3. Discrete Time System Representation

EE 274/COE 197E

Rhandley Cajote, Ph. D.
Crisron Lucas, MSc.

Today's Lesson:

1. What are discrete time systems?
2. DT system representation
   a. Difference Equations
   b. Block diagrams
3. DT system classification, properties
4. Examples of DT systems

Discrete Time Systems

- Systems involving **DT signals** and related **processes** and **operations**

- Signals that are fed to the system are called **input** or **excitation** (denoted as x[n])

- Signals that are going out of the system are called **output** or **response** (denoted as y[n])

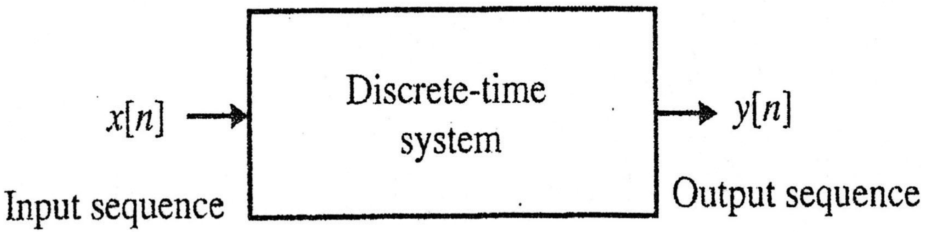


Figure 2.23: Schematic representation of a discrete-time system.

$$y[n] = T(x[n])$$

$$x[n] \xrightarrow{T} y[n]$$

Discrete Time Systems Representation

Difference Equation - a way to describe a system as an equation involving current, past and future values of the input and output.

Solving the difference equation will allow us to determine the output of a system for any given input signal.

Discrete Time Systems Representation

$$y_1(n) = x(n+2)$$ **Time advance**

$$y_2(n) = x(n-5)$$ **Time delay**

$$y_3(n) = 0.2\left[x(n-2) + x(n-1) + x(n) + x(n+1) + x(n+2)\right]$$

$$y_4(n) = median\left[x(n-1), x(n), x(n+1)\right]$$

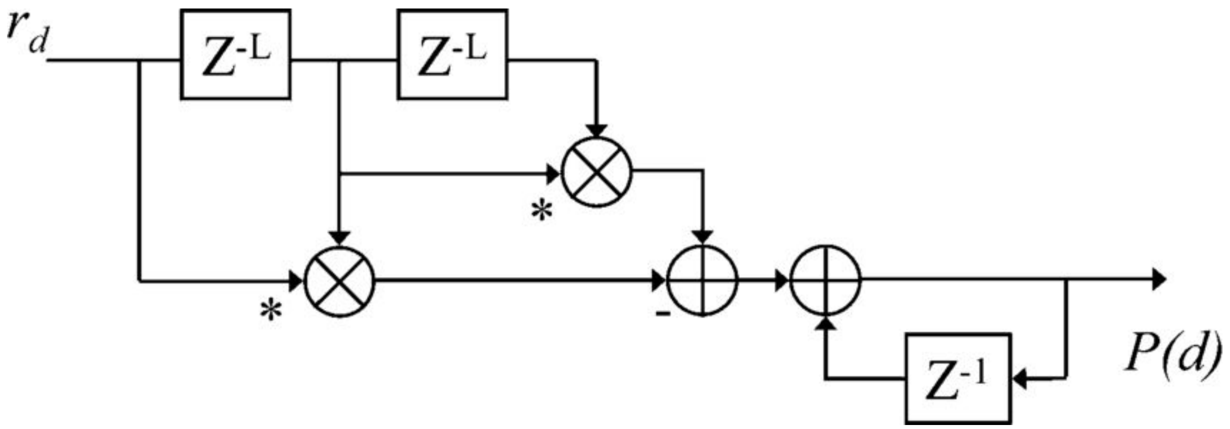
MA filter

median filter

$$y_5(n) = \sum_{k=-\infty}^{n} x(k)$$ **accumulator**

Discrete Time Systems Representation

Block Diagrams- a way to describe a system graphically (involving current, past and future values of the input and output).



