PRECISE summer school
Warsaw

July 03, 2023

hands-on cosmological simulations

session 1: background and the N-body method

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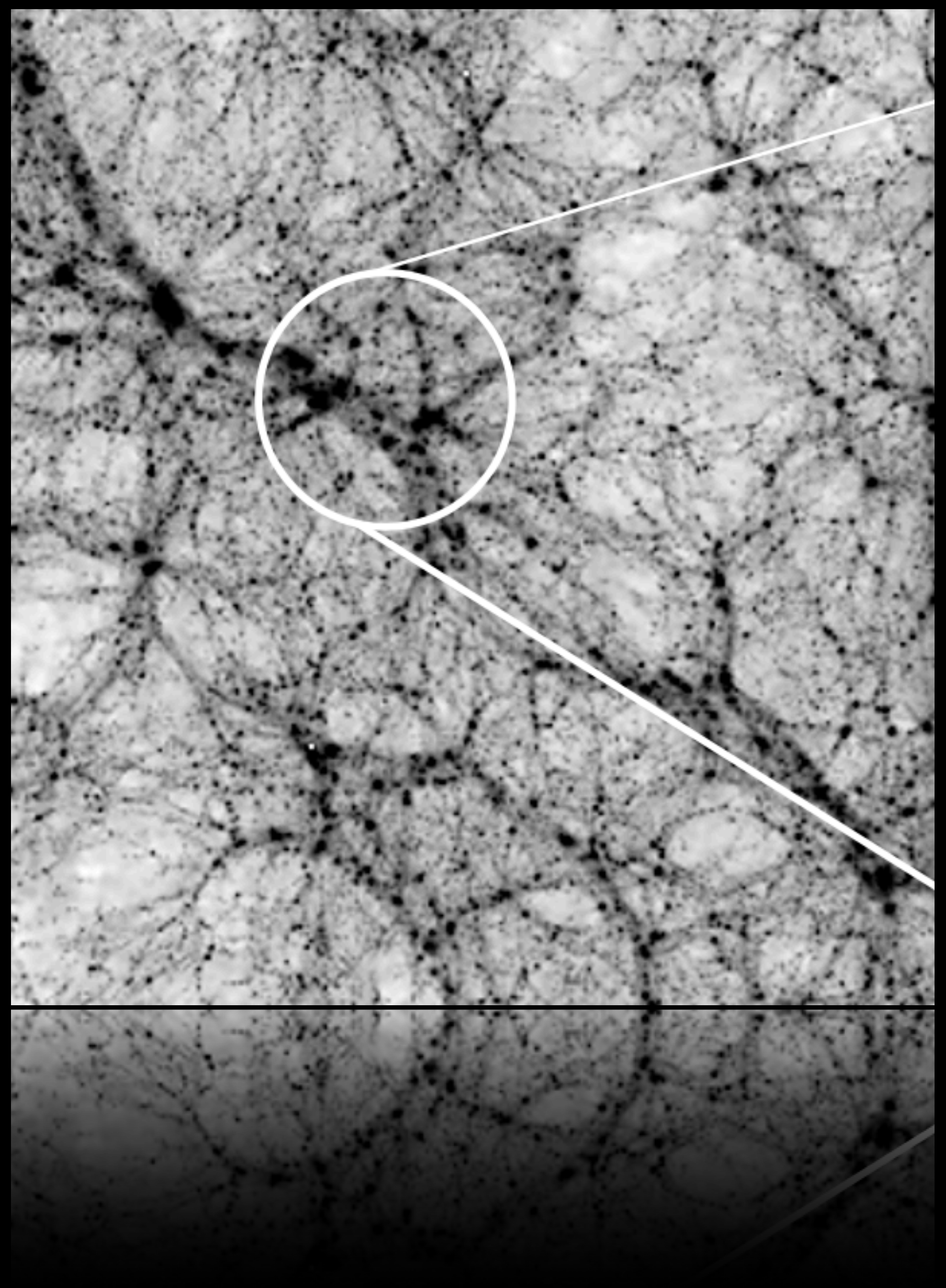
Durham University

structures are **coupled** across multiple scales

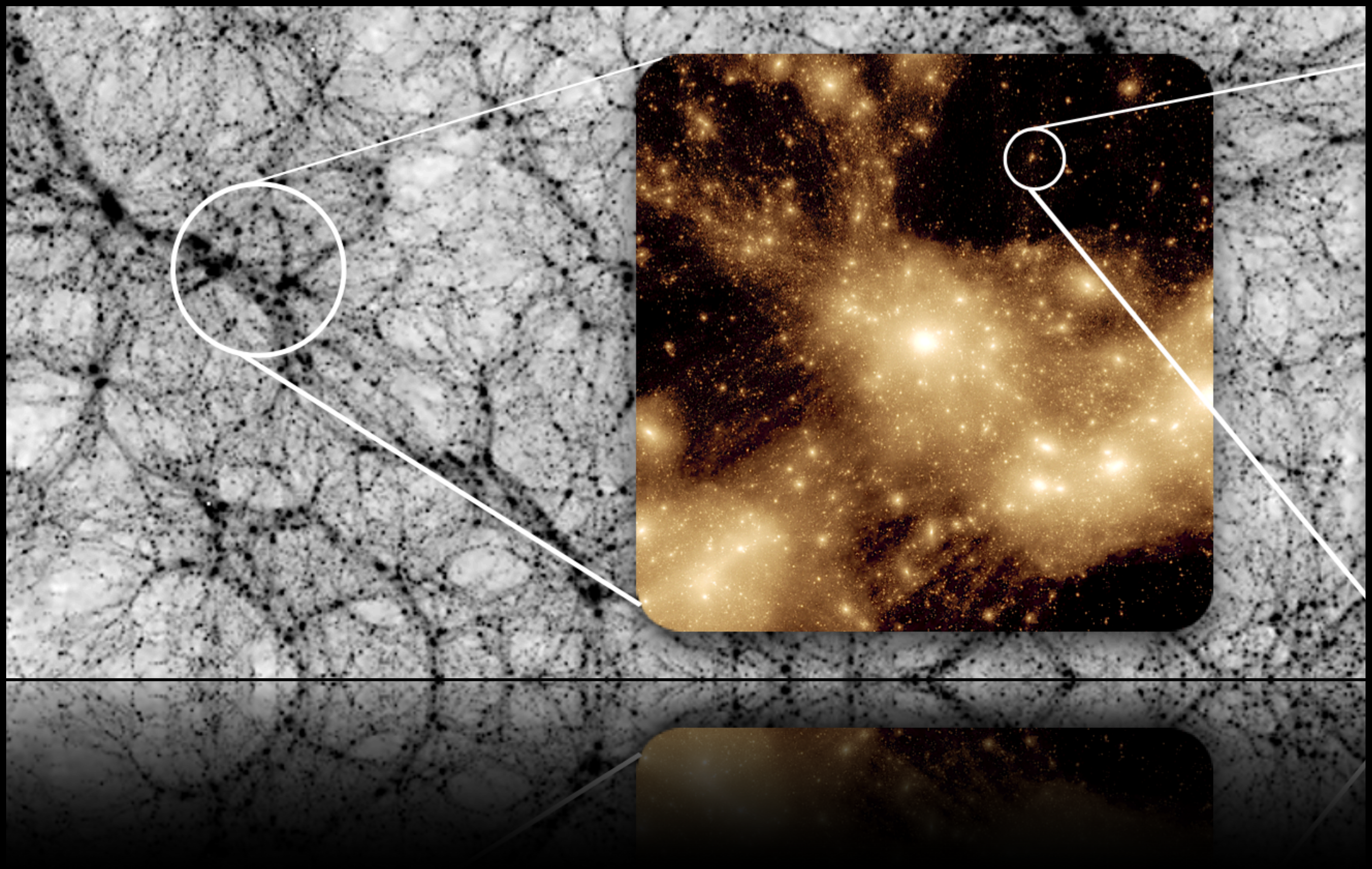
large-scale structure
[~ 10^9 light-years]



structures are coupled across multiple scales

large-scale structure
[~ 10^9 light-years]

dark matter haloes
[~ 10^6 light-years]



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[~ 10^9 light-years]

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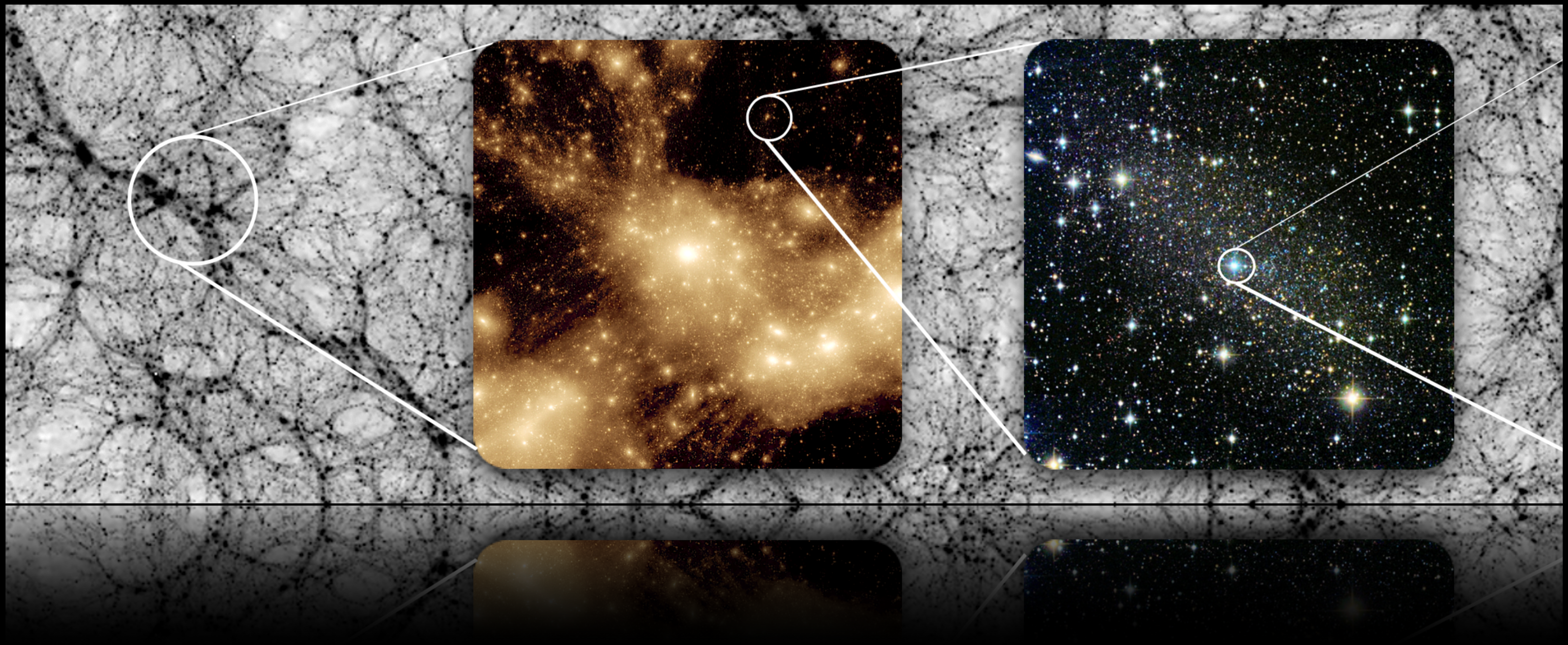
[~ 10^6 light-years]

galaxies
[~ 10^3 light-years]



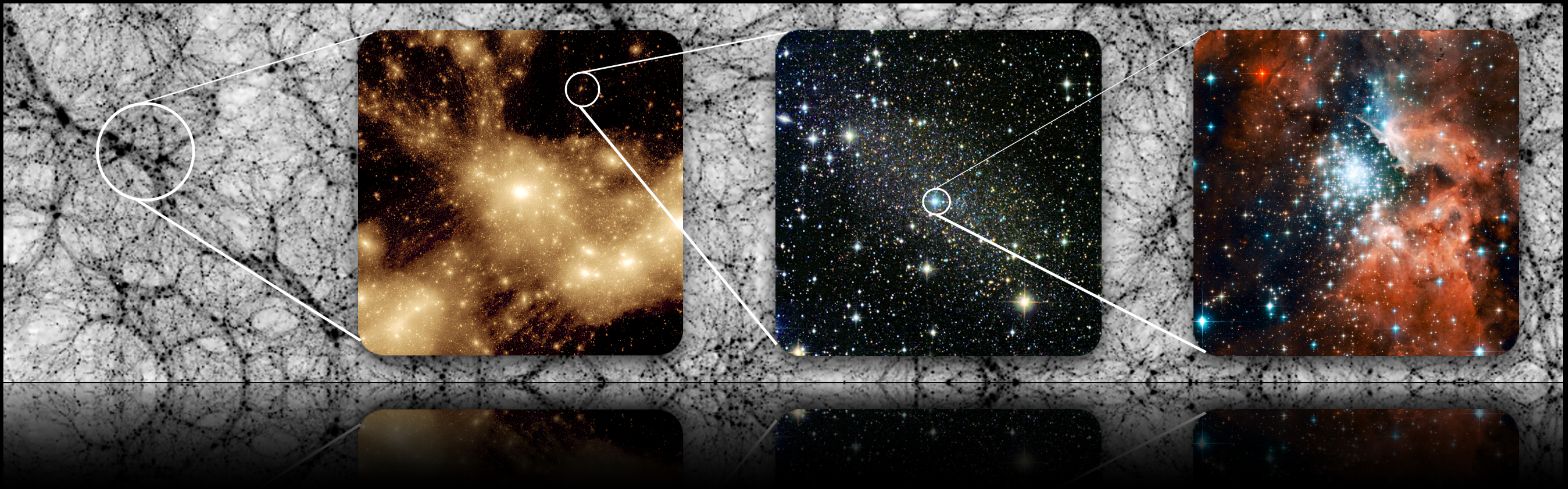
structures are coupled across multiple scales

large-scale structure
[~ 10^9 light-years]

dark matter haloes
[~ 10^6 light-years]

galaxies
[~ 10^3 light-years]

star formation sites
[~ light-years]



