Centro de Investigación en Matemáticas, A.C.

El proceso de Dyson determinista

T E S I S

Que para obtener el grado de

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Presenta

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Dedicatoria . . .

Abstract

Palabras clave:

Agradecimientos

A mis padres . . .

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Chapter 1

Preliminaries

The purpose of this chapter is (1) to introduce the concepts and results that are used along the thesis and (2) to standardize the notation coming from different areas, so it is easier to read the text. In the first section it is given a brief introduction to the objects of study of random matrix theory and a few matrix algebra results that are recalled later in the proofs. The second section introduces stochastic calculus first for $\mathbb{R}$-valued processes and then for processes taking values in spaces of matrices. The theory developed here will be used for the dynamics of eigenvalues in Chapters 2 and 4.The last section covers the essential definition and theorems coming from non-commutative probability theory that constitute a precedent for Chapter 3.

In order to be consistent along the text, we introduce here the notation we will be using. However, the use of several kinds of object commonly denoted with the same symbols, makes it necessary to specify every time what kind of object we are dealing with. The time parameter of a stochastic process will always be shown in parentheses (i.e. $W(t)$). Subindexes will represent the entry of a matrix or vector. An integer interval of length k will be represented by the symbol $[k]$, i.e.

$$[k] \coloneqq \{1, 2, \ldots, k-1, k\}.$$

When we place a set S in the combinations symbol $\binom{S}{k}$, we denote the collection of all of the subsets of S that have exactly k elements.

The determinant of a matrix A will be denoted by $[A]$ and if S, T are sets of integers, $[A]_{S,T}$ represents the determinant of the submatrix $A_{S,T}$. The transpose of A is denoted as A^T and if it has complex entries, its adjoint element is A^*. The space of $n \times m$ matrices with entries in a field $\mathbb{F}$ is denoted by $\mathcal{M}_{n,m}(\mathbb{F})$. The space of $n \times n$ symmetric matrices is denoted by $\mathcal{H}_{n,n}(\mathbb{R})$ and the space of $n \times n$ hermitian by $\mathcal{H}_{n,n}(\mathbb{C})$.

The symbol $\langle X, Y \rangle(t)$ denotes the quadratic covariation between X and Y and sometimes it will also be represented by $\mathrm{d}X\mathrm{d}Y$. The derivative with respect to a given variable x will be represented either using $\frac{\mathrm{d}}{\mathrm{d}x}$, $\frac{\partial}{\partial x}$ or ∂_x and following the convention in the literature we will sometimes denote $\frac{\mathrm{d}}{\mathrm{d}t} f = g$ as $\mathrm{d}f = g\mathrm{d}t$.

In the next section we introduce the basic concepts in the study of random matrices.

1.1 Introduction to main concepts in Random Matrix Theory

This section is meant to present the essential objects we study in random matrix theory and illustrate a few techniques that can be used to derive results. Most of the relevant results, however, are presented later with the tools introduced in the following sections.

1.1.1 Matrix algebra

Before stating the most specific concepts and results related to random matrices, it is important to mention a few purely algebraic results of matrices as they will be useful along the thesis.

The Cauchy-Binet formula allows us to find the minor of a product of matrices in terms of the minors of the individual matrices.

Theorem 1.1.1 (Cauchy-Binet formula)**.** *Let m, n, p, k be integers, A an $m \times n$ matrix, and B an $n \times p$ matrix, then*

$$[AB]_{S,T} = \sum_{|U| \subset \binom{[n]}{k}} [A]_{S,U}[B]_{U,T},$$

where $S \in \binom{[m]}{k}, T \in \binom{[p]}{k}$.

The following Theorem taken from Marcus, Spielman, and Srivastava (2022) can be seen as an equivalent to the Cauchy-Binet formula for sums of matrices.

Theorem 1.1.2. *Let k, n be integers such that $k \leq n$, A, B two $n \times n$ matrices, and $S, T \in \binom{[n]}{k}$. Then*

$$[A+B]_{S,T} = \sum_{i=0}^{k} \sum_{V \in \binom{[k]}{i}} (-1)^{\|U\|_1 + \|V\|_1} [A]_{U(S),V(T)} [B]_{\bar{U}(S),\bar{V}(T)},$$

with $\overline{U} = [k] \setminus U$.

A basic linear algebra theorem that has major relevance is the Spectral Theorem.

Theorem 1.1.3 (Spectral Theorem)**.** *Let A by an $n \times n$ self adjoint matrix. Then there exists an orthonormal basis $v_1, \ldots, v_n \in \mathbb{R}^n$ and real eigenvalues $\lambda_1, \ldots, \lambda_n$ such that, for every $1 \leq i \leq n$.*

$$Av_i = \lambda_i v_i.$$

1.1.2 Random matrix ensembles

A random matrix R is simply a measurable function from a probability space to a space of matrices.

$$R : (\Omega, \mathscr{F}, \mathbb{P}) \to \mathcal{M}_{n,m}(\mathbb{F}).$$

In general Random Matrix Theory, the field for the entries $\mathbb{F}$ can be quite general but for the goals of this work it is enough to consider $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Given any self adjoint $n \times n$ matrix A, we can associate an empirical probability measure $\hat{\mu}$ to its set of eigenvalues $\lambda_1, \ldots, \lambda_n$ given by

$$\hat{\mu}(B) \coloneqq \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_B(\lambda_j).$$

We call $\hat{\mu}$ the empirical spectral measure.

If A is random (i.e. its entries are random variables), $\hat{\mu} : \Omega \times \mathscr{F} \to [0,1]$ is a random measure, which means that for every ω, $\hat{\mu}(\omega, \cdot)$ is a probability measure and for every measurable set $B \in \mathscr{F}$, $\hat{\mu}(\cdot, B)$ is a real random variable. In this case, it is possible to define a deterministic empirical measure associated to A by simply taking the expectation of $\hat{\mu}$ on the measure of A.

$$\hat{\nu}(B) \coloneqq \mathbb{E}\left[\frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_B(\lambda_j)\right] = \frac{1}{n} \sum_{j=1}^{n} P\left(\lambda_j \in B\right).$$

We call $\hat{\nu}$ the mean spectral measure.

We are usually interested on knowing if $\hat{\mu}$ and $\hat{\nu}$ converge to a given law when $n \to \infty$. The following examples show that this happens in some cases.

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Example 1.1.1.

Using that the trace equals the sum of eigenvalues, we have that the expectation over $\hat{\nu}$ is equal to

$$\int_{\mathbb{C}} z\hat{\nu}(\mathrm{d}z) = \mathbb{E}\left[\frac{1}{n}Tr(A)\right].$$

The next identity allows us to compute similar moments of A.

Theorem 1.1.4 (Trace identity). *Let A be a normal $n \times n$ matrix ($A^*A = AA^*$), then*

$$\frac{1}{n}Tr(A^k A^{*j}) = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^k \overline{\lambda}_i^j = \int_{\mathbb{C}} z^k \bar{z}^j \hat{\mu}(\mathrm{d}x),$$

where $\hat{\mu}$ is the empirical spectral measure associated to A. If A is random and we take expectation over its probability law, we have

$$\int_{\mathbb{C}} x^k \bar{z}^j \hat{\nu}(\mathrm{d}z) = \frac{1}{n}\mathbb{E}\left[Tr(A^k A^{*j})\right].$$

In random matrix theory it is common to work with matrix ensembles. An ensemble is a set of matrices with a probability measure associated to them.

Example 1.1.2 (Independent identically distributed entries ensemble). If A is an $n \times n$ matrix whose entries $A_{ij}, 1 \leq i \leq n, 1 \leq j \leq n$ are all independent identically distributed random variables, we say that A is an i.i.d. ensemble.

Example 1.1.3 (Diagonal i.i.d. ensemble). If D is a diagonal $n \times n$ matrix whose every entry is an i.i.d. random variable, then we say that D is a diagonal i.i.d. ensemble.

Example 1.1.4 (Gaussian invariant ensembles). Let $\mathbb{H}$ denote the field of quaternions and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. If R is an hermitian matrix whose entries are standard normal random variables in $\mathbb{F}$ independent except for symmetries, then we say that R is a Gaussian invariant ensemble. Depending on $\mathbb{F}$ we have particular names for each ensemble.

- If $\mathbb{F} = \mathbb{R}$, we call R the Gaussian orthogonal ensemble.
- If $\mathbb{F} = \mathbb{C}$, we call R the Gaussian unitary ensemble.
- If $\mathbb{F} = \mathbb{H}$, we call R the Gaussian symplectic ensemble.

The specific names are given because the distribution of the eigenvalues of R remains unchanged under a conjugation by an orthogonal (respectively unitary, symplectic) matrix. This property is analogous to the property of vector of independent normal variables that preserve their distribution after being transformed by an orthogonal matrix.

Example 1.1.5 (Haar unitary ensemble). If we consider $\mathcal{U}_{n,n}(\mathbb{C})$ the group of complex unitary matrices ($U^*U = I_n = UU^*$). We can define in $\mathbb{U}_{n,n}(\mathbb{C})$ a Haar measure that is unique up to a constant. If we normalize this measure we have the only Haar probability measure μ_U in $\mathbb{U}_{n,n}(\mathbb{C})$. A Haar unitary ensemble is a matrix sampled from μ_U.

Example 1.1.6 (Wigner example). Let W be a random self-adjoint matrix whose every entry is a i.i.d. random variable except for the symmetries. Then we say that W is a Wigner ensemble. Notice in particular that the Gaussian invariant ensembles are Wigner ensembles. This is one of the first studied ensembles.

Example 1.1.7 (Wishart ensemble). Let R be an $n \times n$ i.i.d. standard Gaussian ensemble and define $E := R^T R$, then we say E is a Wishart ensemble. This ensemble is used to model covariance matrices.

1.1.3 Asymptotic results for random matrices

The next results gives the convergence of an empirical spectral measure to a continuous probability measure in $\mathbb{R}$ when the matrix dimension n tends to infinity. The limit only depends on the first two moments of the variables involved and not on the whole distribution, so it can be seen as a matrix analogous of the Central Limit Theorem.

Theorem 1.1.5 (Wigner's semicircle law Mingo and Speicher (2017)). *For each $n \in \mathbb{N}$, let $W^{(n)}$ be a Wigner ensemble and its entries $W_{ij}^{(n)}$ satisfy the following conditions.*

1. $\mathbb{E}\left[|W_{ij}^{(n)}|^k\right] < \infty$ *for all* $k \in \mathbb{N}$.
2. $\mathbb{E}\left[W_{ij}\right] = 0$ *for every* $1 \leq i \leq n, 1 \leq j \leq n$.
3. $\mathbb{E}\left[W_{ij}^{(n)2}\right] = 1/\sqrt{n}$.

Then both $\hat{\mu}$ and $\hat{\nu}$ converge in distribution to the semicircle distribution, i.e. the absolutely continuous distribution with density

$$f(x) = \frac{\sqrt{4-x^2}}{2\pi}$$

1.2 Stochastic Calculus

When we work with continuous time processes, one of the most important tools is stochastic calculus. The results provided by the Itô integral and related concepts allow to study dynamical systems with a random behaviour. In this section we introduce some stochastic calculus that are used along the thesis. The first part deals with the definition and main properties of Itô and Stratonovich integrals in $\mathbb{R}$-valued processes, we later generalize the definition to $\mathbb{R}^d$-valued processes and finally to processes taking values in spaces of matrices.

1.2.1 Stochastic calculus for $\mathbb{R}^n$-valued processes

We begin by defining the Itô integral for $\mathbb{R}$-valued processes and stating its main properties. In all of the next definitions, consider we are working on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_t, P)$ and we use the convention that a continuous time stochastic process is a stochastic process indexed by $\mathbb{R}^+ = [0, \infty)$. Most of the definition and results here are taken from Klebaner (2012), a more general approach to the subject can be found in Karatzas and Shreve (2014) or Revuz and Yor (2013).

There are several definitions of the Itô integral, some of them are more general. For the sake of simplicity, we will use one that resembles the definition of Lebesgue integral. Let us consider a simple process $X = (X(t))_{t\geq 0}$ adapted to the filtration $\mathscr{F}_t$, i.e.

$$X(t) = \xi_0 \delta_{0,t} + \sum_{j=0}^{n-1} \xi_j \mathbb{1}_{(t_j, t_{j+1}]}(t),$$

for some $0 = t_0 < t_1 < \cdots < t_n = T$ and variables ξ_i that are $\mathscr{F}_{t_i}$-measurables. For simple adapted processes, the Itô integral with respect to a Brownian motion can be defined as

Definition 1.2.1. Let X be a simple processes adapted to $\mathscr{F}_t$ and square integrable. Let $B = (B(t))_{t\geq 0}$ be a Brownian motion adapted to $\mathscr{F}_t$. The Itô integral of X with respect to B on $[0, T]$ is

$$\int_0^T X(s)\mathrm{d}B(s) := \sum_{j=0}^{n-1} \xi_j (B(j+1) - B(j)).$$

It is not hard to verify from the definition that the integral is linear and has zero mean. Also, it can be proven that it satisfies the so called Itô isometry,

$$\mathbb{E}\left[\left(\int_0^T X(s)\mathrm{d}B(s)\right)^2\right] = \int_0^T \mathbb{E}\left[X^2(s)\right]\mathrm{d}s.$$

Another important fact is that the Itô integral is a martingale. These last two happen to be the most important properties of this integral and one usually wants to preserve them when defining a stochastic integral for more general processes.

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If we have a square integrable continuous time process $X(t)$ adapted to $\mathscr{F}_t$ and a sequence of simple adapted processes $X_n(t)$ that are also square integrable and converge in probability to $X(t)$, then under certain conditions, one can prove that the integrals $\int_0^T X_n(s)\mathrm{d}B(s)$ also have a limit in probability. We define the Itô integral for general adapted processes as this limit.

cuáles son estas condiciones?

Definition 1.2.2. Let X be a square integrable adapted process to $\mathscr{F}_t$ and $(X_n(t))_{n\geq 0}$ a sequence of simple adapted, square integrable processes converging in probability to $X(t)$. We define the integral of $X(t)$ with respect to a Brownian motion $B(t)$ as

$$\int_0^T X(s)\mathrm{d}B(s) = \overset{P}{\lim_{n\to\infty}} \int_0^T X_n(s)\mathrm{d}B(s).$$

As we have mentioned before, when extending the definition of the Itô integral to more general processes, one wishes to preserve the good properties it has for simple processes. The next Theorem allows us to characterize when the integral has these properties

Theorem 1.2.1. *If $X = (X(t), t \geq 0)$ and $Y = (Y(t), t \geq 0)$ are regular adapted process and satisfy*

$$P\left(\int_0^T X^2(s)\mathrm{d}s < \infty\right) = 1, \qquad P\left(\int_0^T Y^2(s)\mathrm{d}s < \infty\right) = 1,$$

then the integrals $\int_0^T X(s)\mathrm{d}B(s), \int_0^T Y(s)\mathrm{d}B(s)$ exist and it satisfy the following properties,

1. *Linearity. For $\alpha, \beta \in \mathbb{R}$,*

$$\int_0^T (\alpha X(s) + \beta Y(s))\mathrm{d}B(s) = \alpha \int_0^T X(s)\mathrm{d}B(s) + \beta \int_0^T Y(s)\mathrm{d}B(s). \tag{1.1}$$

If additionally, the process X satisfies

$$\int_0^T \mathbb{E}\left[X^2(s)\right]\mathrm{d}s < \infty,$$

then the integral $\int_0^T X(s)\mathrm{d}B(s)$ has the following properties

2. *Martingale property. For $t \leq T$*

$$\mathbb{E}\left[\int_0^T X(s)\mathrm{d}B(s)\middle|\mathscr{F}_t\right] = \int_0^t X(s)\mathrm{d}B(s) = \int_0^T X(s)\mathbb{1}_{[0,t]}(s)\mathrm{d}B(s).$$

3. *Itô's isometry.*

$$\mathbb{E}\left[\int_0^T X(s)\mathrm{d}B(s)\right]^2 = \int_0^T \mathbb{E}\left[X^2(s)\right]\mathrm{d}s.$$

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If we consider, for a given adapted process X, the Itô integral process $I = (I(t), t \geq 0)$ defined as

$$I(t) := \int_0^t X(s)\mathrm{d}B(s),$$

we know that this process is a martingale. We can also define its quadratic variation $\mathrm{d}\langle I, I\rangle(t)$ as a limit in probability,

$$\mathrm{d}\langle I, I\rangle(t) := \overset{P}{\lim_{\delta_n \to 0}} \sum_{i=0}^{n-1} (Y(t_{j+1,n}) - Y(t_{j,n}))^2,$$

with $\{t_{j,n}\}_{j=1}^n$ a partition of $[0,t]$ for every n and $\delta_n = \sup_j \{t_{j+1,n} - t_{j,n}\} \to 0$ as $n \to \infty$.

The following Theorem gives us a way to find explicitly the quadratic variation of an Itô integral process,

Theorem 1.2.2. *The quadratic variation of an Itô integral is*

$$\mathrm{d}\langle I(t), I(t)\rangle(t) = \mathrm{d}\langle \int_0^t X(s)\mathrm{d}B(s), \int_0^t X(s)\mathrm{d}B(s)\rangle(t) = \int_0^t X^2(s)\mathrm{d}s.$$

We can notice in Definition 1.2.1 that when we define the Itô integral, we care about in which point of the interval $(t_i, t_{i+1}]$ we evaluate $X(t)$. This is not casual and we want to take the lower value so that the resulting process is a martingale. However, this causes that several results from classical calculus are not generalized in Itô calculus. One of the main results in Itô calculus is the following theorem, that allows us to prove some other important facts.

Theorem 1.2.3 (Itô formula for Brownian motion)**.** *Let f be a twice differentiable function and B a Brownian motion, then*

$$f(B(t)) = f(0) + \int_0^t f'(B(s))\mathrm{d}B(s) + \frac{1}{2}\int_0^t f''(B(s))\mathrm{d}s.$$

The proof is done using a Taylor expansion of second order for f and taking a limit. We will omit it here, but it is important to notice that (under suitable conditions) the Itô formula allows us to write a function of a Brownian motion as a sum of a martingale and a finite variation process.

We can define the Itô integral with respect to processes other than the Brownian motion. The class of processes for which we can define an Itô integral is rather large, so we will define a smaller class that is of interest for this work.

Definition 1.2.3 (Itô process)**.** We say that an $\mathscr{F}_t$-adapted process $Y = (Y(t), 0 \leq t \leq T)$ is an Itô process if there exist $\mu = (\mu(t), 0 \leq t \leq T)$, $\sigma = (\sigma(t), 0 \leq t \leq T)$ adapted processes such that $\int_0^T |\mu(s)|\mathrm{d}s < \infty$, $\int_0^T \sigma^2(s)\mathrm{d}s < \infty$, and $Y(0)$ an $\mathscr{F}_0$-measurable variable that satisfy

$$Y(t) = Y(0) + \int_0^t \mu(s)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}B(s). \tag{1.2}$$

It is an usual convention to write the "differential" of an Itô process as

$$\mathrm{d}Y(s) = \mu(s)\mathrm{d}s + \sigma(s)\mathrm{d}B(s).$$

This notation only means that Y satisfies (1.2). We usually call $\int_0^t \mu(s)\mathrm{d}s$ the finite variation part of Y and $\int_0^t \sigma(s)\mathrm{d}B(s)$ the martingale part of Y.

Using the fact that the covariation of any function with a finite variation function is zero, we can find that the quadratic variation of an Itô process is given by

$$\mathrm{d}\langle Y, Y\rangle(t) = \int_0^t \sigma^2(s)\mathrm{d}s.$$

Now we can define the integral of an adapted process X with respect to an Itô process Y.

Definition 1.2.4 (Itô integral with respect to an Itô process)**.** Let Y be an adapted process such that its Itô integral exists for every t in $[0,T]$. Let Y be an Itô process $\mathrm{d}Y = \mu \mathrm{d}s + \sigma \mathrm{d}B$ and X, Y satisfy

$$\int_0^T |X(s)\mu(s)|\mathrm{d}s < \infty,$$
$$\int_0^T X^2(s)\sigma^2(s)\mathrm{d}s < \infty.$$

Then, the integral of X with respect to Y is defined, for $0 \leq t \leq T$ as

$$\int_0^t X(s)\mathrm{d}Y(s) = \int_0^t X(s)\mu(s)\mathrm{d}s + \int_0^t X(s)\sigma(s)\mathrm{d}B(s).$$

Although the definition of an Itô integral with respect to an Itô process can be given in a more direct way, it happens to coincide with the last one. In a similar spirit as in the definition of the Itô integral with respect to an Itô process, we can also extend Theorem 1.2.3 for Itô processes.

Theorem 1.2.4 (Itô formula for Itô processes)**.** *Let Y be an Itô process satisfying* $\mathrm{d}Y = \mu \mathrm{d}s + \sigma \mathrm{d}B$ *and f be a twice continuously differentiable function, then the stochastic differential of $f(Y(t))$ is well defined and is given by*

$$\begin{aligned}\mathrm{d}f(Y(t)) &= f'(Y(t))\mathrm{d}Y(t) + \frac{1}{2}f''(Y(t))\mathrm{d}\langle Y, Y\rangle(t),\\ &= \left(f'(Y(t))\mu(t) + \frac{1}{2}f''(Y(t))\sigma^2(t)\right)\mathrm{d}t + f'(Y(t))\sigma(t)\mathrm{d}B(t).\end{aligned}$$

Although the Itô integral is the most common one, it is not the only notion of stochastic integration and some others can be used in certain contexts. One of the most used alternatives is the Stratanovich integral, that preserves several properties of standard calculus. The Stratanovich integral is useful when we deal with random matrix calculus because it allows to simplify calculations.

Definition 1.2.5 (Stratanovich integral)**.** Let X and Y be two continuous adapted processes. The Stratanovich integral of X with respect to Y denoted as $\int_0^t X(s)\partial Y(s)$ is the L^2 limit of the sums

$$\sum_{i=0}^{n-1} \frac{1}{2}(X(t_{i+1,n}) + X(t_{i,n}))(Y(t_{i+1,n}) - Y(t_{i,n})).$$

as $\delta_n = \delta_n = \sup_j \{t_{j+1,n} - t_{j,n}\} \to 0$.

The main difference between the Itô and Stratanovich integrals is the point we take for the evaluation of the integrand process in the interval $(t_{i,n}, t_{i+1,}]$. While we take the left point in the Itô integral, we take the average between the extremes for the Stratanovich one. Both integrals happen to be related by the following result.

Theorem 1.2.5 (Relationship between Itô and Stratanovich integrals)**.** *Let X, Y be two continuous adapted processes such that the Itô integral of X with respect to Y is well-defined. The Stratanovich integral of X with respect to Y is*

$$\int_0^t X(s)\partial Y(s) = \int_0^t X(s)\mathrm{d}Y(s) + \frac{1}{2}\langle X, Y\rangle(t).$$

By the last Theorem we can write the Stratanovich differential similarly to the Itô differential as

$$Y(s)\partial X(s) = Y(s)\mathrm{d}X(s) + \frac{1}{2}\mathrm{d}\langle X, Y\rangle(t).$$

Sometimes we write this diferential as $Y(s)\partial X(s) = Y(s) \circ \mathrm{d}X(s)$. This notation is especially helpful when we work with matrix-valued processes.

Perhaps the main situation when the Stratanovich integral is used instead of the Itô version is when we want to preserve the classical integration by parts formula. The next Theorem uses the relationship between both integrals to compute the differential of the product XY.

Theorem 1.2.6 (Integration by parts for Itô and Stratanovich integral Revuz and Yor (2013))**.** *Let X, Y be two adapted processes such that the integrals $\int_0^t X(s)\mathrm{d}Y(s)$ and $\int_0^t Y(s)\mathrm{d}X(s)$ are well defined, then*

$$\begin{aligned} \mathrm{d}(XY) &= X\mathrm{d}Y + Y\mathrm{d}X + \mathrm{d}\langle X, Y\rangle, \\ &= X\mathrm{d}Y + \frac{1}{2}\mathrm{d}\langle X, Y\rangle + Y\mathrm{d}X + \frac{1}{2}\mathrm{d}\langle X, Y\rangle = X\partial Y + Y\partial X, \\ &= X \circ \mathrm{d}Y + Y \circ \mathrm{d}X. \end{aligned}$$

Notice that in the case of the Stratanovich integral, we recover the classical integration by parts formula.

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The next results are technical but they are needed for the proofs in Chapter 2. The first one states the existence of a process called local time and the stochastic differential equation it satisfies, the second one gives a way to prove when this local time process is zero.

o de Revuz-Yor

Theorem 1.2.7 (Tanaka's formula (Revuz and Yor (2013)))**.** *Let X be a continuous semimartingale. For any real number a, there exists an increasing continuous process L^a called the local time of X in a such that,*

$$\begin{aligned} |X(t) - a| &= |X(0) - a| + \int_0^t \operatorname{sgn}(X(s) - a)\ \mathrm{d}X(s) + L^a(t), \\ (X(t) - a)^+ &= (X(0) - a)^+ + \int_0^t \mathbb{1}_{\{X(s) > a\}}\ \mathrm{d}X(s) + \frac{1}{2}L^a(t), \\ (X(t) - a)^- &= (X(0) - a)^- - \int_0^t \mathbb{1}_{\{X(s) \leq a\}}\ \mathrm{d}X(s) + \frac{1}{2}L^a(t). \end{aligned}$$

o de Revuz-Yor.
9 Lemma 3.3.

Theorem 1.2.8 (Revuz and Yor (2013))**.** *Let $\rho : (0, \infty) \to (0, \infty)$ a measurable function that satisfies*

$$\int_{0+} \frac{\mathrm{d}s}{\rho(s)} = \infty.$$

If X is a continuous semimartingale such that, for some $\epsilon > 0$ and every t, the process

$$A_t = \int_0^t \mathbb{1}_{\{0 < X(s) \leq \epsilon\}} \rho(X(s))^{-1}\ \mathrm{d}\langle X, X\rangle(s) < \infty \qquad a.s.,$$

then $L^0(X) = 0$.

Gronwall's lemma allows us to bound a function satsifying a differential inequality by the solution of the associated differential equation. It will be useful for the multidimensional version of the Yamada-Watanabe Theorem.

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Lemma 1.2.9 (Gronwall's lemma (Le Gall (2016)))**.** *Let $T > 0$ and let g be any nonnegative bounded measurable function on $[0, T]$. Assume that there exists two constants $a \geq 0$ and $b \geq 0$ such that for every $t \in [0, T]$,*

$$g(t) \leq a + b\int_0^t g(s)\ \mathrm{d}s.$$

Then we also have, for every $t \in [0,T]$,

$$g(t) \leq a\exp(bt).$$

Below it is stated a generalized version of the well-known Mckean's principle. This result gives solutions for a stopping time to be infinte a.s. In particular, it is used when working with eigenvalue processes to conclude the non-collision of the eigenvalues. This generalization and its proof appear in Mayerhofer, Pfaffel, and Stelzer (2011).

Lemma 1.2.10 (Generalized McKean's argument). *Let $Z = (Z_s)_{s\in\mathbb{R}_+}$ be an adapted càdlàg $\mathbb{R}^+ \setminus \{0\}$-valued stochastic process on a stochastic interval $[0,\tau_0)$ such that $Z_0 > 0$ a.s. and $\tau_0 = \inf\{0 < s \leq \tau_0 : Z_{s-} = 0\}$. Suppose that $h : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$ is continuous and satisfies the following:*

1. *For all $t \in [0,\tau_0)$, we have $h(Z_t) = h(Z_0) + M_t + P_t$, where*
   - *(a) P is an adapted càdlàg process on $[0,\tau_0)$ such that $\inf_{t\in[0,\tau_0\wedge T]} P_t > -\infty$ a.s. for each $T \in \mathbb{R}_+ \setminus \{0\}$,*
   - *(b) M is a continuous local martingale on $[0,\tau_0)$ with $M_0 = 0$,*
2. *$\lim_{z\to 0} h(z) = -\infty$.*

Then $\tau_0 = \infty$ a.s.

Stochastic Calculus for $\mathbb{R}^n$-valued processes

If we have a continuous time stochastic process taking values in $\mathbb{R}^n$, it is possible to give a definition of the Itô integral with respect to a multivariate Brownian motion. Based on this, we can extend several results of univariate stochastic calculus, this can then be used for introducing stochastic calculus for matrix-valued processes.