Discrete Time Systems Representation

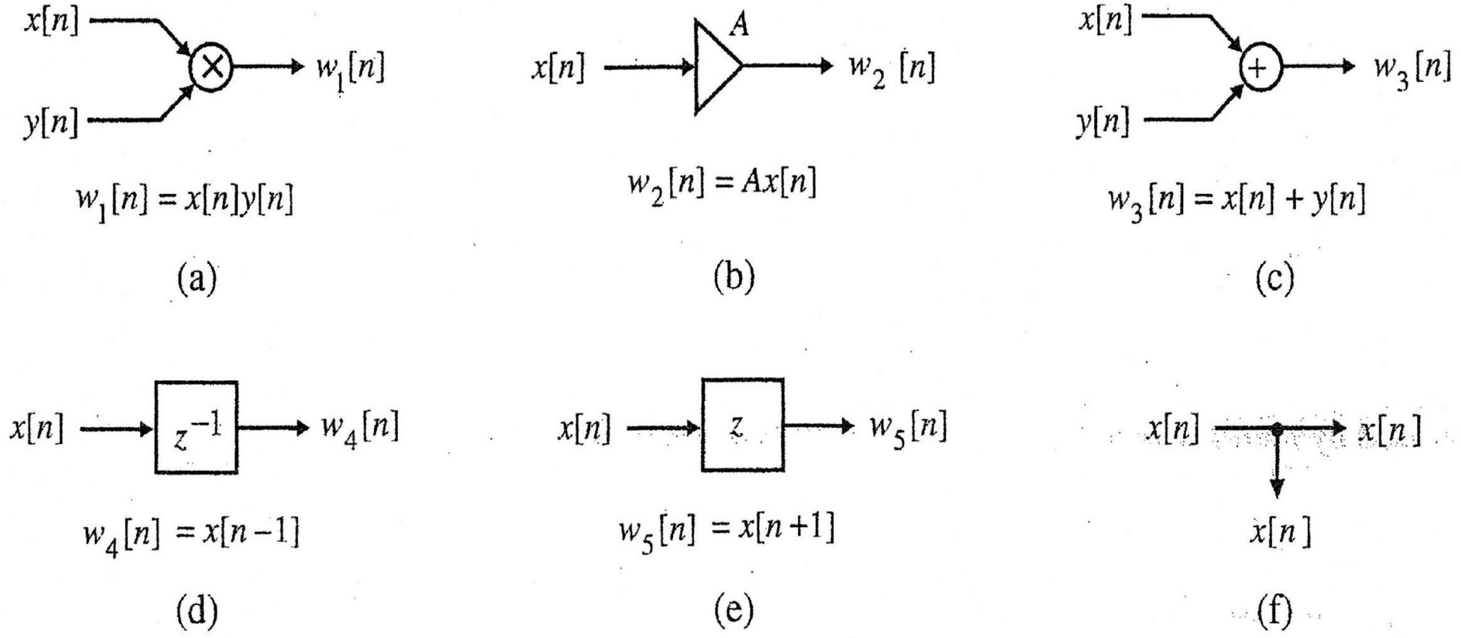


Figure 2.5: Schematic representations of basic operations on sequences: (a) modulator, (b) multiplier, (c) adder, (d) unit delay, (e) unit advance, and (f) pick-off node.

Discrete Time Systems Representation

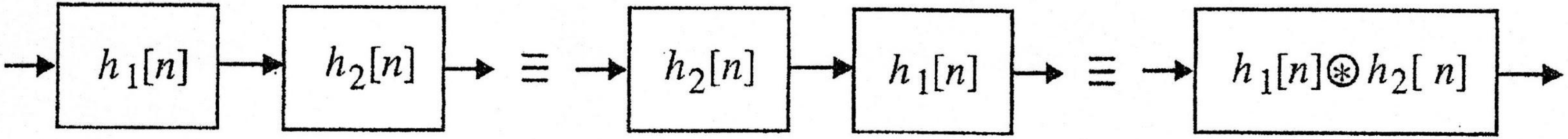


Figure 2.33: The cascade connection.

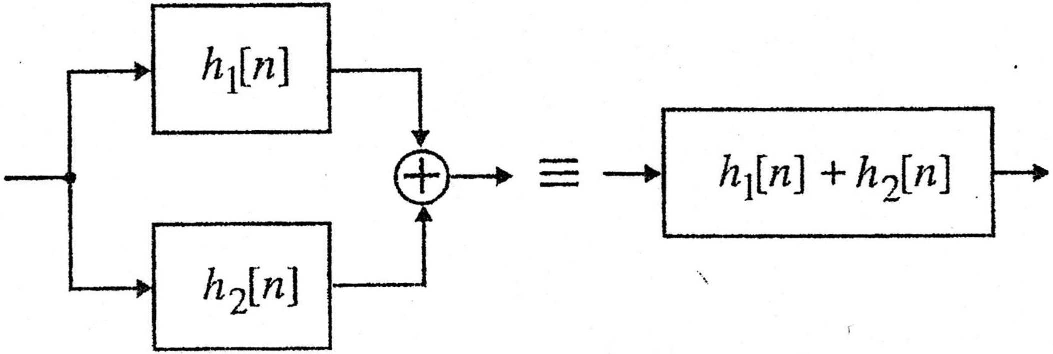


Figure 2.34: The parallel connection.

