6. Inverse & Unilateral z-transform

EE 274/COE 197E

Rhandley Cajote, Ph. D.
Crisron Lucas, MSc.

Today’s Lesson:

1. The Inverse Z-Transform
2. Power Series Expansion
3. Partial Fraction Expansion and Table Look-up
4. The Unilateral Z-Transform
5. Solving DT System Responses using Z-Transform

The Inverse Z-Transform

- The inverse z-transform is defined as follows:

$$x_n = \frac{1}{2\pi j}\int_c X(z)z^{n-1}dz$$

- Note that the function X(z) is integrated along a complex plane (z-plane)

- Solving this requires contour integration

  Luckily, we can solve this using other mathematical techniques

Method 1: Power Series Expansion

- In cases where X(z) can be represented as a **proper rational expression**,

$$X(z) = \frac{1 - z^{-1}}{1 + 3z^{-1} + 2z^{-2}}, \quad ROC: \ |z| > 2$$

- We can use **polynomial division** to rewrite the rational expression as a **sum of terms in z/z^{-1}**

Notes on Method 1:

- For a **causal sequence**, the polynomials in X(z) are arranged in **decreasing powers of z**

- For an **anticausal sequence**, the polynomials in X(z) are arranged in **increasing powers of z**

- Not advised for a two-sided sequence, X(z) must first be decomposed to its causal and anti causal parts
  - Requires partial fraction expansion

Method 1: Power Series Expansion

- Determine the Inverse z-transform of the following:

$$X(z) = \frac{1 - z^{-1}}{1 + 3z^{-1} + 2z^{-2}}, \quad ROC: \; |z| > 2$$

- Since the ROC is a region **outside a circle of radius 2**, then we expect a **causal(right-sided)** time-domain sequence

Method 1: Power Series Expansion

- Perform long division

$$X(z) = \frac{1 - z^{-1}}{1 + 3z^{-1} + 2z^{-2}}, \quad ROC: \ |z| > 2$$

					1	-	$4z^{-1}$	+	$10z^{-2}$	-	$22z^{-3}$	+	...	
1	+	$3z^{-1}$	+	$2z^{-2}$	1	-	z^{-1}							
				- (	1	+	$3z^{-1}$	+	$2z^{-2}$	)				
						-	$4z^{-1}$	-	$2z^{-2}$					
					- (	-	$4z^{-1}$	-	$12z^{-2}$	-	$8z^{-3}$	)		
									$10z^{-2}$	+	$8z^{-3}$			
								- (	$10z^{-2}$	+	$30z^{-3}$	+	$20z^{-4}$	)
										-	$22z^{-3}$	-	$20z^{-4}$	

Method 1: Power Series Expansion

- Perform long division

$$X(z) = \frac{1 - z^{-1}}{1 + 3z^{-1} + 2z^{-2}}, \quad ROC: \ |z| > 2$$

					1	-	$4z^{-1}$	+	$10z^{-2}$	-	$22z^{-3}$	+	...	
1	+	$3z^{-1}$	+	$2z^{-2}$	1	-	z^{-1}							
				- (	1	+	$3z^{-1}$	+	$2z^{-2}$	)				
						-	$4z^{-1}$	-	$2z^{-2}$					
					- (	-	$4z^{-1}$	-	$12z^{-2}$	-	$8z^{-3}$	)		
									$10z^{-2}$	+	$8z^{-3}$			
								- (	$10z^{-2}$	+	$30z^{-3}$	+	$20z^{-4}$	)
										-	$22z^{-3}$	-	$20z^{-4}$	

$$\boxed{x[n] = \{\underset{\uparrow}{1}, -4, 10, -22, \ldots\}}$$

Method 1: Power Series Expansion

- Suppose we have the anticausal form

$$X(z) = \frac{1 - z^{-1}}{1 + 3z^{-1} + 2z^{-2}} \quad ROC: \; |z| < 2$$

- We rearrange the terms in **increasing order of z**

Method 1: Power Series Expansion

➢ Suppose we have the anticausal form $X(z) = \dfrac{1 - z^{-1}}{1 + 3z^{-1} + 2z^{-2}}$ $ROC: \ |z| < 2$

