5. Z-Transform for DT systems

EE 274/COE 197E

Rhandley Cajote, Ph. D.
Crisron Lucas, MSc.

Today's Lesson:

1. The DT complex exponential
2. Z Transform definition, examples
3. Region of Convergence
4. Properties of ROC
5. Properties of z Transform

The complex exponential

- Most of the signals that we see in DT systems can be represented by a complex exponential:

$$\boxed{z^n = re^{j\omega n}}$$

A. $5(2^n)$ → exponentially increasing
B. $3(-0.1^n)$ → exponentially decreasing, oscillating
C. $\cos(0.5n) = 0.5(e^{j0.5n}) + 0.5(e^{-j0.5n})$ → oscillating
D. $5(1^n)$ → constant

The complex exponential

- Consider a DT system, with $x[n] = z^n$ as an input

$$
\begin{aligned}
y[n] &= h[n] * x[n], \quad x[n] = z^n \\
&= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]z^{n-k}
\end{aligned}
$$

The complex exponential

- Consider a DT system, with $x[n] = z^n$ as an input

$$y[n] = h[n] * x[n], \quad x[n] = z^n$$

$$= \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]z^{n-k}$$

$$y[n] = z^n \boxed{\sum_{k=-\infty}^{\infty} h[k]z^{-k}} = z^n H(z)$$

In the perspective of z, DT convolution becomes multiplication

The z Transform

- For a discrete-time, LTI system

$$y[n] = z^n \left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right) = z^n H(z)$$

- Output to a complex exponential is also a complex exponential of the **same frequency**
- Output is scaled by a complex factor H(z)
- We define H(z) as the **z-transform** of the impulse response h[n].

Examples

➢ Determine the z-transform of the following sequences.

$$x_1[n] = \{\underset{\uparrow}{1}, 2, 3, 4, 5\}$$

$$x_2[n] = a^n u[n]$$

$$x_3[n] = -a^n u[-n-1]$$

(a is a real number)

Examples

$$X_1(z) = \sum_{k=-\infty}^{\infty} x_1[k]z^{-k}$$

$$x_1[n] = \{\underset{\uparrow}{1}, 2, 3, 4, 5\}$$

$$= \sum_{k=0}^{4} x_1[k]z^{-k}$$

$$= 1z^{-0} + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}$$

$$X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}$$

$$\text{for all } z \text{ except } z = 0$$

Examples

$$x_2[n] = a^n u[n]$$

$$X_2(z) = \sum_{k=-\infty}^{\infty} x_2[k] z^{-k}$$

$$= \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= 1 + a^1 z^{-1} + \ldots + a^k z^{-k} + \ldots$$

$$= \frac{1}{1 - az^{-1}}, \text{ if } \left|az^{-1}\right| < 1$$

$$X_2(z) = \frac{1}{1 - az^{-1}}, \text{ if } |z| > |a|$$

Examples

$$x_3[n] = -a^n u[-n-1]$$

$$\begin{aligned} X_3(z) &= \sum_{k=-\infty}^{\infty} x_3[k] z^{-k} \\ &= -\sum_{k=-\infty}^{-1} a^k z^{-k} \\ &= -\sum_{m=0}^{\infty} \left(az^{-1}\right)^{-m-1}, \text{ where } m = -k-1 \\ &= -\left(az^{-1}\right)^{-1} \sum_{m=0}^{\infty} \left(a^{-1}z\right)^{m} \end{aligned}$$

Examples

$$x_3[n] = -a^n u[-n-1]$$

$$X_3(z) = \sum_{k=-\infty}^{\infty} x_3[k] z^{-k}$$

$$= -\left(az^{-1}\right)^{-1} \sum_{m=0}^{\infty} \left(a^{-1}z\right)^m$$

$$= -\frac{\left(az^{-1}\right)^{-1}}{1 - a^{-1}z}, \text{ if } \left|a^{-1}z\right| < 1$$

$$X_3(z) = \frac{1}{1 - az^{-1}}, \text{ if } |z| < |a|$$

Examples

$x_1[n] = \{\underset{\uparrow}{1}, 2, 3, 4, 5\},$	$X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4},$	for all z except $z = 0$
$x_2[n] = a^n u[n],$	$X_2(z) = \frac{1}{1-az^{-1}},$	if $\lvert z\rvert > \lvert a\rvert$
$x_3[n] = -a^n u[-n-1],$	$X_3(z) = \frac{1}{1-az^{-1}},$	if $\lvert z\rvert < \lvert a\rvert$

- X(z) is usually a rational expression in z^{-1}.
- The expression for X(z) is accompanied by a restriction on the value of z. We call this restriction the **region of convergence** (ROC).

Examples

$$x_1[n] = \{\underset{\uparrow}{1}, 2, 3, 4, 5\}, \qquad X_1(z) = \begin{array}{l} 1 + 2z^{-1} + 3z^{-2} \\ +4z^{-3} + 5z^{-4}, \end{array} \qquad \begin{array}{l} \text{for all } z \\ \text{except } z = 0 \end{array}$$

$$x_2[n] = a^n u[n], \qquad X_2(z) = \frac{1}{1-az^{-1}}, \qquad \text{if } |z| > |a|$$

$$x_3[n] = -a^n u[-n-1], \qquad X_3(z) = \frac{1}{1-az^{-1}}, \qquad \text{if } |z| < |a|$$

➢ Note that two or more different signal can have the same expression for X(z). The **complete z-domain representation** of a signal x[n] **includes the expression for X(z) and the ROC!**

Examples

	Sequence $x[n]$	z-Transform $X(z)$	ROC
1.	$\delta[n]$	1	All z
2.	$u[n]$	$\dfrac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
3.	$a^n u[n]$	$\dfrac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4.	$-a^n u[-n-1]$	$\dfrac{1}{1 - az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
5.	$na^n u[n]$	$\dfrac{az^{-1}}{(1 - az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$