Discrete Time Systems Representation

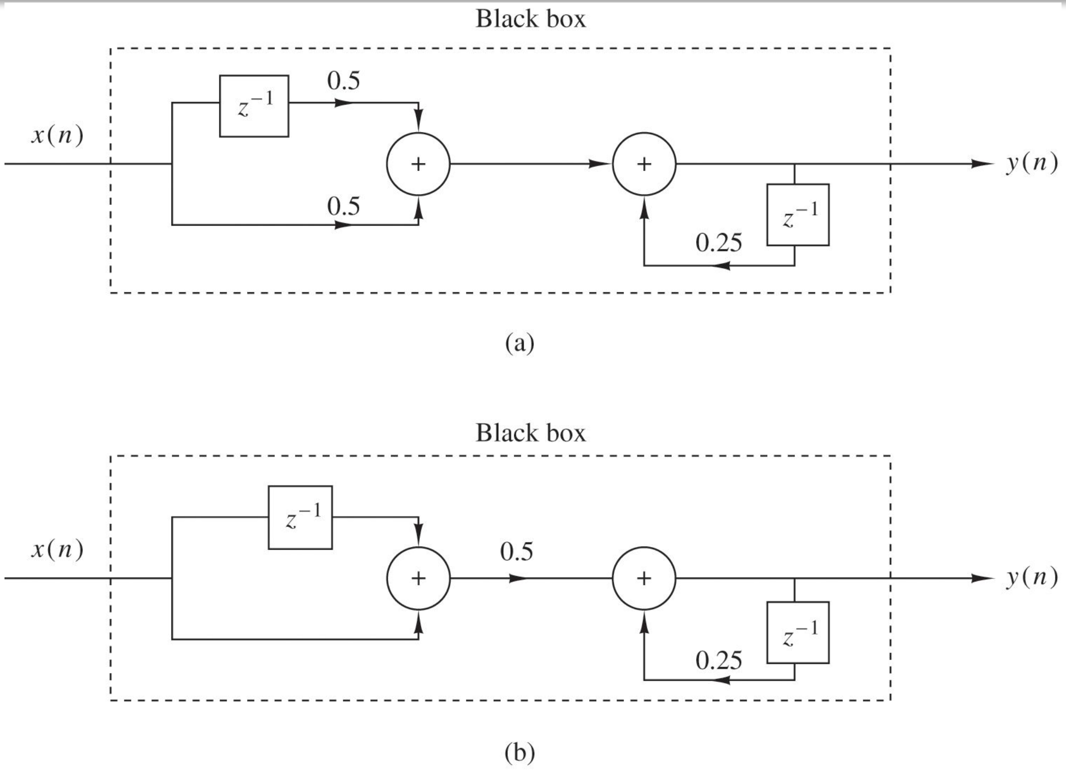


Figure 2.2.7 Block diagram realizations of the system $y(n) = 0.25y(n-1) + 0.5x(n) + 0.5x(n-1)$.

Discrete Time Systems Classification

1. System Memory
2. Recursion
3. Time invariance
4. Linearity
5. Casuality
6. Stability
7. Passivity
8. Energy Loss

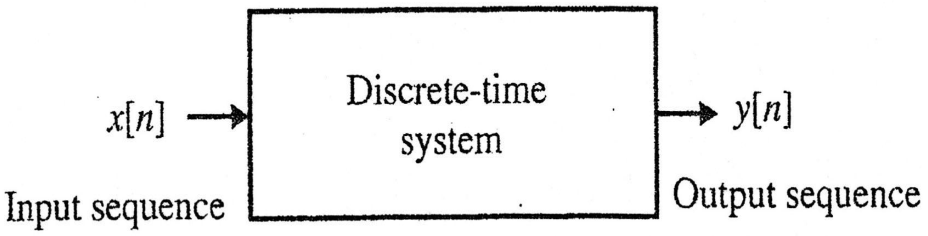


Figure 2.23: Schematic representation of a discrete-time system.

Discrete Time Systems Classification

1. System Memory

➢ **Static/Memoryless** – no memory, i.e. The current output depends only on the current input and sample index

➢ **Dynamic** – with memory, i.e. the current output depends on the past outputs and inputs, and possibly future outputs

Discrete Time Systems Classification

1. <u>System Memory</u>

   a. $y[n] = x[n-1] + 2x[n-2]$
   b. $y[n] = x[n] + 3y[n-1] + y[n-2]$
   c. $y[n] = x[n]$
   d. $y[n] = x[-n]$

Discrete Time Systems Classification

1. <u>System Memory</u>

   a. $y[n] = x[n-1] + 2x[n-2] \rightarrow$ dynamic
   b. $y[n] = x[n] + 3y[n-1] + y[n-2] \rightarrow$ dynamic
   c. $y[n] = x[n] \rightarrow$ static/memoryless
   d. $y[n] = x[-n] \rightarrow$ static/memoryless

Discrete Time Systems Classification

Dynamic: Recursive Systems

- A system is **recursive** if the output is dependent on the input **and past output values**.

Discrete Time Systems Classification

Dynamic: Recursive Systems

a. $y[n] = x[n-1] + 2x[n-2] \rightarrow$ dynamic, non-recursive
b. $y[n] = x[n] + 3y[n-1] + y[n-2] \rightarrow$ dynamic, recursive
c. $y[n] = x[n] \rightarrow$ static/memoryless
d. $y[n] = x[-n] \rightarrow$ static/memoryless

Discrete Time Systems Classification

2. Time invariance

- **Time-Invariant** – a time-delay on the input directly equates to a time-delay of the output function
- **Time-Variant** – a time-delay on the input does not equate to a time-delay of the output function; input and output characteristics change with time

Discrete Time Systems Classification

2. <u>Time invariance</u>

A. Time scaling and Folding

$$y[n] = x[-2n]$$

$$y[n+k] \stackrel{?}{=} x[-2(n+k)]$$

$$x[-2n+k] \neq x[-2n-2k)]$$ → **time variant**

Discrete Time Systems Classification

2. <u>Time invariance</u>

B. Modulator

$$y[n] = x[n]\cos(0.25\pi n)$$

$$y[n+k] \stackrel{?}{=} x[n+k]\cos(0.25\pi n)$$

$$x[n+k]\cos(0.25\pi(n+k)) \neq x[n+k]\cos(0.25\pi n)$$

→ **time variant**

Discrete Time Systems Classification

2. Time invariance

C. Moving average

$$y[n] = 0.5x[n] + 0.5x[n-1]$$

$$y[n+k] \stackrel{?}{=} 0.5x[n+k] + 0.5x[(n+k)-1]$$

$$y[n+k] = 0.5x[n+k] + 0.5x[(n+k)-1]$$

→ **time invariant**

Discrete Time Systems Classification

3. <u>Linearity</u>

➢ A system , H, is **linear** if and only if:

$$\boxed{\mathbf{H\,[\,a_1x_1(n) + a_2x_2(n)\,] = a_1\,H\,[x_1(n)] + a_2\,H\,[x_2(n)]}}$$

