Dynamic Programming

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Summary

- Policy Evaluation and Iteration
- Value Iteration
- Dynamic programming

DYNAMIC PROGRAMMING

- A process that aim to optimize a program (i.e. policy) exploiting the sequential or the temporal component of the problem

  - A method for solving complex problems
    - Breaking them into subproblems
    - Subproblems are usually easier to solve
    - Once solved the solutions is just a combination of the subproblems’ solutions

DYNAMIC PROGRAMMING

- Dynamic Programming is a very general solution method.
- Problems must have some properties:
    - Optimal structure
        - Optimality principle
        - Optimal solution can be decomposed into subproblems
    - Overlapping subproblems
        - Subproblems recur many times
        - Solutions can be kept and reused
    - Markov decision processes satisfy both properties
        - Bellman equation dives recursive decomposition
        - Value function can be stored and reused

DYNAMIC PROGRAMMING

- We assume the full knowledge of the MDP
- It is used for planning in an MPD
  - For prediction

    $$input = MDP\ \langle S, A, P, R, \gamma \rangle\ and\ \pi$$

    $$input = MRP\ \langle S, P^{\pi}, R^{\pi}, \gamma \rangle$$

    $$output = v_{\pi}$$

  - For control

    $$input = MDP\ \langle S, A, P, R, \gamma \rangle$$

    $$output = v_{*}\ and\ \pi_{*}$$

DYNAMIC PROGRAMMING

- It is used to solve many other problems:
  - Scheduling
  - String algorithms
  - Graph algorithms
  - Graphical Models
  - Bioinformatics

ITERATIVE POLICY EVALUATION

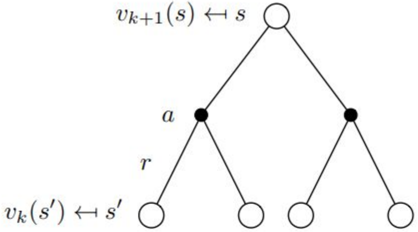
- **Problems:**

  Evaluate a policy π

- **Solution:**

  Iterative application of Bellman expectation backup

  - Synchronous backups:
    - At each iteration $k+1$
    - For all states $s \in S$
    - Update $v_{k+1}(s)$ $from$ $v_k(s')$
    - Where s' is a successor state of s



$$v_{k+1}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

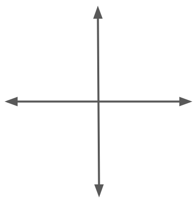
$$\mathbf{v}^{k+1} = \boldsymbol{\mathcal{R}}^{\boldsymbol{\pi}} + \gamma \boldsymbol{\mathcal{P}}^{\boldsymbol{\pi}} \mathbf{v}^k$$

