



---

# Novel Principles for Grand Unification Theories

---

**Written by**

Sophie Lund Wagner – sowag21@student.sdu.dk

**Under the guidance of**

Francesco Sannino – sannino@cp3.sdu.dk

Department of Physics, Chemistry and Pharmacy  
University of Southern Denmark  
Odense, Denmark  
29th of July 2025

## Abstract

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Etiam lobortis facilisis sem. Nullam nec mi et neque pharetra sollicitudin. Praesent imperdiet mi nec ante. Donec ullamcorper, felis non sodales commodo, lectus velit ultrices augue, a dignissim nibh lectus placerat pede. Vivamus nunc nunc, molestie ut, ultricies vel, semper in, velit. Ut porttitor. Praesent in sapien. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Duis fringilla tristique neque. Sed interdum libero ut metus. Pellentesque placerat. Nam rutrum augue a leo. Morbi sed elit sit amet ante lobortis sollicitudin. Praesent blandit blandit mauris. Praesent lectus tellus, aliquet aliquam, luctus a, egestas a, turpis. Mauris lacinia lorem sit amet ipsum. Nunc quis urna dictum turpis accumsan semper.

# Contents

<b>1 Mathematical theory</b>	<b>5</b>
1.1 Group theory . . . . .	5
1.1.1 Groups . . . . .	5
1.1.2 Representation . . . . .	6
1.1.3 Subgroups . . . . .	6
1.1.4 Schur's Lemma . . . . .	7
1.1.5 Characters . . . . .	7
1.1.6 Young Tableaux . . . . .	8
1.1.7 Lie groups . . . . .	8
1.1.8 Roots . . . . .	9
1.1.9 Dynkin Diagrams . . . . .	9
<b>2 Physics theory</b>	<b>11</b>
2.1 Effective potential . . . . .	11
2.2 The renormalisation group . . . . .	13
2.2.1 The Callan-Symanzik Equation . . . . .	13
2.2.2 The beta function and anomalous dimensions . . . . .	13
2.2.3 Fixed point analysis . . . . .	13
2.3 Chiral theories . . . . .	13
2.3.1 Calculation of the effective potential . . . . .	15
2.4 GG model with one scalar . . . . .	16
2.4.1 Beta functions of the Lagrangian . . . . .	17
2.4.2 BY model with $p=0$ and one scalar . . . . .	17
2.4.3 Beta functions . . . . .	19
2.4.4 CAF . . . . .	19
<b>3 Papers</b>	<b>21</b>
3.1 Paper: Conformal Phase Diagram of Complete Asymptotically Free Theories . . . . .	21
3.2 Paper: Asymptotically safe and free chiral theories with and without scalars . . . . .	23



# Chapter 1

## Mathematical theory

I will write some notes here [1]

### 1.1 Group theory

#### 1.1.1 Groups

Groups are used as a fundamental mathematical language to describe, understand and predict symmetries and physical phenomena within physics particular in quantum field theory. A group  $G$  is a set of elements  $\{g\}$  with an operation that assigns to every pair of elements a third element within that same set. The operation must be

1. Closure: If  $f, g \in G$  then  $h = fg \in G$ .
2. Associativity: For all  $g, h, k \in G$ ,  $(gh)k = g(hk)$ .
3. Identity element: There exist an element,  $e \in G$  such that for all  $f \in G$  we have  $ef = fe = f$ .
4. Inverse element: For every  $f \in G$  there exist an inverse element such that  $gg^{-1} = g^{-1}g = e$ .

Groups can be interpreted more visual as a multiplication table, as seen in table ?? (for discrete group elements). An example of this for the group  $Z_3$  see table ??.

A group can be both finite and infinite.  $Z_3$  is a finite group because it has a finite number of group elements, where an infinite group will have an infinite number of group elements. The number of elements in a finite group is called the order of the group. Thus,  $Z_3$  is of order 3. Groups can also be categorized by being abelian or non-abelian. The operation of an abelian group has an extra axiom, restricting the multiplication law

1. Communicative: For all  $g, h \in G$ ,  $gh = hg$ .

From the multiplication table for  $Z_3$  ?? it can be seen that the group is abelian as well.

### 1.1.2 Representation

A representation of a group  $G$  is a mapping,  $D$  of the elements of the group onto a set of linear operators acting on a vector space  $V$ . This makes the abstract group easier to realize, hence representations are matrices (for finite groups). This is useful because without this a mathematical abstract object like a group, can't directly be understood but with a representation of the group, we can realize it using one of the things we understand well: matrices. The group representations are useful because they live in linear space. This also means that we can always change the representation by performing a linear transformation. The mapping  $D$  need to uphold two conditions

- The representation of the identity element  $e \in G$  must yield the identity operator in the respective vector space:  $D(e) = 1$ .
- The groups multiplication law is mirrored by the multiplication of their corresponding linear operators on the vector space:  $D(g_1)D(g_2) = D(g_1g_2)$ .

An example of this is the 3D rotation group:  $SO(3)$ . The group has a representation that consist of  $3 \times 3$  matrices. The dimension of a representation is defined by the dimensions of the space that it acts upon. In this example the dimensions is thus 3. Each group has multiple different representations and they don't have to have the same dimensions. A representation is regular if the dimensions of the representation is the order of the group.

Representations can also be described by being reducible, completely reducible and irreducible. To understand these definitions, we need to first understand invariant subspaces. An invariant subspace is when all mappings  $D(g)$  on any vector  $w$  in the subspace  $W \subset V$  maps the vector back into the subspace  $W$ :

$$D(g)w \in W \quad \forall g \in G, w \in W. \quad (1.1)$$

A representaiton is reducible if it has a non-trivial invariant subspace. Non-trivial meaning that it has to be an invariant subspace which is not just  $\{0\}$  or the entire set of  $V$ . An irreducible representation (irrep) is when the representation has no non-trivial invariant subspaces. Thus you cannot "break" the representation further down. A completely reducible representation is on block diagonal form and can thus be written as a direct sum of irreducible representations

$$D_1 \oplus D_2 \oplus \dots . \quad (1.2)$$

An important property of finite groups is that their representations always is completely reducible [2].

