Columbia University
IEOR 4732: Computational Methods in Finance

Case Study 3

1 Descriptions of Stochastic Volatility Models

1-1 Geometric Brownian Motion with Stochastic Arrival (GBMSA) – Heston SVM

In Heston stochastic volatility model, stock price follows the process:

$$\begin{aligned} dS_t &= (r-q)S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^{(2)}, \end{aligned}$$

where the two Brownian components $W_t^{(1)}$ and $W_t^{(2)}$ are correlated with rate ρ. The parameters κ, θ, and σ have certain physical meanings: κ is the mean reversion speed, θ is the long run variance, and σ is the volatility of the volatility. The characteristic function for the log of stock price process as shown on Page 18 is given by

$$\begin{aligned} \phi(u) &= \mathbb{E}(e^{iu\ln S_t}) \\ &= \frac{\exp\{iu\ln S_0 + i(r-q)tu + \frac{\kappa\theta t(\kappa - i\rho\sigma u)}{\sigma^2}\}}{(\cosh\frac{\gamma t}{2} + \frac{\kappa - i\rho\sigma u}{\gamma}\sinh\frac{\gamma t}{2})^{\frac{2\kappa\theta}{\sigma^2}}} \exp\left\{-\frac{(u^2+iu)v_0}{\gamma\coth\frac{\gamma t}{2} + \kappa - i\rho\sigma u}\right\} \end{aligned}$$

where $\gamma = \sqrt{\sigma^2(u^2+iu) + (\kappa - i\rho\sigma u)^2}$, and S_0 and v_0 are the initial values for the price process and the volatility process, respectively. Derivation of Heston characteristic function is given in the notes. With a closed form for Heston characteristic function for the log price, one can employ various techniques to price European options as done in Case Study 1. The resulting model may be used to estimate parameter values consistent with market option prices for vanilla options across the entire strike and maturity spectrum. The Heston parameter set is $\Theta_{\text{Heston}} = \{\kappa, \theta, \sigma, \rho, v_0\}$

1-2 Variance Gamma with Stochastic Arrival (VGSA) Process

To obtain the VGSA process, we take the VG process which is a homogeneous Lévy process and build in stochastic volatility by evaluating it at a continuous time change given by the integral of a Cox, Ingersoll and Ross (CIR) process. The mean reversion of the CIR process introduces the clustering phenomena often referred to as volatility persistence. This enables us to calibrate to market price surfaces that go across strike and maturity simultaneously. This process is tractable in the analytical expressions for its characteristic function. Formally we define the CIR process $y(t)$ as the solution to the stochastic differential equation

$$dy_t = \kappa(\eta - y_t)dt + \lambda\sqrt{y_t}dW_t$$

where $W(t)$ is a Brownian motion, η is the long-term rate of time change, κ is the rate of mean reversion, and λ is the volatility of the time change. The process $y(t)$ is the instantaneous rate of time change and so the time change is given by $Y(t)$ where

$$Y(t) = \int_0^t y(u)du$$

The SDE of the log of the market variable is the same as the VG process with the above time change. The characteristic function for the time change $Y(t)$ is given by

$$\begin{aligned} \mathbb{E}(e^{iuY(t)}) &= \phi(u, t, y(0), \kappa, \eta, \lambda) \\ &= A(t, u)e^{B(t,u)y(0)} \end{aligned}$$

where

$$\begin{aligned} A(t, u) &= \frac{\exp\left(\frac{\kappa^2\eta t}{\lambda^2}\right)}{\left(\cosh(\gamma t/2) + \frac{\kappa}{\gamma}\sinh(\gamma t/2)\right)^{2\kappa\eta/\lambda^2}} \\ B(t, u) &= \frac{2iu}{\kappa + \gamma\coth(\gamma t/2)} \end{aligned}$$

with

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2 iu}$$

The stochastic volatility Lévy process, termed the VGSA process, is defined by

$$Z_{VGSA}(t) = X_{VG}(Y(t); \sigma, \nu, \theta)$$

where σ, ν, θ, κ, η, and λ are the six parameters defining the process. Its characteristic function is given by

$$\mathbb{E}(e^{iuZ_{VGSA}(t)}) = \phi(-i\Psi_{VG}(u), t, \frac{1}{\nu}, \kappa, \eta, \lambda)$$

where Ψ_{VG} is the log characteristic function of the variance gamma process at unit time, namely,

$$\Psi_{VG}(u) = -\frac{1}{\nu}\log\left(1 - iu\theta\nu + \sigma^2\nu u^2/2\right)$$

We define the asset price process at time t as follows:

$$S(t) = S(0)\frac{e^{(r-q)t+Z(t)}}{\mathbb{E}[e^{Z(t)}]}$$

We note that

$$\mathbb{E}[e^{Z(t)}] = \phi(-i\Psi_{VG}(-i), t, \frac{1}{\nu}, \kappa, \eta, \lambda)$$

Therefore the characteristic function of the log of the asset price at time t as shown on Page 27 is given by

$$\begin{aligned}
\Phi(u) &= \mathbb{E}(e^{iu\ln S_t}) \\
&= \exp(iu(\log S_0 + (r-q)t)) \times \frac{\phi(-i\Psi_{VG}(u), t, \frac{1}{\nu}, \kappa, \eta, \lambda)}{\phi(-i\Psi_{VG}(-i), t, \frac{1}{\nu}, \kappa, \eta, \lambda)^{iu}}
\end{aligned}$$

With a closed form for the VGSA characteristic function for the log price, one can employ various techniques to price European options as done in Case Study 1. The resulting model may be used to estimate parameter values consistent with market option prices for vanilla options across the entire strike and maturity spectrum. The VGSA parameter set is $\Theta_{\text{VGSA}} = \{\sigma, \nu, \theta, \kappa, \eta, \lambda\}$

2 What needs to be done in this Case Study

1. Obtain Heston and VGSA parameters via calibration to S&P 500 options (for each snapshot separately)
   (a) with equal weights
   (b) with weights inversely proportional to bid-ask spread

2. Having parameters of Heston and VGSA, generate a call option premium surface for various strikes and maturities and construct a local volatility surface for each model utilizing the following equation

$$\sigma(K,T) = \left(2\frac{\frac{\partial C}{\partial T} + q(T)C + (r(T) - q(T))K\frac{\partial C}{\partial K}}{K^2\frac{\partial^2 C}{\partial K^2}} \right)^{\frac{1}{2}}$$

3. Write down your findings/observations on comparing the local volatility surface implied from Heston call surface with the one implied form VGSA call surface.