4. Discrete Time System Responses

EE 274/COE 197E

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Today's Lesson:

1. What are the different types of DT system responses?
2. Solution to LCCDEs for LTI Systems
3. Natural and Forced Response
4. Zero-Input and Zero-State response
5. The Impulse Response
6. Step response and Sinusoidal Response

DT System Responses

- DT Systems are commonly characterized by the input-output relationship
- LTI System responses can be different depending on the input
- System Responses can be mathematically calculated using direct and indirect methods

Common system responses:

a. Natural Response
b. Forced Response
c. Zero-input
d. Zero-State Response
e. Impulse Response
f. Step Response
g. Sinusoidal Response

DT System Responses

- Input and Output relationship of **linear time-invariant systems** can be described by **Linear Constant Coefficient Difference Equations** (LCCDEs)

- Direct method is similar to finding the solution of differential equations (indirect uses Z-transform similar to Laplace)

Natural and Forced Response

Natural Response

- Output of the system when there is no input (x[n] = 0)
- Also referred to as homogeneous solution of LCCDE
- Commonly form is an exponential DT signal:

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_N\lambda_N^n$$

Forced Response

- Output of the system when there is an input ($x[n] \neq 0$)
- Also referred to as particular solution of LCCDE
- There are many forms of the solution depending on the input (can be found in table of characteristic equations)

Natural and Forced Response

Table of characteristic equations:

Input Signal, x[n]	Particular Solution y [n]
A (constant)	K
AM^n	KM^n
An^M	$K_0n^M + K_1n^{M-1} + \ldots + K_M$
A^nn^M	$A^n(K_0n^M + K_1n^{M-1} + \ldots + K_M)$
$A\cos(\omega_0 n)$	$K_1\cos(\omega_0 n) + K_2\sin(\omega_0 n)$
$A\sin(\omega_0 n)$	

Natural and Forced Response

General steps:

1. Write the LCCDE representing the system
2. Set the input x[n] to zero
3. Solve the roots λ of the homogeneous solution
4. Without solving the coefficients C, solve the forced response (see table of characteristic equation)
5. Add the homogeneous and particular solutions to get the form of the total solution
6. Solve the coefficients using initial conditions and system of linear equations

Example

- Consider the system represented by the following difference equation:

$$y[n] + 1.4y[n-1] + 0.45y[n-2] = x[n] - x[n-1]$$

- Determine the response of the system at $n \geq 0$ when
    - $x[n] = 0.1^n u[n]$, $y[-1] = 0$ and $y[-2] = 1$ (initial conditions)

Example

- Solve for λ

$$y[n] + 1.4y[n-1] + 0.45y[n-2] = 0$$

$$\lambda^n + 1.4\lambda^{n-1} + 0.45\lambda^{n-2} = 0$$

$$\lambda^2 + 1.4\lambda + 0.45 = 0$$

$$(\lambda + 0.9)(\lambda + 0.5) = 0$$

$$y_h[n] = C_1(-0.9)^n + C_2(-0.5)^n, \;\; n \geq 0$$

Example

- Solve for particular solution (a= 0.1)

$$y[n] + 1.4y[n-1] + 0.45y[n-2] = x[n] - x[n-1]$$

$$Ka^n u[n] + 1.4Ka^{n-1}u[n-1] + 0.45Ka^{n-2}u[n-2] = a^n u[n] - a^{n-1}u[n-1]$$

- Solve at n = 2 (all terms turned ON)

$$Ka^2 + 1.4Ka + 0.45K = a^2 - a$$

$$K = \frac{a^2 - a}{a^2 + 1.4a + 0.45}$$

K = -3/20 = -0.15

Example

- Solve for total response

$$y_{tot}[n] = y_h[n] + y_p[n]$$
$$= C_1(-0.9)^n + C_2(-0.5)^n + Ka^n,\ n \geq 0$$

$$y_{tot}[n] = C_1(-0.9)^n + C_2(-0.5)^n + Ka^n,\ n \geq 0$$
$$y_{tot}[0] = C_1 + C_2 + K$$
$$y[n] = x[n] - x[n-1] - 1.4y[n-1] - 0.45y[n-2]$$
$$y[0] = x[0] - x[-1] - 1.4y[-1] - 0.45y[-2]$$
$$y[0] = 1 - 0 - 0 - 0.45 = 0.55$$