**[Comment: Need to write about unitary representations]**

### 1.1.3 Subgroups

A subgroup  $H$  is a group with a set of elements which all is a part of a group  $G$ . Then  $H$  will be a subgroup of  $G$ . There are two trivial subgroups of  $G$ , namely the identity and the group itself. Subgroups thus capture a smaller symmetry structure inside a larger symmetry. Using subspaces we

can define cosets. A coset can both be a right- or a left-coset. A right-coset of a subgroup  $H$  in a group  $G$  is defined to be all the elements on the form  $Hg$ , with  $g \in G$ . This means that all elements of the subgroup multiplied with a group element to the right is a right-coset. The set which holds all right-cosets of the subgroup is denoted as  $G/H$ . This is called a coset-space. From the definition of cosets normal subgroups can be defined as having their right-coset and left-coset being equal

$$gH = Hg . \quad (1.3)$$

#### 1.1.4 Schur's Lemma

We won't prove Schur's Lemma here, merely state it. Take two irreducible representations  $D_1$  and  $D_2$  of the group  $G$ . Now consider a linear map that take one representation to the other

$$T : V \rightarrow W , \quad (1.4)$$

while respecting the group rules. This is also called a homomorphism of representations.

#### 1.1.5 Characters

A character is the trace of a representation

$$\chi(g) = \text{tr } D(g) , \quad (1.5)$$

The characters of a group representation are useful for two main reasons. First, using the cyclic property of the trace  $\text{tr } AB = \text{tr } BA$ , means that under a similarity transformation<sup>I</sup> the characters will not differ. This also means that characters are constant on conjugate classes<sup>II</sup>. Second, character tables are useful to describe the representation quite well. Let us look at an example of  $S_3$ . This group is a permutation group of the elements 1, 2, 3. This means that it has six elements

$$\{e, (12), (13), (23), (123), (132)\} , \quad (1.7)$$

and the conjugate classes are easy to find. They preserve the shape of the permutations, so the three groups are

$$C_1 = \{e\}, C_2 = \{(12), (13), (23)\}, C_3 = \{(123), (132)\} . \quad (1.8)$$

The trivial representation  $D_0$  is defined as  $D_0(g) = 1 \forall g$ . Thus, this representation has characters  $\chi_0(g) = 1$ . We know from the number of conjugate classes that there exist three irreducible representations. We also know that the dimensions of the representations depend of the number of elements in the group

$$\sum_a n_a^2 = N , \quad (1.9)$$

---

<sup>I</sup>A similarity transformation of a matrix  $M$  is given by

$$M' = PMP^{-1} , \quad (1.6)$$

where the matrices  $M$  and  $M'$  is said to be *similar*. They have the same trace, determinant and eigenvalues but in different bases. [missing source]

<sup>II</sup>For  $g_1 \in G$  the conjugate class is  $\{g^{-1}g_1gg \in G\}$ , so for a matrix, the conjugate will be a similar matrix.

	e	(12), (13), (23)	(123), (132)
0	1	1	1
1	1	-1	1
2	2	0	-1

in our case we must thus have the dimensions of the two remaining representations to be one and two. The one second one dimensional representations is almost the same as the first, only the sign differs. Even permutations will have +1 and odd -1. Therefore, the second conjugate class is odd (only one permutation) and the third one is even (two permutations). Now the two-dimensional trivial representation must have a character with the value 2. To find the remaining characters, the orthogonality:

$$\frac{1}{N} \sum_g \chi_{D_a}(g)^* \chi_{D_b}(g) = \delta_{ab} \quad (1.10)$$

now, using these two equations can be made, and the result gives us the character table for the group.

### 1.1.6 Young Tableaux

### 1.1.7 Lie groups

#### Non-abelian group representations

Lie groups are continuously generated groups, where continuously indicates that using infinitesimal elements of the group a general element can be reached. This also requires, that the group elements are arbitrarily close to the identity. An infinitesimal element can be written

$$g(\alpha) = 1 + i\alpha^a T^a + \mathcal{O}^a, \quad (1.11)$$

where  $T^a$  is the generators of the group and  $\alpha^a$  is the infinitesimal group parameters. The generators span the space of the infinitesimal group transformations and they obey the commutation relation

$$[T^a, T^b] = if^{abc}T^c \quad (1.12)$$

which is a linear combination of generators. The vector space spanned by the generators are called the Lie algebra [1]. The structure constants  $f^{abc}$  obey the Jacobi identity

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0. \quad (1.13)$$

[Comment: der skal skrives noget mere forklarende her, kap 70 og 15.4 i peskin skal bare ind her] Two useful characteristics of representations are the quadratic Casimir operator  $C_2(R)$  and the Dynkin index  $T(R)$  [3]. The Dynkin index is defined

$$\text{tr } T_R^a T_R^b = T(R) \delta^{ab}. \quad (1.14)$$

[Comment: mere mat]

$$\text{tr } T_R^a T_R^b = \frac{1}{2} \delta^{ab}. \quad (1.15)$$

**[Comment: mere mat]** If we now consider anti-commutator relations we have another invariant symbol

$$A(R)d^{abc} \equiv \frac{1}{2} \text{tr} T_R^a \{T_R^b, T_R^c\} , \quad (1.16)$$

with  $A(R)$  being the anomaly coefficient [3]. Using the cyclic property of the trace and that  $(T_R^a)_j^i = -(T_R^a)^i_j$ , we have

$$A(R) = -A(\bar{R}) . \quad (1.17)$$

For real or pseudoreal representation the anomaly coefficient is thus  $A(R) = 0$ .