Examples

6.	$-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7.	$(\cos\omega_0 n)u[n]$	$\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
8.	$(\sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
9.	$(r^n\cos\omega_0 n)u[n]$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
10.	$(r^n\sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

The Region of Convergence

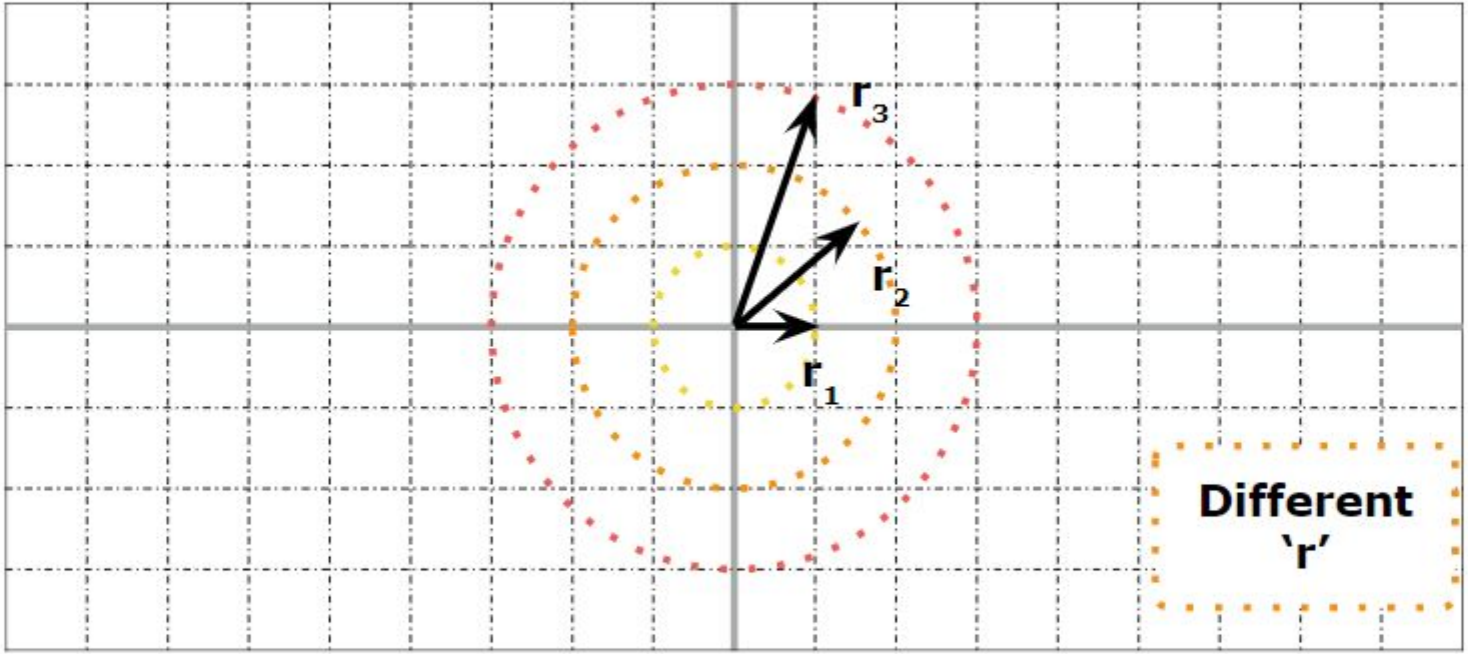
- **<u>For convergence of X(z):</u>**

$$|X(z)| \leq \sum_{n=-\infty}^{\infty} \left|x[n]z^{-n}\right| = \sum_{n=-\infty}^{\infty} \left|x[n]r^{-n}\right| < \infty$$

- Since $z = re^{j\omega n}$, the magnitude of **r** at different **ω** affects the convergence of X(z)
- Defined by **circular regions**

The Region of Convergence

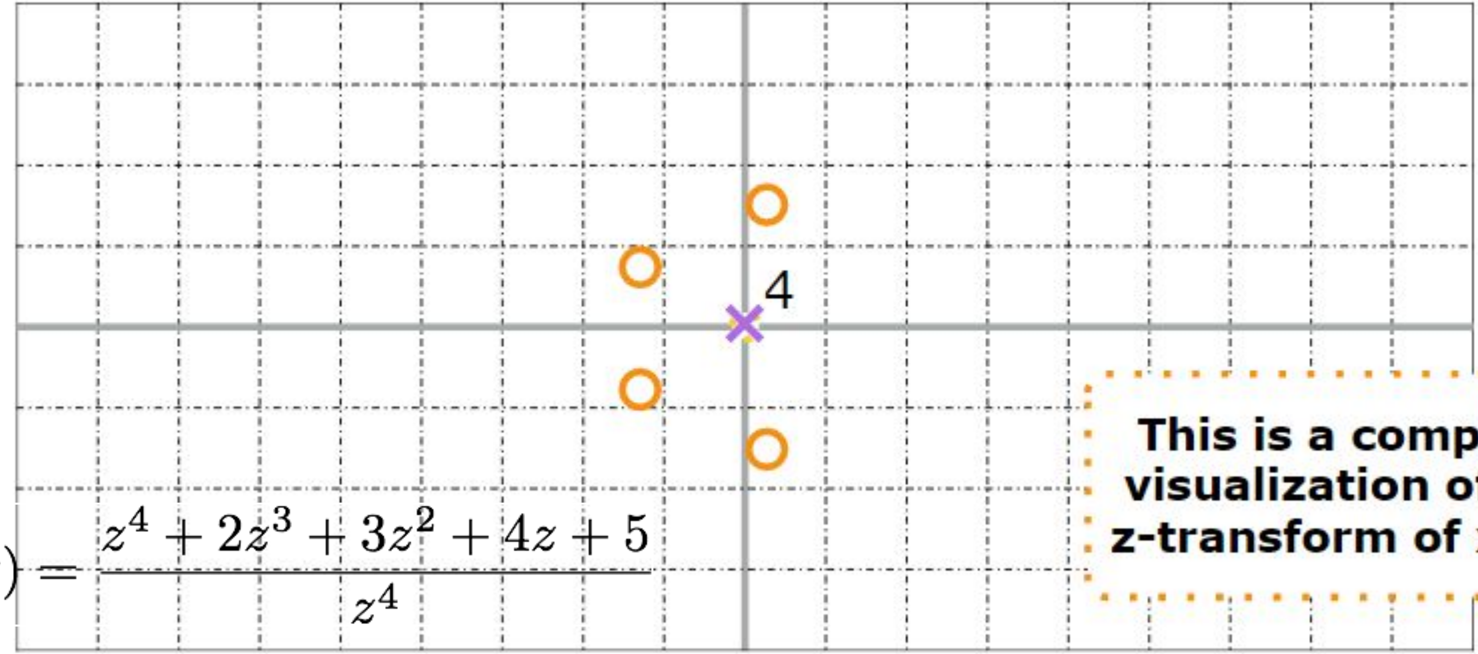
➢ <u>Illustration of $z = re^{j\omega n}$</u>