					$-0.5z$	+	$1.25z^2$	-	$1.625z^3$	+	...	
$2z^{-2}$	+	$3z^{-1}$	+	1	$-z^{-1}$	+	1					
				- (	$-z^{-1}$	-	1.5	-	$0.5z$	)		
							2.5	+	$0.5z$			
						- (	2.5	+	$3.75z$	+	$1.25z^2$	)
								-	$3.25z$	-	$1.25z^2$	

$$\boxed{x[n] = \{\ldots, -1.625, 1.25, -0.5, \underset{\uparrow}{0}\}}$$

Method 2: Partial Fraction Expansion

➢ Goal: Express X(z) as a linear combination of $X_k(z)$'s with known inverse transforms

$$X(z) = a_1 X_1(z) + a_2 X_2(z) + \ldots + a_k X_k(z) + \ldots$$

➢ We use the **linearity** property to get the sequence x[n]

$$x[n] = \sum_{k=1}^{m} a_k x_k[n] \xleftrightarrow{z} X(z) = \sum_{k=1}^{m} a_k X_k(z)$$

Method 2: Partial Fraction Expansion

➢ For a proper rational expression X(z), there are several possible cases:

<u>**Case 1**</u>: real and distinct poles

<u>**Case 2**</u>: complex conjugate poles

<u>**Case 3**</u>: multiple order poles

Method 2: Partial Fraction Expansion

<u>Case 1</u>: real and distinct poles

- ➢ We can rewrite the rational expression X(z) as follows:

$$X(z) = \frac{N(z)}{D(z)} = \sum_{k=1}^{m} \frac{A_k}{1 - p_k z^{-1}}$$

- ➢ The inverse z-transform will depend on the ROC
  - ○ Recall the table of common z-transform pairs

Method 2: Partial Fraction Expansion

<u>Case 1</u>: real and distinct poles

	Sequence $x[n]$	z-Transform $X(z)$	ROC
1.	$\delta[n]$	1	All z
2.	$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3.	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4.	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
5.	$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
6.	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$

Method 2: Partial Fraction Expansion

Case 2: complex conjugate poles

➢ We get a sinusoidal sequence in the time domain.

$$X(z) = \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}}$$

$$x[n] = Ap^n u[n] + A^*(p^*)^n u[n]$$

$$= |A|e^{j\angle A}(|p|e^{j\angle p})^n u[n] + |A|e^{-j\angle A}(|p|e^{-j\angle p})^n u[n]$$

$$x[n] = 2|A|(|p|)^n \cos((\angle p)n + \angle A)u[n]$$

Method 2: Partial Fraction Expansion

<u>Case 2</u>: complex conjugate poles

7.	$(\cos \omega_0 n)u[n]$	$\dfrac{1-(\cos \omega_0)z^{-1}}{1-2(\cos \omega_0)z^{-1}+z^{-2}}$	$\lvert z \rvert > 1$
8.	$(\sin \omega_0 n)u[n]$	$\dfrac{(\sin \omega_0)z^{-1}}{1-2(\cos \omega_0)z^{-1}+z^{-2}}$	$\lvert z \rvert > 1$
9.	$(r^n \cos \omega_0 n)u[n]$	$\dfrac{1-(r\cos \omega_0)z^{-1}}{1-2(r\cos \omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z \rvert > r$
10.	$(r^n \sin \omega_0 n)u[n]$	$\dfrac{r(\sin \omega_0)z^{-1}}{1-2(r\cos \omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z \rvert > r$

Method 2: Partial Fraction Expansion

Case 3: multiple order poles

➢ If X(z) has a pole p with multiplicity k, we can rewrite X(z) as follows:

$$X(z) = \frac{A}{(1-pz^{-1})^k(1-qz^{-1})}$$
$$= \frac{A_1}{1-pz^{-1}} + \frac{A_2}{(1-pz^{-1})^2} + \ldots + \frac{A_k}{(1-pz^{-1})^k}$$
$$+ \frac{B}{1-qz^{-1}}$$

Method 2: Partial Fraction Expansion

Case 3: multiple order poles

- The numerators $A_1, A_2, \ldots, A_k$ are determined using differentiation

$$X(z) = \frac{A}{(1-pz^{-1})^k(1-qz^{-1})}$$
$$= \frac{A_1}{1-pz^{-1}} + \frac{A_2}{(1-pz^{-1})^2} + \ldots + \frac{A_k}{(1-pz^{-1})^k}$$
$$+ \frac{B}{1-qz^{-1}}$$

Example

- Consider a sequence x[n] with z-transform

$$X(z) = \frac{1 + z^{-1}}{\left(1 - z^{-1}\right)\left(1 - 0.5z^{-1}\right)}$$

- Find x[n] via partial fraction expansion.

