral theories which are completely asymptotically free. Before doing that, we'll briefly consider what a chiral theory is. Chirality comes from the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \ , \tag{2.15}$$

where γ^μ is the gamma matricies. ***[Comment: I use the chiral rep / weyl rep]*** Rewriting the equation using the momentum operator $\not{p} = \gamma^\mu(i\partial_\mu)$ and the 2×2 form of the gamma matricies

$$\left[\begin{pmatrix} 0 & p^0 \\ p^0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\right]\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0 \ . \tag{2.16}$$

From this it can be seen that massless particles come in two types, left- and right-handed and that they do not mix [**for dummies**]. In this case an additional symmetry also applies for the field, namely the chiral symmetry $\psi \rightarrow e^{i\gamma^5\phi}\psi$ [**zee**]. Chiral fermions are fermions which transform differently under the gauge group. This also means that you cannot combine left- and right-handed fermions and form a gauge invariant mass term in the lagrangian. Thus, in chiral theories fermions obtain mass at spontantous symmetry breaking like the Higgs mechanism [1].

To investegate chiral theories and how the gauge coupling in the theory change with energy scale, we want to consider the beta function of the lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + iT_{ij}\sigma^\mu D_\mu \bar{T}^{ij} + i\tilde{F}_k\sigma^\mu D_\mu \bar{\tilde{F}}^k + iF_m\sigma^\mu D_\mu \bar{F}^m \ , \tag{2.17}$$

with Weyl fermions in fundamental F, anti-fundamental $\tilde{F}$, and two-index symmetric and antisymmetric representations. The lagrangian is thus a Yang-Mills gauge theory with Weyl fermions in different representations. The flavor indices on the anti-fundamental and the fundamental representation is $k = 1, 2, \dots, (N \pm 4 + p)$ and $m = 1, 2, \dots, p$. The interval of these indicies will make sense later. The Weyl fermion representations in the lagrangian are from two chiral theory models. The Georgi-Glashow (anti-symmetric two-index representation) and the Bars-Yankielowicz (symmetric two-index representation) models [4].

The beta function for Yang-Mills theories to first loop order found in Equation 2.14

$$\beta_{g_{1-loop}} = \frac{-g^3}{16\pi^2}\left[\frac{11}{3}C_2(H) - \frac{4}{3}n_f C(R)\right] \ , \tag{2.18}$$

this is generic and without a specific fermion representation. First, we want to consider the beta function of the fine structure constant $\alpha_g = \frac{g^2}{16\pi^2}$ instead of the coupling itself

$$\beta_{\alpha_g} = \frac{\mathrm{d}\alpha_g}{\mathrm{d}\ln\mu} = \frac{2g}{16\pi^2}\frac{\mathrm{d}g}{\mathrm{d}\ln\mu} \ . \tag{2.19}$$

Next we can use that we are considering SU(N). This results in the Casimir operator $C_2(H)$ to be equal to N [1].***[Comment: maybe do the calculation earlier]***. Thus, the betafunction can be written as

$$\beta_{\alpha_g} = -\alpha_g^2\left[\frac{11}{3}N - \frac{4}{3}n_f C(R)\right] \ , \tag{2.20}$$

Next lets consider the second term which depends on the number of fermions for each representation. This is where chiral theories differ from vector-like theories. We consider the same fermionic representations as we had in the lagrangian in Equation 2.17. To do so, it is necessary to specify the flavors of each fermionic representation. This depends on gauge anomalies, also called $[\mathrm{SU}(N)]^3$ anomalies [1]. This anomaly appears in triangle diagrams of three gauge currents. For the theory to be valid the anomaly has to vanish. This happens in the SM. For SU$(3)_c$ it is trivial because the theory is vector-like and thus have an equal amount of left- and righthanded fermions in the same representation. However, for SU$(2)_L$ it is not as trivial because it is chiral. For three SU$(2)_L$ gauge currents the anomaly cancels because the representation is pseudoreal. However, the anomaly can still happen between SU$(2)_L$ gauge currents and U(1) or gravity. These combinations still cancel due to the definition of the hypercharge and ***[Comment: why for gravity]*** [1].

Thus, to ensure that our chiral theories are anomaly free, it is important that the different representations are build so their anomaly coefficients $A(r)$ cancel

$$\sum_{fermions} A(r) = \mathcal{A} \stackrel{!}{=} 0 \ , \tag{2.21}$$

For a symmetric representation $A(\Psi_{sym}) = N + 4$ and antisymmetric representation $A(\Psi_{Anti-sym}) = N - 4$. The fundamental and anti-fundamental representation is given in Equation 1.17.

