7. Stability and Frequency Response

EE 274/COE 197E

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Today's Lesson:

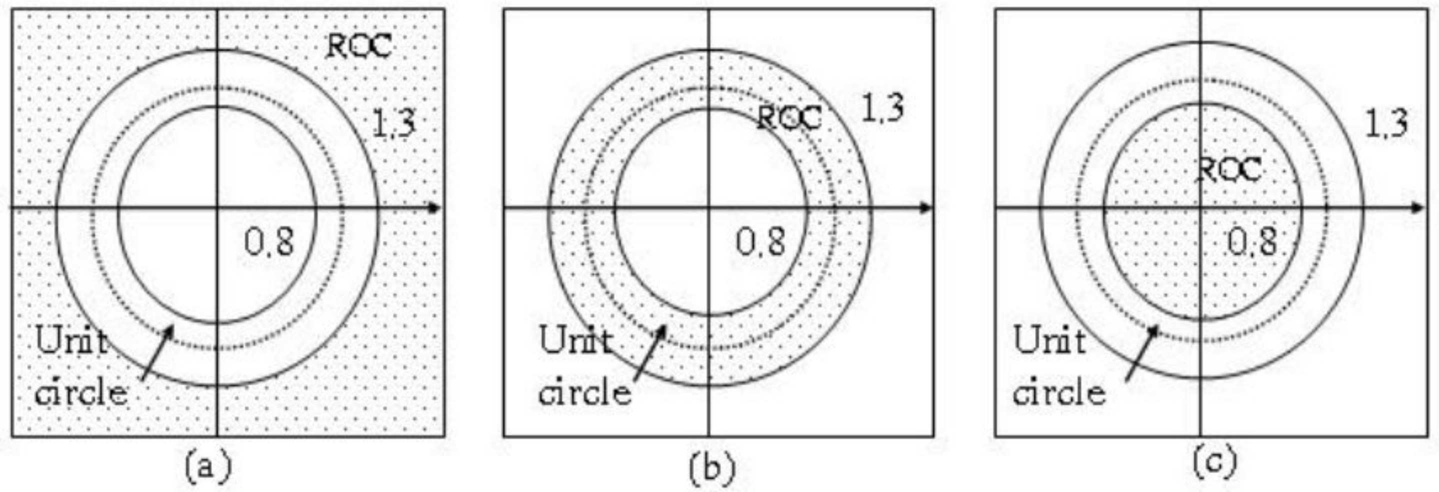
1. ROC and BIBO stability
2. DT Signals along the unit circle
3. Frequency Response of DT signals
4. Magnitude Plot
5. Phase Plot

ROC and Stability

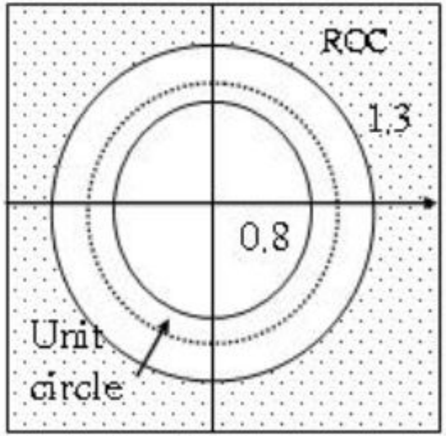
- A DT signal is stable if the ROC includes the unit circle
  - For causal signals, rightmost pole should be within the unit circle
  - For anti-causal signals, leftmost pole should be outside the unit circle

Example

$$X(z) = \frac{1}{1 - 0.8z^{-1}} + \frac{1}{1 - 1.3z^{-1}} = \frac{2 - 2.1z^{-1}}{1 - 2.1z^{-1} + 1.04z^{-2}}$$

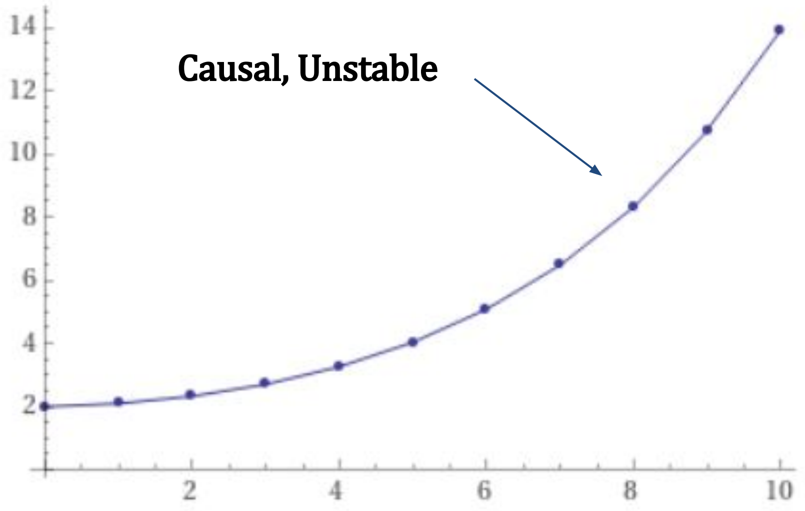


Example



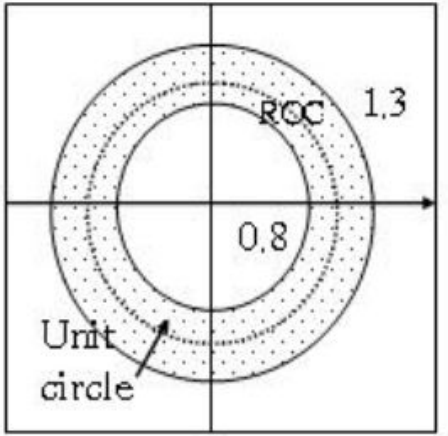
(a)