RANDOM POLICY IN A SMALL WORLD

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

Terminal states

Actions



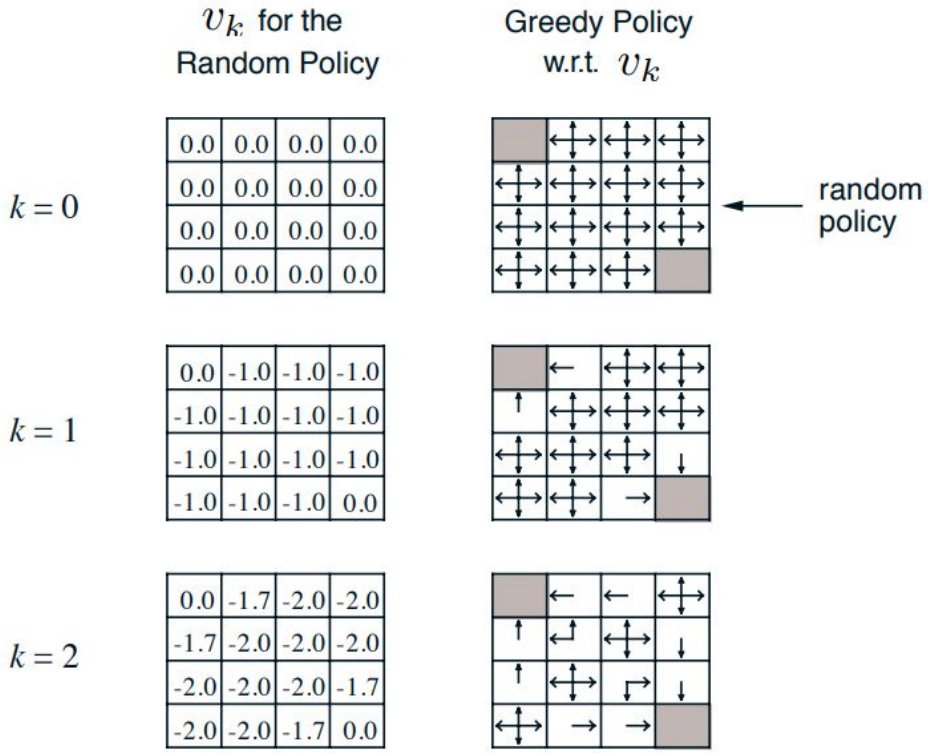
Properties

$r = -1$ on every action
$\gamma = 1$ undiscounted

$$\pi(n|\cdot) = \pi(e|\cdot) = \pi(s|\cdot) = \pi(w|\cdot) = 0.25$$

Agent follows an uniform random policy

RANDOM POLICY IN A SMALL WORLD



RANDOM POLICY IN A SMALL WORLD

$k = 3$

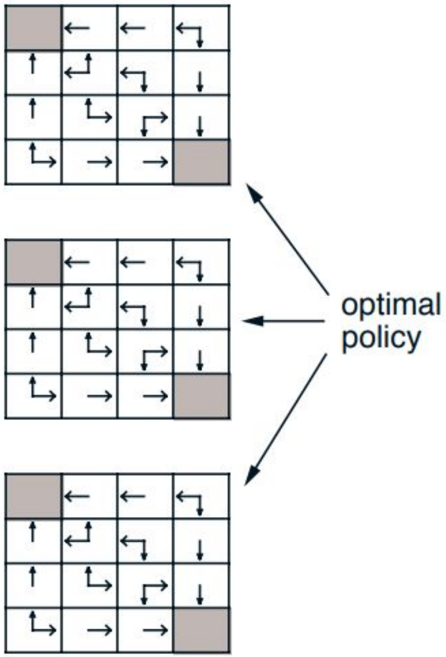
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



