Centro de Investigación en Matemáticas

Maestría en Probabilidad y Estadística

Avance de tesis

Estudio probabilista del proceso de Dyson determinista

Autor:
Juan Esaul González Rangel

Asesores:
Octavio Arizmendi Echegaray
José Luis Ángel Pérez Garmendia

13 de marzo de 2024

Contents

1 Preliminaries

1.1 Introduction to main concepts in Random Matrix Theory

Mi idea es usar esta sección para hacer algo así como una presentación a grandes rasgos del estudio de las matrices aleatorias (qué es un ensamble, ensambles comunes, qué se hace, el nombre de algunas técnicas) y tal vez un repaso histórico. También lo quiero usar para aclarar la notación que voy a usar a lo largo del texto y tal vez incluir algún resultado de álgebra matricial que necesite después.

1.1.1 Matrix algebra

1.1.2 Random matrix ensembles

1.1.3 Asymptotic results for random matrices

Laws of large numbers, Wigner's semicircle law.

1.2 Stochastic calculus

En esta sección quiero primero incluir resultados clásicos y muy importantes de cálculo estocástico (por ejemplo, fórmula de Itô o fórmula de Tanaka), así como resultados particualres que se usan en las pruebas en los siguientes capítulos (criterio de McKean, lema de Gronwall). Después, en una subsección hablar de la generalización de estos reusltados a procesos matriciales.

1.2.1 Stochastic calculus for $\mathbb{R}^n$-valued processes

Agregar definición de integral de Itô y Strata- novich, fórmula de Itô para matrices

Agregar demostración de que los resultados son generalizables a semimartingalas matriciales

Theorem 1.1 (Integration by parts for Itô and Stratanovich integral)**.**

Tomado de Revuz-Yor

Let X, Y be two continuous semimartingales, then

$$\begin{aligned} \mathrm{d}(XY) &= X\mathrm{d}Y + Y\mathrm{d}X + \langle X, Y\rangle, \\ &= X\mathrm{d}Y + \frac{1}{2}\langle X, Y\rangle + Y\mathrm{d}X + \frac{1}{2}\langle X, Y\rangle = X \circ \mathrm{d}Y + Y \circ \mathrm{d}X. \end{aligned}$$

Incluir pruebas para todos estos resultados

Theorem 1.2 (Tanaka's formula)**.**

Tomado de Revuz-Yor

Let X be a continuous semimartingale. For any real number a, there exists an increasing continuous process L^a called the local time of X in a such that,

$$|X(t)-a| = |X(0)-a| + \int_0^t \operatorname{sgn}(X(s)-a)\,\mathrm{d}X(s) + L^a(t),$$
$$(X(t)-a)^+ = (X(0)-a)^+ + \int_0^t \mathbb{1}_{\{X(s)>a\}}\,\mathrm{d}X(s) + \frac{1}{2}L^a(t),$$
$$(X(t)-a)^- = (X(0)-a)^- - \int_0^t \mathbb{1}_{\{X(s)\le a\}}\,\mathrm{d}X(s) + \frac{1}{2}L^a(t).$$

Theorem 1.3.

Tomado de Revuz-Yor. Completar hipótesis.

If X is a continuous semimartingale such that , for some $\varepsilon > 0$ and every t, the process

$$A_t = \int_0^t \mathbb{1}_{\{0<X(s)\le\varepsilon\}}\rho(X(s))^{-1}\,\mathrm{d}\langle X,X\rangle(s) < \infty \qquad a.s.,$$

then $L^0(X) = 0$.

Lemma 1.4 (Gronwall's lemma). *Let $T > 0$ and let g be any nonnegative bounded measurable function on $[0,T]$. Assume that there exists two constants $a \ge 0$ and $b \ge 0$ such that for every $t \in [0,T]$,*

Tomado de Le Gall, pág 213

$$g(t) \le a + b\int_0^t g(s)\,\mathrm{d}s.$$

Then we also have, for every $t \in [0,T]$,

$$g(t) \le a\exp(bt).$$

Lemma 1.5 (McKean's argument). *Let $Z = (Z_s)_{s\in\mathbb{R}_+}$ be an adapted càdlàg $\mathbb{R}^+ \setminus \{0\}$-valued stochastic process on a stochastic interval $[0,\tau_0)$ such that $Z_0 > 0$ a.s. and $\tau_0 = \inf\{0 < s \le \tau_0 : Z_{s-} = 0\}$. Suppose that $h : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$ is continuous and satisfies the following:*

Tomado de Mayerhofer (2011)

1. *For all $t \in [0,\tau_0)$, we have $h(Z_t) = h(Z_0) + M_t + P_t$, where*
   - (a) *P is an adapted càdlàg process on $[0,\tau_0)$ such that $\inf_{t\in[0,\tau_0\wedge T]} P_t > -\infty$ a.s. for each $T \in \mathbb{R}_+ \setminus \{0\}$,*
   - (b) *M is a continuous local martingale on $[0,\tau_0)$ with $M_0 = 0$,*
2. *$\lim_{z\to 0} h(z) = -\infty$.*

Then $\tau_0 = \infty$ a.s.

1.2.2 Stochastic calculus for matrix-valued processes

1.3 Non-commutative probability

Como aún no leo la parte de probabilidad libre finita, no sé qué preliminares sea conveniente incluir aquí, pero me imagino que lo impresindible serían la definición de un espacio de probabilidad no conmutativo, tipos de independencia, las convoluciones asociadas a ellas y las transformadas y cumulantes que linealizan cada convolución.

1.3.1 Non-commutative probability space

1.3.2 Notions of independence

1.3.3 Convolution

1.3.4 Classical and non-commutative central limit theorems

1.3.5 Asymptotic freeness for random matrices

2 Eigenvalue processes for matrix-valued processes

2.1 Dyson Brownian motion

Unificar notación

Al principio de este capítulo (en el borrador es una sección) escribiría una breve introducción sobre el proceso de Dyson, en qué contexto surge y tal vez explicar un poco la intuición de la forma de la ecuación (esto de repulsión couloumbica) y los casos GUE y GOE. Todo esto para motivar los teoremas.
Después de eso vendrían los teoremas que tomé de Graczyk y Małecki para aplicarlos a mostrar los resultados de Dyson y de otros proecesos matriciales. También está pendiente incluir algo de texto entre cada teorema, explicando para que se usa cada uno en la tesis.

Theorem 2.1 (First part of Multidimensional Yamada-Watanabe). *Let $p \in \mathbb{N}$ and*

$$b_i : \mathbb{R}^p \to \mathbb{R}, \qquad i = 1, \dots, p,$$

be real-valued continuous functions satisfying the following Lipschitz conditions for $C > 0$,

$$|b_i(y_1) - b_i(y_2)| \leq C \|y_1 - y_2\|, \quad i = 1, \dots, p,$$

for every $y_1, y_2 \in \mathbb{R}^p$.

Further, let $\sigma_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, p$ be a set of measurable functions such that

$$|\sigma_i(x) - \sigma_i(y)|^2 \leq \rho_i(|x - y|), \quad x, y \in \mathbb{R},$$

where $\rho_i : (0, \infty) \to (0, \infty)$ are measurable functions such that

$$\int_{0^+} \rho_i^{-1}(x)\, \mathrm{d}x = \infty.$$

Then the pathwise uniqueness holds for the following system of stochastic differential equations

$$\mathrm{d}Y_i = \sigma_i(Y_i)\mathrm{d}B_i + b_i(Y)\mathrm{d}t, \qquad i = 1, \dots, p, \tag{2.1}$$

where $B_1, \dots, B_p$ are independent Brownian motions.

Proof. Let Y and $\hat{Y}$ be two solutions with respect to the same multidimensional Brownian motion $B = (B_i)_{i\leq p}$ such that $Y(0) = \hat{Y}(0)$ a.s., for $i \leq p$ we have

$$Y_i(t) - \hat{Y}_i(t) = \int_0^t \sigma_i(Y_i) - \sigma_i(\hat{Y}_i)\,\mathrm{d}B_i(s) + \int_0^t b_i(Y_i) - b_i(\hat{Y}_i)\,\mathrm{d}s. \tag{2.2}$$

We can then see that

$$\int_0^t \frac{\mathbb{1}_{\{Y_i(s)>\hat{Y}_i(s)\}}}{\rho_i(Y_i(s)-\hat{Y}_i(s))}\mathrm{d}\langle Y_i - \hat{Y}_i, Y_i - \hat{Y}_i\rangle = \int_0^t \frac{(\sigma_i(Y_i(s)) - \sigma_i(\hat{Y}_i(s)))^2}{\rho_i(Y_i(s)-\hat{Y}_i(s))}\mathbb{1}_{\{Y_i(s)>\hat{Y}_i(s)\}}\,\mathrm{d}s \leq t.$$

Applying Theorem 1.3 we have that the local time of $Y_i - \hat{Y}_i$ at 0 is 0. Then, we can use the Tanaka's formula to find

$$\begin{aligned}
|Y_i(t) - \hat{Y}_i(t)| &= \int_0^t \mathrm{sgn}(Y_i - \hat{Y}_i)\,\mathrm{d}(Y_i(s) - \hat{Y}_i(s)),\\
&= \int_0^t \mathrm{sgn}(Y_i(t) - \hat{Y}_i(t))(\sigma_i(Y_i) - \sigma_i(\hat{Y}_i))\,\mathrm{d}B_i(s)\\
&\qquad + \int_0^t \mathrm{sgn}(Y_i(t) - \hat{Y}_i(t))(b_i(Y_i(t)) - b_i(\hat{Y}_i(t)))\,\mathrm{d}t.
\end{aligned}$$

Since σ_i is bounded, we have that $\mathrm{sgn}(Y_i(t) - \hat{Y}_i(t))(\sigma_i(Y_i(t)) - \sigma_i(\hat{Y}_i(t)))$ is bounded and therefore the first integral in the last expression is a martingale with mean 0, which in turns implies that

$$|Y_i(t) - \hat{Y}_i(t)| - \int_0^t \mathrm{sgn}(Y_i(s) - \hat{Y}_i(s))(b_i(Y_i(s)) - b_i(\hat{Y}_i(s)))\,\mathrm{d}t,$$

is a zero-mean martingale. Then, by using the Lipschitz properties of b_i we have

$$\begin{aligned}
\mathbb{E}\left(|Y_i(t) - \hat{Y}_i(t)|\right) &= \mathbb{E}\left(\int_0^t \mathrm{sgn}(Y_i(s) - \hat{Y}_i(s))(b_i(Y_i(s)) - b_i(\hat{Y}_i(s)))\,\mathrm{d}s\right),\\
&\leq \mathbb{E}\left(\int_0^t |b_i(Y_i(s)) - b_i(\hat{Y}_i(s))|\,\mathrm{d}s\right),\\
&= \int_0^t \mathbb{E}\left(|b_i(Y_i(s)) - b_i(\hat{Y}_i(t))|\right)\,\mathrm{d}s \leq C\int_0^t \mathbb{E}\left(|Y_i(t) - \hat{Y}_i(s)|\right)\,\mathrm{d}s.
\end{aligned}$$

Summing for every i we get

$$\mathbb{E}\left(|Y(t) - \hat{Y}(t)|\right) \leq Cp\int_0^t \mathbb{E}\left(|Y(t) - \hat{Y}(s)|\right)\,\mathrm{d}s.$$

Using Gronwall's lemma (1.4) we get that

$$\mathbb{E}\left(|Y(t) - \hat{Y}(t)|\right) = 0,$$

which implies $Y(t) = \hat{Y}(t)$ a.s. for every $t > 0$, ending the proof. □

2.1.1 Real case

2.1.2 Complex case

2.1.3 Non-collision of the eigenvalues

2.2 Wishart processes

2.3 Jacobi processes

Theorem 2.2. *Let X_t be a symmetric $p\times p$ matrix-valued stochastic process satisfying the stochastic differential equation*

$$\mathrm{d}X_t = g(X_t)\mathrm{d}B_t h(X_t) + h(X_t)\mathrm{d}B_t^T g(X_t) + b(X_t)\mathrm{d}t, \tag{2.3}$$

where g, h, b are real functions acting spectrally, and X_0 is a symmetric $p \times p$ matrix with p different eigenvalues.

Let $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$, and

$$\tau = \inf\{t : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j\}. \tag{2.4}$$

Then, for $t < \tau$ the eigenvalue process Λ_t verifies the following stochastic differential equations:

$$\mathrm{d}\lambda_i = 2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i + \left(b(\lambda_i) + \sum_{k\neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k}\right)\mathrm{d}t, \tag{2.5}$$

where $(W_i)_i$ are independent Brownian motions.

Proof. Recall that for every t, the process $X(t)$ admits a decomposition of the form

$$X(t) = H\Lambda H^T,$$

where both Λ and H are matrix-valued stochastic processes, $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_p)$ is the diagonal matrix of ordered eigenvalues of $X(t)$ and H is the corresponding matrix of eigenvectors.

