8. Fourier Series/Transform for DT systems

EE 274/COE 197E

Rhandley Cajote, Ph. D.
Crisron Lucas, MSc.

Today's Lesson:

1. Dirichlet Conditions for DTFS/DTFT
2. Discrete Time Fourier Series
3. Discrete Time Fourier Transform
4. DT Operations in the Frequency Domain
5. DT Systems in the Frequency Domain

Frequency Response

- Frequency Response of DT systems can be directly obtained from their Fourier Transform:

  a. Discrete Time Fourier Series (for periodic signals)
  b. Discrete Time Fourier Transform (for aperiodic signals)

- There are specific conditions for the existence of the Fourier Series also known as **Dirichlet Conditions**

  → derived for CT signals

Dirichlet Conditions

1. The DT signal should be absolutely summable
2. There should be finite discontinuities within the signal duration
3. There should be finite extrema (minimum and maximum) within the signal duration

Discrete Time Fourier Series

- DT signals can be expressed as a combination of harmonically related complex exponentials (or sinusoids)
- This is the **synthesis equation** for the discrete-time Fourier series (DTFS) → inverse transform

$$x[n] = \sum_{<N>} c_k \phi_k[n]$$

$$x[n] = \sum_{<N>} c_k e^{jk\omega_0 n} = \sum_{<N>} c_k e^{jk(2\pi/N)n}$$

Discrete Time Fourier Series

- Manipulating the synthesis equation will give us the **analysis equation** of the DTFS
  - analysis = "time to frequency"

$$c_k = \frac{1}{N} \sum_{<N>} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{<N>} x[n] e^{-jk(2\pi/N)n}$$

Example

- Consider the signal: $x[n] = \sin(\omega_0 n)$

- Determine the Discrete Time Fourier Series of x[n].

Example

➢ Using the Euler's formula:

$$
\begin{aligned}
x[n] &= \sin(\omega_0 n), \;\; \omega_0 = \frac{2\pi}{N} \\
&= \frac{1}{j2} e^{j\frac{2\pi}{N}n} - \frac{1}{j2} e^{-j\frac{2\pi}{N}n} \\
&= \frac{1}{j2} e^{j\frac{2\pi}{N}n} - \frac{1}{j2} e^{j\frac{2\pi}{N}(N-1)n} = \sum_{<N>} c_k e^{jk(2\pi/N)n}
\end{aligned}
$$

→ **No need to use Analysis Equation!**

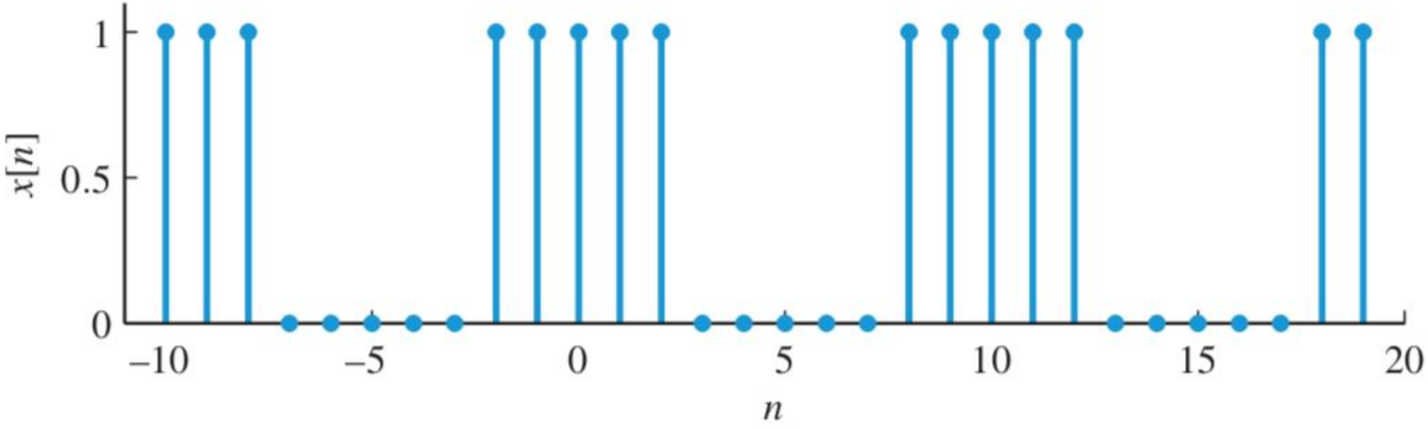
Example

$$x[n] = \frac{1}{j2} e^{j\frac{2\pi}{N}n} - \frac{1}{j2} e^{j\frac{2\pi}{N}(N-1)n} = \sum_{<N>} c_k e^{jk(2\pi/N)n}$$

k	0	1	2	…	N-2	N-1
c_k	0	1/(j2)	0	…	0	-1/(j2)