the pathway to simulating the cosmos

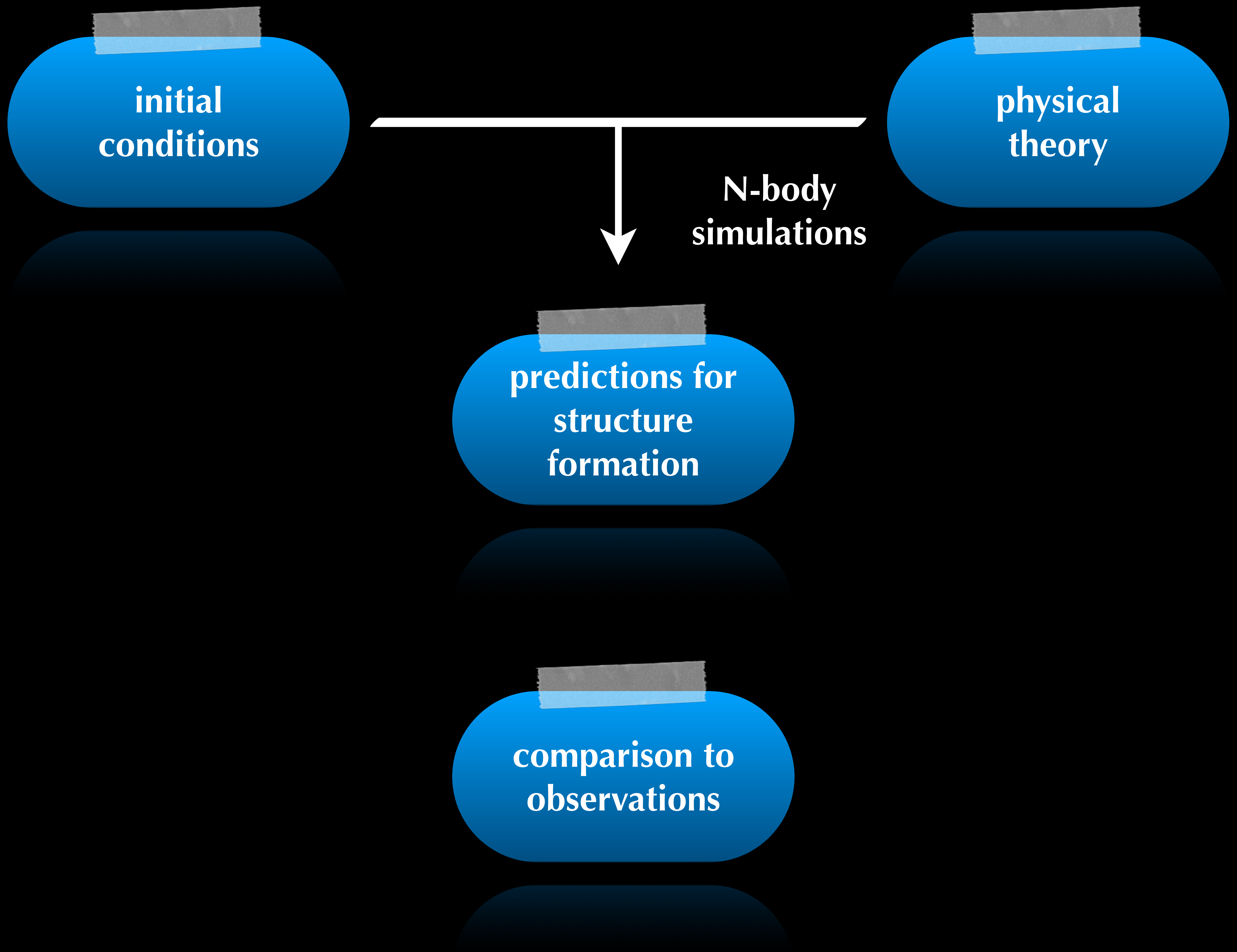
initial conditions

physical theory

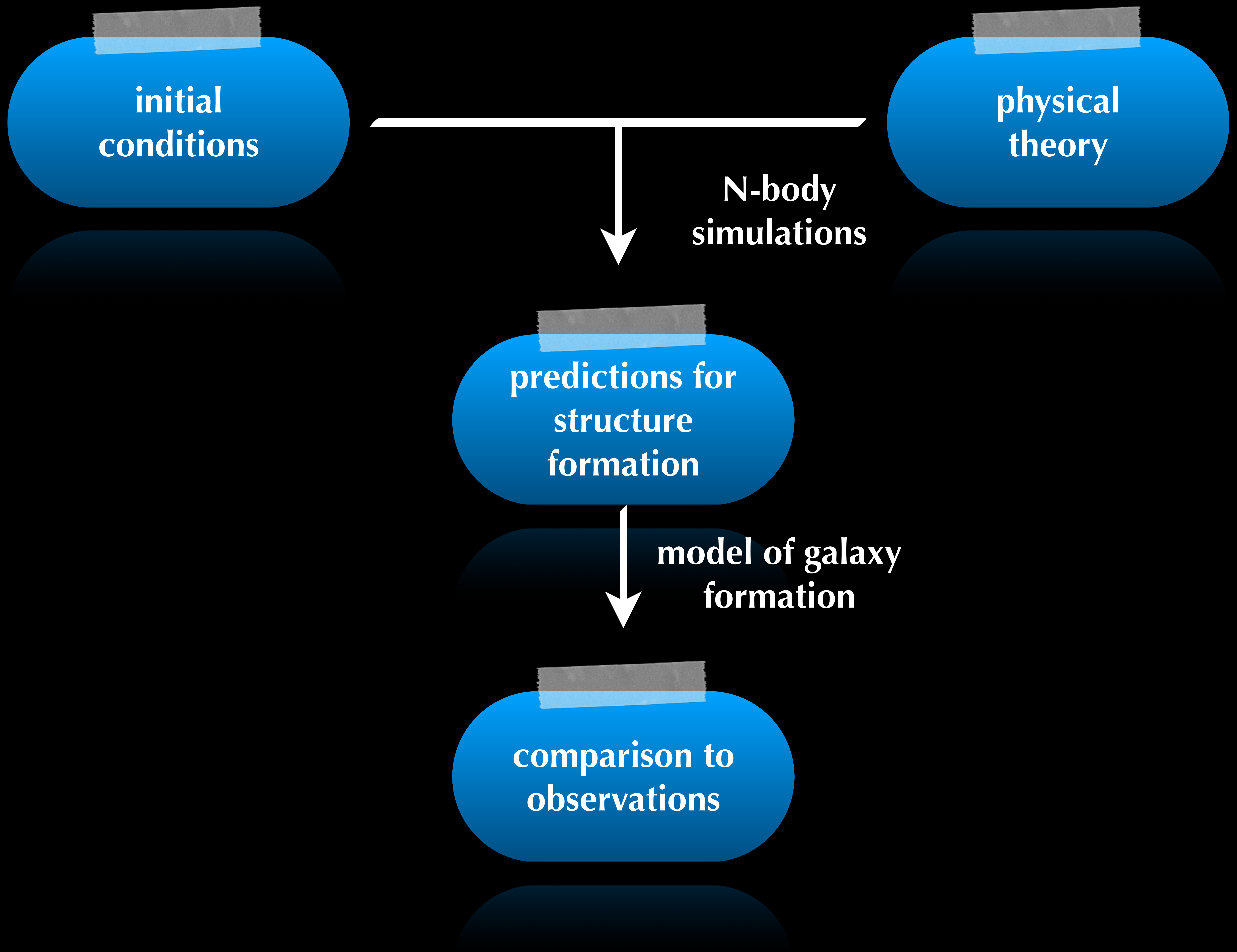
predictions for structure formation

comparison to observations

the pathway to simulating the cosmos

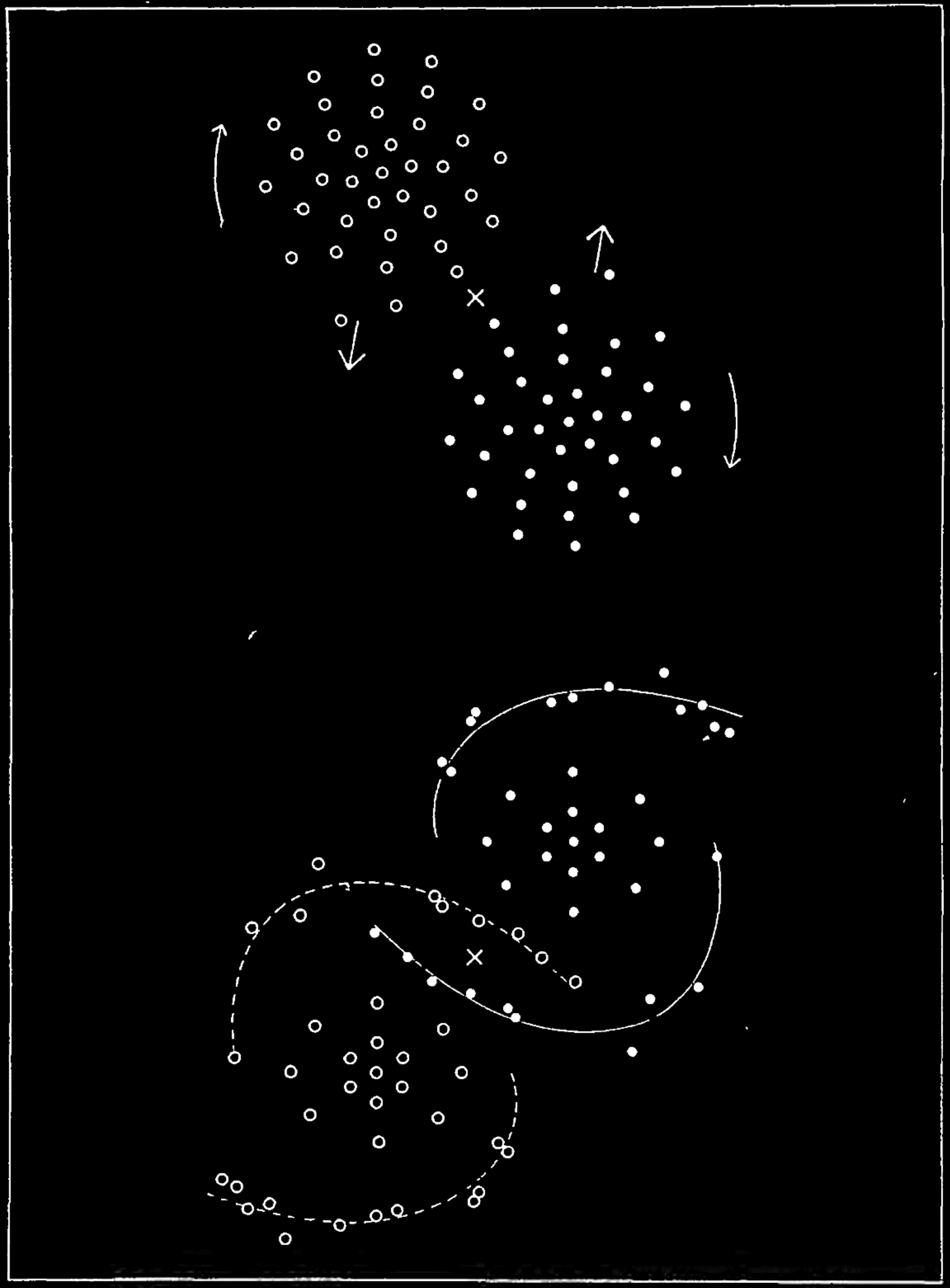


the pathway to simulating the cosmos



a little bit of history

Holmberg (1941)
the tidal encounter of two
“extragalactic nebulae”



White (1977);
700 particles

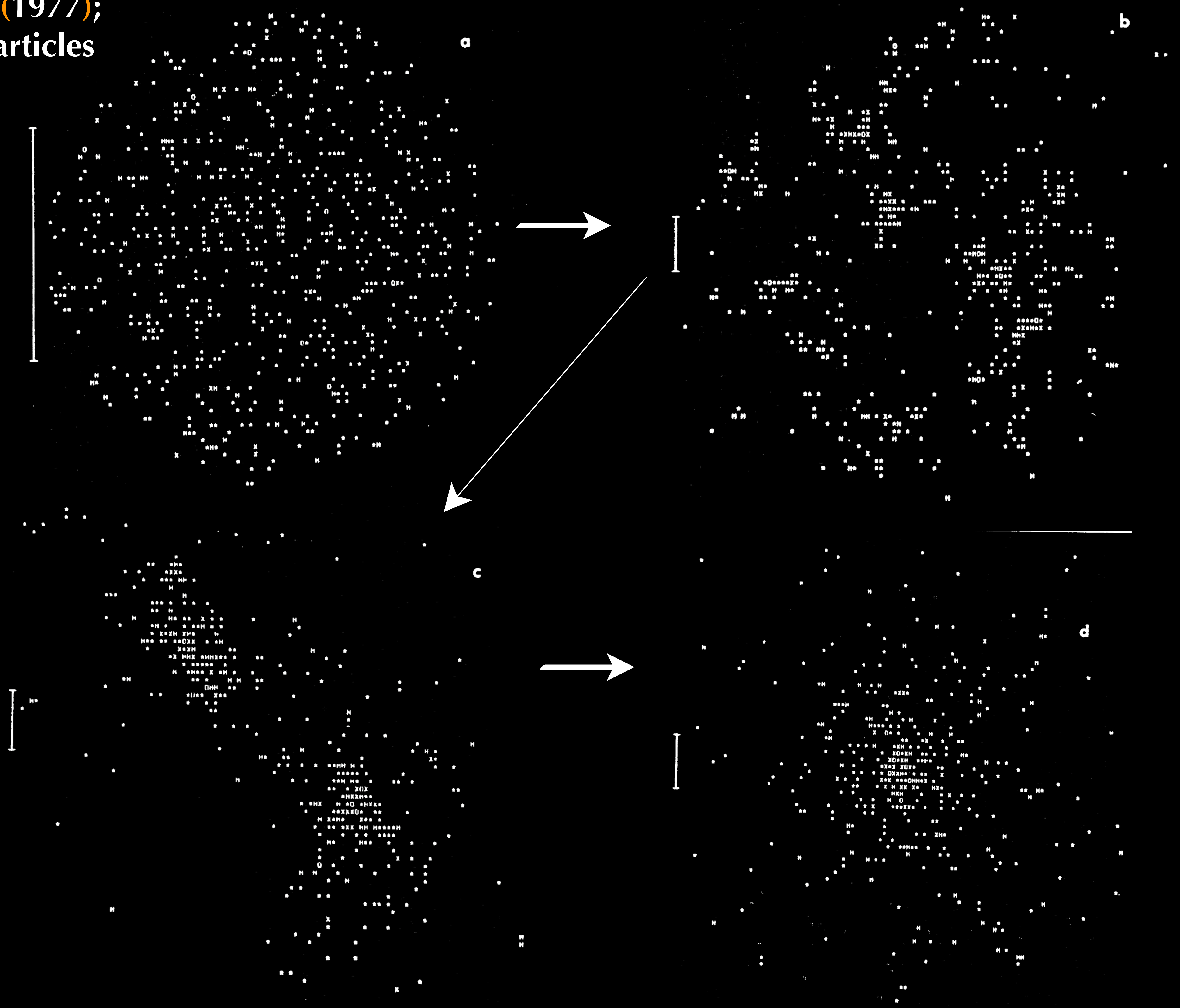


Fig. 1(c,d)

White (1977);
700 particles

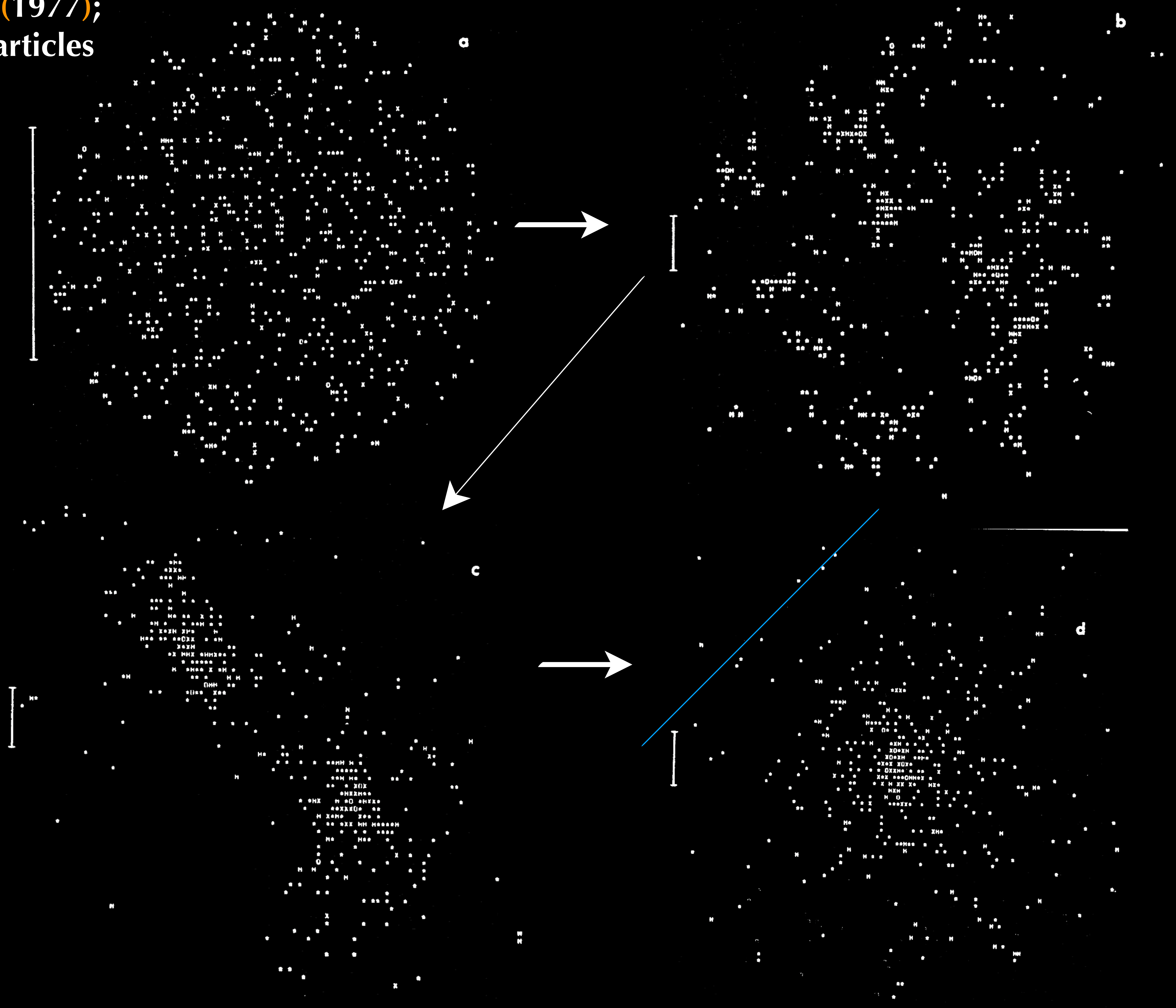
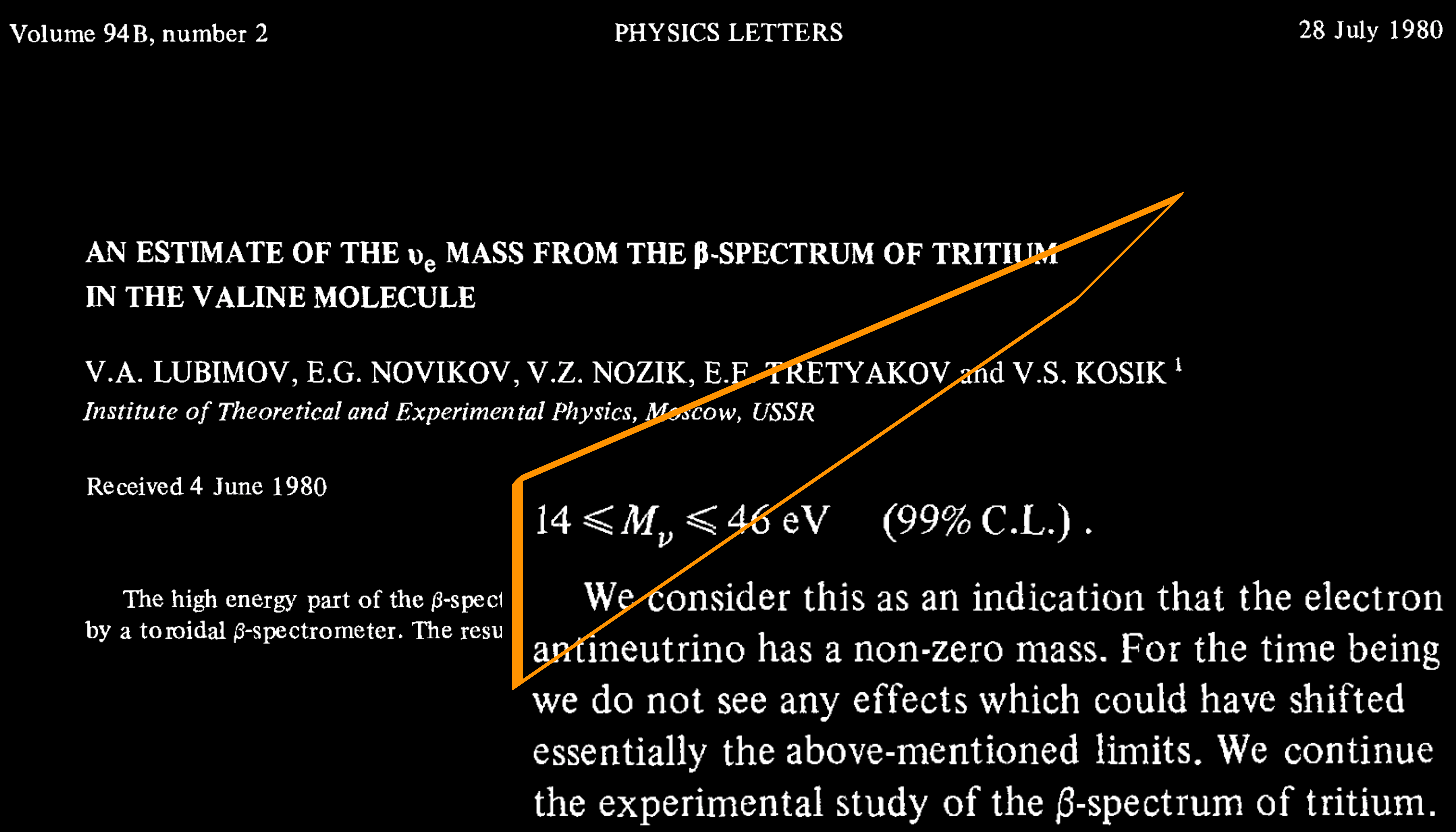


Fig. 1(c,d)


d

the rise of particle dark matter



Volume 94B, number 2 PHYSICS LETTERS 28 July 1980

AN ESTIMATE OF THE υ_e MASS FROM THE β-SPECTRUM OF TRITIUM IN THE VALINE MOLECULE

V.A. LUBIMOV, E.G. NOVIKOV, V.Z. NOZIK, E.F. TRETYAKOV and V.S. KOSIK [1]
Institute of Theoretical and Experimental Physics, Moscow, USSR

