unit - III

Continuous Distributions and sampling distributions

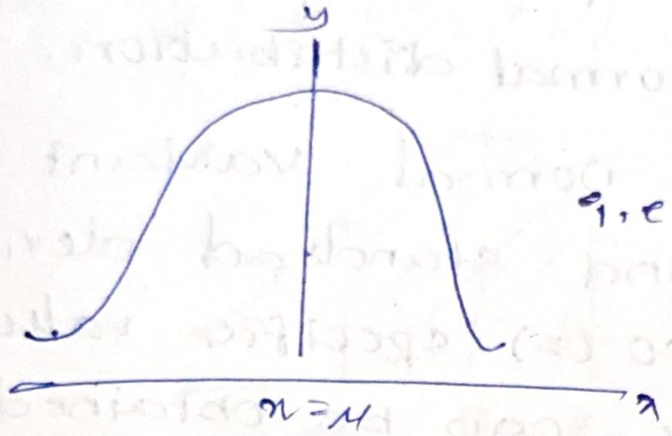
The normal Distribution (or) Gaussian Distribution

$$f(x, \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$$

where $-\infty < x < \infty$

$-\infty < \mu < \infty$

$\sigma > 0$



i.e., $\int_{-\infty}^{\infty} f(x)dx = 1$

⇒ a continous variable x is said to be have a normal distribution, if its Density function or probability function is given by

$$f(x, \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$$

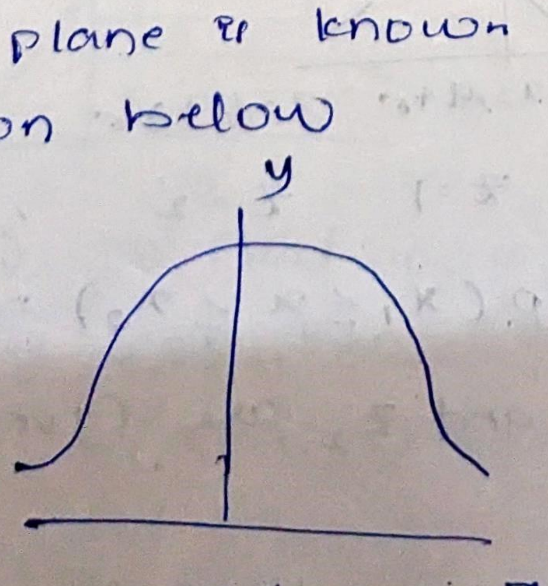
where $-\infty < x < \infty$

$-\infty < \mu < \infty$ and $\sigma > 0$

(increasing no. of trails infinity)

properties of Normal curve

The graph of the normal distribution $y = f(x)$ in xy plane is known as normal curve as shown below



i.e., $\int_{x=-\infty}^{\infty} f(x)dx = 1$

→ The normal curve is bell shaped symme-trical about the line $x = \mu$.

→ The area under the normal curve is equivalent to 1

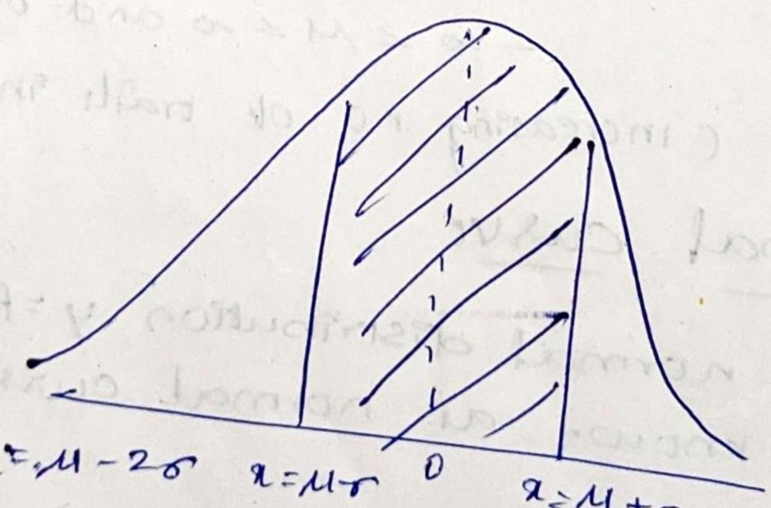
→ Mean, Median, Mode coincides in the normal distribution

→ The x axis is asymptote to normal curve

Characteristics of Normal distribution.