$$x(n) = 0.8^n u(n) + 1.3^n u(n)$$



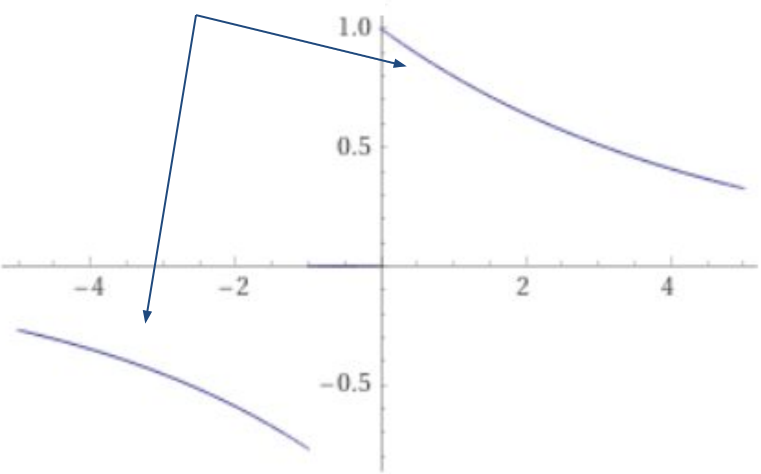
Example

$$x(n) = 0.8^n u(n) - 1.3^n u(-n-1)$$



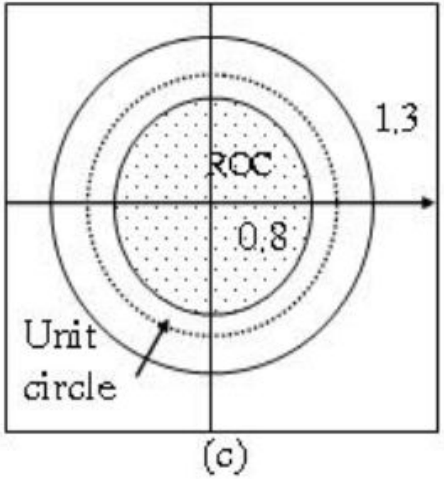
(b)

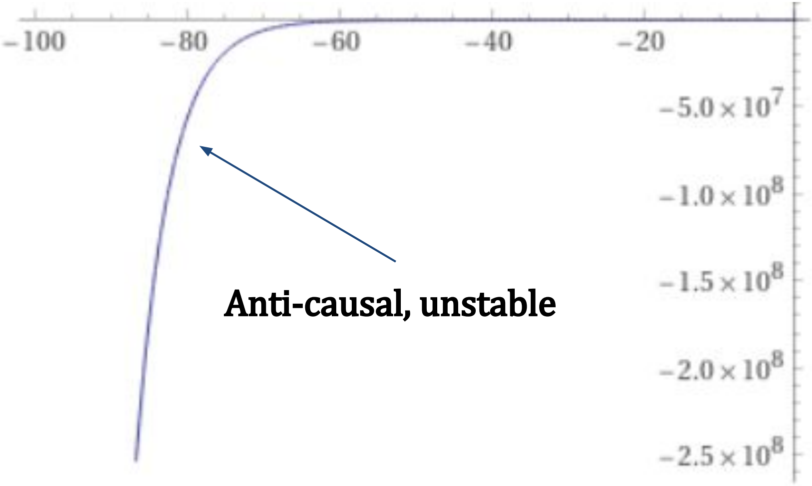
Two-sided, stable



Example

$$x(n) = -0.8^n u(-n-1) - 1.3^n u(-n-1)$$





The unit circle

- What happens when the pole is at the unit circle?
  - Recall: ROC should not contain poles

1. $A(1)^n u(n)$ or $Au(n) \rightarrow$ BIBO stable
2. $\cos(0.5\pi n) \rightarrow$ BIBO stable
3. $e^{j0.2n} \rightarrow$ BIBO stable

- DT Systems with poles at the unit circle are **marginally stable** (BIBO stable but unstable in a strict sense since the signal does not decay to zero)

Frequency Response

- The Frequency Response describes the **magnitude** and **phase** of a DT signal/system with respect to the frequency (either Hz or rad/s)

- Can be obtained from the z-transform of the **impulse response** (transfer function) by setting $z = e^{j\omega n}$

- Similar to a time-domain frequency sweep / AC analysis
  - testing the outputs of a DT system using sinusoidal inputs of varying frequencies