## SU(2)

### 1.1.8 Roots

### 1.1.9 Dynkin Diagrams



# Chapter 2

## Physics theory

### 2.1 Effective potential

The effective potential is a function which to lowest order in perturbation theory is equal to the classical potential energy and to higher orders would include quantum corrections [1]. These corrections are generally not finite and would require renormalisation. The effective potential can be used for studying spontaneous symmetry breaking, which is based on minimizing the classical potential  $V(\phi)$ . However, the vacuum at a quantum level isn't static, but rather fluctuating. Thus, using the effective potential these quantum fluctuation is taken into account [zee]. To define the effective potential, we first need to define the effective action. To do this, we consider a scalar field theory  $\phi$  with energy as the connected generating functional  $E[J]$

$$Z[J] = e^{-iE[J]} = \int \mathcal{D}\phi \phi \exp \left[ i \int d^4x (\mathcal{L}[\phi] + J\phi) \right], \quad (2.1)$$

where both the external source  $J$  and the field  $\phi$  depends on the spacetime coordinate  $x$ . The equation is in the functional integral formalism. If we translate it to the canonical formalism we recognize that the right hand side is simply an amplitude on the form

$$\langle \Omega | \exp [-iHT] | \Omega \rangle, \quad (2.2)$$

where  $|\Omega\rangle$  is the vacuum state of the interacting Hamiltonian  $H$ , and  $T$  is the relevant time interval. Thus, it is simply a vacuum to vacuum amplitude, in the presence of  $J$ . So  $E[J]$  must represent the vacuum energy [1]. To define an effective action, we first need to compute the expectation value of the field. This holds information on the state of the vacuum and the systems response to external perturbations

$$\phi_c(x) \equiv \frac{\delta E[J]}{\delta J(x)}, \quad (2.3)$$

the expectation value  $\phi_c(x)$  in the canonical formalism  $\langle \Omega | \phi(x) | \Omega \rangle_J$  is a weighted average of the fluctuations of the vacuum configuration and is sometimes called the classical field [1]. Now evaluating the functional derivative

$$\phi_c(x) = -\frac{1}{Z[J]} \int \mathcal{D}\phi \phi(x) \exp \left[ i \int d^4x (\mathcal{L}[\phi] + J\phi) \right] = \langle \Omega | \phi(x) | \Omega \rangle_J. \quad (2.4)$$

Now we can perform a Legendre transformation. This is done analogous to statistical mechanics, where the Legendre transformation relates free energy to the energy:  $F = E - TS$ , where the free energy is a function of temperature  $T$  and the energy is a function of entropy  $S$ . In this case we have a function  $E$  of  $J$  and we transform it to the effective action  $\Gamma$  which is a function of  $\phi_c$ <sup>1</sup>

$$\Gamma[\phi_c] = -E[J] - \int d^4y J(y) \phi_c(y) . \quad (2.5)$$

To better understand the role of the effective action, we take the functional derivative of it with respect to the classical field. This is analogue to the classic action  $S[\phi] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$  where varying the action with respect to the field  $\phi(x) \rightarrow \phi(x) + \delta\phi(x)$  gives the equations of motion

$$\frac{\delta S[\phi]}{\delta \phi} = 0 . \quad (2.6)$$

Now let us do the same for the quantum corrected action

$$\frac{\delta \Gamma[\phi_c]}{\delta \phi_c(x)} = -\frac{\delta E[J]}{\delta \phi_c(x)} - \int d^4y J(y) \frac{\delta \phi_c(y)}{\delta \phi_c(x)} - \int d^4y \frac{\delta J(y)}{\delta \phi_c(x)} \phi_c(y) , \quad (2.7)$$

using the definition of the functional derivative ?? and that  $E[J]$  depends on the source, we can rewrite the functional derivative in the first term using the chain rule <sup>II</sup>

$$\frac{\delta \Gamma[\phi_c]}{\delta \phi_c(x)} = - \int d^4y \frac{\delta J(y)}{\delta \phi_c(x)} \frac{\delta E[J]}{\delta J(y)} - J(x) - \int d^4y \frac{\delta J(y)}{\delta \phi_c(x)} \phi_c(y) , \quad (2.8)$$

using the definition of the classical field in Equation 2.5, we obtain

$$\frac{\delta \Gamma[\phi_c]}{\delta \phi_c(x)} = - \int d^4y \frac{\delta J(y)}{\delta \phi_c(x)} (-\phi_x(y)) - J(x) - \int d^4y \frac{\delta J(y)}{\delta \phi_c(x)} \phi_c(y) = -J(x) . \quad (2.9)$$

Thus, with no external source  $J$ , the effective action has the same form as the classic action, when computing the equations of motion. The equation can therefore be interpreted as the quantum equation of motion. Using that the effective action, actually is an action we can expand the functional derivative  $\Gamma[\phi_c]$

$$\Gamma[\phi_c] = \int d^4x \mathcal{L}_{\text{eff}}(\phi_c, \partial_\mu \phi_c) = \int d^4x \left[ -V_{\text{eff}}(\phi_c) + Z(\phi_c)(\partial_\mu \phi_c)^2 + \dots \right] , \quad (2.10)$$

where the  $\mathcal{L}_{\text{eff}}$  is the effective lagrangian,  $V_{\text{eff}}(\phi_c)$  is the effective potential, and the dots indicate higher powers of derivatives. Using this expansion of the effective action, the functional derivative can also be written as

$$\frac{\delta \Gamma[\phi_c]}{\delta \phi_c(x)} = -\frac{\delta V_{\text{eff}}(\phi_c)}{\delta \phi_c(x)} - \int d^4x \left[ Z(\phi_c)(\partial_\mu \phi_c)^2 + \dots \right] = -\frac{\delta V_{\text{eff}}(\phi_c)}{\delta \phi_c(x)} , \quad (2.11)$$

<sup>1</sup>The difference in parentheses used () and [] refers to the function of or the functional of.