Examples

$$x_1[n] = \{\underset{\uparrow}{1}, 2, 3, 4, 5\},$$

$$X_1(z) = \begin{array}{l} 1 + 2z^{-1} + 3z^{-2} \\ +4z^{-3} + 5z^{-4}, \end{array} \quad \begin{array}{l} \text{for all } z \\ \text{except } z = 0 \end{array}$$



$$X_1(z) = \frac{z^4 + 2z^3 + 3z^2 + 4z + 5}{z^4}$$

This is a complete visualization of the z-transform of $x_1[n]$

Examples

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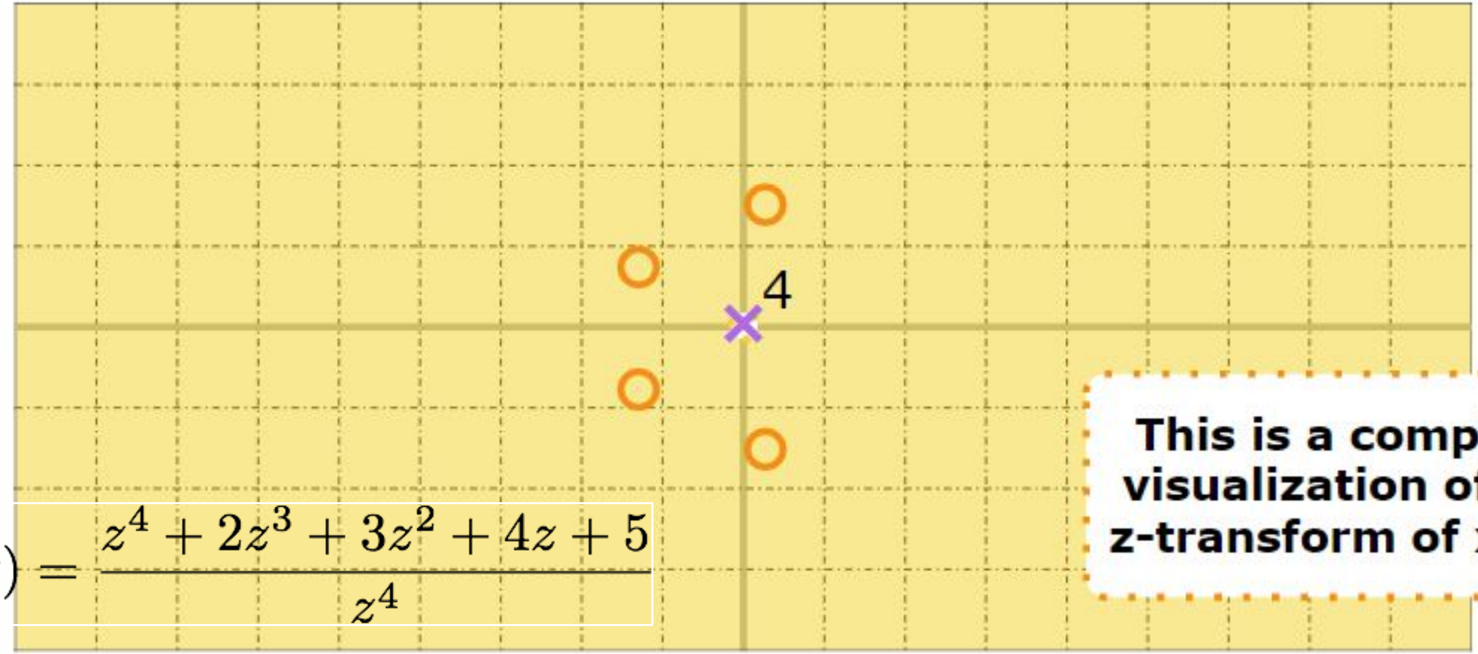
$$X_1(z) = \begin{array}{l} 1 + 2z^{-1} + 3z^{-2} \\ +4z^{-3} + 5z^{-4}, \end{array} \quad \begin{array}{l} \text{for all } z \\ \text{except } z = 0 \end{array}$$