Example

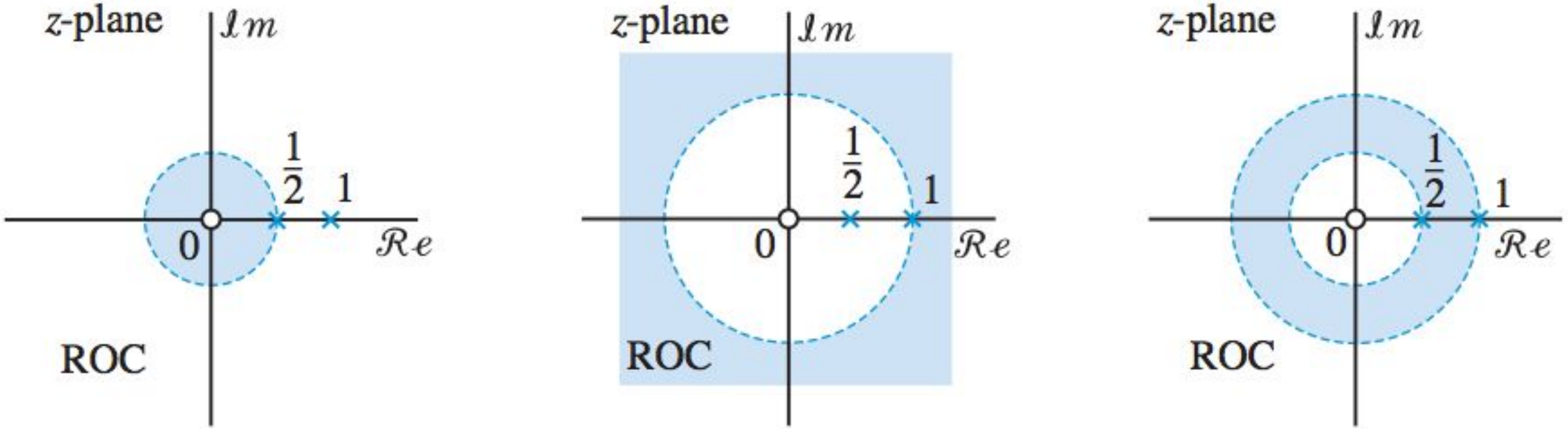
- Since this is a proper rational function

$$X(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})} = \frac{A}{1 - z^{-1}} + \frac{B}{1 - 0.5z^{-1}}$$

- We get A = 4, B = -3

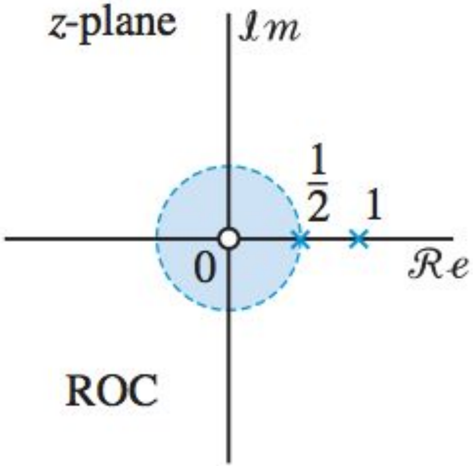
Example

➢ No ROCs given, three possible cases:



Example

➢ No ROCs given, three possible cases:

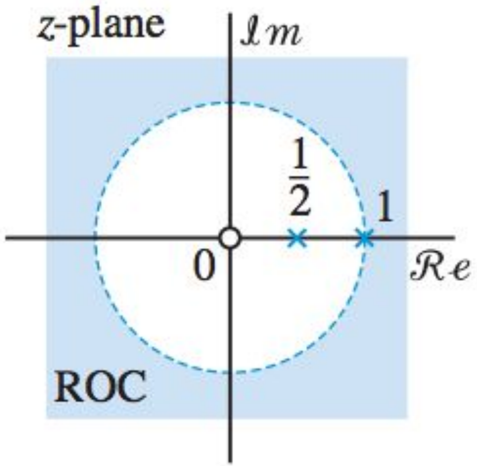


$$X(z) = \frac{4}{1 - z^{-1}} - \frac{3}{1 - 0.5z^{-1}}$$

$$x[n] = -\left\{4 - 3\left(0.5^{n}\right)\right\} u[-n-1]$$

Example

➢ No ROCs given, three possible cases:

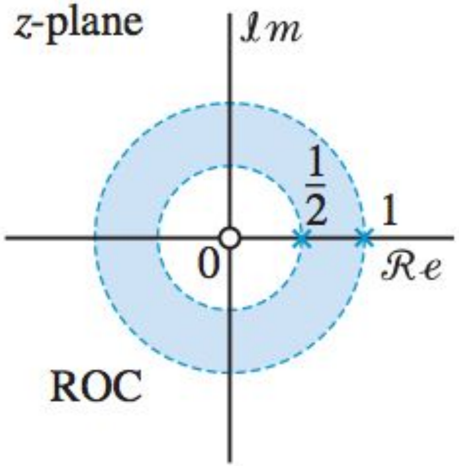


$$X(z) = \frac{4}{1 - z^{-1}} - \frac{3}{1 - 0.5z^{-1}}$$

$$x[n] = \{4 - 3\,(0.5^n)\}\,u[n]$$

Example

➢ No ROCs given, three possible cases:



$$X(z) = \frac{4}{1 - z^{-1}} - \frac{3}{1 - 0.5z^{-1}}$$

$$x[n] = -4u[-n-1] - 3(0.5^n)u[n]$$

Example

- ➢ Consider a causal sequence x[n] with z-transform

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

- ➢ Find x[n] via partial fraction expansion.

Example

- Complex conjugate poles

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}} = \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}}$$

$$p = \frac{1}{2}(1 + j) = \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}}, \quad A = \frac{1}{2} - j\frac{3}{2} = \frac{\sqrt{10}}{2} e^{-j71.56^\circ}$$

$$p^* = \frac{1}{2}(1 - j) = \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}, \quad A^* = \frac{1}{2} + j\frac{3}{2} = \frac{\sqrt{10}}{2} e^{j71.56^\circ}$$

Example

- Complex conjugate poles

$$x[n] = 2Ar^n \cos(\omega n + \theta)u[n]$$

$$\boxed{\begin{array}{ll} r = \dfrac{1}{\sqrt{2}}, & \omega = \dfrac{\pi}{4} \\ A = \dfrac{\sqrt{10}}{2}, & \theta = -71.56^\circ \end{array}}$$

Solving DT System Responses in z- domain

➢ For **causal LTI systems**, we can use the z-Transform, convolution property, and inverse z-transform to solve the different system responses

$$\text{LCCDE} \rightarrow Y^{+}(z) = X^{+}(z)H^{+}(z) \rightarrow \text{PFE} + \text{lookup} \rightarrow y(n)$$

$Y^{+}(z)$, $X^{+}(z)$, and $H^{+}(z)$ are the **unilateral z-transforms**

Unilateral Z-Transform

- For causal systems (events/samples occur at n >0)

$$X^{+}(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

- Index of summation starts at zero

Properties of the Unilateral Z-Transform

- Sequences which are equal for $n \geq 0$, and differ for $n < 0$ have the same $X^{+}(z)$.

- ROC is **always at least the exterior of a circle**.

- All properties of the two-sided z-transform can be used in the one-sided z-transform **except for the time-shifting property.**

- Can be used in analyzing DT systems with nonzero initial conditions

Properties of the Unilateral Z-Transform

Time-shifting for Uni-Z:

- Time Delay, k > 0

$$if \quad x[n] \quad \overset{z^+}{\longleftrightarrow} X^+(z)$$

$$then \quad x[n-k] \quad \overset{z^+}{\longleftrightarrow} z^{-k}\left\{X^+(z) + \sum_{n=1}^{k} x[-n]z^n\right\}, \quad k > 0$$

- Time Advance, k > 0

$$if \quad x[n] \quad \overset{z^+}{\longleftrightarrow} X^+(z)$$

$$then \quad x[n+k] \quad \overset{z^+}{\longleftrightarrow} z^{+k}\left\{X^+(z) - \sum_{n=0}^{k-1} x[n]z^{-n}\right\}, \quad k > 0$$

Example

Consider the causal discrete-time system represented by the following difference equation:

$$y[n] = 7x[n] - 3x[n-1] + \frac{4}{3}y[n-1] - \frac{1}{3}y[n-2]$$

Determine the total response of the system when the input is given by the following expression:

$$x[n] = 0.5^n u[n]$$

Initial conditions: y[-1] = y[-2] = 1

Example

$$y[n] = 7x[n] - 3x[n-1] + \frac{4}{3}y[n-1] - \frac{1}{3}y[n-2]$$

$$\begin{aligned} Y^+(z) = 7X^+(z) - 3z^{-1}\left(X^+(z) + x[-1]z\right) \\ + \frac{4}{3}z^{-1}\left(Y^+(z) + y[-1]z\right) \\ - \frac{1}{3}z^{-2}\left(Y^+(z) + y[-1]z + y[-2]z^2\right) \end{aligned}$$

