

PRECISE summer school
Warsaw

July 05, 2023

hands-on cosmological simulations

session 2: initial conditions and cosmological integration

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&
Shaun Brown

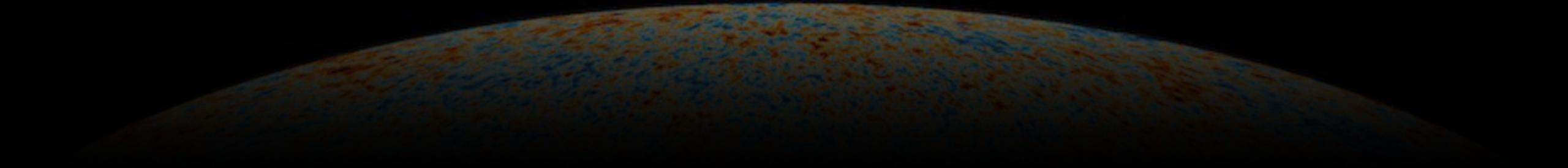
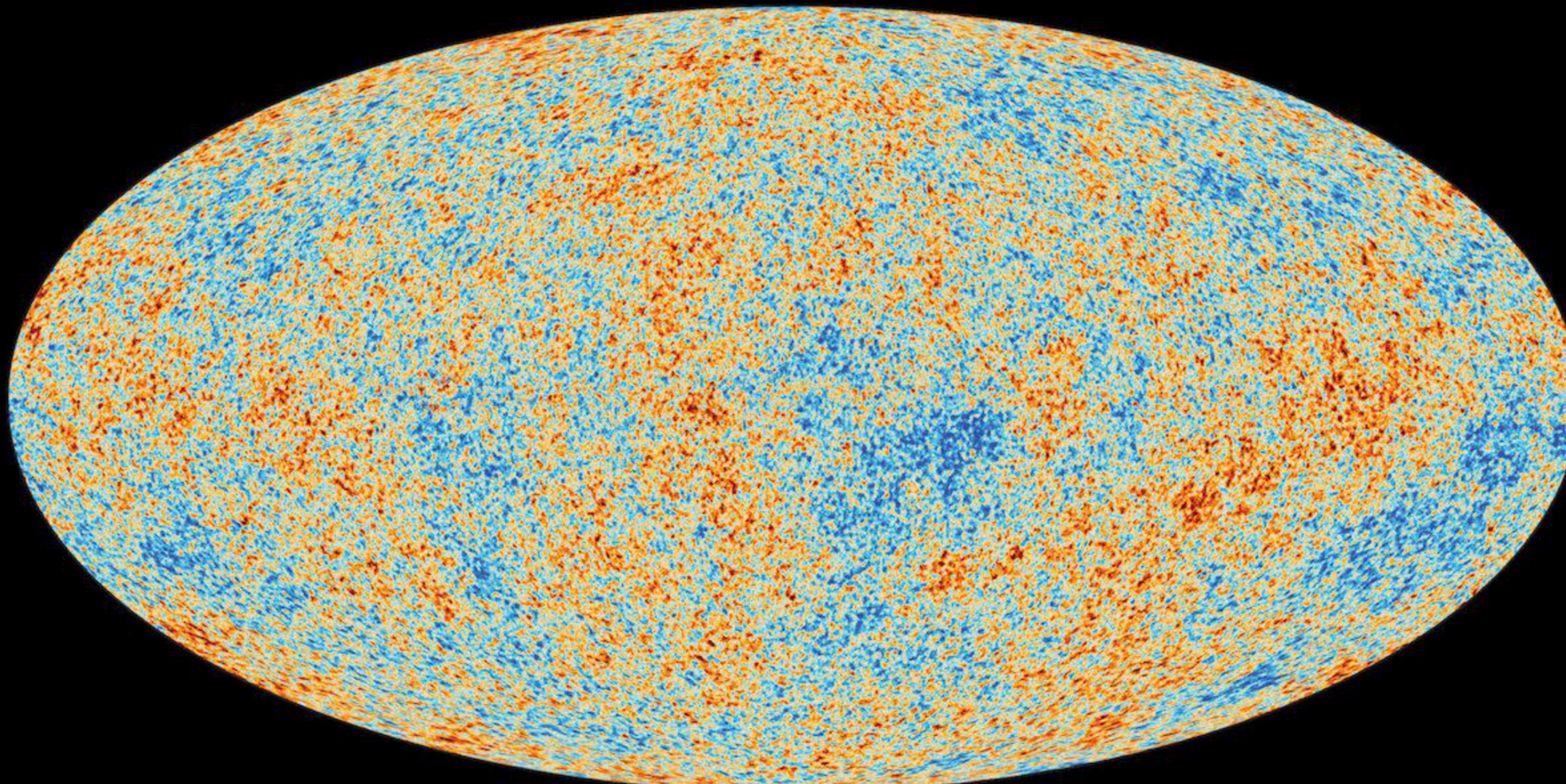
sownak.bose@durham.ac.uk