Let us define the stochastic logarith of H as

$$\mathrm{d}A := H^{-1}\partial H = H^T\partial H = H^T\mathrm{d}H + \frac{1}{2}(\mathrm{d}H^T)\mathrm{d}H.$$

By using Itô's formula on $I = H^T H$ we find

$$0 = \mathrm{d}I = \mathrm{d}(H^T H) = H^T\mathrm{d}H + (\mathrm{d}H)^T H + (\mathrm{d}H)^T\mathrm{d}H = H^T\partial H + (\partial H)^T H = A + A^T.$$

Which means A is skew symmetric. Using that $H^T H = I$, we have $\Lambda = H^T H\Lambda H^T H = H^T X H$, by the matrix Itô formula, we find

Revisar bien esta cuenta

$$\begin{aligned}
\mathrm{d}\Lambda = \mathrm{d}(H^T X H) &= (\partial H^T X)H + H^T X\partial H, \\
&= (\partial H)^T X H + H^T(\partial X)H + H^T X\partial H, \\
&= (\partial H)^T H\Lambda + H^T(\partial X)H + \Lambda H^T\partial H, \\
&= (\partial A)^T\Lambda + H^T(\partial X)H + \Lambda\partial A, \\
&= H^T(\partial X)H - (\partial A)\Lambda + \Lambda\partial A.
\end{aligned}$$

The entries in the diagonals of $(\partial A)\Lambda$ and $\Lambda\partial A$ coincide, and thus the diagonal of $\Lambda\partial A - (\partial A)\Lambda$ is zero. Let us denote $\mathrm{d}N = H^T(\partial X)H$, then

$$\mathrm{d}\lambda_j = \mathrm{d}N_{jj},$$

and, using that Λ is a diagonal matrix, if $i \neq j$,

$$0 = \mathrm{d}N_{i,j} + (\lambda_i - \lambda_j)\mathrm{d}A_{ij}.$$

This leads to the following representation for A_{ij},

$$\mathrm{d}A_{i,j} = \frac{\mathrm{d}N_{i,j}}{\lambda_j - \lambda_i}, \qquad i \neq j. \tag{2.6}$$

From (2.3) we compute the quadratic covariation $\mathrm{d}X_{ij}\mathrm{d}X_{km}$,

$$\begin{aligned}
\mathrm{d}X_{ij}\mathrm{d}X_{km} = \mathrm{d}\big\langle & (g(X(t))\mathrm{d}B(t)h(X(t)))_{ij} + (h(X_t)\mathrm{d}B^{\intercal}(t)(g(X(t))))_{ij}, \\
& (g(X(t))\mathrm{d}B(t)h(X(t)))_{km} + (h(X_t)\mathrm{d}B^{\intercal}(t)(g(X(t))))_{km}\big\rangle, \\
& = \mathrm{d}\big\langle (g(X(t))\mathrm{d}B(t)h(X(t)))_{ij}, (g(X(t))\mathrm{d}B(t)h(X(t)))_{km}\big\rangle \\
& + \mathrm{d}\big\langle (g(X(t))\mathrm{d}B(t)h(X(t)))_{ij}, (h(X_t)\mathrm{d}B^{\intercal}(t)(g(X(t))))_{km}\big\rangle \\
& + \mathrm{d}\big\langle (h(X_t)\mathrm{d}B^{\intercal}(t)(g(X(t))))_{ij}, (h(X_t)\mathrm{d}B^{\intercal}(t)(g(X(t))))_{km}\big\rangle \\
& + \mathrm{d}\big\langle (h(X_t)\mathrm{d}B^{\intercal}(t)(g(X(t))))_{ij}, (g(X(t))\mathrm{d}B(t)h(X(t)))_{km}\big\rangle
\end{aligned}$$

Let us first find $\mathrm{d}\big\langle (g(X(t))\mathrm{d}B(t)h(X(t)))_{ij}, (g(X(t))\mathrm{d}B(t)h(X(t)))_{km}\big\rangle$, the other summands are analogous,

$$\begin{aligned}
& \mathrm{d}\big\langle (g(X(t))\mathrm{d}B(t)h(X(t)))_{ij}, (g(X(t))\mathrm{d}B(t)h(X(t)))_{km}\big\rangle, \\
& \qquad = \mathrm{d}\left\langle \sum_{p,q} g(X(t))_{ip}\mathrm{d}B(t)_{pq}h(X(t))_{qj}, \sum_{r,s} g(X(t))_{kr}\mathrm{d}B(t)_{rs}h(X(t))_{sm}\right\rangle
\end{aligned}$$

using the independence between the entries in the brownian matrix,

$$\begin{aligned}
& = \sum_{p,q} \mathrm{d}\big\langle g(X(t))_{ip}\mathrm{d}B(t)_{pq}h(X(t))_{qj}, g(X(t))_{kp}\mathrm{d}B(t)_{pq}h(X(t))_{qm}\big\rangle \\
& = \sum_{pq} g(X(t))_{ip}h(X(t))_{qj}, g(X(t))_{kp}h(X(t))_{qm}\mathrm{d}t, \\
& = \left(\sum_{p} g(X(t))_{ip}g(X(t))_{kp}\right)\left(\sum_{q} h(X(t))_{qj}h(X(t))_{qm}\right)\mathrm{d}t, \\
& = \big(g(X(t))g(X(t))^{\intercal}\big)_{ik}\big(h(X(t))^{\intercal}h(X(t))\big)_{jm}\mathrm{d}t, \\
& = (Hg(\Lambda)H^{\intercal}Hg(\Lambda)H^{\intercal})_{ik}\,(Hh(\Lambda)H^{\intercal}Hh(\Lambda)H^{\intercal})_{jm}\,\mathrm{d}t, \\
& = \left(Hg^2(\Lambda)H^{\intercal}\right)_{ik}\left(Hh^2(\Lambda)H^{\intercal}\right)_{jm}\mathrm{d}t = g^2(X)_{ik}h^2(X)_{jm}\mathrm{d}t.
\end{aligned}$$

Proceeding similarly with the other four summands we find

$$\mathrm{d}X_{ij}\mathrm{d}X_{km} = (g^2(X)_{ik}h^2(X)_{jm} + g^2(X)_{im}h^2(X)_{jk} + g^2(X)_{jk}h^2(X)_{im} + g^2(X)_{jm}h^2(X)_{ik})\mathrm{d}t.$$

Since $\mathrm{d}N = H^T(\partial X)H$ only differs in a finite variation part of $H^T(\mathrm{d}X)H$, the martingale part of both processes coincide and then the quadratic covariation of the entries of N is

$$\begin{aligned}
\mathrm{d}N_{ij}\mathrm{d}N_{km} &= \mathrm{d}\left\langle (H^\intercal \mathrm{d}XH)_{ij}, (H^\intercal \mathrm{d}XH)_{km}\right\rangle = \sum_{pqrs} \mathrm{d}\left\langle H^\intercal_{ip}\mathrm{d}X_{pq}H_{qj}, H^\intercal_{kr}\mathrm{d}X_{rs}H_{sm}\right\rangle, \\
&= \sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sm}\mathrm{d}X_{pq}\mathrm{d}X_{rs}, \\
&= \sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sm}\big(g^2(X)_{pr}h^2(X)_{qs} + g^2(X)_{ps}h^2(X)_{qr} + g^2(X)_{qs}h^2(X)_{pr} + g^2(X)_{qr}h^2(X)_{ps}\big)\mathrm{d}t.
\end{aligned}$$

We find first $\sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sm}g^2(X)_{pr}h^2(X)_{qs}$ and the other terms are similar,

$$\begin{aligned}
\sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sm}g^2(X)_{pr}h^2(X)_{qs} &= \left(\sum_{pr} H^\intercal_{ip}g^2(X)_{pr}H_{rk}\right)\left(\sum_{qs} H^\intercal_{jq}h^2(X)_{qs}H_{sm}\right), \\
&= \left(H^\intercal Hg^2(\Lambda)H^\intercal H\right)_{ik}\left(H^\intercal Hh^2(\Lambda)H^\intercal H\right)_{jm} = g^2(\Lambda)_{ik}h^2(\Lambda)_{jm}.
\end{aligned}$$

Repeating the analogous procedure with all of the terms we find that the covariation is

$$\mathrm{d}N_{ij}\mathrm{d}N_{km} = (g^2(\Lambda)_{ik}h^2(\Lambda)_{jm} + g^2(\Lambda)_{im}h^2(\Lambda)_{jk} + g^2(\Lambda)_{jk}h^2(\Lambda)_{im} + g^2(\Lambda)_{jm}h^2(\Lambda)_{ik})\mathrm{d}t.$$

It follows that the quadratic variation in the diagonal is

$$\mathrm{d}N_{ii}\mathrm{d}N_{jj} = 4\delta_{ij}g^2(\lambda_i)h^2(\lambda_j)\mathrm{d}t.$$

Now, in order to compute F, the finite variation part of N, we use (2.3),

$$\begin{aligned}
\mathrm{d}F &= H^T b(X)H\mathrm{d}t + \frac{1}{2}(\mathrm{d}H^T\mathrm{d}XH + H^T\mathrm{d}X\mathrm{d}H), \\
&= b(\Lambda)\mathrm{d}t + \frac{1}{2}\left((\mathrm{d}H^T H)(H^T\mathrm{d}XH) + (H^T\mathrm{d}XH)(H^T\mathrm{d}H)\right),
\end{aligned}$$

using that the martingale part of $H^T\mathrm{d}H$ and $H^T\partial H$ coincide and the same with $H^T(\partial X)H$ and $H^T(\mathrm{d}X)H$,

$$= b(\Lambda)\mathrm{d}t + \frac{1}{2}((\mathrm{d}N\mathrm{d}A)^T + \mathrm{d}N\mathrm{d}A).$$

Now we can use (2.6) and (2.3) to find $\mathrm{d}N\mathrm{d}A$,

$$(\mathrm{d}N\mathrm{d}A)_{ij} = \sum_{k\neq j}\mathrm{d}N_{ik}\mathrm{d}A_{kj} = \sum_{k\neq j}\frac{\mathrm{d}N_{ik}\mathrm{d}N_{kj}}{\lambda_j - \lambda_k} = \delta_{ij}\sum_{k\neq j}\frac{g^2(\lambda_i)h^2(\lambda_k) + g^2(\lambda_k)h^2(\lambda_i)}{\lambda_i - \lambda_k}\mathrm{d}t.$$

Recalling that $G(x, y) = g^2(x)h^2(x) + g^2(y)h^2(y)$, we have that

$$(\mathrm{d}N\mathrm{d}A)_{ij} = \delta_{ij}\sum_{k\neq j}\frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k}\mathrm{d}t.$$

From (2.3) we have that the martingale part of N_{ii} has the form $2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i$ for some Brownian motion W_i. Putting together the martingale and finite variation parts of N we have that

$$\mathrm{d}N_{ii} = 2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i + \delta_{ij}\sum_{k\neq j}\frac{G(\lambda_i,\lambda_k)}{\lambda_i-\lambda_k}\mathrm{d}t.$$

Since $\mathrm{d}\lambda_i = \mathrm{d}N_{ii}$, this finishes the proof. □

Theorem 2.3. *Let W_t be a complex $p\times p$ Brownian matrix. Suppose that an $\mathcal{H}_p$ valued stochastic process X_t satisfies the following matrix stochastic differential equation:*

$$\mathrm{d}X_t = g(X_t)\mathrm{d}W_t h(X_t) + h(X_t)\mathrm{d}W_t^* g(X_t) + b(X_t)\mathrm{d}t, \tag{2.7}$$

with $g, h, b : \mathbb{R}\to\mathbb{R}$ and X_0 is a hermitian $p\times p$ random matrix with p different eigenvalues.

Let $G(x,y) = g^2(x)h^2(y) + g^2(y)h^2(x)$, and

$$\tau = \inf\{t : \lambda_i(t) = \lambda_j(t) \text{ for some } i\neq j\}.$$

Then, for $t<\tau$ the eigenvalue process Λ_t verifies the following stochastic differential equations:

$$\mathrm{d}\lambda_i = 2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i + \left(b(\lambda_i) + 2\sum_{k\neq i}\frac{G(\lambda_i,\lambda_k)}{\lambda_i-\lambda_k}\right)\mathrm{d}t, \tag{2.8}$$

where $(W_i)_i$ are independent Brownian motions.