Received 4 June 1980

The high energy part of the β-spect
by a toroidal β-spectrometer. The resu

$$14 \leqslant M_\nu \leqslant 46 \text{ eV} \quad (99\% \text{ C.L.}) .$$

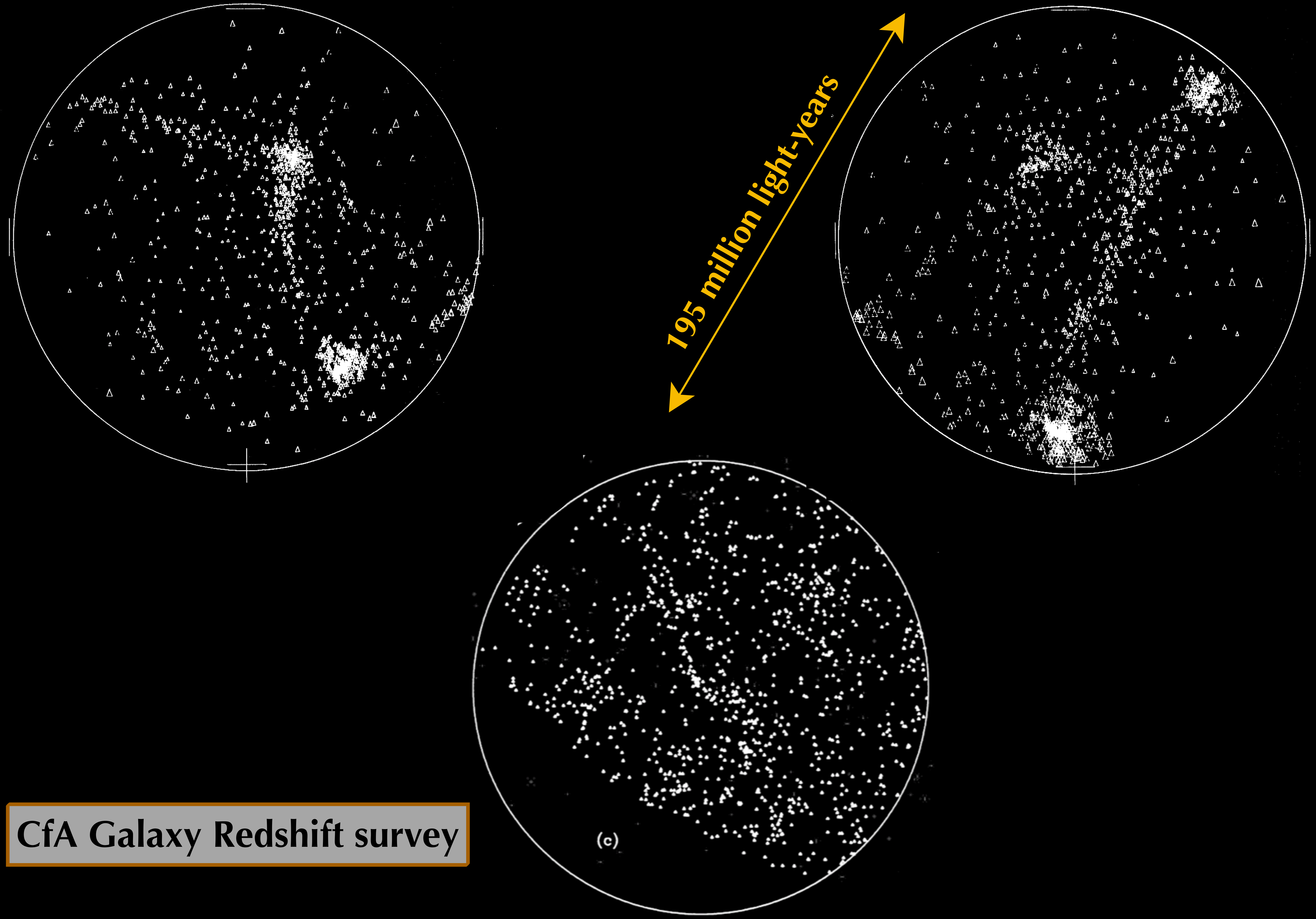
We consider this as an indication that the electron antineutrino has a non-zero mass. For the time being we do not see any effects which could have shifted essentially the above-mentioned limits. We continue the experimental study of the β-spectrum of tritium.

early candidates: the neutrino (~1980s)

weak interactions ✓ not visible ✓ known particle ✓

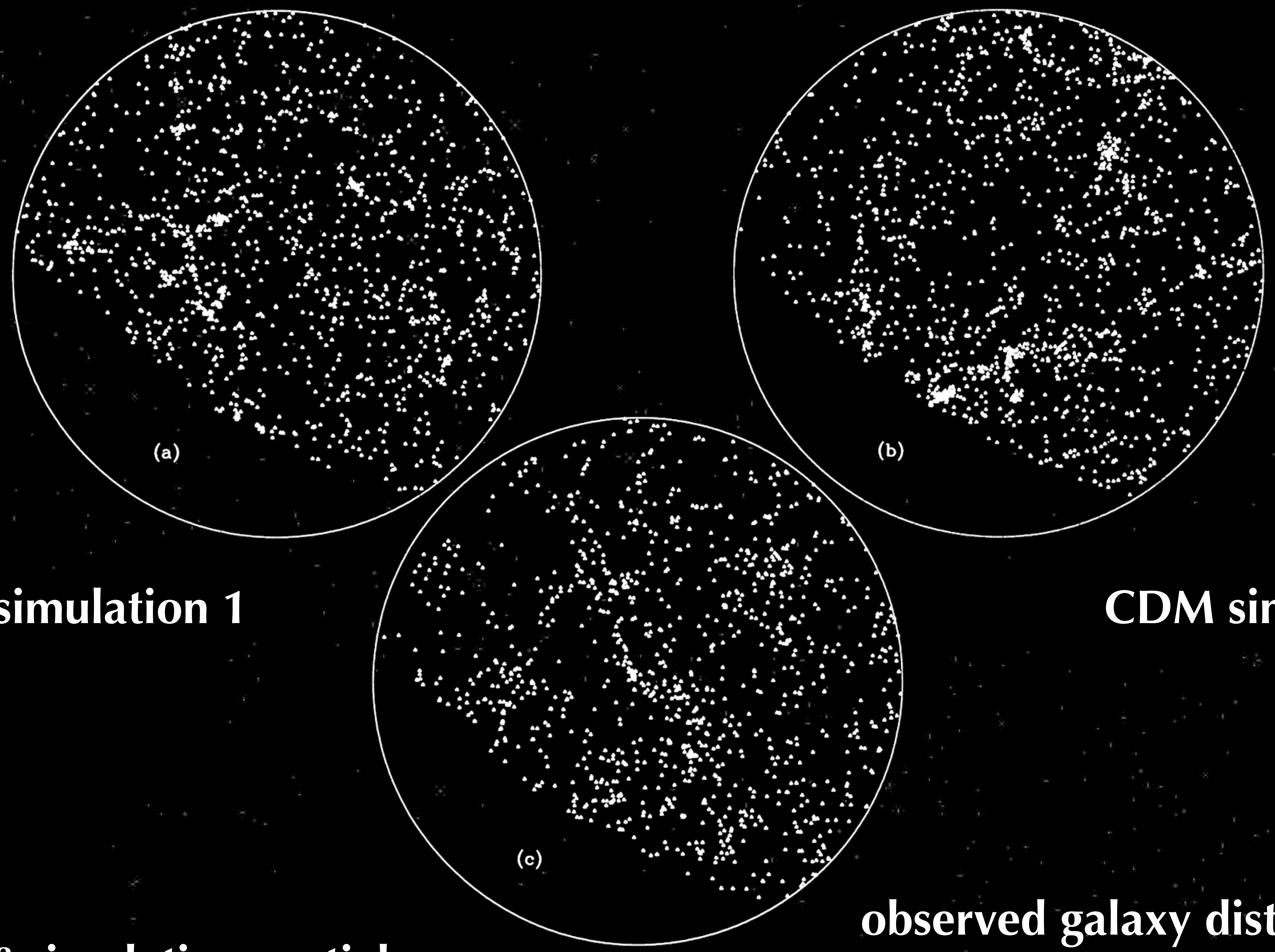
neutrinos are (almost) massless — zip around the Universe near the speed of light

hot dark matter



Klypin & Shandarin (1983); 32^3 simulation particles

the emergence of cold dark matter



CDM simulation 1

CDM simulation 2

observed galaxy distribution

Davis+ (1985); 32^3 simulation particles

Springel+ (2005)
The Millennium Simulation Project

~ 512x larger computational volume
~ 300,000x more resolution elements (2160^3 DM particles)

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z = 48.4
T = 0.05 Gyr
500 kpc


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9 kpc
z = 9.8
log M★ = 7.74
SFR = 0.2 M☉ yr⁻¹
30 kpc
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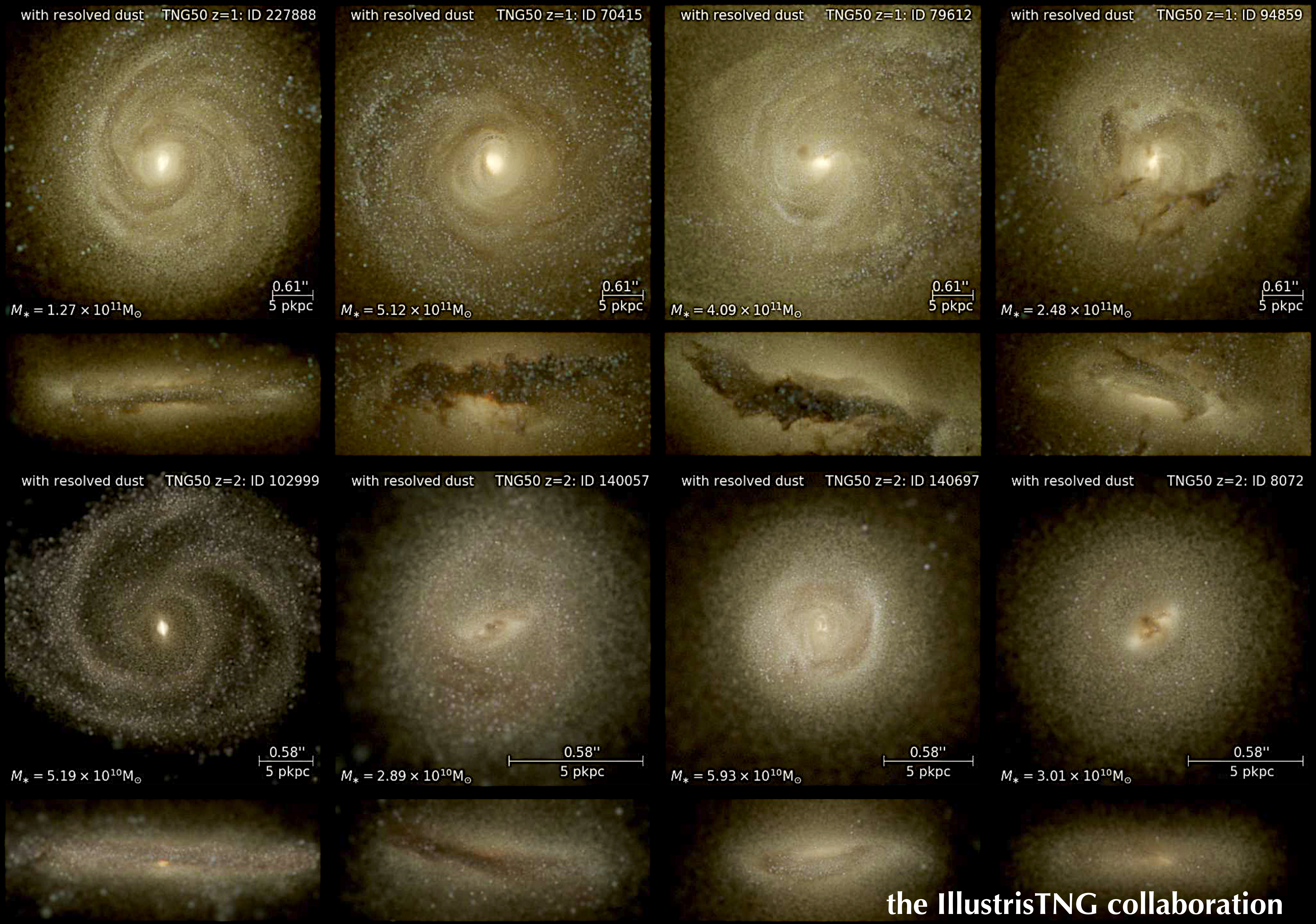
with resolved dust TNG50 z=1: ID 227888
0.61''
5 pkpc
$M_* = 1.27 \times 10^{11} M_\odot$
with resolved dust TNG50 z=1: ID 70415
0.61''
5 pkpc
$M_* = 5.12 \times 10^{11} M_\odot$
with resolved dust TNG50 z=1: ID 79612
0.61''
5 pkpc
$M_* = 4.09 \times 10^{11} M_\odot$
with resolved dust TNG50 z=1: ID 94859
0.61''
5 pkpc
$M_* = 2.48 \times 10^{11} M_\odot$
with resolved dust TNG50 z=2: ID 102999
0.58''
5 pkpc
$M_* = 5.19 \times 10^{10} M_\odot$
with resolved dust TNG50 z=2: ID 140057
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$M_* = 2.89 \times 10^{10} M_\odot$
with resolved dust TNG50 z=2: ID 140697
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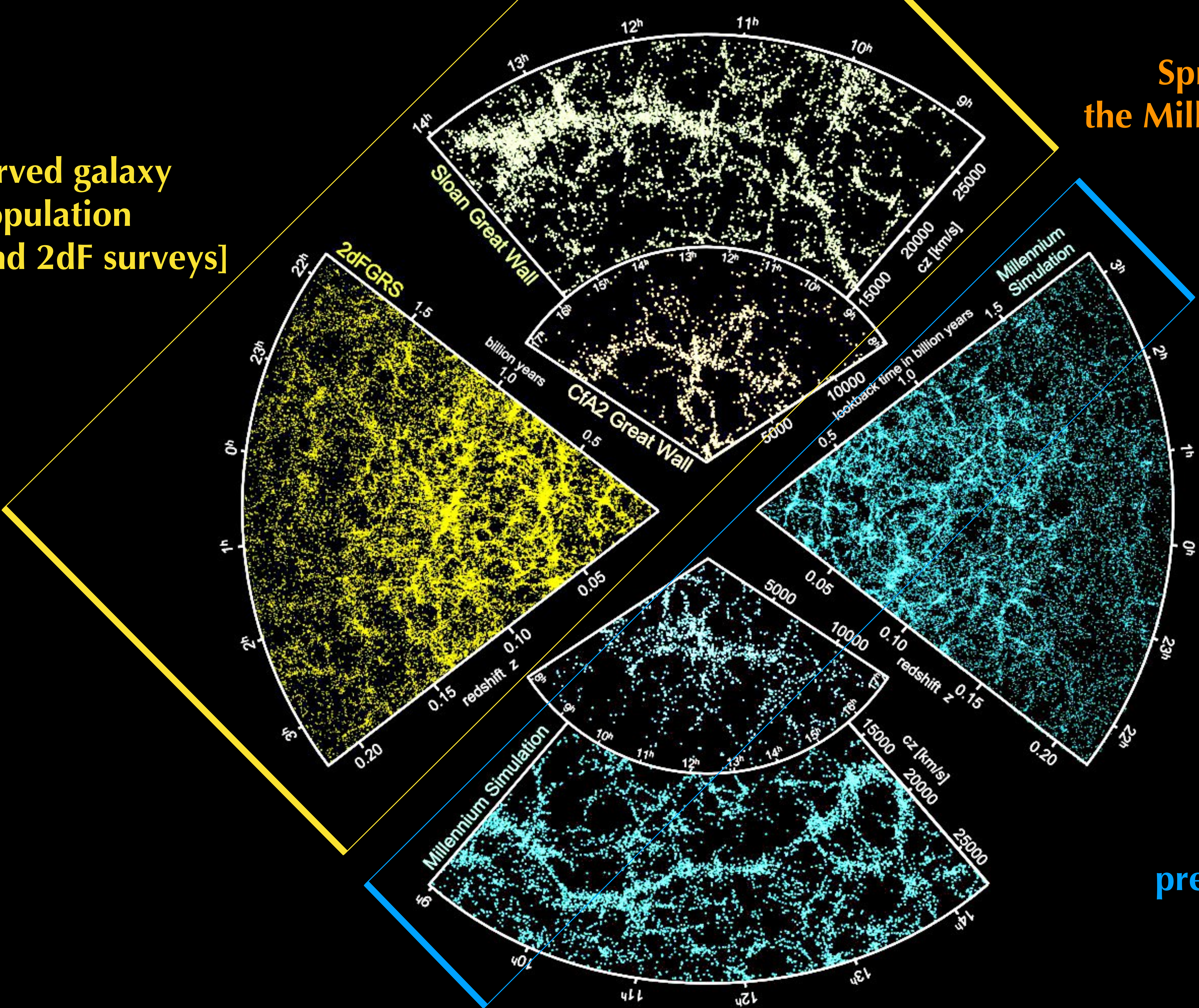
5 pkpc
$M_* = 5.93 \times 10^{10} M_\odot$
with resolved dust TNG50 z=2: ID 8072
0.58''
5 pkpc
$M_* = 3.01 \times 10^{10} M_\odot$
the IllustrisTNG collaboration

Springel+ (2005)
the Millennium simulation

observed galaxy
population
[SDSS and 2dF surveys]



prediction of the ΛCDM
model

the N-body method

Poisson-Vlasov equation
[fluid]

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \mathbf{v} + \frac{\partial f}{\partial \mathbf{v}} \cdot \left(-\frac{\partial \Phi}{\partial \mathbf{x}} \right) = 0$$

$$\nabla^2 \Phi \left(\mathbf{x}, t \right) = 4\pi G \int f(\mathbf{x}, \mathbf{v}, t) \, \mathrm{d}\mathbf{v}$$

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collisionless Boltzmann equation

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collisionless Boltzmann equation

the "distribution function"
[number density of particles at position x with velocity v]

Poisson-Vlasov equation
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macroscopic DM
[particles]

$$\ddot{\mathbf{x}}_i = -\nabla_i \Phi\left(\mathbf{x}_i\right)$$

$$\Phi\,(\mathbf{x}) = -G \sum_{j=1}^{N} \frac{m_j}{\left[\left(\mathbf{x} - \mathbf{x}_j\right)^2 + \epsilon^2\right]^{1/2}}$$

sample the phase-space of dark matter with macroscopic particles

Poisson-Vlasov equation
[fluid]