Discrete Time Systems Classification

3. <u>Linearity</u>

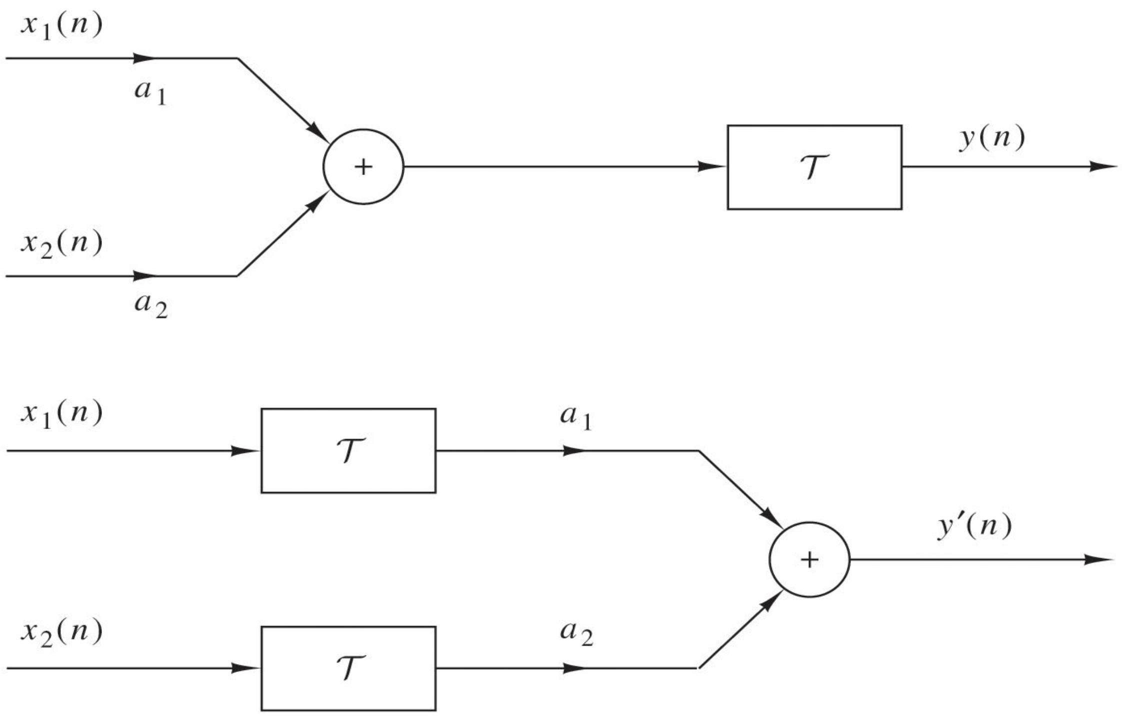


Figure 2.2.9 Graphical representation of the superposition principle. $\mathcal{T}$ is linear if and only if $y(n) = y'(n)$.

Discrete Time Systems Classification

3. <u>Linearity</u>

   1. $y(n) = nx(n)$
   2. $y(n) = x(n^2)$
   3. $y(n) = x^2(n)$
   4. $y(n) = Ax(n) + B$
   5. $y(n) = e^{x(n)}$

Discrete Time Systems Classification

3. <u>Linearity</u>

1. $y(n) = nx(n)$ is **linear**
2. $y(n) = x(n^2)$ is **linear**
3. $y(n) = x^2(n)$ is **nonlinear**
4. $y(n) = Ax(n) + B$ is **linear**
5. $y(n) = e^{x(n)}$ is **nonlinear**

Discrete Time Systems Classification

4. Causality

- **Causal** – the output depends only on the current and past inputs.
- **Non-causal** – the depends on current , past & future inputs.

Discrete Time Systems Classification

4. <u>Causality</u>

$$y(n) = x(n) + 3x(n-2) - 5x(n-20)$$

$$y(n) = 5x(n+10) - 4x(n) + x(n-3)$$

Discrete Time Systems Classification

4. <u>Causality</u>

$$y(n) = x(n) + 3x(n-2) - 5x(n-20)$$

(causal system)

$$y(n) = 5x(n+10) - 4x(n) + x(n-3)$$

(non-causal system)

Discrete Time Systems Classification

4. <u>Causality</u>

A. $y(n) = x(n^2)$
B. $y(n) = x(n) - x(n-1)$
C. $y(n) = \Sigma^{n}_{k=-\infty}\ x(k)$
D. $y(n) = x(-n)$
E. $y(n) = a\ x(n)$
F. $y(n) = x(n) + 3\ x(n+4)$
G. $y(n) = x(2n)$

Discrete Time Systems Classification

4. <u>Causality</u>

   A. $y(n) = x(n^2)$ → **non-causal**
   B. $y(n) = x(n) - x(n-1)$ → **causal**
   C. $y(n) = \Sigma^{n}_{k=-\infty}\, x(k)$ → **causal**
   D. $y(n) = x(-n)$ → **non-causal**
   E. $y(n) = a\, x(n)$ → **causal**
   F. $y(n) = x(n) + 3\, x(n+4)$ → **non-causal**
   G. $y(n) = x(2n)$ → **non-causal**

Discrete Time Systems Classification

5. Stability

- ➢ An arbitrary relaxed system is bounded input – bounded output (BIBO) stable IFF every bounded input produces a bounded output.
- ➢ If for some bounded input the output is unbounded , then the system is unstable .

