PRECISE summer school
Warsaw

July 05, 2023

hands-on cosmological simulations

session 2: cosmological integration and initial conditions

Sownak Bose
&
Shaun Brown

sownak.bose@durham.ac.uk
@Swnk16

Durham University

choice of softening

$$\Phi(x) = -G\sum_{j=1}^{N}\frac{m_j}{\left[\left(\mathbf{x}-\mathbf{x}_j\right)^2+\epsilon^2\right]^{1/2}}$$

gravitational softening, ϵ, is introduced in N-body simulations to avoid large-angle scattering due to close encounters. higher-resolution simulations use smaller softening [and therefore also shorter timesteps]

simple choice: $\epsilon = \alpha \cdot d = \alpha \frac{L_{\mathrm{box}}}{N_p^{1/3}}$

$\alpha = 0.01 - 0.05$

mean inter-particle separation

choice of softening

$$\Phi(x) = -G\sum_{j=1}^{N} \frac{m_j}{\left[\left(\mathbf{x}-\mathbf{x}_j\right)^2 + \epsilon^2\right]^{1/2}}$$

gravitational softening, ϵ, is introduced in N-body simulations to avoid large-angle scattering due to close encounters. higher-resolution simulations use smaller softening [and therefore also shorter timesteps]

simple choice: $\epsilon = \alpha \cdot d = \alpha \frac{L_{\mathrm{box}}}{N_p^{1/3}}$

$\alpha = 0.01 - 0.05$

mean inter-particle separation

stochastic acceleration from close encounters

stricter choice: $\frac{G m_{\mathrm{DM}}}{\epsilon^2} \lesssim \frac{G M_{200}}{r_{200}^2} \Rightarrow \epsilon \gtrsim \frac{r_{200}}{\sqrt{N_{200}}}$

minimum mean-field acceleration in DM halo

Power+ (2003)

higher-order time integration

once forces on particles have even calculated, we need to propagate these forward in time and update their (1) positions and (2) velocities

kick $$\mathbf{v}_j^{n+1/2} = \mathbf{v}_j^n + \frac{1}{2}\mathbf{a}_j^n \cdot \Delta t$$

drift $$\mathbf{x}_j^{n+1} = \mathbf{x}_j^n + \mathbf{v}_j^{n+1/2} \cdot \Delta t$$

kick $$\mathbf{v}_j^{n+1} = \mathbf{v}_j^{n+1/2} + \frac{1}{2}\mathbf{a}_j^{n+1} \cdot \Delta t$$

2nd order
"leapfrog" method

much more stable

"symplectic"
[~energy is conserved]

cosmological time integration

once forces on particles have even calculated, we need to propagate these forward in time and update their (1) positions and (2) momenta, $\mathbf{p} = a\dot{\mathbf{x}}$

cosmological time integration

once forces on particles have even calculated, we need to propagate these forward in time and update their (1) positions and (2) momenta, $\mathbf{p} = a\dot{\mathbf{x}}$

cosmological scale factor

cosmological time integration

once forces on particles have even calculated, we need to propagate these forward in time and update their (1) positions and (2) momenta, $\mathbf{p} = a\dot{\mathbf{x}}$

kick

$$\mathbf{p}_j\left(\tau_n + \Delta\tau\right) = \mathbf{p}_j\left(\tau_n\right) + \mathbf{a}_j\left(\tau_n\right)\int_{\tau_n}^{\tau_n+\Delta\tau}\frac{\mathrm{d}\tau}{aH(a)}$$

cosmological scale factor

Hubble parameter

cosmological time integration

once forces on particles have even calculated, we need to propagate these forward in time and update their (1) positions and (2) momenta, $\mathbf{p} = a\dot{\mathbf{x}}$

kick $$\mathbf{p}_j\left(\tau_n + \Delta\tau\right) = \mathbf{p}_j\left(\tau_n\right) + \mathbf{a}_j\left(\tau_n\right)\int_{\tau_n}^{\tau_n+\Delta\tau} \frac{\mathrm{d}\tau}{aH(a)}$$

drift $$\mathbf{x}_j\left(\tau_n + \Delta\tau\right) = \mathbf{x}_j\left(\tau_n\right) + \mathbf{p}_j\left(\tau_n\right)\int_{\tau_n}^{\tau_n+\Delta\tau} \frac{\mathrm{d}\tau}{a^2H(a)}$$

cosmological scale factor

Hubble parameter

cosmological time integration

once forces on particles have even calculated, we need to propagate these forward in time and update their (1) positions and (2) momenta, $\mathbf{p} = a\dot{\mathbf{x}}$

kick

$$\mathbf{p}_j\left(\tau_n + \Delta\tau\right) = \mathbf{p}_j\left(\tau_n\right) + \mathbf{a}_j\left(\tau_n\right)\int_{\tau_n}^{\tau_n+\Delta\tau}\frac{\mathrm{d}\tau}{aH(a)}$$

drift

$$\mathbf{x}_j\left(\tau_n + \Delta\tau\right) = \mathbf{x}_j\left(\tau_n\right) + \mathbf{p}_j\left(\tau_n\right)\int_{\tau_n}^{\tau_n+\Delta\tau}\frac{\mathrm{d}\tau}{a^2H(a)}$$

kick—drift—kick

$$E\left(\Delta\tau\right) = K\left(\frac{\Delta\tau}{2}\right)\circ D\left(\Delta\tau\right)\circ K\left(\frac{\Delta\tau}{2}\right)$$

cosmological scale factor

Hubble parameter

a note on the Hubble parameter

we define the Hubble parameter as $H(t) \equiv \dfrac{\dot{a}(t)}{a(t)}$.
denoting the present-day value as

$$H_0 \equiv H(t = t_{\rm now}) = 100h\,{\rm km s^{-1} Mpc^{-1}}$$

a note on the **Hubble** parameter

we define the Hubble parameter as $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$, denoting the present-day value as

$$H_0 \equiv H(t = t_{\text{now}}) = 100h \, \text{kms}^{-1}\text{Mpc}^{-1}$$

$$H(a) = H_0 \, E(a)$$

$$E(a) = \sqrt{\Omega_m \, a^{-3} + \Omega_r \, a^{-4} + \Omega_k \, a^{-2} + \Omega_\Lambda \exp\left(3\tilde{w}(a)\right)}$$

$$\tilde{w}(a) = (a-1) \, w_a - \left(1 + w_0 + w_a\right) \log\left(a\right)$$

$$w(a) \equiv w_0 + w_a \, (1 - a)$$

a note on the Hubble parameter

we define the Hubble parameter as $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$. denoting the present-day value as

$$H_0 \equiv H(t = t_{\rm now}) = 100h\,{\rm km s^{-1} Mpc^{-1}}$$

$$H(a) = H_0\,E(a)$$

$$E(a) = \sqrt{\Omega_m\,a^{-3} + \Omega_r\,a^{-4} + \Omega_k\,a^{-2} + \Omega_\Lambda \exp\left(3\tilde{w}(a)\right)}$$

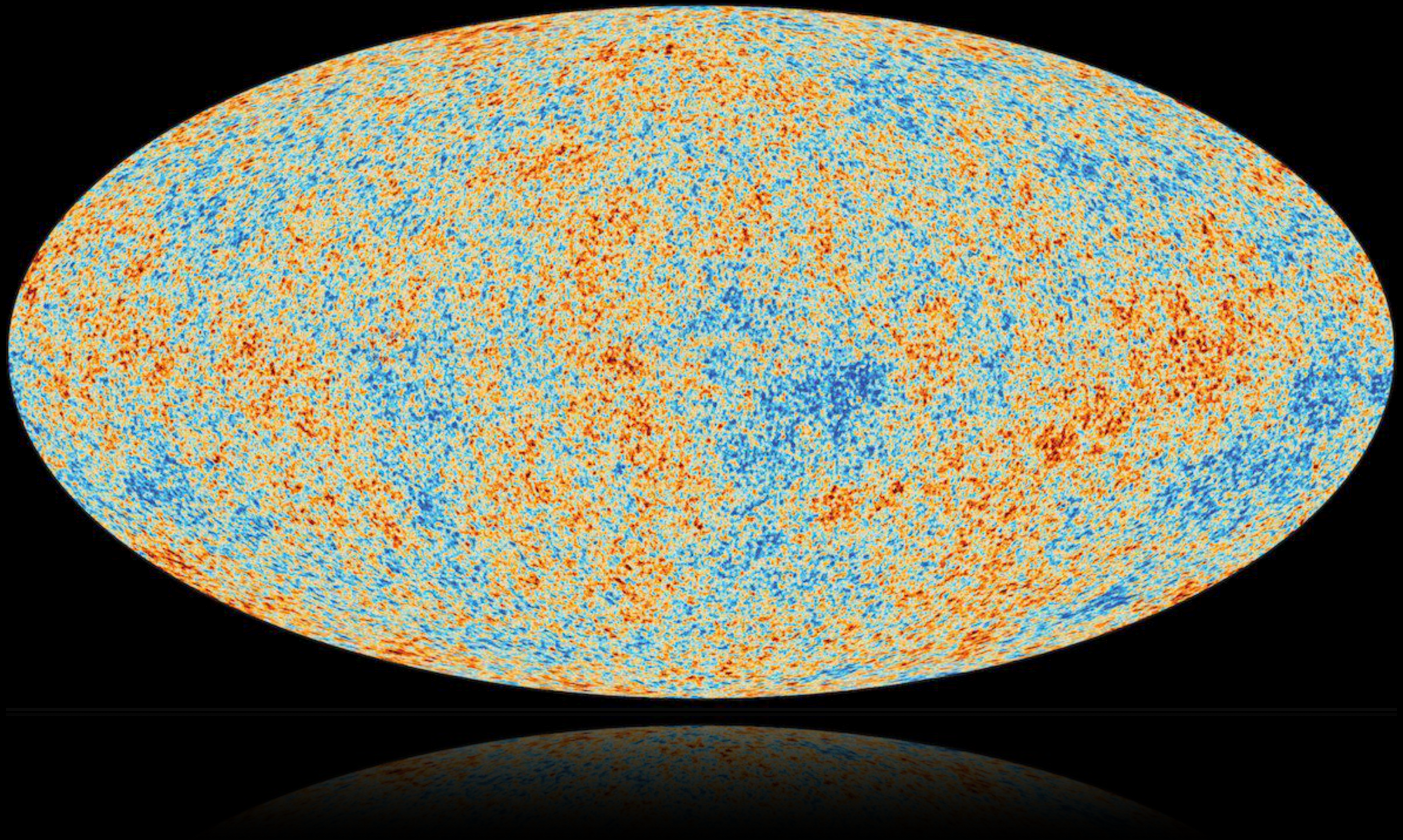
$$\tilde{w}(a) = (a-1)\,w_a - \left(1 + w_0 + w_a\right)\log\left(a\right)$$