→ The probability the normal variant x with the mean μ and standrad deviation "σ" lies between two (2) specific values x_1 and x_2 with $x_1 \le x_2$ can be obtained using area under the standrad normal curve as follows

Step 1 : $z = \frac{x-\mu}{\sigma}$ Find z_1 and z_2 correspond-ing to the values of x_1 and x_2 respectively



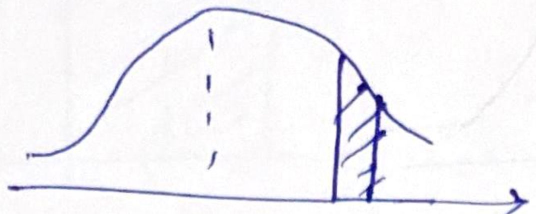
Step 2 : a) To find $P(x_1 \le x \le x_2) = P(z_1 < z < z_2)$

Case(i) if both z_1 and z_2 are (+ve) (or) both (-ve)

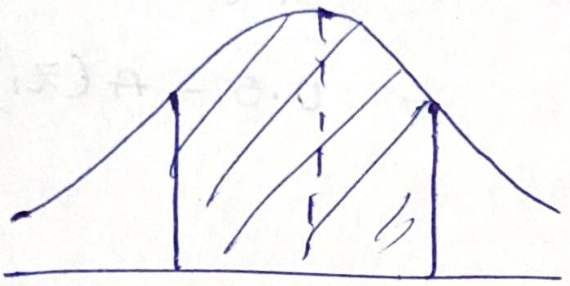
Then

$$P(x_1 \le x \le x_2) = |A(z_2) - A(z_1)|$$

= Area under the normal curve 0 to z_2 -
Area under the normal curve 0 to z_1



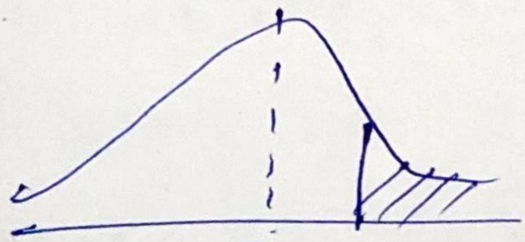
case (ii) if both $z_1 > 0$ and $z_2 < 0$ are opposite,
then $P(x_1 \le X \le x_2) = A(z_2) + A(z_1)$



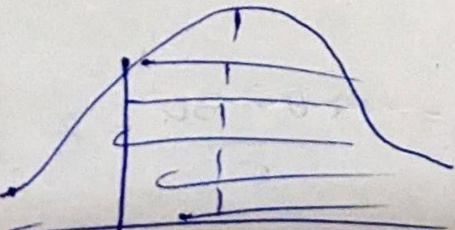
Step 2 (b). To Find $P(z > z_1)$

Case ① If $z_1 > 0$, Then $P(z > z_1) = 0.5 - A(z_1)$

$[\because P(z < 0) = P(z > 0) = \frac{1}{2}]$



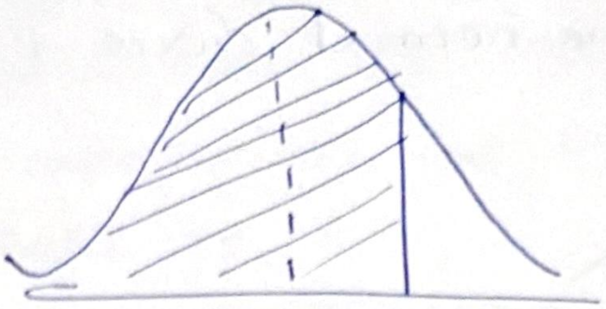
Case 2 if $z_1 < 0$, Then $P(z > z_1) = 0.5 + A(z_1)$