 @Swnk16

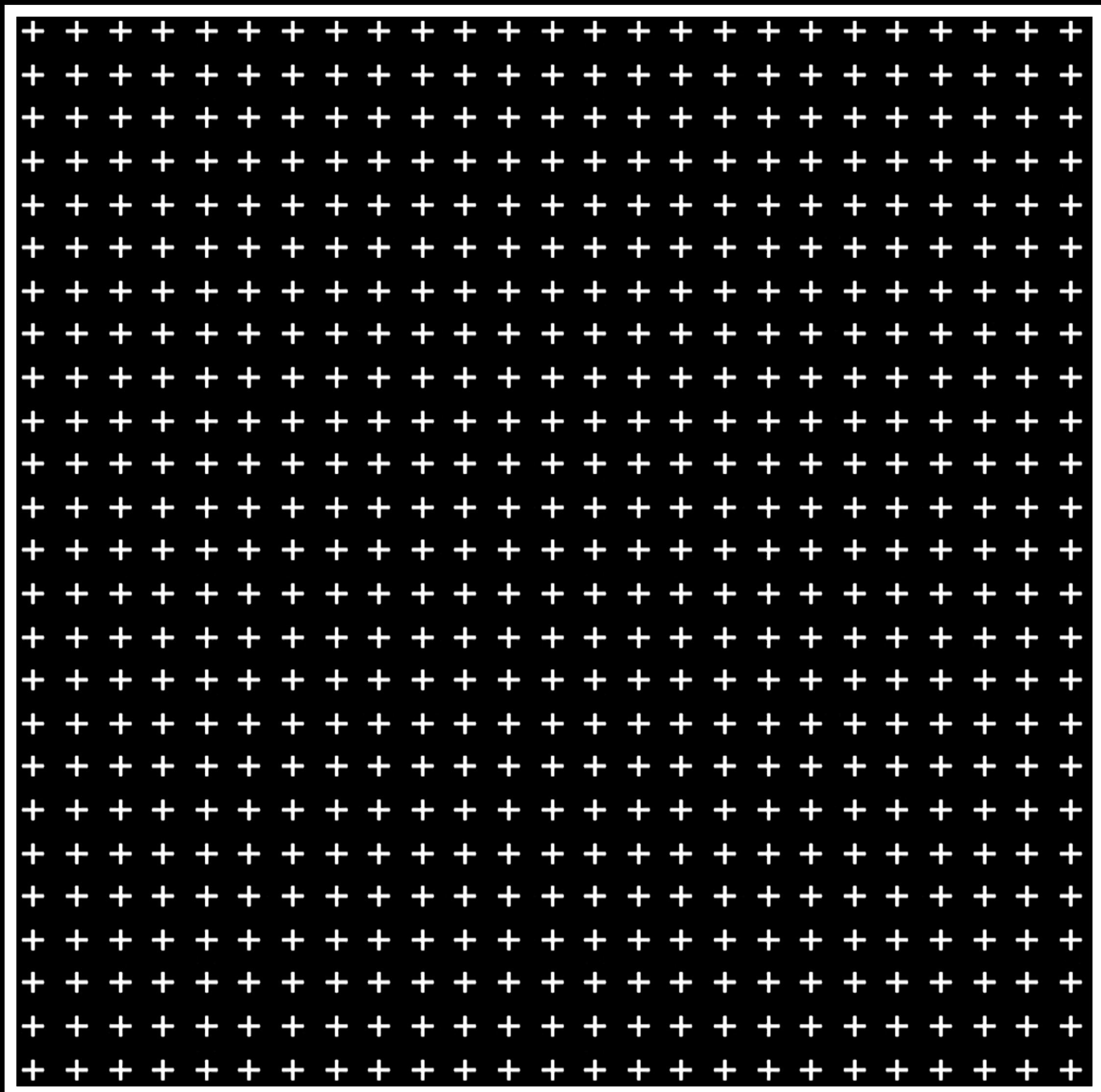


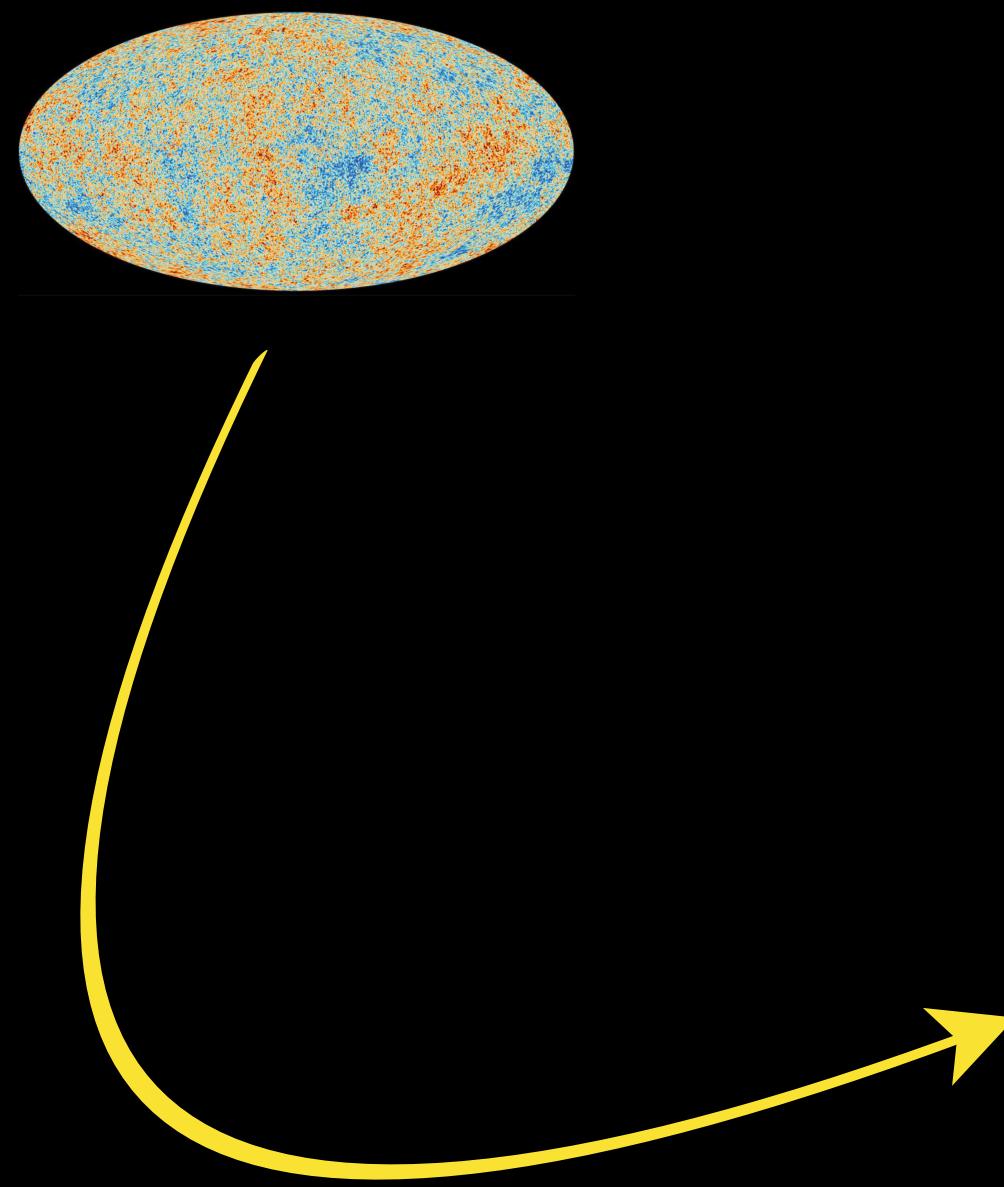
setting up initial conditions

measurement of the **cosmic microwave background radiation** gives us *precise constraints* on cosmological initial conditions