Proof. Recall that for a complex Brownian motion Z we have that

No sé si debo incluir esto en los preliminares.

$$\mathrm{d}Z\mathrm{d}Z = 0, \qquad \mathrm{d}Z\mathrm{d}\overline{Z} = 2\mathrm{d}t.$$

Then we can compute the quadratic covariation $\mathrm{d}X_{ij}\mathrm{d}X_{kl}$ using (2.7),

$$\begin{aligned}
\mathrm{d}X_{ij}\mathrm{d}X_{kl} &= \mathrm{d}\langle X_{ij}, X_{kl}\rangle, &(2.9)\\
&= \mathrm{d}\langle (g(X_t)\mathrm{d}W_t h(X_t) + h(X_t)\mathrm{d}W_t^* g(X_t))_{ij}, (g(X_t)\mathrm{d}W_t h(X_t) + h(X_t)\mathrm{d}W_t^* g(X_t))_{kl}\rangle, &(2.10)\\
&= \mathrm{d}\langle (g(X_t)\mathrm{d}W_t h(X_t))_{ij}, (h(X_t)\mathrm{d}W_t^* g(X_t))_{kl}\rangle + \mathrm{d}\langle (g(X_t)\mathrm{d}W_t h(X_t))_{kl}, (h(X_t)\mathrm{d}W_t^* g(X_t))_{ij}\rangle, &(2.11)\\
&= 2g^2(X_t)_{il}h^2(X_t)_{jk}\mathrm{d}t + 2g^2(X_t)_{kj}h^2(X_t)_{li}\mathrm{d}t, &(2.12)\\
&= 2\left(g^2(X_t)_{il}h^2(X_t)_{kj} + g^2(X_t)_{jk}h^2(X_t)_{il}\right)\mathrm{d}t. &(2.13)
\end{aligned}$$

Analogously to the real case, we define A, the stochastic logarithm of H, as

$$A := H^{-1}\partial H = H^*\partial H.$$

By using Itô's formula we find,

$$0 = \mathrm{d}I = \mathrm{d}(H^*H) = H\partial H^* + (\partial H)H^* = A^* + A,$$

which means A is skew-Hermitian. This implies that the real parto of the terms in the diagonal of A is zero. Let us now apply Itô's formula to $\Lambda = H^*XH$,

$$
\begin{aligned}
\mathrm{d}\Lambda &= \mathrm{d}(H^*XH) = H^*(\mathrm{d}(XH)) + (\mathrm{d}H^*)XH + \mathrm{d}(H^*)\mathrm{d}(XH),\\
&= H^*(\mathrm{d}X)H + H^*X\mathrm{d}H + H^*(\mathrm{d}X\mathrm{d}H) + (\mathrm{d}H^*)XH + \mathrm{d}(H^*)(\mathrm{d}X)H + \mathrm{d}(H^*)X\mathrm{d}H + \mathrm{d}H^*\mathrm{d}X\mathrm{d}H,\\
&= H^*(\partial X)H + H^*X\partial H + (\partial H^*)XH = H^*(\partial X)H + \Lambda H^*\partial H + (\partial H^*)H\Lambda,\\
&= H^*(\partial X)H + \Lambda\partial A + \partial A^{\mathsf{T}}\Lambda = H^*(\partial X)H + \Lambda\partial A - \partial A\Lambda.
\end{aligned}
$$

By the relationship between Itô's and Stratanovich's integrals,

$$
H^*(\partial X)H = H^*(\mathrm{d}X)H + \frac{1}{2}(\mathrm{d}H^*(\mathrm{d}X)H + H^*\mathrm{d}X\mathrm{d}H),
$$

so using that X is hermitian, we have that $H^*(\partial X)H$ is also hermitian and its diagonal elements are real. The process $\Lambda\partial A - (\partial A)\Lambda$ is zero in the diagonal and thus $\mathrm{d}\lambda_i = (H^*(\partial X)H)_{ii}$. If $i \neq j$, we have

$$
0 = (H^*(\partial X)H)_{ij} + \lambda_i\partial A_{ij} - \lambda_j\partial A_{ji} = (H^*(\partial X)H)_{ij} + (\lambda_i - \lambda_j)\partial A_{ij}.
$$

The last part implies $\partial A_{ij} = \frac{(H^*(\partial X)H)_{ij}}{\lambda_j - \lambda_i}$, whenever $i \neq j$.

Define $\mathrm{d}N = \mathrm{d}H^*(\partial X)H$. The martingale part of N and $H^*(\mathrm{d}X)H$ is the same, since they differ only in a finite variation term. We can find $\mathrm{d}N_{ij}\mathrm{d}N_{kl}$ using $\mathrm{d}X_{ij}\mathrm{d}X_{kl}$,

$$
\mathrm{d}N_{ij}\mathrm{d}Nkl = 2(g^2(\Lambda)_{il}h^2(\Lambda)_{jk} + g^2(\Lambda)_{jk}h^2(\Lambda)_{il})\mathrm{d}t.
$$

Then, for the elements in the diagonal we have

$$
\mathrm{d}N_{ii}\mathrm{d}N_{jj} = 4\delta_{ij}(g^2(\lambda_i)h^2(\lambda_i))\mathrm{d}t. \tag{2.14}
$$

Now we compute the finite variation part of $\mathrm{d}N$ from (2.7). Let us denote it as $\mathrm{d}F$.

$$
\begin{aligned}
\mathrm{d}F &= H^*b(X)H\mathrm{d}t + \frac{1}{2}(\mathrm{d}H^*(\mathrm{d}X)H + H^*\mathrm{d}X\mathrm{d}H),\\
&= b(\Lambda)\mathrm{d}t + \frac{1}{2}\big((\mathrm{d}H^*H)(H^*\mathrm{d}XH) + (H^*\mathrm{d}XH)(H^*\mathrm{d}H)\big),\\
&= b(\Lambda)\mathrm{d}t + \frac{1}{2}\big((\mathrm{d}N\mathrm{d}A)^* + \mathrm{d}N\mathrm{d}A\big).
\end{aligned}
$$

Using the quadratic variation of $\mathrm{d}N$ and $\mathrm{d}A$ we find their covariation,

$$
\begin{aligned}
(\mathrm{d}N\mathrm{d}A)_{ij} &= \sum_k (\mathrm{d}N)_{ik}(\mathrm{d}A)_{kj} = \sum_k \frac{(\mathrm{d}N)_{ik}(\mathrm{d}N)_{kj}}{\lambda_j - \lambda_i},\\
&= 2\delta_{ij}\sum_{k\neq j}\frac{g^2(\lambda_i)h^2(\lambda_k) + g^2(\lambda_k)h^2(\lambda_j)}{\lambda_j - \lambda_k} + \mathrm{d}N_{ij}\mathrm{d}A_{jj}.
\end{aligned}
$$

By the properties shown above for $\mathrm{d}N$ and $\mathrm{d}A$, if$i = j$, $\mathrm{d}N_{jj}$ is real and $\mathrm{d}A_{jj}$ is purely imaginary. By independence of the real and imaginary parts of the complex Brownian motion, this implies that $\mathrm{d}N_{jj}\mathrm{d}A_{jj} = 0$. We have

$$
\mathrm{d}F_{ii} = \left(b(\lambda_i) + 2\sum_{k\neq i}\frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_j}\right)\mathrm{d}t,
$$

where $G(x,y) = g^2(x)h^2(y) + g^2(y)h^2(x)$.

Using the quadratic variation of $\mathrm{d}N$, we find that the martingale part of $\mathrm{d}N_{ii}$ is

$$\mathrm{d}M_{ii} = 2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i,$$

for some Brownian motion. Recall that $\mathrm{d}\lambda_i = \mathrm{d}N_{ii}$, then we have that there exist $W_1,\dots,W_p$ independent Brownian motions such that

$$\mathrm{d}\lambda_i = \mathrm{d}N_{ii} = 2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i + \left(b(\lambda_i) + 2\sum_{k\neq i}\frac{G(\lambda_i,\lambda_k)}{\lambda_i-\lambda_j}\right)\mathrm{d}t.$$

This ends the proof. □

Theorem 2.4. *Let $\Lambda = (\lambda_i)_{i=1,\dots,p}$ be a process starting at the open simplex Δ_p and satisfying* (2.5) *with functions $g,h,b:\mathbb{R}\to\mathbb{R}$ such that g^2, h^2, b are Lipschitz continuous and g^2h^2 is convex or is continuously differentiable with derivative uniformly Lipschitz on $\mathbb{R}$. Then the first collision time τ defined as in* (2.4) *is infinite a.s.*

Proof. Define $U := -\sum_{i<j}\log(\lambda_j-\lambda_i)$ for $t\in[0,\tau]$. By Itô's formula and the fact that $\mathrm{d}\lambda_i\mathrm{d}\lambda_j = 4\delta_{ij}g^2(\lambda_i)h^2(\lambda_i)$ we find

$$\begin{aligned}
\mathrm{d}U &= \sum_{i<j}\left[\frac{\mathrm{d}\lambda_i-\mathrm{d}\lambda_j}{\lambda_j-\lambda_i} + \frac{1}{2}\frac{\mathrm{d}\langle\lambda_i,\lambda_i\rangle + \mathrm{d}\langle\lambda_j,\lambda_j\rangle}{(\lambda_j-\lambda_i)^2}\right],\\
&= \sum_{i<j}\left[\frac{\mathrm{d}\lambda_i-\mathrm{d}\lambda_j}{\lambda_j-\lambda_i} + 2\frac{g^2(\lambda_i)h^2(\lambda_i) - g^2(\lambda_j)h^2(\lambda_j)}{(\lambda_j-\lambda_i)^2}\mathrm{d}t\right].
\end{aligned}$$

Now we define the following processes

$$\begin{aligned}
\mathrm{d}M &= 2\sum_{i<j}\frac{g(\lambda_i)h(\lambda_i)\mathrm{d}\nu_i - g(\lambda_)h(\lambda_j)\mathrm{d}\nu_j}{\lambda_j-\lambda_i},\\
\mathrm{d}A_1 &= \sum_{i<j}\frac{b(\lambda_i)-b(\lambda_j)}{\lambda_j-\lambda_i}\mathrm{d}t,\\
\mathrm{d}A_2 &= 2\sum_{i<j}\frac{\left(g^2(\lambda_j)-g^2(\lambda_i)\right)\left(h^2(\lambda_j)-h^2(\lambda_i)\right)}{(\lambda_j-\lambda_i)}\mathrm{d}t,\\
\mathrm{d}A_3 &= \sum_{i<j}\frac{1}{\lambda_j-\lambda_i}\sum_{k\neq i,k\neq j}\left(\frac{G(\lambda_i,\lambda_k)}{\lambda_i-\lambda_k} - \frac{G(\lambda_j,\lambda_k)}{\lambda_j-\lambda_k}\right)\mathrm{d}t,\\
&= \sum_{i<j<k}\frac{G(\lambda_j,\lambda_k)(\lambda_k-\lambda_j) - G(\lambda_i,\lambda_k)(\lambda_k-\lambda_i) + G(\lambda_i,\lambda_j)(\lambda_j-\lambda_i)}{(\lambda_j-\lambda_i)(\lambda_k-\lambda_i)(\lambda_k-\lambda_j)}\mathrm{d}t.
\end{aligned}$$

Using (2.5) we find that $\mathrm{d}U = \mathrm{d}M + \mathrm{d}A_1 + \mathrm{d}A_2 + \mathrm{d}A_3$. Our goal is to use the McKean argument by proving that U is bounded over any bounded interval $[0,t]$. Let us start by showing that the finite variation part of U ($\mathrm{d}A_1$, $\mathrm{d}A_2$, and $\mathrm{d}A_3$) is bounded. Lipschitz continuity of b, g^2 and h^2 implies that $|A_1(t)| \le Kp(p-1)t/2$ and $|A_2(t)| \le K^2p(p-1)t$ with K a constant appearing in the Lipschitz condition. Later, we define a function H as

$$H(x, yz) := \left[(g^2(x) - g^2(z))(h^2(y) - h^2(z)) + (g^2(y) - g^2(z))(h^2(x) - h^2(z))\right](y - x),$$

then

$$H(x, y, z) = (G(x, y) - G(x, z) - G(y, z) + G(z, z))(y - x),$$

and

$$\begin{aligned} H(x, y, z) + H(y, z, x) - H(x, z, y) &= 2(z - y)G(y, z) - 2(z - x)G(x, z) + 2(y - x)G(x, y) \\ &\quad + G(x, x)(z - y) - G(y, y)(z - x) + G(z, z)(y - x). \end{aligned}$$

By the Lipschitz conditions on g^2 and h^2 we find that $|H(x, y, z)| \leq 2K^2|(y-x)(z-y)(z-x)|$. Using the last equality, we can write $2\mathrm{d}A_3 = \mathrm{d}A_4 + \mathrm{d}A_5$, with $0 < A_4(t) < K^2 p(p-1)(p-2)t/6$ and

Escribir detalles de esta cuenta

$$\mathrm{d}A_5(t) = \sum_{i<j<k} \frac{G(\lambda_j, \lambda_j)(\lambda_k - \lambda_i) - G(\lambda_i, \lambda_i)(\lambda_k - \lambda_j) - G(\lambda_k, \lambda_k)(\lambda_j - \lambda_i)}{(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)} \mathrm{d}t, \tag{2.15}$$

$$= \sum_{i<j<k} \left(\frac{G(\lambda_j, \lambda_j) - G(\lambda_i, \lambda_i)}{\lambda_j - \lambda_i} - \frac{G(\lambda_k, \lambda_k) - G(\lambda_j, \lambda_j)}{\lambda_k - \lambda_j} \right) \frac{1}{\lambda_k - \lambda_i} \mathrm{d}t. \tag{2.16}$$

If $G(x, x)$ is convex, then A_5 is non positive. If $G(x, x)$ is continuously differentiable with derivative uniformly Lipschitz, then

$$|G'(x, x) - G'(y, y)| \leq C|x - y|,$$

and the (2.16) is bounded by C, which means $|A_5(t)| \leq Ct$.