Frequency Response

- FIR in the frequency domain

$$H(\omega) = \sum_{k=N_1}^{N_2} h(k) e^{-j\omega k}$$

- IIR in the frequency domain

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{k=0}^{M} p_k e^{-j\omega k}}{\sum_{k=0}^{N} d_k e^{-j\omega k}}$$

Magnitude Response

- Typically represented in **dB** (y-axis) vs. **ω** (x-axis)

- $|H(\omega)|$ or $|H(e^{j\omega})|$

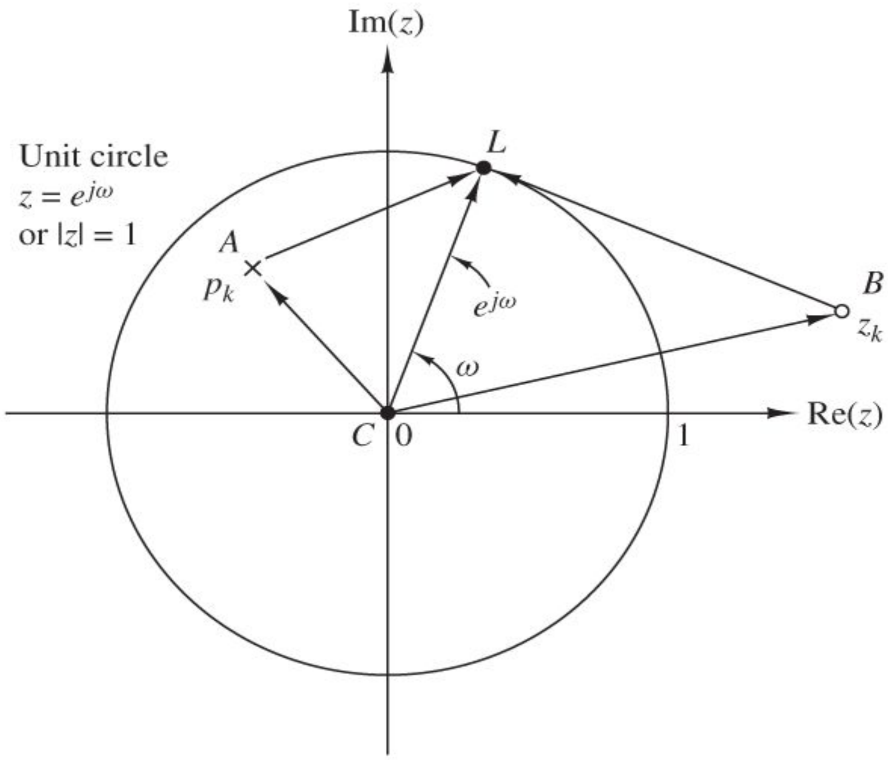
- To get the magnitude plot, we need to get the relative distances of each of the zeros and poles from the unit circle at varying ω

$$H(\omega) = b_0 \frac{\prod_{k=1}^{M}(1 - z_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - p_k e^{-j\omega})}$$

$$= b_0 e^{-j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$

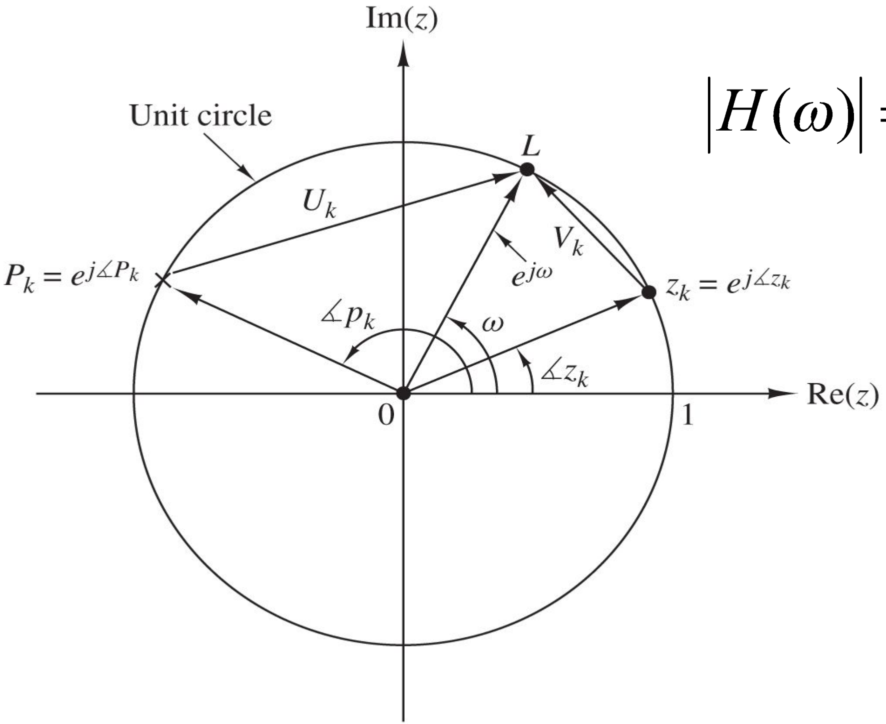
$$= b_0 e^{-j\omega(N-M)} \frac{\prod_{k=1}^{M} V_k(\omega) e^{j\Theta_k(\omega)}}{\prod_{k=1}^{N} U_k(\omega) e^{j\Phi_k(\omega)}}$$

Magnitude Response



(a)

Magnitude Response



$$|H(\omega)| = |b_0| \frac{V_1(\omega) \cdots V_M(\omega)}{U_1(\omega) \cdots U_N(\omega)}$$

Figure 5.2.2 A zero on the unit circle causes $|H(\omega)| = 0$ and $\omega = \measuredangle z_k$. In contrast, a pole on the unit circle results in $|H(\omega)| = \infty$ at $\omega = \measuredangle p_k$.