Discrete Time Systems Classification

5. Stability

➢ Show that:

$$y(n) = y^2(n\text{-}1) + x(n)$$ is **BIBO unstable**

5. <u>Stability</u>

Use $x(n) = u(n)$; assume relaxed system

$$y(n) = y^2(n\text{-}1) + x(n)$$

$$y(0) = y^2(\text{-}1) + x(0) = 0{+}1 = 1$$

$$y(1) = y^2(0) + x(1) = 1{+}1 = 2$$

$$y(2) = y^2(1) + x(2) = 4{+}1 = 5$$

$$y(3) = y^2(2) + x(3) = 25{+}1 = 26$$

Discrete Time Systems Classification

Stable Systems: Passive

- A DT System is passive if $$\sum_{n=-\infty}^{\infty} \left|y[n]\right|^2 \leq \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 \leq \infty$$

- where x[n] is the input and y[n] is the output
- Finite energy input ⇒ at most, same energy in the output
- Passive systems are BIBO stable systems

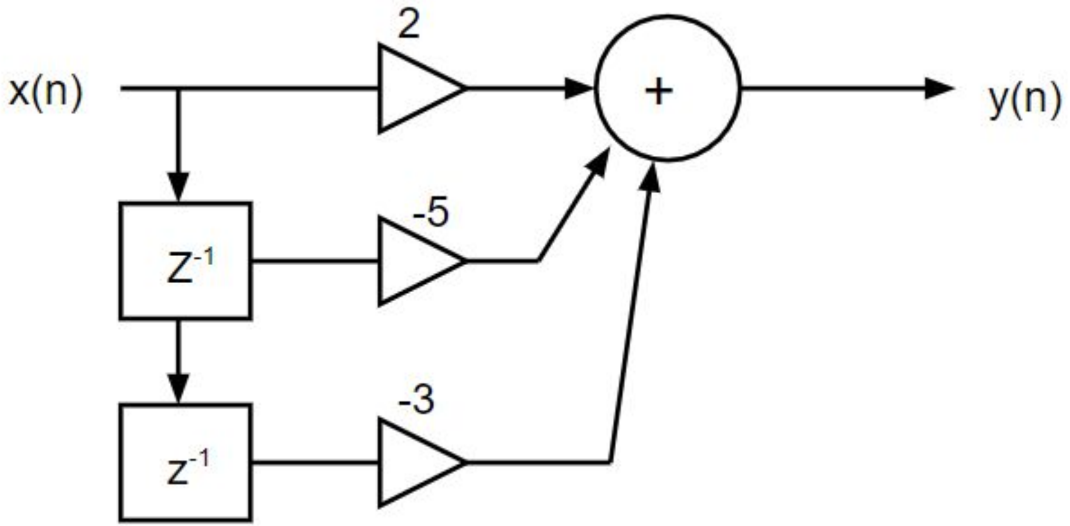
Discrete Time Systems Classification

6. <u>Lossy systems</u>

- ➢ A DT System is lossless if $$\sum_{n=-\infty}^{\infty} |y[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$
- ➢ where x[n] is the input and y[n] is the output
- ➢ e.g. Down-scaling/Down-sampling (lossy)