$$\mathcal{A} = N \pm 4 + \bar{p}A(\bar{\Box}) + pA(\Box) = N \pm 4 - \bar{p} + p = 0 \ , \tag{2.22}$$

where $\bar{p}$ is the amount of Weyl fermions transforming in the anti-fundamental representation, that is the number of fermion flavors transforming under an anti-fundamental. p is equivalent just for the fundamental representation. Thus, $\bar{p}$ is restricted to be equal to:

$$\bar{p} = N \pm 4 + p \ . \tag{2.23}$$

Thus, when summing over the Dynkin index for all fermion representations, the anomaly free expression of $\bar{p}$ is used

$$n_f T(R) = T(\Psi) + \bar{p}T(\bar{\Box}) + pT(\Box) = \frac{1}{2}\left(N \pm 2 + N \pm 4 + p + p\right) = N + p \pm 3 \ , \tag{2.24}$$

where $T(\Box) = T(\bar{\Box})$ is given by equation 1.15 and $T(\Psi) = \frac{N \pm 2}{2}$, where $+$ is for symmetric and $-$ is for anti-symmetric representations. ***[Comment: where does this come from]*** Now inserting this in the beta function, remembering that n_f is the number of Dirac fermions, so we need to divide that term by two, when inserting the amount of Weyl fermions

$$\beta_{\alpha_g} = -\alpha_g^2 \left[\frac{22}{3}N - \frac{4}{3}N \mp \frac{12}{3} - \frac{4}{3}p\right] = -\alpha_g^2 \left[\left(6 - \frac{4}{3}x\right) N \mp 4\right] \ , \tag{2.25}$$

where $x = p/N$. From this the fixed point can be found to one loop order by setting the beta function equal to zero as explained in **??**, yielding two solutions

$$\beta_{\alpha_g} = 0 \Rightarrow \alpha_g = 0 \vee 6N - \frac{4}{3}xN \mp 4 = 0 \ , \tag{2.26}$$

where the first solution is the trivial UV fixed point, also called the gaussian fixed point. The second fixed point is for $x = x_{FP} = \frac{9}{2} \mp \frac{3}{N}$ [4] ***[Comment: need to do the fix point analysis - first the veneziano limit takes p and N to infinity but keep x=p/N constant, this makes the explicit chiral factor disappear. I think I need to do the change of variables also for vector like theories to really see the difference.]***.

Chapter 3

Papers

3.1 Paper: Conformal Phase Diagram of Complete Asymptotically Free Theories

The paper approaches CAF's by considering three main cases. Gauge, gauge and Yukawa and lastly gauge, yukawa and scalar interactions. Their approach is generally to write the RG equation on the form:

$$\mu \frac{\mathrm{d}\alpha_H}{\mathrm{d}\mu} = \alpha_H(c_1\alpha_g + c_2\alpha_H) \ . \tag{3.1}$$

where the coefficients (normally quite complicated) are written as a constant. From previous work the paper considers these coeffecients to be positive or negative. Doing this flows of the gauge and Yukawa couplings are plotted. (this is all done to first order where only the gaussian fix point excists.) After this scalars are introduced making it more complex. This gives constrictions on the constants to ensure asymptotic freedom of the couplings. To study IR fix point, the paper considers the gauge coupling to second order. This gives a new fix point for certain conditions on the coeffecients. Lastly, a stability analysis is made using einvalue equations and the jacobi matrix. This is done to investegate whether the fix points are repulsive or attractive. First it is done without scalars then with scalars.

The paper consideres fix point structure of gauge-Yukawa theories. These theories have scalar, gauge and yukawa-self interactions

$$\alpha_g = \frac{g^2}{(4\pi)^2}, \quad \alpha_y = \frac{y^2}{(4\pi)^2}, \quad \alpha_\lambda = \frac{\lambda^2}{(4\pi)^2} \ . \tag{3.2}$$

First considering a system with only a gauge coupling. That is no Yukawa or self couplings. To do this we consider the beta equation

$$\mu \frac{\mathrm{d}\alpha_g}{\mathrm{d}\mu} = b_0\alpha_g^2 \ . \tag{3.3}$$

which describes how a coupling constant, in this case the gauge self coupling, changes with the energy scale μ. The equation here is to first loop order, which is invariant under different choice of renomalisation. The constant in fron b_0 is obtained by solving the Feynmann diagram ***[Comment: ....]***. Further, it tells ***[Comment: need to look more into this]***. Equation 3.3 can simply be solved by isolating the variables on either side and integrating

$$\int_{\alpha_g(\mu_0)}^{\alpha_g(\mu)} \frac{\mathrm{d}\alpha_g}{\alpha_g^2} = b_0 \int_{\mu_0}^{\mu} \frac{\mathrm{d}\mu}{\mu} \iff -\frac{1}{\alpha_{g0}} + \frac{1}{\alpha_g} = b_0 \ln \frac{\mu}{\mu_0} \ , \tag{3.4}$$

where $\alpha_g(\mu_0) = \alpha_{g_0}$ is a fixed scale[I]. Obtaing the expression for the gaug self coupling:

$$\alpha_g = \frac{\alpha_{g_0}}{1 - b_0\alpha_{g_0} \ln(\frac{\mu}{\mu_0})} \ . \tag{3.5}$$