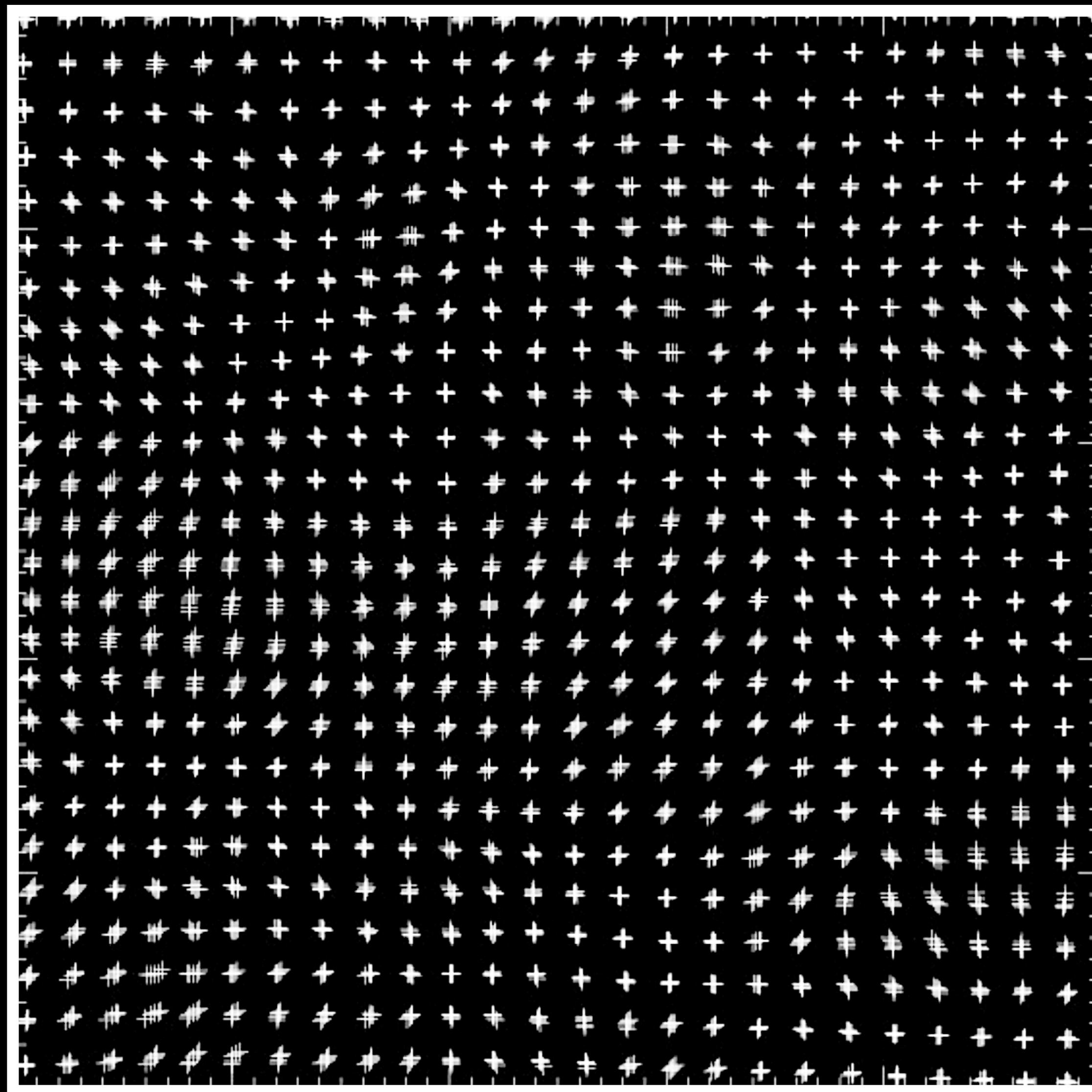
homogeneous & isotropic ICs



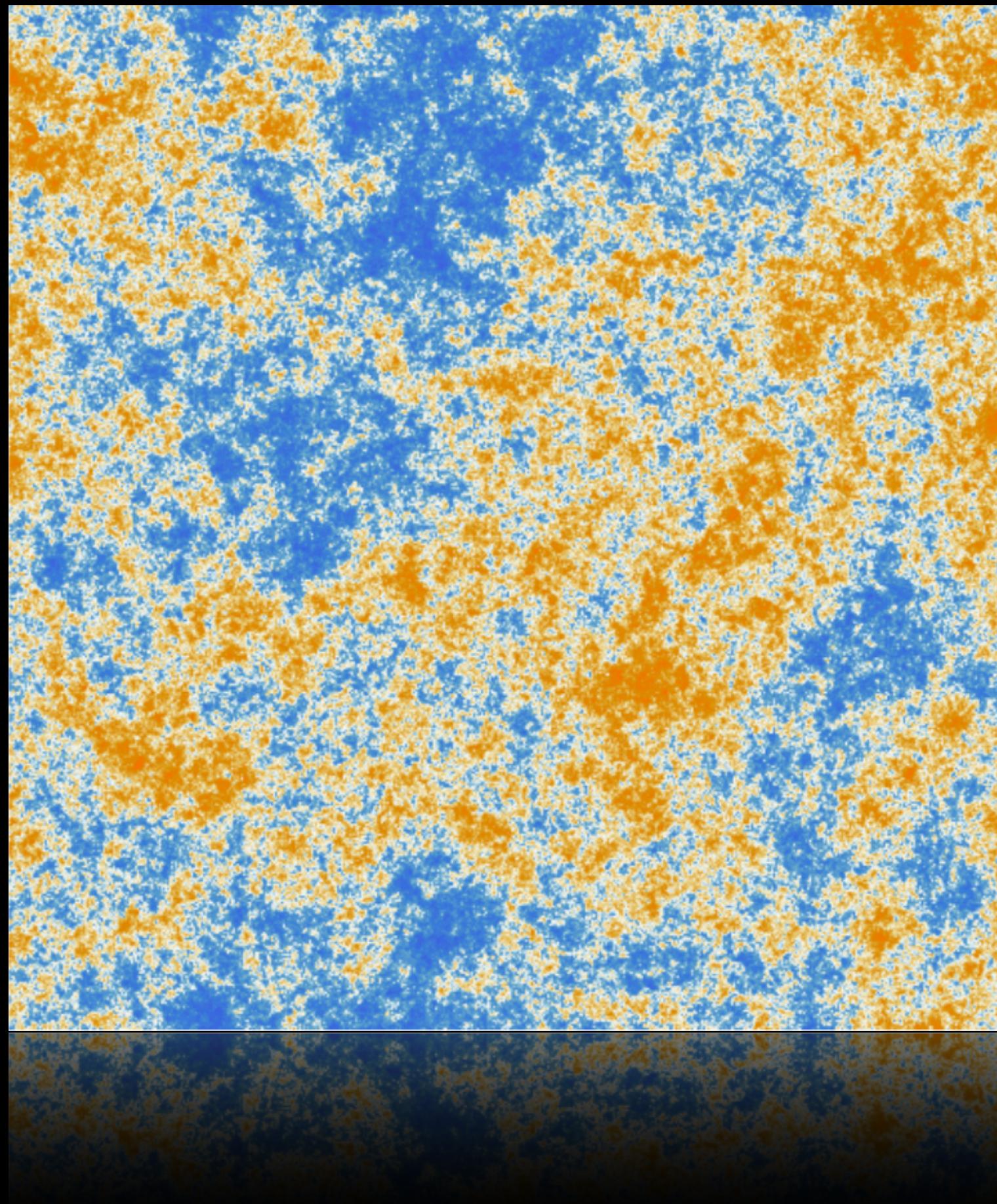


**these tiny fluctuations
will grow into non-linear
structures (e.g. filaments,
walls, haloes, clusters)
through gravitational
instability**