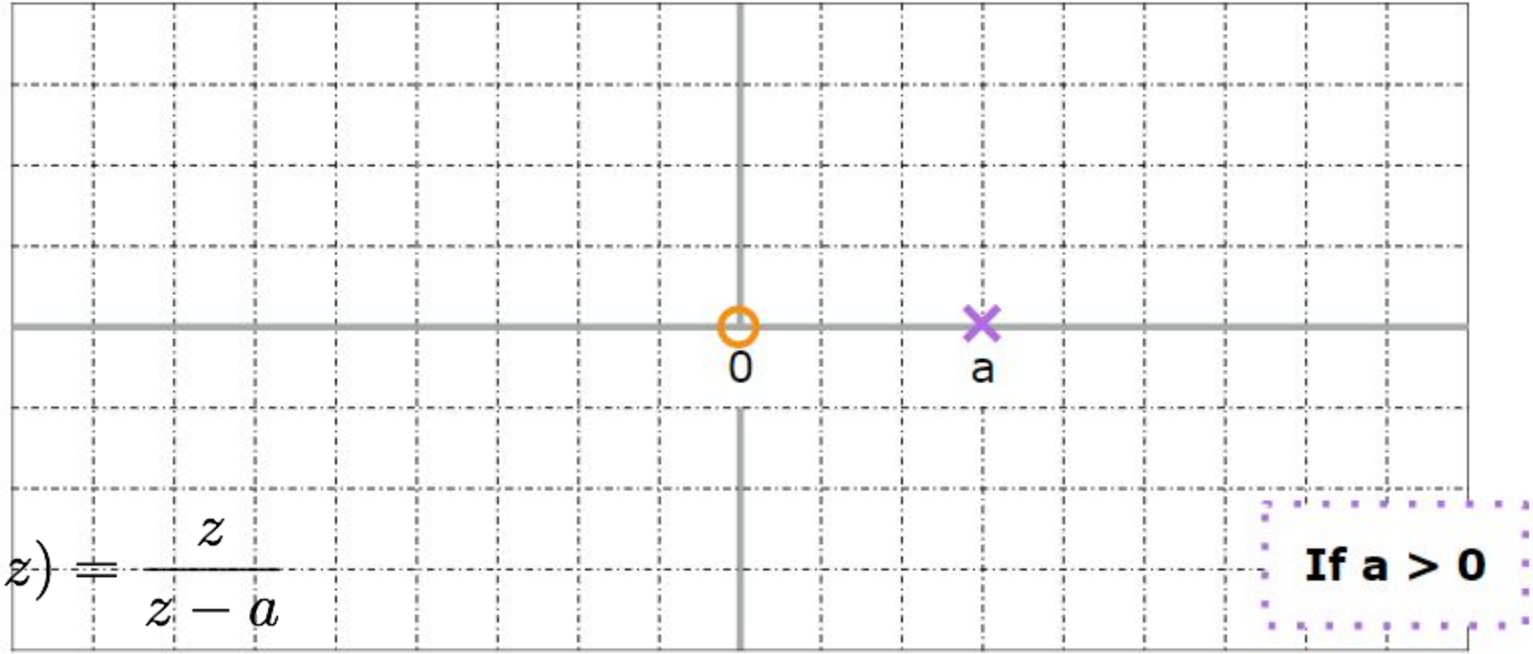


$$X_1(z) = \frac{z^4 + 2z^3 + 3z^2 + 4z + 5}{z^4}$$

This is a complete visualization of the z-transform of $x_1[n]$

Examples

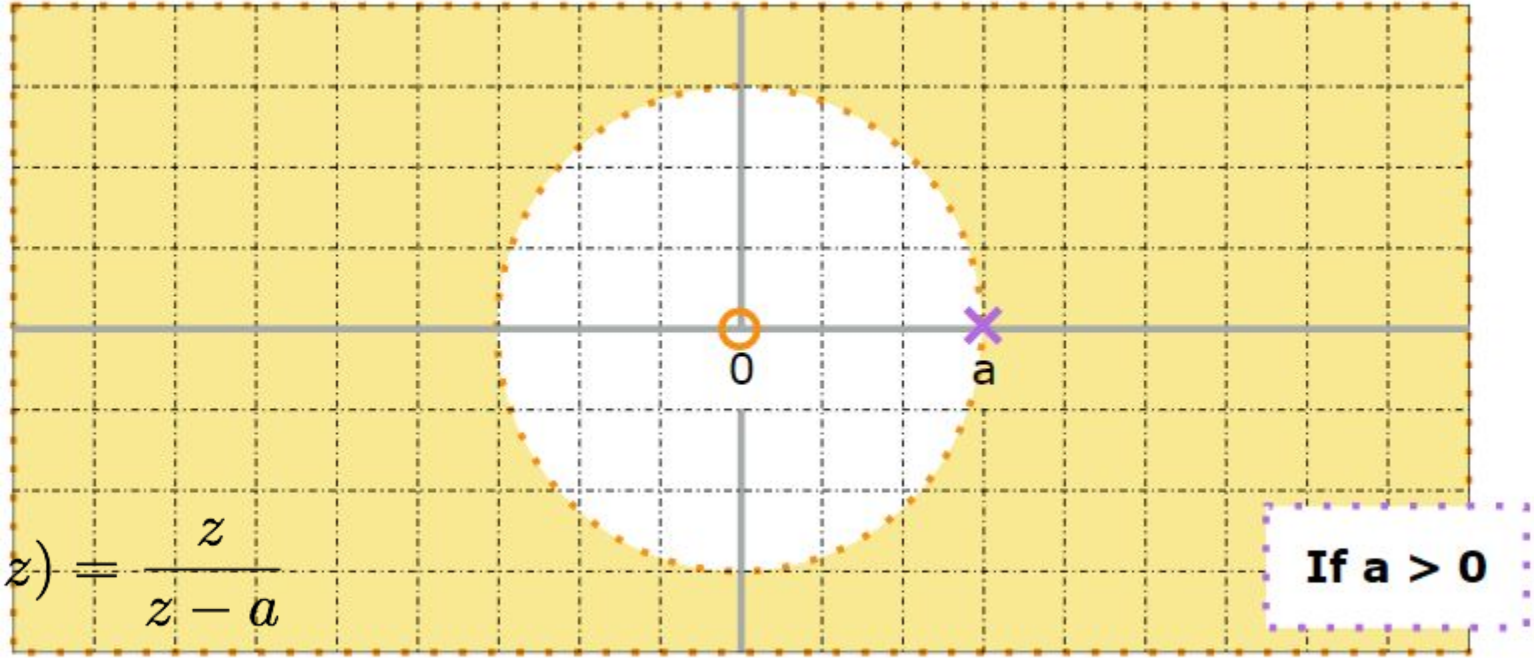
$x_2[n] = a^n u[n],$ $X_2(z) = \frac{1}{1-az^{-1}},$ if $|z| > |a|$