This is asymptotically free for $b_0 < 0$, which is what we want. So from now on we consider only cases where $b_0 < 0$. As $\mu \to \infty$ in Equation 3.5 we approach the UV sector and the trvial UV fix point at $alpha_g \to 0$. It should also be noted that there is unphysical branch ***[Comment: ...]*** Now promoting the system to also having a Yukawa coupling. The renomalisation group equation looks a little different than Equation 3.3 for the Yukawa coupling. Further, it should be noted that the beta equation of the gauge coupling does not change when we add fermion or even scalar interactions, because it is independtent of these. The fermion beta function is ony independtent on the scalar interaction and then gauge gauge interactions but will be affected by the gauge fermion interactions adding an extra term to the function:

$$\mu\frac{\mathrm{d}\alpha_H}{\mathrm{d}\mu} = \alpha_H(c_1\alpha_g + c_2\alpha_H) \ . \tag{3.6}$$

[Comment: Should the H not be a y?] the factors $c_1 < 0$ and $c_2 > 0$ can again be found doing Feynman diagram computations, this time of ***[Comment: ..]***. The c_1 is less than zero, showing the gauge bosons screening properties. Hence the c_2 constant is larger than zero a theory with only a single Yukawa coupling on its own can never be asymptotically free. This can quickly be shown by making the same computation as for the single gauge boson. However, opposite to this case the constant in front is postive. This effects the point at which $\mu = \mu_0 \exp\left(\frac{1}{b_0\alpha_{g_0}}\right)$ for the gauge bosons this wasn't a problem because the pole wpuld be reached in an unphysical branch because α_g would be negative. However, with $b_0 \to c_2$ in the case of a single Yukawa theory it would happen in a physical branch, and the theory would have a Landau pole. After the Landau pole the coupling would get negative and thus be in an unphyscial branch as it approaches the trivial UV fix point. Thus, as stated in the begning of this paragraph, a single Yukawa theory cannot be asymptotically free with out other interactions balacing it out. , it is nesseary to find a theory The beta equation has changed from the previous equation because of the extra interaction

Now having two interactions means that we have a set of coupled differentiel equations to solve. This is becuase for every independtent interaction we will have a beta function, the beta function can depend on all the other cooplings in principal. The solution will be more complicated, but the principle in what we need to analyse is the same. What solutions will be asymptotically free, are there any

[I]To obtain an expression for how the coupling changes with time using the renonormalisation group, we need a fix point, that is a value of the coupling constant and some scale. This is also called the renomalisation scale, and is the required initial conditions possible poles? And then set constraints to avoid poles. Further, we need to ensure that there are physical branches, without negative couplings. This can be plotted together and seen in their Figure 1. The figure depicts the trivial UV fix point (pink), the fixed flow[I] (green) and a asymptotical region below the green line and a non-asymptotical region below the green line. The arrows on the plot point towards the IR region hence they point away from the UV trivial fix point. From these arrows it can be seen that they all orginates from the fix point thus being asymptotically free below the green line. Whereas for above the green line the arrows point towards the uv point and does not point away from it, meaning it is is never reached. Further, regions where the arrows are vertical suggest that the gauge coupling doesn't change when the Yukawa coupling does, this hints a Landau Pole. ***[Comment: unsure about the last part]***

Adding scalar interactions

Lets introduce scalars to the mix. The scalar coupling is not independtent on any of the other interactions ***[Question: Because of lack of constraing symmetries?]***. Thus the beta equation has five terms

$$\mu\frac{\mathrm{d}\alpha_\lambda}{\mathrm{d}\mu} = \alpha_\lambda(d_1\alpha_\lambda + d_2\alpha_g + d_3\alpha_H) + d_4\alpha_g^2 + d_5\alpha_H^2\ , \tag{3.7}$$

where $d_1, d_3, d_4 \geq 0$ and $d_2, d_5 \leq 0$. These relations to zero are again based on Feynman diagram calculations. Solving the three coupled equations for different cases. They obtain a window for the relation of the scalar coupling and the gauge coupling, being constrainted by ensuring no Landau poles at IR or UV.

[Comment: I'm not done writing about all of it]

3.2 Paper: Asymptotically safe and free chiral theories with and without scalars

The paper fouces on two main theories Georgi-Glashow (GG) and Bars-Yankielowicz (BY), both SU(N) theories with two index representations, the GG model han anti-symmetric and the BY symmetric indicies. The paper first considers these models "gauge-fermion" without scalars. This is done to 2nd and 3rd loop order in the gauge coupling. Using Banks-Zacks method they get a constaing value of x=p/N for a asymptotically free theory. Further, they consider the IR fix point or the conformal window for small N. Next the paper considers Meson like scalars (composite particles) and next Higgs like scalars. It is possible to obtain CAF's for both BY and GG for specific values of p and N.

[I] Fixed flow means that the flow between the considered couplings are constant or "fixed" in this case

$$\frac{\alpha_H}{\alpha_g} = \frac{b_0 - c_1}{c_2}$$