Example

- Solve for total response

$$y_{tot}[n] = C_1(-0.9)^n + C_2(-0.5)^n + Ka^n, \;\; n \geq 0$$

$$y_{tot}[1] = -0.9C_1 - 0.5C_2 + aK$$

$$y[n] = x[n] - x[n-1] - 1.4y[n-1] - 0.45y[n-2]$$

$$y[1] = x[1] - x[0] - 1.4y[0] - 0.45y[-1]$$

$$y[1] = a - 1 - 1.4(0.55) - 0 = a - 1.77$$

Example

- Solve for total response

$$n = 0 : C_1 + C_2 = -K + 0.55$$

$$n = 1 : -0.9C_1 - 0.5C_2 = a(1 - K) - 1.77$$

$$C_1 + C_2 = 0.70$$

$$-0.9C_1 - 0.5C_2 = -1.655$$

C_1 = 261/80 = 3.2625, C_2 = -41/16 = -2.5625

Zero-Input Response

- Output of the system when there is no input (x[n] = 0)
- "Subset" of the Natural Response since the coefficient are solved prior to total response
- Same form as Natural Response:

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_N\lambda_N^n$$

Zero-Input and Zero-State Response

Zero-state Response

- Output of the system when there is an input ($x[n] \neq 0$)
- The response when initial conditions are zero (relaxed state)
- Forced response is a subset of Zero-state response
- Still use the table of characteristic equations

Zero-Input and Zero-State Response

General steps:

1. Write the LCCDE representing the system
2. Set the input x[n] to zero
3. Solve the roots λ of the homogeneous solution
4. Solve the coefficients C using initial conditions and systems of linear equation
5. Get the form of the zero-state response
6. Solve the coefficients of zero-state response using initial conditions set to 0 and system of linear equations

*Alternatively, you can subtract the ZI from the total response

Example

- Consider the system represented by the following difference equation:

$$y[n] + 1.4y[n-1] + 0.45y[n-2] = x[n] - x[n-1]$$

- Determine the response of the system at n≥0 when
  - $x[n] = 0.1^n u[n]$, $y[-1] = 0$ and $y[-2] = 1$ (initial conditions)

Example

- Solve for λ

$$y[n] + 1.4y[n-1] + 0.45y[n-2] = 0$$

$$\lambda^n + 1.4\lambda^{n-1} + 0.45\lambda^{n-2} = 0$$

$$\lambda^2 + 1.4\lambda + 0.45 = 0$$

$$(\lambda + 0.9)(\lambda + 0.5) = 0$$

$$y_h[n] = C_1(-0.9)^n + C_2(-0.5)^n, \;\; n \geq 0$$

Example

$$y_{ZI}[n] + 1.4y_{ZI}[n-1] + 0.45y_{ZI}[n-2] = 0$$

$$y_{ZI}[n] = -1.4y_{ZI}[n-1] - 0.45y_{ZI}[n-2]$$

$$y_{ZI}[0] = -1.4y_{ZI}[-1] - 0.45y_{ZI}[-2] = -0.45$$

$$y_{ZI}[1] = -1.4y_{ZI}[0] - 0.45y_{ZI}[-1] = 0.63$$

$$y_{ZI}[0] = C_1(-0.9)^0 + C_2(-0.5)^0 = C_1 + C_2 = -0.45$$

$$y_{ZI}[1] = C_1(-0.9)^1 + C_2(-0.5)^1 = -0.9C_1 - 0.5C_2 = 0.63$$

$$C_1 = -1.0125 + C_2 = 0.5625$$

Example

- Solve for Zero-State

$$y_{ZS}[n] + 1.4y_{ZS}[n-1] + 0.45y_{ZS}[n-2] = x[n] - x[n-1]$$

$$y_{ZS}[n] = (0.1)^n u(n) - (0.1)^{n-1}u(n-1) - 1.4y_{ZS}[n-1] - 0.45y_{ZS}[n-2]$$

$$y_{ZS}[0] = (0.1)^0 u(0) - \cancel{(0.1)^{-1}u(-1)} - \cancel{1.4y_{ZS}[-1] - 0.45y_{ZS}[-2]} = 1$$

$$y_{ZS}[1] = (0.1)^1 u(1) - (0.1)^0 u(0) - 1.4y_{ZS}[0] \cancel{- 0.45y_{ZS}[-1]} = -2.3$$

$$y_{ZS}[2] = (0.1)^2 u(2) - (0.1)^1 u(1) - 1.4y_{ZS}[1] - 0.45y_{ZS}[0] = 2.68$$