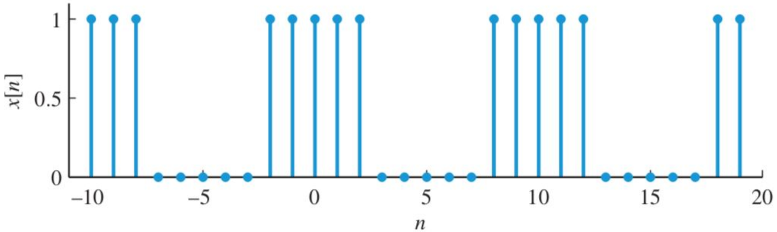
Example

- Use the analysis equation of DTFS to determine the spectrum of the discrete-time periodic pulse train below.



Example

$$c_k = \frac{1}{N}\sum_{<N>} x[n]e^{-jk(2\pi/N)n}, \quad N = 10$$



$$= \frac{1}{10}\sum_{n=-5}^{4} x[n]e^{-jk(2\pi/10)n}$$

$$= \frac{1}{10}\sum_{n=-2}^{2} (1)e^{-jk(2\pi/10)n}$$

$$= \frac{1}{10}\sum_{m=0}^{4} (1)e^{-jk(2\pi/10)(m-2)}, \quad m = n+2$$

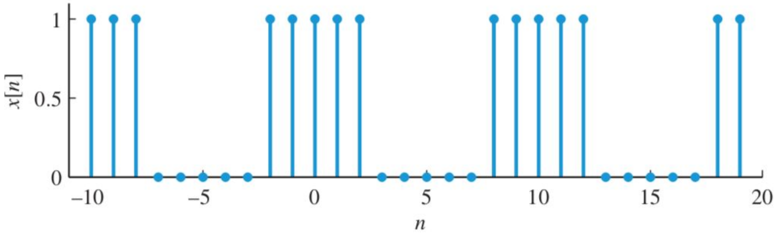
Example

$$c_k = \frac{1}{10}\sum_{m=0}^{4}(1)e^{-jk(2\pi/10)(m-2)}$$

$$= \frac{e^{jk(4\pi/10)}}{10}\sum_{m=0}^{4}\left[e^{-jk(2\pi/10)}\right]^m$$

$$= \frac{e^{jk(4\pi/10)}}{10}\cdot\frac{1-\left[e^{-jk(2\pi/10)}\right]^5}{1-\left[e^{-jk(2\pi/10)}\right]}$$

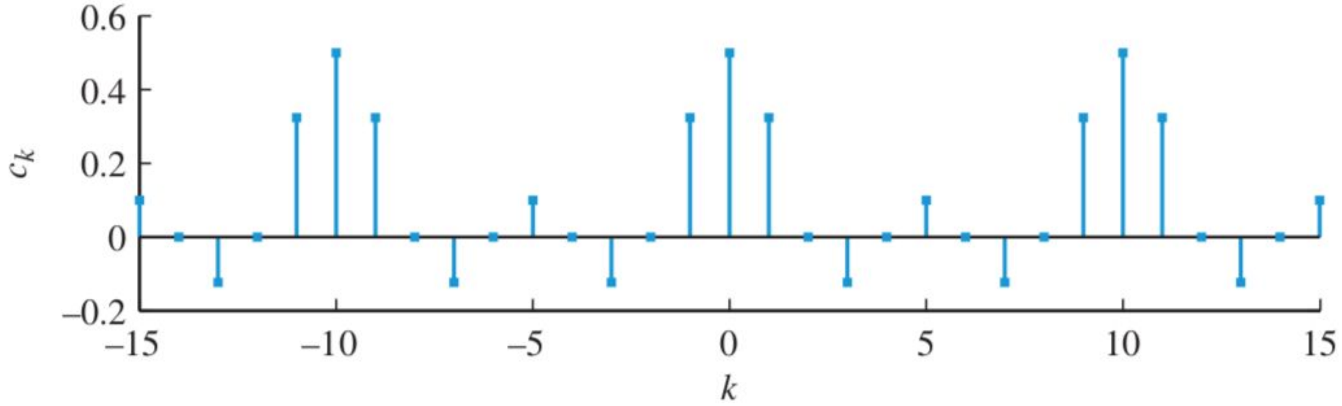
$$c_k = \frac{e^{jk(4\pi/10)}}{10}\cdot\frac{e^{-jk(2\pi/10)(5/2)}}{e^{-jk(2\pi/10)/2}}\cdot\frac{\sin\left[\frac{2\pi}{10}k\cdot\frac{5}{2}\right]}{\sin\left[\frac{2\pi}{10}k\cdot\frac{1}{2}\right]}$$