Example

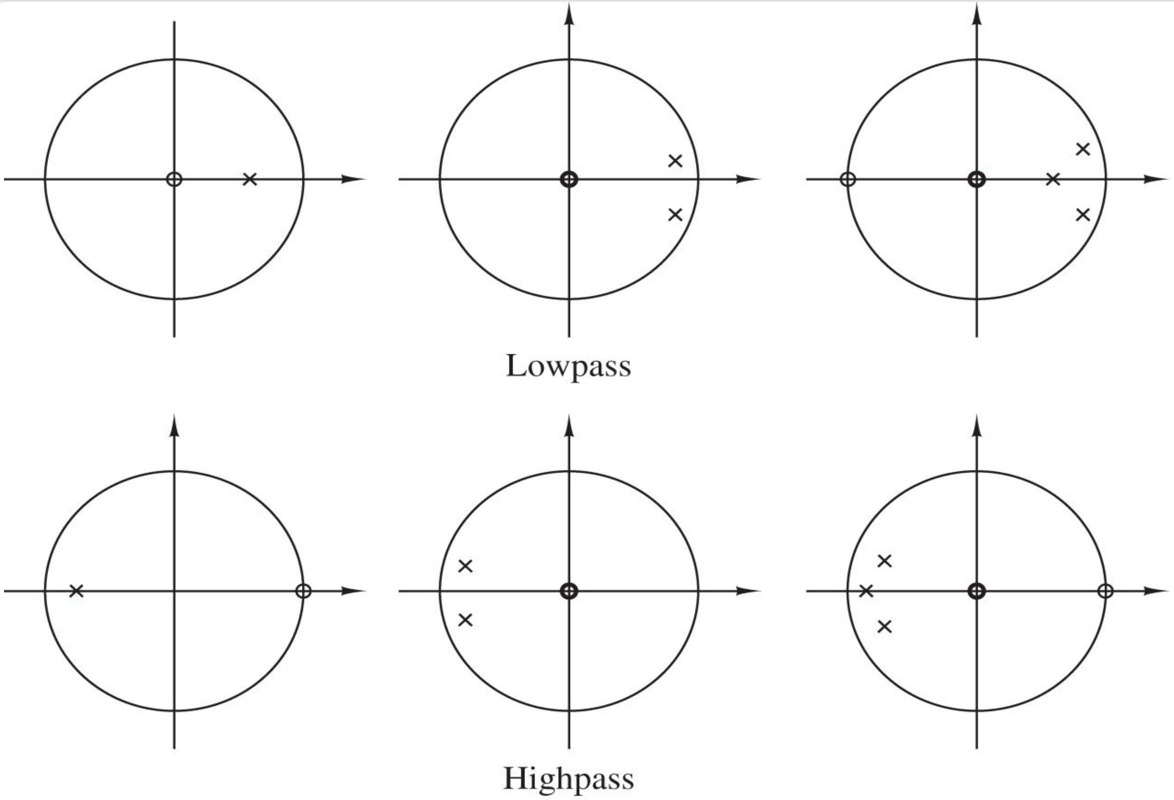
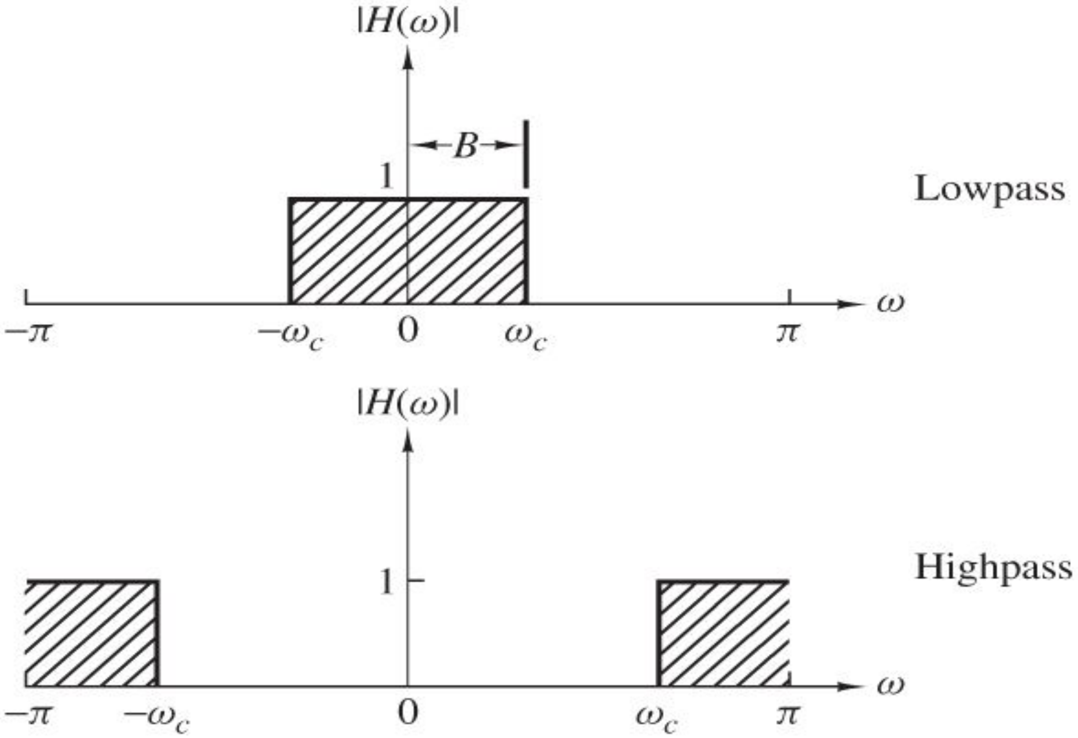
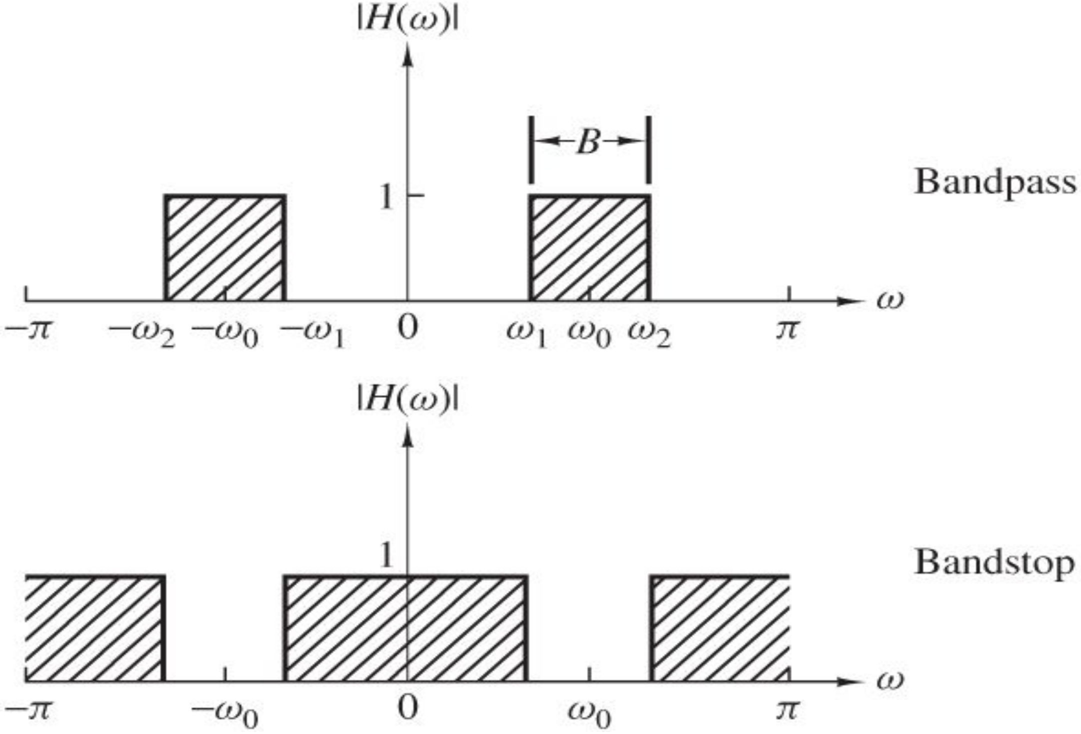