An $\mathbb{R}^n$-valued Brownian motion $\vec{B} = \{(B_1(t),\dots,B_n(t)), t \geq 0\}$ is an n length vector whose every entry B_i is an independent Brownian motion in $\mathbb{R}$. An n-dimensional process $\vec{X}$ is said to be adapted to a filtration $\mathscr{F}$ if each one of its entries is. If X_i is the ith entry of $\vec{X}$ and for every i we have that

$$\int_0^T X_i^2(s)\mathrm{d}s < \infty,$$

then we can define the Itô itegral of $\vec{X}$ with respect to $\vec{B}$ in $0 \leq t \leq T$ as

$$\int_0^t \vec{X}(s)\cdot \mathrm{d}\vec{B}(s) \coloneqq \sum_{j=1}^n \int_0^t X_j(s)\mathrm{d}B_j(s).$$

Notice that the integral notation suggests the similarity with a dot product. Similarly we also denote the multivariate integral as $\vec{X}(s)\cdot \mathrm{d}\vec{B}(s)$.

The process $\sum_{j=1}^n \int_0^t X_j(s)\mathrm{d}B_j(s)$ takes values in $\mathbb{R}$. If we add a finite variation part μ, then we can have a process Y similar to an Itô process but driven by a multidimensional Brownian motion,

$$\mathrm{d}Y(s) = \mu(s)\mathrm{d}s + \sum_{j=1}^n X_j(s)\mathrm{d}B_j(s).$$

If we take $\vec{\mu}(t) = (\mu_1(t),\dots,\mu_n(t))$ to be a vector of integrable functions and for each $i \in [n]$ we consider a vector-valued process $\vec{\sigma}_i(t) = (\sigma_{i1}(t),\dots,\sigma_{in}(t))$, then for each $i \in [n]$ we have a single dimensional Itô process driven by a multivariate Brownian motion,

$$\mathrm{d}Y_i = \mu_i(s)\mathrm{d}s + \sum_{j=1}^n \sigma_{ij}(s)\mathrm{d}B_j(s)$$

By taking $\vec{Y} = (Y_1, \dots, Y_n)$ we have an n-dimensional Itô process, which is denoted in differential form by

$$\mathrm{d}\vec{Y}(s) = \vec{\mu}(s)\mathrm{d}s + \Sigma(s)\mathrm{d}\vec{B}(s),$$

with Σ an $n \times n$ matrix valued function with entries σ_{ij}.

Before stating the Itô formula for multidimensional processes, we need to know the quadratic covariation between entries of a multidimensional Itô process.

Theorem 1.2.11. *Let $\vec{Y}$ be an n-dimensional Itô process, then the quadratic covariation of two of its entries Y_i, Y_j is given by*

$$\langle Y_i, Y_j \rangle(t) = \int_0^t \left(\Sigma\Sigma^T\right)_{ij}(t)\mathrm{d}t.$$

The matrix $\Sigma\Sigma^T$ is often called the diffusion matrix. In the next Theorem, we generalize the Itô formula for multidimensional Itô processes.

Theorem 1.2.12 (Multidimensional Itô formula)**.** *Let $\vec{Y}$ be an n-dimensional Itô process and $f : \mathbb{R}^n \to R^m$ be a C^2 function. The process $f(Y_1(t), \dots, Y_n(t))$ is also an Itô process and has a stochastic differential given by*

$$\mathrm{d}f(Y_1(t), \dots, Y_n(t)) = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(Y_1(t), \dots, Y_n(t)) \\ + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f(Y_1(t), \dots, Y_n(t))\mathrm{d}\langle Y_i, Y_j\rangle(t).$$

In particular when $n = 2$, $Y_1(t) = Y(t)$ for some Itô process $\mathrm{d}Y = \mu\mathrm{d}t + \sigma\mathrm{d}B$, $Y_2(t) = t$ and $f : \mathbb{R}^2 \to \mathbb{R}$, we have that

$$\mathrm{d}f(Y(t), t) = \frac{\partial f}{\partial x}(Y(t), t)\mathrm{d}Y(t) + \frac{\partial f}{\partial t}(Y(t), t)\mathrm{d}t + \frac{1}{2}\sigma^2(t)\frac{\partial^2 f}{\partial x^2}(Y(t), t)\mathrm{d}t. \tag{1.3}$$

Infinitesimal generator and harmonic functions

Every Itô process is markovian and thus it has an associated Markov semigroup and infinitesimal generator. These operators can tell us many of the properties of the processes and in particular the infinitesimal generator can be used to prove that some transformations of an Itô process are martingales.

Definition 1.2.6. Let X be an Itô process with $\mathrm{d}X(t) = \mu(X, t)\mathrm{d}t + \sigma(X, t)\mathrm{d}B(t)$. The infinitesimal generator of X is the second order differential operator $\mathcal{A}_t$,

$$\mathcal{A}_t f(x, t) = (\mathcal{A}_t f)(x, t) = \frac{1}{2}\sigma^2(x, t)\frac{\partial^2 f}{\partial x^2}(x, t) + \mu(x, t)\frac{\partial f}{\partial x}(x, t).$$

With this definition, we can re-write equation (1.3) as

$$\mathrm{d}f(Y(t), t) = \left(\mathcal{A}_t f(Y(t), t) + \frac{\partial f}{\partial t}(Y(t), t)\right)\mathrm{d}t + \frac{\partial f}{\partial x}(Y(t), t)\sigma(Y(t), t)\mathrm{d}B(t).$$

If the integral $\int_0^t \frac{\partial f}{\partial x}(X(s), s)\sigma(X(s), s)\mathrm{d}B(s)$ is a martingale, then the process $f(Y(t), t) - \int_0^t \left(\mathcal{A}_s f(Y(s)) + \frac{\partial f}{\partial s}(Y(s), s)\right)\mathrm{d}s$ is a martingale. This result is stated in the following Theorem

Theorem 1.2.13. *Let $Y(t)$ be an Itô process with differential $\mathrm{d}Y(t) = \mu(Y(t), t)\mathrm{d}t + \sigma(Y(t), t)\mathrm{d}B(t)$ such that $\mu(x, t)$ and $\sigma(x, t)$ are Lipschitz in x with the same constant for every t and satisfy*

$$|\mu(x,t)| + |\sigma(x,t)| \leq K(1+|x|).$$

If $f(x,t)$ is a twice continuously differentiable function in x and once in t with $\partial_x f$ bounded, then the process

$$M^f(t) := f(Y(t),t) - \int_0^t \left(\mathcal{A}_s f(Y(s),s) + \frac{\partial f}{\partial s}(Y(s),s) \right) \mathrm{d}s,$$

is a martingale.

We have in particular that under the same conditions, when $\mathcal{A}_t f(Y(t)) + \frac{\partial f}{\partial t}(Y(t),t) = 0$, $f(Y(t),t)$ is a martingale. If f only depends on x, this is equivalent to asking it to be a solution to $\mathcal{A}_t f = 0$. These functions are known as harmonic functions for the process Y.

Complex stochastic calculus

It is possible to define continuous time stochastic processes in more general fields than $\mathbb{R}$ and then create a notion of stochastic integral for these processes. Particuarly, in the case of random matrix theory, we care about process taking values in the field of complex numbers ($\mathbb{C}$) and the field of quaternions ($\mathbb{H}$). Provided that both spaces can be seen as a vector space with $\mathbb{R}$ as the field of scalars, the extension of the definitions is natural by considering that every process in $\mathbb{C}$ or $\mathbb{H}$ has the form $A + iB$ or $A + iB + jC + kD$, respectively, with A, B, C, D stochastic processes taking values in $\mathbb{R}$.

Example 1.2.1 (Brownian motion in $\mathbb{C}$)**.** Let B_1 and B_2 be two independent Brownian motions taking values in $\mathbb{R}$. We say that $Z = B_1 + iB_2$ is a Brownian motion in $\mathbb{C}$.

Some of the matrix-valued processes in this thesis have entries in $\mathbb{C}$ and thus it is useful to introduce the following result for complex Brownian motions.

Theorem 1.2.14. *Let Z be a complex Brownian motion, then its quadratic covariation and the quadratic variation with respect to its complex conjugate are given by*

$$\langle Z, Z\rangle(t) = \langle B_1 + iB_2, B_2 + iB_2\rangle(t) = \langle B_1, B_1\rangle(t) - \langle B_2, B_2\rangle(t) = 0,$$
$$\langle Z, \overline{Z}\rangle(t) = \langle B_1 + iB_2, B_2 - iB_2\rangle(t) = \langle B_1, B_1\rangle(t) + \langle B_2, B_2\rangle(t) = 2t.$$

1.2.2 Stochastic calculus for matrix-valued processes

In a similar fashion as we can generalize the stochastic calculus results for $\mathbb{R}^n$-valued processes, it is possible to extend the definitions and results to matrix-valued processes. Given a filtered probability space $(\Omega, \mathscr{F}\mathscr{F}_t, P\,())$, an $n \times m$ continuous time matrix valued process M is a function

$$\begin{aligned} M : \mathbb{R}^+ \times \Omega &\to \mathcal{M}_{m,n}(\mathbb{F}), \\ (t,\omega) &\mapsto M(t,\omega), \end{aligned}$$

where for every fixed ω^*, $M(t,\omega^*)$ is a function from $\mathbb{R}^+$ to $\mathcal{M}_{m,n}(\mathbb{F})$ and for every fixed t^*, $M(t^*)$ is a random matrix. $\mathbb{F}$ represents an arbitrary field for the entries, $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ are usual choices, but in this thesis we are only interested in matrix-valued processes with entries in $\mathbb{R}$ and $\mathbb{C}$.

Usually, we need the matrix-valued process to satisfy some symmetry condition such as being symmetric, hermitian or orthogonal. It is common then to restrict the matrix valued process to take values in a smaller subset of $\mathcal{M}_{m,n}$. Along this work, we are only interested in squared matrix-valued processes.

Example 1.2.2 (Brownian motion in $\mathcal{M}_{n,n}(\mathbb{F})$)**.** We say that a matrix-valued process $B = (B(t), t \geq 0)$ is a standard Brownian motion in $M_{n,n}(\mathbb{F})$ if every entry B_{ij} is an independent Brownian motion in the field $\mathbb{F}$.

Example 1.2.3 (Symmetric Brownian motion)**.** Let W be an $n \times n$ symmetric matrix-valued stochastic process. We say that W is a standard Brownian motion in the space of symmetric matrices if every entry W_{ij} is a real Brownian motion independent of all the other entries, except for the symmetries.

Now we show the definition of the Itô integral for matrix-valued processes.

Definition 1.2.7 (Itô integral with respect to a matrix-valued Itô process)**.** Let $W = (W(t), t \geq 0)$ be a matrix-valued Brownian motion in $\mathcal{M}_{n,m}(\mathbb{F})$ and let X and Y be two adapted matrix-valued processes in $\mathcal{M}_{p,n}(\mathbb{F})$ and $\mathcal{M}_{m,q}(\mathbb{F})$, respectively. The ij entry of the Itô integral $\int_0^t (X(s)\mathrm{d}W(s)Y(s))$ is defined as,

$$\left(\int_0^t (X(s)\mathrm{d}W(s)Y(s))\right)_{ij} = \sum_{k,l} \int_0^t X_{ik}(s)Y_{lj}(s)\mathrm{d}W_{kl}(s),$$

where $1 \leq k \leq n$, $1 \leq l \leq m$, $1 \leq i \leq p$ and $1 \leq j \leq q$.

The definition above applies also when we are integrating with respect to a Brownian motion in smaller subspace of $\mathcal{M}_{n,m}(\mathbb{F})$. An interesting property of the stochastic matrix integral is that one can integrate by the left or by the right and this operation need not to be commutative, even if it is well defined in both cases.

Just as in the $\mathbb{R}$ and $\mathbb{R}^n$ case, we can enlarge the class of process with respect we can integrate. It is convenient to define such processes only in spaces of squared matrices.

Definition 1.2.8 (Matrix-valued Itô process)**.** Let B be a Brownian motion in $\mathcal{M}_{n,n}(\mathbb{F})$ and S, R, M be adapted matrix-valued processes taking values in $\mathcal{M}_{n,n}(\mathbb{F})$. Then we say the process X satisfying

$$\mathrm{d}X(t) = S(t)\mathrm{d}B(t)R(t) + M(t)\mathrm{d}t,$$

is an Itô process in $\mathcal{M}_{n,n}(\mathbb{F})$.

In particular, if $\mathbb{F} = \mathbb{R}$ and M is symmetric, we have that Y satisfying

$$\mathrm{d}Y(t) = S(t)\mathrm{d}B(t)R(t) + R(t)\mathrm{d}B(t)^T S(t) + M(t)\mathrm{d}t,$$

is an Itô process in the space of symmetric matrices with real coefficients.

The definition of an Itô integral with respect to an Itô process in $\mathcal{M}_{n,n}(\mathbb{F})$ is a direct extension of the definition of Itô integral with respect to an Itô process in $\mathbb{R}$.

The quadratic covariation between two matrix-valued processes X, Y taking values in $\mathcal{M}_{nm}(\mathbb{F})$ and $\mathcal{M}_{mp}(\mathbb{F})$ is the matrix $\langle X, Y\rangle(t)$ with entries given by

$$\langle X, Y\rangle_{ij}(t) = \sum_{=1}^{m} \langle X_{ik}, Y_{kj}\rangle(t).$$

The same applies when we find the quadratic variation of a matrix-valued process. Now we state the Itô formula for matrix valued processes. This is taken from Trujillo Rivera (2011)

Theorem 1.2.15 (Itô Formula for matrix valued processes)**.** *Let $U \subset \mathcal{M}_{m,n}(\mathbb{R})$ an open set, X a continuous semimartingale taking values in U and $f : U \to \mathbb{R}$ twice continuously differentiable. Then $f(X)$ is a continuous semimartingale and*

$$f(X(t)) = f(X(0)) + \mathrm{Tr}\left(\int_0^t Df(X(s))^T \mathrm{d}X(s)\right) + \frac{1}{2}\sum_{j,l=1}^{n}\sum_{i,k=1}^{m}\int_0^t \frac{\partial^2}{\partial X_{ij}\partial X_{kl}} f(X(s))\mathrm{d}\langle X_{ij}, X_{kl}\rangle(s).$$

The next version of matrix-valued integration by parts formula appears in Bru (1989) and it is extensively used along the thesis.

Theorem 1.2.16 (Integration by parts for matrix-valued processes)**.** *Let X and Y be two matrix-valued semimartingales taking values in $\mathcal{M}_{nm}(\mathbb{F})$ and $\mathcal{M}_{np}(\mathbb{F})$, respectively. Then the differential of the product X^TY is*

$$\mathrm{d}(X^TY) = X^T(\mathrm{d}Y) + (\mathrm{d}X)^TY + (\mathrm{d}X)^T(\mathrm{d}Y).$$

By extending the definition of the Stratanovich integral first to multivariate processes and then to matrix-valued ones, we can see that in general, if X and Y are matrix-valued continuous semimartingales, then

$$Y^T(\partial X) = Y^T(\mathrm{d}X) + \frac{1}{2}(\mathrm{d}Y)^T(\mathrm{d}X).$$

Using this fact, we can write Theorem 1.2.16 in Stratanovich notation as

Theorem 1.2.17 (Integration by parts for matrix-valued Stratanovich integrals)**.** *Let X and Y be two matrix-valued semimartingales taking values in $\mathcal{M}_{nm}(\mathbb{F})$ and $\mathcal{M}_{np}(\mathbb{F})$, respectively. Then the differential of the product X^TY is*

$$\mathrm{d}(X^TY) = X^T(\partial Y) + (\partial X)^TY.$$

1.3 Non-commutative probability

Although random matrices are constructed based on classical definitions of random variables as measurable functions in a given probability space, they differ fundamentally from classical random variables in their basic algebraic properties. Namely, the product of random matrices does not need to commute, whereas the product of real-valued random variables always commutes. This simple fact complicates the use of classical analytical tools in probability to study random matrices. Recently, the emerging field of non-commutative probability has developed several techniques for the algebraic study of non-commutative random variables. These techniques have significantly impacted random matrices theory.

In this section we give a brief introduction to the fundamental concepts and ideas of non-commutative probability and its application to the study of random matrices. These concepts will be later used in Chapter 3.

1.3.1 Non-commutative probability spaces

A classical probability space is defined based on its analytical structure. When we work in non-commutative probability probability spaces, we are mainly concerned about the algebraic relationship between random variables. Before introducing the concept of a non-commutative probability space, we define an unital algebra.

Definition 1.3.1 (Unital algebra)**.** Let $\mathcal{A}$ be a vector space over the field $\mathbb{F}$ equipped with the additional binary operation $\cdot$. We say that $\mathcal{A}$ is a unital algebra if it satisfies the following properties for $a, b, c \in \mathcal{A}$ and $\alpha, \beta \in K$,

1. Right distributivity. $(a+b)\cdot c = a\cdot c + b\cdot c$.
2. Left distributivity. $a\cdot(b+c) = a\cdot b + a\cdot c$.
3. Compatibility with scalars. $(\alpha a)\cdot(\beta b) = (\alpha\beta)(a\cdot b)$.
4. Identity. There is an element $1_{\mathcal{A}}$ such that $a1_{\mathcal{A}} = a = 1_{\mathcal{A}}a$. We call $1_{\mathcal{A}}$ the identity element in $\mathcal{A}$.

Usually we denote $a\cdot b = ab$ to simplify notation. Now we give the definition of non-commutative probability space found in Nica and Speicher (2006).

Definition 1.3.2 (Non-commutative probability space)**.** A non-commutative probability space $(\mathcal{A},\varphi)$ consists of a unital algebra $\mathcal{A}$ over $\mathbb{C}$ and a unital linear functional φ, i.e.

$$\varphi : \mathcal{A} \to \mathbb{C}, \qquad \varphi(1_{\mathcal{A}}) = 1.$$

An element a in $\mathcal{A}$ is called a non-commutative random variable in $(\mathcal{A},\varphi)$ or simply a random variable in $(\mathcal{A},\varphi)$.

A concept related to non-commutative probability is that of $*$-probability space (star probability space). We introduce the definition of a $*$-algebra.

Definition 1.3.3 ($*$-Algebra)**.** We say that a unital algebra $\mathcal{A}$ is a $*$-algebra if it is equipped with an antilinear operation $*$, $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$ that satisfies $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$.

We call a^* the adjoint of a. If we replace "unital algebra" by "$*$-algebra" in Definition 1.3.2 we get the definition of a $*$-probability space.

There are some additional properties we can have for the functional φ in a non-commutative probability space $(\mathcal{A},\varphi)$. If $\varphi(ab) = \varphi(ba)$ for every $a, b \in \mathcal{A}$ we say that φ is tracial. If $(\mathcal{A},\varphi)$ is a $*$-probability space and $\varphi(a^*a) \geq 0$ for every $a \in \mathcal{A}$, we say that φ is positive. If $\varphi(a^*a) = 0$ only when $a = 0$, then we say that φ is faithful.

Depending on its algebraic properties, we can distinguish different kinds of non-commutative random variables.

We say that $a \in \mathcal{A}$ is self-adjoint if $a^* = a$, unitary if $a^*a = 1_{\mathcal{A}} = a^*a$ and normal if it commutes with its adjoint $a^*a = aa^*$.

The notion of a non-commutative probability space generalizes the idea of some spaces of random variables, including spaces of random matrices.

Example 1.3.1. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a classical probability space and $\mathcal{A} = L^\infty(\Omega,\mathbb{P},\mathbb{R})$ the set of bounded measurable functions from Ω taking values in $\mathbb{R}$. Equip $\mathcal{A}$ with the linear operator φ given by

$$\varphi(a) = \int_\Omega a(\omega)\mathrm{d}P(\omega) = \mathbb{E}[a],$$

for $a \in \mathcal{A}$. Then $(\mathcal{A},\varphi)$ is a non commutative probability space.

If we take the space $\mathcal{A}' = L^\infty(\Omega,\mathbb{P},\mathbb{C})$ of bounded measurable functions taking complex values, then we can have a $*$-operation given by the complex conjugate and $(\mathcal{A}',\varphi)$ is a $*$-probability space.

The hypothesis of the variables being bounded is for them to form an algebra. If we can not guarantee a^n is in $\mathcal{A}$ for every n, then $\mathcal{A}$ is not an algebra. This condition causes our non-commutative probability space not to include all of the classical random variables we can define in a classical probability space. However, we can relax hypotheses to enlarge the class of classical random variables that form an algebra.

Example 1.3.2. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a classical probability space and

$$\mathcal{A} = L^{\infty-}(\Omega,\mathbb{P},\mathbb{F}) := \bigcap_{1\leq p<\infty} L^p(\Omega,\mathbb{P},\mathbb{F})$$

with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Clearly $L^{\infty}(\Omega, \mathbb{P}, \mathbb{F}) \subset L^{\infty-}(\Omega, \mathbb{P}, \mathbb{F})$, and since we are asking every variable in $L^{\infty-}(\Omega, \mathbb{P}, \mathbb{F})$ to have all the positive moments, we can assure it is an algebra. With this, we have that the space of the classical random variables with finite moments of every order is a non-commutative probability space.

A definition of a non-commutative probability space that includes random variables which do not necessarily have finite moments of every order can be done, but it is not needed for the purposes of this work.

Example 1.3.3. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathcal{M}_{n,n}(\mathbb{F})$ be the algebra of $n \times n$ matrices with entries in $\mathbb{F}$, where the operations are $+$ the sum of matrices and $\cdot$ the usual matrix product. Denote by tr the normalized trace, i.e.

$$\operatorname{tr}(A) \coloneqq \frac{1}{n} Tr(A) = \frac{1}{n} \sum_{j=1}^{n} A_{jj},$$

for $A \in \mathcal{M}_{n,n}(\mathbb{F})$.

Define a $*$-operation by

$$(A^*)_{ij} \coloneqq \overline{A_{ji}}.$$

Then $(\mathcal{M}_{n,n}(\mathbb{F}), \operatorname{tr})$ is a $*$-probability space. In the case $\mathbb{F} = \mathbb{R}$, the $*$-operation is simply taking the transpose.

The entries of a matrix need not to be random for the space of matrices to be a non-commutative probability space, but we can indeed ask every entry to be a random variable and thus we recover the definition of a space of random variables is indeed a non-commutative probability space.

Example 1.3.4. Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathcal{A} = \mathcal{M}_{n,n}(L^{\infty-}(\Omega, \mathbb{P}, \mathbb{F}))$ the space of $n \times n$ matrices with entries in $L^{\infty-}(\Omega, \mathbb{P}, \mathbb{F})$. Equip $\mathcal{A}$ with the functional $\varphi(\cdot)$ given by

$$\varphi(A) = \mathbb{E}\left[\operatorname{tr}(A)\right] = \frac{1}{n} \sum_{j=1}^{n} \int_{\Omega} A_{jj}(\omega) \mathrm{d}\mathbb{P}(\omega).$$

Then $(\mathcal{A}, \varphi)$ is a $*$-probability space with the $*$-operation being the conjugate transpose or simply the transpose if $\mathbb{F} = \mathbb{R}$.

One of the central concepts in classical probability theory is the distribution of a random variable. In a non-commutative probability space $(\mathcal{A}, \varphi)$ we can define the distribution of a random variable a as the way in which φ acts on a, in analogy to classical probability, we define the moments of a non-commutative random variable.

Definition 1.3.4. Let a be a random variable in a $*$-probability space $(\mathcal{A}, \varphi)$. A $*$-moment of a is any expression of the form

$$\varphi(a^{e_0} \cdots a^{e_k}),$$

with $k \in \mathbb{N}$ and $e_0, \ldots, e_k \in \{1, *\}$.

Denote by $\mathbb{C}\langle x, x^* \rangle$ the unital algebra freely generated by the indeterminates x and x^*, i.e. $\mathbb{C}\langle x, x^* \rangle$ is the algebra over $\mathbb{C}$ generated by all of the monomials of the form

$$x^{e_0} \cdots x^{e_k},$$

with k and $e_1, \ldots, e_k$ as in Definition 1.3.4. Now we present the definition of a $*$-distribution.

Definition 1.3.5. Let a be a random variable in a $*$-probability space $(\mathcal{A}, \varphi)$. A $*$-distribution is a linear functional

$$\mu : \mathbb{C}\langle x, x^* \rangle \to \mathbb{C},$$

that satisfies

$$\mu(x^{e_0} \cdots x^{e_k}) = \varphi(a^{e_0} \cdots a^{e_k}).$$

with $k \in \mathbb{N}$ and $e_0, \ldots, e_k \in \{1, *\}$.

In the particular case in which a is normal, the distribution is determined by the moments of the form $\varphi(a^k(a^*)^l)$ for $k, l \geq 0$, and moreover, if a is self-adjoint it is enough to consider the moments $\varphi(a^k)$ for $k \geq 0$.

For the normal case, it is possible to link the concept of distribution in classical probability with the definition in the non-commutative case.

Definition 1.3.6. Let a be a random variable in a non-commutative probability space $(\mathcal{A}, \varphi)$. If there exists a compactly supported probability measure μ on $\mathbb{C}$ such that for every $k, l \in \mathbb{N}$,

$$\int_{\mathbb{C}} z^k \bar{z}^l \mathrm{d}\mu(z) = \varphi(a^k(a^*)^l),$$

then the distribution of a is uniquely determined by μ and we call μ the analytic distribution of a.

In the case $\mathcal{A} = L^\infty(\Omega, \mathbb{P}, \mathbb{F})$ and $\varphi(\cdot) = \mathbb{E}[\cdot]$, the analytical distribution of $a \in \mathcal{A}$ is the measure on the Borel sets of $\mathbb{F}$ induced by a, i.e.

$$\mu(B) = P\left(\{\omega \in \Omega : a(\omega) \in B\}\right),$$

for B a Borel set of $\mathbb{F}$.

Example 1.3.5 (Haar unitary random variable)**.** Let u be a random variable in a non-commutative probability space $(\mathcal{A}, \varphi)$. We say that u is a Haar unitary element if it is unitary ($u^*u = 1_{\mathcal{A}} = uu^*$) and for every $k \in \mathbb{Z} \setminus \{0\}$ we have

$$\varphi(u^k) = 0.$$

This definition generalizes the one of Haar unitary ensemble for random matrices.

1.3.2 Notions of independence

The definition of independence in classical probability assumes that the product of random variables is commutative. This property does not hold in general for non-commutative probability spaces and particularly in the case of random matrix spaces. Instead, we have four different notions of independence defined in terms of the moments.

Formally, a notion of independence is a rule that allows to compute mixed moments of the form

$$\varphi(a^{m_1} b^{n_1} \cdots a^{m_k} b^{n_k}), \qquad k \in \mathbb{N}, \quad m_i, n_j \in \{1, *\},$$

in terms of the individual moments of a and b.

The classical notion of independence corresponds to what we call tensor independence in non-commutative probability.

Definition 1.3.7 (Tensor independence)**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and I a set of indexes. A set of unital subalgebras $(\mathcal{A}_i)_{i \in I}$ is called tensor independent if

1. For every $a \in \mathcal{A}_i$ and $b \in \mathcal{A}_j$, a and b commute.

2. For all the finite subsets $J \subset I$ and all $a_j \in \mathcal{A}_j$ we can compute $\varphi\left(\prod_{j\in J} a_j\right)$ as

$$\varphi\left(\prod_{j\in J} a_j\right) = \prod_{j\in J} \varphi\left(a_j\right)$$

When we say that two random variables a, b are tensor independent, we mean that the unitary subalgebras $\mathcal{A}_1, \mathcal{A}_2$ generated by a and b, respectively are tensor independent.

Tensor independence is a symmetric relationship. If a is tensor independent of b, then b is tensor independent of a, this is not the case for every notion of independence. We can distinguish three additional notions of independence, for the purposes of this work, we introduce only the notion of "Free independence", the definition of the other notions of independence can be found in Perales Anaya (2016).

Definition 1.3.8 (Free independence)**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and I a set of indexes. A set of unital subalgebras $(\mathcal{A}_i)_{i\in I}$ is called tensor independent if $\varphi(a_1 \cdots a_k) = 0$ whenever we have that

1. $k \in \mathbb{Z}^+$,
2. $a_j \in \mathcal{A}_{i(j)}$ with $i(j) \in I$ for every $j \in [k]$,
3. $\varphi(a_j) = 0$ for every $j \in [k]$,
4. Consecutive elements in $a_1 \cdots a_k$ come from different algebras, i.e.

$$i(1) \neq i(2), i(2) \neq i(3), \ldots, i(k-1) \neq i(k).$$

When we say that two random variables a, b are freely independent, we mean that the unitary subalgebras $\mathcal{A}_1, \mathcal{A}_2$ generated by a and b, respectively are freely independent.

1.3.3 Convolution

In classical probability, when we need to compute the distribution of a sum of (tensor) independent random variables, we use a convolution. In non-commutative probability spaces, adding two independent random variables in any notion of independence gives place to a different kind of convolution.