Step 2 (c) To Find $P(z < z_1) = 1 - P(z > z_1)$

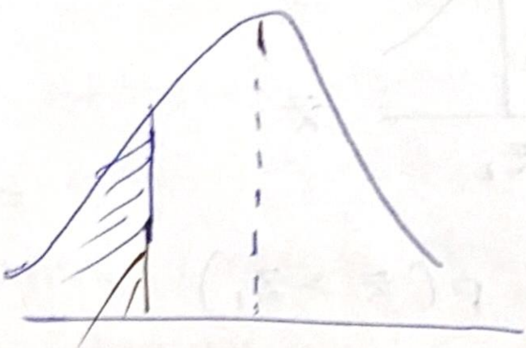
Case (i): if $z_1 > 0$, Then

$$P(z < z_1) = 1 - P(z > z_1)$$
$$= 1 - [0.5 - A(z_1)]$$
$$= 0.5 + A(z_1)$$



Case ② If $z_1 < 0$, then

$$P(z < z_1) = 1 - P(z > z_1)$$
$$= 1 - [0.5 + A(z_1)]$$
$$= 0.5 - A(z_1)$$



Q) A x is a normal variant with mean 30 and standrad deviation 5 find the probability that

(i) $26 \le x \le 40$

(ii) $x \ge 45$

Soln Given $\mu = 30$, $\sigma = 5$

(i) when $x = 26$, $z = \frac{x-\mu}{\sigma} = \frac{26-30}{5} = -0.8$ (z_1 say)

$x = 40$, $z = \frac{x-\mu}{\sigma} = \frac{40-30}{5} = 2$ (z_2 say)

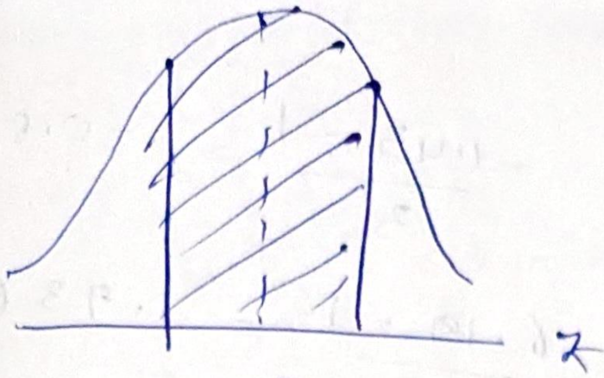
$\therefore P(26 \le x \le 40) = P(\underset{z_1}{-0.8} \le z \le \underset{z_2}{2})$ z_1 & z_2 are o[illegible] in sign

$= |A(z_2)| + |A(z_1)|$

$= A(2) + A(0.8)$

$= 0.4772 + 0.2881$

$= 0.7653$ (from normal distri [illegible]



ii) when $x = 45$, $z = \frac{x-\mu}{\sigma} = \frac{45-30}{5} = 3$ (z_1 say)

$P(X \geq 45) = P(z > 3)$ $\quad z(+ve)$

$= 0.5 - A(z_1)$

$= 0.5 - A(3)$

$= 0.5 - 0.4487$

$= 0.00135$

Q) For a normal distributed variant with mean 1 and standard deviation 3 find the probability that

i) $3.43 \leq x \leq 6.19$

ii) $-1.43 \leq x \leq 6.19$

Solⁿ Given $\mu = 1$, $\sigma = 3$

(i) when $x = 3.43$, $z = \frac{x-\mu}{\sigma} = \frac{3.43-1}{3} = 0.81$ (z_1 say)

$x = 6.19$, $z = \frac{x-\mu}{\sigma} = \frac{6.19-1}{3} = 1.73$ (z_2 say)

$\therefore P(3.43 \leq x \leq 6.19) = P(\underset{z_1}{0.81} \leq z \leq \underset{z_2}{1.73})$

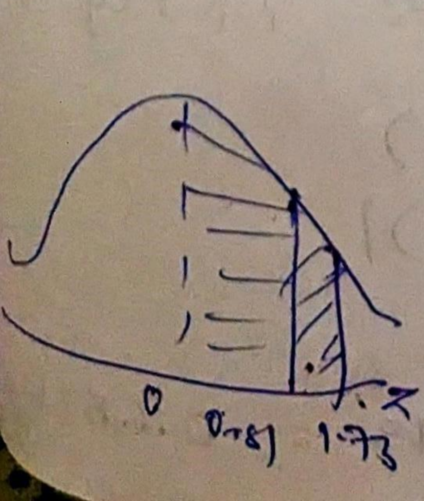
$= |A(z_2) \cancel{+} A(z_1)|$

$= |A(1.73) + A(0.81)|$

$= \cancel{0.4582 + 0.2910}$

$= |0.4582 - 0.2910|$

$= \cancel{0.1472}$ 0.1672



(ii) $-1.43 \le x \le 6.19$

when $x = -1.43$ $\quad z = \dfrac{-1.43 - 1}{3} = -0.81$ (z_1 say)

