EE 274 / CoE 121

1st Semester 2009-2010

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Structures for the Realization of Discrete-Time Systems

Linear time-invariant discrete-time system

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Rational System Function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Structures for FIR Systems

$$y(n)=\sum_{k=0}^{M-1} b_k x(n-k) \qquad 9.2.1$$

$$H(z)=\sum_{k=0}^{M-1} b_k z^{-k} \qquad 9.2.2$$

$$h(n)=\begin{cases} b_n & 0 \le n \le M-1 \\ 0 & otherwise \end{cases} \qquad 9.2.3$$

Direct-Form Structure
Tapped-delay-line Filter or Transversal Filter

Non-recursive difference equation

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

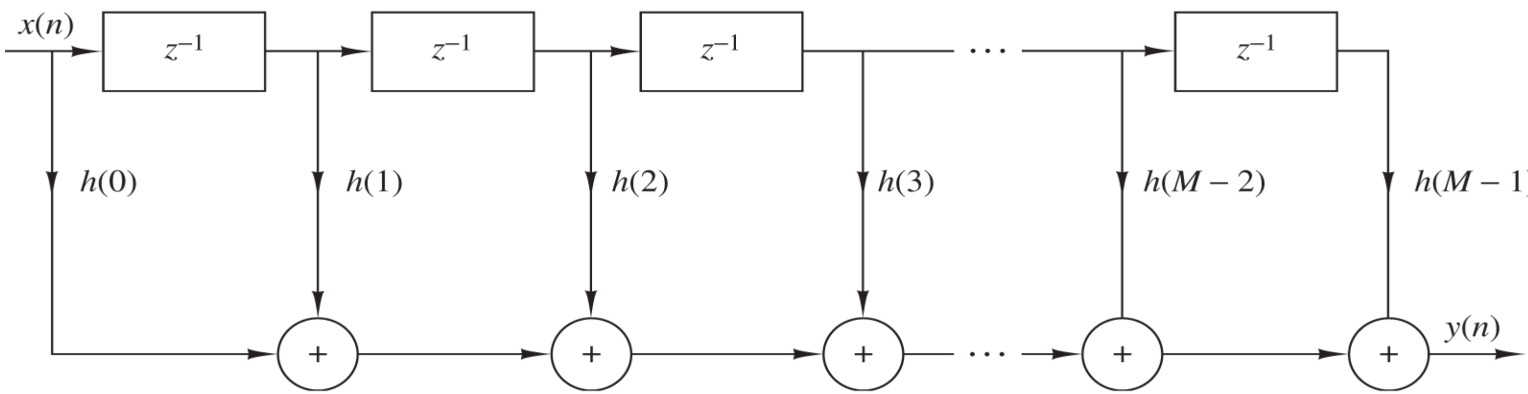


Figure 9.2.1 Direct-form realization of FIR system.

M- memory locations, M multiplications, M-1 additions per output point

Direct-Form Structure

Linear phase FIR: symmetric or asymmetric

$$h(n) = \pm h(M-1-n)$$

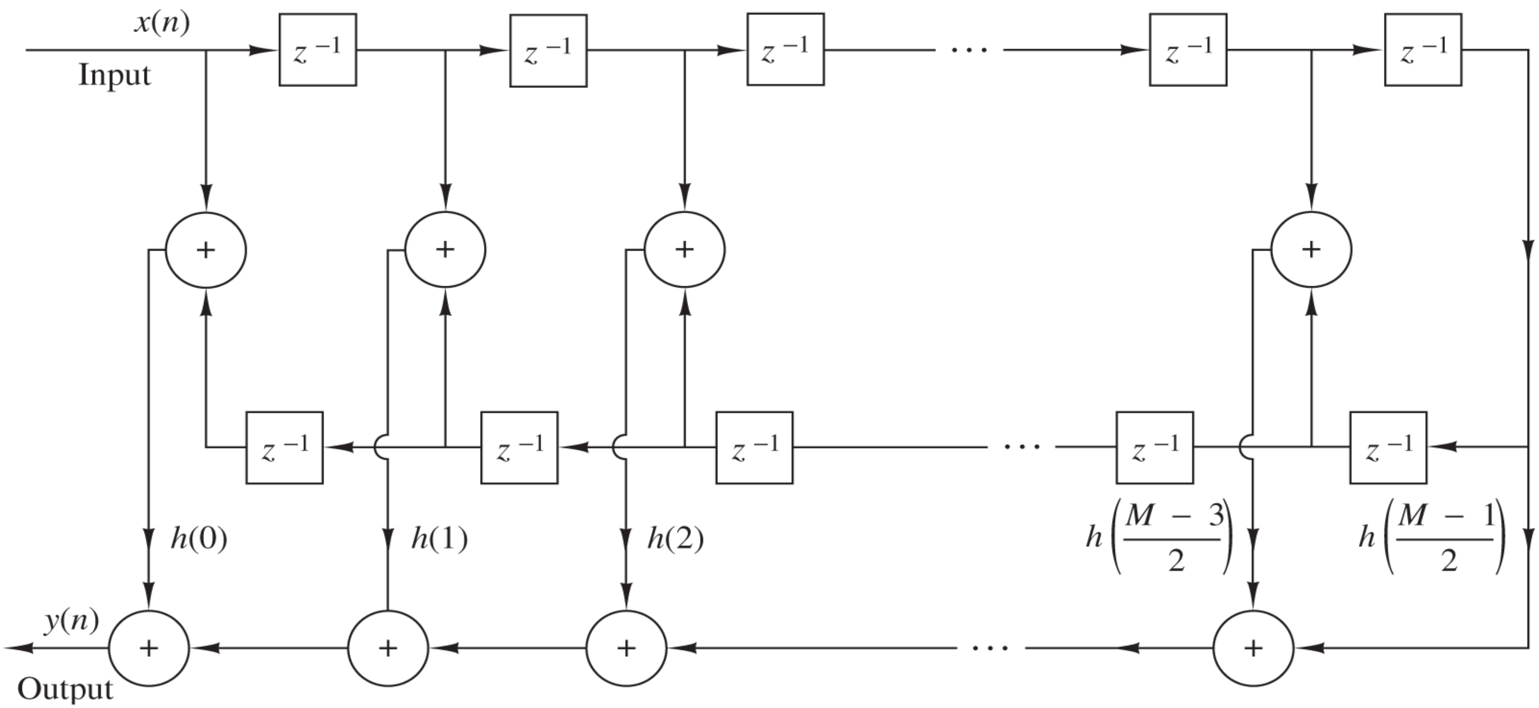
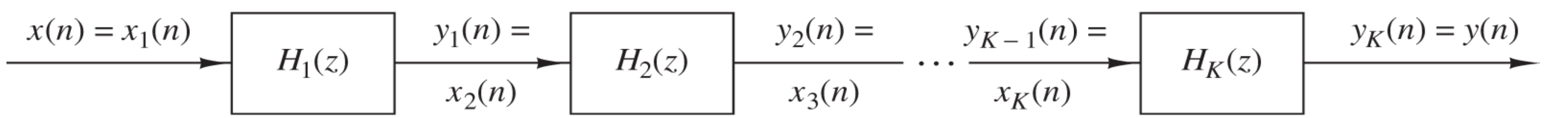