Example

➢ Solve for Zero-State

$$y_{ZS}[n] = C_1(-0.9)^n + C_2(-0.5)^n + K(0.1)^n u(n)$$

$$y_{ZS}[0] = C_1(-0.9)^0 + C_2(-0.5)^0 + K(0.1)^0 u(0) = 1$$

$$y_{ZS}[1] = C_1(-0.9)^1 + C_2(-0.5)^1 + K(0.1)^1 u(1) = -2.3$$

$$y_{ZS}[2] = C_1(-0.9)^2 + C_2(-0.5)^2 + K(0.1)^2 u(2) = 2.68$$

$$C_1 = 4.275;\ C_2 = -3.125\ ;\ K = -0.15$$

→ Adding C1 and C2 for ZS and ZI will result to the total response coefficients solved previously

Notes on ZI and ZS Responses

A DT system is linear if:

- ➢ The total response is also the sum of Zero Input and Zero State responses
- ➢ The principle of superposition applies individually to the Zero Input and the Zero State Responses

→ Otherwise, the DT system is **non-linear**

Transient and Steady-State Response

Transient response: response of the DT system as n approaches 0; "switching states" (e.g. decaying exponential e.g. $y[n] = (2)0.5^{n}$)

Steady-state: response of the DT system as n approaches infinity; (e.g. constant value $y[n] = 15u[n]$)

Transient and Steady-State Response

$$y[n] = \underbrace{C_1(-0.9)^n + C_2(-0.5)^n + K(0.1)^n u(n)}$$

Since all terms are exponentially decreasing, the total response is also the transient response; The steady-state response is zero

Impulse Response and Step Response

Impulse response: response of the DT system when the input x[n] is an impulse signal

→ denoted by h[n]; once this is obtained, we can just convolve any x[n] to obtain any y[n]

Step response: response of the DT system when the input x[n] is a unit step signal

→ used to check BIBO stability and for binary logic DT systems

The Impulse Response

General steps:

1. Write the LCCDE representing the system
2. Set the input x[n] to an **impulse** function
3. Get the form of the zero-state response
4. Solve the coefficients of zero-state response using initial conditions set to 0 and system of linear equations

*Solving the impulse response assumes a relaxed system (IC are set to 0, form is similar to the Zero State Response)

The Impulse Response

$$y[n] + 1.4y[n-1] + 0.45y[n-2] = x[n] - x[n-1]$$

$$h[n] + 1.4h[n-1] + 0.45h[n-2] = \delta[n] - \delta[n-1]$$

$$h[n] = -1.4h[n-1] - 0.45h[n-2] + \delta[n] - \delta[n-1]$$

$$h[0] = \cancel{-1.4h[-1] - 0.45h[-2]} + \delta[0] \cancel{- \delta[-1]} = 1$$

$$h[1] = -1.4h[0] - \cancel{0.45h[-1]} + \cancel{\delta[1]} - \delta[0] = -1.4 - 1 = -2.4$$

$$h[2] = -1.4h[1] - 0.45h[0] + \cancel{\delta[2] - \delta[1]} = -1.4(-2.4) - 0.45(1) = 2.91$$

The Impulse Response

$$y[n] + 1.4y[n-1] + 0.45y[n-2] = x[n] - x[n-1]$$

$$h[n] = C_1(-0.9)^n + C_2(-0.5)^n + K(0.1)^n u(n)$$

$$h[0] = C_1(-0.9)^0 + C_2(-0.5)^0 + K(0.1)^0 u(0) = 1$$

$$h[1] = C_1(-0.9)^1 + C_2(-0.5)^1 + K(0.1)^1 u(1) = -2.4$$

$$h[2] = C_1(-0.9)^2 + C_2(-0.5)^2 + K(0.1)^2 u(2) = 2.91$$

$$C_1 = 4.75; C_2 = -3.75; K = 0$$

The Impulse Response

Impulse Response of the system:

$$h[n] = 4.75(-0.9)^n - 3.75(-0.5)^n$$

- To get the **step response**, just convolve h[n] with a step input x[n] = u[n]
- To get the **sinusoidal steady-state response**, just convolve h[n] with the sinusoidal input x[n] = cos(ωn)

Notes on the Impulse Response

- If h[n] is a finite sequence (has a finite memory length), then we say h[n] is a **Finite Impulse Response (FIR)**