We have found that the finite variation part of U is bounded for finite t, then we can apply McKean's argument 1.5 to conclude that U can not explode in finite time and thus $\tau = \infty$ a.s. $\square$

Theorem 2.5 (Spectral matrix Yamada-Watanabe theorem). *Let $X(t)$ be a $p \times p$ symmetric matrix-valued process satisfying the equation* (2.3) *with initial condition $X(0)$ that is a symmetric $p \times p$ matrix with p different eigenvalues. Suppose further that*

$$|g(x)h(x) - g(y)h(y)|^2 \leq \rho(|x - y|), \qquad x, y \in \mathbb{R}, \tag{2.17}$$

with $\rho : (0, \infty) \to (0, \infty)$ a measurable function satisfying

$$\int_{0^+} \rho^{-1},$$

that $G(x, y) := g^2(x)h^2(y) + g^2(y)h^2(x)$ is locally Lipschitz and strictly positive on the set $\{x \neq y\}$ and that b is locally Lipschitz. Then if τ is defined as in (2.4), *for $t < \tau$, the process of eigenvalues satisfying* (2.5) *has a pathwise unique solution.*

Proof. Let $PNP^\intercal$ be a diagonalization for $X(0)$. We need to show that a unique strong solution exists for (2.5) when $\Lambda(0) = N$. The functions

$$b_i(\lambda_1, \ldots, \lambda_p) = b(\lambda_i) + \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k},$$

are locally Lipschitz continuous on Δ_p so they can be extended from the compact sets

$$D_m = \{0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_p < m, \lambda_{i+1} - \lambda_i \geq 1/m\},$$

to bounded Lipschitz functions on $\mathbb{R}^p$. Let b_i^m denote such extension for $m \in \mathbb{N} \setminus \{0\}$. For $i = 1, \ldots, p$, we consider the system of SDEs,

$$\mathrm{d}\lambda_i = 2g(\lambda_i^m)h(\lambda_i^m)\mathrm{d}W_i + b_i^m(\Lambda^m)\mathrm{d}t.$$

We have that $|g(x)h(x) - g(y)h(y)|^2 \leq \rho(|x-y|)$ and $\int_{0^+} \rho(x)^{-1}\mathrm{d}x = \infty$, and using Theorem 2.1 we get that there is a unique strong solution for the system of SDEs. Since $D_m \subset D_{m+1}$, we have that $\lim_{m\to\infty} D_m = \Delta_p$, so there exists a unique strong solution $\Lambda(t)$ for the SDEs system up to the first exit time from Δ_p. This time is τ, the first collision time of the eigenvalues. □

Corollary 2.6. *Suppose that b, g^2, h^2 are Lipschitz continuous, g^2h^2 is convex or or continuously differentiable with derivative uniformly Lipschitz on $\mathbb{R}$ and that $G(x,y) := g^2(x)h^2(y) + g^2(y)h^2(x)$ is strictly positive on $\{x \neq y\}$. Then the system of SDEs* (2.5) *for the eigenvalue process satisfying* (2.3) *has a unique strong solution on* $[0, \infty)$.

Proof. Recall that if f is a non-negative Lipschitz continuous function, then $\sqrt{f}$ is $1/2$-Hölder continuous. Since g^2 and h^2 are Lipschitz continuous, then g^2h^2 is locally Lipschitz continuous and gh is $1/2$-Hölder continuous. Then

$$|g(x)h(x) - g(y)h(y)|^2 \leq \left(K|x-y|^{\frac{1}{2}}\right)^2 = K^2|x-y|.$$

Taking $\rho(|x-y|) = K^2|x-y|$ we see that the conditions of Theorem 2.5 are satisfied and then the uniqueness and existence of the strong solution applies on $[0, \tau)$. By Theorem 2.4 we have that $\tau = \infty$ a.s., and thus the existence and uniqueness is satisfied on $[0, \infty)$. □

Después de exponer estos teoremas, restaría aplicarlos para exponer primero los resultados de Dyson sobre el proceso de eigenvalores de un movimiento Browniano en el espacio de matrices ortogonales y autoadjuntas, y después aplicarlos para encontrar ecuaciones de procesos de eigenvalores más generales (Wishart, Jacobi, Laguerre). Estos ejemplos vienen en Graczyk y Małecki, así que incluirlos no debería ser tan difícil.

3 Finite Free Probability

Esta parte de la tesis será la última en trabajarse. Aunque es un área relativamente nueva, los resultados que se usarán están bien documentados, por lo que se espera que se pueda trabajar más rápidamente.

3.1 Convolution of polynomials

3.1.1 Symmetric additive convolution

Definition 1 (Symmetric additive convolution)**.** Let $p(x)$, $q(x)$ be two complex polynomials of x, with degree less or equal to d,

$$p(x) = \sum_{j=0}^{d} x^{d-j}(-1)^j a_j,$$
$$q(x) = \sum_{j=0}^{d} x^{d-j}(-1)^j b_j.$$

The dth symmetric additive convolution of p and q is

$$\begin{aligned} p(x) \boxplus_d q(x) &:= \sum_{k=0}^{d} x^{d-k}(-1)^k \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} a_i b_j, \\ &= \frac{1}{d!} \sum_{k=0}^{d} D^k p(x) D^{d-k} q(0), \\ &= \frac{1}{d!} \sum_{k=0}^{d} D^k q(x) D^{d-k} p(0), \end{aligned}$$

with D denoting the differentiation with respect to x.

3.1.2 Symmetric multiplicative convolution

Definition 2 (Symmetric multiplicative convolutions)**.** Let p and q be as in Definition 1 with degree at most d, the dth symmetric multiplicative convolution of p and q is

$$p(x) \boxtimes_d q(x) := \sum_{i=0}^{d} x^{d-i}(-1)^i \frac{a_i b_i}{\binom{d}{i}}.$$

3.1.3 Linearization of convolutions

Tomarlo de la tesis de Daniel Perales

3.2 Finite free convolutions and random matrices

Theorem 3.1. *Let R be a $d \times d$ matrix and R_{ij} be independent Gaussian random variables with mean 0 and variance 1, then*

$$E\left[\chi_x\left(\frac{R+R^T}{\sqrt{2}}\right)\right] = H_d(x),$$

where $H_d(x) = e^{D^2/2}x^d$ is the dth Hermite polynomial.

Proof. □

Theorem 3.2. *Let R be a $d \times d$ matrix and R_{ij} be independent Gaussian random variables with mean 0 and variance 1, then*

$$E\left[\chi_x\left(RR^\mathsf{T}\right)\right] = L_d(x),$$

where $L_d(x) = \left(1 - \frac{\mathrm{d}}{\mathrm{d}x}\right)^d x^d$ is the dth Laguerre polynomial.

Proof. □

3.2.1 Minor orthogonality

Theorem 3.3 (Cauchy-Binet formula)**.**

Tal vez debería mover este a preliminares

Let m, n, p, k be integers, A an $m \times n$ matrix, and B a $n \times p$ matrix, then

$$[AB]_{S,T} = \sum_{|U| \subset \binom{[n]}{k}} [A]_{S,U}[B]_{U,T},$$

Hay que ver si voy a dejar la misma notación para el determinante o cambiarla.

where $S \in \binom{[m]}{k}$, $T \in \binom{[p]}{k}$.

Definition 3 (Minor orthogonality)**.** Let R be an $m \times n$ random matrix. We say R is minor orthogonal if for every $k, l \in \mathbb{Z}$ such that $k, l \leq \max\{m.n\}$ and all sets S, T, U, V with $|S| = |T| = k$ and $|U| = |V| = l$, it satisfies

$$E_R\left[[R]_{S,T}[R^*]_{U,V}\right] = \frac{\delta_{S,V}\delta_{T,U}}{\binom{\max\{m,n\}}{k}}.$$

Lemma 3.4. *If R is minor orthogonal and Q is a constant matrix such that $QQ^* = I$, then Q is minor orthogonal. If $Q^*Q = I$, then RQ is minor orthogonal.*

Proof. Recall that by the Cauchy-Binet formula, for $|S| = |T| = k$ we have

$$[QR]_{S,T} = \sum_{|W|=k} [Q]_{S,W}[R]_{W,T},$$

so with $|S| = |T| = k$, $|U| = |V| = l$,

$$\begin{aligned}
E_R\left[[QR]_{S,T}[R^*Q^*]_{U,V}\right] &= E_R\left[\sum_{|W|=k}\sum_{|Z|=l}[Q]_{S,W}[R]_{W,T}[R^*]_{U,Z}[Q^*]_{Z,V}\right],\\
&= \sum_{|W|=k}\sum_{|Z|=l}[Q]_{S,W}[Q^*]_{Z,V}E_R\left[[R]_{W,T}[R^*]_{U,Z}\right],\\
&= \sum_{|W|=k}\sum_{|Z|=l}[Q]_{S,W}[Q^*]_{Z,V}E_R\frac{\delta_{W,Z}\delta_{T,U}}{\binom{\max\{m,n\}}{k}},\\
&= \sum_{|W|=k}[Q]_{S,W}[Q^*]_{W,V}\frac{\delta_{T,U}}{\binom{\max\{m,n\}}{k}} = [QQ^*]_{S,V}\frac{\delta_{T,U}}{\binom{\max\{m,n\}}{k}},\\
&= [I]_{S,V}\frac{\delta_{T,U}}{\binom{\max\{m,n\}}{k}},
\end{aligned}$$

Notice that $[I]_{S,V} = 1$ if and only if $S = V$, so we conclude that

$$E_R\left[[QR]_{S,T}[R^*Q^*]_{U,V}\right] = \frac{\delta_{S,V}\delta_{T,U}}{\binom{\max\{m,n\}}{k}}.$$

The case $Q^*Q = I$ is proven in the same way. □

Definition 4 (Signed permutation matrix)**.** A signed permutation matrix is a matrix that can be written EP where E is a diagonal matrix with entries ± 1 and P is a permutation matrix.

Lemma 3.5. *A random matrix sampled uniformly from the set of signed permutation matrices is minor-orthogonal.*

Proof. Let Q be a signed permutation matrix, we can write $Q = EP$, where E is a diagonal random matrix with entries ± 1 taken uniformly and P is a matrix chosen uniformly from the permutation matrices, and both are independent. Then for $|S| = |T| = k$ and $|U| = |U| = l$,

$$\begin{aligned}
E_Q\left[[Q]_{S,T}[Q^*]_{U,V}\right] &= E_{E,P}\left[[EP]_{S,T}[P^*E]_{U,V}\right],\\
&= \sum_{|W|=k}\sum_{|Z|=l}E_{E,P}\left[[E]_{S,W}[P]_{W,T}[P^*]_{U,Z}[E]_{Z,V}\right],
\end{aligned}$$

every $[E]_{S,W}$ is diagonal and the determinant would be zero if $S \neq W$, so

$$= E_{E,P}\left[[E]_{S,S}[P]_{W,T}[P^*]_{U,Z}[E]_{V,V}\right],$$

Let $\{\chi_i\}_{1\leq i\leq n}$ be the diagonal entries of E, then

$$= E_E\left[\prod_{i\in S}\chi_i\prod_{j\in V}\chi_j\right]E_P\left[[P]_{S,T}[P^*]_{U,V}\right].$$

Now we use that the variables chi_i are independent and uniform in $\{-1,1\}$, so that $E[\chi_i] = 0$, but $E[\chi_i^2] = 1$ for all i, and this means

$$E_E\left[\prod_{i\in S}\chi_i\prod_{j\in V}\chi_j\right]=\delta_{S,V}.$$

This last equality leads to

$$\begin{aligned}E_R\left[[QR]_{S,T}[R^*Q^*]_{U,V}\right]&=\delta_{S,V}E_P\left[[P]_{S,T}[P^*]_{U,V}\right],\\&=\delta_{S,V}E_P\left[[P]_{S,T}[P^*]_{U,S}\right],\\&=\delta_{S,V}E_P\left[[P]_{S,T}[P]_{S,U}\right].\end{aligned}$$

The submatrix $P_{S,T}$ can be transformed in a diagonal matrix by a permutation matrix because it has at most a non zero entry for each row and each column. If the diagonal matrix has a zero entry in the diagonal, then the determinant $[P]_{S,T}$ is zero, in other case, it is different that zero. The only case when all of the diagonal entries of the diagonal matrix are not zero is when $T=\pi(S)$ with π the permutation function corresponding to P. This means that in order to have a non-zero determinant we need $T=\pi(S)=U$, and $[P]_{S,U}=\in\{-1,1\}$, so

$$\begin{aligned}E_R\left[[QR]_{S,T}[R^*Q^*]_{U,V}\right]&=\delta_{S,V}\delta_{T,U}E_P\left[[P]_{S,T}[P]_{S,T}\right],\\&=\delta_{S,V}\delta_{T,U}E_P\left[[P]^2_{S,T}\right],\\&=\delta_{S,V}\delta_{T,U}E_P\left[\delta_{T=\pi(S)}\right],\\&=\delta_{S,V}\delta_{T,U}\mathbb{P}\left(T=\pi(S)\right).\end{aligned}$$

We are supposing that we are sampling uniformly from the permutation matrices of size $n\times n$, so the probability that $T=\pi(S)$ when π is a permutation of n elements and $|S|=|T|=k$ is $1/\binom{n}{k}$. So, we can conclude

$$E_R\left[[QR]_{S,T}[R^*Q^*]_{U,V}\right]=\frac{\delta_{S,V}\delta_{T,U}}{\binom{n}{k}}.$$

This is the definition of being minor-orthogonal.