Figure 9.2.2 Direct-form realization of linear-phase FIR system (M odd).

Cascade-Form Structures

Factor H(z) in second-order FIR systems

$$H(z) = \prod_{k=1}^{K} H_k(z)$$



(a)

where $H_k(z) = b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}, \qquad k = 1, 2, \ldots, K$

K is the integer part of (M+1)/2

Cascade-Form Structures

The second-order FIR systems in the cascade structure

$$H_k(z) = b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}, \qquad k = 1, 2, \ldots, K$$

Form pairs of complex-conjugate roots for real-valued coefficients $\{b_{ki}\}$

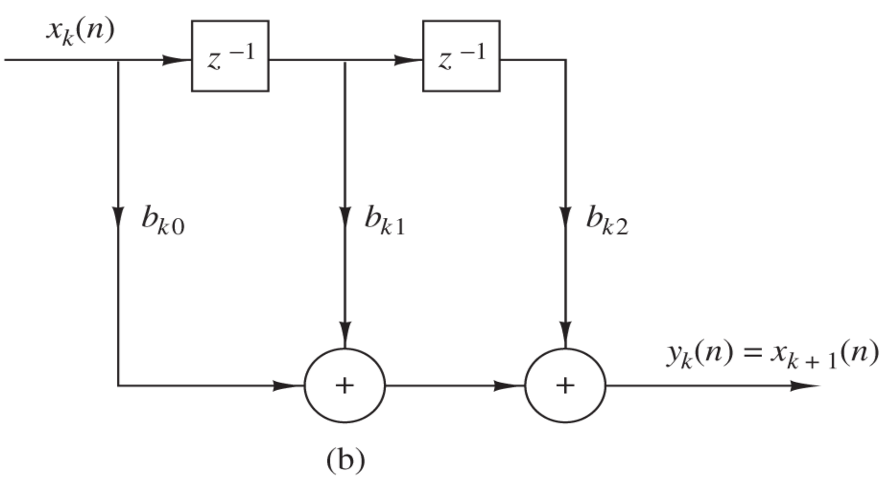


Figure 9.2.3 Cascade realization of an FIR system.

Cascade-Form Structures

For linear phase FIR, complex conjugate pairs of zeros imply reciprocal complex conjugate pairs of zeros. Use fourth-order FIR filter structure.

$$H(z) = \prod_{k=1}^{K} H_k(z)$$

where $$H_k(z) = c_{k0}\left(1 - z_k z^{-1}\right)\left(1 - z_k^* z^{-1}\right)\left(1 - z^{-1}/z_k\right)\left(1 - z^{-1}/z_k^*\right)$$

$$= c_{k0} + c_{k1}z^{-1} + c_{k2}z^{-2} + c_{k1}z^{-3} + c_{k0}z^{-4}$$

Cascade-Form Structures

$$H_k(z) = c_{k0}\left(1 - z_k z^{-1}\right)\left(1 - z_k^* z^{-1}\right)\left(1 - z^{-1}/z_k\right)\left(1 - z^{-1}/z_k^*\right)$$

$$= c_{k0} + c_{k1}z^{-1} + c_{k2}z^{-2} + c_{k1}z^{-3} + c_{k0}z^{-4}$$

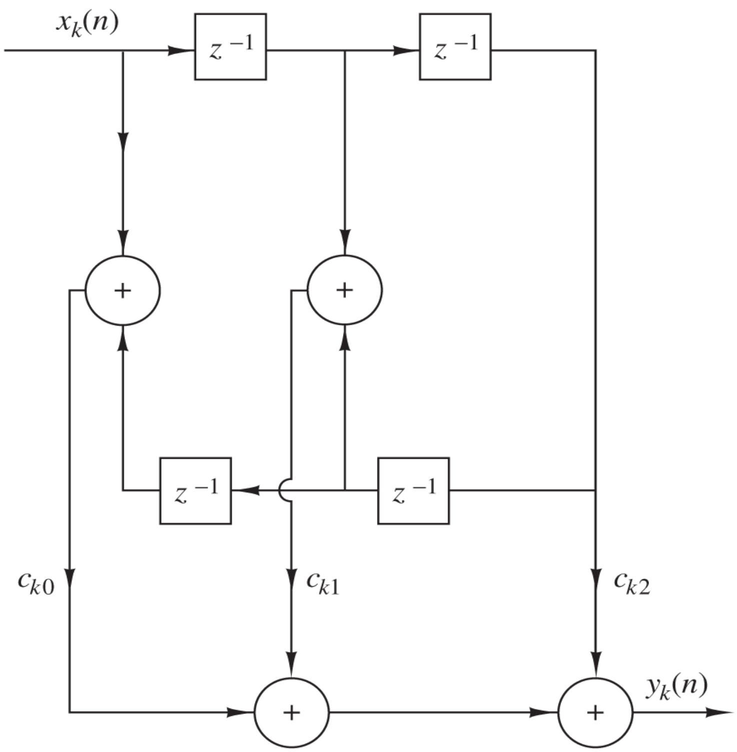


Figure 9.2.4 Fourth-order section in a cascade realization of an FIR system.