$x = 6.19$ $\quad z = \dfrac{6.19 - 1}{3} = 1.73$ (z_2 say)

[one +ve & one -ve]

$P(-1.43 \le x \le 6.19) = P(-0.81 \le z \le 1.73)$

$= |A(z_2)| + |A(z_1)|$

$= A(1.73) + A(0.81)$

[since from n.D table] $\leftarrow$ $= 0.4582 + 0.2910$

$= 0.7492$

VV IMP **

Q) If the masses of 300 students are normally distributed with mean 68 kg and standard deviation 3 kg. How many students

i) Between 65 and 71 kg inclusive (in b/w)

ii) Greater than 72 kg

iii) Less than or equal to 64 kg

Given that

$\mu = 68$ kg $\quad \sigma = 3$ kg

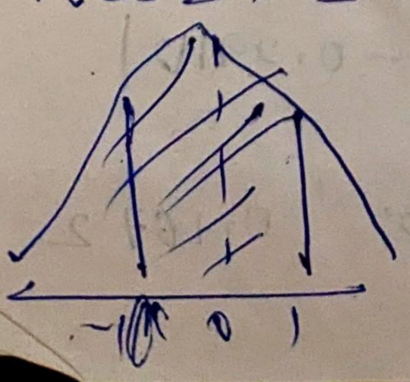
x denotes the masses of students

(i) soln.

$x = 65, \quad \dfrac{x-\mu}{\sigma} = \dfrac{65-68}{3} = -1$ (say z_1)

$x = 71, \quad \dfrac{x-\mu}{\sigma} = \dfrac{71-68}{3} = 1$ (say z_2)

$P(65 \le x \le 71) = P(-1 \le z \le 1)$



$= |A(z_2)| + |A(z_1)|$

$= A(1) + A(1)$

$= 0.3413 + 0.3413 = 0.6826$

$\therefore$ Required no. of students $= 300 \times 0.6826$

$= 205$ (approx)

ii) Greater than 72 kg

at $x = 72$, $\dfrac{72-68}{3} = \dfrac{4}{3} = 1.33$ (say z_1)

$P(x > 72) = P(z > 1.33)$

$= 0.5 - A(z_1)$

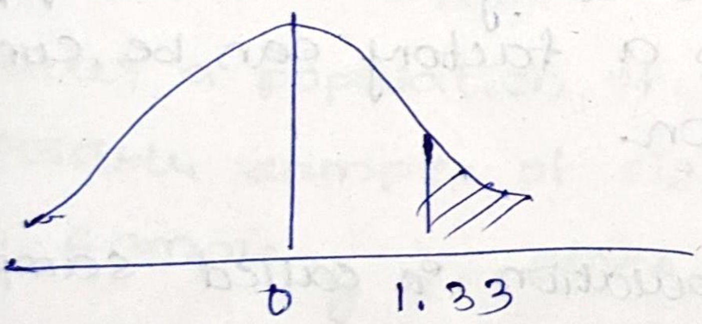
$= 0.5 - A(1.33)$

$= 0.5 - 0.4082$

$= 0.0918$

$\therefore$ no. of students more than 72 kg

$= 300 \times 0.0918 = 28$ (Approx)



iii) Less than or equal to 64 kg

$x = 64 = \dfrac{64-68}{3} = \dfrac{-4}{3} = -1.33$ (say z_1)

$P(x \le 64) = P(z > z_1) = P(z > -1.33)$

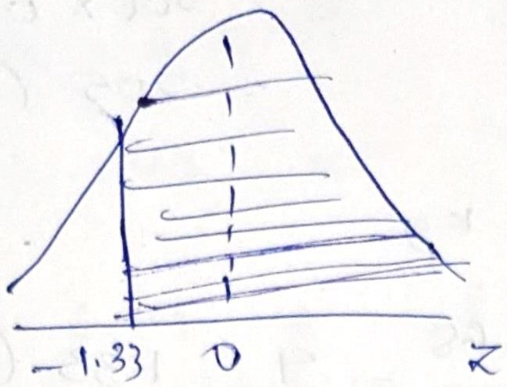
$= 0.5 + A(z_1)$

$= 0.5 + A(1.33)$

$= 0.5 + 0.4082$

$= 0.9082$

$\therefore$ no. of students less than or equal to 64 kg : $300 \times 0.9082 = 272$ (Approx)



22/4/24

Fundamental sample distribution:

Population:

The collection of objects is called population the size of the population is no. of objects and it is denoted by N

eg:

the no. of students in a college is a finite population the goods prepared in a factory can be considered as an infinite population.