Definition 1.3.9 (Non-commutative convolution)**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and $a, b \in \mathcal{A}$ be independent (in some notion), then the convolution (in some notion) of a and b is the algebraic distribution of $a + b$, i.e. the linear operator characterizing the moments

$$\varphi\left((a+b)^m\right),$$

with m a sequence of the form $m = e_1 e_2 \cdots e_k$, $k \in \mathbb{N}$ and $e_j \in \{1, *\}$, for all $j \in [k]$.

If a is a classical random variable and its moment generating function $\mathbb{E}\left[e^{ta}\right]$ exists, we can define $K_a(t)$ the cumulant generating function of a as

$$K_a(t) \coloneqq \log \mathbb{E}\left[e^{ta}\right].$$

This function is analytic (at least in a neighborhood of zero) and has a Taylor expansion given by

$$K_a(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \kappa_n(a),$$

with $\kappa_n(a)$ being the nth cumulant of a. This function linearizes the (tensor) convolution in the sense that if a, b are (tensor) independent, then

$$K_{a+b}(t) = K_a(t) + K_b(t).$$

This ultimately implies that the cumulants $\kappa_a(n)$ linearize the (tensor) convolution if a, b are (tensor) independent, i.e.

$$\kappa_{a+b}(n) = \kappa_a(n) + \kappa_b(n).$$

Analogous coefficients can be defined for the other notions of independence. The main tool to study them is the Cauchy-Stieltjes transform sometimes called just “Cauchy transform”.

Definition 1.3.10 (Cauchy-Stieltjes transform)**.** Let μ be a probability measure on $\mathbb{R}$, its Cauchy-Stieltjes transform, $G_\mu(t)$ is

$$G_\mu(z) \coloneqq \int_{\mathbb{R}} \frac{1}{z-t}\mu(\mathrm{d}t),$$

for $z \in \mathbb{C}^+ \coloneqq \{z \in \mathbb{C} : Im(z) > 0\}$. The Cauchy transform takes values in $\mathbb{C}^- \coloneqq \{z \in \mathbb{C} : Im(z) > 0\}$.

When μ is characterized by its moments $m_k(\mu)$, there is a relationship between them and the Cauchy-Stieltjes transform μ given by

$$\begin{aligned} G_\mu(z) &= z^{-1}\int_{\mathbb{R}} \frac{1}{1-\frac{t}{z}}\mu(\mathrm{d}t) = z^{-1}\int_{\mathbb{R}} \sum_{k=0}^{\infty} \left(\frac{t}{z}\right)^k \mu(\mathrm{d}t), \\ &= \sum_{k=0}^{\infty} z^{-(k+1)} \int_{\mathbb{R}} t^k \mu(\mathrm{d}t) = \sum_{k=0}^{\infty} \frac{m_k(\mu)}{z^{k+1}}, \end{aligned}$$

with $m_k(\mu)$ being the kth moment of μ and for $|z| \geq \sup\{t : t \in \mathrm{supp}(\mu)\}$.

The collection of functions $f_z(t) = 1/(z-t)$ parametrized by $z \in \mathbb{C}^+$ forms a separating test family for the space of probability measures. This implies that the Cauchy-Stieltjes transform characterizes uniquely the probability measure. Due to the relevance of this proposition we state it in the next Theorem.

Theorem 1.3.1. *Given μ and ν two probability measures on $\mathbb{C}$ with Cauchy-Stieltjes transforms G_μ and G_ν respectively. Then*

$$\mu = \nu \Leftrightarrow G_\mu = G_\nu.$$

Now we compute a few examples of Cauchy-Stieltjes transforms

ejemplos

Example 1.3.6.

If μ is absolutely continuous with respect to the Lebesgue measure and has density f_μ, then it is possible to recover f_μ from the Cauchy transform G_μ by using the Stieltjes inversion formula.

Theorem 1.3.2. *Let μ be an absolutely continuous probability measure in $\mathbb{R}$ with density h_μ and Cauchy transform G_μ. Then*

$$f_\mu(x) = \lim_{y \to 0} -\frac{1}{\pi} Im[G_\mu(x+iy)].$$

Using the Cauchy-Stieltjes transform it is possible to define other transforms that linearize convolution in different notions of independence. We include here the ones that are useful for this work

Definition 1.3.11 (Non-commutative linearizing transforms)**.** Let μ be a probability measure on $\mathbb{R}$ with Cauchy-Stieltjes transform G_μ. We can define the following transforms for μ.

1. The $\mathcal{K}$ transform $\mathcal{K}_\mu$ is the compositional inverse of G_μ,

$$\mathcal{K}_\mu(z) := G_\mu^{-1}(z).$$

2. The $\mathcal{R}$ transform $\mathcal{R}_\mu$ is defined in terms of the $\mathcal{K}$ transform

$$\mathcal{R}_\mu(z) := \mathcal{K}_\mu(z) - \frac{1}{z}.$$

3. The S transform S_μ is defined in terms of the inverse under composition of the $\mathcal{R}$ transform,

$$S_\mu(z) := \frac{1}{z}\mathcal{R}_\mu^{-1}(z).$$

Let a and b be non-commutative random variables with real support and analytic distributions μ and ν, respectively. The distribution of the sum $a + b$ is said to be the convolution in a notion of independence if a and b satisfy a notion of independence.

1. If a and b are tensor independent, the distribution of $a+b$ is the classical (or tensor) convolution of μ and ν denoted by $\mu * \nu$.

2. If a and b are freely independent, the distribution of $a + b$ is the free convolution of μ and ν denoted by $\mu \boxplus \nu$.

We say that μ and ν satisfy an independence relationship if the associated random variables satisfy it.

The relationship between the transforms defined above and the convolution of non-commutative random variables is stated in the following theorem.

Theorem 1.3.3. *Let μ, ν be two compactly supported probability measures on $\mathbb{R}$. If they are freely independent, then*

$$\mathcal{R}_{\mu\boxplus\nu}(z) = \mathcal{R}_\mu(z) + \mathcal{R}_\nu(z).$$

The fact that the notion of non-commutative convolution generalizes the sum of independent random variables allows us to get results that are similar to the Central Limit Theorem. In the next section we state this result for the classical and free independence cases.

1.3.4 Classical and free central limit theorems

Each notion of independence and convolution gives in turn a new limit for sums of the form

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}},$$

for X_i independent (in some sense) and identically distributed random variables with mean 0 and variance 1.

In classical probability, the variables are tensor independent and the limit is the Gaussian distribution. The next Theorem gives us the generalization for the case of free independence. A proof can be found in Nica and Speicher (2006).

Theorem 1.3.4. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and $(a_i)_{i\in\mathbb{N}}$ be a sequence of random variables in $\mathcal{A}$ with common distribution and that have zero mean ($\varphi(a_1) = 0$) and variance 1 ($\varphi(a_1^2) = 1$).*

Denote by $\xrightarrow{D}$ the convergence in distribution and define S_n as

$$S_n := \frac{\sum_{j=1}^n a_j}{\sqrt{n}}.$$

1. *If the random variables are tensor independent, then*

$$S_n \xrightarrow[n\to\infty]{D} N,$$

with N a standard normal random variable.

2. *If the random variables are freely independent, then*

$$S_n \xrightarrow[n\to\infty]{D} s,$$

with s a standard semicircular random variable.

Chapter 2

Eigenvalue processes for matrix-valued processes

In this chapter we prove the form of the dynamic equation for the Dyson Brownian motion in the real and complex cases. We then show an extension of the result to matrix-valued diffusion processes. Most of the material in the chapter is taken from Graczyk and Małecki (2011) which in turn uses techniques appearing in Bru (1989) for the study of Wishart processes. In the first section, the essential tools are introduced and then the result is proven for the Dyson Brownian motion. In the second section, the results are generalized for matrix-valued Itô processes and the particular cases of the Wishart and Jacobi processes are given.

At the end of the chapter we will prove that if $B = (B(t), t \geq 0)$ is a Brownian motion in $\mathcal{M}_{n,n}(\mathbb{R})$ and X is a matrix-valued process satisfying the following system of stochastic differential equations

$$\mathrm{d}X(t) = g(X(t))\mathrm{d}B(t)h(X(t)) + h(X(t))\mathrm{d}B(t)^T g(X(t)) + b(X(t))\mathrm{d}t, \tag{2.1}$$

then its eigenvalues are semimartingales that satisfy the system of stochastic differential equations given by

$$\mathrm{d}\lambda_i = 2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i + \left(b(\lambda_i) + \sum_{k\neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k}\right)\mathrm{d}t, \tag{2.2}$$

until the time of first collision, which means whenever $\lambda_i = \lambda_j$ for any $i \neq j$. We will show, however that the first collision is impossible in finite time a. s. The analogous result holds when B is a Brownian motion in $\mathcal{M}_{n,n}(\mathbb{C})$ and we take $B(t)^*$ instead of $B(t)^T$ in (2.1).

In order to illustrate the techniques and introduce the most basic process of this kind, we will first prove the particular cases of the real and complex Dyson Brownian motion, which are simply the eigenvalue processes of Brownian motion in $\mathcal{H}_{n,n}(\mathbb{R})$ and $\mathcal{H}_{n,n}(\mathbb{C})$, respectively. In both cases, the processes are particular cases of (2.2) with $b \equiv 0$ and $g \equiv h \equiv 1/\sqrt[4]{2}$. Notice that this implies that X is a process in $\mathcal{H}_{n,n}(\mathbb{R})$ ($\mathcal{H}_{n,n}(\mathbb{C})$) whose off-diagonal entries are standard real-valued (complex-valued) Brownian motions and the diagonal entries would are real-valued (complex-valued) Brownian motions with variance 2. In the later case (2.2) would turn into

$$\mathrm{d}\lambda_i = \sqrt{2}\mathrm{d}W_i + \sum_{k\neq i} \frac{\mathrm{d}t}{\lambda_i - \lambda_k},$$

which exactly the system of stochastic differential equations we expect to have for the Dyson Brownian motion.

2.1 Dyson Brownian Motion

This section is mainly used to show the results for the Dyson Brownian motion, but some of the general results will be used also to prove the analogous equations for the spectrum of more general matrix diffusions in later sections. In the first subsection we derive the stochastic differential equations for the eigenvalues of a symmetric real Brownian matrix up to the collison time. In the second subsection we do the same for a self adjoint complex Brownian matrix. In the third section we prove that the collision time is almost surely infinite. No proof of the existence and uniqueness of a solution for the equations is given in this section, but it is later given in a more general case in section 2.2.

The first description of the Dyson Brownian motion was given in Dyson (2013) as a model for a Couloumb gass executing each one a Brownian motion and the repuslive forces between them. In generality, the Dyson Brownian motion is the process that models the spectrum of a Brownian motion in $\mathcal{H}_{n,n}(\mathbb{F})$ and it is described by the system of stochastic differential equations

$$\mathrm{d}\lambda_i = \sqrt{\frac{2}{\beta}}\mathrm{d}W_i + \sum_{k\neq i} \frac{\mathrm{d}t}{\lambda_i - \lambda_k},$$

where the $W_i, 1 \leq i \leq n$ are independent standard Brownian motions and β parametrizes the field in which the entries of the matrix take values. If $\beta = 1$, then $\mathbb{F} = \mathbb{R}$, if $\beta = 2$, then $\mathbb{F} = \mathbb{C}$ and if $\beta = 4$, $\mathbb{F} = \mathbb{H}$. Other matrix models for different values of β have been proposed in Allez and Guionnet (2013) and Holcomb and Paquette (2017). In the first one for $\beta \in [0, 2]$ and for $\beta \in (0, \infty)$ in the second one.

In this section, we only prove the result for $\beta \in \{1, 2\}$ and we will notice that the difference is essentially the covariance between entries of the matrix, so the extension to $\beta = 4$ would be quite similar.

2.1.1 Real case

The properties of the one dimensional Brownian motion allow to easily extend the definition to Brownian motions in different spaces, such as $\mathbb{R}^n$ or $\mathcal{M}_{n,n}(\mathscr{F})$. In the particular case that B is a Brownian motion in the space of symmetric matrices with real entries, for every t, the process is a multiple of a GOE, i.e. $B(t) = \sqrt{t}R$ with R a GOE. This means the law of B keeps invariant under orthogonal transformations and this property is essential for the results in this case. For the sake of clearness, we give a precise definition of a Brownian motion in the space of symmetric matrices.

Definition 2.1.1 (Brownian motion in $\mathcal{H}_{n,n}(\mathbb{R})$)**.** Let $B = (B(t), t \geq 0)$ be a stochastic process taking values in $\mathcal{M}_{n,n}(\mathbb{R})$ whose entries are standard Brownian motions $\{B_{ij}(t)\}_{1\leq i\leq n, 1\leq j\leq n}$ such that,

$$\mathrm{d}\langle B_{ij}, B_{kl}\rangle(t) = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,\mathrm{d}t.$$

Then we say that B is a Brownian motion in $\mathcal{H}_{n,n}(\mathbb{R})$.

A process B as defined above clearly has real eigenvalues, so we can order them. Let $\lambda_1(t) > \lambda_2(t) > \cdots > \lambda_n(t)$ be the eigenvalues. Notice they are also time-dependent functions, we are interested in knowing if for some t_0 the order of some of them is changed. Due to the continuity of the paths, this happens only if at some point the eigenvalues changing the order are equal. The next stopping time gives us the first time of collision of the eigenvalues.

Definition 2.1.2 (First time of collision)**.** Let $\lambda_1(t) > \lambda_2(t) > \cdots > \lambda_n(t)$ be the ordered eigenvalues of a matrix-valued stochastic process. We define the first collision time τ as

$$\tau := \inf\{t : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j\}. \tag{2.3}$$

In the following Theorem, we derive a stochastic differential equation for the behaviour of a Brownian symmetric matrix's spectrum on $[0, \tau)$.

Theorem 2.1.1. *Let $B = (B(t), t \geq 0)$ be a symmetric $n \times n$ matrix-valued Brownian motion in $\mathcal{M}_{n,n}(\mathbb{R})$ with diagonalization $B = H\Lambda H^T$ and eigenvalues $\lambda_1, \ldots, \lambda_n$. Define τ the first time of collision of the eigenvalues as in* (2.3).

Then, for $t < \tau$ the eigenvalue process $\Lambda = (\Lambda(t), t \geq 0)$ verifies the following stochastic differential equations:

$$\mathrm{d}\lambda_i = \sqrt{2}\mathrm{d}W_i + \sum_{k \neq i} \frac{\mathrm{d}t}{\lambda_i - \lambda_k} \tag{2.4}$$

where $(W_i)_{i\in[n]}$ are independent Brownian motions on $\mathbb{R}$.

Proof. For a fixed $t \geq 0$ the matrix $B(t)$ is equal in law to $\sqrt{t}R$ with R a Gaussian orthogonal ensemble. Using that R is invariant under orthogonal transformations this implies that for every orthogonal matrix O, we have

$$OB(t)O^T \overset{d}{=} O\sqrt{t}RO^T \overset{d}{=} \sqrt{t}ORO^T \overset{d}{=} \sqrt{t}R \overset{d}{=} B(t).$$

In particular, $B(t)$ is also invariant under orthogonal transformations.

We have that $H^{-1} = H^T$, its stochastic logarithm L is defined by a stochastic differential equation as

$$\mathrm{d}L \coloneqq H^{-1}\partial H = H^T\partial H = H^T\mathrm{d}H + \frac{1}{2}(\mathrm{d}H^T)\mathrm{d}H.$$

We use the matrix Itô formula on $I = H^T H$,

$$0 = \mathrm{d}I = \mathrm{d}(H^T H) = H^T\mathrm{d}H + (\mathrm{d}H)^T H + (\mathrm{d}H)^T\mathrm{d}H = H^T\partial H + (\partial H)^T H = \mathrm{d}L + \mathrm{d}L^T.$$

This implies that $\mathrm{d}L^T = -\mathrm{d}L$ and the stochastic logarithm of H is skew-symmetric. Now we use that $\Lambda = H^T BH$ and the matrix Itô formula again to get

$$\begin{aligned}
\mathrm{d}\Lambda = \mathrm{d}(H^T BH) &= (\partial H^T B)H + H^T B\partial H = (\partial H)^T BH + H^T(\partial B)H + H^T B\partial H, \\
&= (\partial H)^T H\Lambda + H^T(\partial B)H + \lambda H^T\partial H = (\partial L)^T\Lambda + H^T(\partial B)H + \Lambda\partial L, \\
&= H^T(\partial B)H - (\partial L)\Lambda + \Lambda\partial L.
\end{aligned}$$

The diagonals of $(\partial L)\Lambda$ and $\Lambda\partial L$ coincide, so $\mathrm{d}\Lambda_{ii} = \big(H^T(\partial B)H\big)_{ii}$. Let $\mathrm{d}N \coloneqq H^T(\partial B)H,$. Now, for $i \neq j$, we use that Λ is diagonal to get

$$0 = \mathrm{d}N_{ij} + (\lambda_i - \lambda_j)\mathrm{d}L_{ij}.$$

We can then conclude that $\mathrm{d}L_{ij} = \mathrm{d}N_{ij}/(\lambda_j - \lambda_i)$ whenever $i \neq j$. Now we need to find a more explicit representation for $\mathrm{d}N$. We see that $\mathrm{d}N$ and $H^T(\mathrm{d}B)H$ differ only in a finite variation part, so the martingale term must coincide and it is characterized by the quadratic covariation that we can find using the covariation of B.

$$\begin{aligned}
\mathrm{d}N_{ij}\mathrm{d}N_{kl} &= \mathrm{d}\langle (H^T(\partial B)H)_{ij}, (H^T(\partial B)H)_{kl}\rangle(t) = \mathrm{d}\langle (H^T(\mathrm{d}B)H)_{ij}, (H^T(\mathrm{d}B)H)_{kl}\rangle(t), \\
&= \sum_{p,q,r,s} \mathrm{d}\langle H_{pi}\mathrm{d}B_{pq}H_{qj}, H_{rk}\mathrm{d}B_{rs}H_{sl}\rangle(t) = \sum_{p,q,r,s} H_{pi}H_{qj}H_{rk}H_{sl}\mathrm{d}B_{pq}\mathrm{d}B_{rs},
\end{aligned}$$

Recall that $\mathrm{d}\langle B_{ij}, B_{kl}\rangle(t) = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,\mathrm{d}t$,

$$
\begin{aligned}
&= \sum_{p,q,r,s} H_{pi}H_{qj}H_{rk}H_{sl}(\delta_{rp}\delta_{sq} + \delta_{rq}\delta_{sp})\mathrm{d}t, \\
&= \left(\sum_p H^T_{ip}\delta_{rp}H_{rk}\right)\left(\sum_q H^T_{jq}\delta_{qs}H_{sl}\right)\mathrm{d}t + \left(\sum_p H^T_{ip}\delta_{sp}H_{sl}\right)\left(\sum_q H^T_{jq}\delta_{rq}H_{rk}\right)\mathrm{d}t \\
&= (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mathrm{d}t.
\end{aligned}
$$

We had previously got that $\mathrm{d}\Lambda_{ii} = \mathrm{d}N_{ii}$, and thus using the covariation of N we can find the martingale term of λ_i,

$$
\mathrm{d}\lambda_i\mathrm{d}\lambda_j = \mathrm{d}\Lambda_{ii}\mathrm{d}\Lambda_{jj} = \mathrm{d}N_{ii}\mathrm{d}N_{jj} = 2\delta_{ij}\mathrm{d}t.
$$

Then the martingale term of every eigenvalue is $\sqrt{2}$ times a Brownian motion that is independent of the martingale term of any other eigenvalue. Now we need to find the finite variation part of λ_i. Let us call F the finite variation part of N, then

$$
\begin{aligned}
\mathrm{d}F &= \frac{1}{2}\left(\mathrm{d}H^T\mathrm{d}BH + H^T\mathrm{d}B\mathrm{d}H\right) = \frac{1}{2}\left((\mathrm{d}H^TH)(H^T\mathrm{d}BH) + (H^T\mathrm{d}Bh)(H^T\mathrm{d}H)\right), \\
&= \frac{1}{2}\left((\mathrm{d}N\mathrm{d}L)^T + \mathrm{d}N\mathrm{d}L\right).
\end{aligned}
$$

The last equality is because $\mathrm{d}N$ and $H^T(\mathrm{d}B)H$ only differ in a finite variation term. Using the previous results we find

$$
(\mathrm{d}N\mathrm{d}L)_{ij} = \sum_k \mathrm{d}N_{ik}\mathrm{d}L_{kj} = \sum_{k\neq j}\frac{\mathrm{d}N_{ik}\mathrm{d}N_{kj}}{\lambda_j - \lambda_k} = \delta_{ij}\sum_{k\neq j}\frac{\mathrm{d}t}{\lambda_j - \lambda_k}.
$$

We can conclude that

$$
\mathrm{d}F_{ii} = \frac{1}{2}\left((\mathrm{d}N\mathrm{d}L)^T_{ii} + (\mathrm{d}N\mathrm{d}L)_{ii}\right) = (\mathrm{d}N\mathrm{d}L)_{ii} = \sum_{k\neq i}\frac{\mathrm{d}t}{\lambda_i - \lambda_k}.
$$

Now we know the martingale and finite variation terms of λ_i and we can write the explicit expression for it.

$$
\mathrm{d}\lambda_i = \sqrt{2}\mathrm{d}W_i + \sum_{k\neq i}\frac{\mathrm{d}t}{\lambda_i - \lambda_k}
$$

where $W_1, \ldots, W_n$ are independent standard Brownian motions. □

2.1.2 Complex case

Before proceeeding with the result, we define a Brownian motion in $\mathcal{H}_{n,n}(\mathbb{C})$, which is totally analogous to the Brownian motion in $\mathcal{H}_{n,n}(\mathbb{R})$.

Definition 2.1.3 (Brownian motion in $\mathcal{H}_{n,n}(\mathbb{C})$)**.** Let $B = (B(t), t \geq 0)$ be a stochastic process taking values in $\mathcal{M}_{n,n}(\mathbb{C})$ whose off-diagonal entries are complex Brownian motions $\{B_{ij}(t)\}_{1\leq i\leq n, 1\leq j\leq n}$ such that,

$$
\mathrm{d}\langle B_{ij}, B_{kl}\rangle(t) = 2\delta_{ik}\delta_{jl}\mathrm{d}t, \tag{2.5}
$$

and the diagonal entries are n independient real-valued Brownian motions with variance 2, which means

$$\mathrm{d}\langle B_{ii}, B_{jj}\rangle(t) = 2\delta_{ij}\mathrm{d}t. \tag{2.6}$$

Then we say that B is a Brownian motion in $\mathcal{H}_{n,n}(\mathbb{C})$.

Notice that (2.5) implies that $B_{ij}(t) = \overline{B_{ji}}$ and together with (2.6) this means that B is effectively a process taking values in $\mathcal{H}_{n,n}(\mathbb{C})$.

For the case of a Brownian motion in $\mathcal{H}_{n,n}(\mathbb{C})$ there are basically two equivalent formulations of the Dyson Brownian motion. If we take the eigenvalue process defined as in 2.1.3, then the eigenvalues satisfy the system of SDE

$$\mathrm{d}\lambda_i = \mathrm{d}W_i + 2\sum_{k\neq i}\frac{\mathrm{d}t}{\lambda_i - \lambda_k},$$

if instead we consider the process $B' = \sqrt{\frac{1}{2}}B$, then the eigenvalues obey the system

$$\mathrm{d}\lambda_i = \mathrm{d}W_i + \sum_{k\neq i}\frac{\mathrm{d}t}{\lambda_i - \lambda_k}.$$

We are more interested in the latter re-scaled process since it generalizes the $\mathcal{H}_{n,n}(\mathbb{R})$ case in the following sense. If we take $\beta \in \{1, 2\}$, and study the eigenvalues of the process $\sqrt{\frac{1}{\beta}}B(t)$ with B a Brownian motion in $\mathcal{H}_{n,n}(\mathbb{R})$ (resp. $\mathcal{H}_{n,n}(\mathbb{C})$), then before the time of first collision they satisfy

$$\mathrm{d}\lambda_i = \sqrt{\frac{2}{\beta}}\mathrm{d}W_i + \sum_{k\neq i}\frac{\mathrm{d}t}{\lambda_i - \lambda_k}. \tag{2.7}$$

Although we do not prove it here, it is a well known fact that equation (2.7) holds also in the case $\beta = 4$ which is for a self-adjoint random matrix whose entries are quaternionic Brownian motions Dyson (2013).

Theorem 2.1.2. *Let $B' = (B(t), t \geq 0)$ be a matrix-valued Brownian motion in $\mathcal{H}_{n,n}(\mathbb{C})$ and $B \coloneqq \frac{1}{2}B'$ have diagonalization $B = H\Lambda H^*$ and eigenvalues $\lambda_1, \dots, \lambda_n$. Define τ the first time of collision of the eigenvalues as in* (2.3)

Then, for $t < \tau$ the eigenvalue process $\Lambda = (\Lambda(t), t \geq 0)$ verifies the following system of stochastic differential equations:

$$\mathrm{d}\lambda_i = \mathrm{d}W_i + \sum_{k\neq i}\frac{\mathrm{d}t}{\lambda_i - \lambda_k}, \tag{2.8}$$

where $(W_i)_{i\in[n]}$ are independent Brownian motions.