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softening

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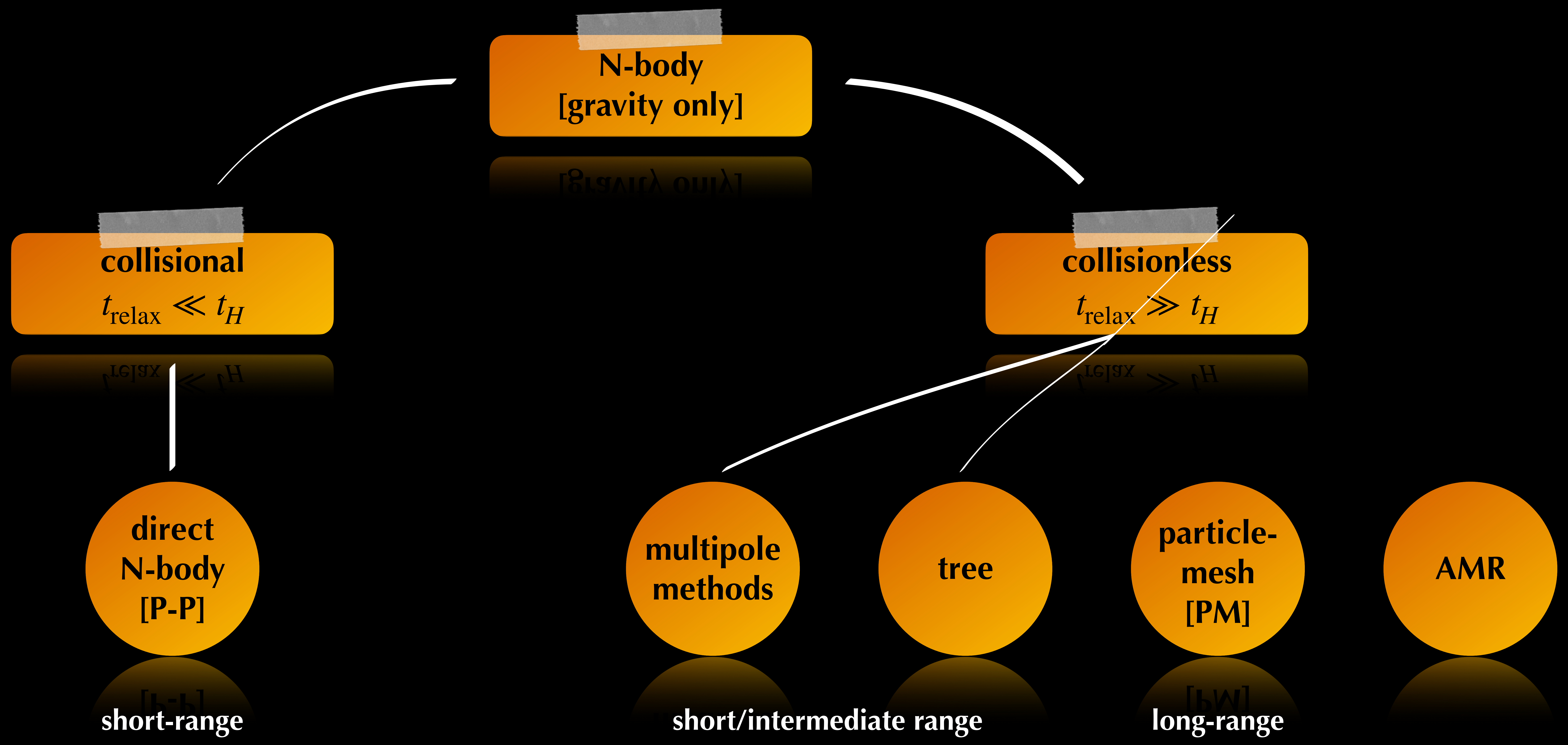
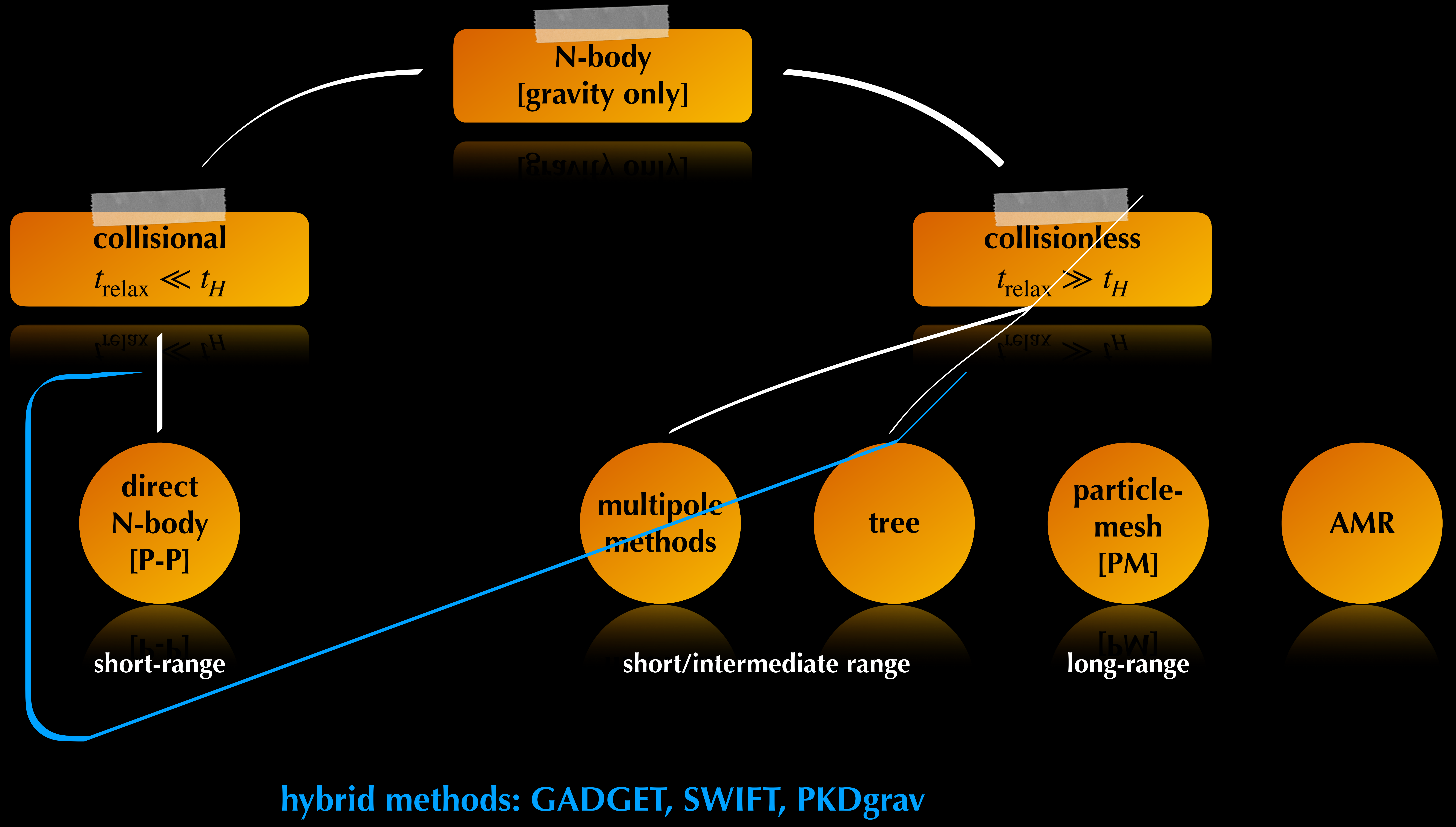
sample the phase-space of dark matter with macroscopic particles

→ need large N to be valid!


N-body
[gravity only]
collisional
$t_{\mathrm{relax}} \ll t_H$
collisionless
$t_{\mathrm{relax}} \gg t_H$
direct
N-body
[P-P]
multipole
methods
tree
particle-
mesh
[PM]
AMR
short-range
short/intermediate range
long-range



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hybrid methods: GADGET, SWIFT, PKDgrav

direct N-body: particle-particle method

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this is the simplest N-body algorithm one can cook up: simply compute the force on particle j due to all $N-1$ other particles, and repeat over all particles

direct N-body: particle-particle method

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super-easy to understand

general

highly accurate

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computationally *extremely* expensive

$\mathcal{O}(N^2)$

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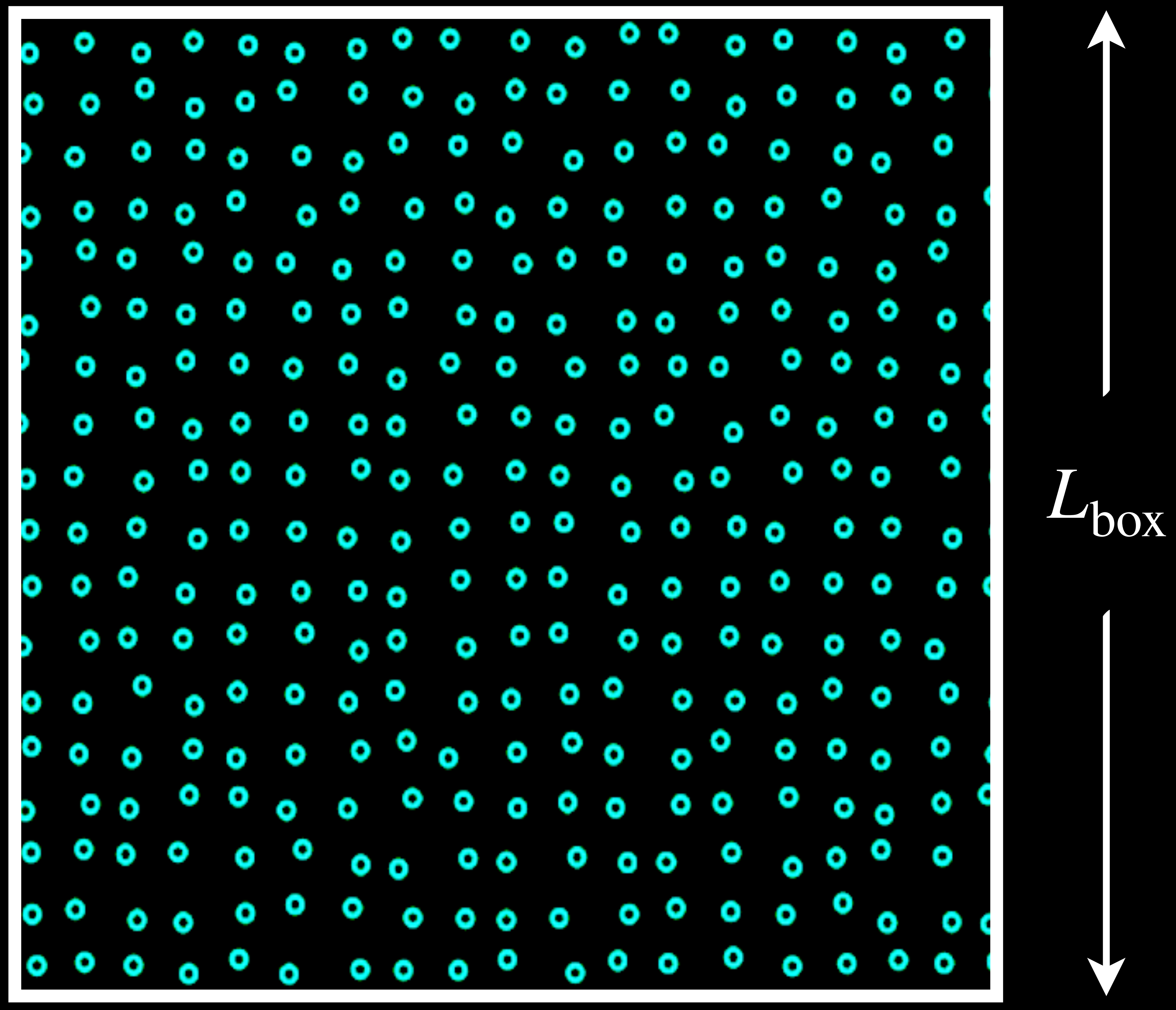
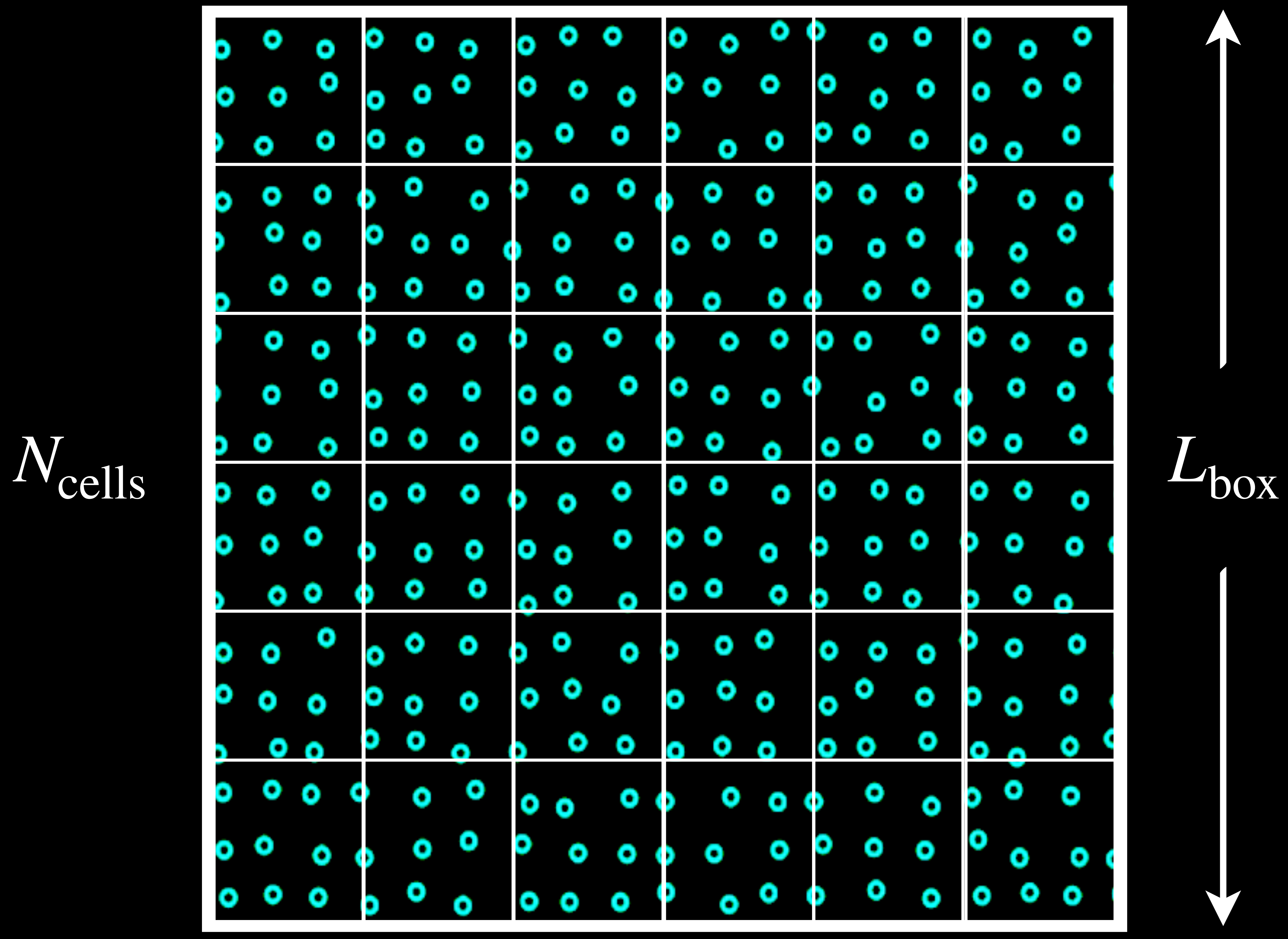
particle-mesh method



use the properties of Fourier transforms + periodic boundary conditions to solve: $\Phi(\mathbf{k}) = G(\mathbf{k}) \cdot \rho(\mathbf{k})$

particle-mesh method



1. divide computational volume into a grid of size N_{cells} x N_{cells} x N_{cells}

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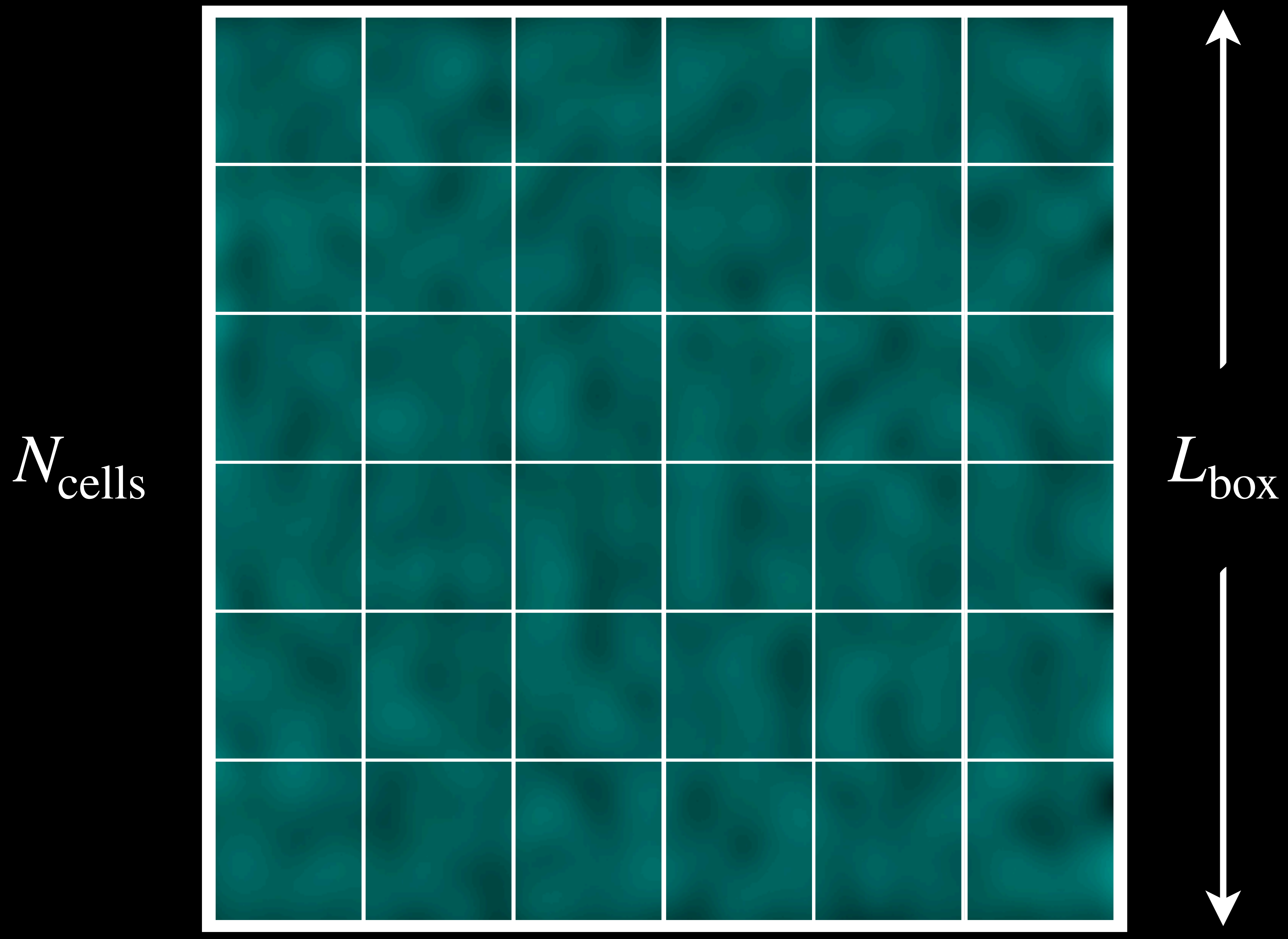
particle-mesh method



1. divide computational volume into a grid of size N_{cells} x N_{cells} x N_{cells}

2. smooth the discrete particle distribution onto mesh using an interpolation scheme (CIC, NGP, TSC etc): density field $\rho(\mathbf{x})$