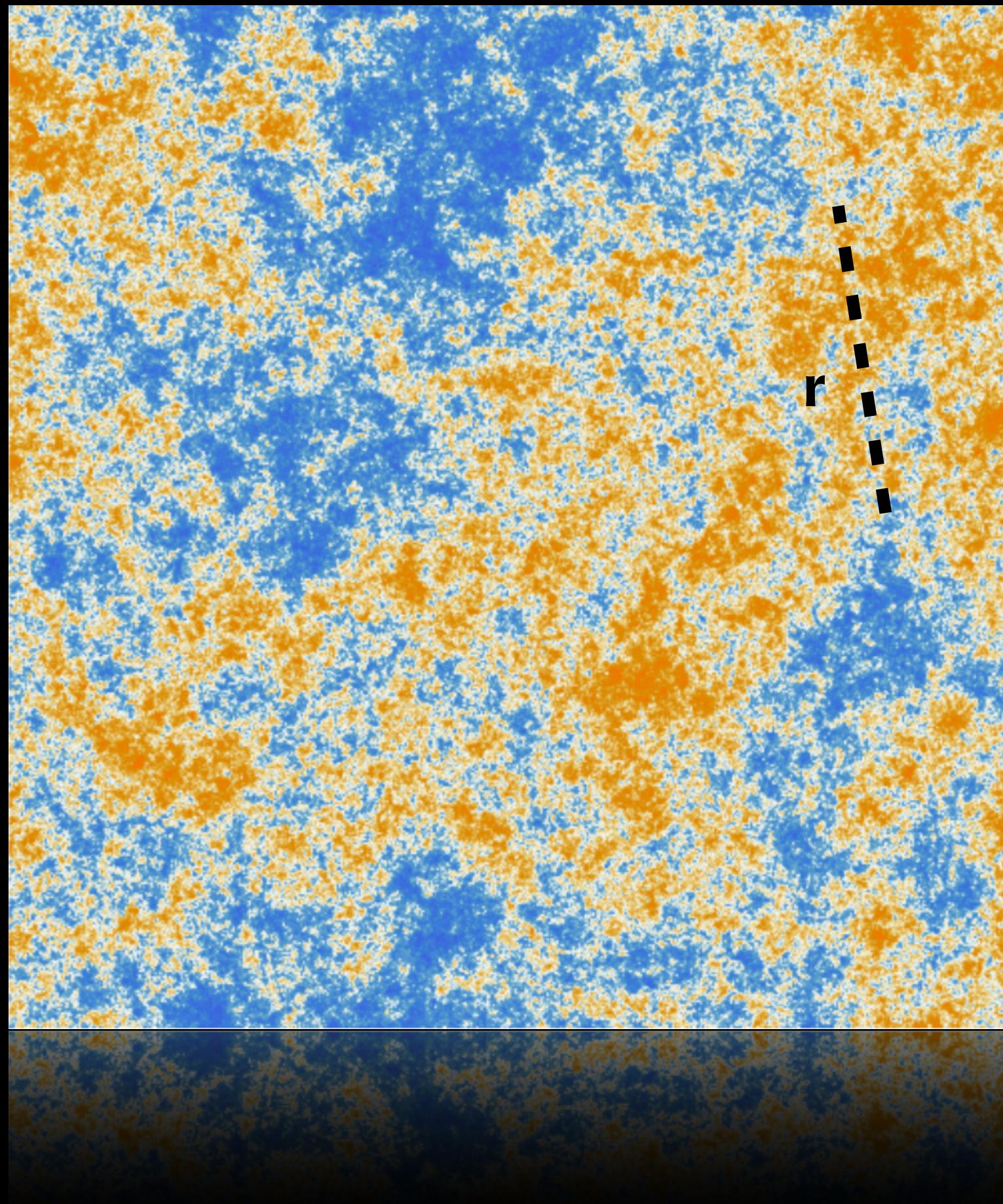
perturbed ICs



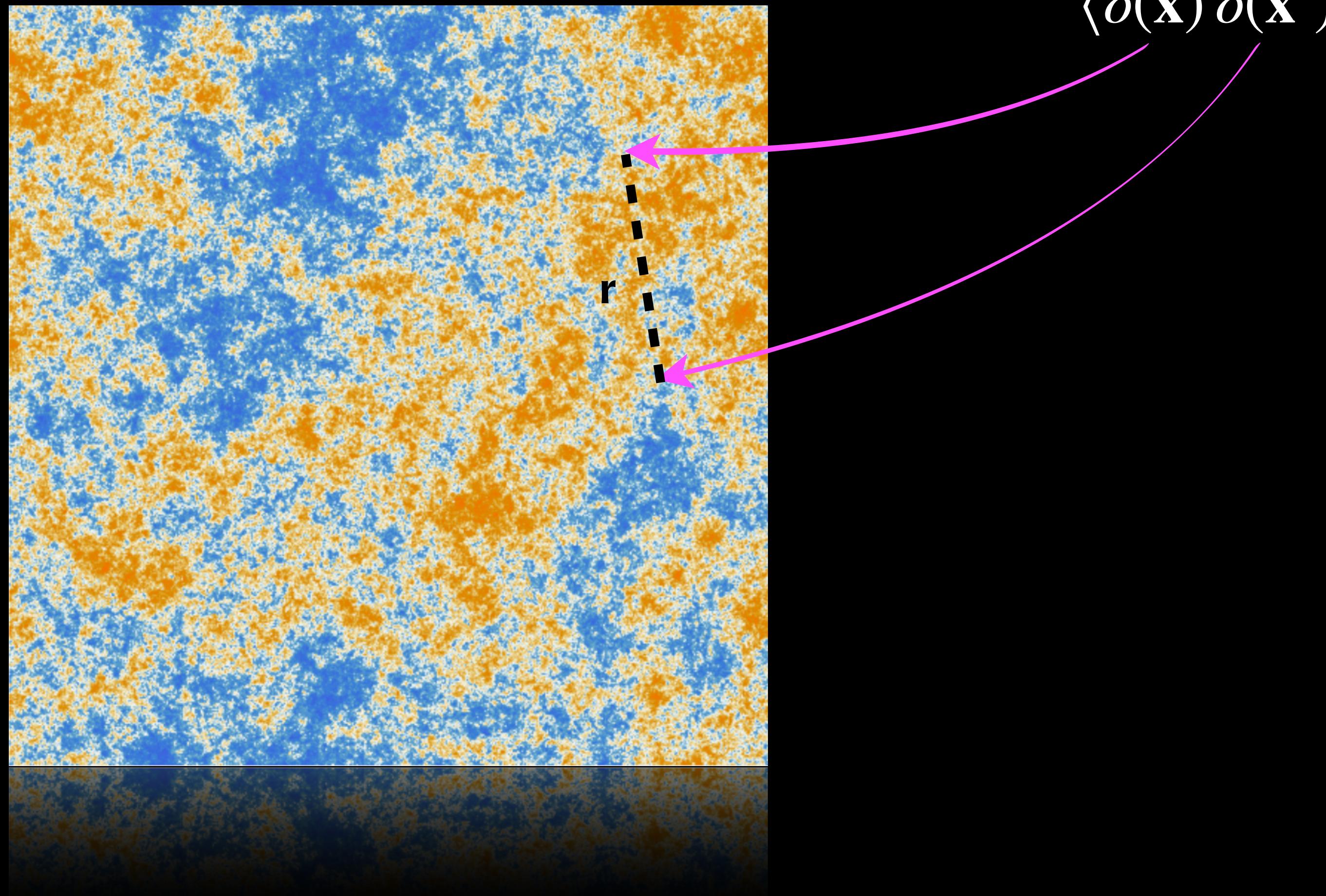
goal: create a **Gaussian random distribution** that is statistically consistent with the fluctuations measured in the CMB