Proof. The proof is the same as the real case, but in this case, the entries are complex Brownian motions outside the diagonal and real Brownian motions in the diagonal. Recalling the covariation for a Brownian motion in $\mathcal{H}_{n,n}(\mathbb{C})$ and the re-scaling, we have

$$\mathrm{d}\langle B_{ij}, B_{kl}\rangle(t) = \frac{1}{2}\mathrm{d}\langle B'_{ij}, B'_{kl}\rangle = \delta_{il}\delta_{jk}\mathrm{d}t.$$

Similarly to the real case, for a fixed $t \geq 0$ the matrix $B(t)$ is equal in law to $\sqrt{t}R$ with R a Gaussian unitary ensemble. Using that R is invariant under unitary transformations this implies that for every unitary matrix U, we have

$$UB(t)U^* \overset{d}{=} U\sqrt{t}RU^* \overset{d}{=} \sqrt{t}URU^* \overset{d}{=} \sqrt{t}R \overset{d}{=} B(t).$$

We have that $H^{-1} = H^*$, so repeating the procedure in the real case we define equally the stochastic logarithm as $\mathrm{d}L \coloneqq H^*\partial H$ and use Itô's formula for $I = H^*H$,

$$0 = \mathrm{d}I = \mathrm{d}(H^*H) = H^*\mathrm{d}H + (\mathrm{d}H)^*H + (\mathrm{d}H)^*\mathrm{d}H = H^*\partial H + (\partial H)^*H = \mathrm{d}L + \mathrm{d}L^*.$$

We have that $\mathrm{d}L^* = -\mathrm{d}L$ and L is skew-hermitian. Now we use Itô fomrula for $\Lambda = H^*BH$,

$$\begin{aligned}
\mathrm{d}\Lambda = \mathrm{d}(H^*BH) &= (\partial H^*B)H + H^*B\partial H = (\partial H)^*BH + H^*(\partial B)H + H^*B\partial H, \\
&= (\partial H)^*H\Lambda + H^*(\partial B)H + \Lambda H^*\partial H = (\mathrm{d}L)^*\Lambda + H^*(\partial B)H + \Lambda\mathrm{d}L, \\
&= H^*(\partial B)H - \mathrm{d}L\Lambda + \Lambda\mathrm{d}L.
\end{aligned}$$

The processes $\mathrm{d}L\Lambda$ and $\Lambda\mathrm{d}L$ have the same diagonal entries, so the diagonal of Λ coincides with the one of $H^*(\partial B)H$. Define $\mathrm{d}N \coloneqq H^*(\partial B)H$. Outside the diagonal, Λ has zero entries, so we can equate this to the corresponding entries of $\mathrm{d}N$ and $\mathrm{d}L\Lambda, \Lambda\mathrm{d}L$ to get for every $i \neq j$,

$$0 = \mathrm{d}N_{ij} + (\lambda_i - \lambda_j)\mathrm{d}L_{ij},$$

which in turn implies

$$\mathrm{d}L_{ij} = \frac{\mathrm{d}N_{ij}}{\lambda_j - \lambda_i}, \qquad i \neq j.$$

Now, repeating the real case, we find the quadratic covariation of $\mathrm{d}N$ using that $\mathrm{d}N$ and $H^*\mathrm{d}BH$ coincide up to a finite variation term, but using $\mathrm{d}\langle B_{ij}, \overline{B_{kl}}\rangle(t) = 2\delta_{il}\delta_{jk}\mathrm{d}t$.

$$\begin{aligned}
\mathrm{d}N_{ij}\mathrm{d}N_{kl} &= \mathrm{d}\langle (H^*(\partial B)H)_{ij}, (H^*(\partial B)H)_{kl}\rangle(t) = \mathrm{d}\langle (H^*(\mathrm{d}B)H)_{ij}, (H^*(\mathrm{d}B)H)_{kl}\rangle(t), \\
&= \sum_{p,q,r,s} \mathrm{d}\langle H^*_{ip}\mathrm{d}B_{pq}H_{qj}, H^*_{kr}\mathrm{d}B_{rs}H_{sl}\rangle(t) = \sum_{p,q,r,s} H^*_{ip}H_{qj}H^*_{kr}H_{sl}\mathrm{d}B_{pq}\mathrm{d}B_{rs}, \\
&= \sum_{p,q,r,s} H^*_{ip}H_{qj}H^*_{kr}H_{sl}\delta_{rq}\delta_{sp}\mathrm{d}t = \left(\sum_p H^*_{ip}\delta_{sp}H_{sl}\right)\left(\sum_q H^*_{kr}\delta_{rq}H_{qj}\right)\mathrm{d}t, \\
&= \delta_{il}\delta_{kj}\mathrm{d}t.
\end{aligned}$$

We can use this and the previous result that $\mathrm{d}\Lambda$ and $\mathrm{d}N$ coincide in the diagonal to find the covariation of the eigenvalues and this way we find their martingale term

$$\mathrm{d}\lambda_i\mathrm{d}\lambda_j = \mathrm{d}\Lambda_{ii}\mathrm{d}\Lambda_{jj} = \mathrm{d}N_{ii}\mathrm{d}N_{jj} = \delta_{ij}\mathrm{d}t.$$

Just as in the real case, the martingale term of every eigenvalue is a (real) Brownian motion independent of any other eigenvalue. Again, call F the finite variation part of N and use that $\mathrm{d}N$ and $H^T(\mathrm{d}B)H$ only differ in a finite variation term to find,

$$\begin{aligned}
\mathrm{d}F &= \frac{1}{2}\left(\mathrm{d}H^*\mathrm{d}BH + H^*\mathrm{d}B\mathrm{d}H\right) = \frac{1}{2}\left((\mathrm{d}H^*H)(H^*\mathrm{d}BH) + (H^*\mathrm{d}BH)(H^*\mathrm{d}H)\right), \\
&= \frac{1}{2}\left((\mathrm{d}N\mathrm{d}L)^* + \mathrm{d}N\mathrm{d}L\right).
\end{aligned}$$

We have an expression for L in terms of N. We recall it to find the covariation $\mathrm{d}\langle N, L\rangle(t)$,

$$(\mathrm{d}N\mathrm{d}L)_{ij} = \sum_k \mathrm{d}N_{ik}\mathrm{d}L_{kj} = \sum_{k\neq j} \frac{\mathrm{d}N_{ik}\mathrm{d}N_{kj}}{\lambda_j - \lambda_k} = \delta_{ij}\sum_{k\neq j}\frac{\mathrm{d}t}{\lambda_j - \lambda_k}.$$

Thus F is a diagonal matrix, and the ith diagonal term is given by

$$\mathrm{d}F_{ii} = \frac{1}{2}\left((\mathrm{d}N\mathrm{d}L)^*_{ii} + (\mathrm{d}N\mathrm{d}L)_{ii}\right) = (\mathrm{d}N\mathrm{d}L)_{ii} = \sum_{k\neq i} \frac{\mathrm{d}t}{\lambda_i - \lambda_k}.$$

With the martingale and finite variation terms of λ_i, we conclude the stated result.

$$\mathrm{d}\lambda_i = \mathrm{d}W_i + \sum_{k\neq i} \frac{\mathrm{d}t}{\lambda_i - \lambda_k}$$

where $W_1, \ldots, W_n$ are independent standard Brownian motions. □

2.1.3 Non-collision of the eigenvalues

We have found that the eigenvalues of a Brownian motion in $\mathcal{M}_{n,n}(\mathbb{R})$ and $\mathcal{M}_{n,n}(\mathbb{C})$ satisfy Dyson's equation until the first time of collision. Now we prove that this time τ is infinite a.s.

The proof we give is taken from Graczyk and Małecki (2011) and makes use of the so called McKean's argument appearing first in McKean (1969) and used in Bru (1989) and Trujillo Rivera (2011). The result is rather general because it is used for the generalization to matrix-valued diffusion processes.

Theorem 2.1.3. *Let $\Lambda = (\lambda_i)_{i=1,\ldots,p}$ be a process starting at the open Weyl chamber Δ_p and satisfying* (2.13) *with functions $g, h, b : \mathbb{R} \to \mathbb{R}$ such that g^2, h^2, b are Lipschitz continuous and g^2h^2 is convex or is continuously differentiable with derivative uniformly Lipschitz on $\mathbb{R}$. Then the first collision time τ defined as in* (2.12) *is infinite a.s.*

Proof. Define $U \coloneqq -\sum_{i<j} \log(\lambda_j - \lambda_i)$ for $t \in [0, \tau]$. By Itô's formula and the fact that $\mathrm{d}\lambda_i \mathrm{d}\lambda_j = 4\delta_{ij} g^2(\lambda_i) h^2(\lambda_i)$ we find

$$\begin{aligned}
\mathrm{d}U &= \sum_{i<j} \left[\frac{\mathrm{d}\lambda_i - \mathrm{d}\lambda_j}{\lambda_j - \lambda_i} + \frac{1}{2}\frac{\mathrm{d}\langle \lambda_i, \lambda_i\rangle + \mathrm{d}\langle \lambda_j, \lambda_j\rangle}{(\lambda_j - \lambda_i)^2}\right], \\
&= \sum_{i<j} \left[\frac{\mathrm{d}\lambda_i - \mathrm{d}\lambda_j}{\lambda_j - \lambda_i} + 2\frac{g^2(\lambda_i)h^2(\lambda_i) - g^2(\lambda_j)h^2(\lambda_j)}{(\lambda_j - \lambda_i)^2}\mathrm{d}t\right].
\end{aligned}$$

Now we define the following processes

$$\begin{aligned}
\mathrm{d}M &= 2\sum_{i<j} \frac{g(\lambda_i)h(\lambda_i)\mathrm{d}\nu_i - g(\lambda_)h(\lambda_j)\mathrm{d}\nu_j}{\lambda_j - \lambda_i}, \\
\mathrm{d}A_1 &= \sum_{i<j} \frac{b(\lambda_i) - b(\lambda_j)}{\lambda_j - \lambda_i}\mathrm{d}t, \\
\mathrm{d}A_2 &= 2\sum_{i<j} \frac{\left(g^2(\lambda_j) - g^2(\lambda_i)\right)\left(h^2(\lambda_j) - h^2(\lambda_i)\right)}{(\lambda_j - \lambda_i)}\mathrm{d}t, \\
\mathrm{d}A_3 &= \sum_{i<j} \frac{1}{\lambda_j - \lambda_i} \sum_{k\neq i, k\neq j} \left(\frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} - \frac{G(\lambda_j, \lambda_k)}{\lambda_j - \lambda_k}\right)\mathrm{d}t, \\
&= \sum_{i<j<k} \frac{G(\lambda_j, \lambda_k)(\lambda_k - \lambda_j) - G(\lambda_i, \lambda_k)(\lambda_k - \lambda_i) + G(\lambda_i, \lambda_j)(\lambda_j - \lambda_i)}{(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}\mathrm{d}t.
\end{aligned}$$

Using (2.13) we find that $\mathrm{d}U = \mathrm{d}M + \mathrm{d}A_1 + \mathrm{d}A_2 + \mathrm{d}A_3$. Our goal is to use the McKean argument by proving that U is bounded over any bounded interval $[0, t]$. Let us start by showing that the finite variation part of U ($\mathrm{d}A_1, \mathrm{d}A_2$, and $\mathrm{d}A_3$) is bounded. Lipschitz continuity of b, g^2 and h^2 implies that

$|A_1(t)| \leq Kp(p-1)t/2$ and $|A_2(t)| \leq K^2p(p-1)t$ with K a constant appearing in the Lipschitz condition. Later, we define a function H as

$$H(x, yz) \coloneqq \left[(g^2(x) - g^2(z))(h^2(y) - h^2(z)) + (g^2(y) - g^2(z))(h^2(x) - h^2(z))\right](y - x),$$

then

$$H(x, y, z) = (G(x, y) - G(x, z) - G(y, z) + G(z, z))\,(y - x),$$

and

$$\begin{aligned} H(x, y, z) + H(y, z, x) - H(x, z, y) &= 2(z - y)G(y, z) - 2(z - x)G(x, z) + 2(y - x)G(x, y) \\ &\quad + G(x, x)(z - y) - G(y, y)(z - x) + G(z, z)(y - x). \end{aligned}$$

By the Lipschitz conditions on g^2 and h^2 we find that $|H(x, y, z)| \leq 2K^2|(y - x)(z - y)(z - x)|$.

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Using the last equality, we can write $2\mathrm{d}A_3 = \mathrm{d}A_4 + \mathrm{d}A_5$, with $0 \leq A_4(t) \leq K^2p(p-1)(p-2)t/6$ and

$$\mathrm{d}A_5(t) = \sum_{i<j<k} \frac{G(\lambda_j, \lambda_j)(\lambda_k - \lambda_i) - G(\lambda_i, \lambda_i)(\lambda_k - \lambda_j) - G(\lambda_k, \lambda_k)(\lambda_j - \lambda_i)}{(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}\mathrm{d}t, \tag{2.9}$$

$$= \sum_{i<j<k} \left(\frac{G(\lambda_j, \lambda_j) - G(\lambda_i, \lambda_i)}{\lambda_j - \lambda_i} - \frac{G(\lambda_k, \lambda_k) - G(\lambda_j, \lambda_j)}{\lambda_k - \lambda_j}\right) \frac{1}{\lambda_k - \lambda_i}\mathrm{d}t. \tag{2.10}$$

If $G(x, x)$ is convex, then A_5 is non positive. If $G(x, x)$ is continuously differentiable with derivative uniformly Lipschitz, then

$$|G'(x, x) - G'(y, y)| \leq C|x - y|,$$

and the (2.10) is bounded by C, which means $|A_5(t)| \leq Ct$.

We have found that the finite variation part of U is bounded for finite t, then we can apply McKean's argument 1.2.10 to conclude that U can not explode in finite time and thus $\tau = \infty$ a.s. □

2.2 Generalization for matrix-valued diffusion processes

Theorem 2.2.1. *Let $B = (B(t), t \geq 0)$ be a Brownian motion in $\mathcal{M}_{p,p}(\mathbb{R})$ and $X(t)$ be a symmetric $p \times p$ matrix-valued stochastic process satisfying the stochastic differential equation*

$$\mathrm{d}X(t) = g(X(t))\mathrm{d}B(t)h(X(t)) + h(X(t))\mathrm{d}B(t)^T g(X(t)) + b(X(t))\mathrm{d}t, \tag{2.11}$$

where g, h, b are real functions acting spectrally, and $X(0)$ is a symmetric $p \times p$ matrix with p different eigenvalues.

Let $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$, and

$$\tau = \inf\{t : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j\}. \tag{2.12}$$

Then, for $t < \tau$ the eigenvalue process $\Lambda(t)$ verifies the following stochastic differential equations:

$$\mathrm{d}\lambda_i = 2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i + \left(b(\lambda_i) + \sum_{k\neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k}\right)\mathrm{d}t, \tag{2.13}$$

where $(W_i)_i$ are independent Brownian motions.

Proof. Recall that for every t, the process $X(t)$ admits a decomposition of the form

$$X(t) = H\Lambda H^T,$$

where both Λ and H are matrix-valued stochastic processes, $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_p)$ is the diagonal matrix of ordered eigenvalues of $X(t)$ and H is the corresponding matrix of eigenvectors.

Let us define the stochastic logarith of H as

$$\mathrm{d}A \coloneqq H^{-1}\partial H = H^T \partial H = H^T \mathrm{d}H + \frac{1}{2}(\mathrm{d}H^T)\mathrm{d}H.$$

By using Itô's formula on $I = H^T H$ we find

$$0 = \mathrm{d}I = \mathrm{d}(H^T H) = H^T \mathrm{d}H + (\mathrm{d}H)^T H + (\mathrm{d}H)^T \mathrm{d}H = H^T \partial H + (\partial H)^T H = A + A^T.$$

Which means A is skew symmetric. Using that $H^T H = I$, we have $\Lambda = H^T H \Lambda H^T H = H^T X H$, by the matrix Itô formula, we find

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$$\begin{aligned}
\mathrm{d}\Lambda = \mathrm{d}(H^T X H) &= (\partial H^T X)H + H^T X \partial H, \\
&= (\partial H)^T X H + H^T (\partial X) H + H^T X \partial H, \\
&= (\partial H)^T H \Lambda + H^T (\partial X) H + \Lambda H^T \partial H, \\
&= (\partial A)^T \Lambda + H^T (\partial X) H + \Lambda \partial A, \\
&= H^T (\partial X) H - (\partial A)\Lambda + \Lambda \partial A.
\end{aligned}$$

The entries in the diagonals of $(\partial A)\Lambda$ and $\Lambda\partial A$ coincide, and thus the diagonal of $\Lambda\partial A - (\partial A)\Lambda$ is zero. Let us denote $\mathrm{d}N = H^T(\partial X)H$, then

$$\mathrm{d}\lambda_j = \mathrm{d}N_{jj},$$

and, using that Λ is a diagonal matrix, if $i \neq j$,

$$0 = \mathrm{d}N_{i,j} + (\lambda_i - \lambda_j)\mathrm{d}A_{ij}.$$

This leads to the following representation for A_{ij},

$$\mathrm{d}A_{i,j} = \frac{\mathrm{d}N_{i,j}}{\lambda_j - \lambda_i}, \qquad i \neq j. \tag{2.14}$$

From (4.1) we compute the quadratic covariation $\mathrm{d}X_{ij}\mathrm{d}X_{km}$,

$$\begin{aligned}
\mathrm{d}X_{ij}\mathrm{d}X_{km} = \mathrm{d}\big\langle &(g(X(t))\mathrm{d}B(t)h(X(t)))_{ij} + (h(X_t)\mathrm{d}B^T(t)(g(X(t))))_{ij}, \\
&(g(X(t))\mathrm{d}B(t)h(X(t)))_{km} + (h(X_t)\mathrm{d}B^T(t)(g(X(t))))_{km}\big\rangle, \\
= {}&\mathrm{d}\big\langle (g(X(t))\mathrm{d}B(t)h(X(t)))_{ij}, (g(X(t))\mathrm{d}B(t)h(X(t)))_{km}\big\rangle \\
&+ \mathrm{d}\big\langle (g(X(t))\mathrm{d}B(t)h(X(t)))_{ij}, (h(X_t)\mathrm{d}B^T(t)(g(X(t))))_{km}\big\rangle \\
&+ \mathrm{d}\big\langle (h(X_t)\mathrm{d}B^T(t)(g(X(t))))_{ij}, (h(X_t)\mathrm{d}B^T(t)(g(X(t))))_{km}\big\rangle \\
&+ \mathrm{d}\big\langle (h(X_t)\mathrm{d}B^T(t)(g(X(t))))_{ij}, (g(X(t))\mathrm{d}B(t)h(X(t)))_{km}\big\rangle
\end{aligned}$$

Let us first find $\mathrm{d}\big\langle (g(X(t))\mathrm{d}B(t)h(X(t)))_{ij}, (g(X(t))\mathrm{d}B(t)h(X(t)))_{km}\big\rangle$, the other summands are analogous,

$$\begin{aligned}
&\mathrm{d}\big\langle (g(X(t))\mathrm{d}B(t)h(X(t)))_{ij}, (g(X(t))\mathrm{d}B(t)h(X(t)))_{km}\big\rangle, \\
&\qquad = \mathrm{d}\Big\langle \sum_{p,q} g(X(t))_{ip}\mathrm{d}B(t)_{pq}h(X(t))_{qj}, \sum_{r,s} g(X(t))_{kr}\mathrm{d}B(t)_{rs}h(X(t))_{sm}\Big\rangle
\end{aligned}$$

using the independence between the entries in the brownian matrix,

$$\begin{aligned}
&= \sum_{p,q} \mathrm{d}\big\langle g(X(t))_{ip}\mathrm{d}B(t)_{pq}h(X(t))_{qj}, g(X(t))_{kp}\mathrm{d}B(t)_{pq}h(X(t))_{qm}\big\rangle \\
&= \sum_{pq} g(X(t))_{ip}h(X(t))_{qj}, g(X(t))_{kp}h(X(t))_{qm}\mathrm{d}t, \\
&= \Big(\sum_{p} g(X(t))_{ip}g(X(t))_{kp}\Big)\Big(\sum_{q} h(X(t))_{qj}h(X(t))_{qm}\Big)\mathrm{d}t, \\
&= \big(g(X(t))g(X(t))^T\big)_{ik}\big(h(X(t))^T h(X(t))\big)_{jm}\mathrm{d}t, \\
&= \big(Hg(\Lambda)H^T Hg(\Lambda)H^T\big)_{ik}\big(Hh(\Lambda)H^T Hh(\Lambda)H^T\big)_{jm}\mathrm{d}t, \\
&= \big(Hg^2(\Lambda)H^T\big)_{ik}\big(Hh^2(\Lambda)H^T\big)_{jm}\mathrm{d}t = g^2(X)_{ik}h^2(X)_{jm}\mathrm{d}t.
\end{aligned}$$

Proceeding similarly with the other four summands we find

$$\mathrm{d}X_{ij}\mathrm{d}X_{km} = (g^2(X)_{ik}h^2(X)_{jm} + g^2(X)_{im}h^2(X)_{jk} + g^2(X)_{jk}h^2(X)_{im} + g^2(X)_{jm}h^2(X)_{ik})\mathrm{d}t.$$

Since $\mathrm{d}N = H^T(\partial X)H$ only differs in a finite variation part of $H^T(\mathrm{d}X)H$, the martingale part of both processes coincide and then the quadratic covariation of the entries of N is

$$\begin{aligned}
\mathrm{d}N_{ij}\mathrm{d}N_{km} &= \mathrm{d}\big\langle (H^T\mathrm{d}XH)_{ij}, (H^T\mathrm{d}XH)_{km}\big\rangle = \sum_{pqrs}\mathrm{d}\big\langle H^T_{ip}\mathrm{d}X_{pq}H_{qj}, H^T_{kr}\mathrm{d}X_{rs}H_{sm}\big\rangle, \\
&= \sum_{pqrs} H^T_{ip}H_{qj}H^T_{kr}H_{sm}\mathrm{d}X_{pq}\mathrm{d}X_{rs}, \\
&= \sum_{pqrs} H^T_{ip}H_{qj}H^T_{kr}H_{sm}\big(g^2(X)_{pr}h^2(X)_{qs} + g^2(X)_{ps}h^2(X)_{qr} + g^2(X)_{qs}h^2(X)_{pr} \\
&+ g^2(X)_{qr}h^2(X)_{ps}\big)\mathrm{d}t.
\end{aligned}$$

We find first $\sum_{pqrs} H^T_{ip}H_{qj}H^T_{kr}H_{sm}g^2(X)_{pr}h^2(X)_{qs}$ and the other terms are similar,

$$\begin{aligned}
\sum_{pqrs} H^T_{ip}H_{qj}H^T_{kr}H_{sm}g^2(X)_{pr}h^2(X)_{qs} &= \Big(\sum_{pr} H^T_{ip}g^2(X)_{pr}H_{rk}\Big)\Big(\sum_{qs} H^T_{jq}h^2(X)_{qs}H_{sm}\Big), \\
&= \big(H^T Hg^2(\Lambda)H^T H\big)_{ik}\big(H^T Hh^2(\Lambda)H^T H\big)_{jm} = g^2(\Lambda)_{ik}h^2(\Lambda)_{jm}.
\end{aligned}$$

Repeating the analogous procedure with all of the terms we find that the covariation is

$$\mathrm{d}N_{ij}\mathrm{d}N_{km} = (g^2(\Lambda)_{ik}h^2(\Lambda)_{jm} + g^2(\Lambda)_{im}h^2(\Lambda)_{jk} + g^2(\Lambda)_{jk}h^2(\Lambda)_{im} + g^2(\Lambda)_{jm}h^2(\Lambda)_{ik})\mathrm{d}t.$$

It follows that the quadratic variation in the diagonal is

$$\mathrm{d}N_{ii}\mathrm{d}N_{jj} = 4\delta_{ij}g^2(\lambda_i)h^2(\lambda_j)\mathrm{d}t.$$

Now, in order to compute F, the finite variation part of N, we use (4.1),

$$\begin{aligned}\mathrm{d}F &= H^T b(X) H \mathrm{d}t + \frac{1}{2}(\mathrm{d}H^T \mathrm{d}X H + H^T \mathrm{d}X \mathrm{d}H),\\ &= b(\Lambda)\mathrm{d}t + \frac{1}{2}\left((\mathrm{d}H^T H)(H^T \mathrm{d}X H) + (H^T \mathrm{d}X H)(H^T \mathrm{d}H)\right),\end{aligned}$$

using that the martingale part of $H^T \mathrm{d}H$ and $H^T \partial H$ coincide and the same with $H^T(\partial X)H$ and $H^T(\mathrm{d}X)H$,

$$= b(\Lambda)\mathrm{d}t + \frac{1}{2}((\mathrm{d}N\mathrm{d}A)^T + \mathrm{d}N\mathrm{d}A).$$

Now we can use (2.14) and (2.2) to find $\mathrm{d}N\mathrm{d}A$,

$$(\mathrm{d}N\mathrm{d}A)_{ij} = \sum_{k\neq j} \mathrm{d}N_{ik}\mathrm{d}A_{kj} = \sum_{k\neq j} \frac{\mathrm{d}N_{ik}\mathrm{d}N_{kj}}{\lambda_j - \lambda_k} = \delta_{ij} \sum_{k\neq j} \frac{g^2(\lambda_i)h^2(\lambda_k) + g^2(\lambda_k)h^2(\lambda_i)}{\lambda_i - \lambda_k}\mathrm{d}t.$$

Recalling that $G(x,y) = g^2(x)h^2(x) + g^2(y)h^2(y)$, we have that

$$(\mathrm{d}N\mathrm{d}A)_{ij} = \delta_{ij} \sum_{k\neq j} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k}\mathrm{d}t.$$

From (2.2) we have that the martingale part of N_{ii} has the form $2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i$ for some Brownian motion W_i. Putting together the martingale and finite variation parts of N we have that

$$\mathrm{d}N_{ii} = 2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i + \sum_{k\neq j} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k}\mathrm{d}t.$$

Since $\mathrm{d}\lambda_i = \mathrm{d}N_{ii}$, this finishes the proof. $\square$

Theorem 2.2.2. *Let $W(t)$ be a complex $p \times p$ Brownian matrix. Suppose that $X = (X(t), t \geq 0)$ is a matrix-valued process taking values in the group of self adjoint matrices and it satisfies the following matrix stochastic differential equation:*

$$\mathrm{d}X(t) = g(X(t))\mathrm{d}W(t)h(X(t)) + h(X(t))\mathrm{d}W(t)^* g(X(t)) + b(X(t))\mathrm{d}t, \tag{2.15}$$

with $g, h, b : \mathbb{R} \to \mathbb{R}$ and X_0 is a hermitian $p \times p$ random matrix with p different eigenvalues.

Let $G(x,y) = g^2(x)h^2(y) + g^2(y)h^2(x)$, and

$$\tau = \inf\{t : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j\}.$$

Then, for $t < \tau$ the eigenvalue process Λ_t verifies the following stochastic differential equations:

$$\mathrm{d}\lambda_i = 2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i + \left(b(\lambda_i) + 2\sum_{k\neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k}\right)\mathrm{d}t, \tag{2.16}$$

where $(W_i)_i$ are independent Brownian motions.

Proof. Recall that for a complex Brownian motion Z we have that

No sé si debo incluir en los preliminares.

$$\mathrm{d}\langle Z, Z\rangle(t) = 0, \qquad \mathrm{d}\langle Z, \overline{Z}\rangle(t) = 2\mathrm{d}t.$$

Then we can compute the quadratic covariation $\mathrm{d}X_{ij}\mathrm{d}X_{kl}$ using (2.15),

$$
\begin{aligned}
\mathrm{d}X_{ij}\mathrm{d}X_{kl} &= \mathrm{d}\langle X_{ij}, X_{kl}\rangle(t), && (2.17)\\
&= \mathrm{d}\langle (g(X)\mathrm{d}Wh(X) + h(X)\mathrm{d}W^* g(X))_{ij}, (g(X)\mathrm{d}Wh(X) + h(X)\mathrm{d}W^* g(X))_{kl}\rangle(t), && (2.18)\\
&= \mathrm{d}\langle (g(X)\mathrm{d}Wh(X))_{ij}, (h(X)\mathrm{d}W^* g(X))_{kl}\rangle + \mathrm{d}\langle (g(X)\mathrm{d}Wh(X))_{kl}, (h(X)\mathrm{d}W^* g(X))_{ij}\rangle(t), && (2.19)\\
&= 2g^2(X)_{il}h^2(X)_{jk}\mathrm{d}t + 2g^2(X)_{kj}h^2(X)_{li}\mathrm{d}t, && (2.20)\\
&= 2\left(g^2(X)_{il}h^2(X)_{kj} + g^2(X)_{jk}h^2(X)_{il}\right)\mathrm{d}t. && (2.21)
\end{aligned}
$$

Analogously to the real case, we define A, the stochastic logarithm of H, as

$$A := H^{-1}\partial H = H^*\partial H.$$

By using Itô's formula we find,

$$0 = \mathrm{d}I = \mathrm{d}(H^*H) = H\partial H^* + (\partial H)H^* = A^* + A,$$

which means A is skew-Hermitian. This implies that the real parto of the terms in the diagonal of A is zero. Let us now apply Itô's formula to $\Lambda = H^*XH$,

$$
\begin{aligned}
\mathrm{d}\Lambda &= \mathrm{d}(H^*XH) = H^*(\mathrm{d}(XH)) + (\mathrm{d}H^*)XH + \mathrm{d}(H^*)\mathrm{d}(XH),\\
&= H^*(\mathrm{d}X)H + H^*X\mathrm{d}H + H^*(\mathrm{d}X\mathrm{d}H) + (\mathrm{d}H^*)XH + \mathrm{d}(H^*)(\mathrm{d}X)H + \mathrm{d}(H^*)X\mathrm{d}H + \mathrm{d}H^*\mathrm{d}X\mathrm{d}H,\\
&= H^*(\partial X)H + H^*X\partial H + (\partial H^*)XH = H^*(\partial X)H + \Lambda H^*\partial H + (\partial H^*)H\Lambda,\\
&= H^*(\partial X)H + \Lambda\partial A + \partial A^T\Lambda = H^*(\partial X)H + \Lambda\partial A - \partial A\Lambda.
\end{aligned}
$$

By the relationship between Itô's and Stratanovich's integrals,

$$H^*(\partial X)H = H^*(\mathrm{d}X)H + \frac{1}{2}(\mathrm{d}H^*(\mathrm{d}X)H + H^*\mathrm{d}X\mathrm{d}H),$$

so using that X is hermitian, we have that $H^*(\partial X)H$ is also hermitian and its diagonal elements are real. The process $\Lambda\partial A - (\partial A)\Lambda$ is zero in the diagonal and thus $\mathrm{d}\lambda_i = (H^*(\partial X)H)_{ii}$. If $i \neq j$, we have

$$0 = (H^*(\partial X)H)_{ij} + \lambda_i\partial A_{ij} - \lambda_j\partial A_{ji} = (H^*(\partial X)H)_{ij} + (\lambda_i - \lambda_j)\partial A_{ij}.$$

The last part implies $\partial A_{ij} = \frac{(H^*(\partial X)H)_{ij}}{\lambda_j - \lambda_i}$, whenever $i \neq j$.