<sup>II</sup>The chain rule for functional derivatives is a generalization of the standard differentiation chain rule  $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial g} \frac{\partial g}{\partial x}$  for a composito function  $f(g(x))$ . For a functional derivative the difference is that  $g(x) \rightarrow g_i(\vec{x})$ . This means that  $f$  is a functional of  $g$  which is a function of  $x$ , this is why we consider the continuum limit and we have

$$\frac{\delta f[g]}{\delta g(x)} = \int \frac{\delta f[g]}{\delta g(x)} \frac{\delta g(x)}{\delta x(y)}$$

for  $\phi_c(x)$  constant in  $x$ , and thereby  $J$  independent of  $x$ . Thus, we can write the effective potential as

$$\frac{\delta V_{\text{eff}}(\phi_c)}{\delta \phi_c(x)} = J(x) . \quad (2.12)$$

For the case of no source, the condition that the effective action has an extremum is converted into a condition on the effective potential to have a minimum [**Comment: is minimimum correct here? it is not just an extremum**]

$$\frac{\delta V_{\text{eff}}(\phi_c)}{\delta \phi_c(x)} = 0 . \quad (2.13)$$

Consider Equation 2.7 with no source term the vacuum expectation value is determined by minimizing  $V_{\text{eff}}$  [zee]. [**Comment: I think I need an example of how to calculate it**]

## 2.2 The renormalisation group

### 2.2.1 The Callan-Symanzik Equation

### 2.2.2 The beta function and anomalous dimensions

#### An example: Non-Abelian Lagrangian

### 2.2.3 Fixed point analysis

#### Banks-Zacks fixed point

So the result is

$$\beta_{g_{1-loop}} = -\alpha_g^2 b_0 = \frac{-g^3}{16\pi^2} \left[ \frac{11}{3} C_2(H) - \frac{4}{3} n_f C(R) \right] . \quad (2.14)$$

## 2.3 Chiral theories

We want to consider chiral theories which are completely asymptotically free. Before doing that, we'll briefly consider what a chiral theory is. Chirality comes from the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 , \quad (2.15)$$

where  $\gamma^\mu$  is the gamma matrices. [**Comment: I use the chiral rep / weyl rep**] Rewriting the equation using the momentum operator  $\not{p} = \gamma^\mu(i\partial_\mu)$  and the  $2 \times 2$  form of the gamma matrices

$$\left[ \begin{pmatrix} 0 & p^0 \\ p^0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{p} \\ -\vec{\sigma} \cdot \vec{p} & 0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \right] \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0 . \quad (2.16)$$

From this it can be seen that massless particles come in two types, left- and right-handed and that they do not mix [**for dummies**]. In this case an additional symmetry also applies for the field, namely the chiral symmetry  $\psi \rightarrow e^{i\gamma^5 \phi} \psi$  [zee]. Chiral fermions are fermions which transform differently under the gauge group. This also means that you cannot combine left- and right-handed fermions and

form a gauge invariant mass term in the lagrangian. Thus, in chiral theories fermions obtain mass at spontaneous symmetry breaking like the Higgs mechanism [1].

To investigate chiral theories and how the gauge coupling in the theory change with energy scale, we want to consider the beta function of the lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + iT_{ij}\sigma^\mu D_\mu \bar{T}^{ij} + i\tilde{F}_k\sigma^\mu D_\mu \bar{\tilde{F}}^k + iF_m\sigma^\mu D_\mu \bar{F}^m , \quad (2.17)$$

with Weyl fermions in fundamental  $F$ , anti-fundamental  $\bar{F}$ , and two-index symmetric and antisymmetric representations. The lagrangian is thus a Yang-Mills gauge theory with Weyl fermions in different representations. The flavor indices on the anti-fundamental and the fundamental representation is  $k = 1, 2, \dots, (N \pm 4 + p)$  and  $m = 1, 2, \dots, p$ . The interval of these indices will make sense later. The Weyl fermion representations in the lagrangian are from two chiral theory models. The Georgi-Glashow (anti-symmetric two-index representation) and the Bars-Yankielowicz (symmetric two-index representation) models [4].

The beta function for Yang-Mills theories to first loop order found in Equation 2.14

$$\beta_{g_{1-loop}} = \frac{-g^3}{16\pi^2} \left[ \frac{11}{3}C_2(H) - \frac{4}{3}n_f C(R) \right] , \quad (2.18)$$

this is generic and without a specific fermion representation. First, we want to consider the beta function of the fine structure constant  $\alpha_g = \frac{g^2}{16\pi^2}$  instead of the coupling itself

$$\beta_{\alpha_g} = \frac{d\alpha_g}{d \ln \mu} = \frac{2g}{16\pi^2} \frac{dg}{d \ln \mu} . \quad (2.19)$$

Next we can use that we are considering  $SU(N)$ . This results in the Casimir operator  $C_2(H)$  to be equal to  $N$  [1]. **[Comment: maybe do the calculation earlier]**. Thus, the betafunction can be written as

$$\beta_{\alpha_g} = -\alpha_g^2 \left[ \frac{11}{3}N - \frac{4}{3}n_f C(R) \right] , \quad (2.20)$$

Next lets consider the second term which depends on the number of fermions for each representation. This is where chiral theories differ from vector-like theories. We consider the same fermionic representations as we had in the lagrangian in Equation 2.17. To do so, it is necessary to specify the flavors of each fermionic representation. This depends on gauge anomalies, also called  $[SU(N)]^3$  anomalies [1]. This anomaly appears in triangle diagrams of three gauge currents. For the theory to be valid the anomaly has to vanish. This happens in the SM. For  $SU(3)_c$  it is trivial because the theory is vector-like and thus have an equal amount of left- and righthanded fermions in the same representation. However, for  $SU(2)_L$  it is not as trivial because it is chiral. For three  $SU(2)_L$  gauge currents the anomaly cancels because the representation is pseudoreal. However, the anomaly can still happen between  $SU(2)_L$  gauge currents and  $U(1)$  or gravity. These combinations still cancel due to the definition of the hypercharge and **[Comment: why for gravity]** [1]. Thus, to ensure that our chiral theories are anomaly free, it is important that the different representations are build so their anomaly coefficients  $A(r)$  cancel

$$\sum_{fermions} A(r) = \mathcal{A} \stackrel{!}{=} 0 , \quad (2.21)$$

For a symmetric representation  $A(\Psi_{sym}) = N + 4$  and antisymmetric representation  $A(\Psi_{Anti-sym}) = N - 4$ . The fundamental and anti-fundamental representation is given in Equation 1.17.