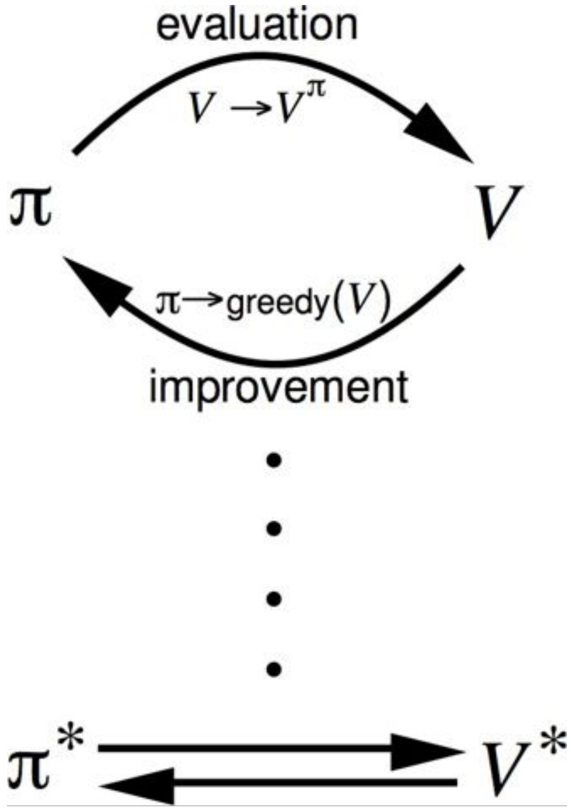
POLICY IMPROVEMENT

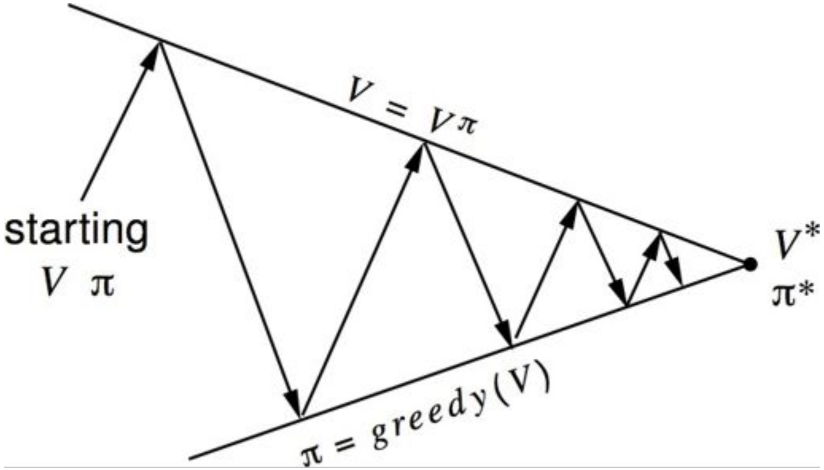
- Give a policy π
  - We want to evaluate: $v_\pi(s) = \mathbb{E}[R_{t+1} + \gamma R_{t+2} | S_t = s]$
  - We want to improve by acting greedly

$$\pi' = greedy(v_\pi)$$

  - In a small gridworld improved policy was optimal, $\pi^1 = \pi^*$
  - We need more iterations of improvement, in this way we can converge to π^*

POLICY ITERATION





Policy evaluation → Estimate v_{π}

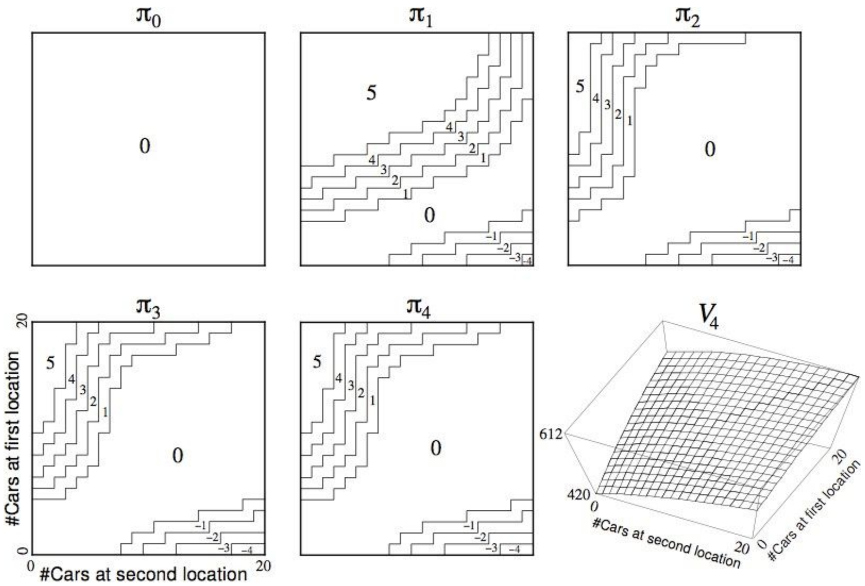
Policy improvement → Generate $\pi' >= \pi$

EXAMPLE – JACK's CAR RENTAL (from sutton)

- The Jack's Car Rental problem is a classic reinforcement learning problem described by Richard Sutton in his book, "Reinforcement Learning: An Introduction."
- The problem involves managing two car rental locations to maximize profit.
- Each location has a limited number of cars that can be rented out or returned each day.
- The number of cars requested and returned at each location is a random variable.
- The goal is to find the optimal policy for transferring cars between the two locations to maximize profit.