Frequency-Sampling Structures

Take samples of the frequency response

$$\omega_k = \frac{2\pi}{M}(k+\alpha), \qquad k = 0,1,\ldots,\frac{M-1}{2}, \qquad M \ odd$$

$$k = 0,1,\ldots,\frac{M}{2}-1, \qquad M \ even$$

$$\alpha = 0 \ or \ \frac{1}{2}$$

Given the frequency response of an FIR filter,

$$H(\omega) = \sum_{n=0}^{M-1} h(n) e^{-j\omega n}$$

Frequency-Sampling Structures

Take samples of the frequency response, H(ω),

$$H(k+\alpha) = H\left(\frac{2\pi}{M}(k+\alpha)\right)$$

$$= \sum_{n=0}^{M-1} h(n) e^{-j2\pi(k+\alpha)n/M}, \qquad k = 0,1,\ldots,M-1$$

Solve for *h(n),*

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k+\alpha) e^{j2\pi(k+\alpha)n/M}, \qquad n = 0,1,\ldots,M-1$$

Frequency-Sampling Structures

From

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k+\alpha)e^{j2\pi(k+\alpha)n/M}, \qquad n = 0,1,\ldots,M-1$$

Solve for the system function,

$$H(z) = \sum_{n=0}^{M-1} h(n)z^{-n}$$

$$= \sum_{n=0}^{M-1}\left[\frac{1}{M}\sum_{k=0}^{M-1} H(k+\alpha)e^{j2\pi(k+\alpha)n/M}\right]z^{-n}$$

Frequency-Sampling Structures

The system function can be written as

$$H(z) = \sum_{k=0}^{M-1} H(k+\alpha)\left[\frac{1}{M}\sum_{n=0}^{M-1}\left(e^{j2\pi(k+\alpha)/M}z^{-1}\right)^n\right]$$

$$= \frac{1 - z^{-M}e^{j2\pi\alpha}}{M}\sum_{k=0}^{M-1}\frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M}z^{-1}}$$

The first term is a comb filter,

$$H_1(z) = \frac{1}{M}\left(1 - z^{-M}e^{j2\pi\alpha}\right)$$

With equally spaced zeros on the unit circle

$$z_k = e^{j2\pi(k+\alpha)/M}, \qquad k = 0, 1, \ldots, M-1$$

Frequency-Sampling Structures

The second term,

$$H_2(z) = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M} z^{-1}}$$

consists of a parallel bank of single-pole filters with resonant frequencies

$$p_k = e^{j2\pi(k+\alpha)/M}, \qquad k = 0, 1, \ldots, M-1$$

Poles and zeros have the same location.

The gains of the parallel bank of resonant filters are

$$H\left(k + \frac{1}{2}\right) = H\left(M - k - \frac{1}{2}\right), \quad for \;\; \alpha = \frac{1}{2}$$

For narrowband filters, most of the {H(k+α)} are zero.

Frequency-Sampling Structures

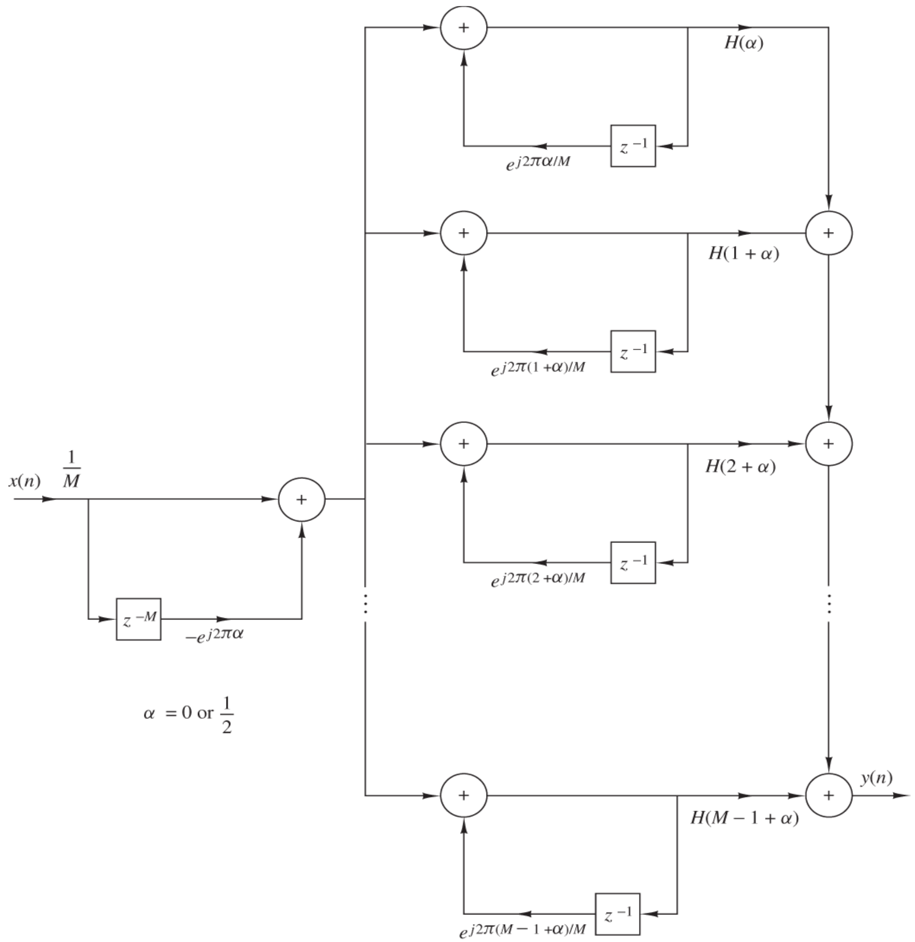


Figure 9.2.5 Frequency-sampling realization of FIR filter.