Example

$$c_k = \frac{e^{jk(4\pi/10)}}{10} \cdot \frac{e^{-jk(2\pi/10)(5/2)}}{e^{-jk(2\pi/10)/2}} \cdot \frac{\sin\left[\frac{2\pi}{10}k \cdot \frac{5}{2}\right]}{\sin\left[\frac{2\pi}{10}k \cdot \frac{1}{2}\right]}$$

$$c_k = \frac{1}{10} \cdot \frac{\sin\left[\frac{2\pi}{10}k \cdot \frac{5}{2}\right]}{\sin\left[\frac{2\pi}{10}k \cdot \frac{1}{2}\right]}, k \notin \{0, \pm N, \pm 2N, \cdots\}$$



Notes on Sampling Period

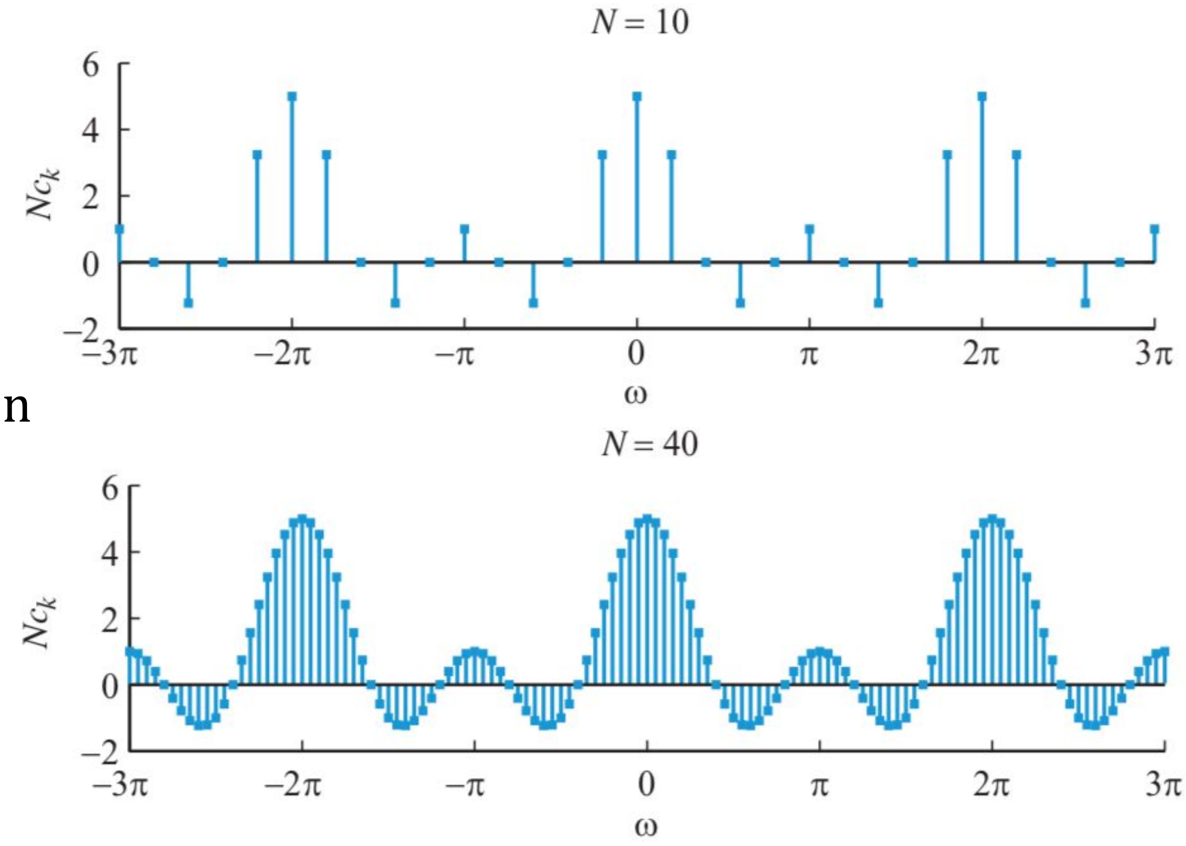
- What happens when **N** is increased?

$$c_k = \frac{e^{jk(4\pi/10)}}{10} \cdot \frac{e^{-jk(2\pi/10)(5/2)}}{e^{-jk(2\pi/10)/2}} \cdot \frac{\sin\left[\frac{2\pi}{10}k \cdot \frac{5}{2}\right]}{\sin\left[\frac{2\pi}{10}k \cdot \frac{1}{2}\right]}$$

$$c_k = \frac{1}{10} \cdot \frac{\sin\left[\frac{2\pi}{10}k \cdot \frac{5}{2}\right]}{\sin\left[\frac{2\pi}{10}k \cdot \frac{1}{2}\right]}, k \notin \{0, \pm N, \pm 2N, \cdots\}$$