$$w(a) \equiv w_0 + w_a\,(1-a)$$

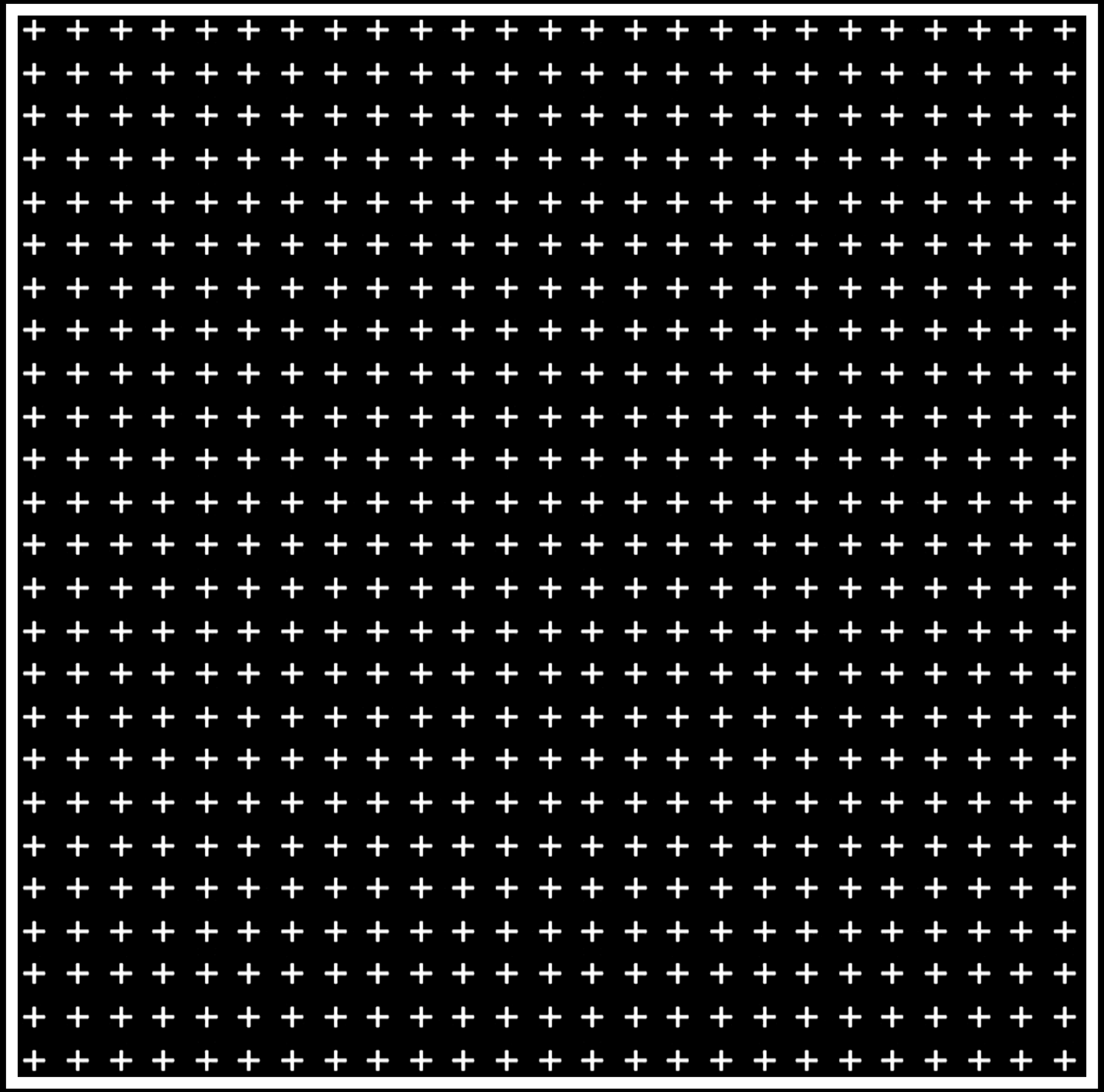
the cosmological model is therefore *fully specified* through Ω_m, Ω_r, Ω_k, Ω_Λ, h, w_0 and w_a

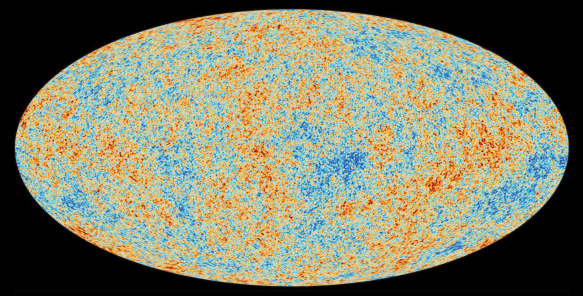
setting up initial conditions

measurement of the cosmic microwave background radiation gives us *precise constraints* on cosmological initial conditions