Examples: Moving Average filter

$$y(n) = 2x(n) - 5x(n-1) - 3x(n-2)$$



Examples: Moving Average filter

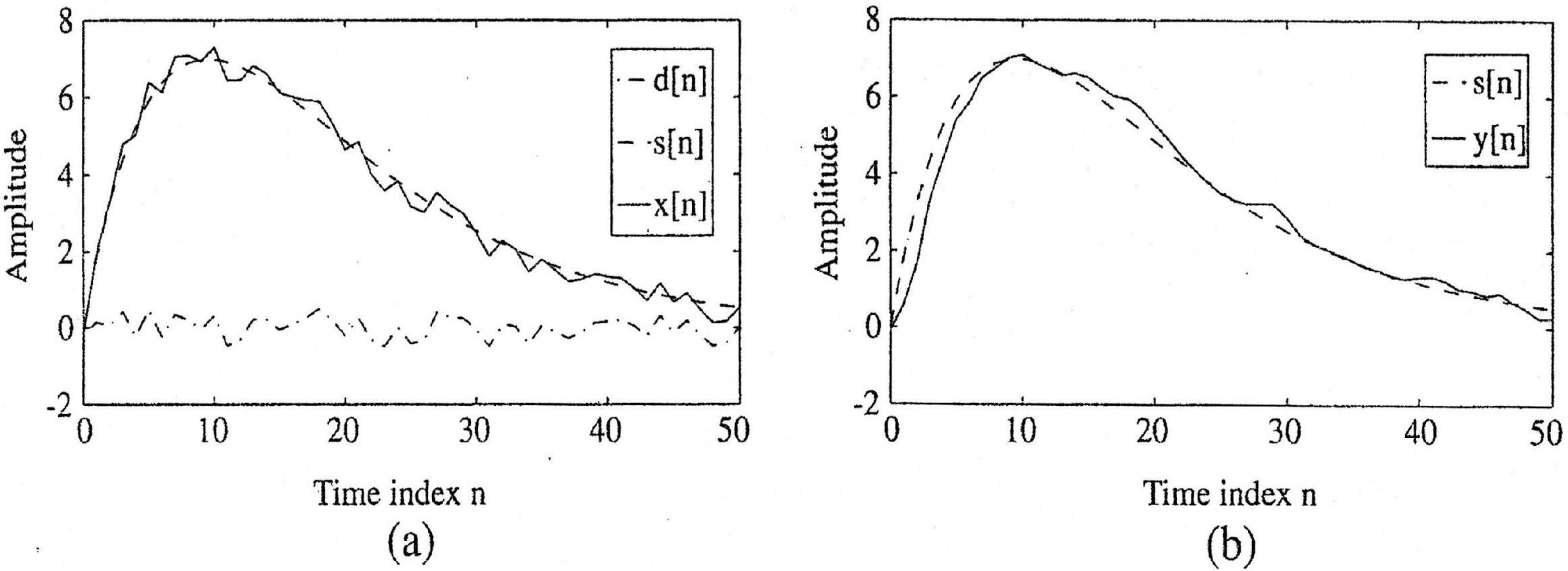
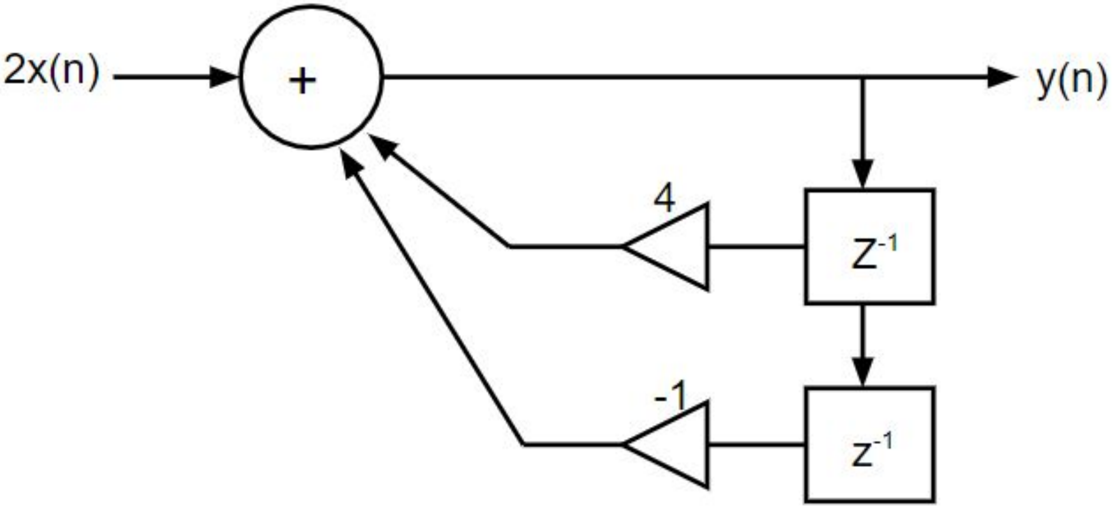


Figure 2.24: Pertinent signals of Example 2.13: $s[n]$ is the original uncorrupted sequence, $d[n]$ is the noise sequence, $x[n] = s[n] + d[n]$, and $y[n]$ is the output of the moving-average filter.