  → DT convolution is practical to use

- If h[n] is an infinite sequence (has an infinite memory length), then it is an **Infinite Impulse Response (IIR)**

  → DT convolution is impractical to use

  → Output is computed recursively

Notes on the Impulse Response

$$y(n) = \frac{1}{2}\left[y(n-1) + \frac{x(n)}{y(n-1)} \right]$$

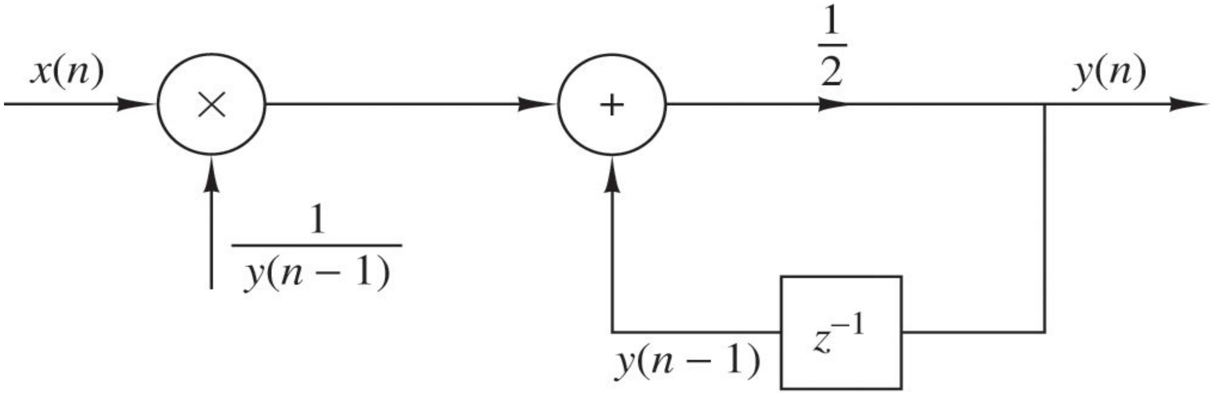


Figure 2.4.2 Realization of the square-root system.

Notes on the Impulse Response

$$y(n) = \frac{1}{2}\left[y(n-1) + \frac{x(n)}{y(n-1)} \right] \qquad s_n = \frac{1}{2}\left(s_{n-1} + \frac{A}{s_{n-1}} \right), \qquad n = 0,..$$

Let A=2 → **sqrt(2) = 1.4142136**

IC: y(-1)= 1
→ y(0)=1.5 → y(1)=1.4166667 → y(2)=1.4142157

IC: y(-1)= 1.5
→ y(0)=1.416667 → y(1)=1.4142157

Notes on the Impulse Response

- If h[n] is absolutely summable (e.g. finite or exponentially decreasing, energy signals), then the DT system is stable
- If h[n] has 0 values for n<0, then the DT system is causal
- In multi-stage systems, the impulse responses follow commutative, associative and distributive property

Notes on the Impulse Response

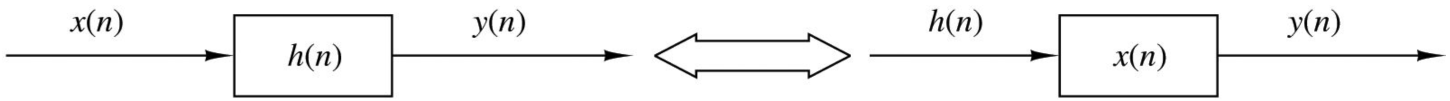


Figure 2.3.4 Interpretation of the commutative property of convolution.