Figure 5.4.2 Pole–zero patterns for several lowpass and highpass filters.

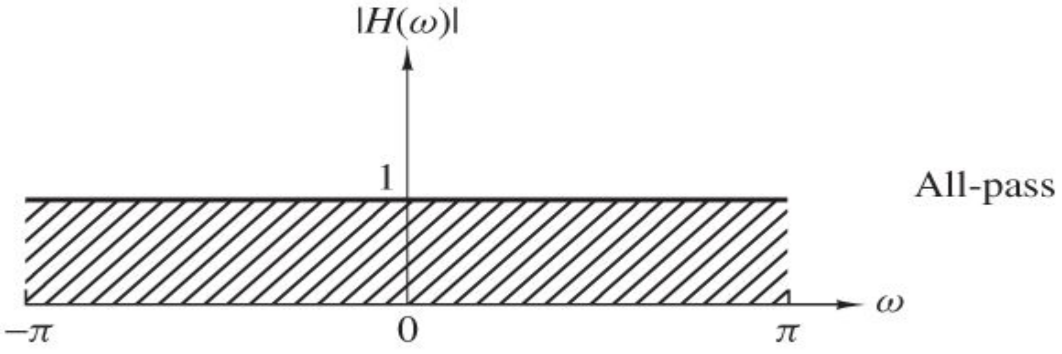
Magnitude Response



Magnitude Response



Magnitude Response



Example

- Find $H(e^{j\omega})$ of the given MA system:

$$y(n) = \frac{1}{3}\left[x(n+1) + x(n) + x(n-1)\right]$$

Example

- Get H(z)

$$y(n) = \frac{1}{3}\left[x(n+1) + x(n) + x(n-1)\right]$$

$Y(z) = [zX(z) + X(z) + z^{-1}X(z)]/3$

$Y(z) = [z + 1 + z^{-1}]X(z)/3$

$H(z) = [z + 1 + z^{-1}]/3$

$H(e^{j\omega}) = [e^{j\omega} + 1 + e^{-j\omega}]/3$

$$\boxed{H(e^{j\omega}) = [1 + 2\cos(\omega)]/3}$$

Example

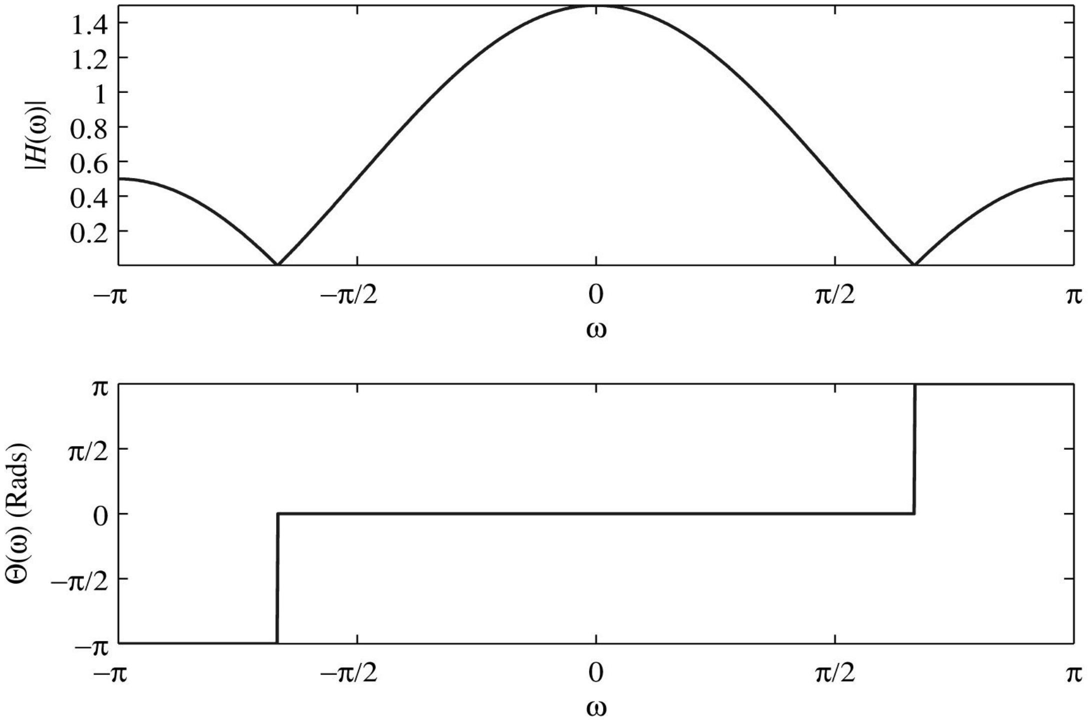


Figure 5.1.1 Magnitude and phase responses for the MA system in Example 5.1.2.

$$|H(\omega)| = \frac{1}{3}|1 + 2\cos\omega|$$

$$\Theta(\omega) = \begin{cases} 0, & 0 \le \omega \le 2\pi/3 \\ \pi, & 2\pi/3 \le \omega < \pi \end{cases}$$

Example

- Find $H(e^{j\omega})$ of the given AR system:

$$y(n) = ay(n-1) + bx(n), \qquad 0 < a < 1$$

Example

- Get H(z)