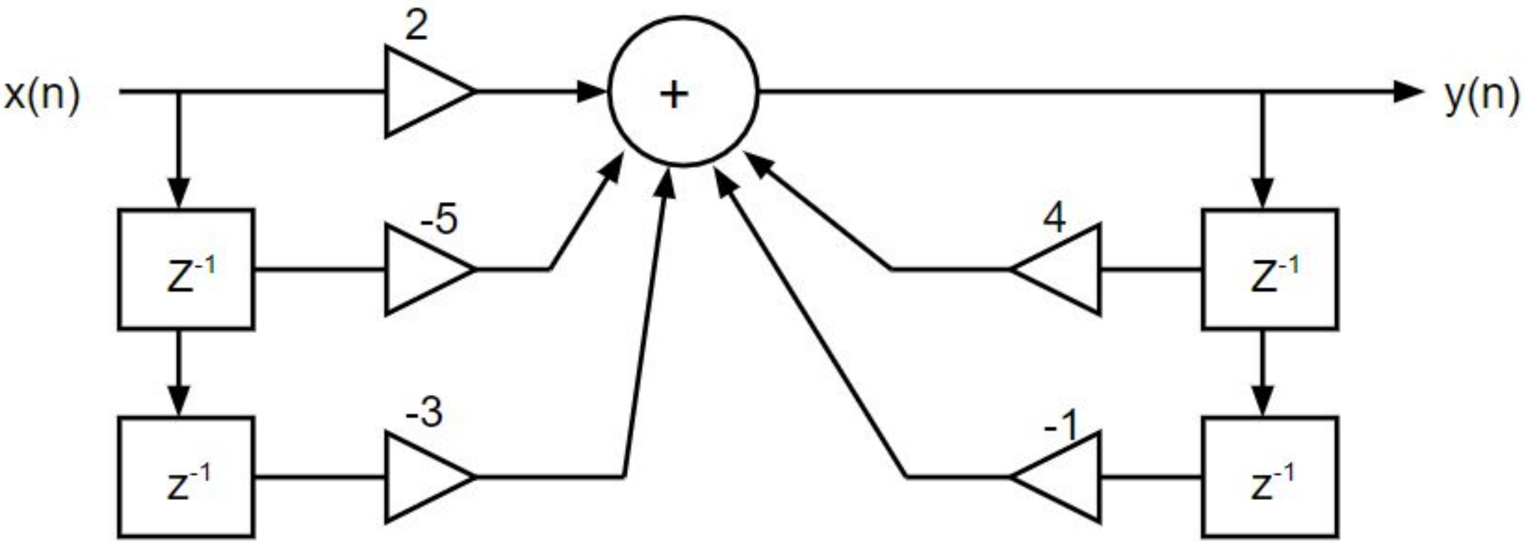
Examples: Autoregressive filter

$$y(n) = 2x(n) + 4y(n-1) - y(n-2)$$



Examples: ARMA filter

$$y(n) = 2x(n) - 5x(n-1) - 3x(n-2) + 4y(n-1) - y(n-2)$$



Examples: Upsampler

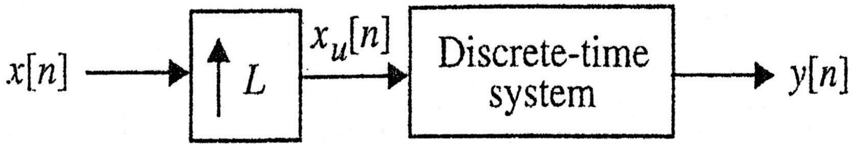
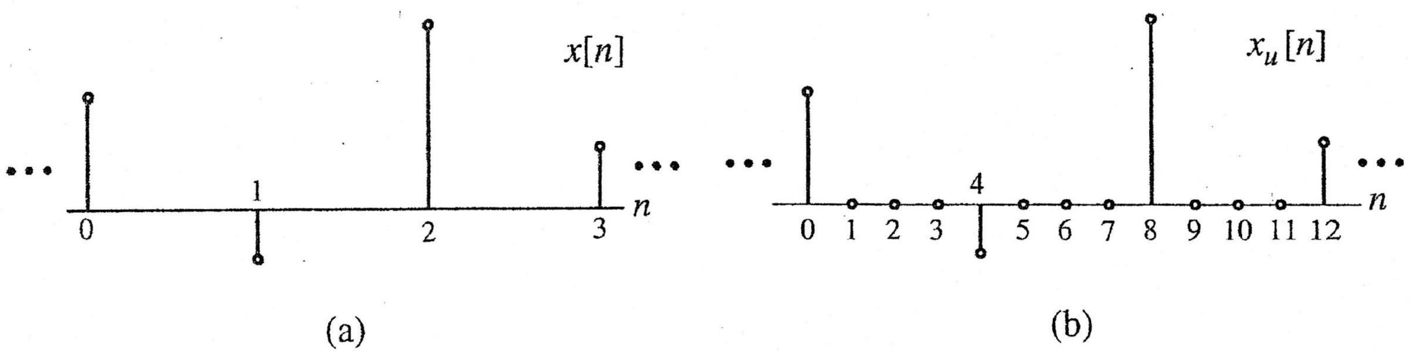


Figure 2.25: A factor-of-L interpolator.