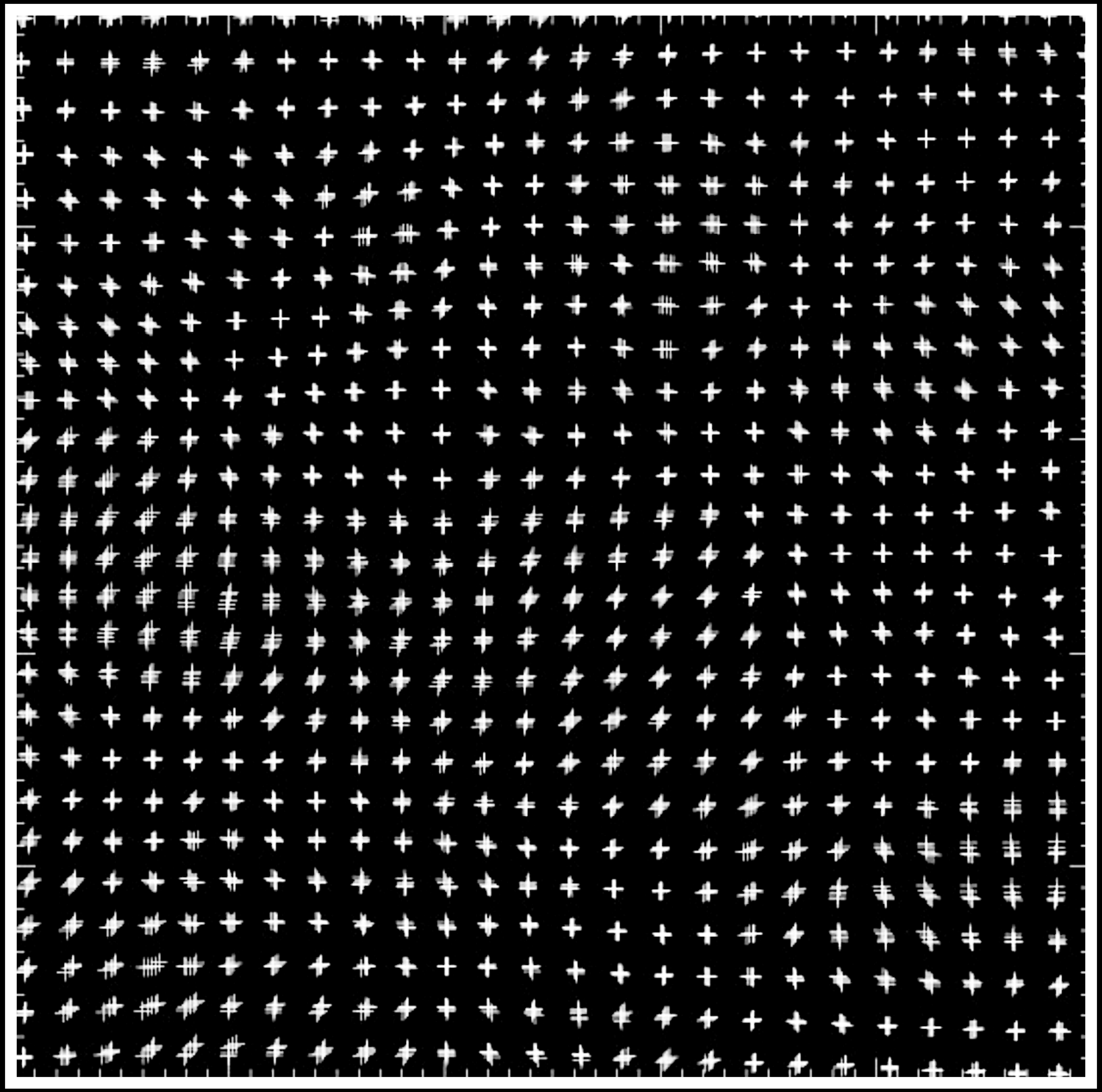


homogeneous & isotropic ICs





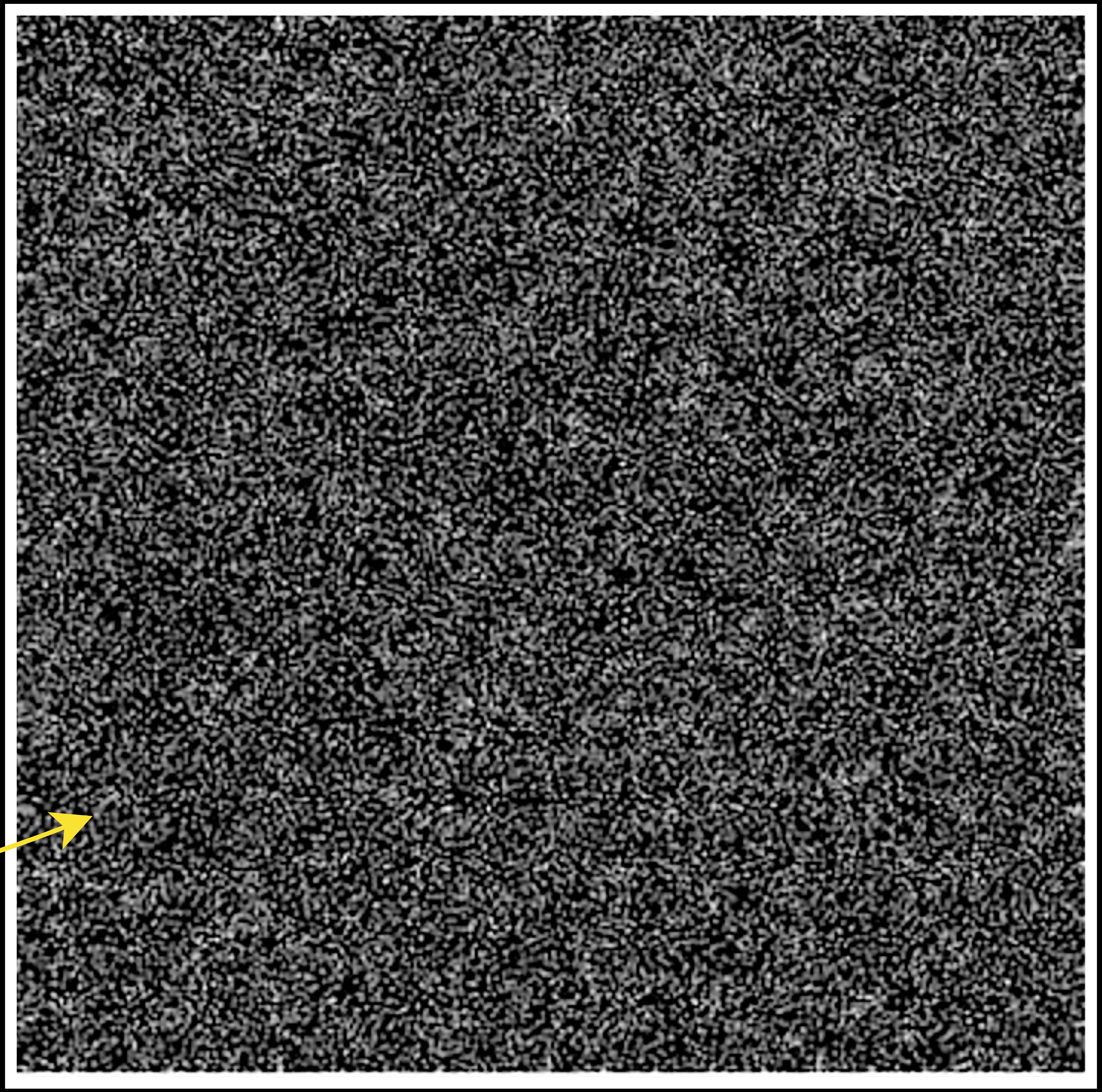
perturbed ICs



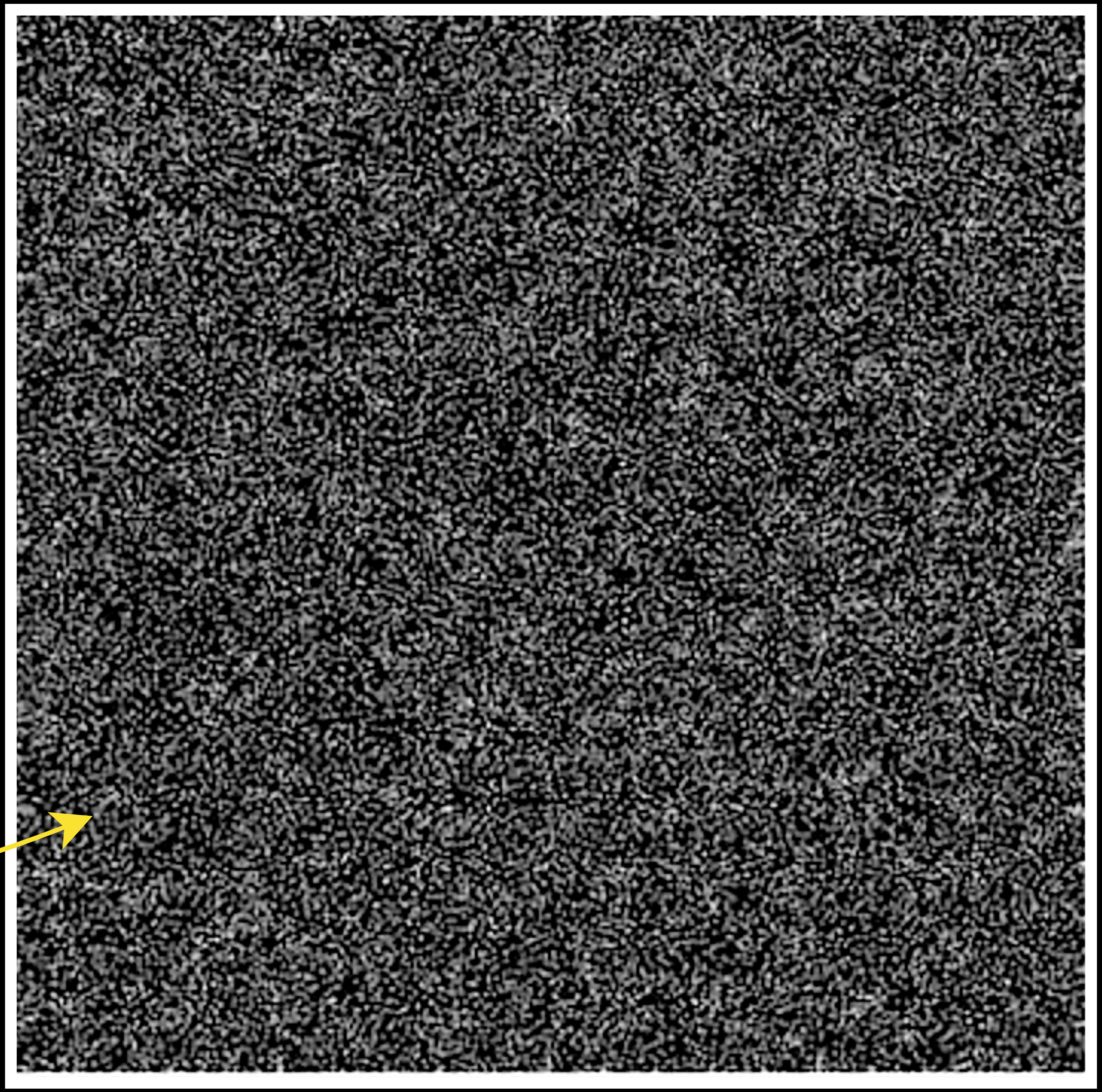
these tiny fluctuations will grow into non-linear structures (e.g. filaments, walls, haloes, clusters) through gravitational instability

glass initial conditions

start with a Poisson [random] particle distribution and evolve this with the sign of gravity “inverted” [repulsive force] until particles “freeze” in comoving coordinates