Define $\mathrm{d}N = \mathrm{d}H^*(\partial X)H$. The martingale part of N and $H^*(\mathrm{d}X)H$ is the same, since they differ only in a finite variation term. We can find $\mathrm{d}N_{ij}\mathrm{d}N_{kl}$ using $\mathrm{d}X_{ij}\mathrm{d}X_{kl}$,

$$\mathrm{d}N_{ij}\mathrm{d}Nkl = 2(g^2(\Lambda)_{il}h^2(\Lambda)_{jk} + g^2(\Lambda)_{jk}h^2(\Lambda)_{il})\mathrm{d}t.$$

Then, for the elements in the diagonal we have

$$\mathrm{d}N_{ii}\mathrm{d}N_{jj} = 4\delta_{ij}(g^2(\lambda_i)h^2(\lambda_i))\mathrm{d}t. \tag{2.22}$$

Now we compute the finite variation part of $\mathrm{d}N$ from (2.15). Let us denote it as $\mathrm{d}F$.

$$
\begin{aligned}
\mathrm{d}F &= H^*b(X)H\mathrm{d}t + \frac{1}{2}(\mathrm{d}H^*(\mathrm{d}X)H + H^*\mathrm{d}X\mathrm{d}H),\\
&= b(\Lambda)\mathrm{d}t + \frac{1}{2}\big((\mathrm{d}H^*H)(H^*\mathrm{d}XH) + (H^*\mathrm{d}XH)(H^*\mathrm{d}H)\big),\\
&= b(\Lambda)\mathrm{d}t + \frac{1}{2}\big((\mathrm{d}N\mathrm{d}A)^* + \mathrm{d}N\mathrm{d}A\big).
\end{aligned}
$$

Using the quadratic variation of $\mathrm{d}N$ and $\mathrm{d}A$ we find their covariation,

$$\begin{aligned}(\mathrm{d}N\mathrm{d}A)_{ij} &= \sum_k (\mathrm{d}N)_{ik}(\mathrm{d}A)_{kj} = \sum_k \frac{(\mathrm{d}N)_{ik}(\mathrm{d}N)_{kj}}{\lambda_j - \lambda_i}, \\ &= 2\delta_{ij} \sum_{k\neq j} \frac{g^2(\lambda_i)h^2(\lambda_k) + g^2(\lambda_k)h^2(\lambda_j)}{\lambda_j - \lambda_k} + \mathrm{d}N_{ij}\mathrm{d}A_{jj}.\end{aligned}$$

By the properties shown above for $\mathrm{d}N$ and $\mathrm{d}A$, if$i = j$, $\mathrm{d}N_{jj}$ is real and $\mathrm{d}A_{jj}$ is purely imaginary. By independence of the real and imaginary parts of the complex Brownian motion, this implies that $\mathrm{d}N_{jj}\mathrm{d}A_{jj} = 0$. We have

$$\mathrm{d}F_{ii} = \left(b(\lambda_i) + 2\sum_{k\neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_j} \right) \mathrm{d}t,$$

where $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$.

Using the quadratic variation of $\mathrm{d}N$, we find that the martingale part of $\mathrm{d}N_{ii}$ is

$$\mathrm{d}M_{ii} = 2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i,$$

for some Brownian motion. Recall that $\mathrm{d}\lambda_i = \mathrm{d}N_{ii}$, then we have that there exist $W_1, \dots, W_p$ independent Brownian motions such that

$$\mathrm{d}\lambda_i = \mathrm{d}N_{ii} = 2g(\lambda_i)h(\lambda_i)\mathrm{d}W_i + \left(b(\lambda_i) + 2\sum_{k\neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_j} \right) \mathrm{d}t.$$

This ends the proof. □

Theorem 2.2.3 (Multidimensional Yamada-Watanabe Theorem Graczyk and Małecki (2011)). *Let $p \in \mathbb{N}$ and*

$$b_i : \mathbb{R}^p \to \mathbb{R}, \qquad i = 1, \dots, p,$$

be real-valued continuous functions satisfying the following Lipschitz conditions for $C > 0$,

$$|b_i(y_1) - b_i(y_2)| \leq C \left\| y_1 - y_2 \right\|, \quad i = 1, \dots, p,$$

for every $y_1, y_2 \in \mathbb{R}^p$.

Further, let $\sigma_i : \mathbb{R} \to \mathbb{R}, i = 1, \dots, p$ be a set of measurable functions such that

$$|\sigma_i(x) - \sigma_i(y)|^2 \leq \rho_i(|x - y|), \quad x, y \in \mathbb{R},$$

where $\rho_i : (0, \infty) \to (0, \infty)$ are measurable functions such that

$$\int_{0^+} \rho_i^{-1}(x)\ \mathrm{d}x = \infty.$$

Then the pathwise uniqueness holds for the following system of stochastic differential equations

$$\mathrm{d}Y_i = \sigma_i(Y_i)\mathrm{d}B_i + b_i(Y)\mathrm{d}t, \qquad i = 1, \dots, p, \tag{2.23}$$

where $B_1, \dots, B_p$ are independent Brownian motions.

Proof. Let Y and $\hat{Y}$ be two solutions with respect to the same multidimensional Brownian motion $B = (B_i)_{i \leq p}$ such that $Y(0) = \hat{Y}(0)$ a.s., for $i \leq p$ we have

$$Y_i(t) - \hat{Y}_i(t) = \int_0^t \sigma_i(Y_i) - \sigma_i(\hat{Y}_i)\ \mathrm{d}B_i(s) + \int_0^t b_i(Y_i) - b_i(\hat{Y}_i)\ \mathrm{d}s. \tag{2.24}$$

We can then see that

$$\int_0^t \frac{\mathbb{1}_{\{Y_i(s) > \hat{Y}_i(s)\}}}{\rho_i(Y_i(s) - \hat{Y}_i(s))} \mathrm{d}\langle Y_i - \hat{Y}_i, Y_i - \hat{Y}_i \rangle = \int_0^t \frac{(\sigma_i(Y_i(s)) - \sigma_i(\hat{Y}_i(s)))^2}{\rho_i(Y_i(s) - \hat{Y}_i(s))} \mathbb{1}_{\{Y_i(s) > \hat{Y}_i(s)\}}\ \mathrm{d}s \leq t.$$

Applying Theorem 1.2.8 we have that the local time of $Y_i - \hat{Y}_i$ at 0 is 0. Then, we can use the Tanaka formula to find

$$\begin{aligned}
|Y_i(t) - \hat{Y}_i(t)| &= \int_0^t \operatorname{sgn}(Y_i(s) - \hat{Y}_i(s))\ \mathrm{d}(Y_i(s) - \hat{Y}_i(s)), \\
&= \int_0^t \operatorname{sgn}(Y_i(t) - \hat{Y}_i(t))(\sigma_i(Y_i) - \sigma_i(\hat{Y}_i))\ \mathrm{d}B_i(s) \\
&\qquad + \int_0^t \operatorname{sgn}(Y_i(s) - \hat{Y}_i(s))(b_i(Y_i(s)) - b_i(\hat{Y}_i(s)))\ \mathrm{d}s.
\end{aligned}$$

Since σ_i is bounded, we have that $\operatorname{sgn}(Y_i(t) - \hat{Y}_i(t))(\sigma_i(Y_i(t)) - \sigma_i(\hat{Y}_i(t)))$ is bounded and therefore the first integral in the last expression is a martingale with mean 0, which in turns implies that

$$|Y_i(t) - \hat{Y}_i(t)| - \int_0^t \operatorname{sgn}(Y_i(s) - \hat{Y}_i(s))(b_i(Y_i(s)) - b_i(\hat{Y}_i(s)))\ \mathrm{d}t,$$

is a zero-mean martingale. Then, by using the Lipschitz properties of b_i we have

$$\begin{aligned}
\mathbb{E}\left[|Y_i(t) - \hat{Y}_i(t)|\right] &= \mathbb{E}\left[\int_0^t \operatorname{sgn}(Y_i(s) - \hat{Y}_i(s))(b_i(Y_i(s)) - b_i(\hat{Y}_i(s)))\ \mathrm{d}s\right], \\
&\leq \mathbb{E}\left[\int_0^t |b_i(Y_i(s)) - b_i(\hat{Y}_i(s))|\ \mathrm{d}s\right], \\
&= \int_0^t \mathbb{E}\left[|b_i(Y_i(s)) - b_i(\hat{Y}_i(s))|\right]\ \mathrm{d}s \leq C \int_0^t \mathbb{E}\left[|Y_i(s) - \hat{Y}_i(s)|\right]\ \mathrm{d}s.
\end{aligned}$$

Summing for every i we get

$$\mathbb{E}\left[|Y(t) - \hat{Y}(t)|\right] \leq Cp \int_0^t \mathbb{E}\left[|Y(t) - \hat{Y}(s)|\right]\ \mathrm{d}s.$$

Using Gronwall's lemma (1.2.9) we get that

$$\mathbb{E}\left[|Y(t) - \hat{Y}(t)|\right] = 0,$$

which implies $Y(t) = \hat{Y}(t)$ a.s. for every $t > 0$, ending the proof. □

Theorem 2.2.4 (Spectral matrix Yamada-Watanabe theorem)**.** *Let $X(t)$ be a $p \times p$ symmetric matrix-valued process satisfying the equation (4.1) with initial condition $X(0)$ that is a symmetric $p \times p$ matrix with p different eigenvalues. Suppose further that*

$$|g(x)h(x) - g(y)h(y)|^2 \leq \rho(|x - y|), \qquad x, y \in \mathbb{R}, \tag{2.25}$$

with $\rho : (0,\infty) \to (0,\infty)$ a measurable function satisfying

$$\int_{0^+} \rho^{-1}(x)\mathrm{d}x = \infty,$$

that $G(x,y) \coloneqq g^2(x)h^2(y) + g^2(y)h^2(x)$ is locally Lipschitz and strictly positive on the set $\{x \neq y\}$ and that b is locally Lipschitz. Then if τ is defined as in (2.12)*, for $t < \tau$, the process of eigenvalues satisfying* (2.13) *has a pathwise unique solution.*

Proof. Let PNP^T be a diagonalization for $X(0)$. We need to show that a unique strong solution exists for (2.13) when $\Lambda(0) = N$. The functions

$$b_i(\lambda_1, \ldots, \lambda_p) = b(\lambda_i) + \sum_{k\neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k},$$

are locally Lipschitz continuous on Δ_p so they can be extended from the compact sets

$$D_m = \{0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_p < m, \lambda_{i+1} - \lambda_i \geq 1/m\},$$

to bounded Lipschitz functions on $\mathbb{R}^p$. Let b_i^m denote such extension for $m \in \mathbb{N} \setminus \{0\}$. For $i = 1, \ldots, p$, we consider the system of SDEs,

$$\mathrm{d}\lambda_i = 2g(\lambda_i^m)h(\lambda_i^m)\mathrm{d}W_i + b_i^m(\Lambda^m)\mathrm{d}t.$$

We have that $|g(x)h(x) - g(y)h(y)|^2 \leq \rho(|x - y|)$ and $\int_{0+} \rho(x)^{-1}\mathrm{d}x = \infty$, and using Theorem 2.2.3 we get that there is a unique strong solution for the system of SDEs. Since $D_m \subset D_{m+1}$, we have that $\lim_{m\to\infty} D_m = \Delta_p$, so there exists a unique strong solution $\Lambda(t)$ for the SDEs system up to the first exit time from Δ_p. This time is τ, the first collision time of the eigenvalues.

□

Corollary 2.2.5. *Suppose that b, g^2, h^2 are Lipschitz continuous, g^2h^2 is convex or or continuously differentiable with derivative uniformly Lipschitz on $\mathbb{R}$ and that $G(x,y) \coloneqq g^2(x)h^2(y) + g^2(y)h^2(x)$ is strictly positive on $\{x \neq y\}$. Then the system of SDEs* (2.13) *for the eigenvalue process satisfying* (4.1) *has a unique strong solution on* $[0,\infty)$.

Proof. Recall that if f is a non-negative Lipschitz continuous function, then $\sqrt{f}$ is 1/2-Hölder continuous. Since g^2 and h^2 are Lipschitz continuous, then g^2h^2 is locally Lipschitz continuous and gh is 1/2-Hölder continuous. Then

$$|g(x)h(x) - g(y)h(y)|^2 \leq \left(K|x-y|^{\frac{1}{2}}\right)^2 = K^2|x-y|.$$

Taking $\rho(|x-y|) = K^2|x-y|$ we see that the conditions of Theorem 2.2.4 are satisfied and then the uniqueness and existence of the strong solution applies on $[0,\tau)$. By Theorem 2.1.3 we have that $\tau = \infty$ a.s., and thus the existence and uniqueness is satisfied on $[0,\infty)$. □

Chapter 3

Finite Free Probability

In this chapter, we introduce the main concepts in finite free probability. In the first section we introduce the definition of polynomial convolutions and some polynomial ensembles. This section is purely algebraic and it is not related to probability. In section two we introduce the concept of minor orthogonality which is central to finite free probability. This is also the section in which we state initial relations between random matrices and convolutions of polynomials. In the third chapter, we introduce the $\mathcal{R}_d$ transform, which relates finite free probability to free probability, we also use this transform to prove the Finite Free Central Limit Theorem.

3.1 Convolution of polynomials

In this section, we present two notions of polynomial convolution, both introduced around a century ago Walsh (1922) Szegö (1939). Their study began using tools outside of probability theory, but we do not include any of those results here, instead, we are merely interested in introducing them to relate them to expected characteristic polynomials of random matrices. In the next subsection we introduce three ensembles of orthogonal polynomials, namely the Hermite, Laguerre and Jacobi polynomials. We prove some nice properties these polynomials, especially related to the notions of convolution introduced previously.

Both notions of convolution are defined for monic complex polynomials. Under this conditions, we can express a polynomial $p(z)$ as

$$p(z) = \sum_{j=0}^{d} z^{d-j}(-1)^j a_j.$$

Both convolutions are defined in function of the polynomial degree, but the polynomials need not to have the same degree. In the case the degree is different, we take the convolution with the highest degree.

3.1.1 Symmetric additive convolution

Definition 3.1.1 (Symmetric additive convolution)**.** Let $p(z), q(z)$ be two complex polynomials of z, with degree less or equal to d,

$$p(z) = \sum_{j=0}^{d} z^{d-j}(-1)^j a_j,$$

$$q(z) = \sum_{j=0}^{d} z^{d-j}(-1)^j b_j.$$

The dth symmetric additive convolution of p and q is

$$\begin{aligned} p(z) \boxplus_d q(z) &:= \sum_{k=0}^{d} z^{d-k}(-1)^k \sum_{i+j=k} \frac{(d-i)!(d-j)!}{d!(d-k)!} a_i b_j, \\ &= \frac{1}{d!} \sum_{k=0}^{d} \partial_z^k p(z) \partial_z^{d-k} q(0), \\ &= \frac{1}{d!} \sum_{k=0}^{d} \partial_z^k q(z) \partial_z^{d-k} p(0), \end{aligned}$$

with ∂_z denoting the differentiation with respect to z.

3.1.2 Symmetric multiplicative convolution

Definition 3.1.2 (Symmetric multiplicative convolutions)**.** Let p and q be as in Definition 3.1.1 with degree at most d, the dth symmetric multiplicative convolution of p and q is

$$p(z) \boxtimes_d q(z) := \sum_{i=0}^{d} z^{d-i}(-1)^i \frac{a_i b_i}{\binom{d}{i}}.$$

3.1.3 Orthogonal polynomials

Hermite polynomials

The Hermite polynomials are defined by a linear operator

$$H_n(z) := e^{-\frac{\partial_z^2}{2}}(z^n) := \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \frac{\partial^{2k} z^n}{\partial z^{2k}}. \tag{3.1}$$

The generalized Hermite polynomials also known as time dependent Hermite polynomials, or Hermite polynomials with variance are polynomials on z and t defined by the analogous linear operator

$$H_n(z,t) := e^{-\frac{t\partial_z^2}{2}}(z^n) := \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{2^k k!} \frac{\partial^{2k} z^n}{\partial z^{2k}}. \tag{3.2}$$

Note that $H_n(z,1) = H_n(z)$. Using the former definitions we can find explicit expressions for both $H_n(z)$ and $H_n(z,t)$,

$$\begin{aligned} H_n(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \frac{\partial^{2k} z^n}{\partial z^{2k}} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{2^k k!} \frac{n!}{(n-2k)!} z^{n-2k} = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k z^{n-2k}}{2^k k!(n-2k)!}, \\ H_n(z,t) &= \sum_{k=0}^{\infty} \frac{(-t)^k}{2^k k!} \frac{\partial^{2k} z^n}{\partial z^{2k}} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-t)^k}{2^k k!} \frac{n!}{(n-2k)!} z^{n-2k} = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-t)^k z^{n-2k}}{2^k k!(n-2k)!}. \end{aligned}$$

An easy substitution allows to see that the coefficient of x^m in $H_n(z,t)$ is

$$a_m = \begin{cases} \frac{n!(-t)^{\frac{n-m}{2}}}{2^{\frac{n-m}{2}}(\frac{n-m}{2})!m!}, & \text{if } m \text{ and } n \text{ have the same parity,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

The last expression gives us a way to find the first few Hermite polynomials,

$$
\begin{aligned}
H_1(z,t) &= z,\\
H_2(z,t) &= z^2 - t,\\
H_3(z,t) &= z^3 - 3tz,\\
H_4(z,t) &= z^4 - 6tz^2 + 3t^2,\\
H_5(z,t) &= z^5 - 10tz^3 + 15t^2z,\\
H_6(z,t) &= z^6 - 15tz^4 + 45t^2z^2 - 15t^3,\\
H_7(z,t) &= z^7 - 21tz^5 + 105t^2z^3 - 105t^3z,\\
H_8(z,t) &= z^8 - 28tz^6 + 210t^2z^4 - 420t^3z^2 + 105t^4,\\
H_9(z,t) &= z^9 - 36tz^7 + 378t^2z^5 - 1260t^3z^3 + 945t^4z,\\
H_{10}(z,t) &= z^{10} - 45tz^8 + 630t^2z^6 - 3150t^3z^4 + 4725t^4z^2 - 945t^5.
\end{aligned}
$$

By replacing $t = 1$, we can find the corresponding standard Hermite polynomials.

The Hermite polynomials are characterized by the following recursion together with the initial conditions $H_1(x,t)$ and $H_2(x,t)$.

$$H_n(x,t) = xH_{n-1}(x,t) - t(n-1)H_{n-2}(x,t). \tag{3.4}$$

Proposition 3.1.1. *The symmetric additive convolution between two Hermite polynomials with the same order $H_d(z,t_1), H_d(z,t_2)$ is another Hermite polynomial with variance $t_1 + t_2$.*

Proof.

$$
\begin{aligned}
&H_d(z,t_1) \boxplus_d H_d(z,t_2) =\\
&= \sum_{k=0}^{d} z^{d-k}(-1)^k \sum_{i=0}^{k} \frac{(d-i)!(d-k+i)!}{d!(d-k)!} b_i a_{k-i},\\
&= \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} z^{d-2k}(-1)^{2k} \sum_{i=0}^{2k} \frac{(d-i)!(d-2k+i)!}{d!(d-2k)!} d! \frac{(-t_1)^{i/2}}{2^{i/2}(i/2)!(d-i)!} d! \frac{(-t_2)^{2k-i}}{2^{k-i/2}(k-i/2)!(d-2k+i)!},\\
&= \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} z^{d-2k}(-1)^{2k} \frac{d!}{2^k} \sum_{i=0}^{2k} \frac{(-t_1)^{i/2}(-t_2)^{k-i/2}}{(i/2)!(k-i/2)!} = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} z^{d-2k}(-1)^{2k} \frac{d!}{k!2^k} \sum_{i=0}^{2k} \frac{k!(-t_1)^{i/2}(-t_2)^{k-i/2}}{(i/2)!(k-i/2)!},\\
&= \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} z^{d-2k}(-1)^{2k} \frac{d!}{k!2^k} \sum_{i=0}^{k} \binom{k}{i} (-t_1)^{i/2}(-t_2)^{k-i/2} = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} z^{d-2k}(-1)^k \frac{d!(t_1+t_2)^k}{k!2^k}\\
&= H_d(z, t_1 + t_2).
\end{aligned}
$$

$\square$

Laguerre polynomials

Jacobi polynomials

3.2 Minor orthogonality

Definition 3.2.1 (Minor orthogonality)**.** Let R be an $m \times n$ random matrix. We say R is minor orthogonal if for every $k, l \in \mathbb{Z}$ such that $k, l \leq \max\{m.n\}$ and all sets S, T, U, V with $|S| = |T| = k$ and $|U| = |V| = l$, it satisfies

$$E_R\left[[R]_{S,T}[R^*]_{U,V}\right] = \frac{\delta_{S,V}\delta_{T,U}}{\binom{\max\{m,n\}}{k}}.$$

Lemma 3.2.1. *If R is minor orthogonal and Q is a constant matrix such that $QQ^* = I$, then Q is minor orthogonal. If $Q^*Q = I$, then RQ is minor orthogonal.*

Proof. Recall that by the Cauchy-Binet formula, for $|S| = |T| = k$ we have

$$[QR]_{S,T} = \sum_{|W|=k} [Q]_{S,W}[R]_{W,T},$$

so with $|S| = |T| = k, |U| = |V| = l$,

$$\begin{aligned}
E_R\left[[QR]_{S,T}[R^*Q^*]_{U,V}\right] &= E_R\left[\sum_{|W|=k}\sum_{|Z|=l}[Q]_{S,W}[R]_{W,T}[R^*]_{U,Z}[Q^*]_{Z,V}\right],\\
&= \sum_{|W|=k}\sum_{|Z|=l}[Q]_{S,W}[Q^*]_{Z,V}E_R\left[[R]_{W,T}[R^*]_{U,Z}\right],\\
&= \sum_{|W|=k}\sum_{|Z|=l}[Q]_{S,W}[Q^*]_{Z,V}E_R\frac{\delta_{W,Z}\delta_{T,U}}{\binom{\max\{m,n\}}{k}},\\
&= \sum_{|W|=k}[Q]_{S,W}[Q^*]_{W,V}\frac{\delta_{T,U}}{\binom{\max\{m,n\}}{k}} = [QQ^*]_{S,V}\frac{\delta_{T,U}}{\binom{\max\{m,n\}}{k}},\\
&= [I]_{S,V}\frac{\delta_{T,U}}{\binom{\max\{m,n\}}{k}},
\end{aligned}$$

Notice that $[I]_{S,V} = 1$ if and only if $S = V$, so we conclude that

$$E_R\left[[QR]_{S,T}[R^*Q^*]_{U,V}\right] = \frac{\delta_{S,V}\delta_{T,U}}{\binom{\max\{m,n\}}{k}}.$$

The case $Q^*Q = I$ is proven in the same way. □

Definition 3.2.2 (Signed permutation matrix)**.** A signed permutation matrix is a matrix that can be written EP where E is a diagonal matrix with entries ±1 and P is a permutation matrix.

Lemma 3.2.2. *A random matrix sampled uniformly from the set of signed permutation matrices is minor-orthogonal.*

Proof. Let Q be a signed permutation matrix, we can write $Q = EP$, where E is a diagonal random matrix with entries ±1 taken uniformly and P is a matrix chosen uniformly from the permutation matrices, and both are independent. Then for $|S| = |T| = k$ and $|U| = |U| = l$,

$$\begin{aligned}
E_Q\left[[Q]_{S,T}[Q^*]_{U,V}\right] &= E_{E,P}\left[[EP]_{S,T}[P^*E]_{U,V}\right],\\
&= \sum_{|W|=k}\sum_{|Z|=l}E_{E,P}\left[[E]_{S,W}[P]_{W,T}[P^*]_{U,Z}[E]_{Z,V}\right],
\end{aligned}$$

every $[E]_{S,W}$ is diagonal and the determinant would be zero if $S \neq W$, so

$$= E_{E,P}\left[[E]_{S,S}[P]_{W,T}[P^*]_{U,Z}[E]_{V,V}\right],$$

Let $\{\chi_i\}_{1\leq i\leq n}$ be the diagonal entries of E, then

$$= E_E\left[\prod_{i\in S}\chi_i\prod_{j\in V}\chi_j\right]E_P\left[[P]_{S,T}[P^*]_{U,V}\right].$$

Now we use that the variables chi_i are independent and uniform in $\{-1,1\}$, so that $E[\chi_i]=0$, but $E[\chi_i^2]=1$ for all i, and this means

$$E_E\left[\prod_{i\in S}\chi_i\prod_{j\in V}\chi_j\right]=\delta_{S,V}.$$

This last equality leads to

$$\begin{aligned}E_R\left[[QR]_{S,T}[R^*Q^*]_{U,V}\right] &= \delta_{S,V}E_P\left[[P]_{S,T}[P^*]_{U,V}\right],\\ &= \delta_{S,V}E_P\left[[P]_{S,T}[P^*]_{U,S}\right],\\ &= \delta_{S,V}E_P\left[[P]_{S,T}[P]_{S,U}\right].\end{aligned}$$

The submatrix $P_{S,T}$ can be transformed in a diagonal matrix by a permutation matrix because it has at most a non zero entry for each row and each column. If the diagonal matrix has a zero entry in the diagonal, then the determinant $[P]_{S,T}$ is zero, in other case, it is different that zero. The only case when all of the diagonal entries of the diagonal matrix are not zero is when $T=\pi(S)$ with π the permutation function corresponding to P. This means that in order to have a non-zero determinant we need $T=\pi(S)=U$, and $[P]_{S,U}=\in\{-1,1\}$, so

$$\begin{aligned}E_R\left[[QR]_{S,T}[R^*Q^*]_{U,V}\right] &= \delta_{S,V}\delta_{T,U}E_P\left[[P]_{S,T}[P]_{S,T}\right],\\ &= \delta_{S,V}\delta_{T,U}E_P\left[[P]_{S,T}^2\right],\\ &= \delta_{S,V}\delta_{T,U}E_P\left[\delta_{T=\pi(S)}\right],\\ &= \delta_{S,V}\delta_{T,U}P\left(T=\pi(S)\right).\end{aligned}$$

We are supposing that we are sampling uniformly from the permutation matrices of size $n\times n$, so the probability that $T=\pi(S)$ when π is a permutation of n elements and $|S|=|T|=k$ is $1/\binom{n}{k}$. So, we can conclude

$$E_R\left[[QR]_{S,T}[R^*Q^*]_{U,V}\right]=\frac{\delta_{S,V}\delta_{T,U}}{\binom{n}{k}}.$$

This is the definition of being minor-orthogonal.