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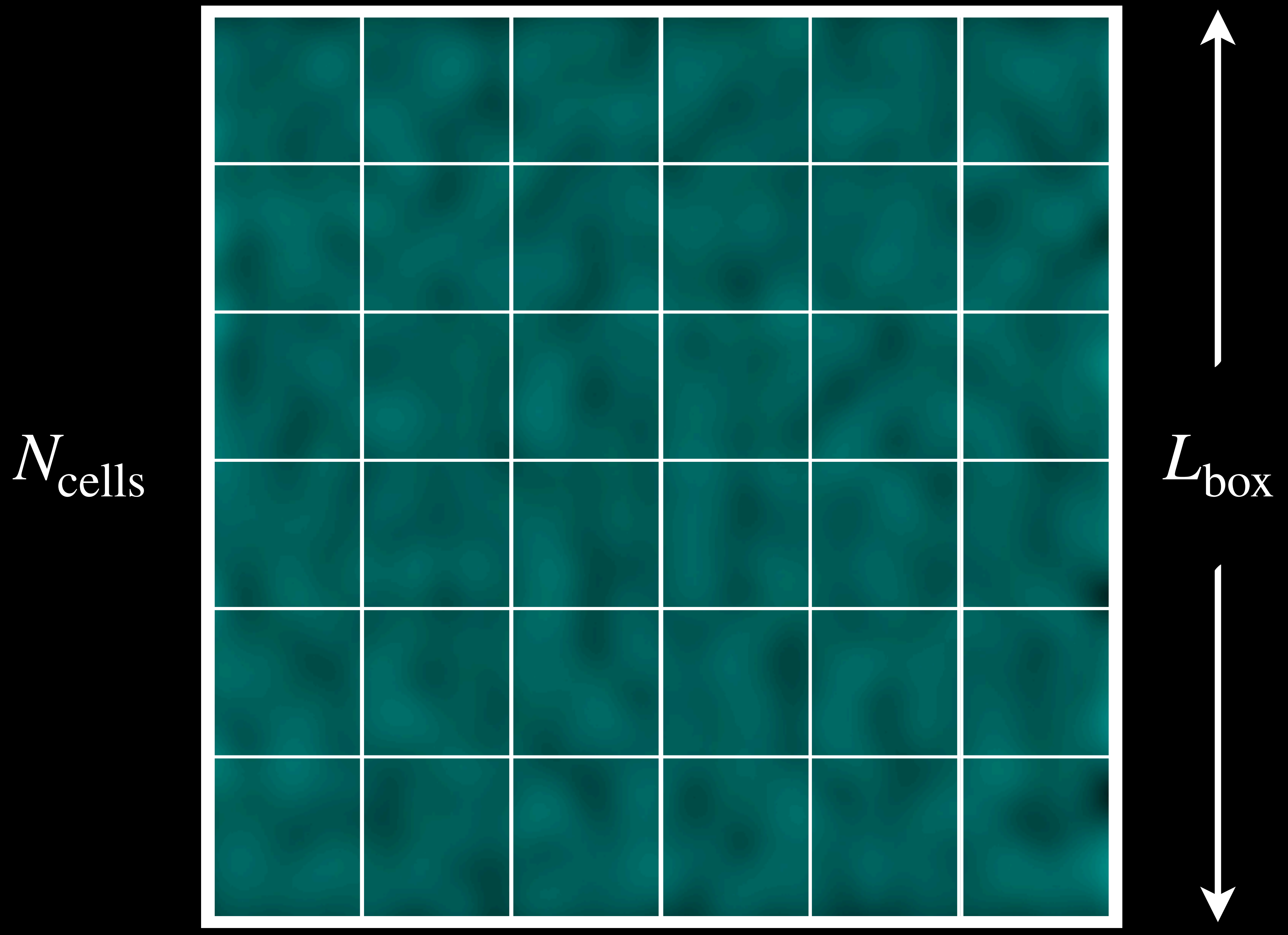

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3. Fourier transform density field: $\rho(\mathbf{x}) \rightarrow \rho(\mathbf{k})$

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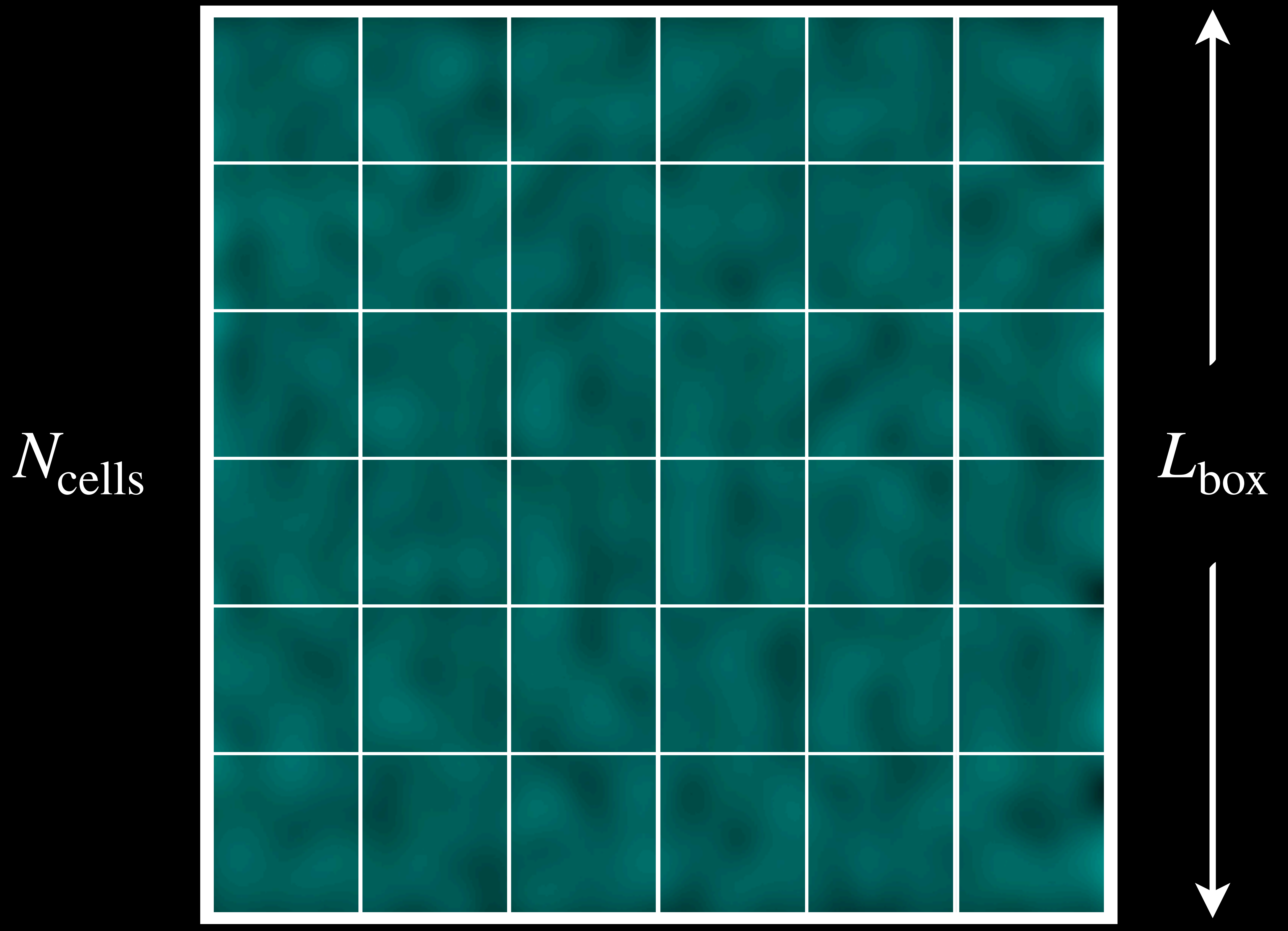
particle-mesh method



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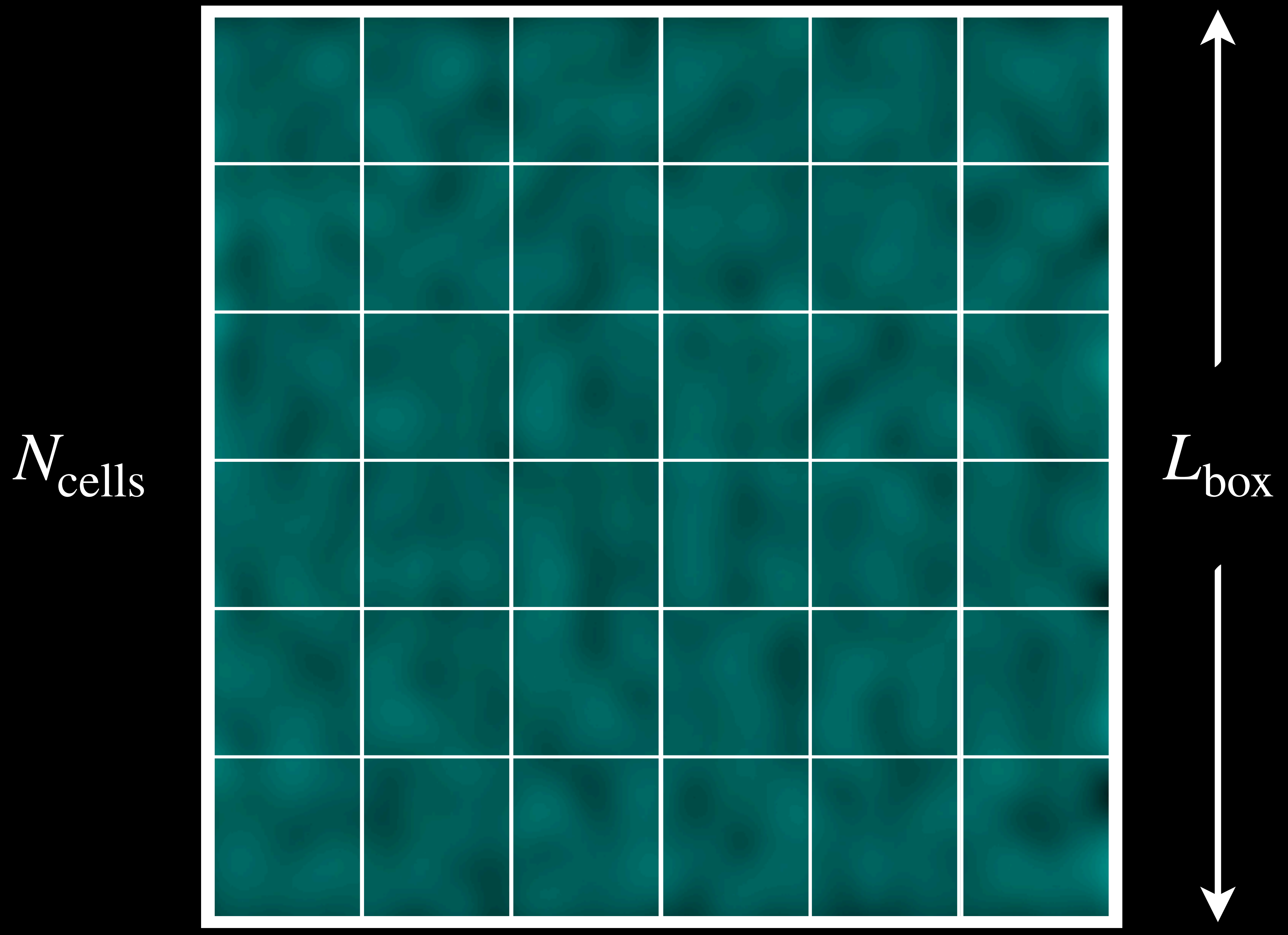
4. solve Poisson's equation to get $\Phi(\mathbf{k})$

use the properties of Fourier transforms + periodic boundary conditions to solve: $\Phi(\mathbf{k}) = G(\mathbf{k}) \cdot \rho(\mathbf{k})$

particle-mesh method



1. divide computational volume into a grid of size $N_{\mathrm{cells}} \times N_{\mathrm{cells}} \times N_{\mathrm{cells}}$

2. smooth the discrete particle distribution onto mesh using an interpolation scheme (CIC, NGP, TSC etc): density field $\rho(\mathbf{x})$

3. Fourier transform density field: $\rho(\mathbf{x}) \rightarrow \rho(\mathbf{k})$

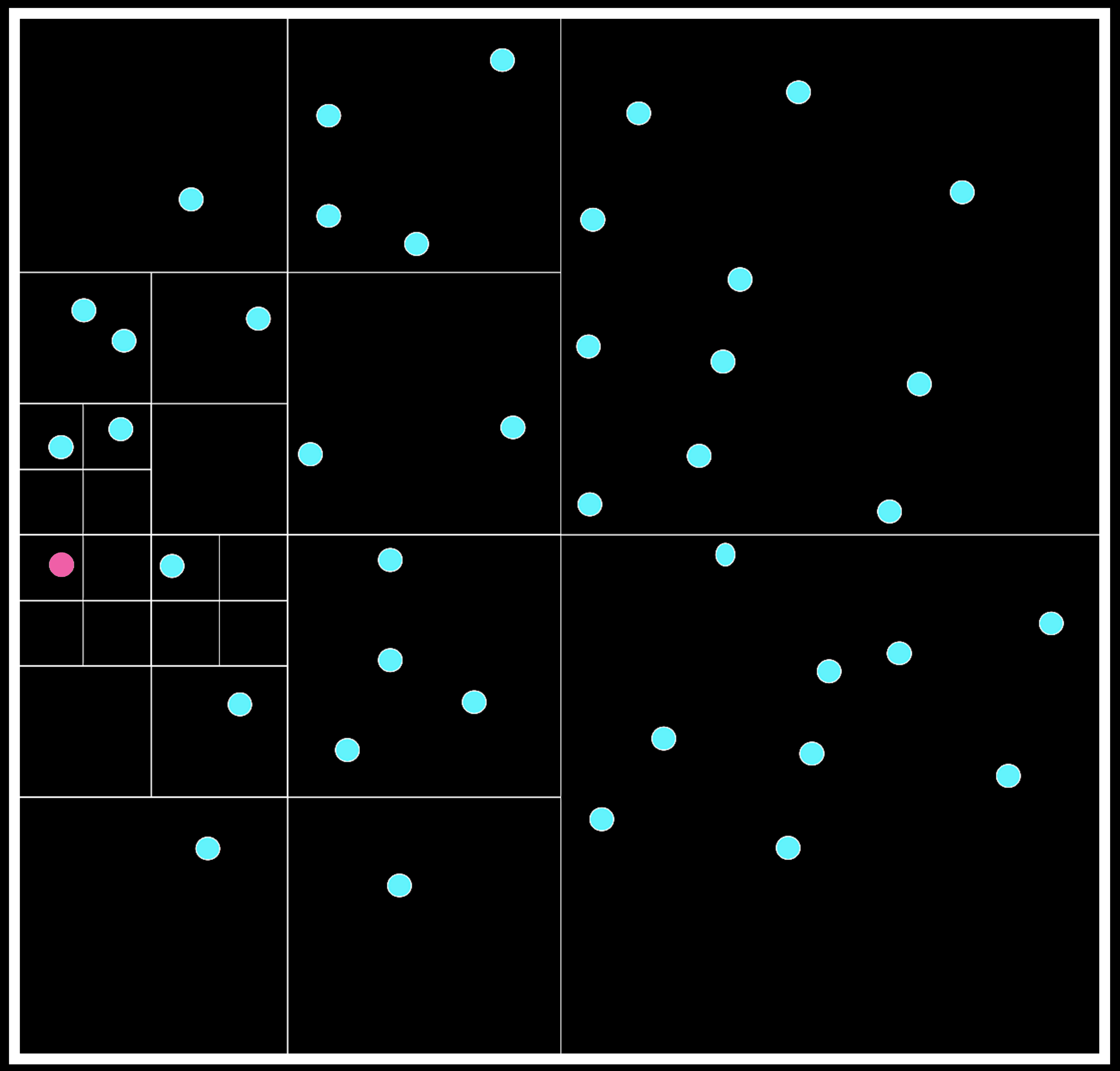
4. solve Poisson's equation to get $\Phi(\mathbf{k})$

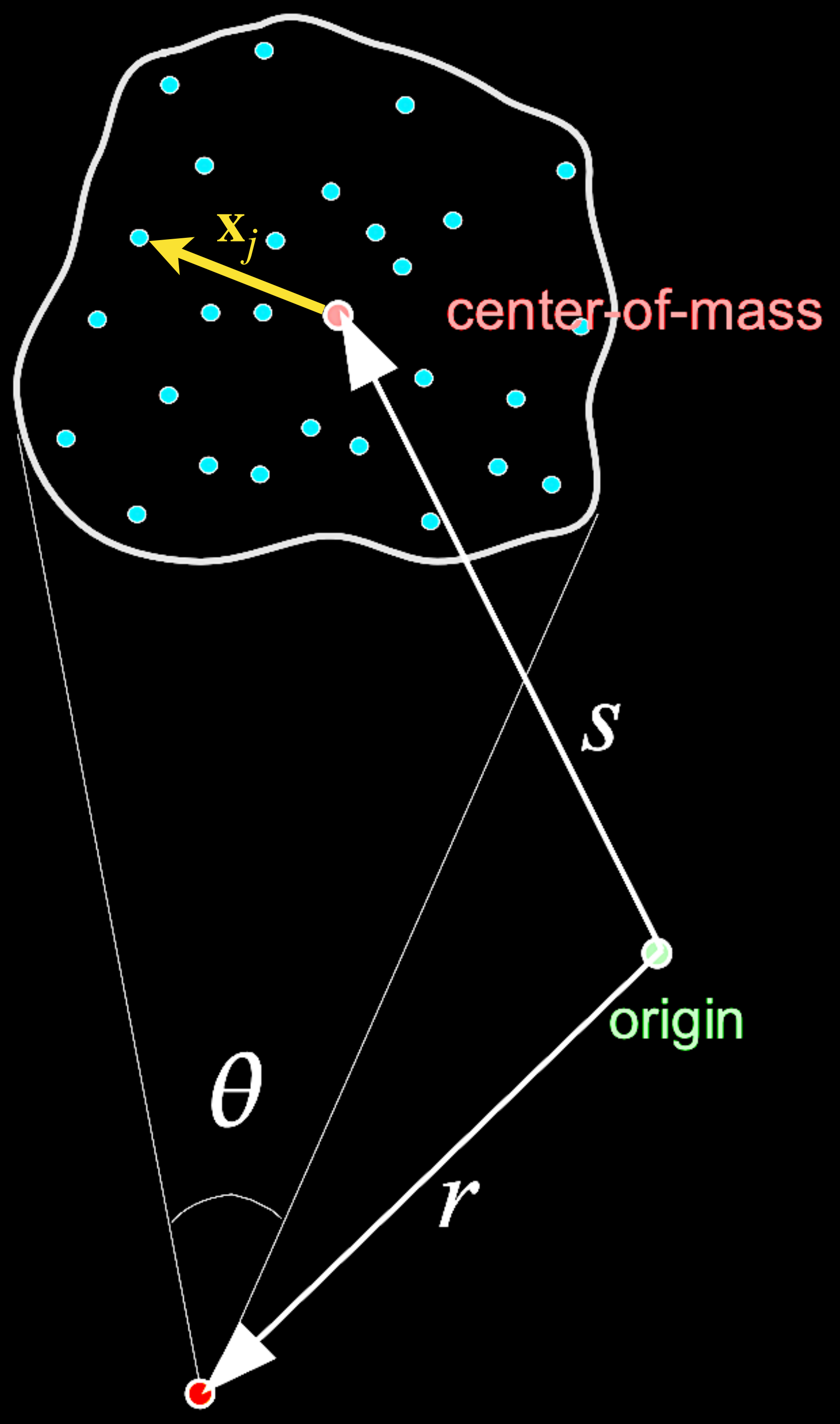
5. inverse Fourier transform to get $\Phi(\mathbf{x})$

use the properties of Fourier transforms + periodic boundary conditions to solve: $\Phi(\mathbf{k}) = G(\mathbf{k}) \cdot \rho(\mathbf{k})$

tree algorithms

based on the idea that you can group together particles that are a long way away from the particle of interest (pink)





Springel (2005)