$$\mathcal{A} = N \pm 4 + \bar{p}A(\bar{\square}) + pA(\square) = N \pm 4 - \bar{p} + p = 0 , \quad (2.22)$$

where  $\bar{p}$  is the amount of Weyl fermions transforming in the anti-fundamental representation, that is the number of fermion flavors transforming under an anti-fundamental.  $p$  is equivalent just for the fundamental representation. Thus,  $\bar{p}$  is restricted to be equal to:

$$\bar{p} = N \pm 4 + p . \quad (2.23)$$

Thus, when summing over the Dynkin index for all fermion representations, the anomaly free expression of  $\bar{p}$  is used

$$n_f T(R) = T(\Psi) + \bar{p}T(\bar{\square}) + pT(\square) = \frac{1}{2} (N \pm 2 + N \pm 4 + p + p) = N + p \pm 3 , \quad (2.24)$$

where  $T(\square) = T(\bar{\square})$  is given by equation 1.15 and  $T(\Psi) = \frac{N \pm 2}{2}$ , where  $+$  is for symmetric and  $-$  is for anti-symmetric representations. **[Comment: where does this come from]** Now inserting this in the beta function, remembering that  $n_f$  is the number of Dirac fermions, so we need to divide that term by two, when inserting the amount of Weyl fermions

$$\beta_{\alpha_g} = -\alpha_g^2 \left[ \frac{22}{3}N - \frac{4}{3}N \mp \frac{12}{3} - \frac{4}{3}p \right] = -\alpha_g^2 \left[ \left( 6 - \frac{4}{3}x \right) N \mp 4 \right] , \quad (2.25)$$

where  $x = p/N$ . From this the fixed point can be found to one loop order by setting the beta function equal to zero as explained in ??, yielding two solutions

$$\beta_{\alpha_g} = 0 \Rightarrow \alpha_g = 0 \vee 6N - \frac{4}{3}xN \mp 4 = 0 , \quad (2.26)$$

where the first solution is the trivial UV fixed point, also called the gaussian fixed point. The second fixed point is for  $x = x_{FP} = \frac{9}{2} \mp \frac{3}{N}$  [4] **[Comment: need to do the fix point analysis - first the veneziano limit takes p and N to infinity but keep x=p/N constant, this makes the explicit chiral factor disappear. I think I need to do the change of variables also for vector like theories to really see the difference.]**

### 2.3.1 Calculation of the effective potential

$$V_1(h, s) = \sum_i \pm n_i \frac{m_i^4(h, s)}{64\pi^2} \left( \log \left[ \frac{m_i^2(h, s)}{Q^2} \right] - C_i \right) \quad (2.27)$$

and

$$V_{\text{eff}}(\varphi) = \frac{\lambda}{4!} \varphi^4 + \frac{\lambda^2 \varphi^4}{256\pi^2} \left( \log \frac{\varphi^2}{Q^2} - \frac{25}{6} \right) + O(\lambda^3) \quad (2.28)$$

For  $\ln 2/\lambda = 8/3$  and  $m^2 = \lambda\phi^2/2$

Fields	$[SU(N)]$	$SU(N - 4)$	$U(1)$
$\Psi$	$\square \square$	1	$N - 4$
$\psi$	$\bar{\square}$	$\bar{\square}$	$-(N - 2)$
$\phi$	$\bar{\square}$	1	2

Table 2.1: OBS squares need to be ontop of each other

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} + i\bar{\Psi}_{ij}\not{D}\Psi_{ij} + i\bar{\psi}_i^a \not{D}\psi_i^a + (D_\mu\phi^* D^\mu\phi) + (y^a\Psi_{ij}\psi_i^a H^j + h.c.) + \lambda H_i^\dagger H_i H_j^\dagger H_j .$$

## 2.4 GG model with one scalar

$$SU(N) \times SU(N - 4) \times U(1) \quad (2.29)$$

where the transformation of the scalar, anti-fundamental fermion, and two index antisymmetric fermion under each group is denoted in Table ???. To write the Lagrangian down, we need to ensure that all terms will be invariant under  $U(1)$  and that no indices remains open. To do this we start by writing down the kinetic terms for all particles. Let us first consider the free theory:

$$\mathcal{L}_{free} = i\bar{\Psi}_{ij}\sigma^\mu\partial_\mu\Psi_{ij} + i\bar{\psi}_i^a\sigma^\mu\partial_\mu\psi_i^a + \partial_\mu(\phi^*)_i\partial^\mu\phi_i , \quad (2.30)$$

where  $\sigma^\mu$  is used because the fermions are chiral and thus are massless. Next, we want to consider the interacting theory, which we do by promoting the global  $U(1)$  symmetry of the Lagrangian to a local symmetry. To ensure that ... we introduce the covariant derivative  $\partial_\mu \rightarrow D_\mu = \partial_\mu - ie(T^a)_j^i A_\mu^a$

$$\mathcal{L}_{kinetic} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} + i\bar{\Psi}_{ij}\not{D}\Psi_{ij} + i\bar{\psi}_i^a \not{D}\psi_i^a + (D_\mu\phi^* D^\mu\phi) , \quad (2.31)$$

where  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$ , with  $A_\mu$  being the connection. This gives us our first coupling constant,  $g$ . If we write out the first term, we obtain

$$F_{\mu\nu}^a F^{\mu\nu a} \approx (\partial_\mu A_\nu^a)^2 + gf^{abc}(\partial_\mu A_\nu^a)A_\mu^b A_\nu^c + g^2 f^{abc}f^{def}A_\mu^b A_\nu^c A_\mu^d A_\nu^e , \quad (2.32)$$

which gives us the gauge boson propagator and the two gauge bosons vertices [Comment: feynman diagrams] The three remaining kinetic terms gives us the scalar and two fermion propagators [Comment: feynman diagram] Next, we consider all the possible interaction terms, which we can write in the Lagrangian. Consider first, the possible Yukawa interactions. Terms like  $\Psi_{ij}\bar{A}_{ij}H^i$  and  $\psi\bar{\psi}H^i$  are not invariant under  $U(1)$ . However, mixing the two fermion representations, is possible, this however gives an open gauge index from the anti-fundamental fermion. To fix this we take the coupling with the same gauge index, giving us  $2a = 2(N - 4)$  Yukawa terms including the hermitian conjugated terms