Examples: Upsampler

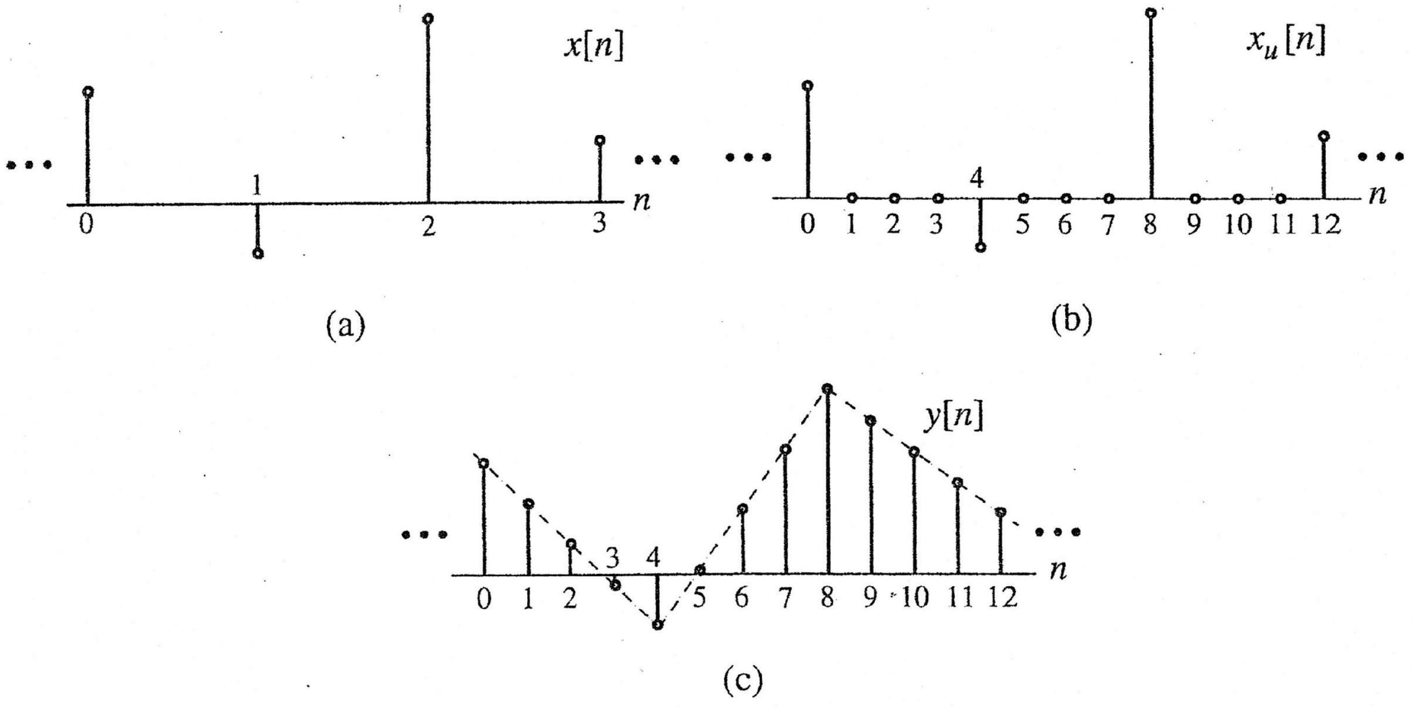


Figure 2.26: Illustration of the linear interpolation method.

Examples: Upsampler

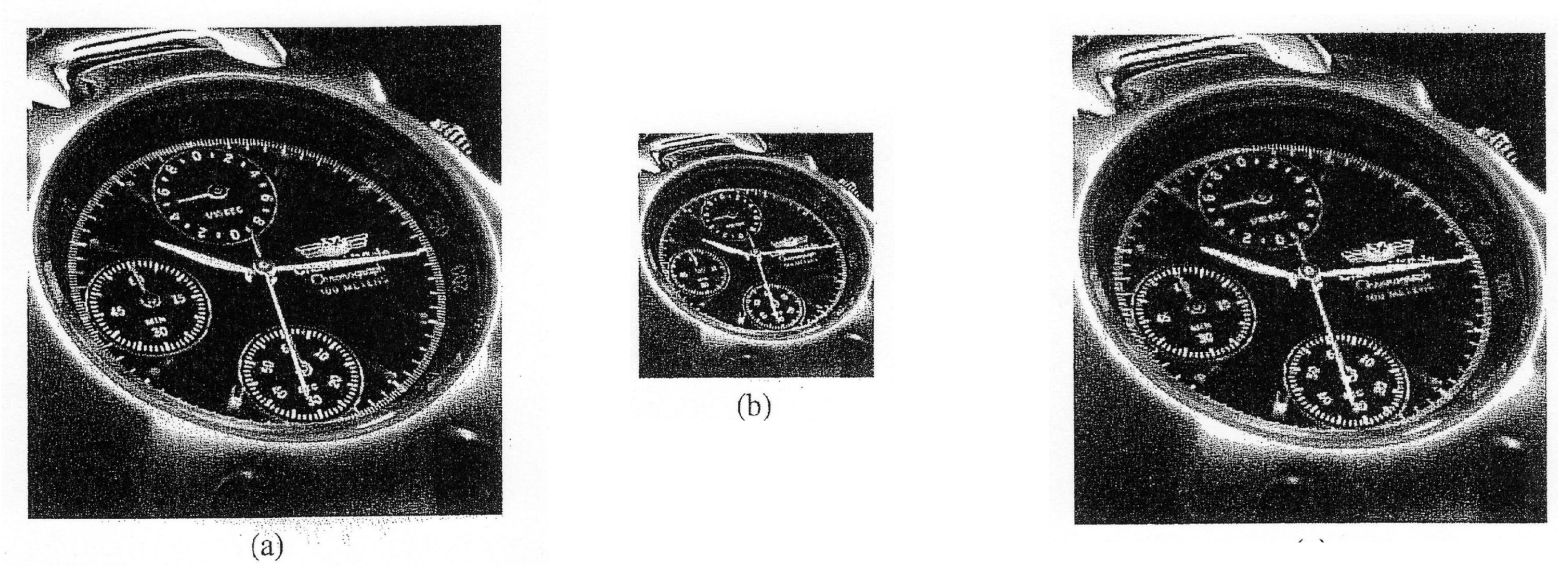


Figure 2.27: (a) Original 512×512 size gray-level image, (b) the down-sampled image of size 256×256, and (c) the zoomed version obtained using the bilinear interpolation.

Summary

- Discrete time systems are represented mathematically by difference equations or graphically by block diagrams
- Discrete time systems can be classified by several properties such as linearity, time-invariance, recursion, causality, stability, energy losses
- Some examples of DT systems are moving average filters, autoregressive systems, ARMA systems, and interpolator filters

For further reading...

- Chapters 2.3-2.5

  “Digital Signal Processing, 3rd ed. By Proakis, J. & Manolakis, D.”

- Chapters 2.1-2.3

  “Signals and Systems by Oppenheim, A & Willsky, A.”

3. Discrete Time System Representation

EE 274/COE 197E

Rhandley Cajote, Ph. D.
Crisron Lucas, MSc.