En el paper no lo pr
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mente tomando que
de tamaño $m \times m$ y
tamaño $m \times n$, todo
resultados se siguen

$\square$

Corollary 3.2.3. *An $m\times n$ random matrix sampled from the Haar measure on $\mathbb{C}_n^m$ is minor-orthogonal.*

Proof. Let R be a Haar distributed random $m\times n$ matrix with $m\leq n$ and Q a random permutation matrix. Any random permutation matrix is unitary, so RQ is Haar distributed for fixed Q, and by Lemmas 3.2.1 and 3.2.2 we have that it is also minor-orthogonal. Then, if Q is uniformly sampled from the signed permutation matrices,

$$E_R\left[[R]_{S,T}[R^*]_{U,V}\right]=E_{R,Q}\left[[RQ]_{S,T}[(RQ)^*]_{U,V}\right].$$

Since Q is minor orthogonal, RQ is also minor orthogonal for fixed R and

$$E_R\left[E_Q\left[[RQ]_{S,T}[(RQ)^*]_{U,V}\right]\right] = E_R\left[\frac{\delta_{S,V}\delta_{T=U}}{\binom{n}{k}}\right] = \frac{\delta_{S,V}\delta_{T=U}}{\binom{n}{k}},$$

where $k = |S| = |T|$. □

esto debería ir en
nares

We denote by $\sigma_k(A)$ the coeficient of of $(-1)^k x^{d-k}$ in the characteristic polynomial of a d-dimensional matrix A. We will use the fact that

$$\sigma_k(A) = \sum_{|S|=k} [A]_{S,S}.$$

Lemma 3.2.4. *Let $m \leq n$, B an $n \times n$ random matrix and R an $m \times n$ minor-orthogonal matrix independent from B. For all sets $S, T \subset \binom{[m]}{k}$ we have*

$$E_{B,R}\left[[RBR^*]_{S,T}\right] = E_B\left[\delta_{S,T}\frac{\sigma_k(B)}{\binom{n}{k}}\right].$$

Proof. Using the Cauchy-Binet formula we have

$$\begin{aligned}
E_{B,R}\left[[RBR^*]_{S,T}\right] &= E_B\left[\sum_{X,Y\in\binom{[n]}{k}} E_R\left[[R]_{S,X}[B]_{X,Y}[R^*]_{Y,T}\right]\right], \\
&= E_B\left[\sum_{X,Y\in\binom{[n]}{k}} [B]_{X,Y} E_R\left[[R]_{S,X}[R^*]_{Y,T}\right]\right], \\
&= E_B\left[\sum_{X,Y\in\binom{[n]}{k}} [B]_{X,Y}\frac{\delta_{S,T}\delta_{X,Y}}{\binom{n}{k}}\right], \\
&= E_B\left[\sum_{X\in\binom{[n]}{k}} [B]_{X,X}\frac{\delta_{S,T}}{\binom{n}{k}}\right], \\
&= E_B\left[\delta_{S,T}\frac{\sigma_k(B)}{\binom{n}{k}}\right].
\end{aligned}$$

□

Lemma 3.2.5. *Let $a > d$, A an $a \times a$ random matrix and Q a random $a \times d$ matrix sampled from the Haar measure on $\mathbb{C}_a^d$, then*

$$E_A\left[E_Q\left[\chi_x\left(QAQ^*\right)\right]\right] = E_A\left[\frac{d!}{a!}\frac{\mathrm{d}^{(a-d)}}{\mathrm{d}x}\chi_x(Q)\right].$$

Proof. Let A be a fixed matrix and Q a Haar unitary matrix on $\mathbb{C}_a^d$, the kth coefficient of the expected characterictic polynomial of QAQ^* is

$$
\begin{aligned}
E_Q\left[\sigma_k(QAQ^*)\right] &= \sum_{|S|=k} E_Q\left[[QAQ^*]_{S,S}\right], \\
&= \sum_{|S|=k} \frac{\sigma_k(A)}{\binom{a}{k}}, \\
&= \frac{\binom{d}{k}\sigma_k(A)}{\binom{a}{k}}.
\end{aligned}
$$

Taking expectaton on the last expression we find

$$
E_A\left[E_Q\left[\chi_x\left(QAQ^*\right)\right]\right] = E_A\left[\frac{\binom{d}{k}\sigma_k(A)}{\binom{a}{k}}\right] = \frac{\binom{d}{k}\sigma_k(A)}{\binom{a}{k}},
$$

which is the kth coefficient of $\frac{d!}{a!}\frac{\mathrm{d}^{(a-d)}}{\mathrm{d}x} E_A\left[\chi_x(A)\right]$. ☐

Theorem 3.2.6. *Let A, B be $d \times d$ random matrices and R a $d \times d$ minor-orthogonal matrix, such that A, B, R are jointly independent, then we have*

$$
E_{A,B,R}\left[\sigma_k(A + RBR^*)\right] = \sum_{i=0}^{k} \frac{\binom{d-i}{k-i}}{\binom{d}{k-i}} E_A\left[\sigma_i(A)\right] E_A\left[\sigma_{k-i}(B)\right].
$$

Proof. We use

$$
\sigma_k(A) = \sum_{|S|=k} [A]_{S,S},
$$

together with Theorem 1.1.2 and Lemma 3.2.4 to get

Hay que aclarar las mas de matrices.

$$
\begin{aligned}
E_{A,B,R}\left[\sigma_k(A + RBR^*)\right] &= \sum_{S\in\binom{[d]}{k}} E_{A,B,R}\left[[A + RBR^*]_{S,S}\right], \\
&= \sum_{S\in\binom{[d]}{k}} \sum_{i=0}^{k} \sum_{U,V\in\binom{[k]}{i}} (-1)^{\|U\|_1+\|V\|_1} E_A\left[[A]_{U(S),V(S)}\right] E_{B,R}\left[[RBR^*]_{\overline{U}(S),\overline{V}(S)}\right], \\
&= \sum_{S\in\binom{[d]}{k}} \sum_{i=0}^{k} \sum_{U,V\in\binom{[k]}{i}} (-1)^{\|U\|_1+\|V\|_1} E_A\left[[A]_{U(S),V(S)}\right] \delta_{\overline{U}(S),\overline{V}(S)} \frac{E_B\left[\sigma_{k-i}(B)\right]}{\binom{d}{k-i}},
\end{aligned}
$$

using that $U(S) = V(S)$ if and only if $\overline{U}(S) = \overline{V}(S)$,

$$
= \sum_{i=0}^{k} \frac{E_B\left[\sigma_{k-i}(B)\right]}{\binom{d}{k-i}} \sum_{S\in\binom{[d]}{k}} \sum_{U,V\in\binom{[k]}{i}} E_A\left[[A]_{U(S),U(S)}\right].
$$

To finish the proof we need to find

$$
\sum_{S\in\binom{[d]}{k}} \sum_{U\in\binom{[k]}{i}} E_A\left[[A]_{U(S),U(S)}\right]. \tag{3.5}
$$

Clearly, we are summing over all of the sets $V \in \binom{[d]}{i}$, but they appear more than once in the sum. To find the number of times every element $V \in \binom{[d]}{i}$ appears in the sum, we can count the total number of terms we are summing in (3.5) and divide by the total number of elements in $\binom{[d]}{i}$. We have that $|\binom{[d]}{i}| = \binom{d}{i}$ and the number of summands is $\binom{d}{k}\binom{k}{i}$, so

$$\frac{\binom{d}{k}\binom{k}{i}}{\binom{d}{i}} = \frac{\frac{d!}{k!(d-k)!}\frac{k!}{i!(k-i)!}}{\frac{d!}{i!(d-i)!}} = \frac{(d-i)!}{(d-k)!(k-i)!} = \binom{d-i}{k-i}.$$

So, we have

$$\sum_{S\in\binom{[d]}{k}} \sum_{U\in\binom{[k]}{i}} E_A\left[[A]_{U(S),U(S)}\right] = \binom{d-i}{k-i} \sum_{V\in\binom{[d]}{i}} E_A\left[[A]_{V,V}\right] = \binom{d-i}{k-i} E_A\left[\sigma_i(A)\right].$$

Thus we can conclude

$$E_{A,B,R}\left[\sigma_k(A+RBR^*)\right] = \sum_{i=0}^{k} \frac{\binom{d-i}{k-i}}{\binom{d}{k-i}} E_A\left[\sigma_i(A)\right] E_A\left[\sigma_{k-i}(B)\right].$$

e aclarar qué sig- $U(S)$.

□

Theorem 3.2.7. *If $p(x)$ is the characteristic polynomial of A and $q(x)$ is the characteristic polynomial of B, where A and B are $d\times d$ normal matrices with complex entries, then*

$$p(x) \boxplus_d q(x) = E_Q\left[\chi_x(A+QBQ^*)\right],$$

where $\chi_x(\cdot)$ denotes the characteristic polynomial of $\cdot$ with x as a variable and E_Q denotes taking expectation over Q where Q is sampled from the Haar measure on the unitary complex $d\times d$ matrices.

Proof. It follows directly from Theorem 3.2.6 and definition of the symmetric additive convolution. □

Theorem 3.2.8. *Let A and B be $d\times d$ random matrices and R a minor-orthogonal $d\times d$ matrix, such that A, B, R are jointly independent, then*

$$E_{A,B,R}\left[\sigma_k(ARBR^*)\right] = \frac{E_A[\sigma_k(A)]E_B[\sigma_k(B)]}{\binom{d}{k}}.$$

Proof.

$$E_{A,B,R}\left[\sigma_k(ARBR^*)\right] = \sum_{S\in\binom{[d]}{k}} E_{A,B,R}\left[[ARBR^*]_{S,S}\right],$$

By the Cauchy-Binet formula and independence

$$= \sum_{S,T\in\binom{[d]}{k}} E_A\left[[A]_{S,T}\right] E_{B,R}\left[[RBR^*]_{T,S}\right],$$

By Lemma 3.2.4

$$\begin{aligned} &= \sum_{S,T\in\binom{[d]}{k}} E_A\left[[A]_{S,T}\right] \delta_{T,S} \frac{E_B\left[\sigma_k(B)\right]}{\binom{d}{k}}, \\ &= \frac{E_A[\sigma_k(A)]E_B[\sigma_k(B)]}{\binom{d}{k}}. \end{aligned}$$

□

Theorem 3.2.9. *Let $p(x)$ be the characteristic polynomial of A and $q(x)$ be the characteristic polynomial of B where A and B are $d\times d$ normal matrices with complex entries, then*

$$p(x) \boxtimes_d q(x) = E_Q \left[\chi_x(AQBQ^*) \right],$$

with χ_x and E_Q as in Theorem 3.2.7.

Proof. It follows directly from Theorem 3.2.8 and definition of the symmetric multiplicative convolution. □

3.3 The $\mathcal{R}_d$ transform

Given any polynomial $p(z)$ with order p we can associate an empirical measure μ_p to its roots z_i given by

$$\mu_p(\{x\}) = \frac{1}{n} \sum_{j=1}^{p} \delta_{x,z_j}.$$

This measure is similar to the spectral empirical measure of a random matrix. We can find its Cauchy transform in terms of the polynomial with the following lemma.

Lemma 3.3.1. *Let p be a monic polynomial of order p with roots $\{z_i\}_{i=1}^n$, then the Cauchy transform of the empirical measure associated to the roots z_i is given by*

$$G_{\mu_p}(z) \coloneqq \frac{1}{n} \sum_{j=1}^{n} \frac{1}{z - z_j} = \frac{\partial_z p}{np}(z).$$

Proof. $p(z)$ is a monic polynomial with roots $\{z_j\}_{j\in[n]}$, then we can write,

$$p(z) = \prod_{j=1}^{n} (z - z_j).$$

By the Leibnitz rule we find

$$\partial_z p(z) = \sum_{j=1}^{n} \prod_{k\neq j} (z - z_k).$$

Using the last equation we have

$$\frac{\partial_z p}{np}(z) = \frac{1}{n} \sum_{j=1}^{n} \frac{\prod_{k\neq j}(z - z_k)}{\prod_{l=1}^{n}(z - z_l)} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{z - z_j} \eqqcolon G_{\mu_p}(z).$$

□

Definition 3.3.1 (The $\mathcal{K}_d$ transform Perales Anaya (2016))**.** Let A be a $d \times d$ symmetric matrix with real entries. We define the $\mathcal{K}_d$ transform of its empirical spectral measure μ_A as

$$\mathcal{K}_d^{\mu_A}(s) \coloneqq -\frac{\partial}{\partial s} \ln \left\| e^{xs}[xI - A] \right\|_d,$$

where $[xI - A]$ represents the determinant of $xI - A$ and the integration domain for the norm is (ρ_A, ∞).

Definition 3.3.2 (The $\mathcal{R}_d$ transform)**.** Let A be a $d \times d$ symmetric matrix with real entries. We define the $\mathcal{R}_d$ transform of its empirical spectral measure μ_A as

$$\mathcal{R}_d^{\mu_A}(s) = \mathcal{K}_d^{\mu_A} - \left(1 + \frac{1}{d}\right) \frac{1}{s}.$$

Theorem 3.3.2. *Let A be a self-adjoint $d \times d$ matrix with empirical spectral distribution μ_A, then*

$$\lim_{d\to\infty} \mathcal{K}_d^{\mu_A}(s) = G_{\mu_A}^{-1}(s),$$

with $s \in (\rho_A, \infty)$ and where $G_{\mu_A}^{-1}(s)$ is the inverse under composition of $G_{\mu_A}(s)$.

Theorem 3.3.3 (Finite central Limit Theorem Marcus (2021))**.** *Let $p_1, p_2, \ldots$ be a sequence of degree d real rooted polynomials with $p_i = \prod_j (x - r_{i,j})$ such that*

$$\sum_j r_{i,j} = 0, \qquad \frac{1}{d}\sum_j r_{i,j}^2 = \sigma^2,$$

for all i. Define $q_i(x) = n^{-m/2} p_i(\sqrt{n}x)$, then

$$\lim_{n\to\infty} (q_1 \boxplus_d \cdots \boxplus_d q_n) = \left(\frac{d-1}{\sigma^2}\right)^{-d/2} H_d\left(x\sqrt{\frac{d-1}{\sigma^2}}\right),$$

with H_d represents the dth Hermite polynomial and the constants work to make it monic.

Chapter 4

Deterministic eigenvalue processes for matrix valued processes

In this last chapter we give a relationship between finite free probability and the eigenvalues of matrix-valued stochastic processes. In the first section construct a matrix process whose eigenvalues evolve according to the dynamics of the Dyson Brownian motion without the martingale part. We call this process the deterministic Dyson Brownian motion. In the second section we construct a similar matrix-valued process whose spectrum evolves according to the finite variation part of equation (2.13). We are especially interested in the deterministic versions of the Wishart and Jacobi processes. In the third section, we relate these processes with finite free probability by showing that the convolution of certain polynomials satisfies differential equations that ultimately leads to conclude that their roots follow a similar dynamics to the deterministic version of the eigenvalue processes. This constitutes the main result in this work.

4.1 Deterministic Dyson Brownian motion

In this section we prove that a given matrix-valued stochastic process has a deterministic spectrum and follows the dynamics of the finite variation part in the Dyson Brownian motion. The proof uses the same techniques as the former results for the stochastic differential equations of eigenvalue processes.

Theorem 4.1.1. *Let Z be a process with covariation $\mathrm{d}Z_{ij}\mathrm{d}Z_{kl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}\delta_{ik})\mathrm{d}t$ and no finite variation part, which means Z is a symmetric matrix with independent Brownian motions in its entries, except for the diagonal, where $Z_{ii} = 0$. Let X be a matrix valued process such that $X = H^T \Lambda H$ and it satisfies the stochastic differential equation*

$$H^T \mathrm{d}X H = \mathrm{d}Z.$$

Then the eigenvalue process Λ satisfies

$$\mathrm{d}\lambda_i = \sum_{k \neq i} \frac{\mathrm{d}t}{\lambda_i - \lambda_k}.$$

Proof. Define $\mathrm{d}A = H^T \partial H$ and $\mathrm{d}N = H^T \partial Z H$.

The same procedure as in 2.2.1 leads to

$$\mathrm{d}\Lambda = \mathrm{d}N + \Lambda \mathrm{d}A - \mathrm{d}A\Lambda.$$

We conclude that $\mathrm{d}\lambda_i = \mathrm{d}N_{ii}$ and for $i \neq j$,

$$\begin{aligned} 0 &= \mathrm{d}N_{ij} + \lambda_i \mathrm{d}A_{ij} - \lambda_j \mathrm{d}A_{ij}, \\ \Rightarrow \mathrm{d}A_{ij} &= \frac{\mathrm{d}N_{ij}}{\lambda_j - \lambda_i}. \end{aligned}$$

The quadratic covariation of N is the same as the one for Z because they only differ in a finite variation term, so

$$\mathrm{d}N_{ij}\mathrm{d}N_{kl} = \mathrm{d}\big\langle (H^T \mathrm{d}XH)_{ij}, (H^T \mathrm{d}XH)_{kl} \big\rangle(t) = \mathrm{d}\big\langle Z_{ij}, Z_{kl} \big\rangle = \big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj} - 2\delta_{ij}\delta_{kl}\delta_{ik}\big)\mathrm{d}t.$$

Particularly, we have that $\mathrm{d}N_{ii}\mathrm{d}N_{jj} = 0$ for every choice of j and i. Thus every entry in the diagonal of N is a finite variation process and so it is λ_i. Let us finally compute the finite variation part F of N.

$$\begin{aligned} \mathrm{d}F &= \frac{1}{2}\big(H^T \mathrm{d}X \mathrm{d}H + \mathrm{d}H^T \mathrm{d}XH\big), \\ &= \frac{1}{2}\big(H^T \mathrm{d}XHH^T \mathrm{d}H + \mathrm{d}H^T HH^T \mathrm{d}XH\big), \\ &= \frac{1}{2}\big(\mathrm{d}N\mathrm{d}A + (\mathrm{d}N\mathrm{d}A)^T\big). \end{aligned}$$

For $\mathrm{d}N\mathrm{d}A$ we have

$$\begin{aligned} (\mathrm{d}N\mathrm{d}A)_{ij} &= \sum_{k\neq j} \mathrm{d}N_{ik}\mathrm{d}A_{kj} = \sum_{k\neq j} \frac{\mathrm{d}N_{ik}\mathrm{d}A_{kj}}{\lambda_j - \lambda_k}, \\ &= \sum_{k\neq j} \frac{\delta_{ik}\delta_{kj} + \delta_{ij}\delta_{kk} - 2\delta_{ik}\delta_{kj}\delta_{ij}}{\lambda_j - \lambda_k}\mathrm{d}t = \delta_{ij}\sum_{k\neq j}\frac{\mathrm{d}t}{\lambda_j - \lambda_k}. \end{aligned}$$

Then F is diagonal with $\mathrm{d}F_{ii} = \sum_{k\neq i}\frac{\mathrm{d}t}{\lambda_i - \lambda_k}$. We conclude that

$$\mathrm{d}\lambda_i = \sum_{k\neq i}\frac{\mathrm{d}t}{\lambda_i - \lambda_k}.$$

□

4.2 Deterministic eigenvalue processes for matrix-valued diffusions

Now that we have shown the construction of a matrix-valued process whose eigenvalue is the deterministic Dyson Brownian motion, we generaliza the result to get processes with a deterministic spectrum that can follow the dynamics of any eigenvalue process with the form (2.13).

Theorem 4.2.1. *Let $B = (B(t), t \geq 0)$ be a Brownian motion in $\mathcal{M}_{p,p}(\mathbb{R})$ and $Y(t) = QMQ^T$ be a symmetric $p \times p$ matrix-valued stochastic process satisfying the stochastic differential equation*

$$\mathrm{d}Y(t) = g(Y(t))\mathrm{d}B(t)h(Y(t)) + h(Y(t))\mathrm{d}B(t)^T g(Y(t)) + b(Y(t))\mathrm{d}t, \tag{4.1}$$

where g, h, b are real functions acting spectrally, and $Y(0)$ is a symmetric $p \times p$ matrix with p different eigenvalues.

Let $G(x, y) = g^2(x)h^2(y)+g^2(y)h^2(x)$, τ be defined as in (2.12), and take a process $X = (X(t), t \geq 0)$ with diagonalization $X = H\Lambda H^T$ such that $H^T(\mathrm{d}\Lambda)H$ has the same off-diagonal entries as

$Q^T(\mathrm{d}M) \circ Q$ and has diagonal entries equal to zero.

Then, for $t < \tau$ the eigenvalue process $\Lambda(t)$ verifies the following system of stochastic differential equations:

$$\mathrm{d}\lambda_i = \left(b(\lambda_i) + \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} \right) \mathrm{d}t. \tag{4.2}$$

Proof. We define again L to be the stochastic logarithm of H, $L := H^T \circ \mathrm{d}H$ and using the same techniques as in Theorem 2.2.1 we have that,

$$\mathrm{d}\Lambda = H^T \circ (\partial X) \circ H - (\partial L) \circ \Lambda + \Lambda \circ \partial L.$$

Using that $\Lambda \circ \partial L - (\partial L) \circ \Lambda$ has zero diagonal, we get that $\mathrm{d}lambda_i = (H^T \circ (\partial X) \circ H)_{ii}$ and by hypothesis, we know that the martingale part of this diagonal is zero.

Define $\mathrm{d}N := H^T \circ (\partial X) \circ H$. For $i \neq j$ we have that $\mathrm{d}L_{ij} = \mathrm{d}N_{ij}/(\lambda_j - \lambda_i)$.

Finally, we compute the finite variation $\mathrm{d}F$ part of $\mathrm{d}N$,

$$\begin{aligned} \mathrm{d}F &= H^T b(X) H \mathrm{d}t + \frac{1}{2}(\mathrm{d}H^T \mathrm{d}X H + H^T \mathrm{d}X H), \\ &= b(\Lambda)\mathrm{d}t + \frac{1}{2}((\mathrm{d}N \mathrm{d}A)^T + \mathrm{d}N \mathrm{d}A). \end{aligned}$$

For $\mathrm{d}N\mathrm{d}A$ we find

$$(\mathrm{d}N\mathrm{d}A)_{ij} = \sum_{k \neq j} \mathrm{d}N_{ik}\mathrm{d}A_{kj} = \sum_{k \neq j} \frac{\mathrm{d}N_{ik}\mathrm{d}N_{kj}}{\lambda_j - \lambda_k}.$$

Now we use that the martingale part of $\mathrm{d}N$ has the same entries as $Q^T M Q$ and by the results in Theorem 2.2.1 we know that

$$(Q^T M Q)_{ik}(Q^T M Q)_{kj} = \delta_{ij} G(\lambda_i, \lambda_k)\mathrm{d}t,$$

so substituting the last result we get

$$\mathrm{d}\lambda_i = \mathrm{d}F_{ii} = b(\lambda_i)\mathrm{d}t + \sum_{k \neq j} \frac{G(\lambda_i, \lambda_k)\mathrm{d}t}{\lambda_j - \lambda_k},$$

which is the desired result.

□

These results can be particularized for any matrix-valued diffusions. Especially, we are interested in the Wishart and Jacobi processes. We give the proofs for these as a corollary to last Theorem.

4.2.1 Wishart process

Corollary 4.2.2.

4.2.2 Jacobi process

Corollary 4.2.3.

4.3 Connections with finite free probability

In this section we relate finite free probability to eigenvalue processes. To do so, we first find the expected characteristic polynomial of the random matrices. We show that in some cases they are well-known polynomials whose convolutions satisfy good properties. We start with the case of a self-adjoint Brownian matrix and then we replicate the results for the Wishart and Jacobi processes.

Lemma 4.3.1. *Let A_β denote an $n \times n$ matrix from the GOE, GUE or GSE, for $\beta = 1, 2, 4$, respectively, then the eigenvalues of the tridiagonal matrix H_β have the same joint law as the eigenvalues of A_β, with H_β defined as*

$$H_\beta = \frac{1}{\sqrt{2}} \begin{bmatrix} N_1 & \xi_2 & 0 & \cdots & 0 \\ \xi_2 & N_2 & \xi_3 & \cdots & 0 \\ 0 & \xi_3 & N_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \xi_n & N_n \end{bmatrix}. \tag{4.3}$$

The diagonal entries $N_i, 0 \leq i \leq n$ are independent normal random variables with mean 0 and variance 2, while the subdiagonal entries ξ_i are independent random variables distributed as

$$\xi_i \sim \chi_{\beta(n+1-i)}.$$

Note that in this case χ_ν denotes the chi distribution which is the squared root of a chi-squared random variable or the absolute value of a normal random variable in the case ν is integer.

We call H_β the tridiagonal β-Hermite ensemble.

Proof. Write A_β as

$$A_\beta = \begin{bmatrix} N_1 & \vec{x}^T \\ \vec{x} & B_\beta \end{bmatrix},$$

with N_1 a normal random variable, $\vec{x}$ an $n-1$-dimensional gaussian vector with independent entries in $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, depending on β, and B_β an $(n-1) \times (n-1)$ matrix from the GOE, GUE or GSE, respectively. All of the elements are independent from each other.

Now we take H to be any $(n-1) \times (n-1)$ orthogonal (or unitary, symplectic, according to β) matrix such that $H\vec{x}^T = \|\vec{x}\|_2 e_1$, where $e_1 = (1, 0, \cdots, 0)$ is the first element in the canonical basis of $\mathbb{R}^{n-1}$. Then we have

$$\begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & H \end{bmatrix} \begin{bmatrix} N_1 & \vec{x}^T \\ \vec{x} & B_\beta \end{bmatrix} \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & H^T \end{bmatrix} = \begin{bmatrix} N_1 & \vec{x}^T \\ \|\vec{x}\|_2 e_1 & HB_\beta \end{bmatrix} \begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & H^T \end{bmatrix} = \begin{bmatrix} N_1 & \|x\|_2 e_1^T \\ \|x\|_2 e_1 & HB_\beta H^T \end{bmatrix}.$$

Now we can find the distribution of each of the blocks of the new matrix. The variable N_1 has not changed, it is a standard normal variable. The term $\|\vec{x}\|_2$ is the norm of a Gaussian vector of length $n-1$ with real (complex or quaternionic) entries, non-correlated and with variance 1/2, so it is distributed like a $\frac{1}{\sqrt{2}}\chi_{\beta(n-1)}$ random variable, where β indicates the number of normal variables in each entry of the matrix. Since B_β is a GOE (GUE, GSE), it is invariant under orthogonal (unitary, symplectic) transformations and thus $HB_\beta H^T$ is a GOE (GUE, GSE).

The matrix $\begin{bmatrix} 1 & \vec{0}^T \\ \vec{0} & H \end{bmatrix}$ is orthogonal (unitary, symplectic), so the eigenvalue distribution of A_β remains unchanged under this transformation. By repeating the procedure with B_β, we find the tridiagonal matrix (4.3), which finishes the proof. ☐

Lemma 4.3.2. *Let W_β be an $n \times n$ matrix from the β-Wishart ensemble. The eigenvalues of W_β has the same joint law as those of the tridiagonal matrix $L_\beta = B_\beta B_\beta^T$ with B_β is an $m \times n$ bidiagonal matrix defined as*

$$B_\beta = \begin{bmatrix} \xi_{n\beta} & 0 & 0 & \cdots & 0 \\ \xi_{\beta(m-1)} & \xi_{n\beta-\beta} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \xi_\beta & \xi_{n\beta-\beta(m-1)} \end{bmatrix}. \tag{4.4}$$

The variables ξ_j being independent random variables with distribution $\xi_j \sim \chi_j$.
We call L_β the tridiagonal β-Laguerre ensemble.

Proof. Take G to be an $m \times n$ matrix with independent standard β-Gaussian random variables as entries (real if $\beta = 1$, complex if $\beta = 2$ and quaternions if $\beta = 4$). Write G as

$$\begin{bmatrix} \vec{x}^T \\ G_1 \end{bmatrix}.$$

We have that $\vec{x}$ is a vector distributed as a multivariate β-normal variable with mean $\vec{0}$ and covariance matrix $\Sigma = I$, while G_1 is an $(m-1) \times n$ matrix of independent standard β-Gaussian random variables.

Take R to be a "right reflector" corresponding to $\vec{x}$ independent of G_1, which means $\vec{x}^T R = \|\vec{x}\|_2 e_1$. Due to being a reflector, R is orthogonal (unitary, simplectic) and this means that $G_1 R$ is an $(m-1)\times n$ matrix with independent standard β-Gaussian matrices as entries.

Now take $G_1 R = [\vec{y}, G_2]$ with $\vec{y}$ being a β-Gaussian vector with mean $\vec{0}$ and covariance matrix $\Sigma = I$. Then G_2 is an $(m-1)\times(n_1)$ matrix of independent standard β-Gaussian random variables. Let L be a left reflector corresponding to $\vec{y}$ ($Ly = \|\vec{y}\|_2 e_1$) independent of G_2. Again by the orthogonality (unitarity, simplecticity) of L, LG_2 is still an $(m-1) \times (n-1)$ matrix of independent standard β-Gaussian random variables. This means that

$$\begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix} GR = \begin{bmatrix} \|\vec{x}\|_2 & 0 \\ \|\vec{y}\|_2 e_1 & LG_2 \end{bmatrix}.$$

We proceed with this procedure now for LG_2. The product by an orthogonal (unitary, simplectic) matrix does not affect the singular values of a matrix, so the singular values of the bidiagonal matrix B_β and the original matrix G are the same. The eigenvalues of $W = GG^T$ are the squares of the singular values of G and the same happens with the eigenvalues of $L_\beta = B_\beta B_\beta^T$ which is a tridiagonal matrix.