$$\mathcal{L}_{Yukawa} = y^a\Psi_{ij}\psi_i^a H^j + h.c. , \quad (2.33)$$

Lastly, we have to consider possible quartic scalar terms. Again, ensuring  $U(1)$  invariant terms, we can write terms like

$$\mathcal{L}_{quartic} = \lambda_1(H_i^\dagger H_j)^2 + \lambda_2(\bar{\Psi}_{ij}\Psi_{ij})^2 + \lambda_3(\bar{\psi}_i\psi_j)^2 + \lambda_4(H_i^\dagger H_j)(\bar{\psi}_i\psi_j) + \lambda_5(H_i^\dagger H_j)(\bar{\Psi}_{ij}\Psi_{ij}) + \lambda_2(\bar{\psi}_i\psi_j)(\bar{\Psi}_{ij}\Psi_{ij}) + \lambda_2(\bar{\Psi}_{ij}\Psi_{ij}) \quad (2.34)$$

and similar terms. However, we will impose another condition namely that the theory needs to be renormalizable. This means that the coupling, shouldn't have negative dimensions. The quartic scalar lagrangian thus simplifies to one term

$$\mathcal{L}_{quartic} = \lambda H_i^\dagger H_i H_j^\dagger H_j , \quad (2.35)$$

thus we also have one scalar coupling, with the coupling constant  $\lambda$ . From the scalar and the yukawa terms we obtain three new vertices. The full Lagrangian thus reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + i\bar{\Psi}_{ij} \not{D} \Psi_{ij} + i\bar{\psi}_i^a \not{D} \psi_i^a + (D_\mu \phi^* D^\mu \phi)^2 + (y^a \Psi_{ij} \psi_i^a H^j + h.c.) + \lambda H_i^\dagger H_i H_j^\dagger H_j . \quad (2.36)$$

### 2.4.1 Beta functions of the Lagrangian

To analyze whether the theory is asymptotically free, we will use the Callan-Symanzik Equation ???. The beta function will first be computed for the gauge coupling

$$\beta(\alpha_g) \mu \frac{d\alpha_g}{d \ln \mu} = b_0 \alpha_g^2 , \quad (2.37)$$

where  $\alpha_g = \frac{g^2}{4\pi}$  and  $\mu$  is an arbitrary energy-scale. The beta equation for the gauge interaction can be computed by calculating the two, one-loop diagrams [Comment: insert feynman diagrams that contribute]. To compute this we first need to renormalize the Lagrangian.

For the gauge coupling the betafunction is thus (without scalars)

$$\beta(g) = -\frac{g^3}{(4\pi)^2} \left[ \frac{11}{3} C_2(G) - \frac{2}{6} T(R) \right] , \quad (2.38)$$

This took quite a lot of computation, and I realized so would the remaining beta functions. I then chose to use a numerical approach, the beta function with scalars was then looked up:

$$\beta(g) = -\frac{g^3}{(4\pi)^2} \left[ \frac{11}{3} C_2(G) - \frac{2}{6} T(F) - \frac{1}{3} T(S) \right] , \quad (2.39)$$

### 2.4.2 BY model with p=0 and one scalar

Fields	$[SU(N)]$	$SU(N-4)$	$U(1)$
$\chi$	$\square\square$	$1$	$N+4$
$\psi$	$\bar{\square}$	$\bar{\square}$	$-(N+2)$
$\phi$	$\bar{\square}$	$1$	$-2$

Table 2.2: caption

Considering a chiral model with scalars, we now focus on the Bars-Yankielowicz model. As seen in table 2.2, the main difference from the GG model is that instead of having a two index antisymmetric

representation, we have a two-index symmetric representation. That is the fermionic field  $\chi_{ij} = \chi_{ji}$ . The lagrangian is similar to the one of GG and can be expressed as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} + i\bar{\chi}^{ij}\not{D}\chi_{ij} + i\bar{\psi}_i{}_a\not{D}\psi_i^a + (D_\mu\phi^*{}^i)(D^\mu\phi_i) + (y^a\chi^{ij}\psi_i^a\phi_j + h.c.) + \lambda(\phi_i^\dagger\phi^i)(\phi_j^\dagger\phi^j) .$$

There is another scalar quartic term however for (anti-)fundamental representations it is not linearly independent on the quartic scalar term in the lagrangian. Let us start by proving that

$$(\phi_i^\dagger T_i^{Aj}\phi^j)(\phi_k^\dagger T_k^{Al}\phi^l) , \quad (2.40)$$

where the generators  $T^A$  upholds the Fierz identity

$$T_i^{Aj}T_k^{Al} = \frac{1}{2} \left( \delta_i^l \delta_k^j - \frac{1}{N} \delta_i^j \delta_k^l \right) . \quad (2.41)$$

The Fierz identity can be shown to be true for fundamental representations of  $SU(N)$  by considering ...The generators of  $SU(N)$  can be written as  $n \times n$  traceless Hermitian matrices  $T^a$ . The noramlisation of theses matrices is first defined to be

$$\text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab} , \quad (2.42)$$

using this we can write a general matrix  $M$  as a linear combination of the identity matrix and the generators of the Lie algebra

$$M = n_0 I + n_a T^a , \quad (2.43)$$

where  $n_0$  and  $n_a$  are coefficients. Now let us take the trace on both sides and simplify using that  $\text{tr}(T^a) = 0$  and  $\text{tr}(I) = N$

$$\text{tr}(M) = n_0 \text{tr}(I) + n_a \text{tr}(T^a) = n_0 N . \quad (2.44)$$

Next using the definition of the normalisation of the generators

$$\text{tr}(MT^b) = n_0 \text{tr}(IT^b) + n_a \text{tr}(T^a T^b) = \frac{n_b}{2} . \quad (2.45)$$

Now using Equation ??, the constants  $n_0, n_a$  can be replaced, and the matrix  $M$  can be written as

$$M = \frac{\text{tr}(M)}{N} I + 2 \text{tr}(MT^a) T^a , \quad (2.46)$$

the matrix decomposition can be written in index notation

$$M_{ij} = \frac{\text{tr}(M)}{N} \delta_{ij} + 2 \text{tr}(MT^a) T_{ij}^a , \quad (2.47)$$