Sample

The subset of the population is called sample and the size of sample are no. of objects. denoted by 'n'

eg:

Suppose a buyer agree to buy goods in batches of 100 items - each and out of each each batch, he takes 5 items to test the quality here the batch size 100 and sample size 5

* if n is less than 30 $n < 30$ sample is called small sample

* if $n \geq 30$ then sample is called large sample

Sampling Distribution 25/4/24

where let $x_1, x_2, x_3 \dots x_n$ observations:

population having mean, variance

$$\mu = \frac{\Sigma x_i}{N} \quad N \rightarrow \text{no. of observations}$$

$$\sigma^2 = \frac{\Sigma (x-\mu)^2}{N}$$

Sampling Distribution mean, variance

$$\mu_{\bar{x}} = \frac{\Sigma x f(x)}{\Sigma f(x)}$$

$$\sigma^2_{\bar{x}} = \frac{\Sigma (x-\mu_{\bar{x}})^2 f(x)}{\Sigma f(x)}$$

Consider a population of size 'N', draw the all possible samples of size 'n' from population.

⇒ Then sampling is done without replacement

⇒ NC_n

⇒ if sampling is done with replacement

⇒ N^n

⇒ sampling distribution of statistics

Statistics	S_1	S_2	..	S_k
no of sample	x	y		z

$= k$

Verification

Without replacement

→ $\mu_{\bar{x}} = \mu$

→ $\sigma^2_{\bar{x}} = \frac{\sigma^2}{n}\left[\frac{N-n}{N-1}\right]$

with replacement

→ $\mu_{\bar{x}} = \mu$

→ $\sigma^2_{\bar{x}} = \frac{\sigma^2}{n}$

Here $\frac{N-n}{N-1}$ → Correction Factor

Note : $\mu, \sigma^2, \sigma_{\bar{x}}^2$

① A population consisting of 1, 5, 6, 8 consider the all possible samples of size '2' with replacement from the population.

a) Find the mean of the population

b) Variance of the population

c) ~~Consider~~ construct the sampling distribution of means

d) Find mean of the sampling distribution of mean

e) Find variance of sampling distribution of means

f) verify with the formula

$N = 4, n = 2$

a) $\mu = \frac{\sum x_i}{N} = \frac{1+5+6+8}{4} = 5$

b) $\sigma^2 = \frac{\sum (x_i - \mu)^2}{N} = \frac{(1-5)^2 + (5-5)^2 + (6-5)^2 + (8-5)^2}{4}$

$= 6.5$

c) sampling with replacement

$N^n = 4^2 = 16$

(1,1) (1,5) (1,6) (1,8)

(5,1) (5,5) (5,6) (5,8)

(6,1) (6,5) (6,6) (6,8)

(8,1) (8,5) (8,6) (8,8)

sampling Means :

1	3	3.5	4.5
3	5	5.5	6.5
3.5	5.5	6	7
4.5	6.5	7	8

mean (x)	1	3	3.5	4.5	5	5.5	6.	6.5	7	8
no. of samples $f(x)$	1	2	2	2	1	2	1	2	2	1

$= 16$

d) $$\mu_{\bar{x}} = \frac{\Sigma x_i f(x_i)}{\Sigma f(x_i)}$$

$$= \frac{(1\times1)+(3\times2)+(3.5\times2)+(4.5\times2)+(5\times1)+(5.5\times2)+(6\times1)+(6.5\times2)+(7\times2)+(8\times1)}{16}$$

$$= \frac{1+6+7+9+5+11+6+13+14+8}{16}$$

$$= \frac{\cancel{80}^{5}}{\cancel{16}} = 5$$

e) variance of sampling distribution of mean

$$\sigma^2_{\bar{x}} = \frac{\Sigma(x_i - \mu_{\bar{x}})^2 f(x_i)}{\Sigma f(x_i)} =$$

$$\frac{((1-5)^2\times1)+((3-5)^2\times2)+((3.5-5)^2\times2)+(4.5-5)^2\times2+(5-5)^2\times1)+((6-1)^2\times1)+((6.5-5)^2\times2)+(7-5)^2\times2+(8-5)^2\times1}{16}$$