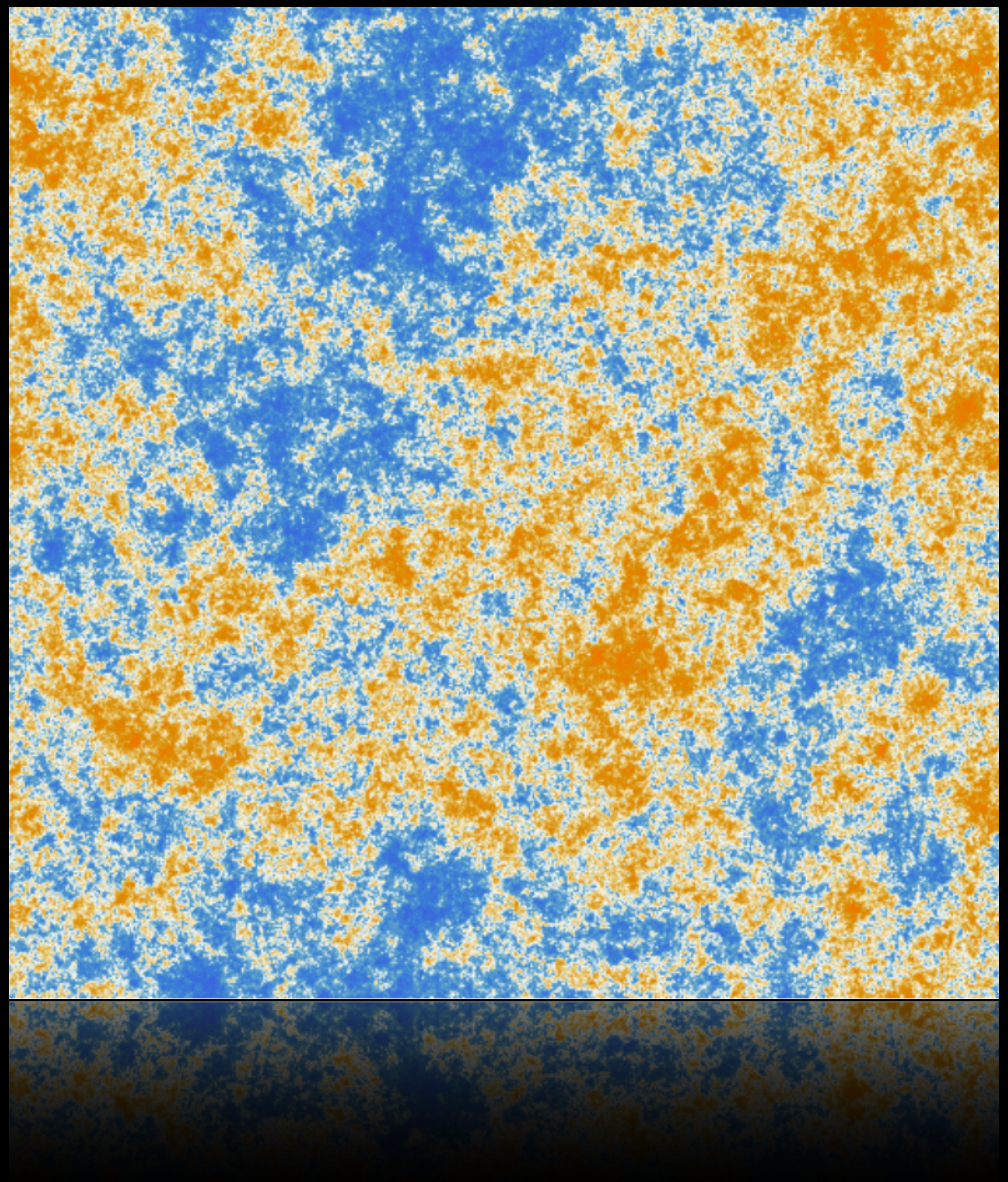


helps wash out the memory of the initial lattice start with a Poisson [random] particle distribution and evolve this with the sign of gravity “inverted” [repulsive force] until particles “freeze” in comoving coordinates