$$y(n) = ay(n-1) + bx(n), \qquad 0 < a < 1$$

$$Y(z) = [az^{-1}Y(z) + bX(z)]$$

$$[1+az^{-1}]Y(z) = bX(z)$$

$$H(z) = b/[1+az^{-1}]$$

$$\boxed{H(e^{j\omega}) = b/[1+ae^{-j\omega}]}$$

-

Example

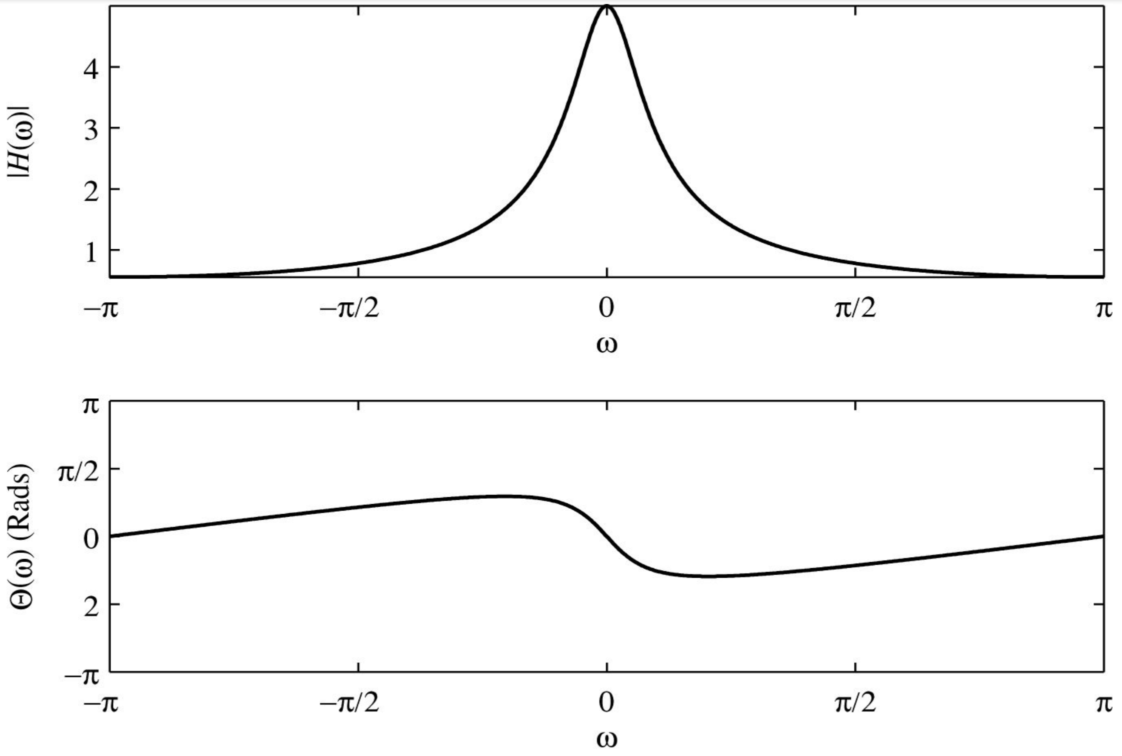


Figure 5.1.2 Magnitude and phase responses for the system in Example 5.1.4 with $a = 0.9$.

$$|H(\omega)| = \frac{|b|}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

$$\Theta(\omega) = \angle b - \tan^{-1}\frac{a\sin\omega}{1 - a\cos\omega}$$

Phase Response

- Typically represented in **degrees** (y-axis) vs. **ω** (x-axis)

- $\angle H(\omega)$ or $\angle H(e^{j\omega})$

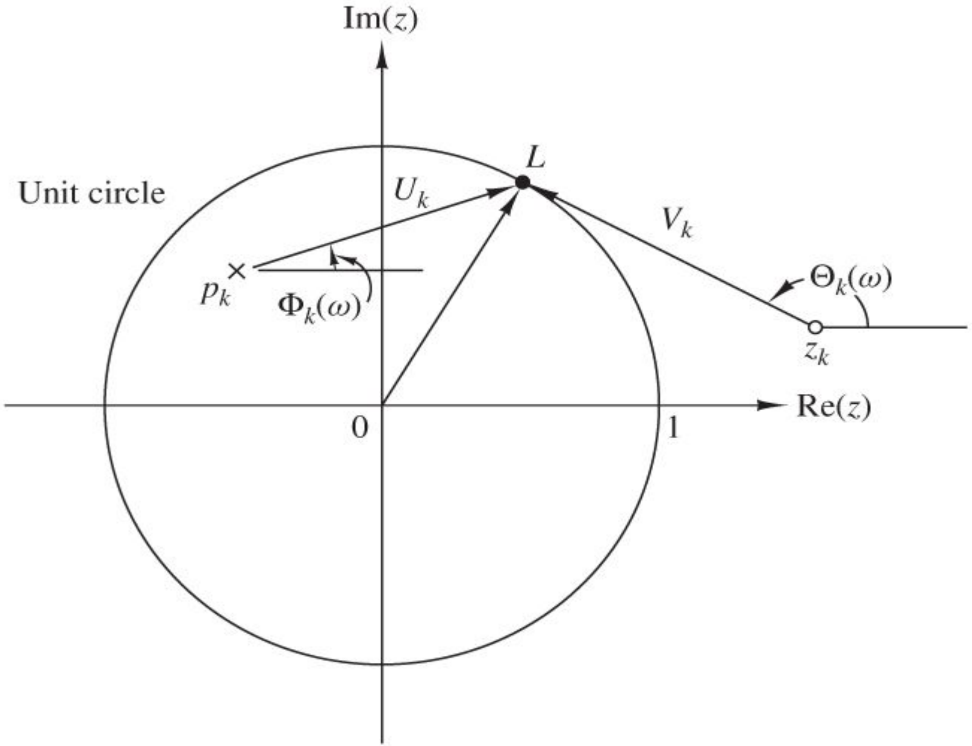
- To get the phase plot, we need to get the relative angle of each of the zeros and poles from the unit circle at varying ω

$$H(\omega) = b_0 \frac{\prod_{k=1}^{M} (1 - z_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - p_k e^{-j\omega})}$$

$$= b_0 e^{-j\omega(N-M)} \frac{\prod_{k=1}^{M} (e^{j\omega} - z_k)}{\prod_{k=1}^{N} (e^{j\omega} - p_k)}$$