Notes on Sampling Period

As **N** is increased, the spacing between frequency domain samples becomes narrower. If **N → ∞**, then the DT signal becomes **aperiodic**, and the frequency domain becomes, **continuous**



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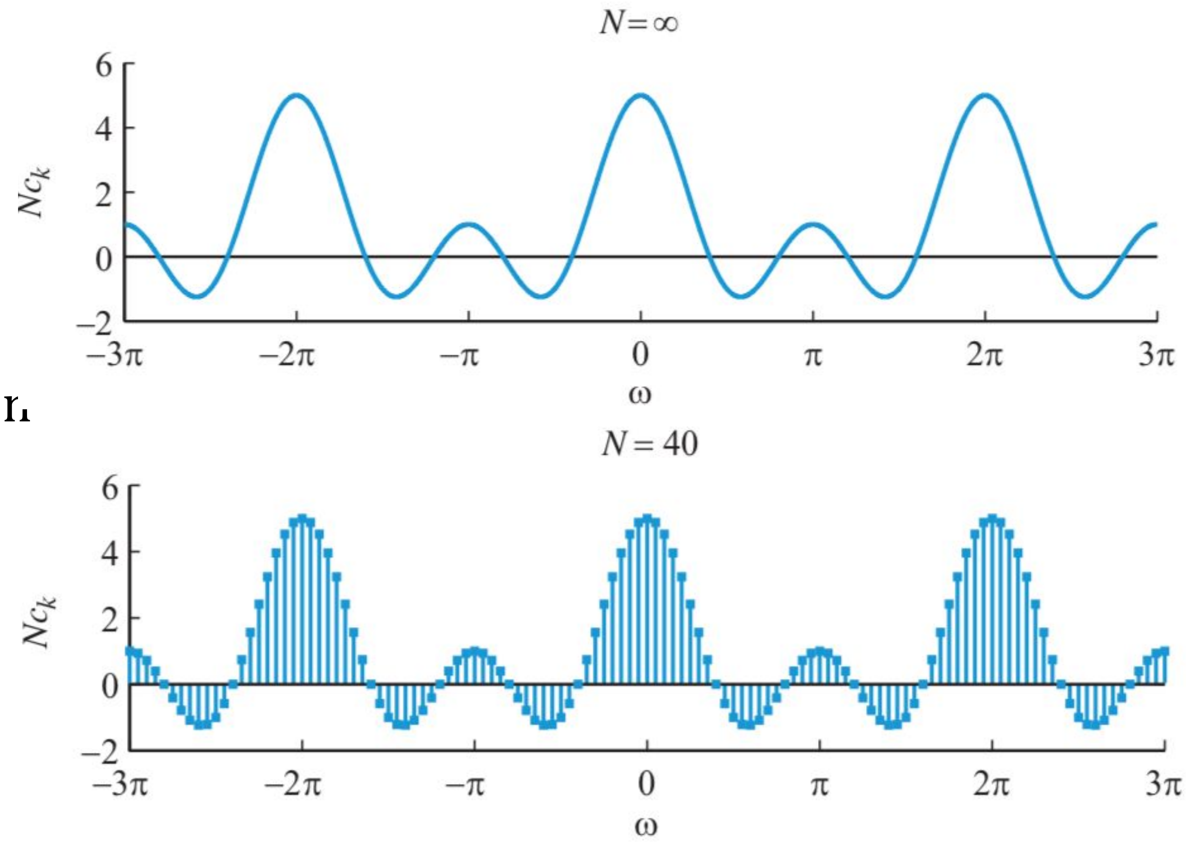


Discrete Time Fourier Transform

- For an aperiodic sequence x[n]:
  - Analysis equation:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- The spectrum $X(e^{j\omega})$ is continuous and periodic
  - Period of the spectrum is 2π

Discrete Time Fourier Transform

- Discrete Time Fourier Transform (aperiodic)
  - Analysis equation:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Discrete Time Fourier Series (periodic)

$$c_k = \frac{1}{N}\sum_{<N>} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{<N>} x[n]e^{-jk(2\pi/N)n}$$

Discrete Time Fourier Transform

- Discrete Time Fourier Transform (aperiodic)
  - Synthesis equation:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

→ Integration over a single period (continuous) spectrum

Example

- Determine the spectrum of a three-point pulse sequence.

$$x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$$

Example

- Use the analysis equation: $x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