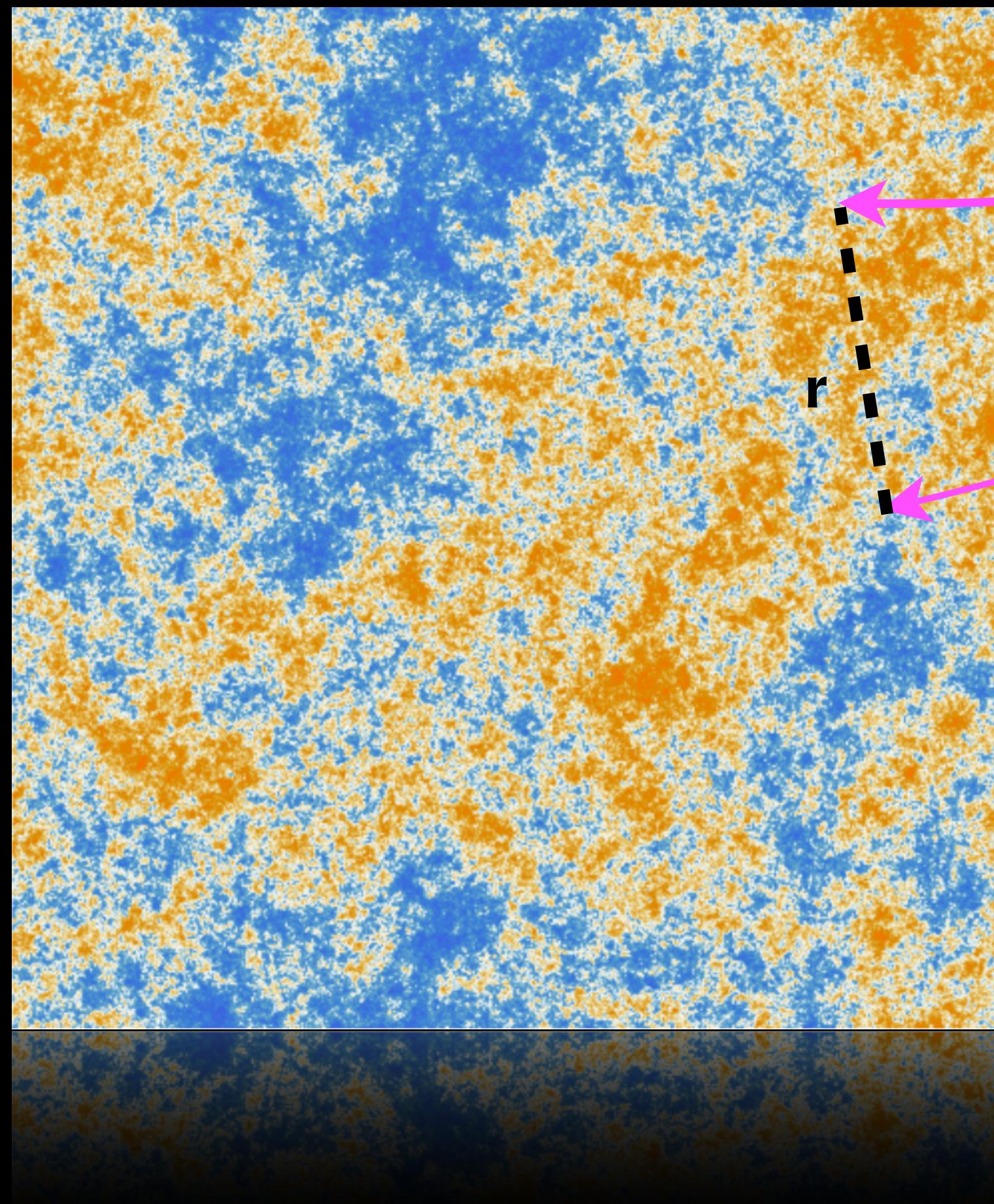
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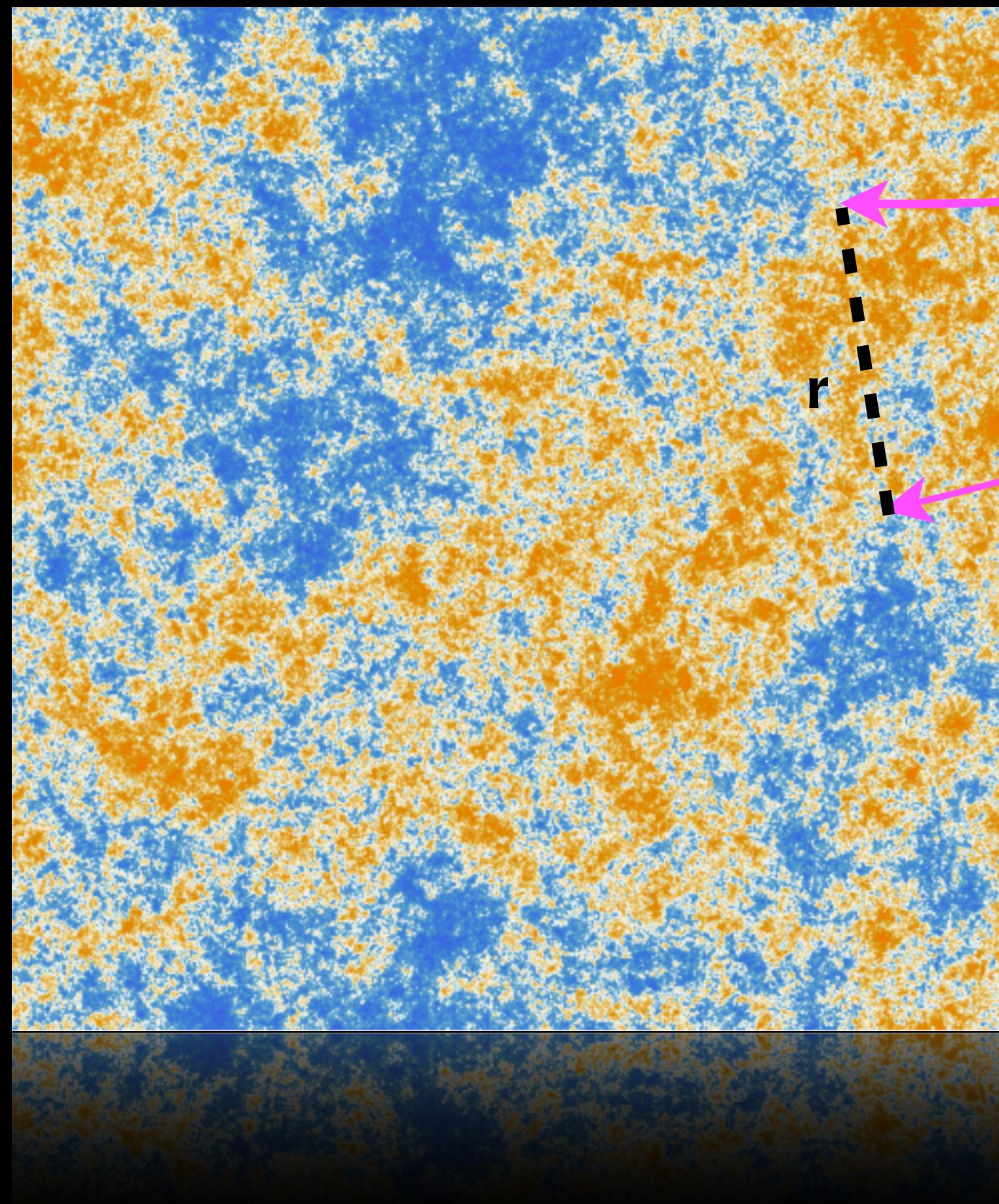
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$$\langle \delta(\mathbf{x}) \delta(\mathbf{x}') \rangle = \xi(r)$$

correlation function

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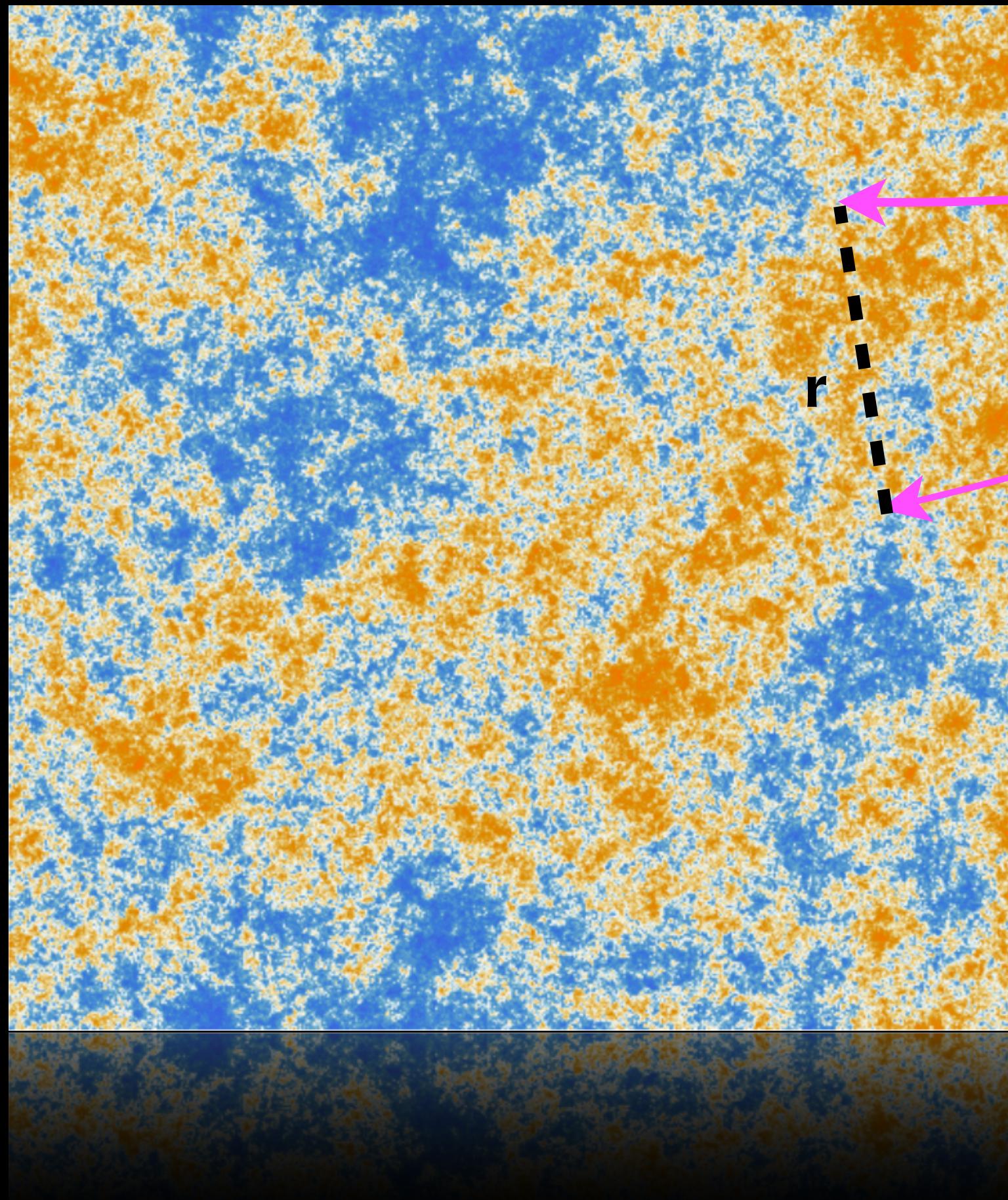
$$\langle \delta(\mathbf{x}) \delta(\mathbf{x}') \rangle = \xi(r)$$