En el paper no lo prueban, pero el resultado se tiene también para una matriz de permutación con signos rectangular. En la siguiente prueba se usa esto, así que falta completar eso. La prueba es igual, simplemente tomando que E es de tamaño $m\times m$ y P de tamaño $m\times n$, todos los resultados se siguen.

□

Corollary 3.6. *An $m\times n$ random matrix sampled from the Haar measure on $\mathbb{C}^m_n$ is minor-orthogonal.*

Proof. Let R be a Haar distributed random $m\times n$ matrix with $m\leq n$ and Q a random permutation matrix. Any random permutation matrix is unitary, so RQ is Haar distributed for fixed Q, and by Lemmas **??** and 3.5 we have that it is also minor-orthogonal. Then, if Q is uniformly sampled from the signed permutation matrices,

$$E_R\left[[R]_{S,T}[R^*]_{U,V}\right]=E_{R,Q}\left[[RQ]_{S,T}[(RQ)^*]_{U,V}\right].$$

Since Q is minor orthogonal, RQ is also minor orthogonal for fixed R and

$$E_R\left[E_Q\left[[RQ]_{S,T}[(RQ)^*]_{U,V}\right]\right] = E_R\left[\frac{\delta_{S,V}\delta_{T=U}}{\binom{n}{k}}\right] = \frac{\delta_{S,V}\delta_{T=U}}{\binom{n}{k}},$$

where $k = |S| = |T|$. □

Theorem 3.7. *Let k, n be integers such that $k \leq n$, A, B two $n \times n$ matrices, and $S, T \in \binom{[n]}{k}$. Then*

$$[A+B]_{S,T} = \sum_{i=0}^{k} \sum_{V \in \binom{[k]}{i}} (-1)^{\|U\|_1+\|V\|_1}[A]_{U(S),V(T)}[B]_{\bar{U}(S),\bar{V}(T)},$$

with $\bar{U} = [k] \setminus U$.

Es un hecho conocido, si da tiempo lo pruebo.

We denote by $\sigma_k(A)$ the coeficient of of $(-1)^k x^{d-k}$ in the characteristic polynomial of a d-dimensional matrix A. We will use the fact that

Tal vez esto debería ir en preliminares

$$\sigma_k(A) = \sum_{|S|=k} [A]_{S,S}.$$

Lemma 3.8. *Let $m \leq n$, B an $n \times n$ random matrix and R an $m \times n$ minor-orthogonal matrix independent from B. For all sets $S, T \subset \binom{[m]}{k}$ we have*

$$E_{B,R}\left[[RBR^*]_{S,T}\right] = E_B\left[\delta_{S,T}\frac{\sigma_k(B)}{\binom{n}{k}}\right].$$

Proof. Using the Cauchy-Binet formula we have

$$\begin{aligned}
E_{B,R}\left[[RBR^*]_{S,T}\right] &= E_B\left[\sum_{X,Y \in \binom{[n]}{k}} E_R\left[[R]_{S,X}[B]_{X,Y}[R^*]_{Y,T}\right]\right], \\
&= E_B\left[\sum_{X,Y \in \binom{[n]}{k}} [B]_{X,Y} E_R\left[[R]_{S,X}[R^*]_{Y,T}\right]\right], \\
&= E_B\left[\sum_{X,Y \in \binom{[n]}{k}} [B]_{X,Y} \frac{\delta_{S,T}\delta_{X,Y}}{\binom{n}{k}}\right], \\
&= E_B\left[\sum_{X \in \binom{[n]}{k}} [B]_{X,X} \frac{\delta_{S,T}}{\binom{n}{k}}\right], \\
&= E_B\left[\delta_{S,T}\frac{\sigma_k(B)}{\binom{n}{k}}\right].
\end{aligned}$$

□

Lemma 3.9. *Let $a > d$, A an $a \times a$ random matrix and Q a random $a \times d$ matrix sampled from the Haar measure on $\mathbb{C}_a^d$, then*

$$E_A \left[E_Q \left[\chi_x \left(QAQ^* \right) \right] \right] = E_A \left[\frac{d!}{a!} \frac{\mathrm{d}^{(a-d)}}{\mathrm{d}x} \chi_x(Q) \right].$$

Proof. Let A be a fixed matrix and Q a Haar unitary matrix on $\mathbb{C}_a^d$, the kth coefficient of the expected characterictic polynomial of QAQ^* is

$$\begin{aligned} E_Q \left[\sigma_k(QAQ^*) \right] &= \sum_{|S|=k} E_Q \left[[QAQ^*]_{S,S} \right], \\ &= \sum_{|S|=k} \frac{\sigma_k(A)}{\binom{a}{k}}, \\ &= \frac{\binom{d}{k} \sigma_k(A)}{\binom{a}{k}}. \end{aligned}$$

Taking expectaton on the last expression we find

$$E_A \left[E_Q \left[\chi_x \left(QAQ^* \right) \right] \right] = E_A \left[\frac{\binom{d}{k} \sigma_k(A)}{\binom{a}{k}} \right] = \frac{\binom{d}{k} \sigma_k(A)}{\binom{a}{k}},$$

which is the kth coefficient of $\frac{d!}{a!} \frac{\mathrm{d}^{(a-d)}}{\mathrm{d}x} E_A \left[\chi_x(A) \right]$. ☐

Theorem 3.10. *Let A, B be $d \times d$ random matrices and R a $d \times d$ minor-orthogonal matrix, such that A, B, R are jointly independent, then we have*

$$E_{A,B,R} \left[\sigma_k(A + RBR^*) \right] = \sum_{i=0}^{k} \frac{\binom{d-i}{k-i}}{\binom{d}{k-i}} E_A \left[\sigma_i(A) \right] E_A \left[\sigma_{k-i}(B) \right].$$

Proof. We use

$$\sigma_k(A) = \sum_{|S|=k} [A]_{S,S},$$

together with Theorem 3.7 and Lemma 3.8 to get

Hay que aclarar las normas de matrices.

$$\begin{aligned} E_{A,B,R} \left[\sigma_k(A + RBR^*) \right] &= \sum_{S \in \binom{[d]}{k}} E_{A,B,R} \left[[A + RBR^*]_{S,S} \right], \\ &= \sum_{S \in \binom{[d]}{k}} \sum_{i=0}^{k} \sum_{U,V \in \binom{[k]}{i}} (-1)^{\|U\|_1 + \|V\|_1} E_A \left[[A]_{U(S),V(S)} \right] E_{B,R} \left[[RBR^*]_{\overline{U}(S),\overline{V}(S)} \right], \\ &= \sum_{S \in \binom{[d]}{k}} \sum_{i=0}^{k} \sum_{U,V \in \binom{[k]}{i}} (-1)^{\|U\|_1 + \|V\|_1} E_A \left[[A]_{U(S),V(S)} \right] \delta_{\overline{U}(S),\overline{V}(S)} \frac{E_B \left[\sigma_{k-i}(B) \right]}{\binom{d}{k-i}}, \end{aligned}$$

using that $U(S) = V(S)$ if and only if $\overline{U}(S) = \overline{V}(S)$,

$$= \sum_{i=0}^{k} \frac{E_B\left[\sigma_{k-i}(B)\right]}{\binom{d}{k-i}} \sum_{S\in\binom{[d]}{k}} \sum_{U,V\in\binom{[k]}{i}} E_A\left[[A]_{U(S),U(S)}\right].$$

To finish the proof we need to find

$$\sum_{S\in\binom{[d]}{k}} \sum_{U\in\binom{[k]}{i}} E_A\left[[A]_{U(S),U(S)}\right]. \tag{3.1}$$

Clearly, we are summing over all of the sets $V \in \binom{[d]}{i}$, but they appear more than once in the sum. To find the number of times every element $V \in \binom{[d]}{i}$ appears in the sum, we can count the total number of terms we are summing in (3.1) and divide by the total number of elements in $\binom{[d]}{i}$. We have that $|\binom{[d]}{i}| = \binom{d}{i}$ and the number of summands is $\binom{d}{k}\binom{k}{i}$, so

$$\frac{\binom{d}{k}\binom{k}{i}}{\binom{d}{i}} = \frac{\frac{d!}{k!(d-k)!}\frac{k!}{i!(k-i)!}}{\frac{d!}{i!(d-i)!}} = \frac{(d-i)!}{(d-k)!(k-i)!} = \binom{d-i}{k-i}.$$

So, we have

$$\sum_{S\in\binom{[d]}{k}} \sum_{U\in\binom{[k]}{i}} E_A\left[[A]_{U(S),U(S)}\right] = \binom{d-i}{k-i} \sum_{V\in\binom{[d]}{i}} E_A\left[[A]_{V,V}\right] = \binom{d-i}{k-i} E_A\left[\sigma_i(A)\right].$$

Thus we can conclude

$$E_{A,B,R}\left[\sigma_k(A+RBR^*)\right] = \sum_{i=0}^{k} \frac{\binom{d-i}{k-i}}{\binom{d}{k-i}} E_A\left[\sigma_i(A)\right] E_A\left[\sigma_{k-i}(B)\right].$$

□

Hay que aclarar qué signofica $U(S)$.

Theorem 3.11. *If $p(x)$ is the characteristic polynomial of A and $q(x)$ is the characteristic polynomial of B, where A and B are $d \times d$ normal matrices with complex entries, then*

$$p(x) \boxplus_d q(x) = E_Q\left[\chi_x(A+QBQ^*)\right],$$

where $\chi_x(\cdot)$ denotes the characteristic polynomial of $\cdot$ with x as a variable and E_Q denotes taking expectation over Q where Q is sampled from the Haar measure on the unitary complex $d \times d$ matrices.

Proof. It follows directly from Theorem 3.10 and definition of the symmetric additive convolution. □

Theorem 3.12. *Let A and B be $d \times d$ random matrices and R a minor-orthogonal $d \times d$ matrix, such that A, B, R are jointly independent, then*

$$E_{A,B,R}\left[\sigma_k(ARBR^*)\right] = \frac{E_A[\sigma_k(A)]E_B[\sigma_k(B)]}{\binom{d}{k}}.$$

Proof.

$$E_{A,B,R}\left[\sigma_k(ARBR^*)\right] = \sum_{S\in\binom{[d]}{k}} E_{A,B,R}\left[[ARBR^*]_{S,S}\right],$$

By the Cauchy-Binet formula and independence

$$= \sum_{S,T\in\binom{[d]}{k}} E_A\left[[A]_{S,T}\right] E_{B,R}\left[[RBR^*]_{T,S}\right],$$

By Lemma 3.8

$$\begin{aligned} &= \sum_{S,T\in\binom{[d]}{k}} E_A\left[[A]_{S,T}\right] \delta_{T,S} \frac{E_B\left[\sigma_k(B)\right]}{\binom{d}{k}}, \\ &= \frac{E_A[\sigma_k(A)]E_B[\sigma_k(B)]}{\binom{d}{k}}. \end{aligned}$$

□

Theorem 3.13. *Let $p(x)$ be the characteristic polynomial of A and $q(x)$ be the characteristic polynomial of B where A and B are $d\times d$ normal matrices with complex entries, then*

$$p(x) \boxtimes_d q(x) = E_Q\left[\chi_x(AQBQ^*)\right],$$

with χ_x and E_Q as in Theorem 3.11.