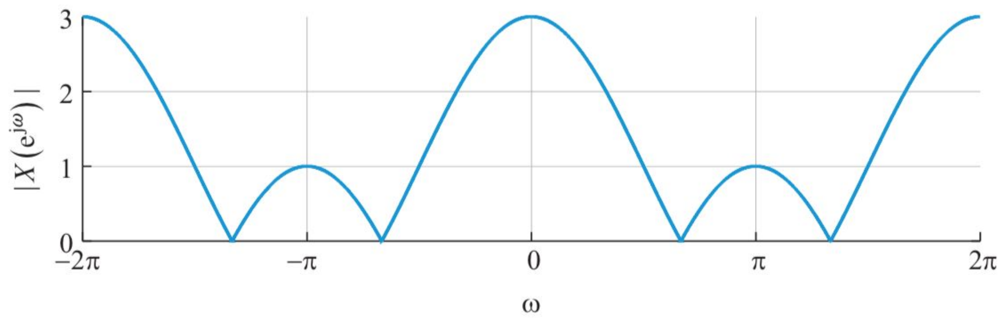
$$= \sum_{n=-1}^{1} x[n]e^{-j\omega n}$$

$$= e^{-j\omega} + 1 + e^{j\omega}$$

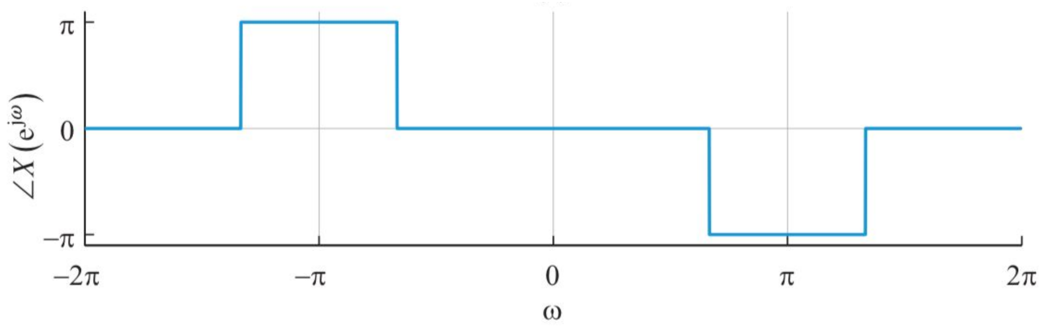
$$X\left(e^{j\omega}\right) = 1 + 2\cos\omega$$

Example

- Use the analysis equation: $x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$



(a)



(b)