Frequency-Sampling Structures

$$H_2(z)=\frac{H(0)}{1-z^{-1}}+\sum_{k=1}^{(M-1)/2}\frac{A(k)+B(k)z^{-1}}{1-2\cos(2\pi k/M)z^{-1}+z^{-2}},\qquad M\ odd$$

$$H_2(z)=\frac{H(0)}{1-z^{-1}}+\frac{H(M/2)}{1+z^{-1}}+\sum_{k=1}^{(M/2)-1}\frac{A(k)+B(k)z^{-1}}{1-2\cos(2\pi k/M)z^{-1}+z^{-2}},\qquad M\ even$$

9.2.15

$$A(k)=H(k)+H(M-k)$$
$$B(k)=H(k)e^{-j2\pi k/M}+H(M-k)e^{j2\pi k/M}$$

9.2.16

$$H\left(\frac{2\pi k}{32}\right)=\begin{cases}1, & k=0,1,2\\ \frac{1}{2}, & k=3\\ 0, & k=4,5,\ldots,15\end{cases}$$

Frequency-Sampling Structures Example M=31, α=0

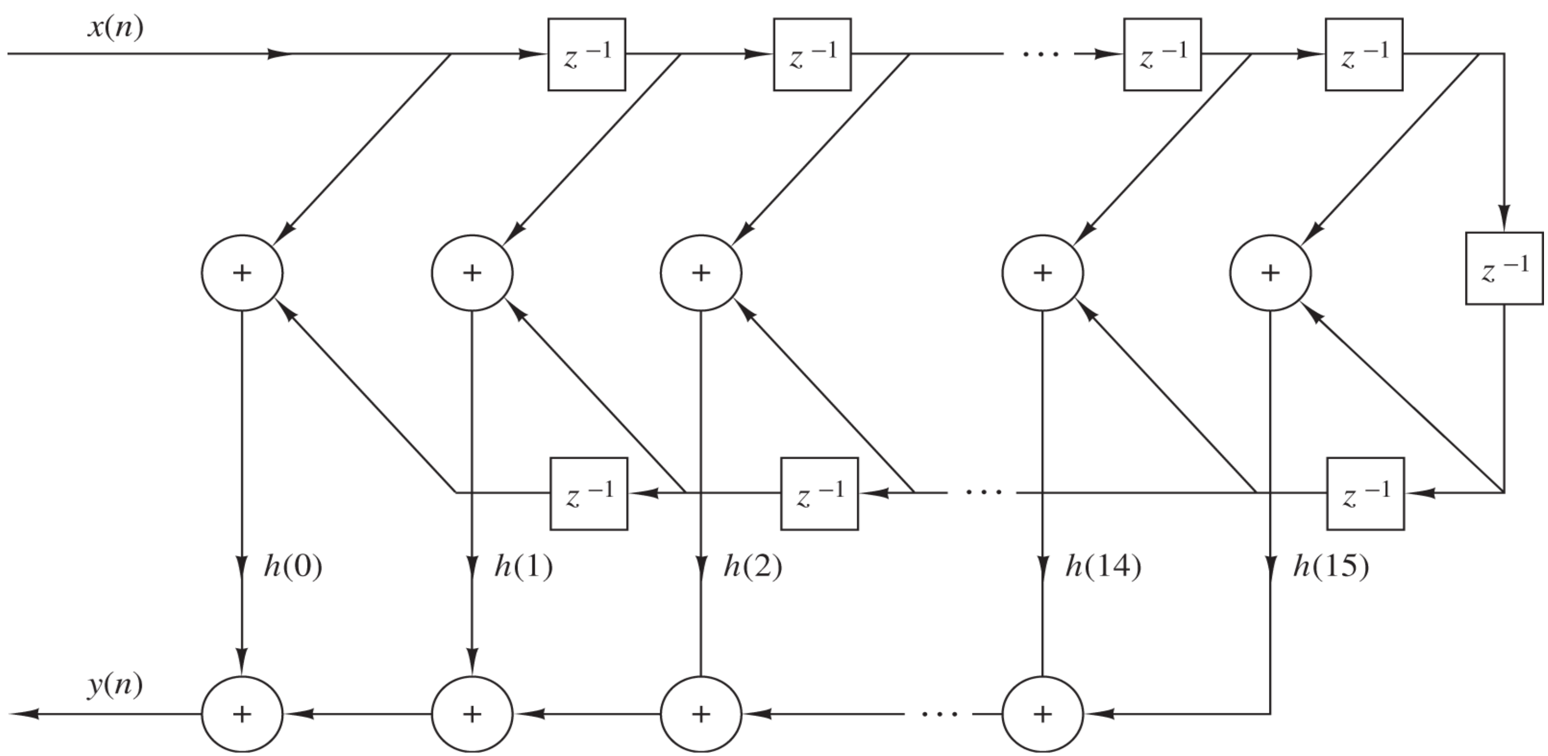
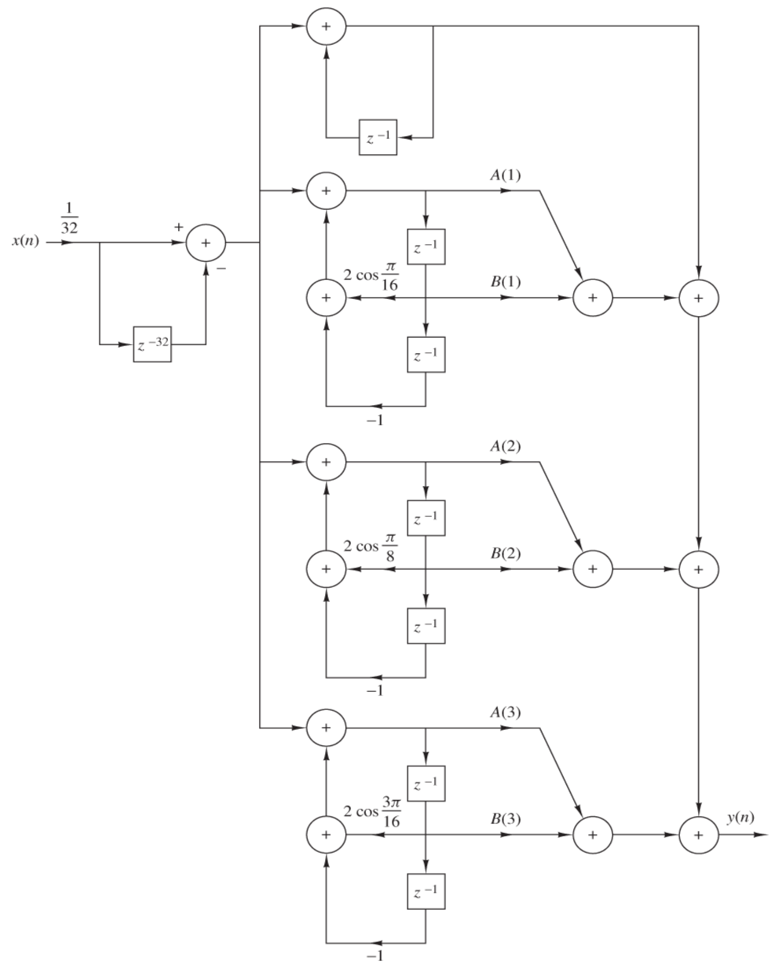


Figure 9.2.6 Direct-form realization of $M = 32$ FIR filter.