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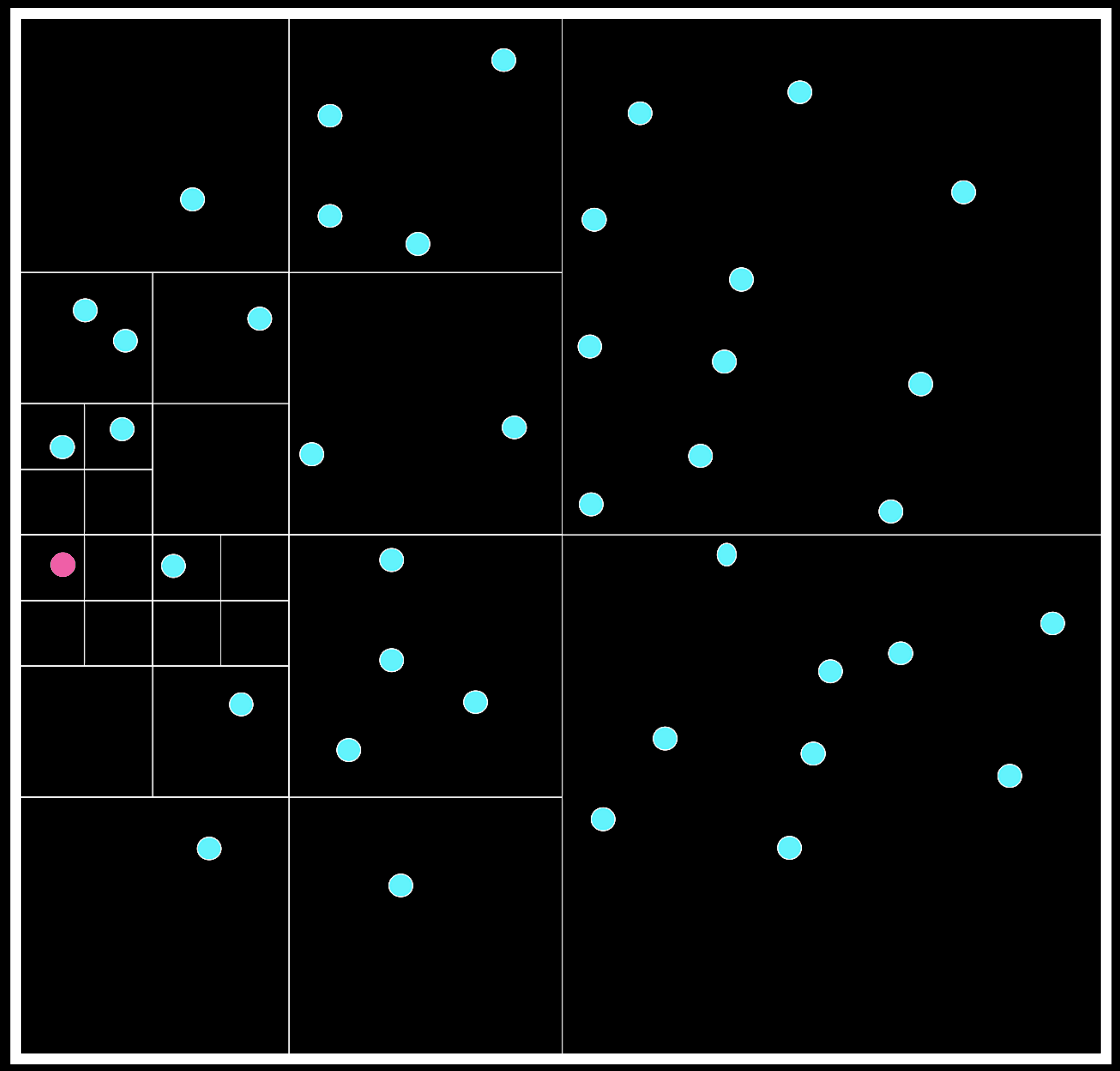
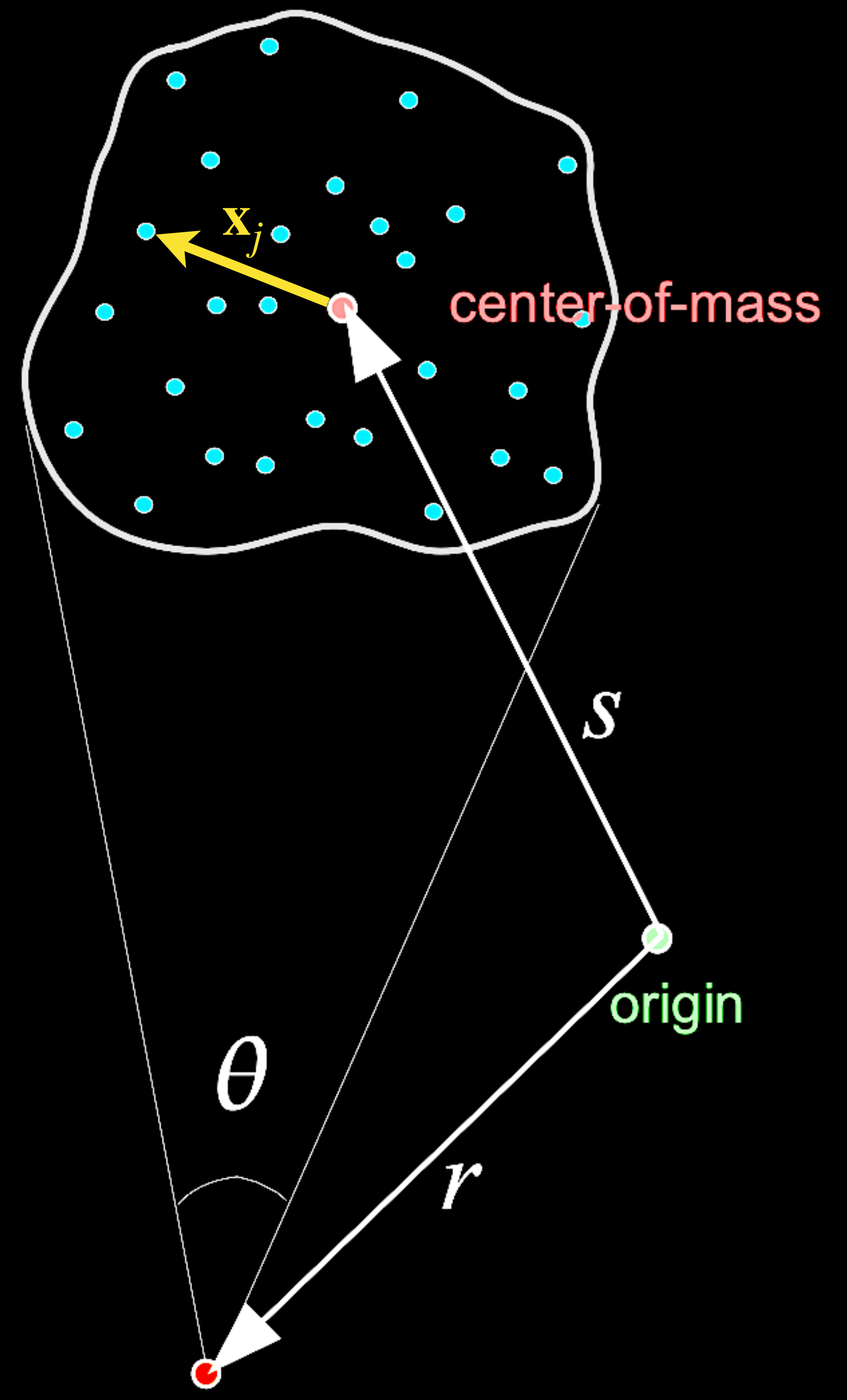
if $\theta < \theta_{\rm crit}$: treat particles as a group contributing a monopole term (+ higher order terms if desired)





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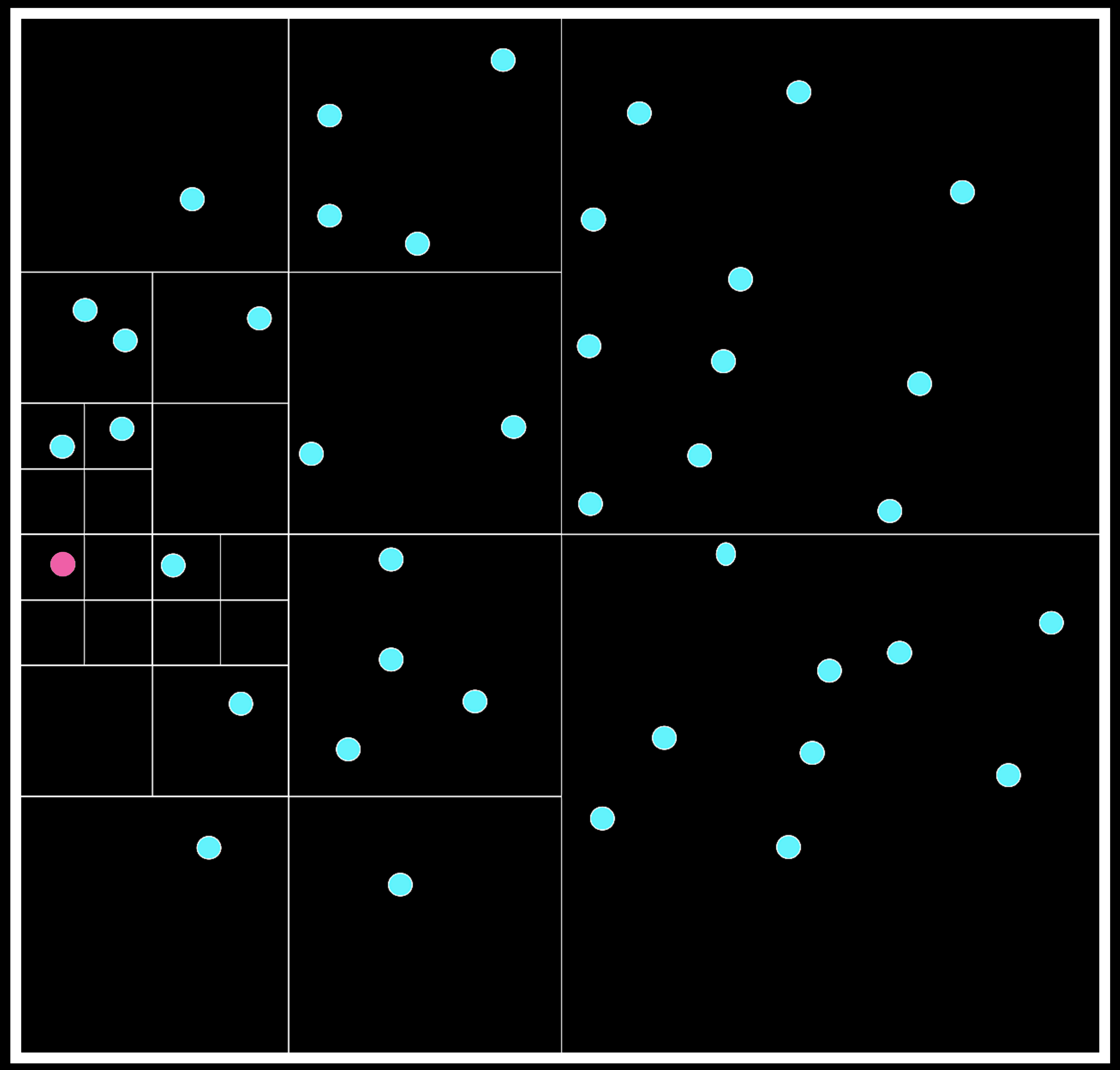
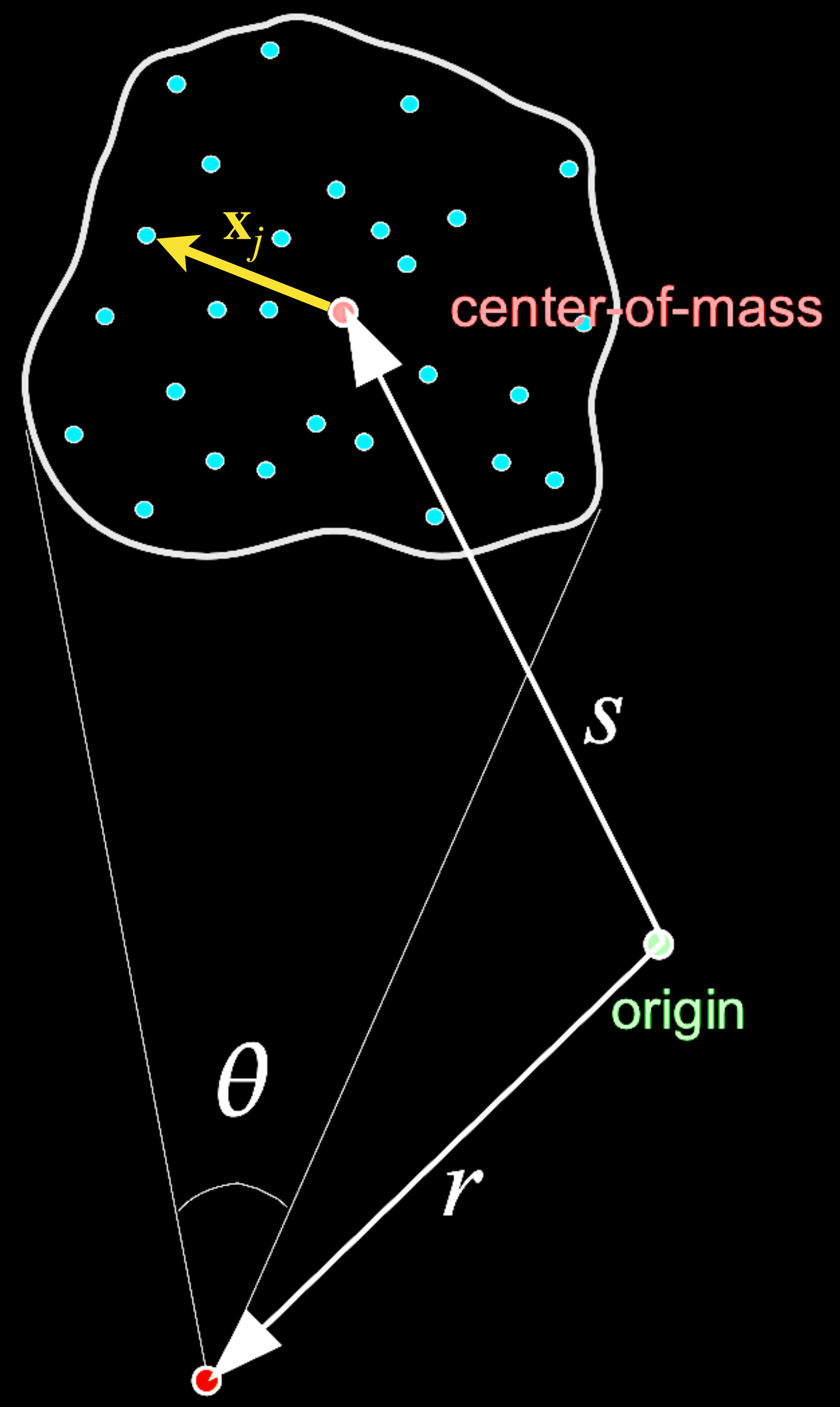
if $\theta > \theta_{\rm crit}$: open up parent cell and process sub-cells individually





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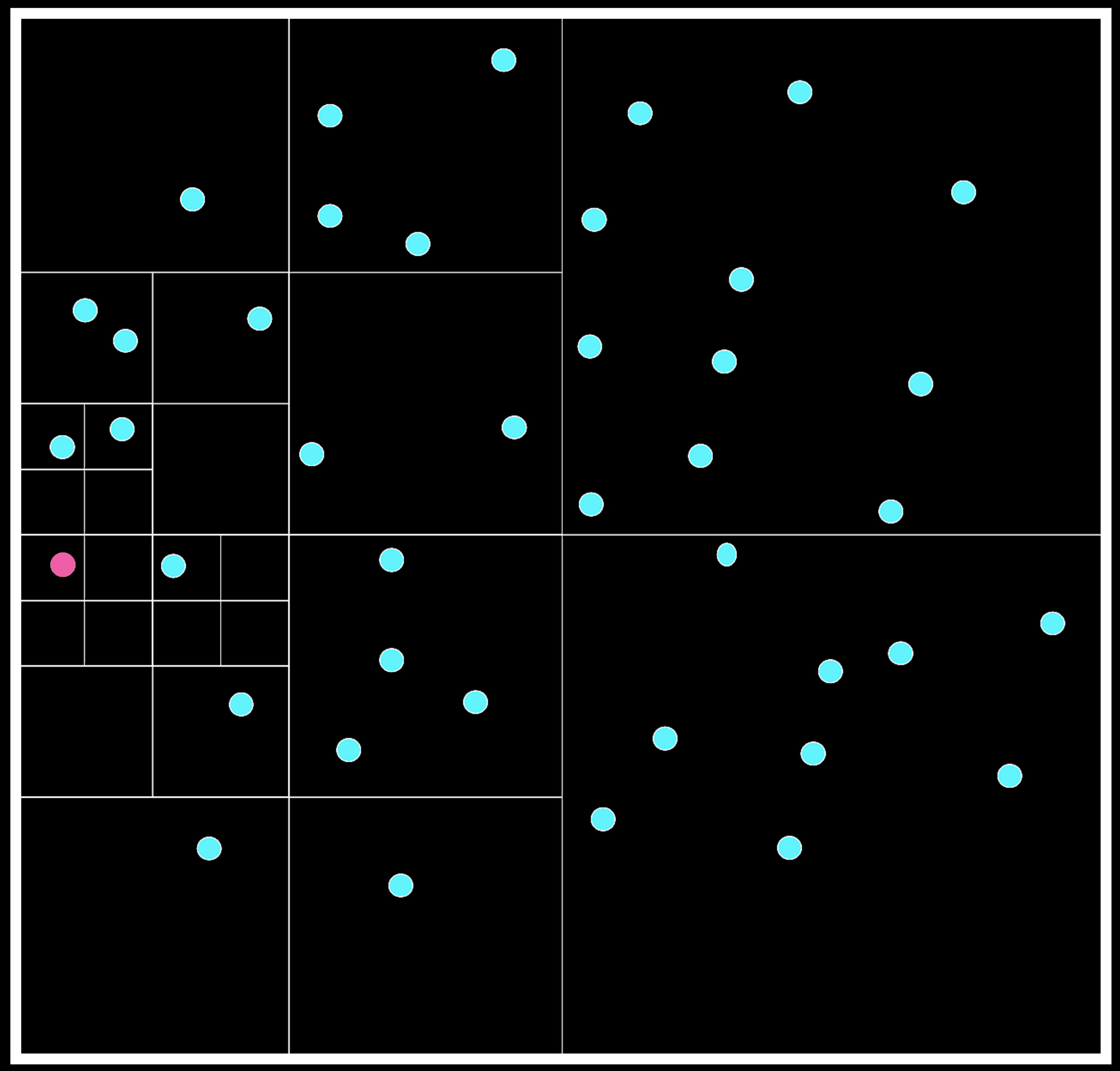
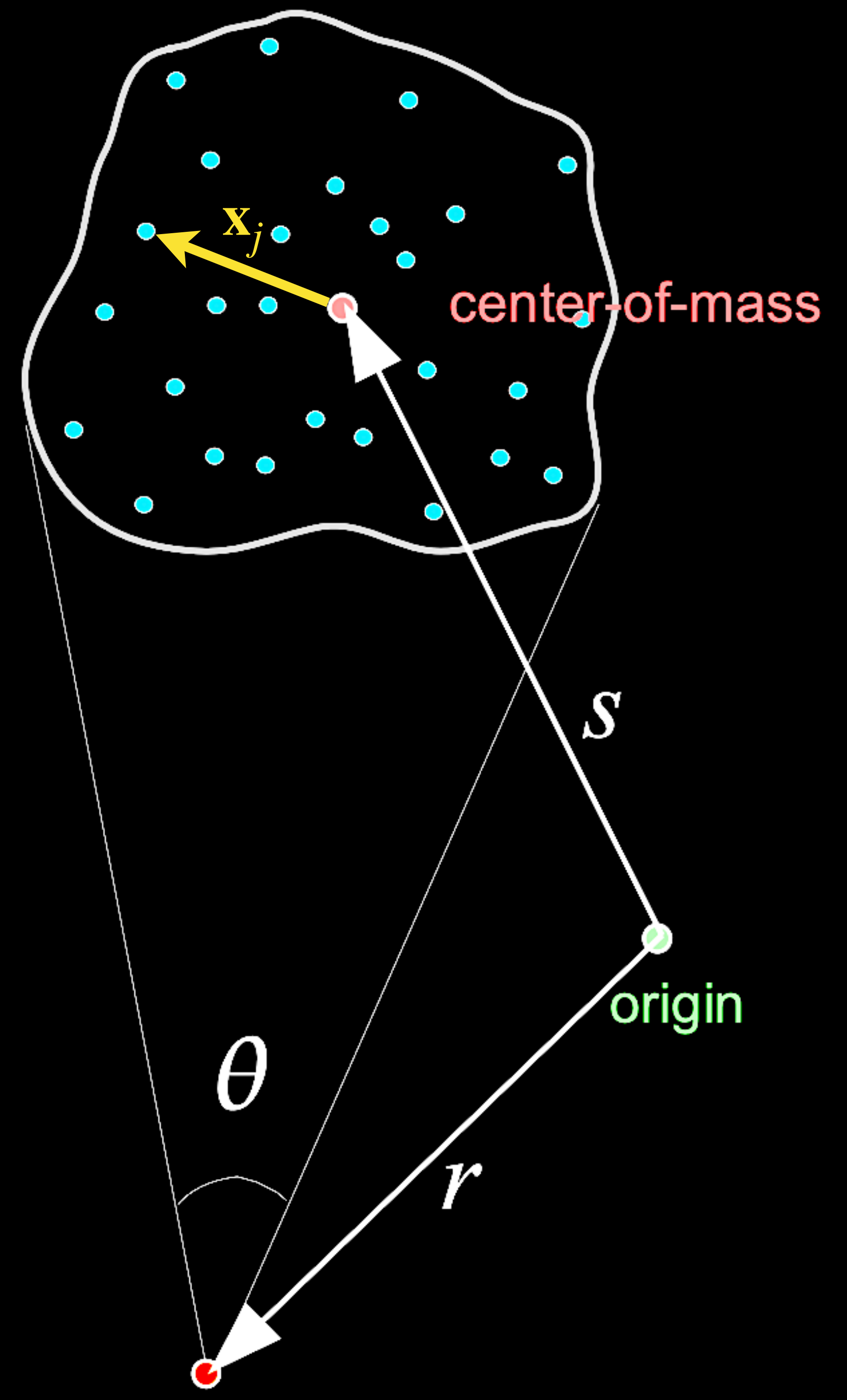