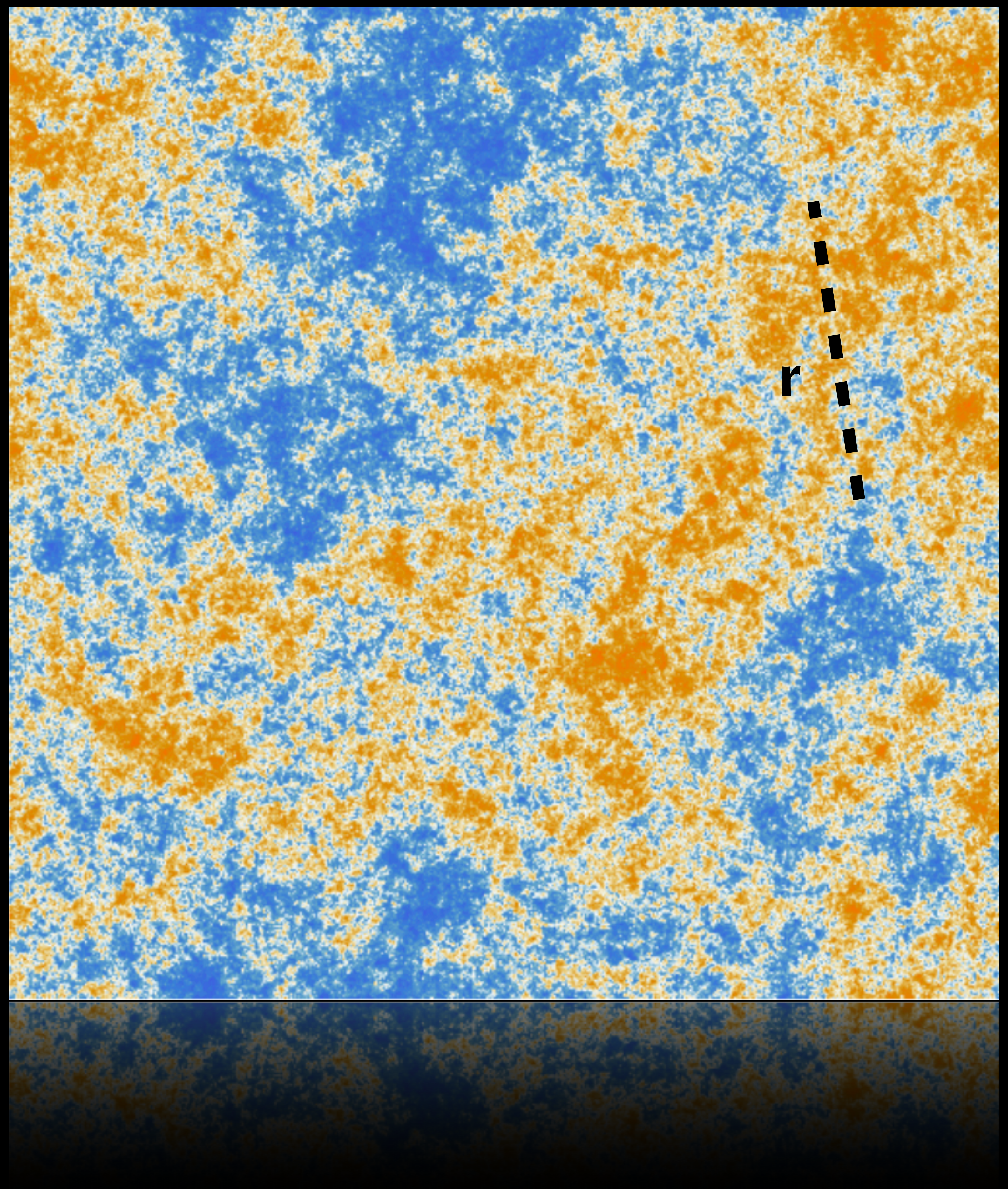


helps wash out the memory of the initial lattice

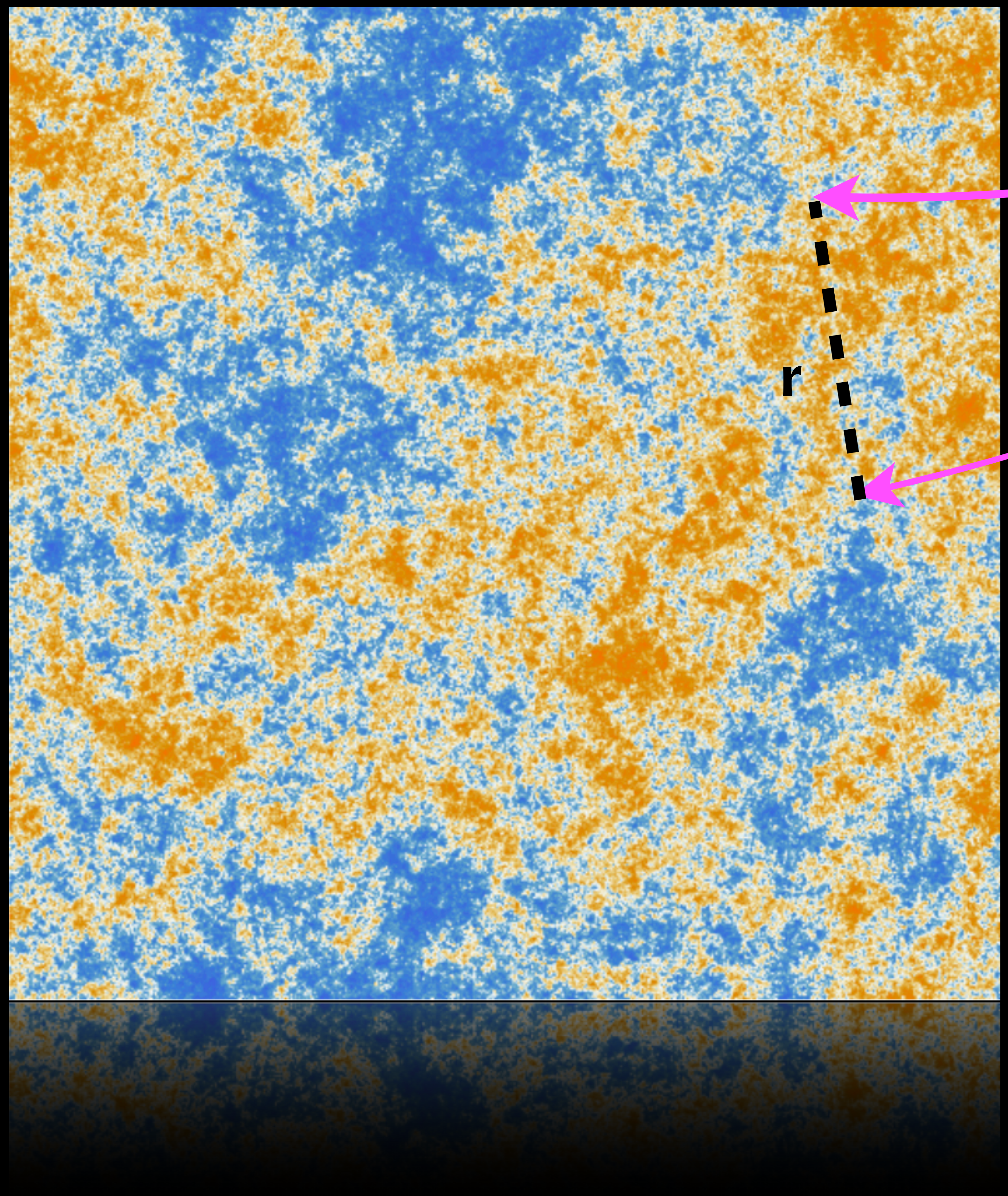
goal: create a **Gaussian random distribution** that is statistically consistent with the fluctuations measured in the CMB



goal: create a Gaussian random distribution that is statistically consistent with the fluctuations measured in the CMB

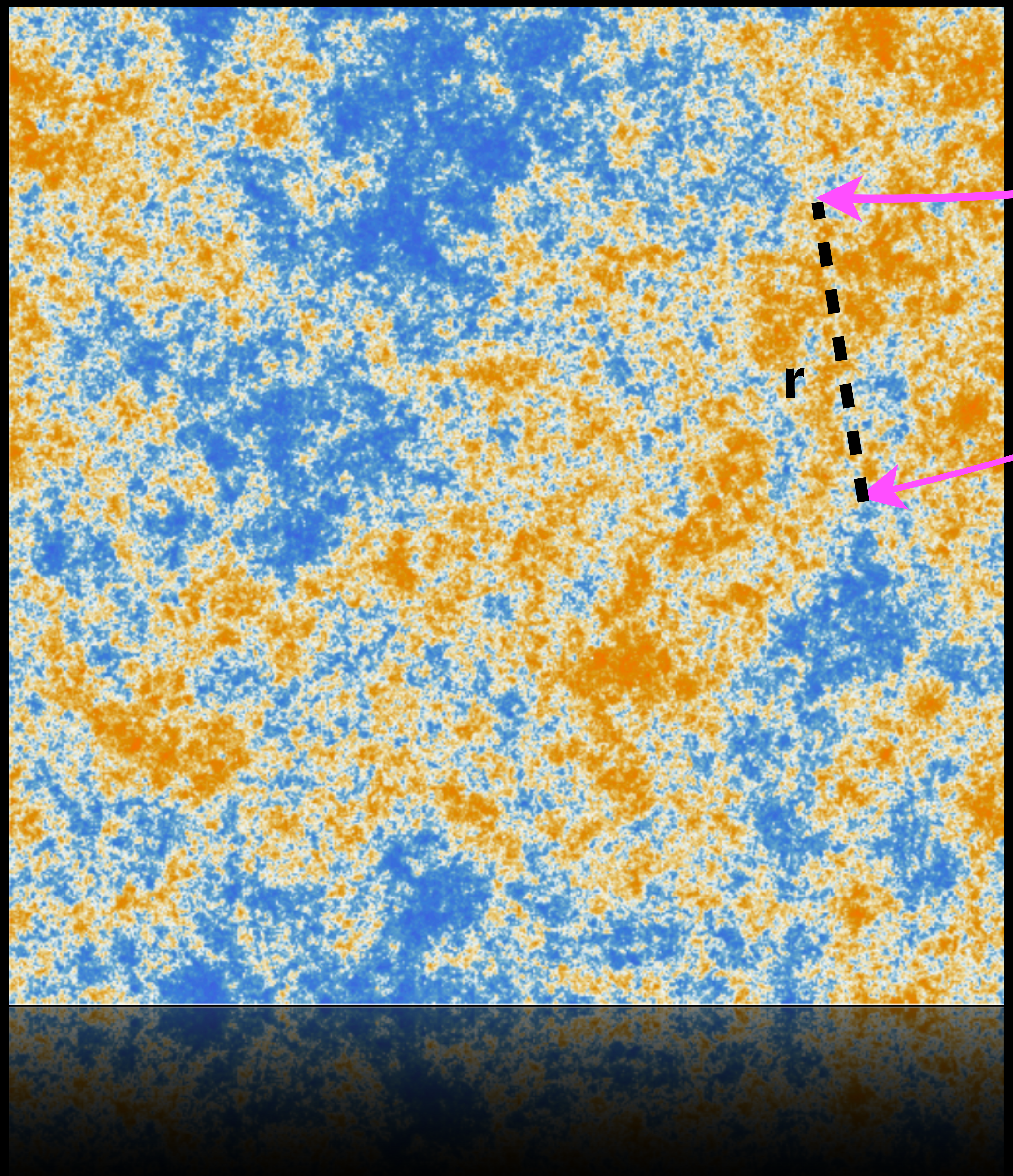


goal: create a **Gaussian random distribution** that is statistically consistent with the fluctuations measured in the CMB