16 multiplications,
31 additions

Frequency-Sampling Structures
Example M=31, α=0



9 multiplications,
13 additions

Figure 9.2.7 Frequency-sampling realization for the FIR filter in Example 9.2.1.

Primary Motivation for Lattice Filters

1. One-to-one correspondence between lattice filter equations and physics of waves in homogenous or stratified media.

2. Lattice filters are robust with respect to coefficient quantization.

Lattice Structure

Consider a sequence of FIR filters with system functions

$$H_m(z) = A_m(z), \qquad m = 0, 1, 2, \ldots, M-1$$

where

$$A_m(z) = 1 + \sum_{k=1}^{m} \alpha_m(k) z^{-k}, \qquad m \geq 1$$

$$h_m(0) = 1$$

$$h_m(k) = \alpha_m(k), \qquad k = 1, 2, \ldots, m$$

m is the degree of the polynomial $A_m(z)$

Lattice Structure

The input-output description of the filter $A_m(z)$ and the implementation are

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)\, x(n-k)$$

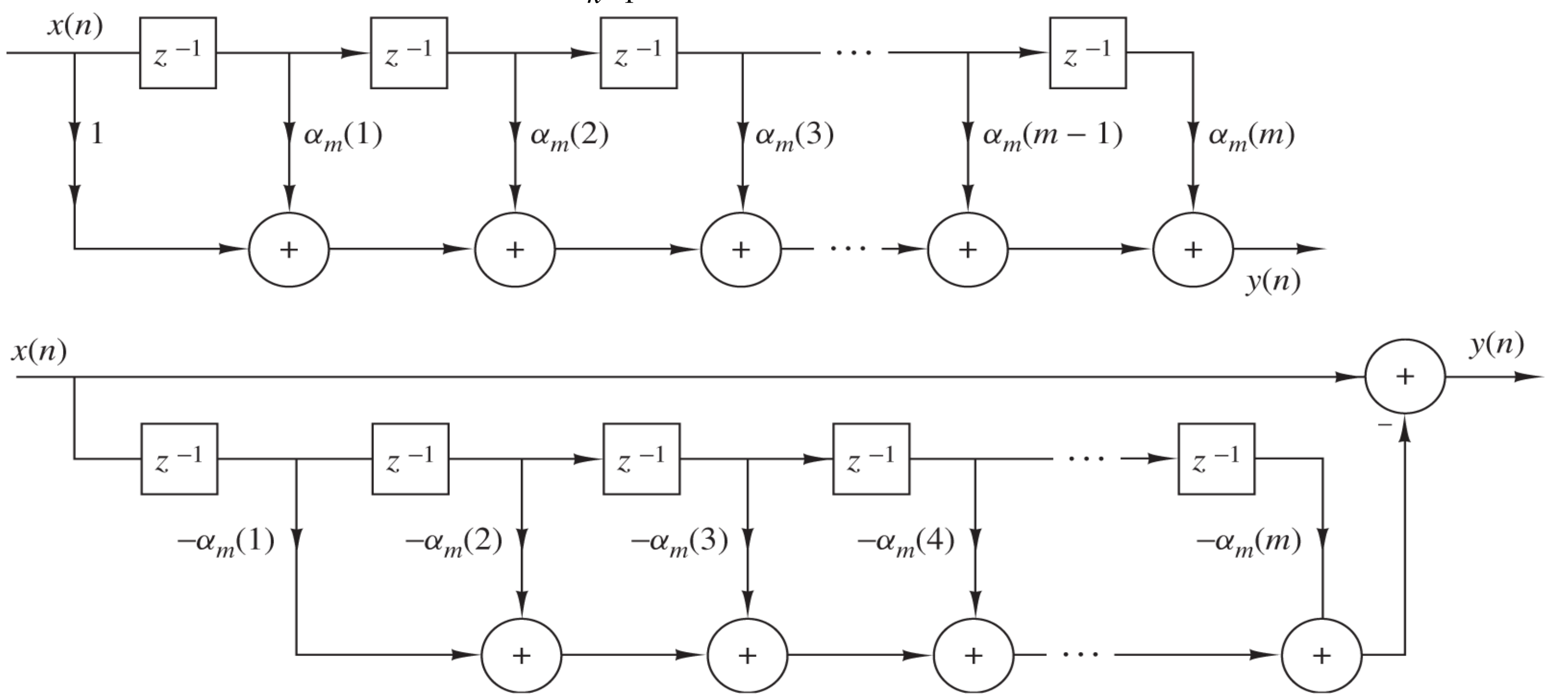
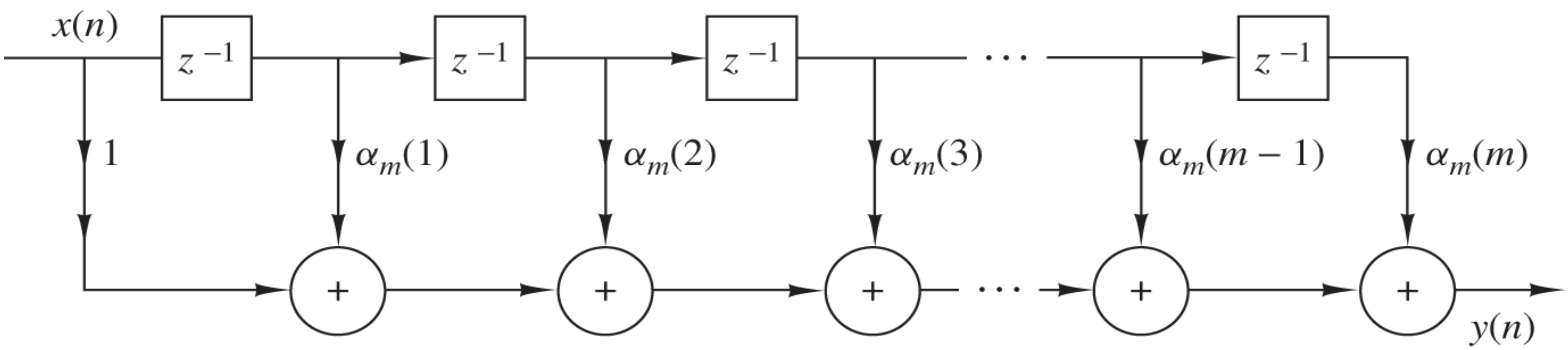


Figure 9.2.8 Direct-form realization of the FIR prediction filter.