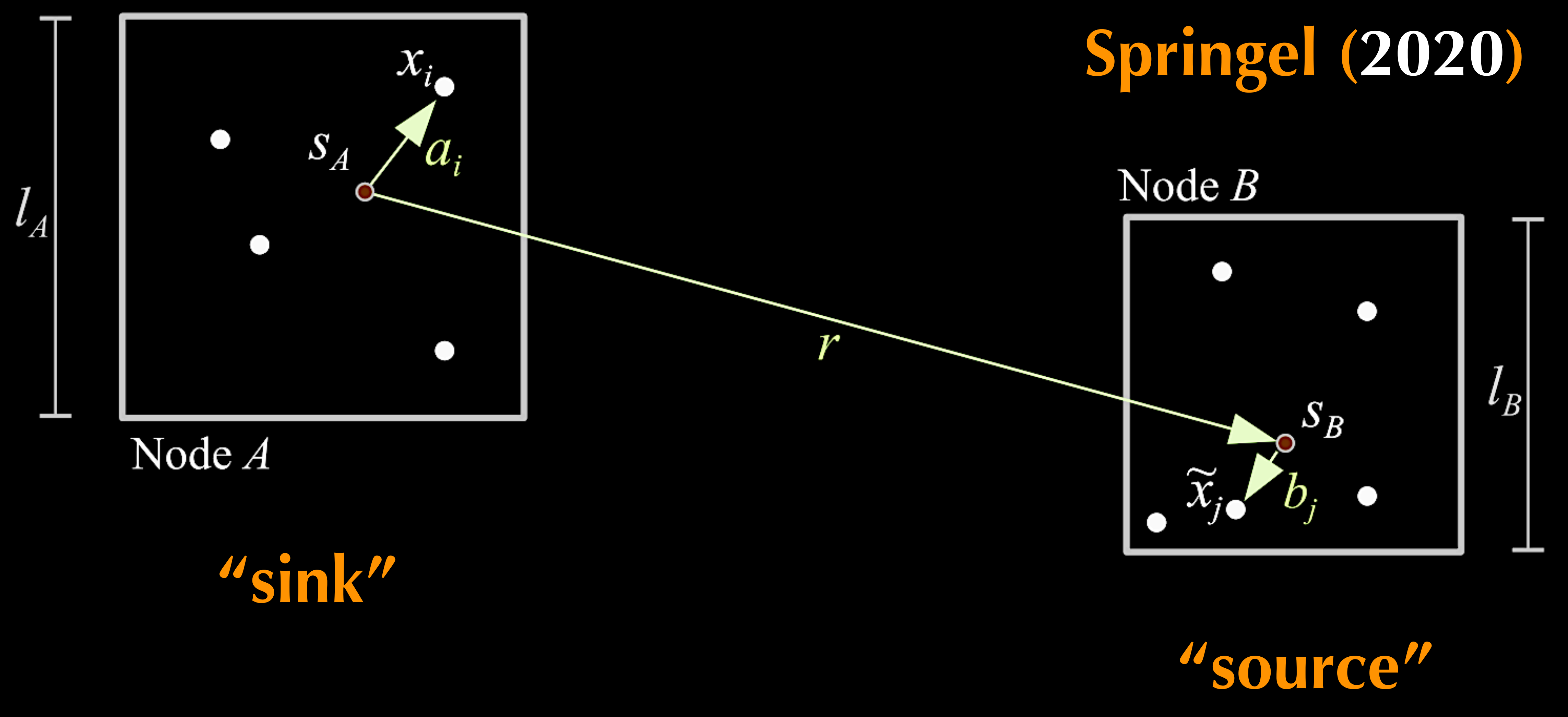


Springel (2005)

gravitational potential generated by points inside a node:

$$\Phi(\mathbf{x}) = -G \sum_{j \in \mathrm{node}} m_j \, g\left(\tilde{\mathbf{x}}_j - \mathbf{x}\right)$$

fast multipole method



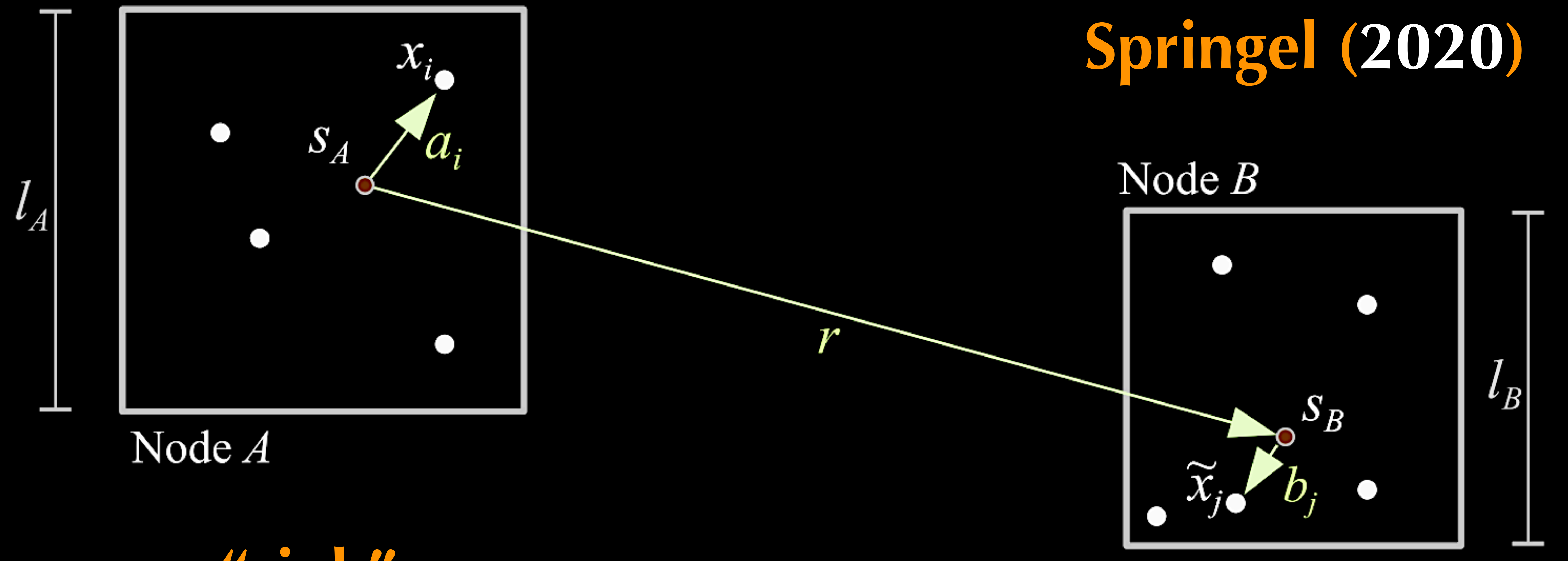
the fast multipole method (FMM) is arguably the most efficient way to compute hierarchical multipole expansions of the gravitational potential with high accuracy.

$\Rightarrow$ perform multipole expansion of potential on both the "source" and "sink" sides just once — this can then be reused for all particles in each node

$$\Phi\left(\mathbf{x}_i\right) = -G \sum_j m_j \, g\left(\tilde{\mathbf{x}}_j - \mathbf{x}_i\right)$$

fast **multipole** method

Springel (2020)



"sink"

"source"

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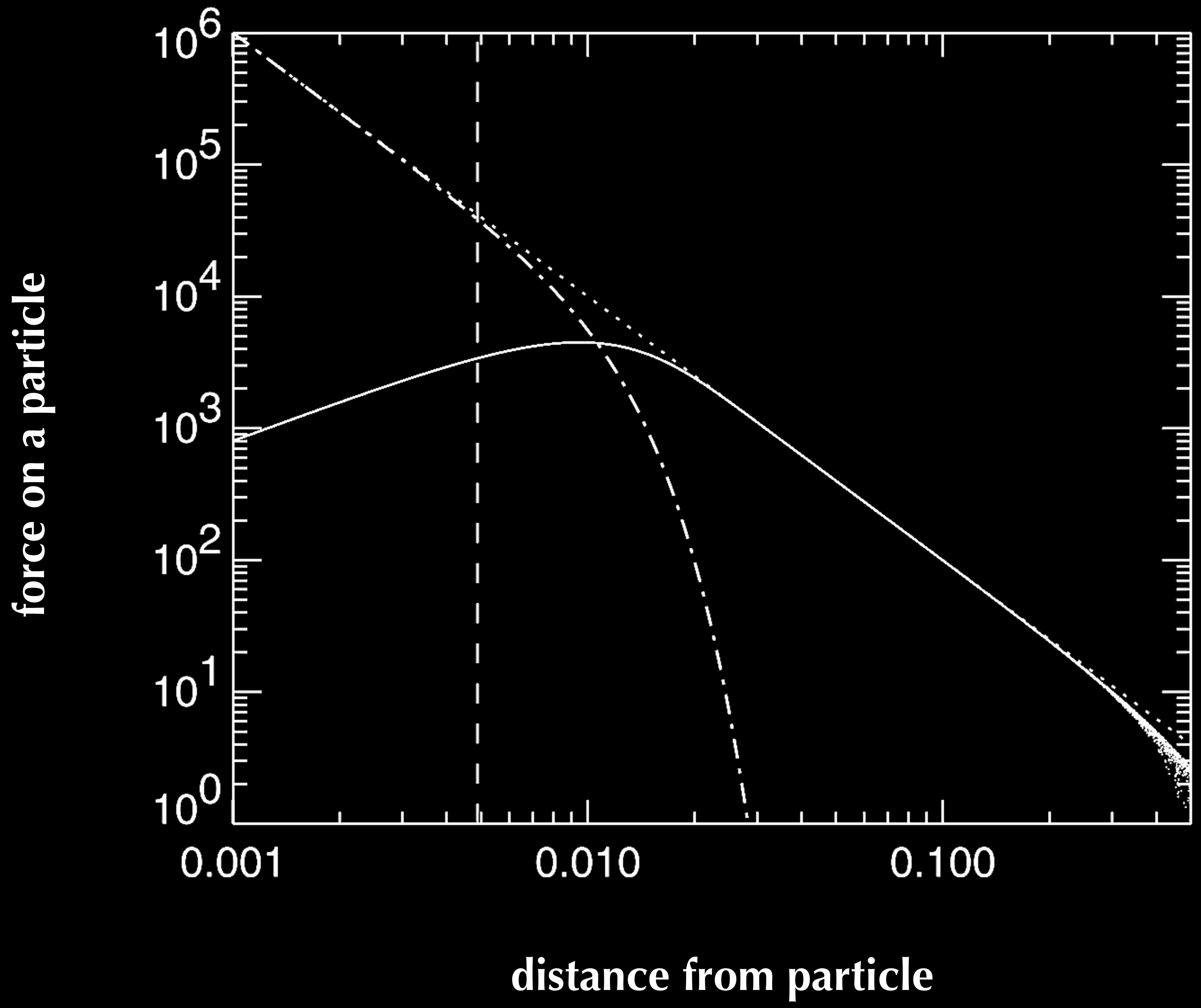
$$\Phi\left(\mathbf{x}_i\right) = -G \sum_j m_j\, g\left(\tilde{\mathbf{x}}_j - \mathbf{x}_i\right)$$

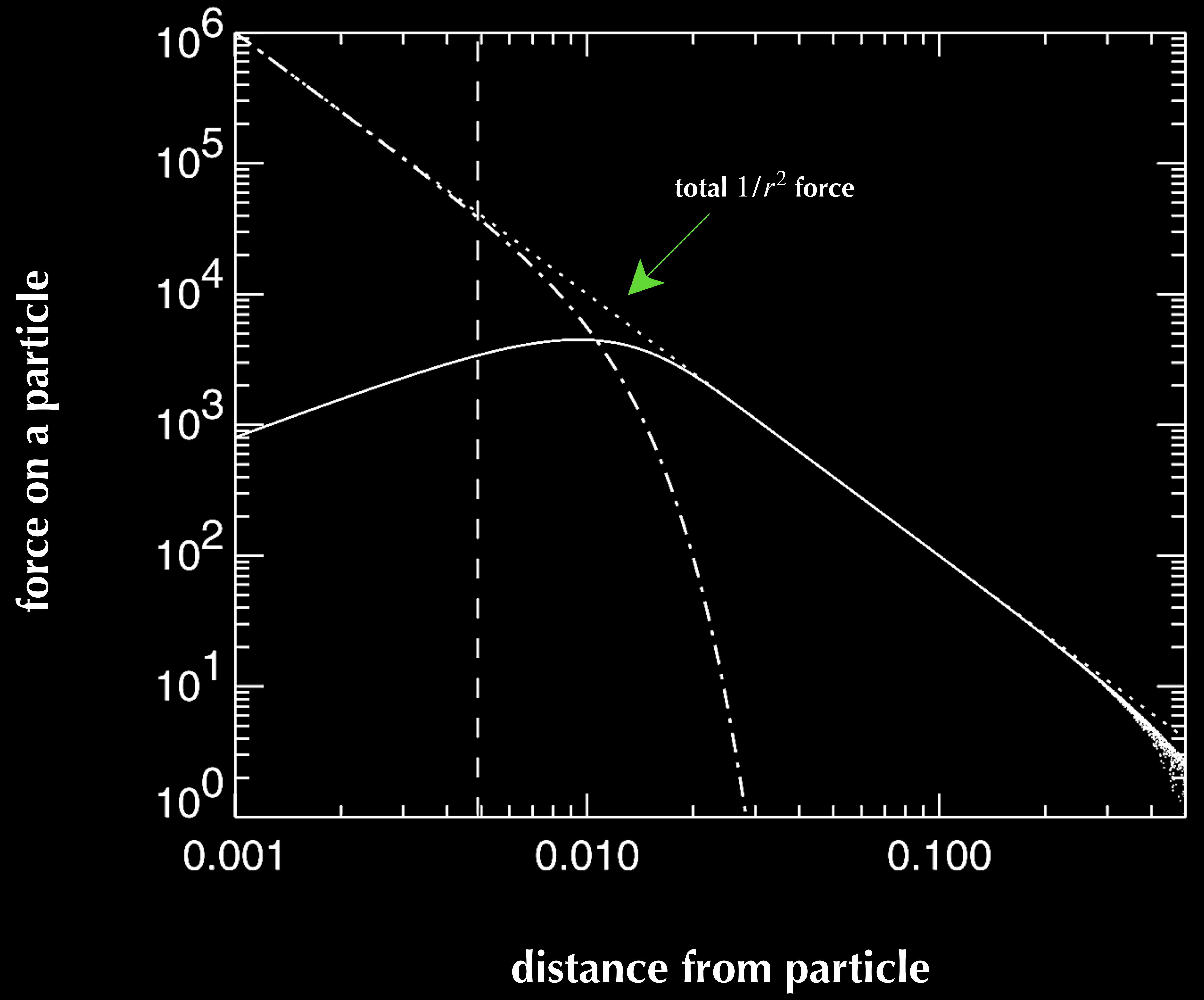
$$g\left(\tilde{\mathbf{x}}_j - \mathbf{x}_i\right) \simeq \sum_{n=0}^{p} \frac{1}{n!} \nabla_{\mathbf{r}}^{(n)} g\left(\mathbf{r}\right) \cdot \left(\mathbf{b}_j - \mathbf{a}_i\right)^{(n)}$$

higher-order multipole expansions $\Rightarrow$ more accurate forces

$p = 1$ (**dipole**)
$p = 2$ (**quadrupole**)
$p = 3$ (**octupole**)
$p = 4$ (**hexadecapole**)
$p = 5$ (**triakontadipole**)



Springel (2005)