$$\langle \delta(\mathbf{k}) \delta^*(\mathbf{k}') \rangle = P(k)$$

correlation function

power spectrum

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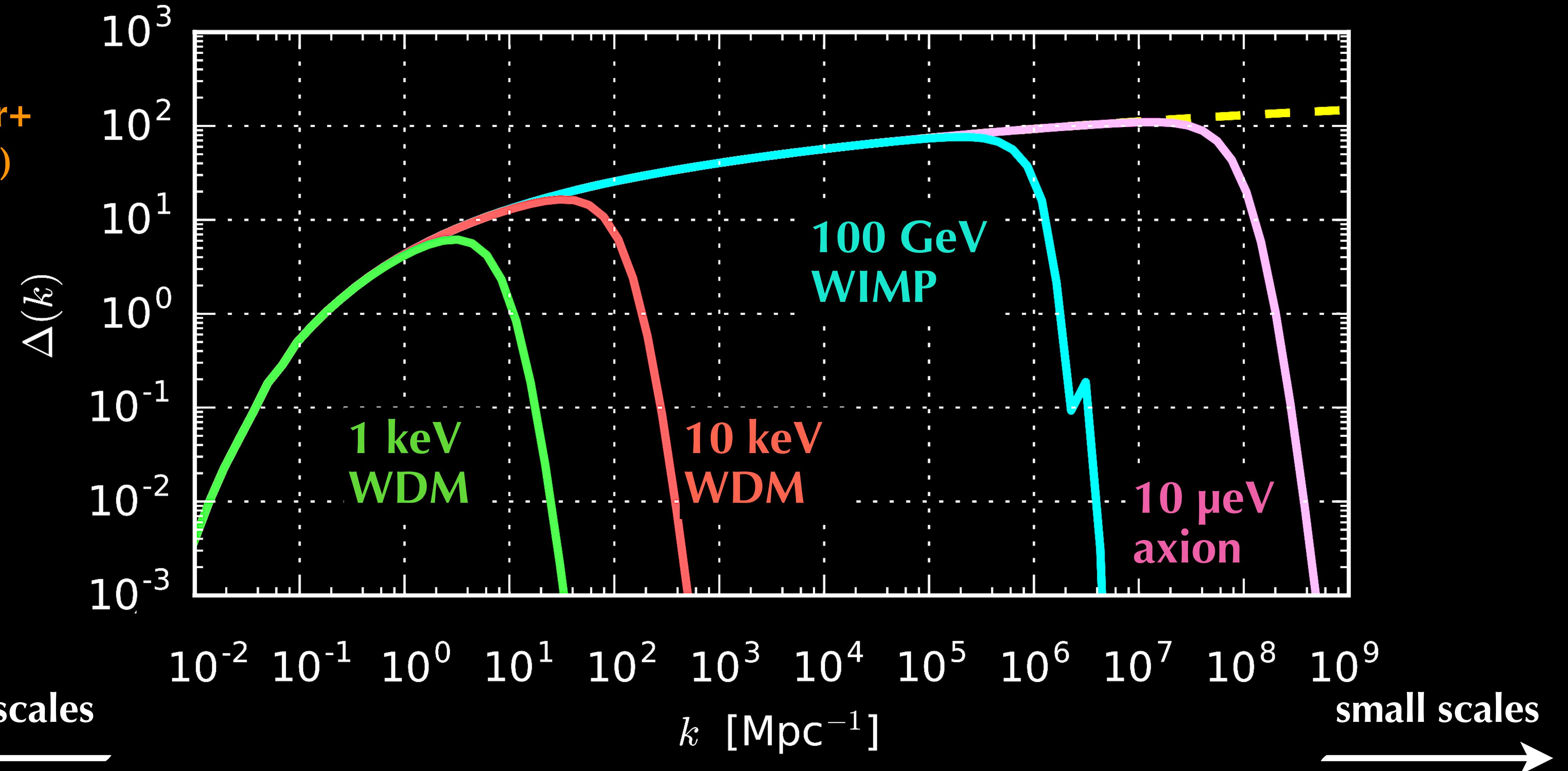
power spectrum

the statistical properties of a Gaussian random field are characterised fully by $P(k)$

fluctuation amplitudes $\propto \sqrt{P(k)}$

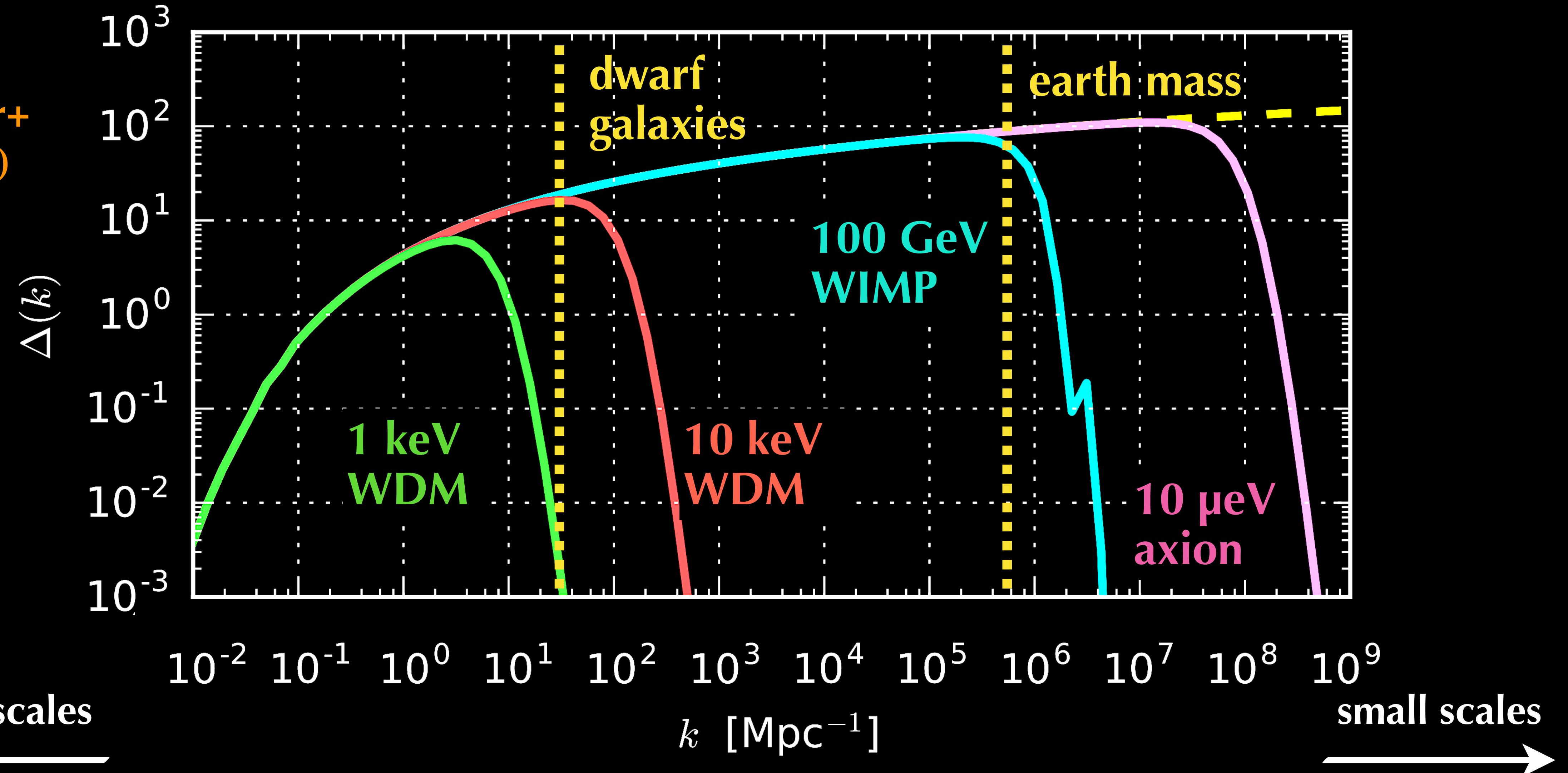
phases $\in [0, 2\pi]$

Stücker+
(2018)



one way to model different DM candidates is to change the $P(k)$ used to generate ICs