Proof. It follows directly from Theorem 3.12 and definition of the symmetric multiplicative convolution. □

4 Deterministic eigenvalue processes for matrix-valued processes

4.1 Model construction

Se ha estado trabajando en la construcción del modelo que permita obtener el proceso de Dyson determinista a partir de la ecuación de Huang y coautores.
En las publicaciones con que contamos no se hace una construcción explícita del modelo, por lo que el trabajo en esta parte es más lento.

In this chapter we study a process M in the space of orthogonal $p\times p$ matrices which satisfies the following stochastic differential equation

$$\mathrm{d}M = [\mathrm{d}K, M] := (\mathrm{d}K)M - M\mathrm{d}K,$$

with K a Brownian matrix in the space of anti-symmetric $p\times p$ matrices.

Aquí empieza la parte de las cuentas que he hecho. En estas primeras, no se usa el supuesto de independencia de las entradas de K.

We proceed in a similar way to the proof of Theorem 2.2. Let $M = H\Lambda H^T$ and define $\mathrm{d}A := H^\intercal \partial H$, by the same argument shown previously, A is an anti-symmetric matrix. Applying the Itô formula to $\mathrm{d}\Lambda = \mathrm{d}(H^\intercal M H)$ and defining $\mathrm{d}N = H^\intercal \partial M H$, we get

$$\mathrm{d}\Lambda = \mathrm{d}N + \Lambda\partial A - (\partial A)\Lambda.$$

The diagonal of $\Lambda\partial A - (\partial A)\Lambda$ is zero, so $\mathrm{d}\lambda_i = \mathrm{d}N_{ii}$. If $i \neq j$, then

$$\begin{aligned} 0 &= \mathrm{d}N_{ij} + \lambda_i \mathrm{d}A_{ij} - \mathrm{d}A_{ij}\lambda_j, \\ \Rightarrow \mathrm{d}A_{ij} &= \frac{\mathrm{d}N_{ij}}{\lambda_j - \lambda_i}. \end{aligned}$$

Estas son las cuentas que aparecen de considerar que K es una matriz anti-hermitiana cuyas entradas son brownianos independientes, excepto por las simetrías.

Let us find the covariation $\mathrm{d}M_{ij}\mathrm{d}M_{kl}$,

$$\begin{aligned} \mathrm{d}M_{ij}\mathrm{d}M_{kl} =& \mathrm{d}\big\langle((\mathrm{d}K)M - M\mathrm{d}K)_{ij}, ((\mathrm{d}K)M - M\mathrm{d}K)_{kl}\big\rangle, \\ =& \mathrm{d}\big\langle((\mathrm{d}K)M)_{ij}, ((\mathrm{d}K)M)_{kl}\big\rangle - \mathrm{d}\big\langle((\mathrm{d}K)M)_{ij}, (M\mathrm{d}K)_{kl}\big\rangle \\ &+ \mathrm{d}\big\langle(M\mathrm{d}K)_{ij}, (M\mathrm{d}K)_{kl}\big\rangle - \mathrm{d}\big\langle(M\mathrm{d}K)_{ij}, ((\mathrm{d}K)M)_{kl}\big\rangle. \end{aligned}$$

For each of the summands we have

$$\begin{aligned} \mathrm{d}\big\langle((\mathrm{d}K)M)_{ij}, ((\mathrm{d}K)M)_{kl}\big\rangle &= \sum_{pq} \mathrm{d}\big\langle \mathrm{d}K_{ip}M_{pj}, \mathrm{d}K_{kq}M_{ql}\big\rangle, \\ &= \sum_{pq} M_{pj}M_{ql}\mathrm{d}K_{ip}\mathrm{d}K_{kq}, \end{aligned}$$

The entries in K are independent, except for $\mathrm{d}K_{ij} = -\mathrm{d}K_{ji}$, so

$$\begin{aligned} &= \delta_{ik}\delta_{pq}\sum_p M_{pj}M_{pl}\mathrm{d}t - \delta_{iq}\delta_{pk}M_{kj}M_{il}\mathrm{d}t, \\ &= \big(\delta_{ik}(MM)_{jl} - M_{kj}M_{il}\big)\mathrm{d}t. \end{aligned}$$

$$\begin{aligned} \mathrm{d}\big\langle(M\mathrm{d}K)_{ij}, (M\mathrm{d}K)_{kl}\big\rangle &= \sum_{pq} \mathrm{d}\big\langle M_{ip}\mathrm{d}K_{pj}, M_{kq}\mathrm{d}K_{ql}\big\rangle, \\ &= \sum_{pq} M_{ip}M_{kq}\mathrm{d}K_{pj}\mathrm{d}K_{ql}, \\ &= \delta_{pq}\delta_{jl}\sum_p M_{ip}M_{kp}\mathrm{d}t - \delta_{pl}\delta_{jq}M_{il}M_{kj}\mathrm{d}t, \\ &= \big(\delta_{jl}(MM)_{ik} - M_{il}M_{kp}\big)\mathrm{d}t. \end{aligned}$$

$$
\begin{aligned}
\mathrm{d}\big\langle (M\mathrm{d}K)_{ij}, (\mathrm{d}KM)_{kl}\big\rangle &= \sum_{pq} \mathrm{d}\big\langle M_{ip}\mathrm{d}K_{pj}, \mathrm{d}K_{kq}M_{ql}\big\rangle, \\
&= \sum_{pq} M_{ip}M_{ql}\mathrm{d}K_{pj}\mathrm{d}K_{kq}, \\
&= \delta_{pk}\delta_{jq}M_{ki}M_{jl}\mathrm{d}t - \delta_{jk}\delta_{pq}\sum_{p} M_{ip}M_{pl}\mathrm{d}t, \\
&= \big(M_{ik}M_{jl}\mathrm{d}t - \delta_{jk}(MM)_{il}\big)\mathrm{d}t.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}\big\langle (\mathrm{d}KM)_{ij}, (M\mathrm{d}K)_{kl}\big\rangle &= \sum_{pq} \mathrm{d}\big\langle \mathrm{d}K_{ip}M_{pj}, M_{kq}\mathrm{d}K_{ql}\big\rangle, \\
&= \sum_{pq} M_{pj}M_{kq}\mathrm{d}K_{ip}\mathrm{d}K_{ql}, \\
&= \delta_{iq}\delta_{pl}M_{lj}M_{ki}\mathrm{d}t - \delta_{pq}\delta_{il}\sum_{p} M_{pj}M_{kp}\mathrm{d}t, \\
&= \big(M_{lj}M_{ki} - \delta_{il}(MM)_{kj}\big)\mathrm{d}t.
\end{aligned}
$$

With these expressions we find

$$
\mathrm{d}M_{ij}\mathrm{d}M_{kl} = \Big(\delta_{ik}(MM)_{jl} + \delta_{jl}(MM)_{ik} + \delta_{jk}(MM)_{il} + \delta_{il}(MM)_{kj} - \big(2M_{il}M_{kj} + M_{ik}M_{jl} + M_{lj}M_{ki}\big)\Big)\mathrm{d}t. \tag{4.1}
$$

Now we can use this to find $\mathrm{d}N_{ij}\mathrm{d}N_{kl}$

$$
\begin{aligned}
\mathrm{d}N_{ij}\mathrm{d}N_{kl} = \mathrm{d}\big\langle (H^{\intercal}\mathrm{d}MH)_{ij}, (H^{\intercal}\mathrm{d}MH)_{kl}\big\rangle &= \sum_{pqrs} \mathrm{d}\big\langle H^{\intercal}_{ip}\mathrm{d}M_{pq}H_{qj}, H^{\intercal}_{kr}\mathrm{d}M_{rs}H_{sl}\big\rangle, \\
&= \sum_{pqrs} H^{\intercal}_{ip}H_{qj}H^{\intercal}_{kr}H_{sl}\mathrm{d}M_{rs}\mathrm{d}M_{pq}.
\end{aligned}
$$

By substituting each of the terms in (4.1) we have

$$
\sum_{pqrs} H^{\intercal}_{ip}H_{qj}H^{\intercal}_{kr}H_{sl}\delta_{pr}(MM)_{qs}\mathrm{d}t = \Big(\sum_{p} H^{\intercal}_{ip}H_{pk}\Big)\Big(\sum_{qs} H^{\intercal}_{jq}(MM)_{qs}H_{sl}\Big)\mathrm{d}t,
$$

using that $H^{\intercal}H = I$,

$$
= \delta_{ik}\big(H^{\intercal}(H\Lambda H^{\intercal}H\Lambda H^{\intercal})H\big)_{jl}\mathrm{d}t = \delta_{ik}\Lambda^2_{jl}\mathrm{d}t.
$$

$$
\begin{aligned}
\sum_{pqrs} H^{\intercal}_{ip}H_{qj}H^{\intercal}_{kr}H_{sl}\delta_{sq}(MM)_{pr}\mathrm{d}t &= \Big(\sum_{q} H^{\intercal}_{jq}H_{ql}\Big)\Big(\sum_{pr} H^{\intercal}_{ip}(MM)_{pr}H_{rk}\Big)\mathrm{d}t, \\
&= \delta_{jl}\big(H^{\intercal}(H\Lambda H^{\intercal}H\Lambda H^{\intercal})H\big)_{ik}\mathrm{d}t = \delta_{jl}\Lambda^2_{ik}\mathrm{d}t.
\end{aligned}
$$

$$\begin{aligned}\sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sl}\delta_{ps}(MM)_{qr}\mathrm{d}t &= \Big(\sum_p H^\intercal_{ip}H_{pl}\Big)\Big(\sum_{qr} H^\intercal_{jq}(MM)_{qr}H_{rk}\Big)\mathrm{d}t,\\ &= \delta_{il}\big(H^\intercal(H\Lambda H^\intercal H\Lambda H^\intercal)H\big)_{jk}\mathrm{d}t = \delta_{il}\Lambda^2_{jk}\mathrm{d}t.\end{aligned}$$

$$\begin{aligned}\sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sl}\delta_{rq}(MM)_{sp}\mathrm{d}t &= \Big(\sum_q H^\intercal_{kq}H_{qj}\Big)\Big(\sum_{sp} H^\intercal_{ls}(MM)_{sp}H_{pi}\Big)\mathrm{d}t,\\ &= \delta_{kj}\big(H^\intercal(H\Lambda H^\intercal H\Lambda H^\intercal)H\big)_{li}\mathrm{d}t = \delta_{kj}\Lambda^2_{li}\mathrm{d}t.\end{aligned}$$

$$\begin{aligned}-2\sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sl}M_{ps}M_{rq}\mathrm{d}t &= -2\Big(\sum_{ps} H^\intercal_{ip}M_{ps}H_{sl}\Big)\Big(\sum_{rq} H^\intercal_{kr}M_{rq}H_{qj}\Big)\mathrm{d}t,\\ &= -2\big(H^\intercal MH\big)_{il}\big(H^\intercal MH\big)_{kj}\mathrm{d}t = -2\Lambda_{il}\Lambda_{kj}\mathrm{d}t.\end{aligned}$$

$$\begin{aligned}-\sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sl}M_{pr}M_{qs}\mathrm{d}t &= -\Big(\sum_{pr} H^\intercal_{ip}M_{pr}H_{rk}\Big)\Big(\sum_{qs} H^\intercal_{ls}M_{sq}H_{qj}\Big)\mathrm{d}t,\\ &= -\big(H^\intercal MH\big)_{ik}\big(H^\intercal MH\big)_{lj}\mathrm{d}t = -\Lambda_{ik}\Lambda_{lj}\mathrm{d}t.\end{aligned}$$

$$\begin{aligned}-\sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sl}M_{sq}M_{rp}\mathrm{d}t &= -\Big(\sum_{pr} H^\intercal_{ip}M_{pr}H_{rk}\Big)\Big(\sum_{qs} H^\intercal_{ls}M_{sq}H_{qj}\Big)\mathrm{d}t,\\ &= -\big(H^\intercal MH\big)_{ik}\big(H^\intercal MH\big)_{lj}\mathrm{d}t = -\Lambda_{ik}\Lambda_{lj}\mathrm{d}t.\end{aligned}$$

Once we know the seven terms, we can write $\mathrm{d}N_{ij}\mathrm{d}N_{kl}$ as

$$\mathrm{d}N_{ij}\mathrm{d}N_{kl} = \Big(\delta_{ik}\Lambda^2_{jl} + \delta_{jl}\Lambda^2_{ik} + \delta_{il}\Lambda^2_{jk} + \delta_{kj}\Lambda^2_{li} - 2\Lambda_{il}\Lambda_{kj} - 2\Lambda_{ik}\Lambda_{lj}\Big)\mathrm{d}t.$$

With this, we find $\mathrm{d}N_{ii}\mathrm{d}N_{jj}$

$$\begin{aligned}\mathrm{d}N_{ii}\mathrm{d}N_{jj} &= \Big(\delta_{ij}\Lambda^2_{ij} + \delta_{ij}\Lambda^2_{ij} + \delta_{ij}\Lambda^2_{ij} + \delta_{ij}\Lambda^2_{ij} - 2\Lambda_{ij}\Lambda_{ij} - 2\Lambda_{ij}\Lambda_{ij}\Big)\mathrm{d}t,\\ &= \big(4\delta_{ij}\Lambda^2_{ii} - 4\Lambda^2_{ij}\big)\mathrm{d}t\end{aligned}$$

Evidently, if $j \neq i$, all of the terms are zero. If $j = l$

$$= \big(4\lambda_i^2 - 4\lambda_i^2\big)\mathrm{d}t = 0.$$

So $\mathrm{d}N_{ii}\mathrm{d}N_{jj} = 0$ for every choice of i and j, which means the diagonal of $\mathrm{d}N$ has no martingale part.