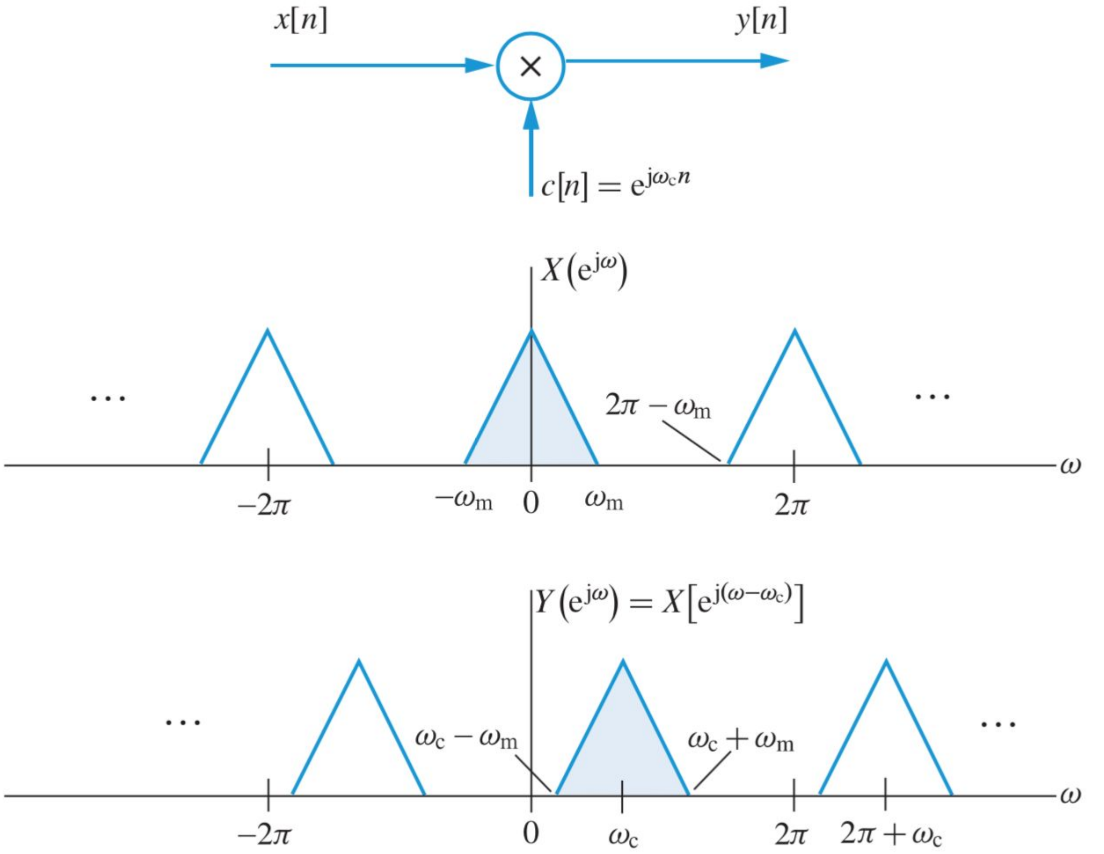
Properties of DTFT

	Property	Sequence	Transform
		$x[n]$	$\mathcal{F}\{x[n]\}$
1.	Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(e^{j\omega}) + a_2X_2(e^{j\omega})$
2.	Time shifting	$x[n-k]$	$e^{-jk\omega}X(e^{j\omega})$
3.	Frequency shifting	$e^{j\omega_0 n}x[n]$	$X[e^{j(\omega-\omega_0)}]$
4.	Modulation	$x[n]\cos\omega_0 n$	$\frac{1}{2}X[e^{j(\omega+\omega_0)}] + \frac{1}{2}X[e^{j(\omega-\omega_0)}]$
5.	Folding	$x[-n]$	$X(e^{-j\omega})$
6.	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$

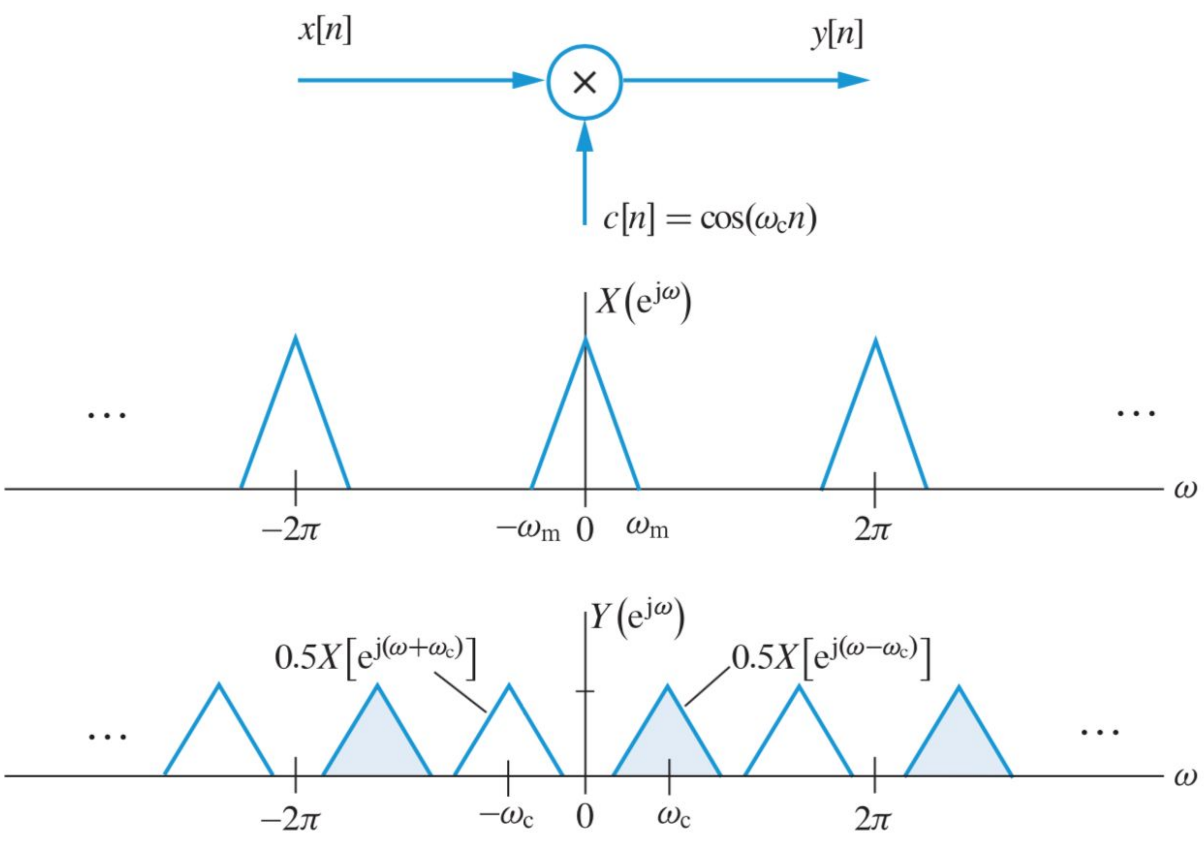
Properties of DTFT

	Property	Sequence	Transform
		$x[n]$	$\mathcal{F}\{x[n]\}$
7.	Differentiation	$nx[n]$	$-j\dfrac{dX(e^{j\omega})}{d\omega}$
8.	Convolution	$x[n]*h[n]$	$X(e^{j\omega})H(e^{j\omega})$
9.	Windowing	$x[n]w[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(e^{j\theta})W[e^{j(\omega-\theta)}]d\theta$
10.	Parseval's theorem	$\displaystyle\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] =$	$\dfrac{1}{2\pi}\displaystyle\int_{2\pi} X_1(e^{j\omega})X_2^*(e^{j\omega})d\omega$