$$X_2(z) = \frac{z}{z-a}$$

Examples

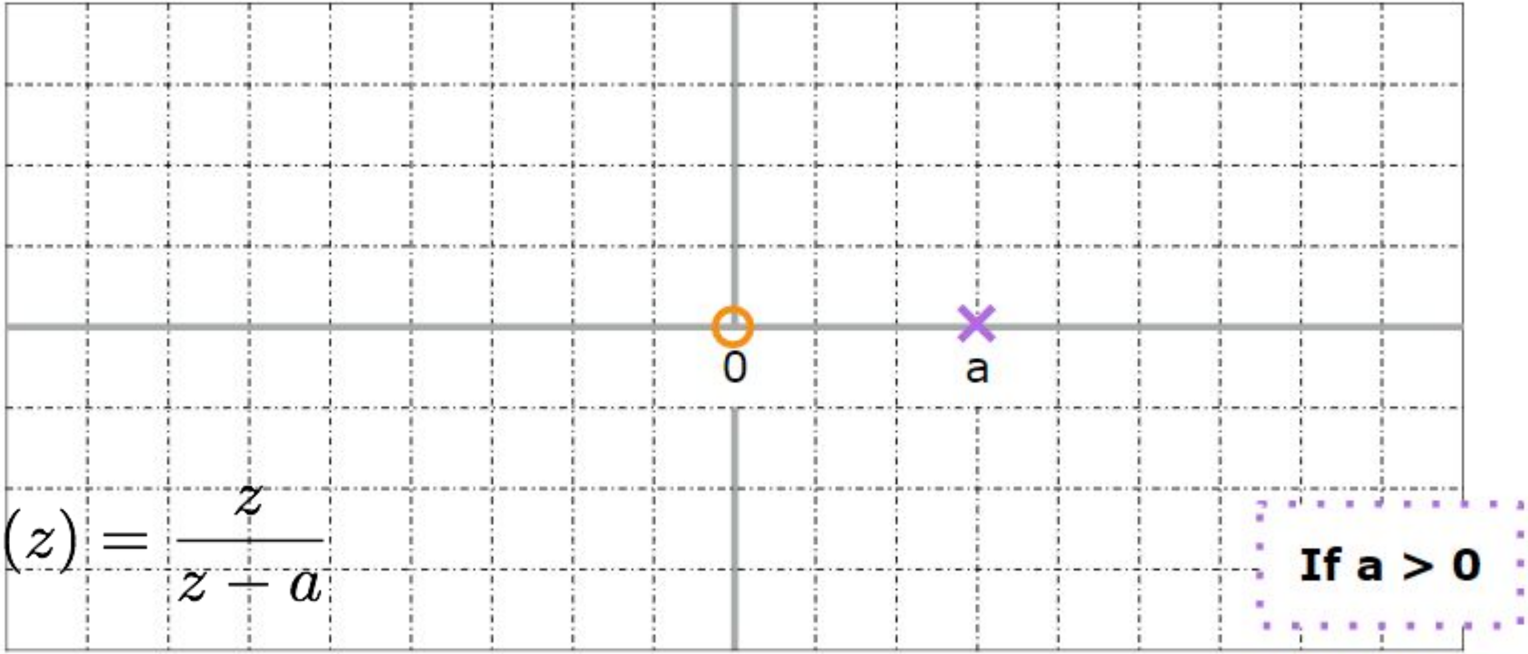
$x_2[n] = a^n u[n],$ $X_2(z) = \frac{1}{1-az^{-1}},$ if $|z| > |a|$