$$\langle \delta(\mathbf{x})\, \delta(\mathbf{x}') \rangle$$

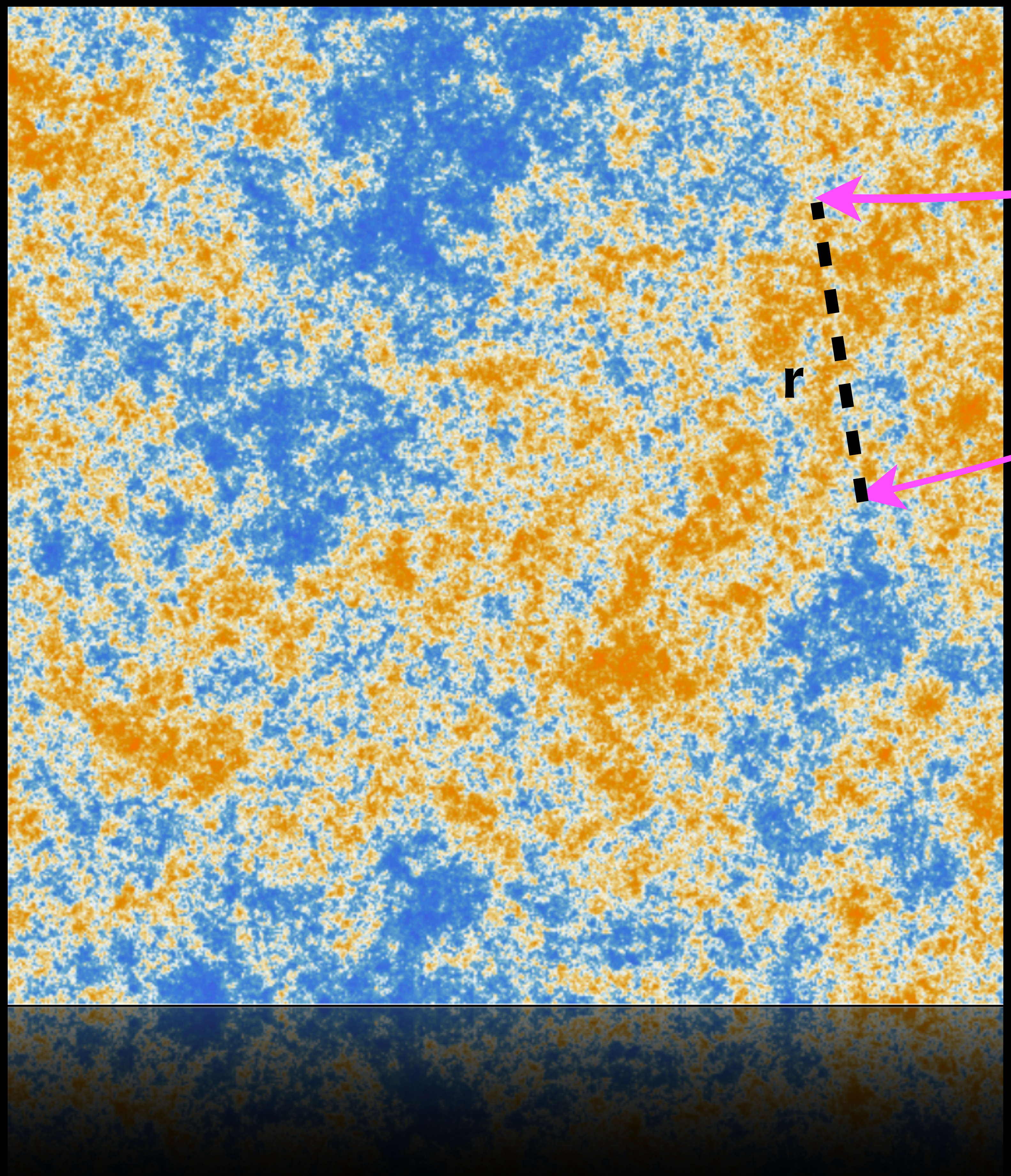
goal: create a **Gaussian random distribution** that is statistically consistent with the fluctuations measured in the CMB



$$\langle \delta(\mathbf{x})\, \delta(\mathbf{x}') \rangle = \xi(r)$$

correlation function

goal: create a **Gaussian random distribution** that is statistically consistent with the fluctuations measured in the CMB



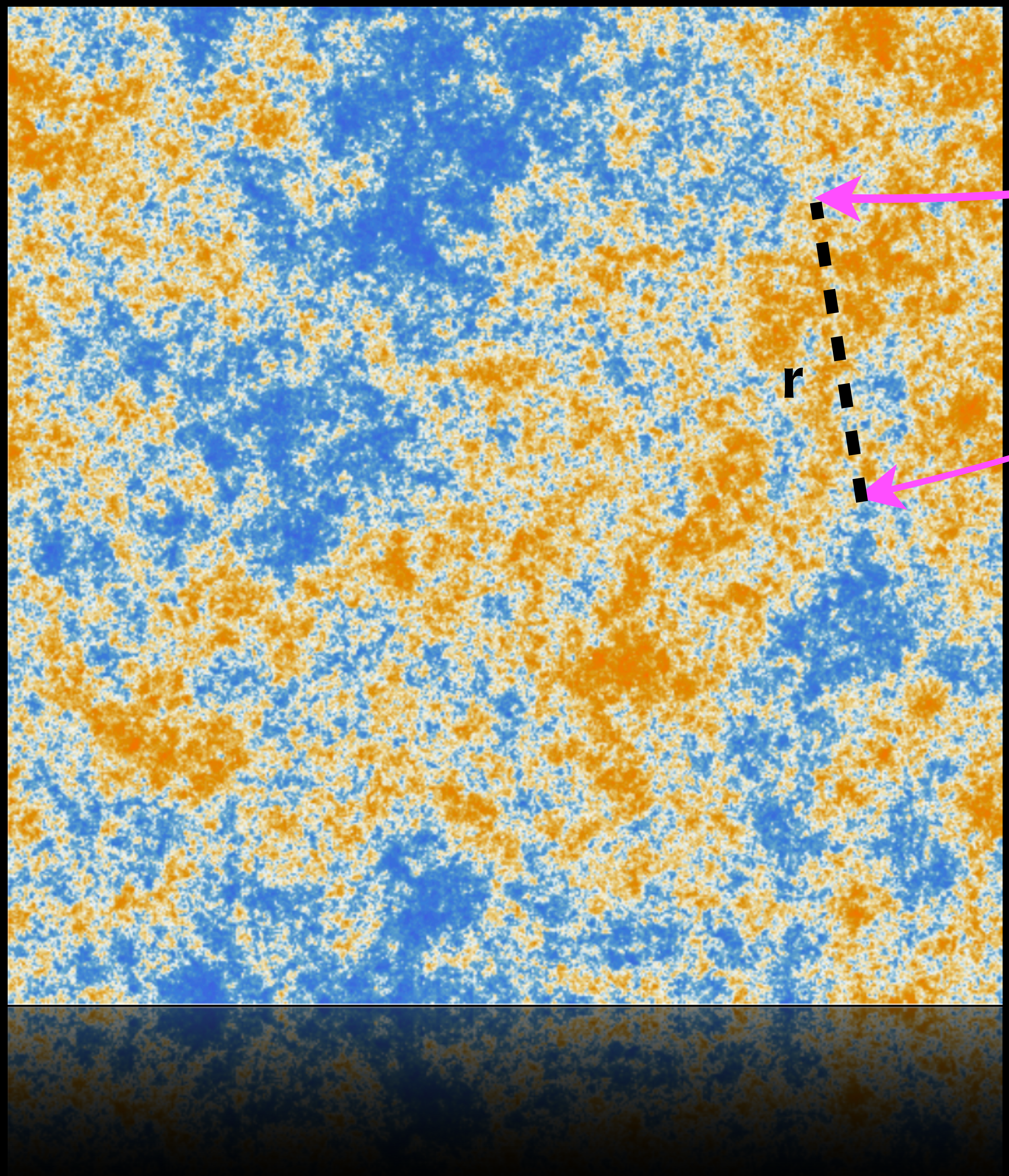
$$\langle \delta(\mathbf{x})\, \delta(\mathbf{x}') \rangle = \xi(r)$$

correlation function

$$\langle \delta(\mathbf{k})\, \delta^{*}(\mathbf{k}') \rangle = P(k)$$

power spectrum

goal: create a **Gaussian random distribution** that is statistically consistent with the fluctuations measured in the CMB



$$\langle \delta(\mathbf{x})\, \delta(\mathbf{x}') \rangle = \xi(r)$$

correlation function

$$\langle \delta(\mathbf{k})\, \delta^{*}(\mathbf{k}') \rangle = P(k)$$

power spectrum

the statistical properties of a Gaussian random field are characterised fully by $P(k)$

fluctuation amplitudes $\propto \sqrt{P(k)}$

phases $\in [0, 2\pi]$

the primordial power spectrum

inflation makes a specific prediction for the primordial density field that is a power-law:

$$P(k) \propto k^{n_s}$$

where $n_s = -1$ corresponds to a scale-invariant power spectrum

the primordial power spectrum

inflation makes a specific prediction for the primordial density field that is a power-law:

$$P(k) \propto k^{n_s}$$

where $n_s = -1$ corresponds to a scale-invariant power spectrum

depending on what the Universe is made of and how fluctuations grow in time and space, the inflationary power spectrum is modified by a transfer function

$$P(k, t) = Ak^{n_s}$$

the primordial power spectrum

inflation makes a specific prediction for the primordial density field that is a power-law:

$$P(k) \propto k^{n_s}$$

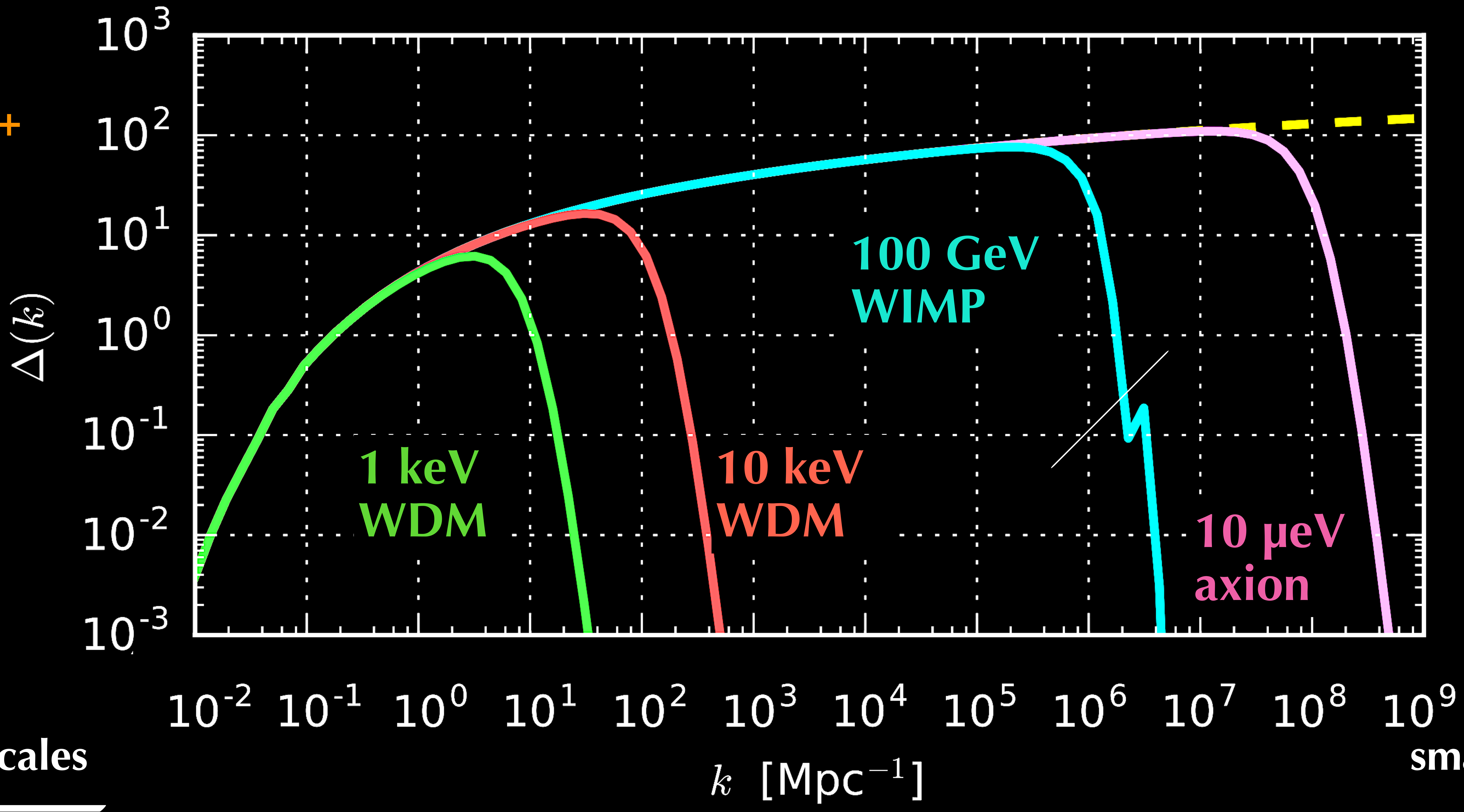
where $n_s = -1$ corresponds to a scale-invariant power spectrum

depending on what the Universe is made of and how fluctuations grow in time and space, the inflationary power spectrum is modified by a transfer function

$$P(k,t) = Ak^{n_s} \left| T(k,t) \right|^2$$

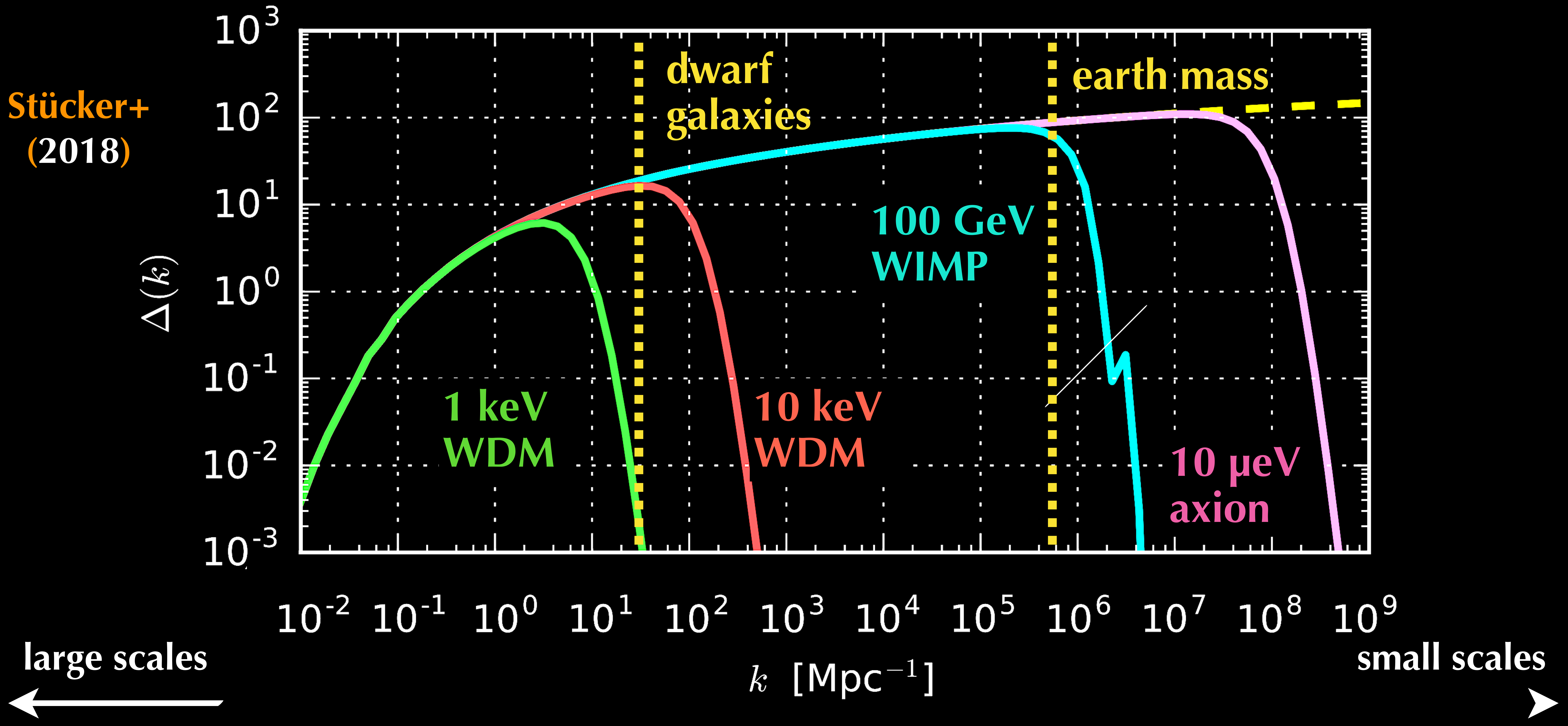
computed by public codes like CAMB, CLASS etc.

Stücker+ (2018)



large scales ← → small scales

one way to model different DM candidates is to change the $P(k)$ used to generate ICs



one way to model different DM candidates is to change the $P(k)$ used to generate ICs

now, we more or less have everything we need: we just need to perturb the initial particle configuration according to these fluctuations. the position of a particle initially at coordinate q after some time t is given by:

$$\mathbf{x}(t) = \mathbf{q} +$$

now, we more or less have everything we need: we just need to perturb the initial particle configuration according to these fluctuations. the position of a particle initially at coordinate q after some time t is given by:

$$\mathbf{x}(t) = \mathbf{q} + \qquad \mathbf{S}(\mathbf{q})$$

now, we more or less have everything we need: we just need to perturb the initial particle configuration according to these fluctuations. the position of a particle initially at coordinate q after some time t is given by:

$$\mathbf{x}(t) = \mathbf{q} + D(t) \cdot \mathbf{S}(\mathbf{q})$$

now, we more or less have everything we need: we just need to perturb the initial particle configuration according to these fluctuations. the position of a particle initially at coordinate q after some time t is given by:

$$\boxed{\mathbf{x}(t) = \mathbf{q} + D(t) \cdot \mathbf{S}(\mathbf{q})}$$

Zel′dovich approximation

now, we more or less have everything we need: we just need to **perturb the initial particle configuration according to these fluctuations**. the position of a particle initially at coordinate q after some time t is given by:

$$\boxed{\mathbf{x}(t) = \mathbf{q} + D(t) \cdot \mathbf{S}(\mathbf{q})}$$

Zel'dovich approximation

Eulerian
[final] position

Lagrangian
[unpertubed]
position

growth
function

displacement field

now, we more or less have everything we need: we just need to perturb the initial particle configuration according to these fluctuations. the position of a particle initially at coordinate $\mathbf{q}$ after some time t is given by:

$$\boxed{\mathbf{x}\,(t) = \mathbf{q} + D(t) \cdot \mathbf{S}(\mathbf{q})}$$

Zel'dovich approximation

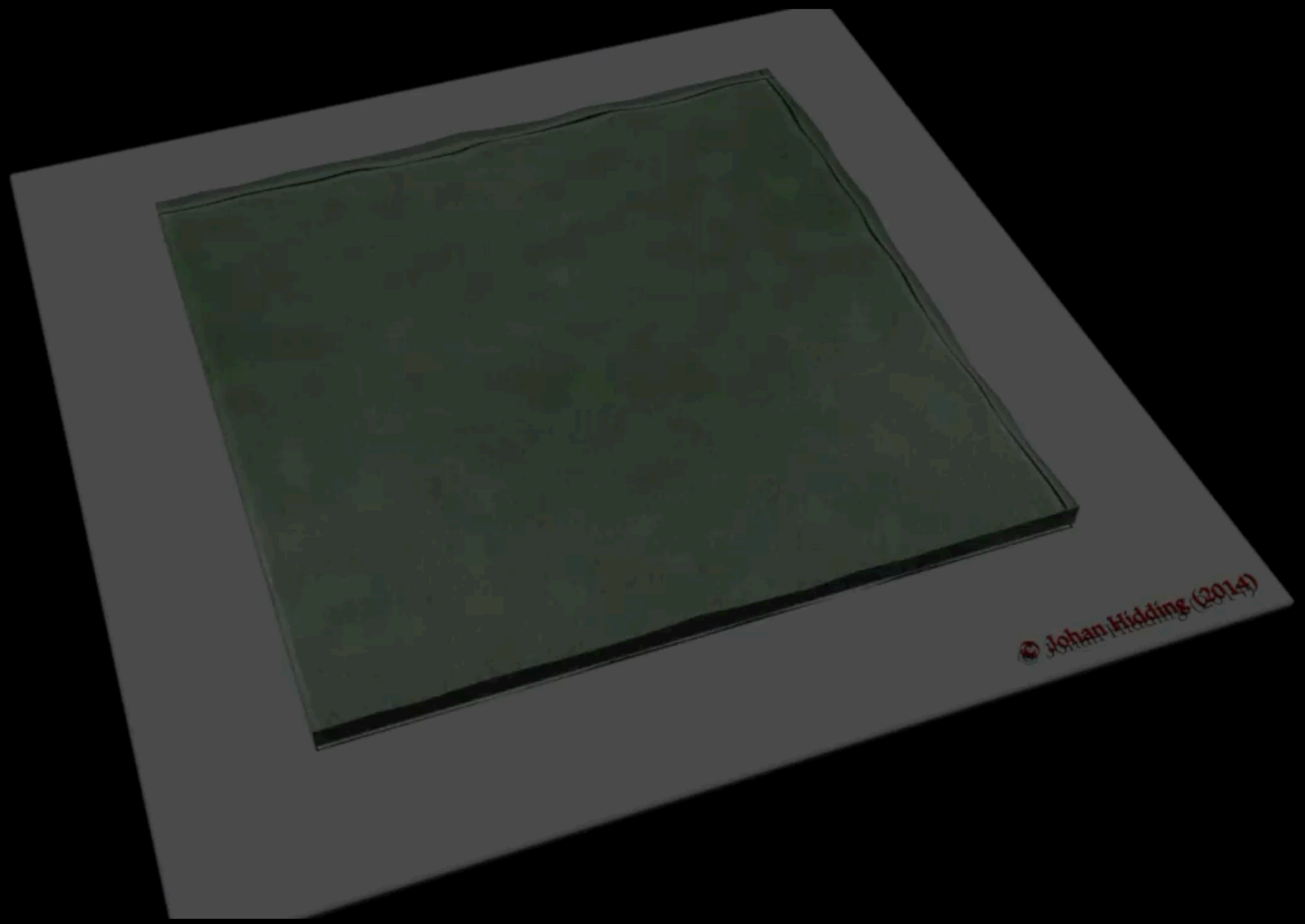
Eulerian [final] position

Lagrangian [unpertubed] position

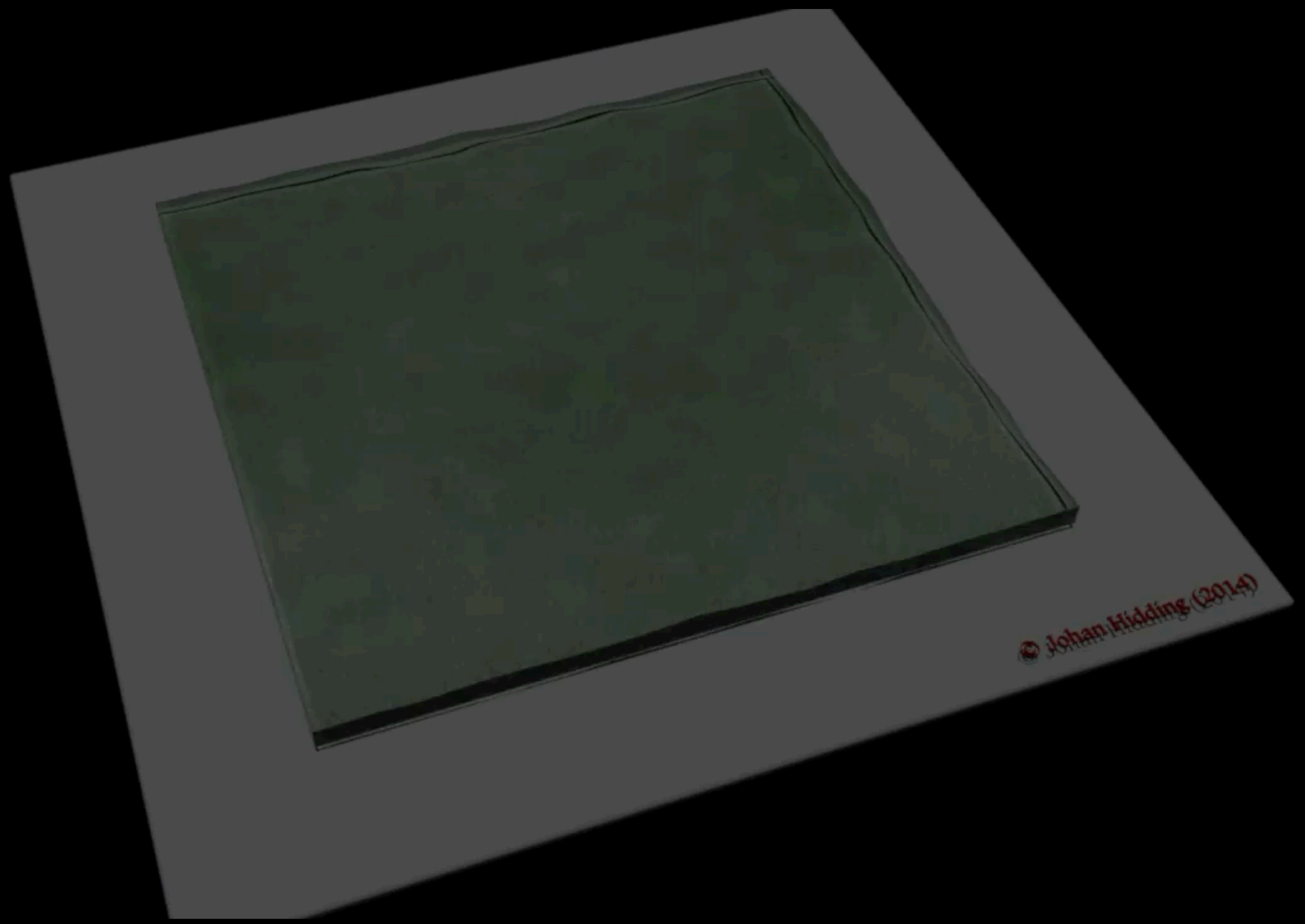
growth function

displacement field

$\mathbf{S}(\mathbf{q}) \propto -\nabla\Phi(\mathbf{q})$, and we know that Φ is related to $\delta(\mathbf{k})$ through Poisson's equation



credit: Johan Hidding



credit: Johan Hidding

the non-linear density field

while the initial conditions of our cosmos may be approximated as a Gaussian random field, the late-time, non-linear density is anything but Gaussian! gravitational instability naturally transitions the matter field away from Gaussianity

$$\boxed{\langle \delta(\mathbf{k}, z)\, \delta^*(\mathbf{k}', z) \rangle = P(k, z)}$$

the non-linear density field

while the initial conditions of our cosmos may be approximated as a Gaussian random field, the late-time, non-linear density is anything but Gaussian! gravitational instability naturally transitions the matter field away from Gaussianity

$$\langle \delta(\mathbf{k}, z)\, \delta^*(\mathbf{k}', z) \rangle = P(k, z)$$

given a density field, $\delta(\mathbf{x})$, you can also quantify the degree of non-Gaussianity by computing moments:

size of the grid used to smooth the density field

$$\langle \delta^n \rangle = \frac{1}{N_g} \sum_i^{N_g} \left(\delta^i - \langle \delta \rangle \right)^n$$

$n = 0$ (mean)
$n = 2$ (variance)
$n = 3$ (skewness)
$n = 4$ (kurtosis)

lecture notes:
https://github.com/sownakbose/PRECISE_Summer_School_Sims

jupyter notebooks:
https://github.com/Shaun-T-Brown/Summer_school