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now, we more or less have everything we need: we just need to **perturb the initial particle configuration according to these fluctuations**. the position of a particle initially at coordinate \mathbf{q} after some time t is given by:

$$\mathbf{x}(t) = \mathbf{q} +$$

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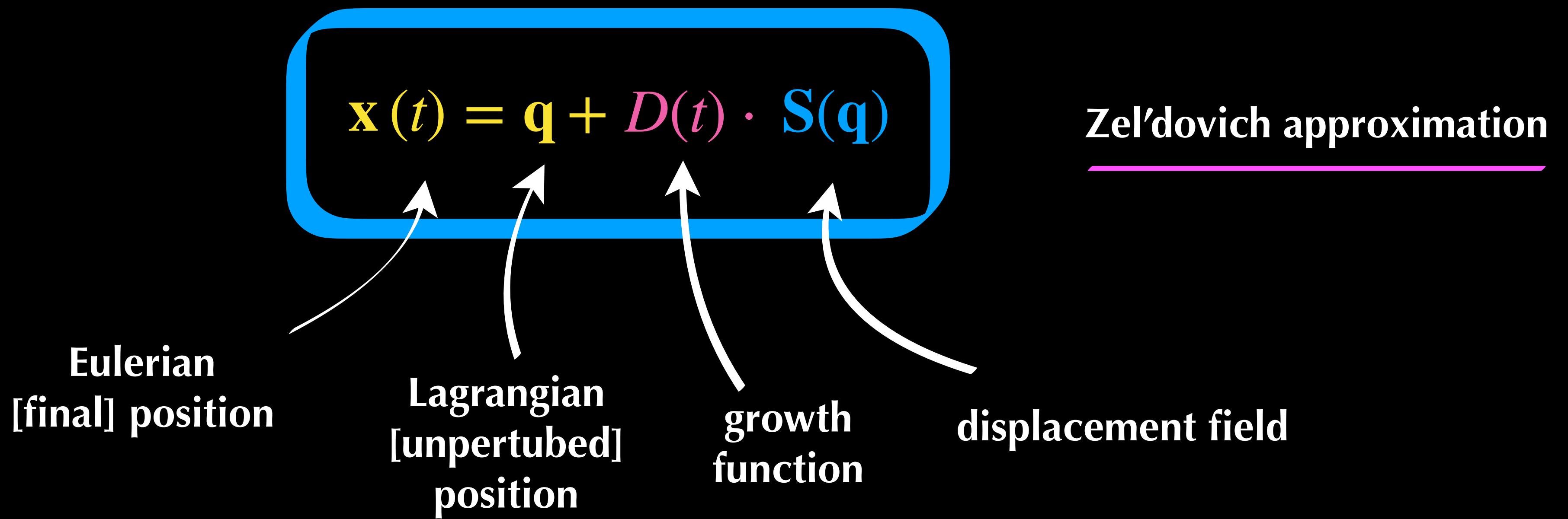
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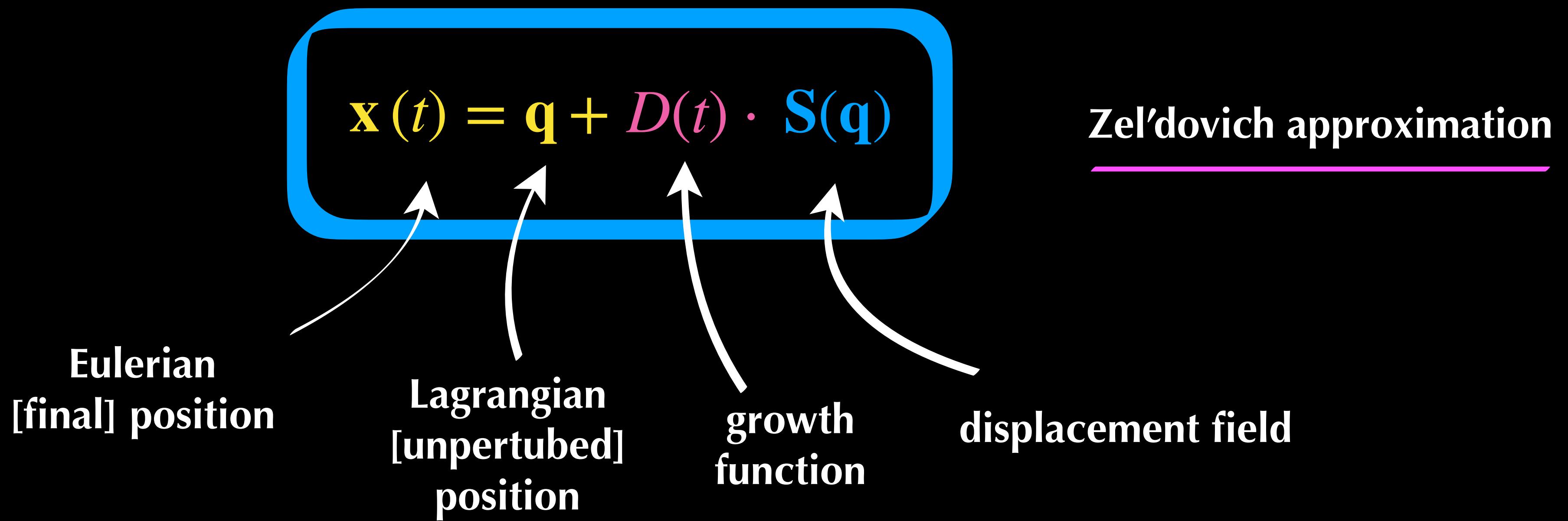
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Zel'dovich approximation