Finally, let us compute $\mathrm{d}F$, the finite variation part of $\mathrm{d}N$. We use that $\mathrm{d}N = H^\intercal(\partial M)H = H^\intercal\mathrm{d}MH + \frac{1}{2}\left((\mathrm{d}H)^\intercal\mathrm{d}MH + H^\intercal\mathrm{d}M\mathrm{d}H\right)$, so

$$
\begin{aligned}
\mathrm{d}F &= \frac{1}{2}\left((\mathrm{d}H)^{\mathsf{T}}\mathrm{d}MH + H^{\mathsf{T}}\mathrm{d}M\mathrm{d}H\right) = \frac{1}{2}\left((\mathrm{d}H)^{\mathsf{T}}HH^{\mathsf{T}}\mathrm{d}MH + H^{\mathsf{T}}\mathrm{d}MHH^{\mathsf{T}}\mathrm{d}H\right),\\
&= \frac{1}{2}\left[((\mathrm{d}H)^{\mathsf{T}}H)(H^{\mathsf{T}}\mathrm{d}MH) + (H^{\mathsf{T}}\mathrm{d}MH)(H^{\mathsf{T}}\mathrm{d}H)\right] = \frac{1}{2}\left[\mathrm{d}A^{\mathsf{T}}\mathrm{d}N + \mathrm{d}N\mathrm{d}A\right].
\end{aligned}
$$

Now we use $\mathrm{d}N_{ij}\mathrm{d}N_{kl}$ to find $(\mathrm{d}N\mathrm{d}A)_{ij}$,

$$
\begin{aligned}
(\mathrm{d}N\mathrm{d}A)_{ij} &= \sum_{k\neq j}\mathrm{d}N_{ik}\mathrm{d}A_{kj} = \sum_{k\neq j}\frac{\mathrm{d}N_{ik}\mathrm{d}N_{kj}}{\lambda_j-\lambda_k} = \sum_{k\neq j}\frac{\delta_{ik}\Lambda_{kj}^2+\delta_{kj}\Lambda_{ik}^2+\delta_{ij}\Lambda_{kk}^2+\delta_{kk}\Lambda_{ij}^2-2\Lambda_{ij}\Lambda_{kk}-2\Lambda_{ik}\Lambda_{jk}}{\lambda_j-\lambda_k}\mathrm{d}t,\\
&= \sum_{k\neq j}\frac{\delta_{ij}\lambda_k^2+\lambda_{ij}^2-2\delta_{ij}\lambda_k}{\lambda_j-\lambda_k}\mathrm{d}t = \delta_{ij}\sum_{k\neq j}\frac{(\lambda_j-\lambda_k)^2}{\lambda_j-\lambda_k} = \delta_{ij}\sum_{k\neq j}\lambda_j-\lambda_k.
\end{aligned}
$$

So in this case the eigenvalues satisfy the equation:

$$
\mathrm{d}\lambda_i = \sum_{k\neq i}\lambda_i-\lambda_k.
$$

Entonces, aunque este proceso si tiene espectro determinista, no satisface la ecuación que queremos.
En la siguiente parte voy a intentar atacarlo de manera más general, usando únicamente que K es anti-hermitiana, y a partir de eso deducir cómo deberían ser las covariaciones para cumplir lo que buscamos.

We know that $\mathrm{d}\lambda_i = \mathrm{d}N_{ii}$. In order to satisfy the desired equation, we need that

$$
\mathrm{d}F_{ii} = \sum_{k\neq i}\frac{\mathrm{d}t}{\lambda_i-\lambda_k}. \tag{4.2}
$$

Due to the previous steps, which are unrelated with the covariation of K, we have that $\mathrm{d}F = \frac{1}{2}\left[\mathrm{d}A^{\mathsf{T}}\mathrm{d}N + \mathrm{d}N\mathrm{d}A\right]$, so the idea is to find a general form for $(\mathrm{d}A\mathrm{d}N)_{ii}$ and set it equal to (4.2).

Let us only assume that $\mathrm{d}K_{ij}\mathrm{d}K_{ij} = c\mathrm{d}t$ and $\mathrm{d}K_{ij}\mathrm{d}K_{ji} = -c\mathrm{d}t$. Recall that $\mathrm{d}N = H^{\mathsf{T}}\mathrm{d}MH$ and $\mathrm{d}A_{ij} = \mathbb{1}_{i\neq j}\mathrm{d}N_{ij}/(\lambda_j-\lambda_i)$.

La condición sobre $\mathrm{d}L_{ij}\mathrm{d}L_{jk}$ que da Menon es equivalente a pedir que $(\mathrm{d}N\mathrm{d}A)_{ii}$ tenga esta forma.

$$
(\mathrm{d}N\mathrm{d}A)_{ii} = \sum_{k\neq i}\frac{\mathrm{d}N_{ik}\mathrm{d}N_{ki}}{\lambda_i-\lambda_k} \quad \Rightarrow \text{(We need that)} \quad \mathrm{d}N_{ik}\mathrm{d}N_{ki} = \mathrm{d}t.
$$

$$
\begin{aligned}
\mathrm{d}t =& \mathrm{d}N_{ik}\mathrm{d}N_{ki} = (H^{\mathsf{T}}\mathrm{d}MH)_{ik}(H^{\mathsf{T}}\mathrm{d}MH)_{ki} = \sum_{pqrs}H^{\mathsf{T}}_{ip}\mathrm{d}M_{pq}H_{qk}H^{\mathsf{T}}_{kr}\mathrm{d}M_{rs}H_{si},\\
=& \sum_{pqrs}H^{\mathsf{T}}_{ip}(\mathrm{d}KM-M\mathrm{d}K)_{pq}H_{qk}H^{\mathsf{T}}_{kr}(\mathrm{d}KM-M\mathrm{d}K)_{rs}H_{si},\\
=& \sum_{nmpqrs}H^{\mathsf{T}}_{ip}\mathrm{d}K_{pm}M_{mq}H_{qk}H^{\mathsf{T}}_{kr}\mathrm{d}K_{rn}M_{ns}H_{si} - \sum_{nmpqrs}H^{\mathsf{T}}_{ip}\mathrm{d}K_{pm}M_{mq}H_{qk}H^{\mathsf{T}}_{kr}M_{rn}\mathrm{d}K_{ns}H_{si}\\
&- \sum_{nmpqrs}H^{\mathsf{T}}_{ip}M_{pm}\mathrm{d}K_{mq}H_{qk}H^{\mathsf{T}}_{kr}\mathrm{d}K_{rn}M_{ns}H_{si} + \sum_{nmpqrs}H^{\mathsf{T}}_{ip}M_{pm}\mathrm{d}K_{mq}H_{qk}H^{\mathsf{T}}_{kr}M_{rn}\mathrm{d}K_{ns}H_{si}.
\end{aligned}
$$

For each of the summands, we will find the cases we already know and a remainder, so that the sum of the four terms equals $\mathrm{d}t$. Let us call the summands ①, ②, ③ and ④.

$$
\begin{aligned}
① &= \delta_{pr}\delta_{mn}c\sum_{mpqs} H^\intercal_{ip}M_{mq}H_{qk}H^\intercal_{kp}M_{ms}H_{si}\mathrm{d}t - \delta_{mr}\delta_{pn}c\sum_{mpqs} H^\intercal_{ip}M_{mq}H_{qk}H^\intercal_{km}M_{ps}H_{si}\mathrm{d}t + R_1\mathrm{d}t,\\
&= c(H^\intercal H)_{ik}(H^\intercal M^2 H)_{ki}\mathrm{d}t - c(H^\intercal MH)_{ii}(H^\intercal MH)_{kk}\mathrm{d}t + R_1\mathrm{d}t,\\
&= c\big(\delta_{ik}\Lambda^2_{ii} - \Lambda_{ii}\Lambda_{kk}\big)\mathrm{d}t + R_1\mathrm{d}t.\\
② &= \delta_{pn}\delta_{ms}c\sum_{mpqr} H^\intercal_{ip}M_{mq}H_{qk}H^\intercal_{kr}M_{rp}H_{mi}\mathrm{d}t - \delta_{ps}\delta_{mn}c\sum_{mpqr} H^\intercal_{ip}M_{mq}H_{qk}H^\intercal_{kr}M_{rm}H_{pi}\mathrm{d}t + R_2\mathrm{d}t,\\
&= c(H^\intercal H)_{ik}(H^\intercal M^2 H)_{ki}\mathrm{d}t - c(H^\intercal H)_{ii}(H^\intercal M^2 H)_{kk}\mathrm{d}t + R_2\mathrm{d}t,\\
&= c\big(\delta_{ik}\Lambda^2_{ii} - \Lambda^2_{kk}\big)\mathrm{d}t + R_2\mathrm{d}t.\\
③ &= \delta_{mr}\delta_{qn}c\sum_{npqs} H^\intercal_{ip}M_{pr}H_{nk}H^\intercal_{kr}M_{ns}H_{si}\mathrm{d}t - \delta_{mn}\delta_{qr}c\sum_{mpqs} H^\intercal_{ip}M_{pm}H_{qk}H^\intercal_{kq}M_{ms}H_{si}\mathrm{d}t + R_3\mathrm{d}t,\\
&= c(H^\intercal MH)_{ik}(H^\intercal MH)_{ki}\mathrm{d}t - c(H^\intercal M^2 H)_{ii}(H^\intercal H)_{kk}\mathrm{d}t + R_3\mathrm{d}t,\\
&= c\big(\Lambda^2_{ik} - \Lambda^2_{ii}\big)\mathrm{d}t + R_3\mathrm{d}t.\\
④ &= \delta_{mn}\delta_{qs}c\sum_{npqr} H^\intercal_{ip}M_{pn}H_{qk}H^\intercal_{kr}M_{rn}H_{qi}\mathrm{d}t - \delta_{ms}\delta_{qn}c\sum_{mpqr} H^\intercal_{ip}M_{pm}H_{qk}H^\intercal_{kr}M_{rq}H_{mi}\mathrm{d}t + R_4\mathrm{d}t,\\
&= c(H^\intercal M^2 H)_{ik}(H^\intercal H)_{ki}\mathrm{d}t - c(H^\intercal MH)_{ii}(H^\intercal MH)_{kk}\mathrm{d}t + R_4\mathrm{d}t,\\
&= c\big(\delta_{ik}\Lambda^2_{ik} - \Lambda_{ii}\Lambda_{kk}\big)\mathrm{d}t + R_4\mathrm{d}t.
\end{aligned}
$$

Equating what we know to what we have, we find that

$$
\begin{aligned}
\mathrm{d}t &= ① - ② - ③ + ④,\\
&= \Big[c\big(\delta_{ik}\Lambda^2_{ii} - \Lambda_{ii}\Lambda_{kk} - \delta_{ik}\Lambda^2_{ii} + \Lambda^2_{kk} - \Lambda^2_{ik} + \Lambda^2_{ii} + \delta_{ik}\Lambda^2_{ik} - \Lambda_{ii}\Lambda_{kk}\big) + R_1 - R_2 - R_3 + R_4\Big]\mathrm{d}t.
\end{aligned}
$$

We conclude that, in order to satisfy the desired equation,

$$
\begin{aligned}
1 &= c\big(\lambda_k^2 + \lambda_i^2 - 2\lambda_i\lambda_k\big) + R_1 - R_2 - R_3 + R_4,\\
\Rightarrow (\lambda_i - \lambda_k)^2 &= \frac{1 + R_2 + R_3 - R_1 - R_4}{c}.
\end{aligned}
$$

The form of the elements R_i is

$$
\begin{aligned}
R_1 &= \sum_{\substack{mnpqrs \\ p\neq r \text{ or } m\neq n \\ m\neq r \text{ or } p\neq n}} H^{\intercal}_{ip}\mathrm{d}K_{pm}M_{mq}H_{qk}H^{\intercal}_{kr}\mathrm{d}K_{rn}M_{ns}H_{si},\\
R_2 &= \sum_{\substack{mnpqrs \\ p\neq n \text{ or } m\neq s \\ p\neq s \text{ or } m\neq n}} H^{\intercal}_{ip}\mathrm{d}K_{pm}M_{mq}H_{qk}H^{\intercal}_{kr}M_{rn}\mathrm{d}K_{ns}H_{si},\\
R_3 &= \sum_{\substack{mnpqrs \\ m\neq r \text{ or } q\neq n \\ m\neq n \text{ or } q\neq r}} H^{\intercal}_{ip}M_{pm}\mathrm{d}K_{mq}H_{qk}H^{\intercal}_{kr}\mathrm{d}K_{rn}M_{ns}H_{si},\\
R_4 &= \sum_{\substack{mnpqrs \\ m\neq n \text{ or } q\neq s \\ m\neq s \text{ or } q\neq n}} H^{\intercal}_{ip}M_{pm}\mathrm{d}K_{mq}H_{qk}H^{\intercal}_{kr}M_{rn}\mathrm{d}K_{ns}H_{si}
\end{aligned}
$$

El problema que encuentro es que estos términos son muy grandes y no veo cómo manejarlos para poder encontrar condiciones sobre K.
Las otras ideas que tengo son 1) suponer que $K = GXH - HX^{\intercal}H$ para G, H simétricas y X una matriz que todas sus entradas sean brownianos independientes. Así tal vez se puede encontrar la covariación en términos de G y H para cualquier matriz browniana antisimétrica, y 2) intentar con matrices pequeñas y ver cómo deben ser sus entradas para que satisfagan lo que queremos (en ese caso los términos R_i son más manejables).