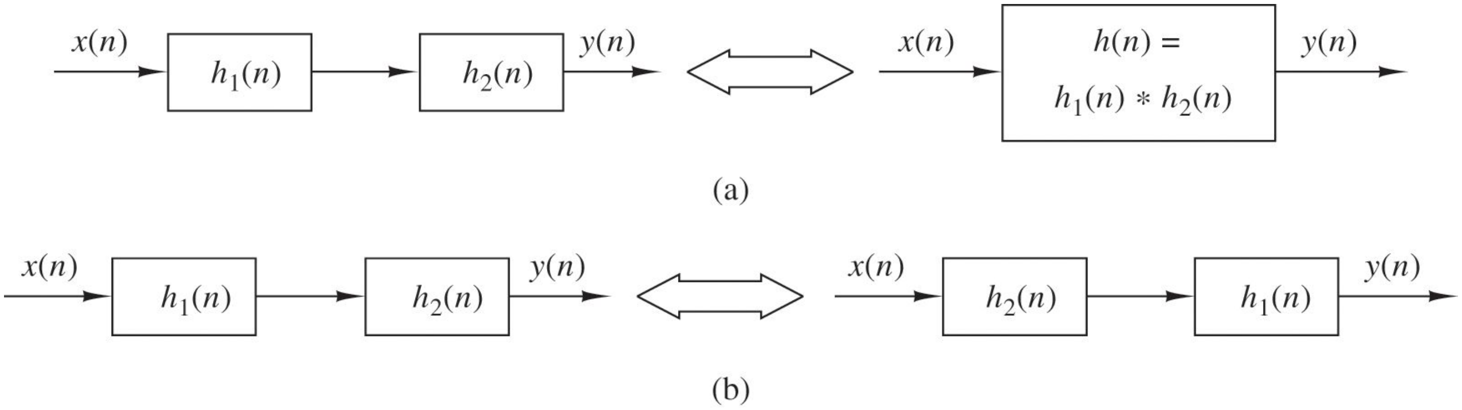


Figure 2.3.5 Implications of the associative (a) and the associative and commutative (b) properties of convolution.

Notes on the Impulse Response

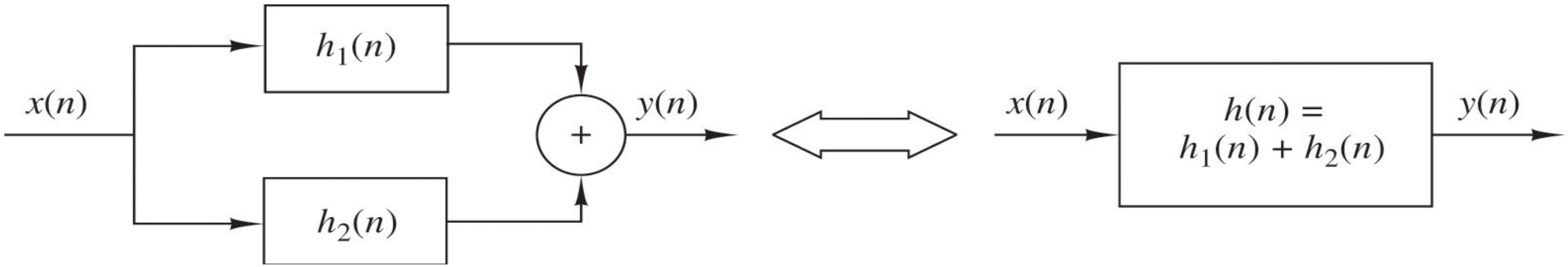
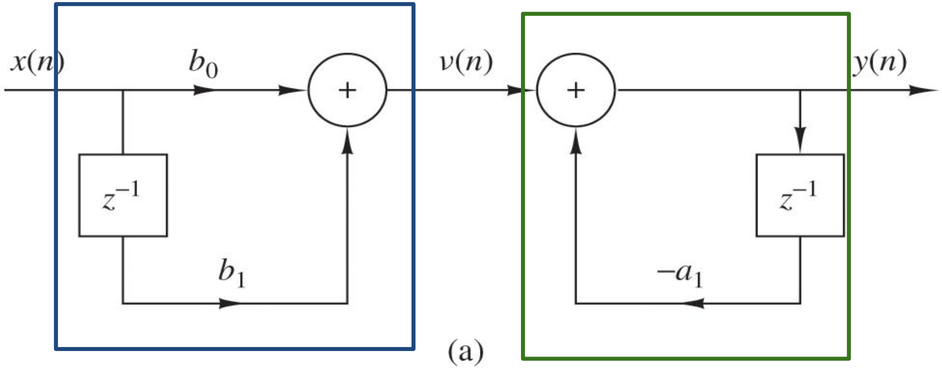