Lattice Structure

The one-step forward predicted value of *x(n)* based on *m* past inputs is

$$\hat{x}(n) = -\sum_{k=1}^{m} \alpha_m(k)\, x(n-k)$$



Prediction Error Filter: $y(n) = x(n) - \hat{x}(n)$

Lattice Filter

<u>Objectives:</u>

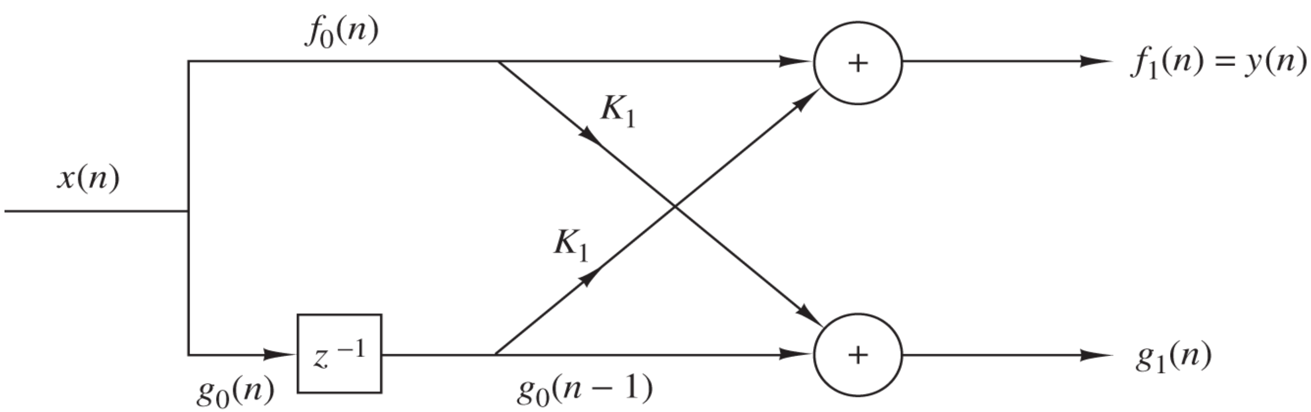
1. Find I/O characteristic
2. Mapping between $\{K_i\}$ & $\{a_M(i)\}$

Lattice Structure

For a first-order filter, *m=1,*

$$y(n) = x(n) + \alpha_1(1)x(n-1)$$

Let $K_1=\alpha_1(1)$



$$f_0(n) = g_0(n) = x(n)$$
$$f_1(n) = f_0(n) + K_1 g_0(n-1) = x(n) + K_1 x(n-1)$$
$$g_1(n) = K_1 f_0(n) + g_0(n-1) = K_1 x(n) + x(n-1)$$

Figure 9.2.9 Single-stage lattice filter.

Lattice Structure

For a second-order filter, *m=2,*

$$y(n) = x(n) + \alpha_2(1)x(n-1) + \alpha_2(2)x(n-2)$$

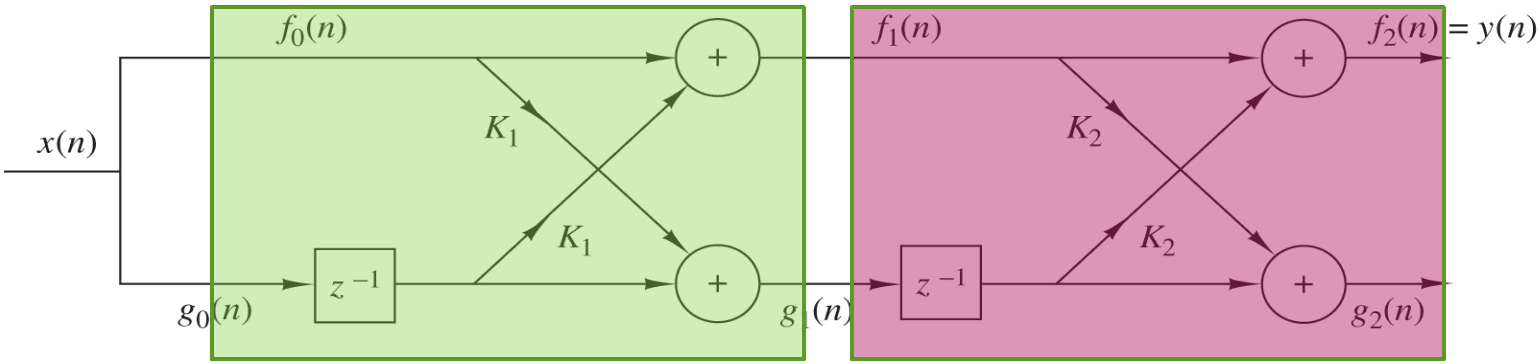


Figure 9.2.10 Two-stage lattice filter.

1st stage:

$$f_1(n) = x(n) + K_1 x(n-1)$$
$$g_1(n) = K_1 x(n) + x(n-1)$$

2nd stage:

$$f_2(n) = f_1(n) + K_2 g_1(n-1)$$
$$g_2(n) = K_2 f_1(n) + g_1(n-1)$$

Lattice Structure

Since $f_2(n)=f_1(n)=g_1(n-1)$

$$\begin{aligned} f_2(n) &= x(n)+K_1x(n-1)+K_2\left[K_1x(n-1)+x(n-2)\right] \\ &= x(n)+K_1(1+K_2)x(n-1)+K_2x(n-2) \end{aligned}$$

Compared to the output of a direct-form FIR filter,

$$\alpha_2(2)=K_2, \qquad \alpha_2(1)=K_1(1+K_2)$$

$$K_2=\alpha_2(2), \qquad K_1=\frac{\alpha_2(1)}{1+\alpha_2(2)}$$

Thus given a direct-form FIR filter, we can compute the equivalent lattice filter.