Example

$$Y^+(z) = 7X^+(z) - 3z^{-1}\left(X^+(z) + x[-1]z\right)$$
$$+ \frac{4}{3}z^{-1}\left(Y^+(z) + y[-1]z\right)$$
$$- \frac{1}{3}z^{-2}\left(Y^+(z) + y[-1]z + y[-2]z^2\right)$$
$$Y^+(z) = 7X^+(z) - 3z^{-1}X^+(z)$$
$$+ \frac{4}{3}z^{-1}Y^+(z) + \frac{4}{3}y[-1]$$
$$- \frac{1}{3}z^{-2}Y^+(z) - \frac{1}{3}y[-1]z^{-1} - \frac{1}{3}y[-2]$$

Example

$$Y^+(z) = 7X^+(z) - 3z^{-1}X^+(z)$$
$$+ \frac{4}{3}z^{-1}Y^+(z) + \frac{4}{3}y[-1]$$
$$- \frac{1}{3}z^{-2}Y^+(z) - \frac{1}{3}y[-1]z^{-1} - \frac{1}{3}y[-2]$$

$$Y^+(z) = \boxed{\frac{7 - 3z^{-1}}{1 - \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}}X^+(z)}$$

Zero-state response

$$+ \boxed{\frac{\frac{4}{3}y[-1]}{1 - \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}} - \frac{\frac{1}{3}y[-1]z^{-1} + \frac{1}{3}y[-2]}{1 - \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}}}$$

Zero-input response

Example

$$Y_{zs}(z) = \frac{7 - 3z^{-1}}{1 - \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}} X^+(z)$$

$$= \frac{7 - 3z^{-1}}{1 - \frac{4}{3}z^{-1} + \frac{1}{3}z^{-2}} \left(\frac{1}{1 - 0.5z^{-1}} \right)$$

$$= \frac{A}{1 - \frac{1}{3}z^{-1}} + \frac{B}{1 - z^{-1}} + \frac{C}{1 - 0.5z^{-1}}$$

$$\boxed{y_{ZS}(n) = -2(⅓)^n + 12u(n) - 3(½)^n}$$

Example

$$Y_{zi}(z) = \frac{\frac{4}{3}y[-1]}{1-\frac{4}{3}z^{-1}+\frac{1}{3}z^{-2}} - \frac{\frac{1}{3}y[-1]z^{-1}+\frac{1}{3}y[-2]}{1-\frac{4}{3}z^{-1}+\frac{1}{3}z^{-2}}$$

$$= \frac{\frac{4}{3}}{1-\frac{4}{3}z^{-1}+\frac{1}{3}z^{-2}} - \frac{\frac{1}{3}z^{-1}+\frac{1}{3}}{1-\frac{4}{3}z^{-1}+\frac{1}{3}z^{-2}}$$

$$= \frac{1-\frac{1}{3}z^{-1}}{1-\frac{4}{3}z^{-1}+\frac{1}{3}z^{-2}}$$

$$Y_{zi}(z) = \frac{1-\frac{1}{3}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-z^{-1}\right)} = \frac{1}{1-z^{-1}}$$

$$\boxed{y_{zi}[n] = u[n]}$$

Example

<u>Total response:</u>

$$
\begin{aligned}
y_{Total}(n) &= y_{ZS}(n) + y_{ZI}(n) \\
&= -2(1/3)^n + 12u(n) - 3(1/2)^n + u(n)
\end{aligned}
$$

$$
= \underbrace{\overbrace{-3(1/2)^n}^{\textbf{Forced Response}} \; \overbrace{-2(1/3)^n \quad}^{\textbf{Natural Response}}}_{\textbf{Transient Response}} \hspace{-1em} \underbrace{+13u(n)}_{\textbf{Steady-State Response}}
$$

Summary

- Inverse z-transform can be easily done via Power Series Expansion, and Partial Fraction Expansion + table lookup
- LCCDEs can be solved indirectly in the z-domain
- Unilateral z-transform can be used to solve causal LTI systems

For further reading...

- Chapter 3.5-3.6

  “Applied Digital Signal Processing, by Manolakis, D & Ingle, V..”

- Chapters 10.7

  “Signals and Systems by Oppenheim, A & Willsky, A.”

6. Inverse & Unilateral z-transform

EE 274/COE 197E

Rhandley Cajote, Ph. D.
Crisron Lucas, MSc.