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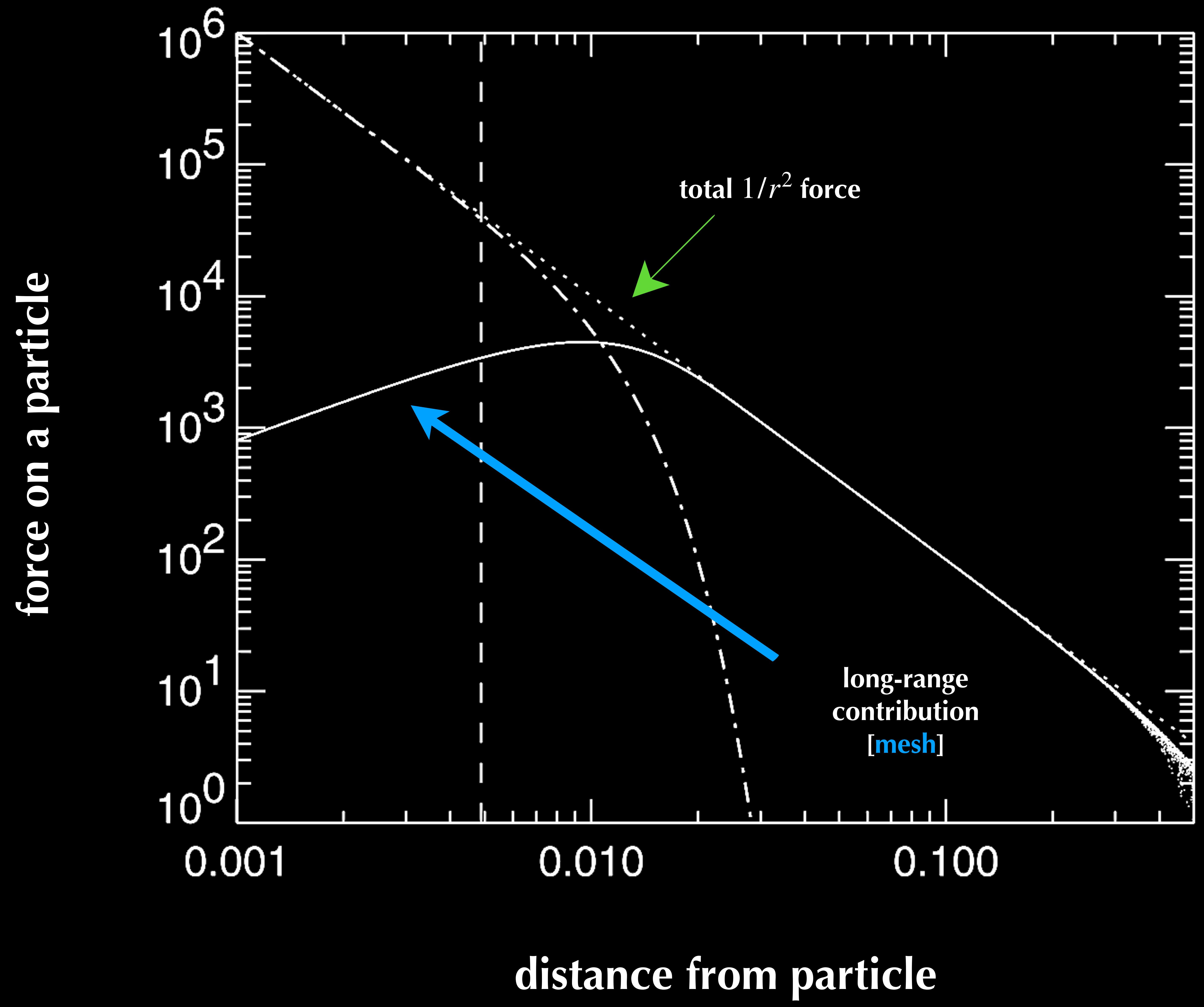

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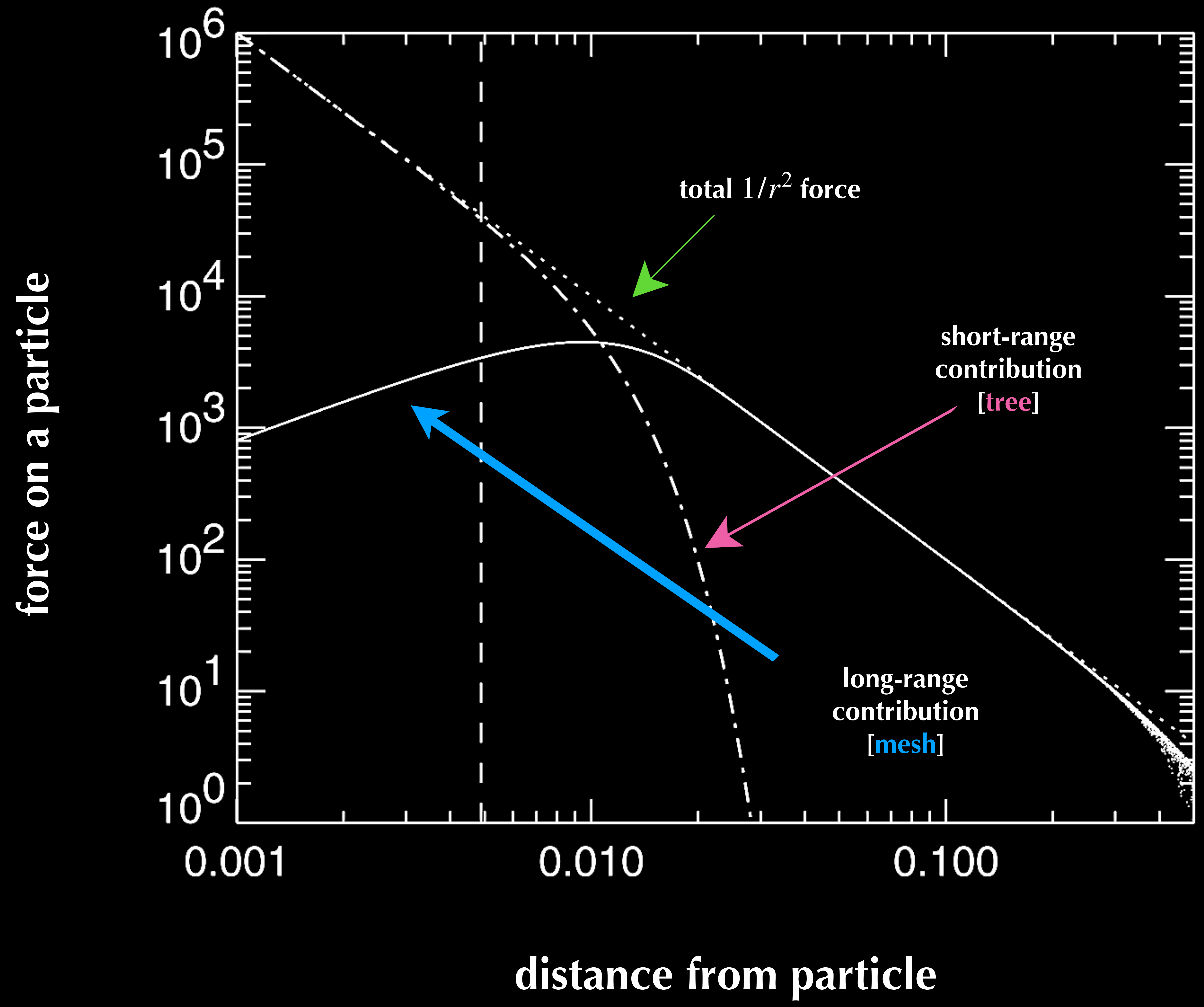

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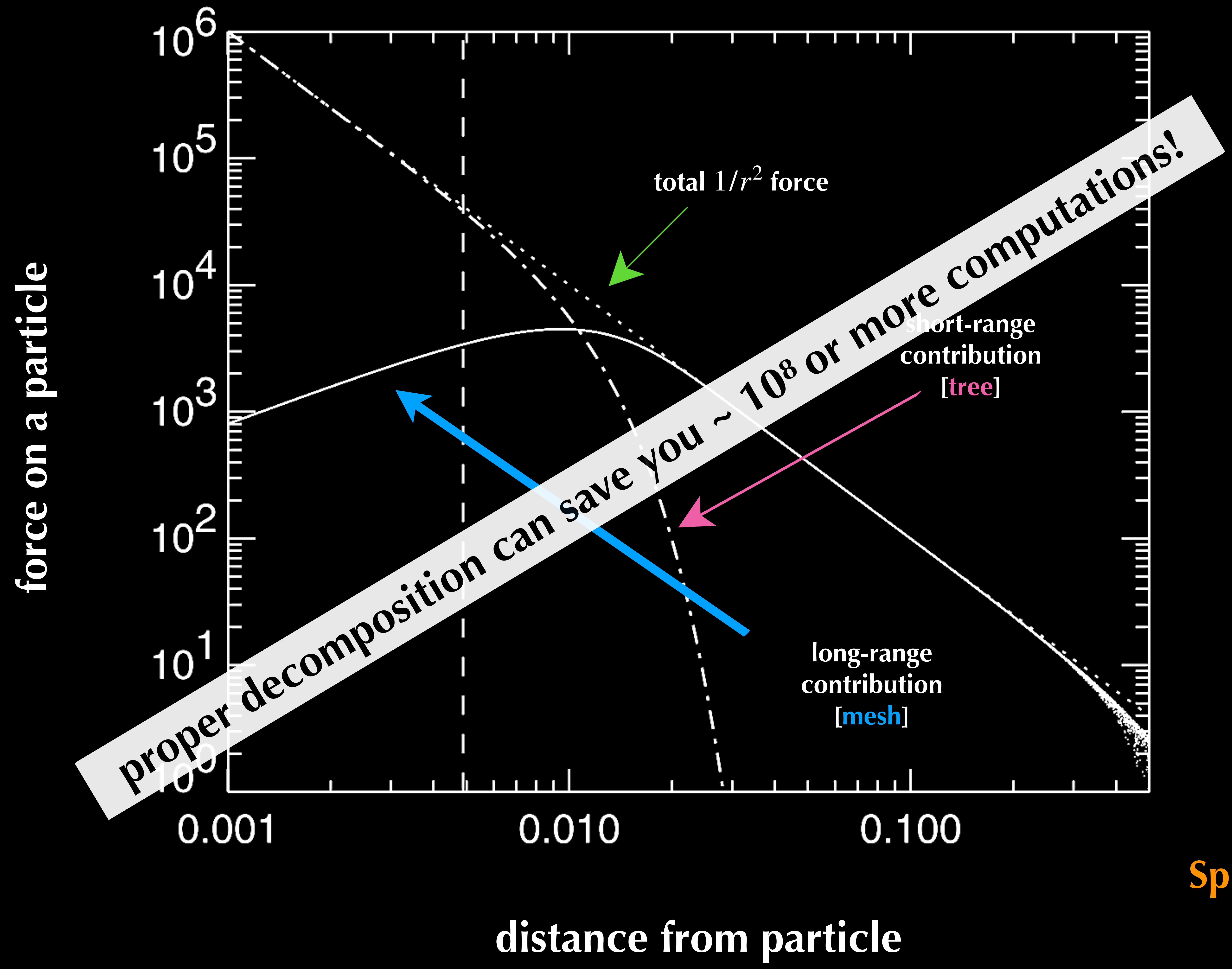

Springel (2005)

basic time integration

once forces on particles have even calculated, we need to propagate these forward in time and update their (1) positions and (2) velocities

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simplest case: the Euler method

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only first-order accurate

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acceleration
[computed from before]

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drift
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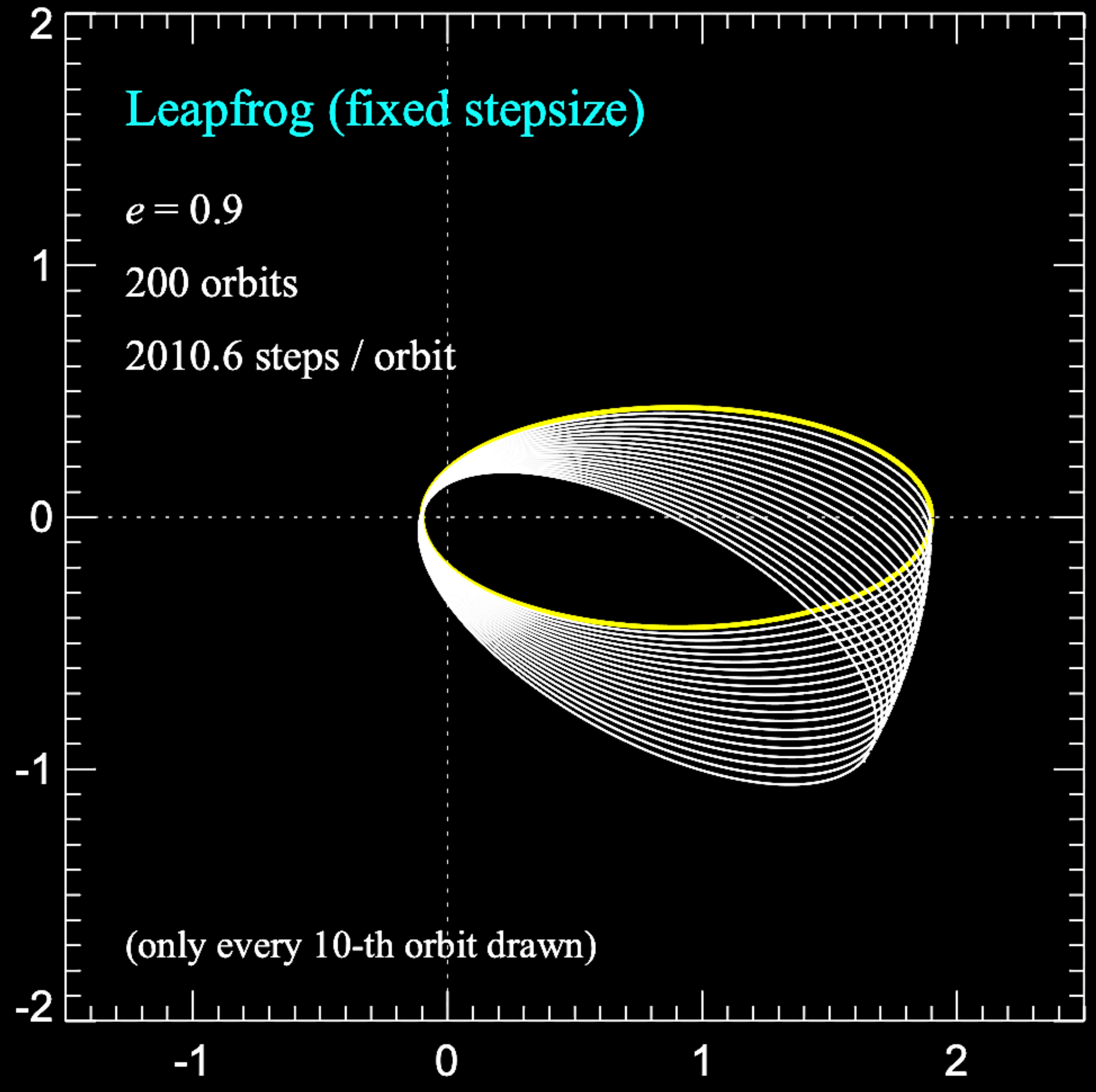
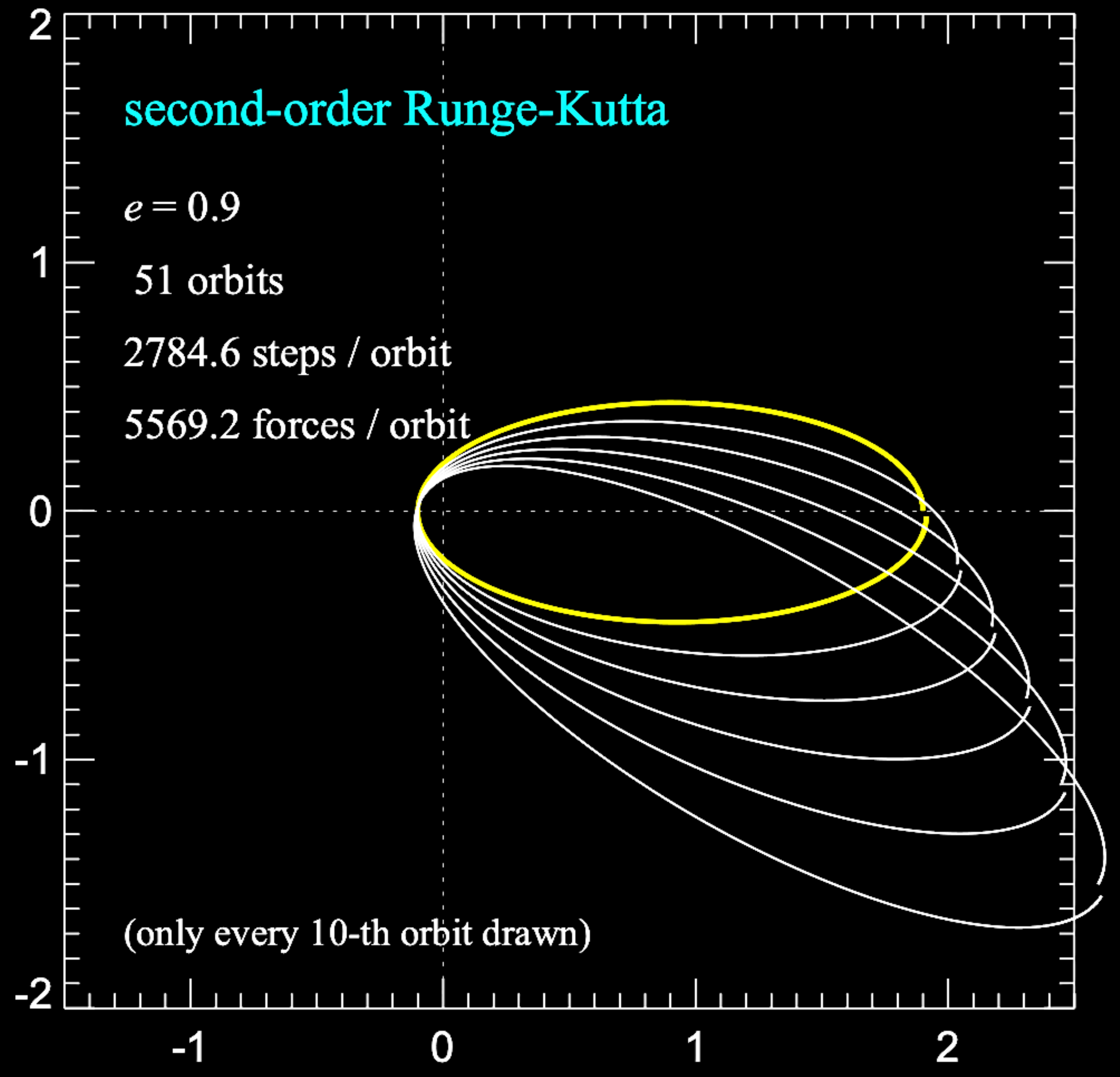
2nd order
"leapfrog" method

much more stable

"symplectic"
[~energy is conserved]





Springel (2005)

choice of timestep

ideally, one would want to choose the size of the timestep, Δt, to be as small as possible

this is impractical from a computational standpoint, but also unnecessary: close-range particle pairs need to be integrated on much shorter timesteps than distant pairs. solution: adopt a hierarchy of timesteps determined by the force acting on the particle

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integration "tolerance" parameter