Frequency shifting

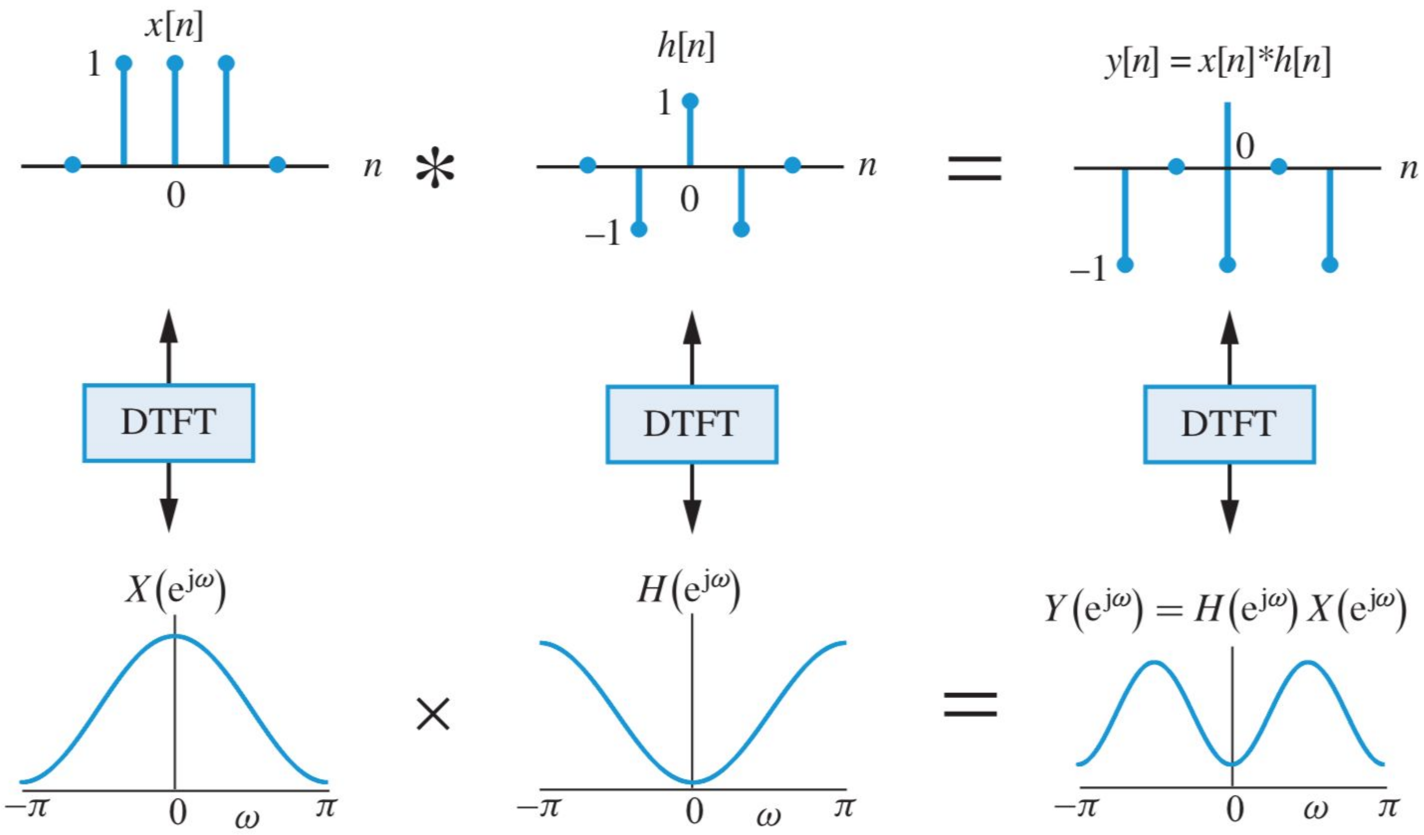


Modulation



$x[n]$
$y[n]$
$\times$
$c[n] = \cos(\omega_c n)$
$X(e^{j\omega})$
...
...
ω
-2π
$-\omega_m$
0
ω_m
2π
$Y(e^{j\omega})$
$0.5X[e^{j(\omega+\omega_c)}]$
$0.5X[e^{j(\omega-\omega_c)}]$
...
...
ω
-2π
$-\omega_c$
0
ω_c
2π

Convolution



Notes on DTFS and DTFT

1. The frequency domain representation of DT signals are periodic (duality of time-frequency)
2. Fourier Series of periodic DT signals are discrete
3. Fourier Transform of aperiodic DT signals are continuous
4. The Fourier transform of the impulse response is the Frequency Response
5. The Fourier transform is also the z-transform evaluated at the unit circle $r = 1$