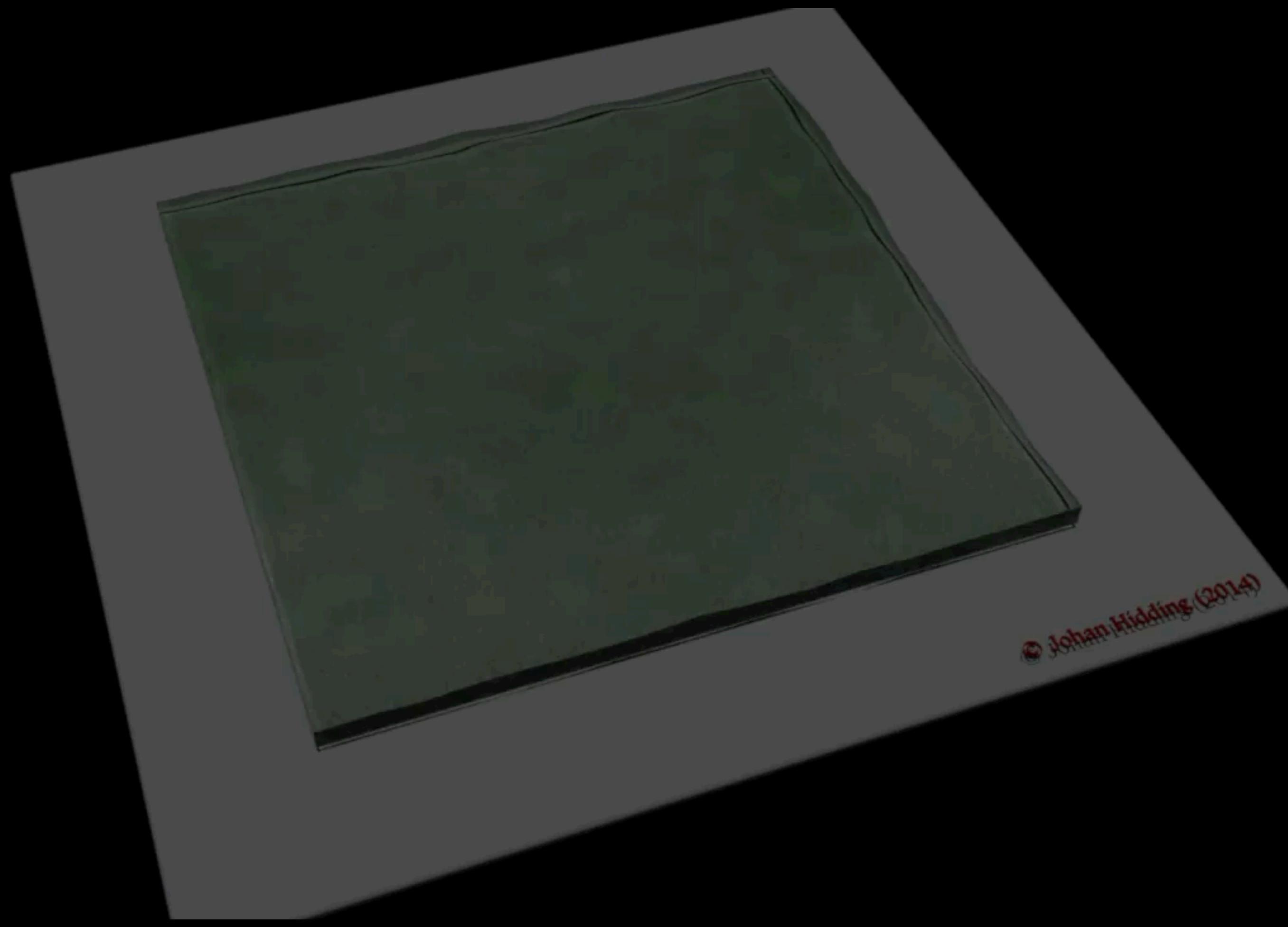
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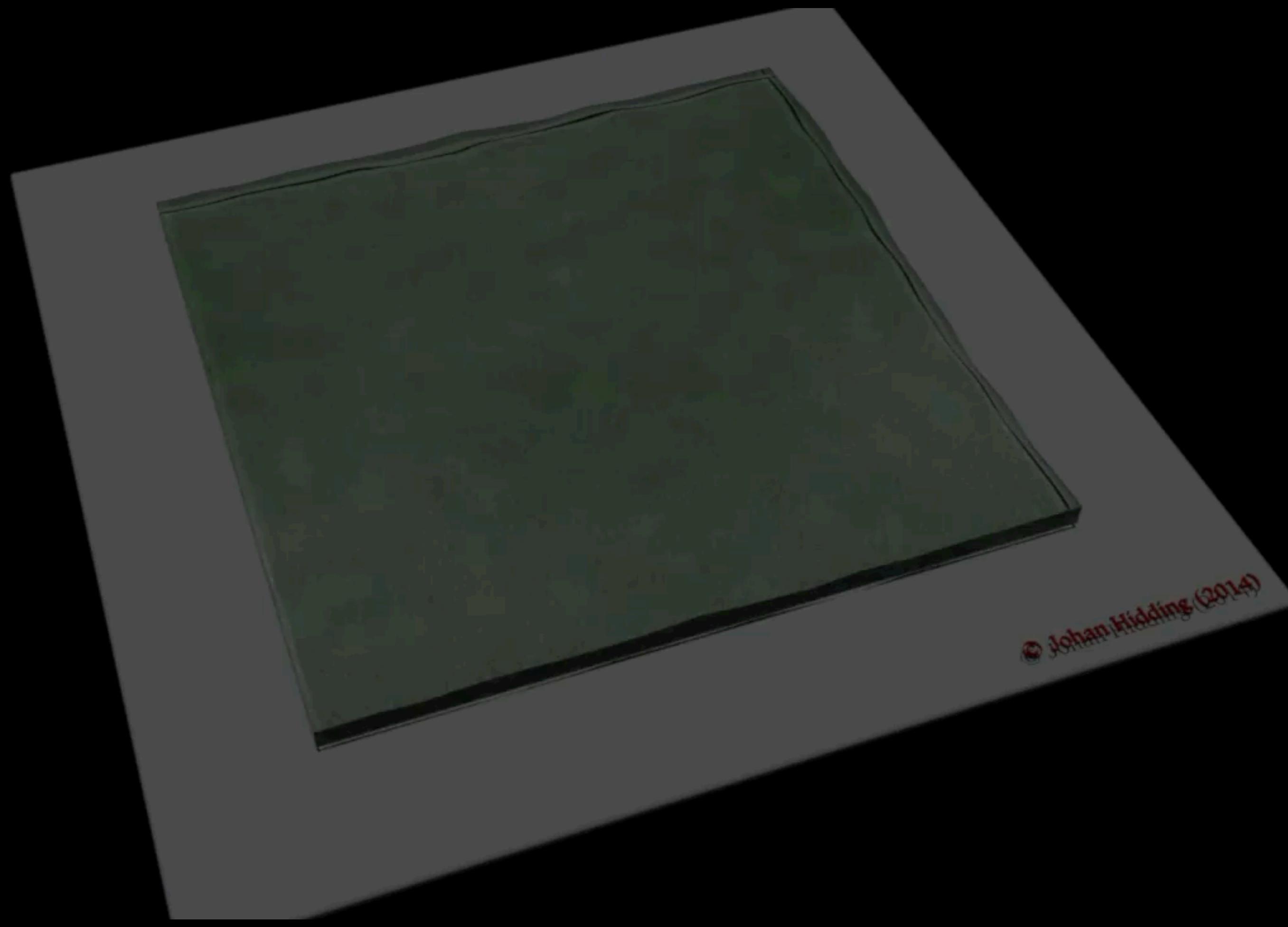
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$\mathbf{S}(\mathbf{q}) \propto -\nabla\Phi(\mathbf{q})$, and we know that Φ is related to $\delta(\mathbf{k})$ through **Poisson's equation**



credit: [Johan Hidding](#)



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choice of softening

$$\Phi(x) = -G \sum_{j=1}^N \frac{m_j}{\left[\left(\mathbf{x} - \mathbf{x}_j\right)^2 + \epsilon^2\right]^{1/2}}$$

gravitational softening, ϵ , is introduced in N-body simulations to avoid large-angle scattering due to close encounters. higher-resolution simulations use smaller softening [and therefore also shorter timesteps]

simple choice: $\epsilon = \alpha \cdot d = \alpha \frac{L_{\text{box}}}{N_p^{1/3}}$

$$\alpha = 0.01 - 0.05$$

mean inter-particle separation

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stochastic acceleration from close encounters

stricter choice: $\frac{Gm_{\text{DM}}}{\epsilon^2} \lesssim \frac{GM_{200}}{r_{200}^2} \Rightarrow \epsilon \gtrsim \frac{r_{200}}{\sqrt{N_{200}}}$

minimum mean-field acceleration in DM halo

Power+ (2003)

higher-order time integration

once forces on particles have even calculated, we need to propagate these forward in time and update their (1) **positions** and (2) **velocities**

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2nd order
“leapfrog” method

much more stable
“symplectic”
[~energy is conserved]

cosmological time integration

once forces on particles have even calculated, we need to propagate these forward in time and update their (1) **positions** and (2) **momenta**, $\mathbf{p} = a\dot{\mathbf{x}}$

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kick

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cosmological scale factor

Hubble parameter

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drift
kick—drift—kick

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$$E(\Delta\tau) = K\left(\frac{\Delta\tau}{2}\right) \circ D(\Delta\tau) \circ K\left(\frac{\Delta\tau}{2}\right)$$

cosmological scale factor

Hubble parameter

a note on the Hubble parameter

we define the Hubble parameter as $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$.
denoting the present-day value as

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$$H(a) = H_0 E(a)$$

$$E(a) = \sqrt{\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_k a^{-2} + \Omega_\Lambda \exp(3\tilde{w}(a))}$$

$$\tilde{w}(a) = (a - 1) w_a - (1 + w_0 + w_a) \log(a)$$

$$w(a) \equiv w_0 + w_a (1 - a)$$

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the cosmological model is therefore **fully specified** through $\Omega_m, \Omega_r, \Omega_k, \Omega_\Lambda, h, w_0$ and w_a