The easiest case is when M and K are 2×2 matrices. In this case, the only matrix that satisfies the hypotheses on K is

$$
\mathrm{d}K = \begin{bmatrix} 0 & \mathrm{d}B(t), \\ -\mathrm{d}B(t) & 0 \end{bmatrix},
$$

with $B(t)$ a Brownian motion with variance c which means ($\mathbb{E}\,(B(t)B(t)) = ct$). Then $R_1 = R_2 = R_3 = R_4 = 0$, and we have that, in order to satisfy the equation, we need that $(\lambda_1-\lambda_2)^2 = \frac{1}{c}$. Without loss of generality, we can suppose $c = 1$ and thus we have $\lambda_1 - \lambda_2 \in \{-1, 1\}$.

Now, the equations found for λ_1 and λ_2 are

$$
\begin{aligned}
\mathrm{d}\lambda_1 &= (\lambda_1 - \lambda_2)\mathrm{d}t,\\
\mathrm{d}\lambda_2 &= (\lambda_2 - \lambda_1)\mathrm{d}t.
\end{aligned}
$$

Taking $\lambda_1 > \lambda_2$ leads to

$$
\begin{aligned}
\mathrm{d}\lambda_1 &= \mathrm{d}t,\\
\mathrm{d}\lambda_2 &= -\mathrm{d}t.
\end{aligned}
$$

But this contradicts that $\lambda_1 - \lambda_2 = 1$.

$$
\lambda_1(0) + \lambda_2(0) = \lambda_1(t) + 1 - \lambda_1(t) = 1.
$$

Hadamard variation formulae

Aquí intenté usar las fórmulas de variación de Hadamard viendo los eigenvalores como función de las entadas para deducir la forma del proceso, pero la verdad es que las partes que dependían de la matriz de eigenvectores se me complicaron (en el Dyson funciona bien por invarianza bajo transformaciones unitarias del GUE). Entonces dejé de hacer esto e intenté volver a usar las técnicas de Graczyk (siguiente subsección).

$$
\begin{aligned}
\frac{\mathrm{d}\lambda_i}{\mathrm{d}M_{jk}} &= \left(U^\intercal \frac{\mathrm{d}M}{\mathrm{d}M_{jk}} U\right)_{ii}, \\
\frac{\mathrm{d}^2\lambda_i}{\mathrm{d}M_{jk}^2} &= 2\sum_{m\neq i} \frac{|\left(U^\intercal \frac{\mathrm{d}M}{\mathrm{d}M_{jk}} U\right)_{im}|^2}{\lambda_i - \lambda_m}, \\
\mathrm{d}\lambda_i &= \sum_{jk} \frac{\mathrm{d}\lambda_i}{\mathrm{d}M_{jk}} \mathrm{d}M_{jk}(t) + \frac{1}{2}\sum_{jk}\sum_{ab} \frac{\mathrm{d}^2\lambda_i}{\mathrm{d}M_{ij}\mathrm{d}M_{ab}} \mathrm{d}\langle M_{jk}, M_{ab}\rangle(t), \\
&= \sum_{jk} \left(U^\intercal \frac{\mathrm{d}M}{\mathrm{d}M_{jk}} U\right)_{ii} \mathrm{d}M_{jk}(t) + \sum_{jk}\sum_{ab}\sum_{m\neq i} \frac{|\left(U^\intercal \frac{\mathrm{d}M}{\mathrm{d}M_{jk}} U\right)_{im}|^2}{\lambda_i - \lambda_m} \mathrm{d}\langle M_{jk}, M_{ab}\rangle(t).
\end{aligned}
$$

In the case of a 2×2 matrix we have for λ_1,

$$
\begin{aligned}
\mathrm{d}\lambda_1 = &\left(U^\intercal \frac{\mathrm{d}M}{\mathrm{d}M_{11}} U\right)_{11} \mathrm{d}M_{11} + 2\left(U^\intercal \frac{\mathrm{d}M}{\mathrm{d}M_{12}} U\right)_{11} \mathrm{d}M_{12} + \left(U^\intercal \frac{\mathrm{d}M}{\mathrm{d}M_{22}} U\right)_{11} \mathrm{d}M_{22} \\
&+ \sum_{m\neq 1} \frac{|\left(U^\intercal \frac{\mathrm{d}M}{\mathrm{d}M_{11}} U\right)_{1m}|^2}{\lambda_1 - \lambda_m} \mathrm{d}\langle M_{11}, M_{11}\rangle(t) + \sum_{m\neq 1} \frac{|\left(U^\intercal \frac{\mathrm{d}M}{\mathrm{d}M_{22}} U\right)_{1m}|^2}{\lambda_1 - \lambda_m} \mathrm{d}\langle M_{22}, M_{22}\rangle(t) \\
&+ 4\sum_{m\neq 1} \frac{|\left(U^\intercal \frac{\mathrm{d}M}{\mathrm{d}M_{12}} U\right)_{1m}|^2}{\lambda_1 - \lambda_m} \mathrm{d}\langle M_{12}, M_{12}\rangle(t), \\
= &(U_{11})^2 \mathrm{d}M_{11} + 4(U_{11}U_{21})\mathrm{d}M_{12} + (U_{21})^2\mathrm{d}M_{22} + \frac{(U_{11}U_{12})^2}{\lambda_1 - \lambda_2} \mathrm{d}\langle M_{11}, M_{11}\rangle(t) \\
&+ \frac{(U_{21}U_{22})^2}{\lambda_1 - \lambda_2} \mathrm{d}\langle M_{22}, M_{22}\rangle(t) + 4\frac{(U_{12}^2 + U_{11}U_{22})^2}{\lambda_1 - \lambda_2} \mathrm{d}\langle M_{12}, M_{12}\rangle(t)
\end{aligned}
$$

4.1.1 Symmetric matrix-valued Brownian motion with zeros on the diagonal

Esto es lo último que intenté y creo que con esto ya se tiene una construcción de un proceso como el que queremos. Es una matriz simétrica con todas sus entradas brownianas independientes (salvo simetría), excepto que tiene ceros en la diagonal, esto hace que la parte de martingala sea cero, pero se mantiene la ecuación del Dyson en la parte de variación finita, si no hay algún error en mis cuentas.

Let X be a process with covariation $\mathrm{d}X_{ij}\mathrm{d}X_{kl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}\delta_{ik})\mathrm{d}t$, which means it is a symmetric matrix with independent brownian entries, except for the diagonal. We assume $X_{ii} = 0$. Write $X = H^\intercal \Lambda H$ and define $\mathrm{d}A = H^\intercal \partial H$ and $\mathrm{d}N = H^\intercal \partial X H$. The same procedure as in former steps leads to

$$\mathrm{d}\Lambda = \mathrm{d}N + \Lambda \mathrm{d}A - \mathrm{d}A\Lambda.$$

We conclude that $\mathrm{d}\lambda_i = \mathrm{d}N_{ii}$ and for $i \neq j$,

$$\begin{aligned} 0 &= \mathrm{d}N_{ij} + \lambda_i \mathrm{d}A_{ij} - \lambda_j \mathrm{d}A_{ij}, \\ \Rightarrow \mathrm{d}A_{ij} &= \frac{\mathrm{d}N_{ij}}{\lambda_j - \lambda_i}. \end{aligned}$$

The quadratic covariation of N is

$$\begin{aligned} \mathrm{d}N_{ij}\mathrm{d}N_{kl} &= \mathrm{d}\big\langle (H^\intercal \mathrm{d}XH)_{ij}, (H^\intercal \mathrm{d}XH)_{kl}\big\rangle = \sum_{pqrs} \mathrm{d}\big\langle H^\intercal_{ip}\mathrm{d}X_{pq}H_{qj}, H^\intercal_{kr}\mathrm{d}X_{rs}H_{sl}\big\rangle, \\ &= \sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sl}\mathrm{d}X_{pq}\mathrm{d}X_{rs} = \sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sl}\big(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr} - 2\delta_{pq}\delta_{rs}\delta_{pr}\big)\mathrm{d}t, \\ &= \left(\sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sl}\delta_{pr}\delta_{qs} + \sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sl}\delta_{ps}\delta_{qr} - 2\sum_{pqrs} H^\intercal_{ip}H_{qj}H^\intercal_{kr}H_{sl}\delta_{pq}\delta_{rs}\delta_{pr}\right)\mathrm{d}t, \\ &= \sum_{pq} H^\intercal_{ip}H_{rk}H^\intercal_{jq}H_{ql}\mathrm{d}t + \sum_{pq} H^\intercal_{ip}H_{pl}H^\intercal_{kq}H_{qj}\mathrm{d}t - 2\sum_{p} H^\intercal_{ip}H_{pj}H^\intercal_{kp}H_{pl}\mathrm{d}t, \\ &= \big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj} - 2\delta_{ij}\delta_{kl}\delta_{ik}\big)\mathrm{d}t. \end{aligned}$$

Particularly, we have that $\mathrm{d}N_{ii}\mathrm{d}N_{jj} = 0$ for every choice of j and i. Thus every entry in the diagonal of N is a finite variation process and so it is λ_i. Let us finally compute the finite variation part F of N.

$$\begin{aligned} \mathrm{d}F &= \frac{1}{2}\big(H^\intercal \mathrm{d}X\mathrm{d}H + \mathrm{d}H^\intercal \mathrm{d}XH\big), \\ &= \frac{1}{2}\big(H^\intercal \mathrm{d}XHH^\intercal \mathrm{d}H + \mathrm{d}H^\intercal HH^\intercal \mathrm{d}XH\big), \\ &= \frac{1}{2}\big(\mathrm{d}N\mathrm{d}A + (\mathrm{d}N\mathrm{d}A)^\intercal\big). \end{aligned}$$

For $\mathrm{d}N\mathrm{d}A$ we have

$$\begin{aligned} (\mathrm{d}N\mathrm{d}A)_{ij} &= \sum_{k\neq j} \mathrm{d}N_{ik}\mathrm{d}A_{kj} = \sum_{k\neq j} \frac{\mathrm{d}N_{ik}\mathrm{d}A_{kj}}{\lambda_j - \lambda_k}, \\ &= \sum_{k\neq j} \frac{\delta_{ik}\delta_{kj} + \delta_{ij}\delta_{kk} - 2\delta_{ik}\delta_{kj}\delta_{ij}}{\lambda_j - \lambda_k}\mathrm{d}t = \delta_{ij}\sum_{k\neq j}\frac{\mathrm{d}t}{\lambda_j - \lambda_k}. \end{aligned}$$

Then F is diagonal with $\mathrm{d}F_{ii} = \sum_{k\neq i}\frac{\mathrm{d}t}{\lambda_i - \lambda_k}$. We conclude that

$$\mathrm{d}\lambda_i = \sum_{k\neq i}\frac{\mathrm{d}t}{\lambda_i - \lambda_k}.$$

4.2 Deterministic eigenvalue processes for matrix-valued diffusions

Aquí planeo escribir un resultado sobre cómo obtener una versión determinista del proceso de eignevalores para cualquier proceso de difusión matricial. Estoy trabajando en ello, pero creo que se puede hacer usando las mismas técnicas de Grackyz y Malecki, pero cambiando el proceso por uno de diagonal cero.

4.3 Connections with finite free probability

La última parte del trabajo de tesis consistirá en establecer una relación entre los procesos matriciales que permiten observar el proceso de Dyson determinista y el polinomio característico esperado de un ensamble G. U.