EXAMPLE – JACK's CAR RENTAL (from Sutton)

- States: Two locations, maximum of 20 cars at each
- Actions: Move up to 5 cars between locations overnight
- Reward: $10 for each car rented (must be available)
- Transitions: Cars returned and requested randomly



POLICY IMPROVEMENT

- Consider a deterministic policy: $a = \pi(s)$
- We can improve this policy with a greedily method:

$$\pi'(s) = argmax_{a \in A} q_\pi(s, a)$$

- This improves the value from any state s over one step,

$$q_\pi(s, \pi'(s)) = max_{a \in A} q_\pi(s, a) \geq q_\pi(s, \pi(s)) = v_\pi(s)$$

- This will improve the value function: $v_{\pi'}(s) \geq v_\pi(s)$

$$\begin{aligned} v_\pi(s) &\leq q_\pi\left(s, \pi'(s)\right) = \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma v_\pi\left(S_{t+1}\right) \mid S_t = s\right] \\ &\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma q_\pi\left(S_{t+1}, \pi'\left(S_{t+1}\right)\right) \mid S_t = s\right] \\ &\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 q_\pi\left(S_{t+2}, \pi'\left(S_{t+2}\right)\right) \mid S_t = s\right] \\ &\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \ldots \mid S_t = s\right] = v_{\pi'}(s) \end{aligned}$$

POLICY IMPROVEMENT

- If the improvement stops,

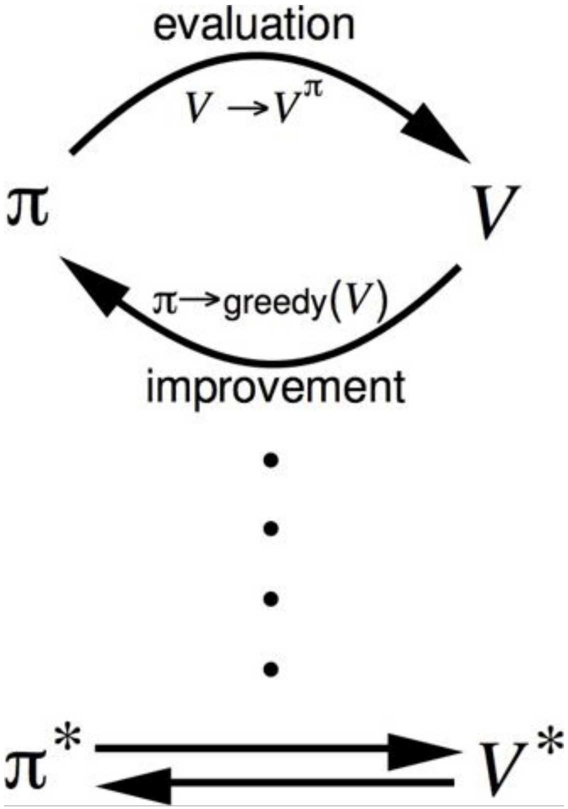
$$q_\pi(s, \pi'(s)) = max_{a \in A} q_\pi(s, a) = q_\pi(s, \pi(s)) = v_\pi(s)$$

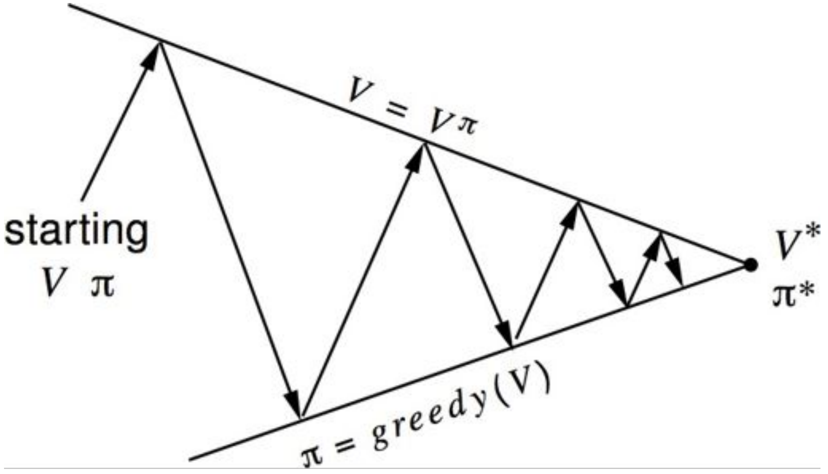
- The Bellman optimality equation is then satisfied

$$v_\pi = \max_{a \in A} q_\pi(s, a)$$

- Hence $v_\pi(s) = v_*(s) \forall s \in S \leftarrow \pi \; optimal \; policy$

POLICY ITERATION





Any Policy evaluation → Estimate v_π

Any Policy improvement → Generate $\pi' >= \pi$

VALUE ITERATION – PRINCIPLE OF OPTIMALITY

- Any optimal policy can be divided into two parts:
  - An optimal first action A_*
  - An optimal policy from successor state S'
- **A policy** $\pi(a|s)$ **achieves optimal value from state ,** $s, v_\pi = v_*(s)$ **if and only if**
  - For any state *s*′ reachable from *s*, π achieve the optimal value from state $s', v_\pi(s') = v_*(s')$