16

$= \frac{51.5}{16} = 3.25$

F) Verification :

$\mu_{\bar{x}} = \mu \qquad 5 = 5$

$\sigma^2_{\bar{x}} = \frac{\sigma^2}{n}$

$3.25 = \frac{6.5}{2} = 3.25$

$3.25 = 3.25$ //

Q) A population consist of 3, 6, 9, 15, 27 consider all possible samples of size 2 that can be drawn without replacement from the population

a) Find the mean of the population

b) variance of the population

c) construct the sampling distribution of means

d) Find mean of the sampling distribution of mean

e) Find variance of the sampling distribution of mean

f) verify with the formula.

a) $\mu = \frac{3+6+9+15+27}{5} = 12$

b) $\sigma^2 = \frac{(3-12)^2 + (6-12)^2 + (9-12)^2 + (15-12)^2 + (27-12)^2}{5}$

$= 72$

$\sigma = 8.48$

c) sampling distribution

without replacement $^{N}C_{n} = {}^{5}C_{2} = \frac{5!}{(5-2)!\,2!} = 10$

(3,6) (3,9) (3,15) (3,27)

(6,9) (6,15) (6,27)

(9,15) (9,27)

(15,27)

means

4.5 6 9 15

7.5 10.5 16.5

12 18

21

Table:

x	4.5	6	9	15	7.5	10.5	16.5	12	18	21
$f(x)$	1	1	1	1	1	1	1	1	1	1

d) $\mu_{\bar{x}} = \frac{\sum x_i f(x_i)}{\sum f(x_i)}$ $= 10$

$= \frac{4.5+6+9+15+7.5+10.5+16.5+12+18+21}{10}$

$\frac{120}{10} = 12$

e) $\frac{((9.5-12)^2\times 1) + (6-12)^2\times 1 + (\ldots + (16.5-12)^2\times 1 + (12-12)^2\times 1 + (9.5-12)^2\times 1 + (10.5-12)^2\times 1 + (18-12)^2\times 1 + (21-12)^2\times 1}{10}$

$= \frac{56.25 + 36 + 9 + 9 + 20.25 + 2.25 + 20.25 + 0 + 36 + 81}{10}$

$= \frac{270}{10} = 27$

e) verification

$\mu_{\bar{x}} = \mu$

$12 = 12$

$\sigma^2_{\bar{x}} = \frac{\sigma^2}{n} =$

Central limit Theorem

Let $\bar{x}$ be the mean of random sample of size of 'n', draw from population of size 'N', ~~adding~~ having the mean μ and variance σ^2 then $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ is a random variable, approaching the standrad normal distribution when $n \to \infty$

Q) a random sample distribution of size 100 is taken from infinite population having

$\mu = 76$ and variance $\sigma^2 = 256$. What is the probability that $\bar{x}$ will be between 75 and 78.

Sol^n Given $n=100$, $\mu = 75$, $\sigma^2 = 256$

$$P(\bar{x}_1 < x < \bar{x}_2) = P(z_1 < z < z_2)$$

$\bar{x}_1 = 75$ $z_1 = \dfrac{\bar{x}_1 - \mu}{\sigma/\sqrt{n}} = \dfrac{-1}{} = -0.625$

$\bar{x}_2 = 78$ $z_2 = \dfrac{\bar{x}_2 - \mu}{\sigma/\sqrt{n}} = 1.25$

$= A(z_2) + A(z_1)$

$= A(1.25) + A(0.625)$

$= 0.3944 + 0.2324$

$= 0.6268$

Q) a random sample size of 64 is taken from infinite population having the mean $\mu = 15.4$ and standrad deviation $\sigma = 68$ what is the probability that $\bar{x}$ be the

a) exceeds 52.9

b) b/w 50.5 and 52.9

c) lessthan 50.6

a) $P(x > 52.9)$

$\bar{x}_1 = 52.9$ $z = \dfrac{52.9 - 15.4}{68/\sqrt{64}} = \dfrac{37.5}{8.5} = 4.41$

$)^2 \times 1 + (15-12)^2 \times$

$12)^2 \times 1 + (12-12)^2 \times$

$5 + 2.25$

σ^2

ple,

ribution

ize 100

aving

H/W

(1) 1, 2, 3, 4, 5, 6

with replacement

a) $\mu = \frac{1+2+3+4+5+6}{6} = 21/6 = 3.5$

b) $\sigma^2 = \frac{(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2}{6}$