The distribution of the entries follows from the definition of the complex and simplectic normal distributions as sums of real normal random variables. $\square$

Lemma 4.3.3. *Let A be an $n \times n$ tridiagonal symmetric matrix with diagonal elements $\{a_i\}_{1\le i\le n}$ and subdiagonal elements $\{b_j\}_{2\le j\le n}$, such that*

$$A = \begin{bmatrix} a_n & b_{n-1} & 0 & \cdots & 0 \\ b_{n-1} & a_{n-1} & b_{n-2} & \cdots & 0 \\ 0 & b_{n-2} & a_{n-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_1 & a_1 \end{bmatrix}.$$

Then its characteristic polynomial $Q_n(x) = \det(xI_n - A)$ satisfies the following recursion

$$Q_n(x) = (x - a_n)Q_{n-1}(x) - b_{n-1}^2 Q_{n-2}(x).$$

Proof. Let us write

$$A = \begin{bmatrix} a_n & b_{n-1}e_1^T \\ b_{n-1}e_1 & B \end{bmatrix},$$

with e_1 the first element in the canonical base of $\mathbb{R}^{n-1}$, and write B as

$$B = \begin{bmatrix} a_{n-1} & b_{n-2}e_1^{(n-2)^T} \\ b_{n-2}e_1^{(n-2)} & C \end{bmatrix}.$$

The determinant of $xI_n - A$ is

$$\begin{aligned}\det(xI_n - A) &= (xI_n - A)_{11}\det(xI_{n-1} - B) - (xI_n - A)_{12}(xI_n - A)_{21}\det(xI_n - C)\\ &= (x - a_n)Q_{n-1}(x) - b_{n-1}^2 Q_{n-2}(x).\end{aligned}$$

□

Theorem 4.3.4. *Let X be an $n \times n$ GUE, then its expected characteristic polynomial, $P_n(x)$ is the nth Hermite polynomial.*

Proof. Let A_2 be a tridiagonalization of X, due to Lemma 4.3.1 we know that the eigenvalues of A_2 have the same joint distribution of those of X, so the expected characteristic polynomial must coincide. We can use Lemma 4.3.3 to find the expected characteristic polynomial of A. Let Q_n denote its characteristic polynomial, then

$$\begin{aligned}P_n(x) &= \mathbb{E}\left(\left(x - \frac{N_1}{\sqrt{2}}\right)Q_{n-1}(x) - \frac{\xi_{2(n-1)}^2}{2}Q_{n-2}(x)\right),\\ &= xP_{n-1}(x) - (n-1)P_{n-2}(x).\end{aligned}$$

So $P_n(x)$ satisfies the recursion that determines the Hermite polynomials. We need to check the initial conditions $P_1(x)$ and $P_2(x)$. For $P_1(x)$ the condition is trivial

$$P_1(x) = \mathbb{E}\left(x - \frac{N_1}{\sqrt{2}}\right) = x.$$

For $P_2(x)$, we have

$$\begin{aligned}P_2(x) &= \mathbb{E}\left(\det\begin{bmatrix} x - N_1 & \xi_2 \\ \xi_2 & x - N_2\end{bmatrix}\right) = \mathbb{E}\left[\left(x - \frac{N_1}{\sqrt{2}}\right)^2 - \frac{\xi_2^2}{2}\right],\\ &= x^2 - 1.\end{aligned}$$

So $P_1(x) = H_1(x)$ and $P_2(x) = H_2(x)$. Using the recursion, we can conclude that $P_n(x) = H_n(x)$. □

Corollary 4.3.5. *Let $B(t)$ be an $n \times n$ self-adjoint complex Brownian matrix, then its expected characteristic polynomial $P_n(x,t)$ is the nth generalized Hermite polynomial with variance t, $H^{[t]_n}$, i.e. the Hermite polynomials which are orthogonal with respect to a Gaussian random variable of variance t.*

Proof. We will prove the result by showing that $P_n(x,t)$ satisfies the following recursion

$$P_n(x,t) = xP_{n-1}(x,t) - t(n-1)P_{n-2}(x,t),$$

with initial conditions $P_1(x,t) = x$ and $P_2(x,t) = x^2 - t$.

For any given t, $B(t)$ has the same law as $\sqrt{t}A_2$ with A_2 a GUE. By Lemma 4.3.1, the expected characteristic polynomial is the same as that of $\sqrt{t}H_2$ with H_2 a 2-Hermite polynomial. Let $Q_n(x,t)$ be the characteristic polyomial of $\sqrt{t}H_2$, applying Lemma 4.3.3 we have

$$
\begin{aligned}
P_n(x,t) &= \mathbb{E}\left(\left(x-\sqrt{\frac{t}{2}}N_1\right)Q_{n-1}(x,t)-\frac{t}{2}\xi_{2(n-1)}^2 Q_{n-2}(x,t)\right),\\
&= xP_{n-1}(x,t)-t(n-1)P_{n-2}(x,t).
\end{aligned}
$$

Now we find the first two polynomials,

$$
\begin{aligned}
P_1(x,t) &= \mathbb{E}\left(x-\sqrt{\frac{t}{2}}N_1\right)=x,\\
P_2(x,t) &= \mathbb{E}\left(\det\begin{bmatrix} x-\sqrt{\frac{t}{2}}N_1 & \sqrt{\frac{t}{2}}\xi_2\\ \sqrt{\frac{t}{2}}\xi_2 & x-\sqrt{\frac{t}{2}}N_2\end{bmatrix}\right),\\
&= \mathbb{E}\left(\left(x-\sqrt{\frac{t}{2}}N_1\right)\left(x-\sqrt{\frac{t}{2}}N_2\right)-\frac{t}{2}\xi_2^2\right),\\
&= x^2-t.
\end{aligned}
$$

□

With the last result, we have found a recursion and initial conditions that are enough to uniquely determine the expected characteristic polynomial of a self-adjoint complex Brownian matrix.

Theorem 4.3.6. *The Hermite polynomials $H_n(z,t)$ solve the differential equation*

$$
\frac{\partial}{\partial t}H_n(z,t)+\frac{1}{2}\frac{\partial^2}{\partial z^2}H_n(z,t)=0, \tag{4.5}
$$

and the Cauchy transform $G_{H_n}(y)$ of the empirical measure associated to its roots $\{z_j(t)\}_{j\in[n]}$,

$$
G_{H_n}(z,t) := \frac{1}{n}\sum_{j=1}^{n}\frac{1}{z_j(t)-z},
$$

satisfies the viscous Burgers equation with diffusion coefficient $-1/2$,

$$
\frac{\partial G_{H_n}(z,t)}{\partial t}+nG_{H_n}(z,t)\frac{\partial G_{H_n}(z,t)}{\partial z}=-\frac{1}{2}\frac{\partial^2 G_{H_n}(z,t)}{\partial z^2}.
$$

Proof. Lets us write $H_n(z,t)=H_n$, $G_{H_n}(z,t)=G$ and $\frac{\partial f}{\partial z}=\partial_z f$ to simplify the notation.

First, we prove that H satisfies (4.5). Write $H_n(z,t)=\exp\left\{-\frac{t\partial_z^2}{2}\right\}(z^n)$, and

$$
\begin{aligned}
\partial_t H_n &= \partial_t \exp\left\{-\frac{t\partial_z^2}{2}\right\}(z^n)=\partial_t\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{t^j\partial_z^{2j}(z^n)}{2^j j!},\\
&= \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{\partial_t(t^j)\partial_z^{2j}(z^n)}{2^j j!}=\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{j(t^{j-1})\partial_z^{2j}(z^n)}{2^j j!},\\
&= \sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j\frac{j(t^{j-1})\partial_z^{2j}(z^n)}{2^j j!}=-\frac{1}{2}\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^{j-1}\frac{(t^{j-1})\partial_z^{2(j-1)}\partial_{zz}(z^n)}{2^{j-1}(j-1)!},\\
&= -\frac{1}{2}\partial_{zz}\sum_{k=0}^{\lfloor n/2-1\rfloor}(-1)^k\frac{(t^k)\partial_z^{2k}(z^n)}{2^k k!}=-\frac{1}{2}\partial_{zz}H_n.
\end{aligned}
$$

For the Cauchy transform part, we recall from Lemma 3.3.1 that since H_n is monic on z, then $G = \frac{\partial_z H_n}{n H_n}$.

Now, let us show that G satisfies (4.3.6), for this we compute $\partial_t G$, $G \partial_z G$ and $\partial_{zz} G$,

$$
\begin{aligned}
\partial_t G &= \frac{1}{n} \partial_t \left(\frac{\partial_z H_n}{H_n} \right) = \frac{1}{n} \frac{H_n \partial_t \partial_z H_n - \partial_z H_n \partial_t H_n}{H_n^2}, \\
\partial_z G &= \frac{1}{n} \partial_z \left(\frac{\partial_z H_n}{H_n} \right) = \frac{1}{n} \frac{H_n \partial_{zz} H_n - (\partial_z H_n)^2}{H_n^2}, \\
\partial_{zz} G &= \frac{1}{n} \partial_z \left(\frac{H_n \partial_{zz} H_n - (\partial_z H_n)^2}{H_n^2} \right), \\
&= \frac{1}{n} \frac{H_n^2 \partial_z H_n \partial_{zz} H_n + H_n^3 \partial_{zzz} H_n - 2 H_n^2 \partial_z H_n \partial_{zz} H_n}{H_n^4} \\
&\qquad - \frac{1}{n} \frac{2 H_n^2 \partial_z H_n \partial_{zz} H_n - 2 H_n (\partial_z H_n)^3}{H_n^4}, \\
&= \frac{1}{n} \frac{-3 H_n \partial_z H_n \partial_{zz} H_n + H_n^2 \partial_{zzz} H_n + 2 (\partial_z H_n)^3}{H_n^3} \\
G \partial_z G &= \frac{\partial_z H_n}{n H_n} \left(\frac{H_n \partial_{zz} H_n - (\partial_z H_n)^2}{H_n^2} \right) = \frac{1}{n^2} \frac{H_n \partial_z H_n \partial_{zz} H_n - (\partial_z H_n)^3}{H_n^3}.
\end{aligned}
$$

Finally, we can use the above results to find

$$
\begin{aligned}
\partial_t G + n G \partial_z G &= \frac{1}{n} \left(\frac{H_n \partial_t \partial_z H_n - \partial_z H_n \partial_t H_n}{H_n^2} + \frac{H_n \partial_z H_n \partial_{zz} H_n - (\partial_z H_n)^3}{H_n^3} \right), \\
&= \frac{1}{n} \left(\frac{-\frac{1}{2} H_n^2 \partial_{zzz} H_n + \frac{1}{2} H_n \partial_z H_n \partial_{zz} H_n + H_n \partial_z H_n \partial_{zz} H_n - (\partial_z H_n)^3}{H_n^3} \right), \\
&= \frac{1}{n} \left(\frac{\frac{3}{2} H_n^2 \partial_z H_n \partial_{zz} H_n - \frac{1}{2} H_n^2 \partial_{zzz} H_n - (\partial_z H_n)^3}{H_n^3} \right) = -\frac{1}{2} \partial_{zz} G.
\end{aligned}
$$

□

Corollary 4.3.7. *The roots $\{z_i(t)\}_{i \leq n}$ of the Hermite polynomials $H_n(z,t)$ satisfy the deterministic Dyson's equation,*

$$
\mathrm{d} z_i = \sum_{k \neq i} \frac{\mathrm{d}t}{z_i - z_k}.
$$

Proof. Let $z_i(t)$ be the roots of $H_n(z,t)$, this means

$$
\begin{aligned}
H_n(z_i(t), t) &= 0, \\
\partial_t H_n(z_i(t), t) &= 0.
\end{aligned}
$$

By the chain rule and the heat kind equation (4.5) in Theorem 4.3.6 we have

$$
0 = \partial_t H_n(z_i(t), t) = (\partial_t(z_i)) \partial_z H_n(z_i(t), t) - \frac{1}{2} \partial_{zz} H_n(z_i(t), t),
$$

by the Leibniz rule

$$\frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = \frac{\partial_{zz} H_n(z_i(t),t)}{2\partial_z H_n(z_i(t),t)} = \frac{2\sum_{k\neq i}\prod_{j\neq i, j\neq k}(z_i - z_j)}{2\prod_{j\neq i}(z_i - z_j)} = \sum_{k\neq i} \frac{1}{z_i - z_k}.$$

□

Theorem 4.3.8. *Let A be a $d \times d$ fixed matrix and W a $d \times d$ self-adjoint Brownian matrix, then $q_A(z,t)$ defined as*

$$q_A(z,t) \coloneqq \mathbb{E}\left[\chi_z(A+W)\right],$$

satisfies the following differential equation

$$\partial_t q_A(z,t) + \frac{1}{2}\partial_{zz} q_A(z,t) = 0.$$

Proof. Let $p(z) = \mathbb{E}\left[\chi_z(A)\right]$ and $r(z,t) = \mathbb{E}\left[\chi_z(W)\right]$. Corollary 4.3.5 tells us that $r(z,t)$ is the dth Hermite polynomial $H_d(z,t)$. First suppose that e write these polynomials as

$$\begin{aligned} p(z) &= \sum_{j=0}^{d} a_j z^j, \\ H_d(z,t) &= \sum_{j=0}^{d} b_j t^{j/2} z^{d-j}. \end{aligned}$$

Notice that if d is *odd* all of the b_j are zero for even j and if d is even, all of the b_j are zero for odd j. Further, using the explicit expression for the coefficients we have the following recursion for b_j

$$\begin{aligned} b_j &= \frac{d!(-1)^{j/2}}{2^{j/2}(j/2)!(d-j)!}, && (4.6)\\ b_{j-2} &= \frac{d!(-1)^{\frac{j-2}{2}}}{2^{\frac{j-2}{2}}\left(\frac{j-2}{2}\right)!(d-j+2)!}, && (4.7)\\ \Rightarrow b_j &= \frac{d!(-1)^{\frac{j-2}{2}}}{2^{\frac{j-2}{2}}\left(\frac{j-2}{2}\right)!(d-j+2)!}\frac{(-1)(d-j+2)(d-j+1)}{j}, && (4.8)\\ &= b_{j-2}\frac{(-1)(d-j+2)(d-j+1)}{j}. && (4.9) \end{aligned}$$

Using the invariance of W under unitary transforms and Theorem 3.2.7 we have that $q_A(z,t) = H_d(z,t) \boxplus_d p(z)$. By definition,

$$q_A(z,t) = H_d(z,t) \boxplus_d p(z) \coloneqq \sum_{k=0}^{d} z^{d-k}(-1)^k \sum_{i=0}^{k} c_{k,i,d} b_i a_{k-i} t^{i/2},$$

with $c_{k,i,d} = \frac{(d-i)!(d-k+i)!}{d!(d-k)!}$.

We compute first $\frac{\partial^2}{\partial z^2} q_A(z,t)$,

$$
\begin{aligned}
\partial_{zz} q_A(z,t) &= \partial_{zz}\left[\sum_{k=0}^{d} z^{d-k}(-1)^k \sum_{i=0}^{k} \frac{(d-i)!(d-k+i)!}{d!(d-k)!} b_i a_{k-i} t^{i/2}\right], \\
&= \sum_{k=0}^{d-2}(d-k)(d-k-1) z^{d-k-2}(-1)^k \sum_{i=0}^{k} \frac{(d-i)!(d-k+i)!}{d!(d-k)!} b_i a_{k-i} t^{i/2}, \\
&= \sum_{k=0}^{d-2} z^{d-k-2}(-1)^k \sum_{i=0}^{k} \frac{(d-i)!(d-k+i)!}{d!(d-k-2)!} b_i a_{k-i} t^{i/2}, \\
&= \sum_{k=2}^{d} z^{d-k}(-1)^k \sum_{i=0}^{k-2} \frac{(d-i)!(d-(k-2)+i)!}{d!(d-(k-2)-2)!} b_i a_{k-2-i} t^{i/2}, \\
&= \sum_{k=2}^{d} z^{d-k}(-1)^k \sum_{i=0}^{k-2} \frac{(d-i)!(d-k+i+2)!}{d!(d-k)!} b_i a_{k-2-i} t^{i/2}.
\end{aligned}
$$

Now the derivative with respect to t is

$$
\begin{aligned}
\partial_t q_A(z,t) &= \partial_t \left[\sum_{k=0}^{d} z^{d-k}(-1)^k \sum_{i=0}^{k} c_{k,i,d} b_i a_{k-i} t^{i/2}\right], \\
&= \sum_{k=0}^{d} z^{d-k}(-1)^k \sum_{i=0}^{k} \frac{(d-i)!(d-k+i)!}{d!(d-k)!} b_i a_{k-i} \frac{i}{2} t^{\frac{i-2}{2}}, \\
&= \sum_{k=0}^{d} z^{d-k}(-1)^k \sum_{j=0}^{k-2} \frac{(d-j-2)!(d-k+j+2)!}{d!(d-k)!} b_{j+2} a_{k-j-2} \left(\frac{j+2}{2}\right) t^{j/2}, \\
&= \sum_{k=0}^{d} z^{d-k}(-1)^k \sum_{j=0}^{k-2} \frac{(d-j)!(d-k+j+2)!}{d!(d-k)!} b_j a_{k-j-2} \frac{(j+2)t^{j/2}}{2},
\end{aligned}
$$

using (4.9) for b_{j+2}

$$
\begin{aligned}
&= \frac{1}{2}\sum_{k=0}^{d} z^{d-k}(-1)^{k+1} \sum_{j=0}^{k-2} \frac{(d-j)!(d-k+j+2)!}{d!(d-k)!} b_j a_{k-j-2} t^{j/2}, \\
&= -\frac{1}{2}\sum_{k=2}^{d} z^{d-k}(-1)^{k} \sum_{j=0}^{k-2} \frac{(d-j)!(d-k+j+2)!}{d!(d-k)!} b_j a_{k-j-2} t^{j/2} = -\frac{1}{2}\partial_{zz} q_A(z,t).
\end{aligned}
$$

□

Theorem 4.3.9. *Let R be a $d \times d$ matrix and R_{ij} be independent Gaussian random variables with mean 0 and variance 1, then*

$$E\left[\chi_x\left(RR^T\right)\right] = L_d(x),$$

where $L_d(x) = \left(1 - \frac{\mathrm{d}}{\mathrm{d}x}\right)^d x^d$ is the dth Laguerre polynomial.

Proof. We use the tridiagonal model defined before together with Lemma 4.3.3. Let $P_n(z)$ be the characteristic polynomial of L_β with L_β being a tridiagonal β-Laguerre ensemble, then

$$P_n(z) = (z - (L_\beta)_{11})P_{n-1}(z) - (L_\beta)_{12}^2 P_{n-2}(z).$$

The corresponding entries are

$$(L_\beta)_{11} = \xi_{n\beta}^2, \qquad (L_\beta)_{12} = \xi_{n\beta}(\xi_{\beta(m-1)} + \xi_{n\beta-\beta}),$$

with this the expected characteristic polynomial satisfies the recursion

$$\begin{aligned}\mathbb{E}\left[P_n(z)\right] &= \mathbb{E}\left[(z-\xi_{n\beta}^2)P_{n-1}(z) - \xi_{n\beta}^2(\xi_{\beta(m-1)}+\xi_{n\beta-\beta})^2 P_{n-2}(z)\right],\\ &= (z-n\beta)\mathbb{E}\left[P_{n-1}(z)\right] - n\beta(\beta(m-1)+n\beta-\beta)2\mathbb{E}\left[\xi_{\beta(m-1)}\xi_{n\beta-\beta}\right])\mathbb{E}\left[P_{n-2}(z)\right]\end{aligned}$$

□

Theorem 4.3.10. *Let $c > 0$ and $P(t,z) := (1-ct\partial_z)^n [z^n]$, with $t \geq 0, z \in \mathbb{C}$, then $P(t,z)$ satisfies the following differential equation*

$$c\partial_z P(t,z) + cz\partial_{zz}P(t,z) + \partial_t P(t,z) = 0.$$

Proof. By definition of $P(t,z)$,

$$P(t,z) = (1-ct\partial_z)^n [z^n] = \sum_{k=0}^{n} \binom{n}{k} (-ct)^k \partial_z^k [z^n] = (1-ct\partial_z)^n [z^n] = \sum_{k=0}^{n} \binom{n}{k} (-ct)^k \frac{n!}{(n-k)!} z^{n-k}.$$

Now we find the derivatives

$$\begin{aligned}\partial_z P(t,z) &= \partial_z \left[\sum_{k=0}^{n} \binom{n}{k} (-ct)^k \frac{n!}{(n-k)!} z^{n-k}\right] = \sum_{k=0}^{n-1} \binom{n}{k} (-ct)^k \frac{n!}{(n-k-1)!} z^{n-k-1},\\ \partial_{zz} P(t,z) &= \partial_z \left[\sum_{k=0}^{n-1} \binom{n}{k} (-ct)^k \frac{n!}{(n-k-1)!} z^{n-k-1}\right] = \sum_{k=0}^{n-2} \binom{n}{k} (-ct)^k \frac{n!}{(n-k-2)!} z^{n-k-2},\\ \partial_t P(t,z) &= \partial_t \left[\sum_{k=0}^{n} \binom{n}{k} (-ct)^k \frac{n!}{(n-k)!} z^{n-k}\right] = \sum_{k=1}^{n} \binom{n}{k} k(-c)^k t^{k-1} \frac{n!}{(n-k)!} z^{n-k},\\ &= \sum_{k=0}^{n-1} \binom{n}{k+1} (k+1)(-c)^{k+1} t^k \frac{n!}{(n-k-1)!} z^{n-k-1}.\end{aligned}$$

For the sum we have

$$\begin{aligned}c\partial_z P(t,z) + cz\partial_{zz} + \partial_t P(t,z) &= \sum_{k=0}^{n-2} \left[c\binom{n}{k} (-ct)^k \frac{n!}{(n-k-1)!} z^{n-k-1} + cz\binom{n}{k} (-ct)^k \frac{n!}{(n-k-2)!} z^{n-k-2}\right.\\ &\qquad \left. + \binom{n}{k+1}(k+1)(-c)^{k+1}t^k \frac{n!}{(n-k-1)!} z^{n-k-1}\right] + cn(-ct)^{n-1}n! + n(-c)^n t^{n-1} n!,\\ &= \sum_{k=0}^{n-2} \binom{n}{k}(-ct)^k n! \left(\frac{cz^{n-k-1} + (n-k-1)cz^{n-k-1} - c(n-k)z^{n-k-1}}{(n-k-1)!}\right),\\ &= \sum_{k=0}^{n-2} \binom{n}{k} \frac{c^{k+1}(-t)^k n! z^{n-k-1}}{(n-k-1)!} (1+n-k-1-n+k) = 0.\end{aligned}$$

□

Theorem 4.3.11. *Let $P(t,z)$ be a monic polynomial with degree n satisfying the equation*

$$c\partial_z P(t,z) + cz\partial_{zz}P(t,z) + \partial_t P(t,z) = 0.$$

Then its roots $(z_i(t))_{i=1}^n$ *satisfy the equation of motion*

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} = c\left(\sum_{k\neq i}\frac{z_i+z_k}{z_i-z_k}+n\right).$$

Proof. Let $z_i(t)$ be such that $P(t,z_i(t))=0$ for every t, this means in particular that $\partial_z P(t,z_i(t))=0$, so

$$\begin{aligned}
0 &= \partial_t P(t,z_i(t)) = \partial_t z_i(t)\partial_z P(t,z)|_{z=z_i} + \partial_t P(t,z)|_{z=z_i},\\
&= \partial_t z_i(t)\partial_z P(t,z)|_{z=z_i} - \left[c\partial_z P(t,z) - cz\partial_{zz}P(t,z)\right]_{z=z_i},
\end{aligned}$$

now we use that $P(t,z)$ is monic and the Leibnitz rule to get

$$\begin{aligned}
\partial_t z_i(t) &= c\left[\frac{\partial_z P(t,z)+z\partial_{zz}P(t,z)}{\partial_z P(t,z)}\right]_{z=z_i} = c\left[1+\frac{2z_i\sum_{k\neq i}\prod_{j\neq i,j\neq k}(z_i-z_j)}{\prod_{j\neq i}(z_i-z_j)}\right] = c\left[1+\sum_{k\neq i}\frac{2z_i}{z_i-z_k}\right],\\
&= c\left[1+\sum_{k\neq i}\frac{2z_i}{z_i-z_k}-\sum_{k\neq i}\frac{z_i-z_k}{z_i-z_k}+\sum_{k\neq i}\frac{z_i-z_k}{z_i-z_k}\right],\\
&= c\left[1+\sum_{k\neq i}\frac{2z_i-z_i+z_k}{z_i-z_k}+n-1\right] = c\left[\sum_{k\neq i}\frac{z_i+z_k}{z_i-z_k}+n\right].
\end{aligned}$$

□

Theorem 4.3.12. *Let* $w_n(t_1,z)=(1-ct_1\partial_z)^n[z^n]$ *and* $w_n(t_2,z)=(1-ct_2\partial_z)^n[z^n]$ *be two Laguerre polynomials of degree* n*, then their asymmetric additive convolution* $w_n(t_1,z)\boxplus\!\boxplus\, w_n(t_2,z)$ *is a Laguerre polynomial of degree* n *with variance* t_1+t_2*,* $w_n(t_1+t_2,z)$*.*

Proof.

$$\begin{aligned}
w_n(t_1,z)\boxplus\!\boxplus\, w_n(t_2,z) &= \sum_{k=0}^n z^{n-k}(-1)^k\sum_{j=0}^k\left(\frac{(n-j)!(n-k+j)!}{n!(n-k)!}\right)^2\binom{n}{j}\frac{(ct_1)^j n!}{(n-j)!}\binom{n}{k-j}\frac{(ct_1)^{k-j}n!}{(n-k+j)!}\\
&= \sum_{k=0}^n z^{n-k}(-1)^k\sum_{j=0}^k\frac{((n-j)!)^2((n-k+j)!)^2(n!)^4(ct_1)^j(ct_2)^{k-j}}{(n!)^2((n-k)!)^2 j!(k-j)!((n-j)!)^2((n-k+j)!)^2},\\
&= \sum_{k=0}^n\binom{n}{k}\frac{(-1)^k z^{n-k}n!}{(n-k)!}\sum_{j=0}^k\binom{k}{j}(ct_1)^j(ct_2)^{k-j},\\
&= \sum_{k=0}^n\binom{n}{k}\frac{(-1)^k z^{n-k}n!}{(n-k)!}[c(t_1+t_2)]^k = w_n(t_1+t_2,z).
\end{aligned}$$

□

Theorem 4.3.13. *Let* $w_n(t,z)$ *be a Laguerre polynomial of order* n *with variance* t *and let* $p(z)$ *be any monic polynomial. Then the asymmetric additive convolution of* $w_n(t,z)$ *and* $p(z)$*,* $w_n(t,z)\boxplus\!\boxplus\, p(z)$ *satisfies the following differential equation*

$$c\partial_z[w_n(t,z)\boxplus\!\boxplus\, p(z)] + cz\partial_{zz}[w_n(t,z)\boxplus\!\boxplus\, p(z)] + \partial_t[w_n(t,z)\boxplus\!\boxplus\, p(z)] = 0.$$

Proof. We find the derivatives first

$$\partial_z[w_n(t,z) \boxplus\!\!\boxplus\, p(z)] = \partial_z\left[\sum_{k=0}^{n} z^{n-k}(-1)^k \sum_{j=0}^{k}\left(\frac{(n-j)!(n-k+j)!}{n!(n-k)!}\right)^2 \binom{n}{j}\frac{(ct_1)^j n!}{(n-j)!}b_{k-j}\right],$$

$$= \sum_{k=0}^{n-1}(n-k)z^{n-k-1}(-1)^k \sum_{j=0}^{k}\left(\frac{(n-j)!(n-k+j)!}{n!(n-k)!}\right)^2 \binom{n}{j}\frac{(ct_1)^j n!}{(n-j)!}b_{k-j},$$

$$\partial_{zz}[w_n(t,z) \boxplus\!\!\boxplus\, p(z)] = \partial_{zz}\left[\sum_{k=0}^{n} z^{n-k}(-1)^k \sum_{j=0}^{k}\left(\frac{(n-j)!(n-k+j)!}{n!(n-k)!}\right)^2 \binom{n}{j}\frac{(ct_1)^j n!}{(n-j)!}b_{k-j}\right],$$

$$= \sum_{k=0}^{n-2}(n-k)(n-k-1)z^{n-k-2}(-1)^k \sum_{j=0}^{k}\left(\frac{(n-j)!(n-k+j)!}{n!(n-k)!}\right)^2 \binom{n}{j}\frac{(ct_1)^j n!}{(n-j)!}b_{k-j},$$

$$\partial_t[w_n(t,z) \boxplus\!\!\boxplus\, p(z)] = \partial_t\left[\sum_{k=0}^{n} z^{n-k}(-1)^k \sum_{j=0}^{k}\left(\frac{(n-j)!(n-k+j)!}{n!(n-k)!}\right)^2 \binom{n}{j}\frac{(ct_1)^j n!}{(n-j)!}b_{k-j}\right],$$

$$= \sum_{k=0}^{n} z^{n-k}(-1)^k \sum_{j=1}^{k}\left(\frac{(n-j)!(n-k+j)!}{n!(n-k)!}\right)^2 \binom{n}{j}\frac{c^j j t_1^{j-1} n!}{(n-j)!}b_{k-j},$$

$$= \sum_{k=1}^{n} z^{n-k}(-1)^k \sum_{j=0}^{k-1}\left(\frac{(n-j-1)!(n-k+j+1)!}{n!(n-k)!}\right)^2 \binom{n}{j+1}\frac{c^{j+1}(j+1)t_1^j n!}{(n-j-1)!}b_{k-j-1},$$

$$= \sum_{k=0}^{n-1} z^{n-k-1}(-1)^{k+1} \sum_{j=0}^{k}\left(\frac{(n-j-1)!(n-k+j)!}{n!(n-k-1)!}\right)^2 \binom{n}{j+1}\frac{c^{j+1}(j+1)t_1^j n!}{(n-j-1)!}b_{k-j},$$

$$= \sum_{k=0}^{n-1}(n-k)^2 z^{n-k-1}(-1)^{k+1} \sum_{j=0}^{k}\left(\frac{(n-j)!(n-k+j)!}{n!(n-k)!}\right)^2 \binom{n}{j}\frac{c^{j+1}t_1^j n!}{(n-j)!}b_{k-j},$$

For the sum we have

$$\begin{aligned}
&(c\partial_z + cz\partial_{zz} + \partial_t)[w_n(t,z) \boxplus\!\!\boxplus\, p(z)] \\
&= \sum_{k=0}^{n-2} z^{n-k-1}(-1)^k \sum_{j=0}^{k}\left(\frac{(n-j)!(n-k+j)!}{n!(n-k-1)!}\right)^2 \binom{n}{j}\frac{c^{j+1}t_1^j n!}{(n-j)!}b_{k-j}\Big(n-k+(n-k)(n-k-1)-(n-k)^2\Big) \\
&\qquad + (-1)^{n-1}\sum_{j=0}^{n-1}\left(\frac{(j+1)!}{n!}\right)^2 \binom{n}{j}\frac{c^{j+1}t_1^j n!}{(n-j)!}b_{n-j+1} + (-1)^n \sum_{j=0}^{n-1}\left(\frac{(j+1)!}{n!}\right)^2 \binom{n}{j}\frac{c^{j+1}t_1^j n!}{(n-j)!}b_{n-j+1} = 0.
\end{aligned}$$

□

As defined in Doumerc (2005) the generalized singular values problem can be expressed by the following matrix

$$W = (M_1^T M_1 + M_2^T M_2)^{-1} M_1^T M_1 (M_1^T M_1 + M_2^T M_2)^{-1},$$

where M_1, M_2 are independent Gaussian matrices with dimensions $n_1 \times k$ and $n_2 \times k$, respectively. We can write $W_i = M_i^T M_i$ for a shorter notation. Now to compute the characteristic polynomial we have,

$$\begin{aligned}
p_W(z) &= \det\left[zI - (W_1+W_2)^{-\frac{1}{2}}W_1(W_1+W_2)^{-\frac{1}{2}}\right], \\
&= \det\left[(W_1+W_2)^{-1}\right]\det\left[z(W_1+W_2) - (W_1+W_2)^{\frac{1}{2}}W_1(W_1+W_2)^{-\frac{1}{2}}\right], \\
&= \det\left[(W_1+W_2)^{-1}\right]\det\left[z(W_1+W_2)(W_1+W_2)^{\frac{1}{2}}(W_1+W_2)^{-\frac{1}{2}} - (W_1+W_2)^{\frac{1}{2}}W_1(W_1+W_2)^{-\frac{1}{2}}\right], \\
&= \det\left[(W_1+W_2)^{-1}\right]\det\left[(W_1+W_2)^{\frac{1}{2}}\right]\det\left[z(W_1+W_2) - W_1\right]\det\left[(W_1+W_2)^{-\frac{1}{2}}\right], \\
&= \det\left[(W_1+W_2)^{-1}\right]\det\left[(z-1)W_1 + zW_2\right].
\end{aligned}$$

Thus finding the expected characteristic polynomial (up to a normalization constant) can be done without the need to use the inverse of $W_1 + W_2$.