Lattice Structure

Equivalence between an m^{th} order direct form FIR filter and an *m*-order lattice filter

$$f_0(n) = g_0(n) = x(n)$$

$$f_m(n) = f_{m-1}(n) + K_m g_{m-1}(n-1), \qquad m = 1, 2, \ldots, M-1$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1), \qquad m = 1, 2, \ldots, M-1$$

$$y(n) = f_{M-1}(n)$$

Lattice Structure

The output of an *m*-stage lattice filter,

$$f_m(n) = \sum_{k=0}^{m} \alpha_m(k) x(n-k), \qquad \alpha_m(0) = 1$$

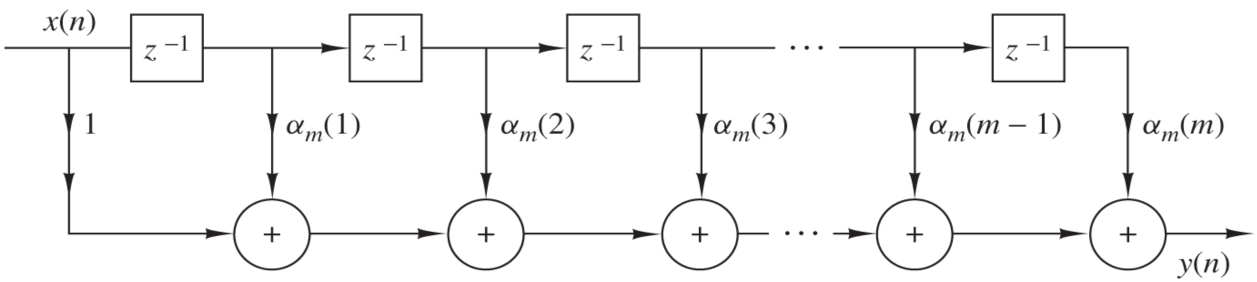
Get the z-transform

$$F_m(z) = A_m(z) X(z)$$

$$A_m(z) = \frac{F_m(z)}{X(z)} = \frac{F_m(z)}{F_0(z)}$$

Lattice Structure

Equivalence between an m^{th} order direct form FIR filter and an m-order lattice filter



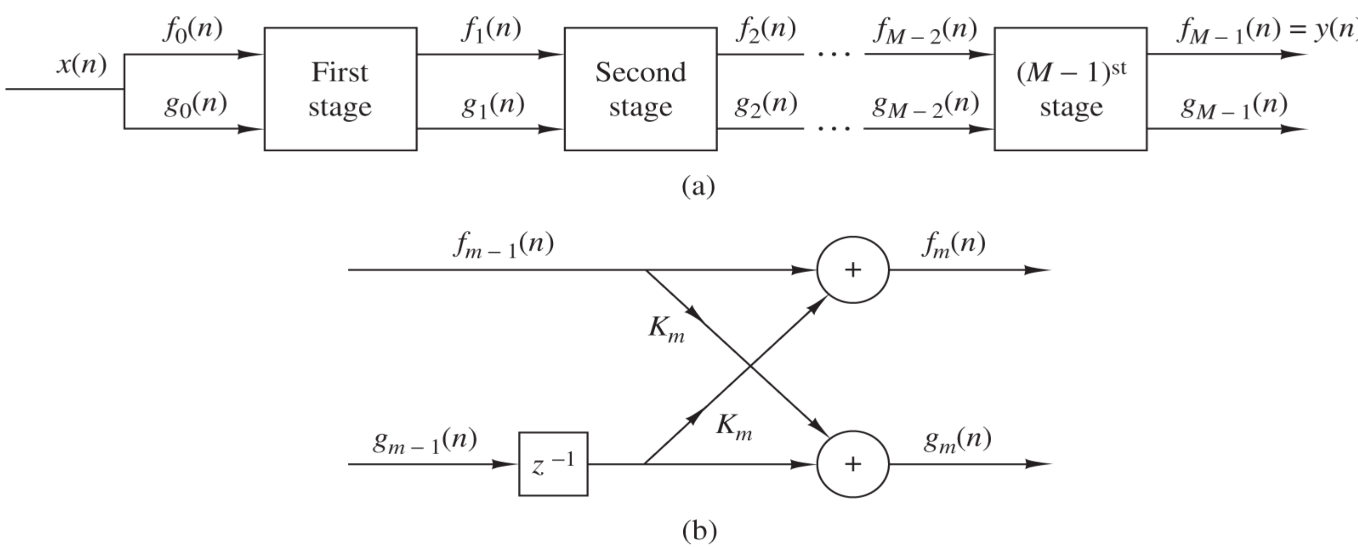


Figure 9.2.11 $(M-1)$-stage lattice filter.

Lattice Structure

For a two-stage lattice filter,

$$
\begin{aligned}
g_2(n) &= K_2 f_1(n) + g_1(n-1) \\
&= K_2\left[x(n) + K_1 x(n-1)\right] + K_1 x(n-1) + x(n-2) \\
&= K_2 x(n) + K_1(1+K_2)x(n-1) + x(n-2) \\
&= \alpha(2)x(n) + \alpha_2(1)x(n-1) + x(n-2)
\end{aligned}
$$

The filter coefficients are: $\{\alpha_2(2), \alpha_2(1), 1\}$

Filter coefficients to p*roduce* $f_2(n)$ $\{1, \alpha_2(1), \alpha_2(2)\}$

Lattice Structure

$$g_m(n) = \sum_{k=0}^{m} \beta_m(k)\, x(n-k) \qquad 9.2.33$$

$$\beta_m(k) = \alpha_m(m-k), \qquad k = 0,1,\ldots,m \qquad 9.2.34$$

$$\hat{x}(n-m) = -\sum_{k=0}^{m-1} \beta_m(k)\, x(n-k) \qquad 9.2.35$$

$$G_m(z) = B_m(z)\, X(z) \qquad 9.2.36$$

$$B_m(z) = \frac{G_m(z)}{X(z)} \qquad 9.2.37$$

Lattice Structure

$$B_m(z) = \sum_{k=0}^{m} \beta_m(k) z^{-k} \qquad 9.2.38$$

$$\begin{aligned} B_m(z) &= \sum_{k=0}^{m} \alpha_m(m-k) z^{-k} \\ &= \sum_{l=0}^{m} \alpha_m(l) z^{l-m} \\ &= z^{-m} \sum_{l=0}^{m} \alpha_m(l) z^{l} \\ &= z^{-m} A_m\left(z^{-1}\right) \end{aligned} \qquad 9.2.39$$