$= \frac{6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25}{6}$

$= 2.91$

c) Sampling with replacement

$N^n = 6^2 = 36$

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)

(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)

(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)

(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)

(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)

(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

Sampling means:

1	1.5	2	2.5	3	3.5
1.5	2	2.5	3	3.5	4
2	2.5	3	3.5	4	4.5
2.5	3	3.5	4	4.5	5

3	3.5	4	4.5	5	5.5
3.5	4	4.5	5	5.5	6

mean ($\bar{x}$)	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
f($\bar{x}$)	1	2	3	4	5	6	5	4	3	2	1

$= 36$

d) $\mu_{\bar{x}} = \dfrac{\Sigma x_i f(x_i)}{\Sigma f(x_i)}$

$$\frac{1+3+6+10+15+21+20+18+15+11+6}{36}$$

$= \dfrac{126}{36} = 3.5$

e) $\sigma^2_{\bar{x}} = \dfrac{\Sigma (x_i - \mu_{\bar{x}})^2 f(x_i)}{\Sigma f(x_i)}$

$$= \frac{(1-3.5)^2\times1 + (1.5-3.5)^2\times2 + (2-3.5)^2\times3 + (2.5-3.5)^2\times4 + (3-3.5)^2\times5 + (3.5-3.5)^2\times6 + (4-3.5)^2\times5 + (4.5-3.5)^2\times4 + (5-3.5)^2\times3 + (5.5-3.5)^2\times2 + (6-3.5)^2\times1}{36}$$

$$= \frac{6.25 + 8 + 6.75 + 4 + 1.25 + 0 + 11.25 + 4 + 6.75 + 8 + 6.25}{36}$$

$= \dfrac{62.5}{36} = 1.74$

f) verification

$\mu_{\bar{x}} = \mu \quad 3.5 = 3.5$

$\sigma^2_{\bar{x}} = \dfrac{\sigma^2}{n} = \dfrac{2.91}{2}$

2) 5, 10, 14, 18, 13, 24 without replacement

a)

$$\mu = \frac{5+10+14+18+13+24}{6} = \frac{84}{6} = 14$$

b)

$$\sigma^2 = \frac{(5-14)^2 + (10-14)^2 + (14-14)^2 + (18-14)^2 + (13-14)^2 + (24-14)^2}{6}$$

$$= \frac{81+16+0+16+1+100}{6} = \frac{214}{6}$$

$$= 35.66$$

c)

$$^{N}C_{n} = {}^{6}C_{2} = \frac{6!}{(6-2)!\,2!} = \frac{720}{24 \times 2} = \frac{720}{48}$$

$$= 15$$

(5,10) (5,14) (5,18) (5,13) (5,24)

(10,14) (10,18) (10,13) (10,24)

(14,18) (14,13) (14,24)

(18,13) (18,24)

(13,24)

<u>means</u>

7.5 9.5 11.5 9 14.5

12 14 11.5 17

16 13.5 19

15.5 21

x	7.5	9	9.5	11.5	12	13.5	14	14.5	15.5	17	19	21	27.5
$f(x)$	1	1	1	2	1	1	1	1	1	1	1	1	1

$= 15$

d)

$$\frac{7.5 + 9 + 9.5 + \cancel{11.5}\ 23 + 12 + 13.5 + 14 + 14.5 + 15.5 + 16 + 17 + 19 + 21 + 27.5}{15}$$

$= 14$

e)

$$\frac{(7.5-14.6)^2 + (9-14.6)^2 + (9.5-14.6)^2 + (11.5-14.6)^2 + (12-14.6)^2 + (13.5-14.6)^2 + (14-14.6)^2 + (14.5-14.6)^2 + (15-14.6)^2 + (16-14.6)^2 + (17-14.6)^2 + (19-14.6)^2 + (21-14.6)^2 + (27.5-14.6)^2}{15}$$

$$= \frac{50.41 + 31.36 + 26.01 + 19.22 + 6.76 + 1.21 + 0.36 + 0.01 + \cancel{2.56} + \cancel{6.76} + \cancel{12.96} + 0.16 + 1.96 + 5.76 + 19.36 + 40.96 + 166.41}{15}$$

$= 24.66$

24.66