In order to find the associated process, we will first generalize the results of the static Wishart and Jacobi matrices with an operator that generalized characteristic polynomial, then we will show that the Wishart process can be built in the base of this operator by adding a time parameter. Finally, we will tray to extend this same procedure to the Jacobi process which, as we will see, is less straightforward. In the process of building this generalizations we will make extensive use of finite free probability.

Definition 4.3.1 (Generalized characteristic polynomial)**.** Let A, B be two random independent Gaussian matrices in $\mathcal{M}_{n_1,k}(\mathbb{C})$ and $\mathcal{M}_{n_2,k}(\mathbb{C})$ respectively. We define the generalized characteristic polynomial $p_{A,B}(x,y,z)$ as

$$p_{A,B}(x,y,z) \coloneqq \det\left[xI + yA^TA + zB^TB\right].$$

Similarly, we define the reciprocal generalized reciprocal polynomial as $q_{A,B}(x,y,z) \coloneqq y^{n_1}z^{n_2}P_{A,B}(x,y,z)$.

Notice that this generalizes the characteristic polynomial of the (static and dynamical) Wishart and the static Jacobi cases.

Theorem 4.3.14. *Let F and G be two-variable polynomials and $A, B \in \mathcal{M}_{k,n_1}, C, D \in \mathcal{M}_{k,n_2}$ such that C and D are invariant under product by signed permutation matrices. Suppose that*

$$\begin{aligned}
q_{A,B}(x,y,z) &= F(\partial_x\partial_y, \partial_x\partial_z)[x^ky^{n_1}z^{n_2}], \\
q_{C,D}(x,y,z) &= G(\partial_x\partial_y, \partial_x\partial_z)[x^ky^{n_1}z^{n_2}].
\end{aligned}$$

Then $q_{A+C,B+D}(x,y,z) = F(\partial_x\partial_y, \partial_x\partial_z)G(\partial_x\partial_y, \partial_x\partial_z)[x^ky^{n_1}z^{n_2}]$

Theorem 4.3.15. *Let $\{A_i\}_{i=0}^{\infty}$ and $\{B_i\}_{i=0}^{\infty}$ be sequences of matrices invariant under transformation by signed permutation matrices such that*

$$\begin{aligned}
\mathbb{E}\left[Tr(A_i^TA_i)\right] &= \sigma_1 n_1 k, \\
\mathbb{E}\left[Tr(B_i^TB_i)\right] &= \sigma_2 n_2 k.
\end{aligned}$$

If we define C_m, D_m as

$$\begin{aligned}
C_m &\coloneqq \sum_{i=0}^{m} \frac{A_i}{\sqrt{m}}, \\
D_m &\coloneqq \sum_{i=0}^{m} \frac{D_i}{\sqrt{m}}.
\end{aligned}$$

Then the matrices C_m, D_m satisfy

$$\lim_{m\to\infty} \mathbb{E}\left[q_{C,D}(x,y,z)\right] = e^{\sigma_1\partial_x\partial_y+\sigma_2\partial_x\partial_z}[x^k y^{n_1} z^{n_2}].$$

Proof. First we use the following expansion for the determinant,

$$\det(A+hB) = \det(A) + h\,Tr\left(\operatorname{adj}(A)\,B\right) + O\left(h^2\right).$$

Agregar un apéndic esto

This means for $q_{A_i/\sqrt{m},B_i/\sqrt{m}}(x,y,z)$ that

$$\begin{aligned}
\mathbb{E}\left[q_{A_i/\sqrt{m},B_i/\sqrt{m}}(x,y,z)\right] &= y^{n_1}z^{n_2}\mathbb{E}\left[\det[xI + \frac{1}{m}\left(y^{-1}A_i^TA_i + z^{-1}B_i^TBI\right)]\right],\\
&= y^{n_1}z^{n_2}\left[x^k + \frac{x^{k-1}y^{-1}}{m}\mathbb{E}\left[Tr(A_i^TA_i)\right] + \frac{x^{k-1}z^{-1}}{m}\mathbb{E}\left[Tr(B_i^TB_i)\right] + O\left(\frac{1}{m^2}\right))\right],\\
&= x^ky^{n_1}z^{n-2} + \frac{\sigma_1n_1k}{m}x^{k-1}y^{n_1-1}z^{n-2} + \frac{\sigma_2n_2k}{m}x^{k-1}y^{n_1}z^{n_2-1} + O\left(\frac{1}{m^2}\right),\\
&= \left(1 + \frac{\sigma_1}{m}\partial_y\partial_x + \frac{\sigma_2}{m}\partial_z\partial_x + O\left(\frac{1}{m^2}\right)\right)[x^ky^{n_1}z^{n_2}]
\end{aligned}$$

Once we have found $q_{A_i/\sqrt{m},B_i/\sqrt{m}}(x,y,z)$ as a polynomial on the operators $\partial_x\partial_y$ and $\partial_x\partial_z$, we apply Theorem 4.3.14 to get, for $q_{C_m,D_m}(x,y,z)$,

$$\mathbb{E}\left[q_{C_m,D_m}(x,y,z)\right] = \left(1 + \frac{\sigma_1}{m}\partial_y\partial_x + \frac{\sigma_2}{m}\partial_z\partial_x + O\left(\frac{1}{m^2}\right)\right)^m [x^ky^{n_1}z^{n_2}]$$

Tanking $m\to\infty$ in the last expression we get

$$\begin{aligned}
\lim_{m\to\infty}\mathbb{E}\left[q_{C_m,D_m}(x,y,z)\right] &= \lim_{m\to\infty}\left(1 + \frac{\sigma_1}{m}\partial_y\partial_x + \frac{\sigma_2}{m}\partial_z\partial_x + O\left(\frac{1}{m^2}\right)\right)^m [x^ky^{n_1}z^{n_2}],\\
&= e^{\sigma_1\partial_x\partial_y+\sigma_2\partial_x\partial_z}[x^ky^{n_1}z^{n_2}],
\end{aligned}$$

which is the desired result. $\square$

Corollary 4.3.16. *Let A and B be equal in law to σ_1N_1 and σ_2N_2 where N_1 and N_2 are matrices with all of the entries being independent standard normal random variables, then*

$$\mathbb{E}\left[q_{A,B}(x,y,z)\right] = e^{\sigma_1\partial_x\partial_y+\sigma_2\partial_x\partial_z}[x^ky^{n_1}z^{n_2}].$$

Proof. Let $\{A_i\}_{i\in\mathbb{N}}$ and $\{B_i\}_{i\in\mathbb{N}}$ be as in Theorem 4.3.15 and define C_m, D_m in the same way. On the one hand we have, as shown previously

$$\lim_{m\to\infty}\mathbb{E}\left[q_{C_m,D_m}(x,y,z)\right] = e^{\sigma_1\partial_x\partial_y+\sigma_2\partial_x\partial_z}[x^ky^{n_1}z^{n_2}].$$

Due to the existence of the second moment of the entries, on the other hand, we have that C_m, D_m converge in law to a Gaussian independent matrix whose entries have variance σ_1^2 and σ_2^2, respectively, i.e.

$$\lim_{m\to\infty}\mathbb{E}\left[q_{C_m,D_m}(x,y,z)\right] = \mathbb{E}\left[q_{\sigma_1N_1,\sigma_2N_2}(x,y,z)\right].$$

$\square$

With this we have an operator that allows us to compute $q_{A,B}(x,y,z)$ for A, B independent Gaussian matrices with zero mean and variances σ_1, σ_2, respectively. Our goal is to be able to use this to study matrix-valued stochastic processes by letting the variances vary linearly in time. Hence, the entries of the matrices have the same law as a standard Brownian motion starting at zero. An interesting question is how this behavior would be affected if we start our processes at points different than zero. The answer is given by Theorem 4.3.14 and noticing that any polynomial $r(x,y,z)$ of orders k, n_1, n_2 can be seen as a polynomial differential operator acting on $x^k y^{n_1} y^{n_2}$, so if $S \in \mathcal{M}_{k,n_1}(\mathbb{C}), T \in \mathcal{M}_{k,n_2}(\mathbb{C})$ are fixed matrices and A, B are Gaussian independent matrices with variances σ_1 and σ_2, respectively, then

$$\mathbb{E}\left[q_{S+A,T+B}(x,y,z)\right] = e^{\sigma_1 \partial_x \partial_y + \sigma_2 \partial_x \partial_z}[q_{S,T}(x,y,z)].$$

We are interested in $p_{A,B}(x,y,)$, the reciprocal polynomial in y and z. Let us denote by $R_k^z(\cdot)$ the reciprocal polynomial operator of order k in the variable z, which means, for $r(x,y,z)$ a polynomial,

$$R_k^z\left(r(x,y,z)\right) = z^k r\left(x, y, \frac{1}{z}\right).$$

Now notice that for A, B Gaussian independent matrices with variances σ_1, σ_2 and given matrices S, T we have $q_{S+A,T+B}(x,y,z) = R_{n_2}^z(R_{n_1}^y(p_{S+A,T+B}(x,y,z))$, so

$$\begin{aligned}
\mathbb{E}\left[p_{A,B}(x,y,z)\right] &= R_{n_2}^z\left(R_{n_1}^y\left(\mathbb{E}\left[q_{S+A,T+B}(x,y,z)\right]\right)\right), \\
&= R_{n_2}^z\left(R_{n_1}^y\left(e^{\sigma_1 \partial_x \partial_y + \sigma_2 \partial_x \partial_z} q_{S,T}(x,y,z)\right)\right), \\
&= R_{n_2}^z\left(R_{n_1}^y\left(e^{\sigma_1 \partial_x \partial_y + \sigma_2 \partial_x \partial_z} R_{n_2}^z(R_{n_1}^y(p_{S,T}(x,y,z)\right)\right), \\
&= \left(R_{n_2}^z \circ R_{n_1}^y \circ e^{\sigma_1 \partial_x \partial_y + \sigma_2 \partial_x \partial_y} \circ R_{n_2}^z \circ R_{n_1}^y\right)[p_{S,T}(x,y,z)].
\end{aligned}$$

We will find an expression for this operator applied to $x^i y^j z^l$ with $j \leq n_1, l \leq n_2$. Then it is extended linearly to a general polynomial.

$$\begin{aligned}
&(R_{n_2}^z \circ R_{n_1}^y \circ e^{\sigma_1 \partial_x \partial_y + \sigma_2 \partial_x \partial_y} \circ R_{n_2}^z \circ R_{n_1}^y)[x^i y^j z^l] \\
&\quad = (R_{n_2}^z \circ R_{n_1}^y \circ e^{\sigma_1 \partial_x \partial_y + \sigma_2 \partial_x \partial_y})[x^i y^{n_1-j} z^{n_2-l}], \\
&\quad = (R_{n_2}^z \circ R_{n_1}^y \circ e^{\sigma_2 \partial_x \partial_z}) \sum_{r=0}^{\infty} \frac{\sigma_1^r}{r!} \partial_x^r \partial_y^r [x^i y^{n_1-j} z^{n_2-l}], \\
&\quad = (R_{n_2}^z \circ R_{n_1}^y \circ e^{\sigma_2 \partial_x \partial_z}) \sum_{r=0}^{\infty} \frac{\sigma_1^r}{r!} \partial_x^r [x^i] \frac{(n_1-j)!}{(n_1-j-r)!} y^{n_1-j-r} z^{n_2-l}, \\
&\quad = (R_{n_2}^z \circ R_{n_1}^y \circ e^{\sigma_2 \partial_x \partial_z}) z^{-l} \sum_{r=0}^{\infty} \binom{n_1-j}{r} y^{n_1-j-r} \sigma_1^r \partial_x^r [x^i],
\end{aligned}$$

doing the same procedure for $e^{\sigma_2 \partial_x \partial_z}$ we get

$$\begin{aligned}
&= (R_{n_2}^z \circ R_{n_1}^y \circ e^{\sigma_2 \partial_x \partial_z}) z^{n_2-l} \left(y + \sigma_1 \partial_x\right)^{n_1-j} [x^i], \\
&= R_{n_2}^z \circ R_{n_1}^y \left\{ \left(z + \sigma_2 \partial_x\right)^{n_2-l} \left(y + \sigma_1 \partial_x\right)^{n_1-j} [x^i] \right\}, \\
&= \left(1 + z\sigma_2 \partial_x\right)^{n_2-l} \left(1 + y\sigma_1 \partial_x\right)^{n_1-j} [x^i y^j z^l].
\end{aligned}$$

In the case when we start the processes at the origin we have that

$$p_{S,T}(x,y,z) = \det[xI + y0_{n_1,k} + z0_{n_2,k}] = x^k \det[I] = x^k,$$

which leads to

$$\mathbb{E}\left[p_{A,B}(x,y,z)\right] = \left(1 + z\sigma_2 \partial_x\right)^{n_2-l} \left(1 + y\sigma_1 \partial_x\right)^{n_1-j} [x^k]. \tag{4.10}$$

If we let $\sigma_1 = \sigma_2 = 1$ and replace $y = -1, z = 0$ in (4.10), we recover the static Wishart matrix and we can conclude that

$$\mathbb{E}\left[\det[xI - A^T A]\right] = (1 - \partial_x)^{n_1}[x^k], \tag{4.11}$$

which is exactly the associated Laguerre polynomial previously found in . cita

Keeping the unitary variances and replacing $x = 0, y = (x-1), z = x$ we get the static Jacobi matrix, as shown previously, thus

$$\mathbb{E}\left[\det[(x-1)W_1 + xW_2]\right] = (1 + x\partial_x)^{n_2}\left(1 + (x-1)\partial_x\right)^{n_1}[x^k] = P_k^{n_2-k, n_1-k}(2x-1).$$

This is the Jacobi polynomial of order k with parameters $n_2 - k$ and $n_1 - k$ evaluated in $2x - 1$. When we normalize to make it monic we get the result appearing in Edelman (1988).

Now let $\sigma_1 = \sigma_2 = t$, then A, B are equal in law to standard Brownian motions in $\mathcal{M}_{n_1,k}(\mathbb{R})$ and $\mathcal{M}_{n_2,k}(\mathbb{R})$, respectively. We should recover the expected characteristic polynomial of the Wishart and Jacobi processes. The Wishart case is evident by replacing -1 with $-t$ in (4.11) and compare it to equation. For the Jacobi process, we have referencia

$$\mathbb{E}\left[\det[(x-1)W_1(t) + xW_2(t)]\right] = (1 + xt\partial_x)^{n_2}\left(1 + (x-1)t\partial_x\right)^{n_1}[x^k]. \tag{4.12}$$

In analogy to what we found for the Dyson Brownian motion and the Wishart process, we need to verify if the roots of the polynomial defined in (4.12) satisfy a differential equation given by the finite variation part of (**??**). Let us work in full generality and define for polynomials of the form Agregar cita $x^i y^j z^l$, the operator

$$Q^t_{n_1,n_2}[x^i y^j z^l] = (1 + zt\partial_x)^{n_2 - l}\left(1 + yt\partial_x\right)^{n_1 - j}[x^i y^j z^l],$$

that is later extended to general polynomials linearly.

Denote the time-dependant polynomial $Q^t_{n_1,n_2}[p(x,y,z)]$ by $\hat{p}(x,y,z,t)$. In analogy to what we have done with the Hernmite and Laguerre cases, we want to find for every t the values $x(t), y(t), z(t)$ such that $\hat{p}(x(t), y(t), z(t), t) = 0$. By derivating with respect to t, we get

$$\begin{aligned}
0 &= \partial_t[\hat{p}(x(t), y(t), z(t), t)] \\
&= \left[(\partial_t x(t))\partial_x \hat{p}(x,y,z,t) + (\partial_t y(t))\partial_y \hat{p}(x,y,z,t) + (\partial_t z(t))\partial_z \hat{p}(x,y,z,t) + \partial_t \hat{p}(x,y,z,t)\right]_{(x,y,z)=(x(t),y(t),z(t))}.
\end{aligned}$$

Lemma 4.3.17. *The time-dependant polynomial $\hat{p}(x,y,z,t)$ satisfies*

$$\partial_t[\hat{p}(x,y,z,t)]_{t=0} = (n_1 y + n_2 z)\partial_x[\hat{p}(x,y,z,t)]_{t=0} - y^2\partial_y\partial_x[\hat{p}(x,y,z,t)]_{t=0} - z^2\partial_z\partial_x[\hat{p}(x,y,z,t)]_{t=0}. \tag{4.13}$$

Proof. A general polynomial p on x, y, z, can be written as

$$p(x,y,z) = \sum_{i,j,l} c_{ijl} x^i y^j z^l,$$

where c_{ijl} is the coefficiente associated to the term $x^i y^j z^l$. When we apply $Q^t_{n_1,n_2}$ to $p(x,y,z)$ we have

$$\begin{aligned}
Q^t_{n_1,n_2}[p(x,y,z)] &= (1+zt\partial_x)^{n_2-l}(1+yt\partial_x)^{n_1-j}\left[\sum_{i,j,l} c_{ijl}x^iy^jz^l\right],\\
&= \sum_{i,j,l} c_{ijl}(1+zt\partial_x)^{n_2-l}(1+yt\partial_x)^{n_1-j}\left[x^iy^jz^l\right]\\
&= \sum_{i,j,l} c_{ijl}\sum_{r=0}^{n_2-l}\binom{n_2-l}{r}(zt\partial_x)^r\sum_{s=0}^{n_1-j}\binom{n_1-j}{s}(yt\partial_x)^{n_1-j}[x^iy^jk^l],\\
&= \sum_{i,j,l} c_{ijl}\left\{(n_2-l)zt\partial_x+(n_1-j)yt\partial_x+O(t^2)\right\}[x^iy^jk^l],\\
&= \sum_{i,j,l} c_{ijl}\left\{(n_2-l)itx^{i-1}y^jz^{l+1}+(n_1-j)itx^{i-1}y^{j+1}z^l\right\}+O(t^2).
\end{aligned}$$

Then, differentiating in t and evaluating at $t=0$ gives us

$$\begin{aligned}
\partial_t\left\{Q^t_{n_1,n_2}[p(x,y,z)]\right\}\big|_{t=0} &= \partial_t\left\{\sum_{i,j,l} c_{ijl}\left\{(n_2-l)itx^{i-1}y^jz^{l+1}+(n_1-j)itx^{i-1}y^{j+1}z^l\right\}+O(t^2)\right\}\Bigg|_{t=0},\\
&= \sum_{ijl} c_{ijl}\{(n_2-l)ix^{i-1}y^jz^{l+1}+(n_1-j)ix^{i-1}y^{j+1}z^l\}.
\end{aligned}$$

We can re-write the last two terms in every summand as

$$\begin{aligned}
(n_2-l)ix^{i-1}y^jz^{l+1} &= n_2z\partial_x[x^iy^jz^l]-z^2\partial_x\partial_z[x^iy^jz^l],\\
(n_1-j)ix^{i-1}y^{j+1}z^l &= n_1y\partial_x[x^iy^jz^l]-y^2\partial_x\partial_y[x^iy^jz^l].
\end{aligned}$$

Extending by linearity to p we have that

$$\partial_t[\hat{p}]_{t=0} = (n_1y+n_2z)\partial_x[p]-y^2\partial_y\partial_x[p]-z^2\partial_z\partial_x[p].$$

This finishes the proof since $p(x,y,z)=\hat{p}(x,y,z,t)|_{t=0}$. ◻

This last result together with (4.13) give us that

$$\begin{aligned}
\partial_t[x(t)]\partial_x[p(x(t),y(t),z(t))]+\partial_t[y(t)]\partial_y[p(x(t),y(t),z(t))]+\partial_t[z(t)]\partial_z[p(x(t),y(t),z(t))\\
= -(n_1y+n_2z)\partial_x[p]+y^2\partial_y\partial_x[p]+z^2\partial_z\partial_x[p].
\end{aligned}$$

Lemma 4.3.18. *Let $r(x,y,z)$ be a homogeneous polynomial of order k in x,y,z, then*

$$y^2\partial_y\partial_xr+z^\partial_z\partial_xr=(y+z)(k-1)\partial_xr-yz(\partial_z\partial_xr+\partial_y\partial_xr)-(y+z)x\partial_{xx}r.$$

Proof. The fact that r is homogeneous with degree k implies that ∂_xr is homogeneus with degree $k-1$, then

$$z\partial_z\partial_xr+y\partial_y\partial_xr+x\partial_{xx}r=(k-1)\partial_xr.$$

The last equality implies

$$y^2\partial_y\partial_x r = y(k-1)\partial_x r - yz\partial_z\partial_x r - yx\partial_{xx}r,$$
$$z^2\partial_z\partial_x r = z(k-1)\partial_x r - zy\partial_y\partial_x r - zx\partial_{xx}r.$$

And summing up these two expressions leads to

$$y^2\partial_y\partial_x r + z^2\partial_z\partial_x r = (y+z)(k-1)\partial_x r - yz(\partial_z\partial_x r + \partial_y\partial_x r) - (y+z)x\partial_{xx}r.$$

□

Then we have for $\hat{p}$

$$\begin{aligned}(\partial_t x)(\partial_x \hat{p}) + (\partial_t y)(\partial_y \hat{p}) + (\partial_t z)(\partial_z \hat{p})|_{t=0} \\ = -(n_1 y + n_2 z)\partial_x\hat{p} + (y+z)(k-1)\partial_x\hat{p} - yz(\partial_z\partial_x\hat{p} + \partial_y\partial_x\hat{p}) - (y+z)x\partial_{xx}\hat{p}|_{t=0}.\end{aligned}$$

Now we can substitute $y = w, z = (w-1)$ and define the polynomial $j(x,w,t)$ as

$$j(x,w,t) = \hat{p}(x(t), w(t), w(t)-1, t).$$

Using the above results, we have for $j(w,t)$

$$\begin{aligned}(\partial_t x)(\partial_x j) + (\partial_t w)(\partial_w j)|_{t=0} \\ = -(n_1 w + n_2 w - n_2))\partial_x j + (2w-1)(k-1)\partial_x j - w(w-1)(\partial_x\partial_w j) - (2w-1)x\partial_{xx}j|_{t=0}.\end{aligned}$$

Letting $x = 0$, we find the evolution equation for the matrix Jacobi process Agregar referencia

$$\begin{aligned}(\partial_t w)\frac{\partial_w j}{\partial_x j}\bigg|_{t=0} &= -(n_1 w + n_2 w) + (2w-1)(k-1) - w(w-1)\frac{\partial_x\partial_w j}{\partial_x j}\bigg|_{t=0}, \\ \Rightarrow \partial_t w &= -[n_1 w + n_2(w-1) + (2w-1)(k-1)]\frac{\partial_x j}{\partial_w j} - w(w-1)\frac{\partial_x\partial_w j}{\partial_w j}, \\ &= [-(n_1-k+1)w - (n_2-k+1)(w-1)]\frac{\partial_x j}{\partial_w j} - w(w-1)\frac{\partial_x\partial_w j}{\partial_w j}.\end{aligned}$$

To further simplify the last expression we need some hypothesis about the relationship between $\partial_x j$ and $\partial_w j$. We recover this from the definition as the generalized characteristic polynomial.

$$\begin{aligned}j(x,w,t) &= \mathbb{E}\left[\det[xI + (u-1)A_1^T(t)A_1(t) + uA_2^T(t)A_2(t)]\right] = \mathbb{E}\left[\det[xI + (u-1)A_1^T(t)A_1(t) + u(I - A_1^T(t)A_1(t))]\right], \\ &= \mathbb{E}\left[\det[(x+u)I - A_1^T(t)A_1(t) + (u-u)A_1^T(t)A_1(t)]\right] = \mathbb{E}\left[\det[(x+u)I - A_1^T(t)A_1(t)]\right].\end{aligned}$$

So this means that the derivatives of j with respect to x are the same as its derivatives with respect to w and then we can equate $\partial_w j$ to $\partial_x j$ to get

$$
\begin{aligned}
\partial_t w_i &= -(n_1 - k + 1)w_i - (n_2 - k + 1)(w_i - 1) - w_i(w_i - 1)\frac{\partial_{xx} j}{\partial_x j}, \\
&= -(n_1 - k + 1)w_i - (n_2 - k + 1)(w_i - 1) - w_i(w_i - 1)\sum_{j \neq i} \frac{2}{w_i - w_j}, \\
&= n_2 - (n_1 + n_2)w_i + (w_i + w_i - 1)(k - 1) - \sum_{j \neq i} \frac{2w_i(w_i - 1)}{w_i - w_j}, \\
&= n_2 - (n_1 + n_2)w_i + (w_i + w_i - 1)\sum_{j \neq i} \frac{w_i - w_j}{w_i - w_j} - \sum_{j \neq i} \frac{2w_i(w_i - 1)}{w_i - w_j}, \\
&= n_2 - (n_1 + n_2)w_i + \sum_{j \neq i} \frac{(w_i + w_i - 1)(w_i - w_j) - 2w_i(w_i - 1)}{w_i - w_j}, \\
&= n_2 - (n_1 + n_2)w_i + \sum_{j \neq i} \frac{w_i^2 + w_i(w_i - 1) - w_i w_j - w_j(w_i - 1) - 2w_i(w_i - 1)}{w_i - w_j}, \\
&= n_2 - (n_1 + n_2)w_i + \sum_{j \neq i} \frac{-w_i w_j - w_j(w_i - 1) + w_i}{w_i - w_j}, \\
&= n_2 - (n_1 + n_2)w_i + \sum_{j \neq i} \frac{w_j(1 - w_i) + w_i(1 - w_j)}{w_i - w_j}.
\end{aligned}
$$

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