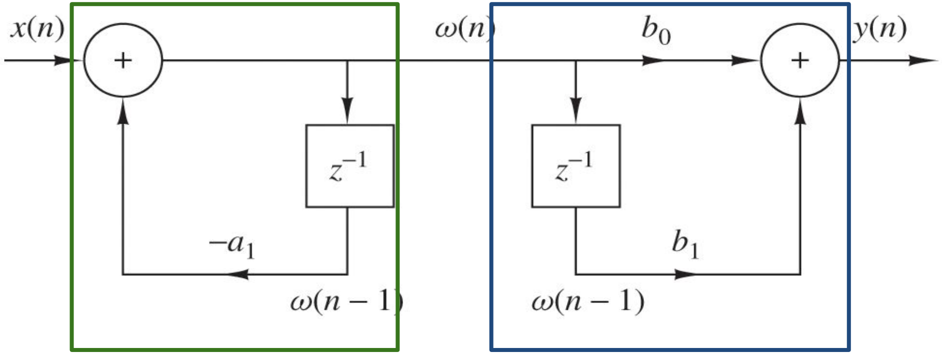
Figure 2.3.6 Interpretation of the distributive property of convolution: two LTI systems connected in parallel can be replaced by a single system with $h(n) = h_1(n) + h_2(n)$.

Notes on LTI System Responses

- Implications of LTI system property:
- Systems blocks can be further optimized (from Direct Form I to Direct Form II)
- Direct Form II represents the same DT system with less blocks



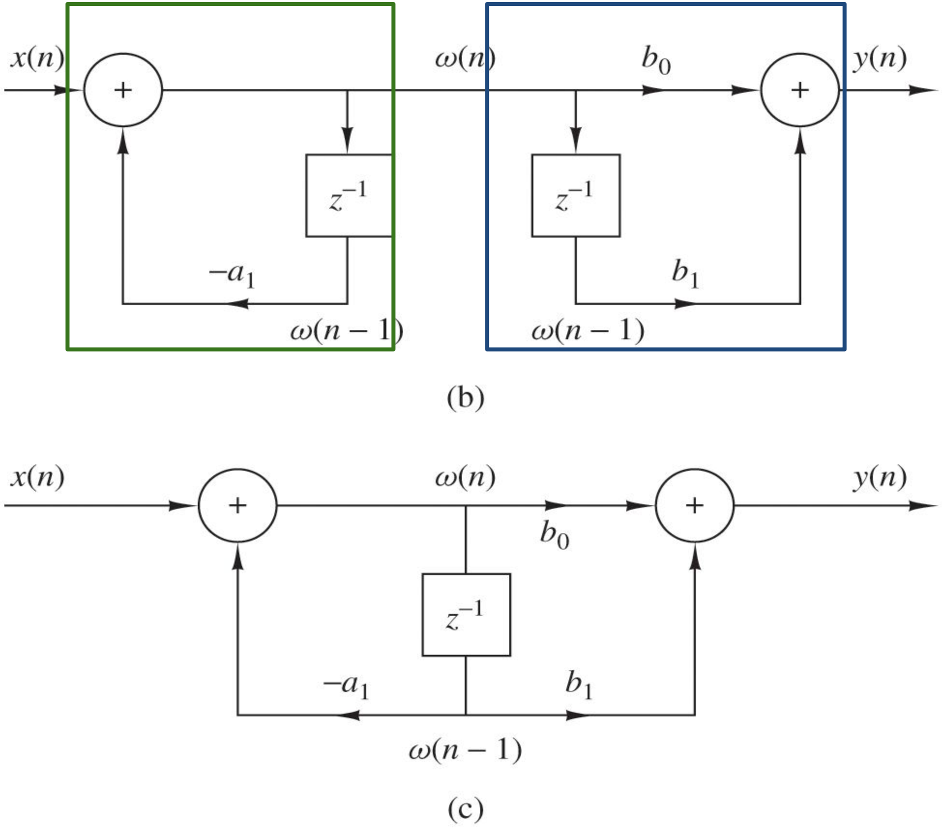
(a)



(b)

Notes on LTI System Responses

- ➢ Implications of LTI system property:
- ➢ Systems blocks can be further optimized (from Direct Form I to Direct Form II)
- ➢ Direct Form II represents the same DT system with less blocks



Summary

- DT systems can be solved directly by solving LCCDEs
- Natural Response and Zero-Input Response denote the system response when there is no input
- The input x[n] affects the Forced Response and the Zero-State response.
- The impulse response represents the system function and can be obtained by solving the LCCDE with x[n] set as an impulse function
- Once obtained, the impulse response can generate many types of responses by means of DT convolution
- LTI system properties can be used to optimize DT systems

For further reading...

- Chapter 2.4

  “Digital Signal Processing, 3rd ed. By Proakis, J. & Manolakis, D.”

- Chapters 2.1-2.3

  “Signals and Systems by Oppenheim, A & Willsky, A.”

- Chapter 10.4

  “Signals, Systems & Transforms by Phillips, C & Parr, J.”

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