The zeros of $B_m(z)$ are the reciprocal of the zeros of $A_m(z)$

$B_m(z)$ is called the reverse polynomial of $A_m(z)$

Lattice Structure

$$F_0(z) = G_0(z) = X(z) \qquad 9.2.40$$

$$F_m(z) = F_{m-1}(z) + K_m z^{-1} G_{m-1}(z), \qquad m = 1, 2, \ldots, M-1 \qquad 9.2.41$$

$$G_m(z) = K_m F_{m-1}(z) + z^{-1} G_{m-1}(z), \qquad m = 1, 2, \ldots, M-1 \qquad 9.2.42$$

$$A_0(z) = B_0(z) = 1 \qquad 9.2.43$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z), \qquad m = 1, 2, \ldots, M-1 \qquad 9.2.44$$

$$B_m(z) = K_m A_{m-1}(z) + z^{-1} B_{m-1}(z), \qquad m = 1, 2, \ldots, M-1 \qquad 9.2.45$$

Lattice Structure

$$\begin{bmatrix} A_m(z) \\ B_m(z) \end{bmatrix} = \begin{bmatrix} 1 & K_m \\ K_m & 1 \end{bmatrix} \begin{bmatrix} A_{m-1}(z) \\ z^{-1}B_{m-1}(z) \end{bmatrix} \qquad 9.2.46$$

$$A_0(z) = B_0(z) = 1 \qquad 9.2.47$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z), \qquad m = 1, 2, \ldots, M-1 \qquad 9.2.48$$

$$B_m(z) = z^{-m} \mathrm{A}_m(z^{-1}), \qquad m = 1, 2, \ldots, M-1 \qquad 9.2.49$$

Direct-Form FIR filter coeff. and Reflection coefficients

Lattice Structure Example

Let K_1=1/4, K_2 = ¼, K_3=1/3, find FIR filter coefficients for DF structure

$$A_1(z) = A_0(z) + K_1 z^{-1} B_0(z)$$
$$= 1 + K_1 z^{-1} = 1 + \tfrac{1}{4} z^{-1}$$

$$B_1(z) = \tfrac{1}{4} + z^{-1}$$

Reverse polynomial of $A_1(z)$

$$A_2(z) = A_1(z) + K_2 z^{-1} B_1(z)$$
$$= 1 + \tfrac{3}{8} z^{-1} + \tfrac{1}{2} z^{-2}$$

$$B_2(z) = \tfrac{1}{2} + \tfrac{3}{8} z^{-1} + z^{-2}$$

Lattice Structure Example

Let K_1=1/4, K_2 = ¼, K_3=1/3, find FIR filter coefficients for DF

$$B_2(z) = \tfrac{1}{2} + \tfrac{3}{8} z^{-1} + z^{-2}$$

$$A_3(z) = A_2(z) + K_3 z^{-1} B_2(z)$$

$$= 1 + \tfrac{13}{24} z^{-1} + \tfrac{5}{8} z^{-2} + \tfrac{1}{3} z^{-3}$$

$$\alpha_3(0) = 1, \qquad \alpha_3(1) = \frac{13}{24}, \qquad \alpha_3(2) = \frac{5}{8}, \qquad \alpha_3(3) = \frac{1}{3}$$

Lattice implementations require less multiplications

Addition of an m-th stage lattice does not alter the output of the previous

Addition of an m-th stage FIR alter the coefficients

Lattice Structure

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z)$$

$$\sum_{k=0}^{m} \alpha_m(k) z^{-k} = \sum_{k=0}^{m-1} \alpha_{m-1}(k) z^{-k} + K_m \sum_{k=0}^{m-1} \alpha_{m-1}(m-1-k) z^{-(k+1)} \qquad 9.2.50$$

$$\alpha_m(0) = 1 \qquad 9.2.51$$

$$\alpha_m(m) = K_m \qquad 9.2.52$$

Recursive equations for the FIR filter coefficients

$$\begin{aligned} \alpha_m(k) &= \alpha_{m-1}(k) + K_m \alpha_{m-1}(m-k) \\ &= \alpha_{m-1}(k) + \alpha_m(m) \alpha_{m-1}(m-k), \qquad 1 \le k \le m-1 \\ & \qquad m = 1, 2, \ldots, M-1 \end{aligned} \qquad 9.2.53$$

Lattice Structure

Conversion of DF FIR filter to Lattice coefficients

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z)$$

$$= A_{m-1}(z) + K_m \left[B_m(z) - K_m A_{m-1}(z) \right]$$ From 9.2.45

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}, \qquad m = M-1,\ M-2, \ldots, 1$$ 9.2.54

$$H(z) = A_3(z) = 1 + \tfrac{13}{24} z^{-1} + \tfrac{5}{8} z^{-2} + \tfrac{1}{3} z^{-3}$$

$K_3 = \alpha(3) = 1/3$

$$B_3(z) = \tfrac{1}{3} + \tfrac{5}{8} z^{-1} + \tfrac{13}{24} z^{-2} + z^{-3}$$

$$A_2(z) = \frac{A_3(z) - K_3 B_3(z)}{1 - K_3^2}$$

$$= 1 + \tfrac{3}{8} z^{-1} + \tfrac{1}{2} z^{-2}$$

$K_2 = \alpha(2) = 1/2$

$$A_1(z) = \frac{A_2(z) - K_2 B_2(z)}{1 - K_2^2}$$

$$= 1 + \tfrac{1}{4} z^{-1}$$

$K_1 = \alpha(1) = 1/4$

Lattice Structure

$$K_m = \alpha_m(m), \qquad \alpha_{m-1}(0) = 1 \qquad 9.2.55$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - K_m\beta_m(k)}{1 - K_m^2}$$

$$= \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2(m)}, \qquad 1 \le k \le m-1 \qquad 9.2.56$$

Recursive equations for computing K_m, begin with m = M-1

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