Using that it is in Euclidian space with the metric  $\delta_{ij}$ , the traces cab be simplified, and the lhs can be rewritten

$$\delta_{jk}\delta_{il}M_{lk} = \frac{1}{N} \delta_{kl}M_{lk}\delta_{ij} + 2\delta_{ln}M_{lk}T_{kn}^a T_{ij}^a , \quad (2.48)$$

which can be simplified to

$$\delta_{jk}\delta_{il} = \frac{1}{N} \delta_{kl}\delta_{ij} + 2T_{kl}^a T_{ij}^a , \quad (2.49)$$

Isolating

$$T_{kl}^a T_{ij}^a = \frac{1}{2} \left( \delta_{jk} \delta_{il} - \frac{1}{N} \delta_{kl} \delta_{ij} \right) \quad (2.50)$$

Inserting this back into the term gives us

$$\frac{1}{2} (\phi_i^\dagger \phi^j)(\phi_k^\dagger \phi^l) \left( \delta_i^l \delta_k^j - \frac{1}{N} \delta_i^j \delta_k^l \right) = \frac{1}{2} \left[ \phi_i^\dagger \phi^j \phi_j^\dagger \phi^i - \frac{1}{N} \phi_i^\dagger \phi^i \phi_k^\dagger \phi^i k \right] = \frac{1}{2} \left( 1 - \frac{1}{N} \right) (\phi_i^\dagger \phi^i)^2, \quad (2.51)$$

so it is not linearly independent on the quartic term in the lagrangian.

### 2.4.3 Beta functions

For the couplings defined as

$$\alpha_g = \frac{g^2}{(4\pi)^2}, \quad \alpha_y = \frac{y^2}{(4\pi)^2}, \quad \alpha_\lambda = \frac{\lambda}{(4\pi)^2}$$

For GG model:

$$\beta_{\alpha_g} = -\frac{11 + 18N}{3} \alpha_g^2 \quad (2.52)$$

$$\beta_{\alpha_y} = \frac{\alpha_y}{2} \left[ (3N - 1)\alpha_y + 6\alpha_g \left( -3N + 2 + \frac{5}{N} \right) \right] \quad (2.53)$$

$$\beta_{\alpha_\lambda} = \alpha_\lambda \left[ 2(N + 1)\alpha_y - \frac{6\alpha_g (N^2 - 1)}{N} + 4\alpha_\lambda (N + 4) \right] - \frac{1}{2}(N + 3)\alpha_y^2 + \frac{3\alpha_g^2 (N^3 + N^2 - 4N + 2)}{4N^2} \quad (2.54)$$

For BY model:

$$\beta_{\alpha_g} = -(1 + 6N)\alpha_g^2 \quad (2.55)$$

$$\beta_{\alpha_y} = \frac{\alpha_y}{2} \left[ (3N + 5)\alpha_y + 6\alpha_g \left( -3N - 2 + \frac{5}{N} \right) \right] \quad (2.56)$$

$$\beta_{\alpha_\lambda} = \alpha_\lambda \left[ 2(N + 1)\alpha_y - \frac{6\alpha_g (N^2 - 1)}{N} + 4\alpha_\lambda (N + 4) \right] - \frac{1}{2}(N + 3)\alpha_y^2 + \frac{3\alpha_g^2 (N^3 + N^2 - 4N + 2)}{4N^2} \quad (2.57)$$

For large N both models reduce to

$$\beta_{\alpha_g} \approx -6N\alpha_g^2 \quad (2.58)$$

$$\beta_{\alpha_y} \approx N\alpha_y \left( -9\alpha_g + \frac{3}{2}\alpha_y \right) \quad (2.59)$$

$$\beta_{\alpha_\lambda} \approx N \left( \alpha_\lambda [2\alpha_y - 6\alpha_g + 4\alpha_\lambda] - \frac{1}{2}\alpha_y^2 + \frac{3}{4}\alpha_g^2 \right) \quad (2.60)$$

### 2.4.4 CAF

After the fixed point analysis we have the following conditions when the Yukawa and gauge coupling is not on fixed flow:

$$b_0 < 0, \quad b_0 - c_1 > 0, \quad (b_0 - d_2)^2 - 4d_1d_4 \geq 0, \quad b_0 - d_2 + \sqrt{(b_0 - d_2)^2 - 4d_1d_4} > 0 \quad (2.61)$$



# Chapter 3

## Papers

### 3.1 Paper: Conformal Phase Diagram of Complete Asymptotically Free Theories

The paper approaches CAF's by considering three main cases. Gauge, gauge and Yukawa and lastly gauge, yukawa and scalar interactions. Their approach is generally to write the RG equation on the form:

$$\mu \frac{d\alpha_H}{d\mu} = \alpha_H(c_1\alpha_g + c_2\alpha_H) . \quad (3.1)$$

where the coefficients (normally quite complicated) are written as a constant. From previous work the paper considers these coefficients to be positive or negative. Doing this flows of the gauge and Yukawa couplings are plotted. (this is all done to first order where only the gaussian fix point exists.) After this scalars are introduced making it more complex. This gives constrictions on the constants to ensure asymptotic freedom of the couplings. To study IR fix point, the paper considers the gauge coupling to second order. This gives a new fix point for certain conditions on the coefficients. Lastly, a stability analysis is made using eigenvalue equations and the jacobian matrix. This is done to investigate whether the fix points are repulsive or attractive. First it is done without scalars then with scalars.