$$X_2(z) = \frac{z}{z-a}$$

Examples

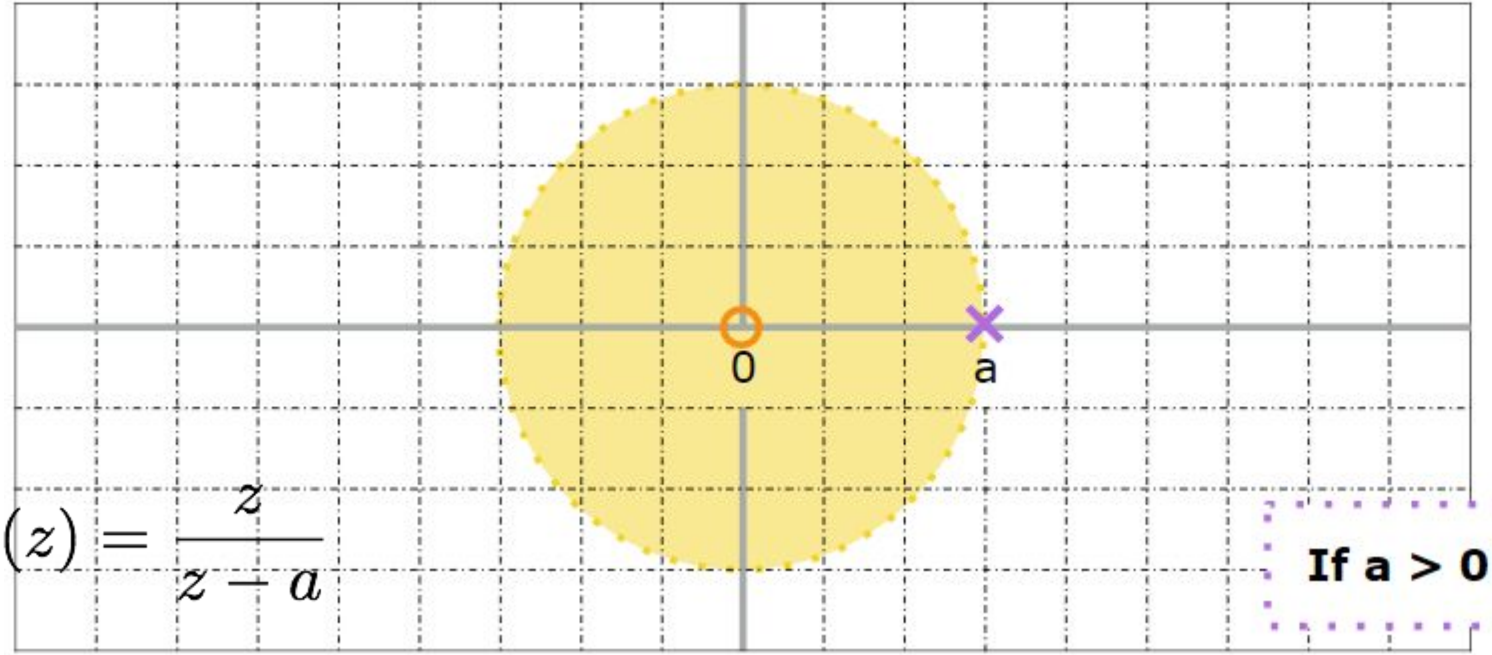
$x_3[n] = -a^n u[-n-1], \quad X_3(z) = \frac{1}{1-az^{-1}}, \quad \text{if } |z| < |a|$



$$X_3(z) = \frac{z}{z-a}$$

Examples

$x_3[n] = -a^n u[-n-1], \quad X_3(z) = \frac{1}{1-az^{-1}}, \quad \text{if } |z| < |a|$

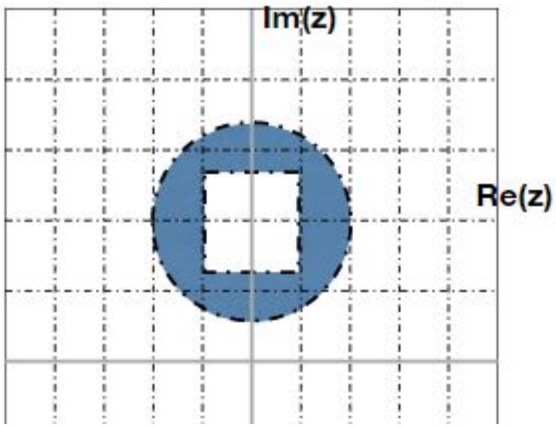


$$X_3(z) = \frac{z}{z-a}$$

If a > 0

Properties of ROC

1. The ROC of X(z) consists of a ring in the z-plane **centered at the origin.**



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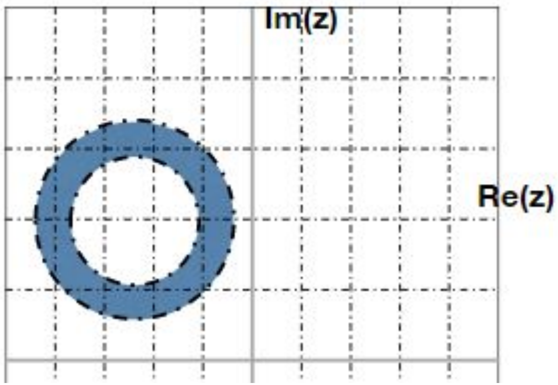

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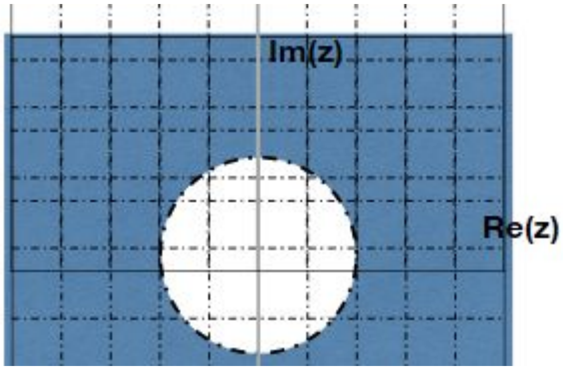
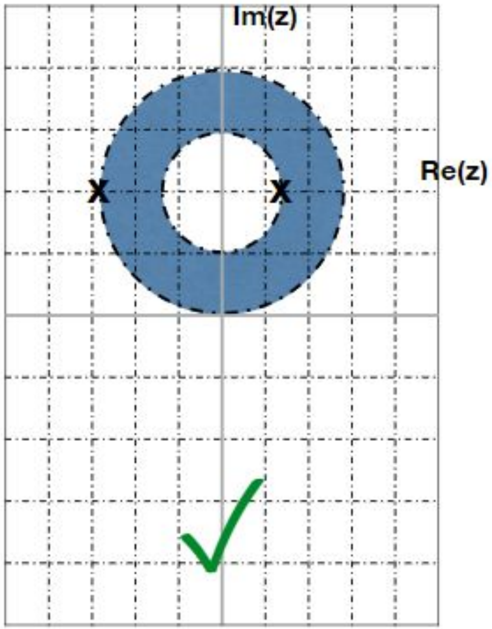
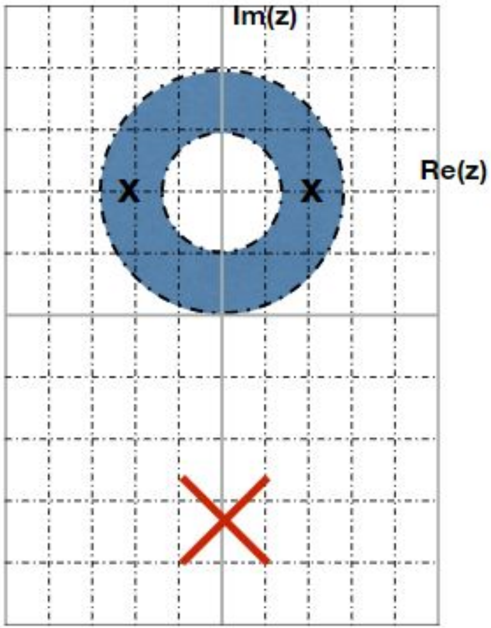