Notes on DTFS and DTFT

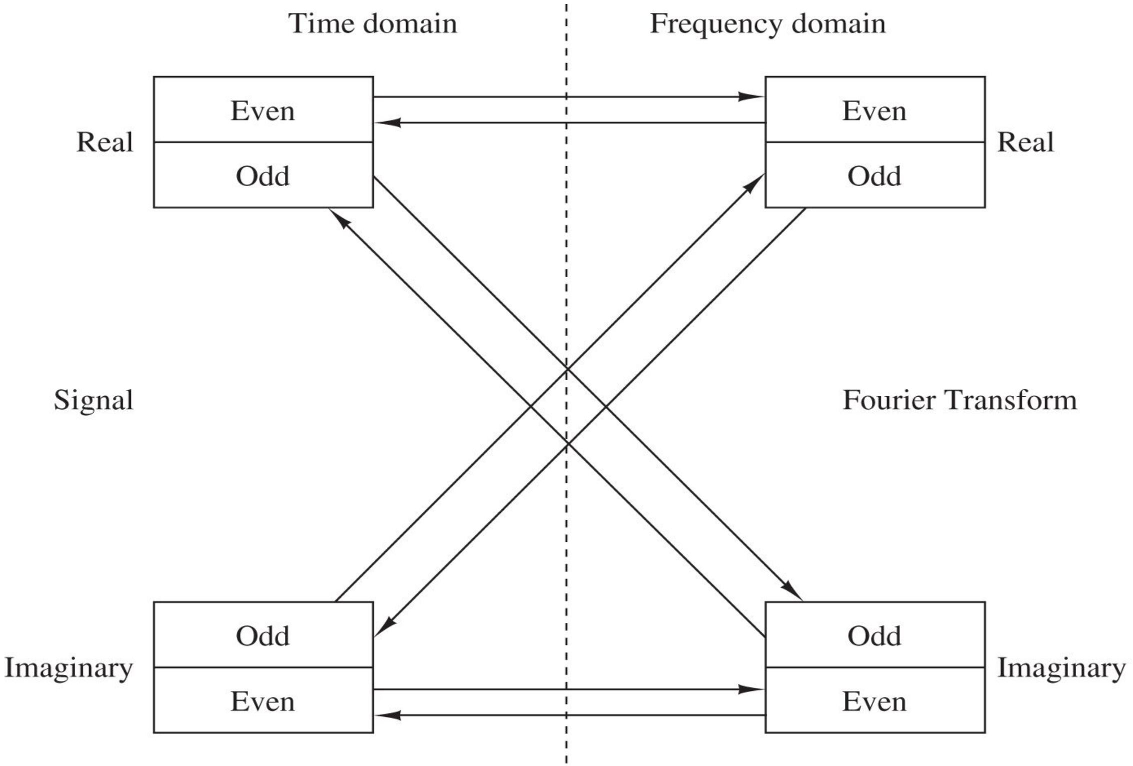


Figure 4.4.2 Summary of symmetry properties for the Fourier transform.

Example

- Determine y(n) when

$$x(n) = Ae^{0.5j\pi n}, \quad -\infty < n < +\infty$$

and

$$h(n) = (0.5)^n u(n)$$

Example

- $h(n) = (0.5)^n u(n)$

$$H(\omega) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

- $Y(\omega) = X(\omega)H(\omega)$

Since x(n) contains one frequency component, we just need to evaluate $H(\omega)$ at $\boldsymbol{\omega = 0.5\pi}$

- $Y(\omega) = |H(\omega{=}0.5\pi)| A e^{0.5j\pi n + \angle H(\omega=0.5\pi)}$

Example

- $h(n) = (0.5)^n u(n)$

$$H(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

- $y(n) = |H(\omega=0.5\pi)| A e^{0.5j\pi n + \angle H(\omega=0.5\pi)}$

$$H\left(\frac{\pi}{2}\right) = \frac{1}{1 - \frac{1}{2}e^{-j\pi/2}} = \frac{2}{\sqrt{5}} e^{-j26.6^o}$$

$$y(n) = \frac{2}{\sqrt{5}} A e^{j(\pi n/2 - 26.6^o)}, \qquad -\infty < n < \infty$$

Summary

- DTFS and DTFT allows us to transform signals into the frequency domain
- The Frequency Response of Systems can be derived from the Fourier Transform of the impulse response
- Sinusoidal steady-state responses can be easily computed by evaluating $H(\omega)$ at the input frequency

For further reading...

- Chapter 4.3-4.5

  “Applied Digital Signal Processing, by Manolakis, D. & Ingle, V.”

- Chapters 3.6-3.7, 5

  “Signals and Systems by Oppenheim, A & Willsky, A.”

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