DETERMINISTIC VALUE ITERATION

- If we know the solution of a generic subproblem $v_*(s')$
- Then the solutions $v_*(s)$ can be found using the formula:

$$v_* \leftarrow \max_{a \in A} R_s^a + \gamma \sum_{s' in S} P_{ss'}^a v_*(s')$$

- The idea is to apply these updates iteratively, start with final rewards and work backwards

EXAMPLE – VALUE ITERATION

g			

Problem

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

V_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1

V_2

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-2
-2	-2	-2	-2

V_3

0	-1	-2	-3
-1	-2	-3	-3
-2	-3	-3	-3
-3	-3	-3	-3

V_4

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-4
-3	-4	-4	-4

V_5

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-5

V_6

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-6

V_7

VALUE ITERATION

- **Problems:**

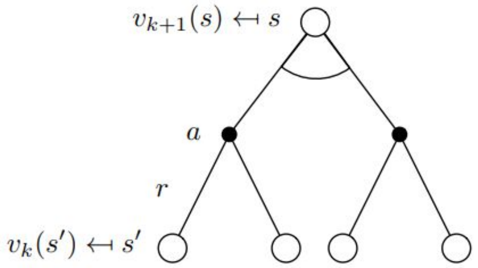
  Find optimal policy π

- **Solution:**

  Iterative application of Bellman optimality backup

  - Synchronous backups:
    - At each iteration $k+1$
    - For all states $s \in S$
    - Update $v_{k+1}(s)\ from\ v_k(s')$

- There is no policy here



$$v_{k+1}(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

$$\mathbf{v}_{k+1} = \max_{a \in \mathcal{A}} \boldsymbol{\mathcal{R}}^{\mathbf{a}} + \gamma \boldsymbol{\mathcal{P}}^{\mathbf{a}} \mathbf{v}_k$$

SYNCHRONOUS DYNAMIC PROGRAMMING

Problem	Bellman Equation	Algorithm
Prediction	Bellman Expectation Equation	Iterative Policy Evaluation
Control	Bellman Expectation Equation + Greedy Policy Improvement	Policy Iteration
Control	Bellman Optimality Equation	Value Iteration

ASYNCHRONOUS DYNAMIC PROGRAMMING

- DP methods described so far used synchronous backups
  - i.e. all states are backed up in parallel

- Asynchronous DP backs up states individually, in any order
- For each selected state, apply the appropriate backup
- Can significantly reduce computation
- Guaranteed to converge if all states continue to be selected
- Three simple ideas for asynchronous dynamic programming:
  - In-place dynamic programming
  - Prioritised sweeping
  - Real-time dynamic programming

IN-PLACE DYNAMIC PROGRAMMING

- Synchronous value iteration stores two copies of value function for all s in S

$$v_{new}(s) \leftarrow \max_{a \in A}(R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v_{old}(s'))$$

- In-place value iteration only stores one copy of value for all s in S

$$v(s) \leftarrow \max_{a \in A}(R_s^a + \gamma \sum_{s' \in S} P_{ss'}^a v(s'))$$

PRIORITISED SWEEPING

- Use magnitude of Bellman error to guide state selection, e.g. max

$$\left| \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v(s') \right) - v(s) \right|$$

- Backup the state with the largest remaining Bellman error
- Update Bellman error of affected states after each backup
- Requires knowledge of reverse dynamics (predecessor states)
- Can be implemented efficiently by maintaining a priority queue

PRIORITISED SWEEPING

- Idea: only states that are relevant to agent
- Use agent's experience to guide the selection of states
- After each time-step S_t, A_t, R_{t+1}
- Backup the state S_t

$$v(S_t) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}^a_{S_t} + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}^a_{S_t s'} v(s') \right)$$