Columbia University
Computational Methods in Finance

Case Study 4

1 Parameter estimation of Heston Stochastic Volatility Model

The goal is to set up parameter estimation of the Heston stochastic volatility model. In Heston stochastic volatility, the underlying process follows the following SDE

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^1 \\ dv_t &= \kappa(\theta - v_t)dt + \lambda \sqrt{v_t} dW_t^2 \end{aligned}$$

where the two Brownian components W_t^1 and W_t^2 are correlated with rate ρ under physical measure. Define $y_t = \ln(S_t)$ and using Itô's lemma we can write it as

$$\begin{aligned} dy_t &= (\mu - \frac{1}{2}v_t)dt + \sqrt{v_t} dW_t^1 \\ dv_t &= \kappa(\theta - v_t)dt + \sigma \sqrt{v_t} dW_t^2 \end{aligned}$$

1-1 via Extended Kalman Filter

For Heston, we assume that the state equation is

$$\begin{aligned} x_k = f(x_{k-1}, u_k) &= \begin{pmatrix} y_k \\ v_k \end{pmatrix} \\ &= \begin{pmatrix} y_{k-1} + (\mu - \frac{1}{2}v_{k-1})\Delta t + \sqrt{v_{k-1}}\sqrt{\Delta t} Z_k^1 \\ v_{k-1} + \kappa(\theta - v_{k-1})\Delta t + \lambda\sqrt{v_{k-1}}\sqrt{\Delta t} Z_k^2 \end{pmatrix} \end{aligned}$$

with system noise

$$u_k = \begin{pmatrix} Z_k^1 \\ Z_k^2 \end{pmatrix}$$

and the covariance matrix

$$Q_k = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

It is easy to see that in the extended Kalman filter, F_k and U_k for Heston stochastic volatility are

$$F_k = \begin{pmatrix} 1 & -\frac{1}{2}\Delta t \\ 0 & 1 - \kappa\Delta t \end{pmatrix}$$

and

$$U_k = \begin{pmatrix} \sqrt{v_{k-1}}\sqrt{\Delta t} & 0 \\ 0 & \lambda\sqrt{v_{k-1}}\sqrt{\Delta t} \end{pmatrix}$$

We assume measurement equation $y_k = \ln(S_k)$, which implies $H_k = (1\ 0)$ and $V_k = (0\ 0)$. For a given set of parameters $\Theta = \{\mu, \kappa, \theta, \lambda, \rho, v_0\}$, we would minimize the following summation as our objective function to obtain the optimal parameter set for the model

$$\sum_{i=1}^{N} \left(\ln(A_k) + \frac{e_k^2}{A_k} \right)$$

where

$$e_k = y_k - h(\hat{x}_{k|k-1}, 0)$$

and

$$A_k = H_k P_{k|k-1} H_k^\top + V_k R_k V_k^\top$$

where $h(\hat{x}_{k|k-1}, 0) = y_{k-1} + (\mu - \frac{1}{2}v_{k-1})\Delta t$. Here θ does not come into the objective function explicitly but implicitly comes in the filtering.

1-2 via Particle Filter

Follow the discussion on construction of likelihood function and particles during the last lecture to set up the objective function for parameter estimation.

1-3 Data

Assume the following parameter set for Heston $\Theta = \{0.02, 1.5, 0.05, 0.18, 0.5, 0.04\}$ too simulate daily prices for ten years. Use the simulated prices as market prices to get the parameters back via filtering.