The paper considers fix point structure of gauge-Yukawa theories. These theories have scalar, gauge and yukawa-self interactions

$$\alpha_g = \frac{g^2}{(4\pi)^2}, \quad \alpha_y = \frac{y^2}{(4\pi)^2}, \quad \alpha_\lambda = \frac{\lambda^2}{(4\pi)^2} . \quad (3.2)$$

First considering a system with only a gauge coupling. That is no Yukawa or self couplings. To do this we consider the beta equation

$$\mu \frac{d\alpha_g}{d\mu} = b_0\alpha_g^2 . \quad (3.3)$$

which describes how a coupling constant, in this case the gauge self coupling, changes with the energy scale  $\mu$ . The equation here is to first loop order, which is invariant under different choice of renormalisation. The constant in front  $b_0$  is obtained by solving the Feynmann diagram [Comment: ....]. Further, it

tells [Comment: need to look more into this]. Equation 3.3 can simply be solved by isolating the variables on either side and integrating

$$\int_{\alpha_g(\mu_0)}^{\alpha_g(\mu)} \frac{d\alpha_g}{\alpha_g^2} = b_0 \int_{\mu_0}^{\mu} \frac{d\mu}{\mu} \iff -\frac{1}{\alpha_{g0}} + \frac{1}{\alpha_g} = b_0 \ln \frac{\mu}{\mu_0}, \quad (3.4)$$

where  $\alpha_g(\mu_0) = \alpha_{g0}$  is a fixed scale<sup>1</sup>. Obtaining the expression for the gauge self coupling:

$$\alpha_g = \frac{\alpha_{g0}}{1 - b_0 \alpha_{g0} \ln(\frac{\mu}{\mu_0})}. \quad (3.5)$$

This is asymptotically free for  $b_0 < 0$ , which is what we want. So from now on we consider only cases where  $b_0 < 0$ . As  $\mu \rightarrow \infty$  in Equation 3.5 we approach the UV sector and the trivial UV fix point at  $\alpha_g \rightarrow 0$ . It should also be noted that there is unphysical branch [Comment: ...] Now promoting the system to also having a Yukawa coupling. The renormalisation group equation looks a little different than Equation 3.3 for the Yukawa coupling. Further, it should be noted that the beta equation of the gauge coupling does not change when we add fermion or even scalar interactions, because it is independent of these. The fermion beta function is only independent on the scalar interaction and then gauge gauge interactions but will be affected by the gauge fermion interactions adding an extra term to the function:

$$\mu \frac{d\alpha_H}{d\mu} = \alpha_H (c_1 \alpha_g + c_2 \alpha_H). \quad (3.6)$$

[Comment: Should the H not be a y?] the factors  $c_1 < 0$  and  $c_2 > 0$  can again be found doing Feynman diagram computations, this time of [Comment: ...]. The  $c_1$  is less than zero, showing the gauge bosons screening properties. Hence the  $c_2$  constant is larger than zero a theory with only a single Yukawa coupling on its own can never be asymptotically free. This can quickly be shown by making the same computation as for the single gauge boson. However, opposite to this case the constant in front is positive. This effects the point at which  $\mu = \mu_0 \exp\left(\frac{1}{b_0 \alpha_{g0}}\right)$  for the gauge bosons this wasn't a problem because the pole would be reached in an unphysical branch because  $\alpha_g$  would be negative. However, with  $b_0 \rightarrow c_2$  in the case of a single Yukawa theory it would happen in a physical branch, and the theory would have a Landau pole. After the Landau pole the coupling would get negative and thus be in an unphysical branch as it approaches the trivial UV fix point. Thus, as stated in the beginning of this paragraph, a single Yukawa theory cannot be asymptotically free without other interactions balancing it out. , it is necessary to find a theory. The beta equation has changed from the previous equation because of the extra interaction

Now having two interactions means that we have a set of coupled differential equations to solve. This is because for every independent interaction we will have a beta function, the beta function can depend on all the other couplings in principle. The solution will be more complicated, but the principle in what we need to analyse is the same. What solutions will be asymptotically free, are there any

---

<sup>1</sup>To obtain an expression for how the coupling changes with time using the renormalisation group, we need a fix point, that is a value of the coupling constant and some scale. This is also called the renormalisation scale, and is the required initial conditions

possible poles? And then set constraints to avoid poles. Further, we need to ensure that there are physical branches, without negative couplings. This can be plotted together and seen in their Figure 1. The figure depicts the trivial UV fix point (pink), the fixed flow<sup>1</sup> (green) and a asymptotical region below the green line and a non-asymptotical region below the green line. The arrows on the plot point towards the IR region hence they point away from the UV trivial fix point. From these arrows it can be seen that they all orginates from the fix point thus being asymptotically free below the green line. Whereas for above the green line the arrows point towards the uv point and does not point away from it, meaning it is never reached. Further, regions where the arrows are vertical suggest that the gauge coupling doesn't change when the Yukawa coupling does, this hints a Landau Pole. [Comment: unsure about the last part]

### Adding scalar interactions

Lets introduce scalars to the mix. The scalar coupling is not independent on any of the other interactions [Question: Because of lack of constraining symmetries?]. Thus the beta equation has five terms

$$\mu \frac{d\alpha_\lambda}{d\mu} = \alpha_\lambda(d_1\alpha_\lambda + d_2\alpha_g + d_3\alpha_H) + d_4\alpha_g^2 + d_5\alpha_H^2 , \quad (3.7)$$

where  $d_1, d_3, d_4 \geq 0$  and  $d_2, d_5 \leq 0$ . These relations to zero are again based on Feynman diagram calculations. Solving the three coupled equations for different cases. They obtain a window for the relation of the scalar coupling and the gauge coupling, being constrained by ensuring no Landau poles at IR or UV.

[Comment: I'm not done writing about all of it]

## 3.2 Paper: Asymptotically safe and free chiral theories with and without scalars

The paper fouces on two main theories Georgi-Glashow (GG) and Bars-Yankielowicz (BY), both  $SU(N)$  theories with two index representations, the GG model han anti-symmetric and the BY symmetric indicies. The paper first considers these models "gauge-fermion" without scalars. This is done to 2nd and 3rd loop order in the gauge coupling. Using Banks-Zacks method they get a constaing value of  $x=p/N$  for a asymptotically free theory. Further, they consider the IR fix point or the conformal window for small  $N$ . Next the paper considers Meson like scalars (composite particles) and next Higgs like scalars. It is possible to obtain CAF's for both BY and GG for specific values of  $p$  and  $N$ .

---

<sup>1</sup>Fixed flow means that the flow between the considered couplings are constant or "fixed" in this case

$$\frac{\alpha_H}{\alpha_g} = \frac{b_0 - c_1}{c_2}$$