valid

Properties of ROC

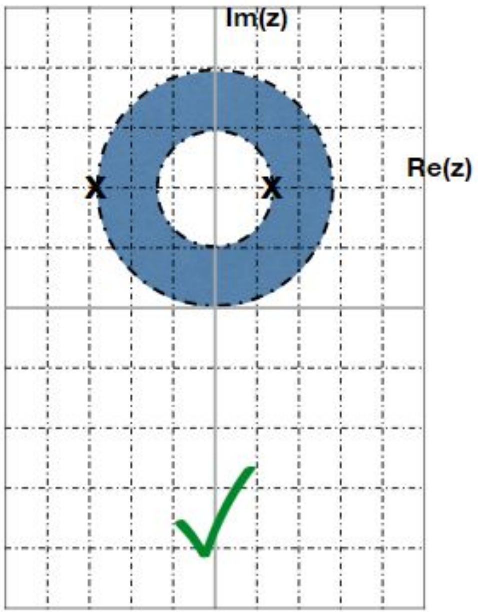
2. The ROC does not contain any poles.

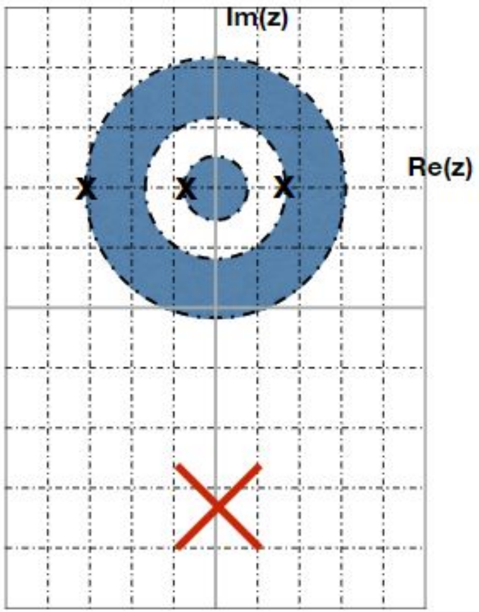




Properties of ROC

3. The ROC is a connected (a single contiguous) region.





Properties of ROC

4. If x[n] is of finite duration, then the ROC is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$. (depends on causality)

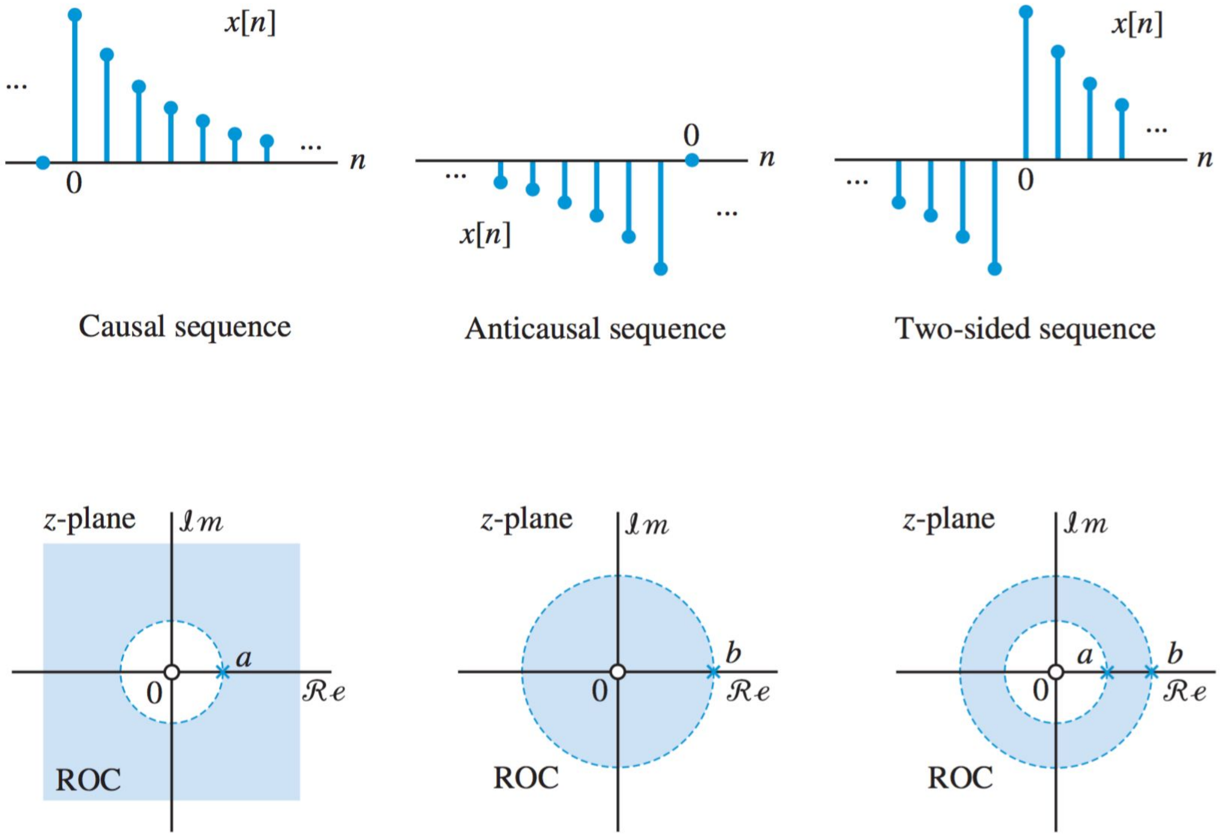
Examples:

1. $X_1 = [1\ 2\ 3\ 4\ 5\ \underline{6}] \rightarrow$ except $z = \infty$
2. $X_2 = [\underline{-1}\ 1\ -1\ 1\ -1\ 1] \rightarrow$ except $z = 0$
3. $X_3 = [0\ 1\ \underline{0}\ 1\ 1\ 0] \rightarrow$ except $z = 0, \infty$

Properties of ROC

5. If x[n] is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC.
6. If x[n] is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| < r_0$ will also be in the ROC.
7. If x[n] is two sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$.

Properties of ROC



DT operations in the z-domain

	Property	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1.	Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(z) + a_2X_2(z)$	At least $R_{x_1} \bigcap R_{x_2}$
2.	Time shifting	$x[n-k]$	$z^{-k}X(z)$	R_x except $z = 0$ or ∞
3.	Scaling	$a^n x[n]$	$X(a^{-1}z)$	$\|a\|R_x$
4.	Differentation	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5.	Conjugation	$x^*[n]$	$X^*(z^*)$	R_x

DT operations in the z-domain

	Property	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
6.	Real-part	$\mathrm{Re}\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	At least R_x
7.	Imaginary part	$\mathrm{Im}\{x[n]\}$	$\frac{1}{2}[X(z)-X^*(z^*)]$	At least R_x
8.	Folding	$x[-n]$	$X(1/z)$	$1/R_x$
9.	Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least $R_{x_1}\bigcap R_{x_2}$
10.	Initial-value theorem	$x[n]=0$ for $n<0$	$x[0]=\lim\limits_{z\to\infty} X(z)$	

Some Notes on Linearity

- The ROC of the linear combination is at least the **intersection of the ROCs** of the individual sequences

$$ROC: \quad R_{x_1} \cap R_{x_2}$$

- If the linear combination yields a finite-duration sequence, the ROC becomes dependent on this resulting sequence.
- If there is no intersection on the ROCs, then there is no z-transform

Example

- Determine the z-transform of the following sequence:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$$

Example

$$x[n] = \boxed{\left(\frac{1}{2}\right)^n u[n]} + \boxed{2^n u[-n-1]}$$

$$\boxed{a^n u[n] \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1-az^{-1}}, \quad |z| > |a|}$$

$$-a^n u[-n-1] \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1-az^{-1}}, \quad |z| < |a|$$

$$\boxed{-a^n u[-n-1] \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1-az^{-1}}, \quad |z| < |a|}$$

Example

$$\left(\frac{1}{2}\right)^n u[n] \quad \overset{z}{\longleftrightarrow} \quad \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2} \qquad 2^n u[-n-1] \quad \overset{z}{\longleftrightarrow} \quad -\frac{1}{1-2z^{-1}}, \quad |z| < 2$$

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$$

$$\boxed{\begin{aligned} X(z) &= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}} \\ ROC &: \left(|z| > \frac{1}{2}\right) \cap (|z| < 2) = \frac{1}{2} < |z| < 2 \end{aligned}}$$

Example

➢ Determine the z-transform of the following sequence:

$$x[n] = 2^n u[n+3]$$

Example

➢ Determine the z-transform of the following sequence:

$$x[n] = 2^n u[n+3]$$

$$a^n u[n] \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \qquad ROC : |z| > |a|$$

$$\begin{aligned} x[n] &= 2^n u[n+3] \\ &= 2^{-3} 2^{n+3} u[n+3] \end{aligned}$$

Example

$$x[n] = 2^n u[n+3]$$
$$= 2^{-3} 2^{n+3} u[n+3]$$

$$2^n u[n] \leftrightarrow \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

$$2^{n+3} u[n+3] \leftrightarrow \frac{z^3}{1-2z^{-1}}, \quad \infty > |z| > 2$$

$$\boxed{2^n u[n+3] \leftrightarrow \frac{2^{-3} z^3}{1-2z^{-1}}, \quad \infty > |z| > 2}$$

Example

- Convolve the two signals using z-transform:

$$x[n] = 3^{n+1}u[n], \quad w[n] = 3^n u[n-1]$$

Example

$$x[n] = 3^{n+1}u[n] = (3)3^n u[n]$$

$$w[n] = 3^n u[n-1] = (3)3^{n-1}u[n-1]$$

$$(3)3^n u[n] \longleftrightarrow \frac{3}{1-3z^{-1}}, ROC : |z| > 3$$

$$(3)3^{n-1}u[n-1] \longleftrightarrow \frac{3z^{-1}}{1-3z^{-1}}, ROC : |z| > 3$$

Example

$$x[n] * w[n] \longleftrightarrow \left(\frac{3}{1-3z^{-1}}\right)\left(\frac{3z^{-1}}{1-3z^{-1}}\right), ROC: |z| > 3$$

$$\longleftrightarrow \frac{9z^{-1}}{(1-3z^{-1})^2}, ROC: |z| > 3$$

5.	$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$

$$\boxed{x[n] * w[n] = (3)n3^n u[n] = n3^{n+1}u[n]}$$

Summary

- Z Transform is a mathematical tool to analyze discrete time signals in a new domain
- Z Transform existence is dictated by the Region of Convergence
- Z Transform properties can be useful in analyzing and solving the responses in a DT system

For further reading...

- Chapter 3.1-3.2, 3.4

  "Applied Digital Signal Processing, by Manolakis, D. & Ingle, V."

- Chapters 10.0-10.2, 10.5

  "Signals and Systems by Oppenheim, A & Willsky, A."

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