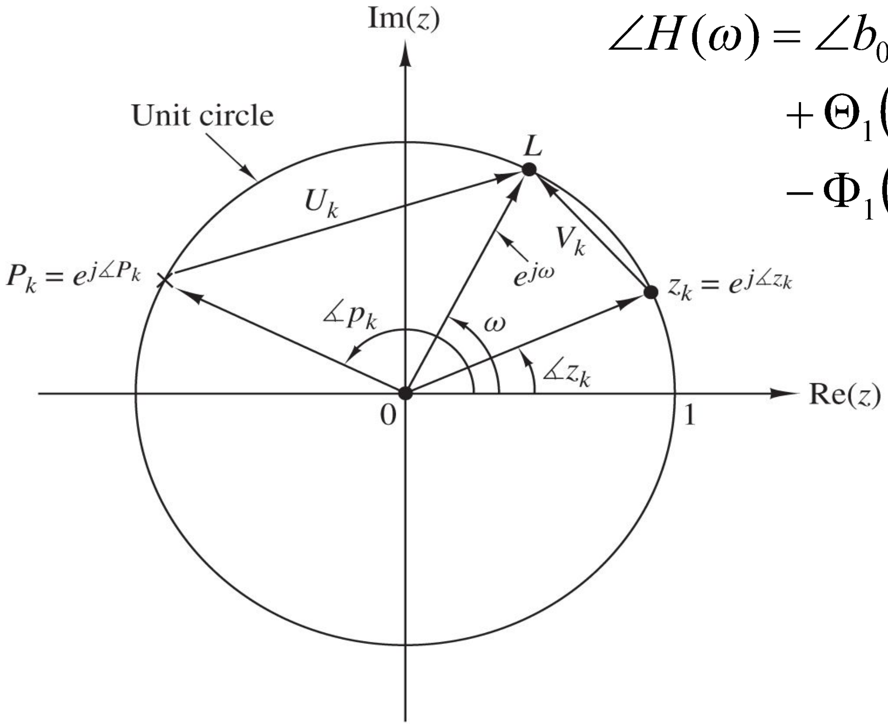
$$= b_0 e^{-j\omega(N-M)} \frac{\prod_{k=1}^{M} V_k(\omega) e^{j\Theta_k(\omega)}}{\prod_{k=1}^{N} U_k(\omega) e^{j\Phi_k(\omega)}}$$

Phase Response



(b)

Phase Response



$$
\begin{aligned}
\angle H(\omega) = \angle b_0 + \angle\omega(N-M) \\
+\Theta_1(\omega)+\Theta_2(\omega)+\cdots+\Theta_M(\omega) \\
-\Phi_1(\omega)-\Phi_2(\omega)\cdots-\Phi_N(\omega)
\end{aligned}
$$

Figure 5.2.2 A zero on the unit circle causes $|H(\omega)| = 0$ and $\omega = \measuredangle z_k$. In contrast, a pole on the unit circle results in $|H(\omega)| = \infty$ at $\omega = \measuredangle p_k$.

Phase Response

- An **ideal filter** has a linear phase response $\Theta(\omega)=-\omega n_0$

- Let x(n) be band-limited in $[\omega_1\ \omega_2]$. Let it pass through a system with

$$H(\omega)=\begin{cases} Ce^{-j\omega n_0}, & \omega_1<\omega<\omega_2 \\ 0, & \text{otherwise} \end{cases}$$

- Then

$$Y(\omega)=X(\omega)H(\omega)$$

$$=CX(\omega)e^{-j\omega n_0}, \quad \omega_1<\omega<\omega_2$$

$\rightarrow$ band-limited response, amplitude scaled & delayed

Phase Response

- The **group delay**, $\tau_g(\omega)$, is the time delay that a signal component of frequency ω undergoes as it passes from the input to the output of the system.

$$\tau_g(\omega) = -\frac{d\Theta}{d\omega}$$

- For linear phase filters, the group delay is constant

Minimum phase systems

- For causal LTI DT systems:
  - Minimum-phase Systems have all zeros inside the unit circle
  - Maximum-phase Systems have all zeros outside the unit circle

- Minimum phase → Stable system

- Among all pole-zero systems having the same magnitude response, the minimum-phase system has the smallest group delay.
- Among all pole-zero systems having the same magnitude response and the same total energy $E(\infty)$, the minimum-phase system has the largest partial energy.

Example

- Given two systems with the same magnitude response:

$$H_1(z) = 1 + \frac{1}{2}z^{-1} = z^{-1}(z + \frac{1}{2})$$

$$H_2(z) = \frac{1}{2} + z^{-1} = z^{-1}(\frac{1}{2}z + 1)$$

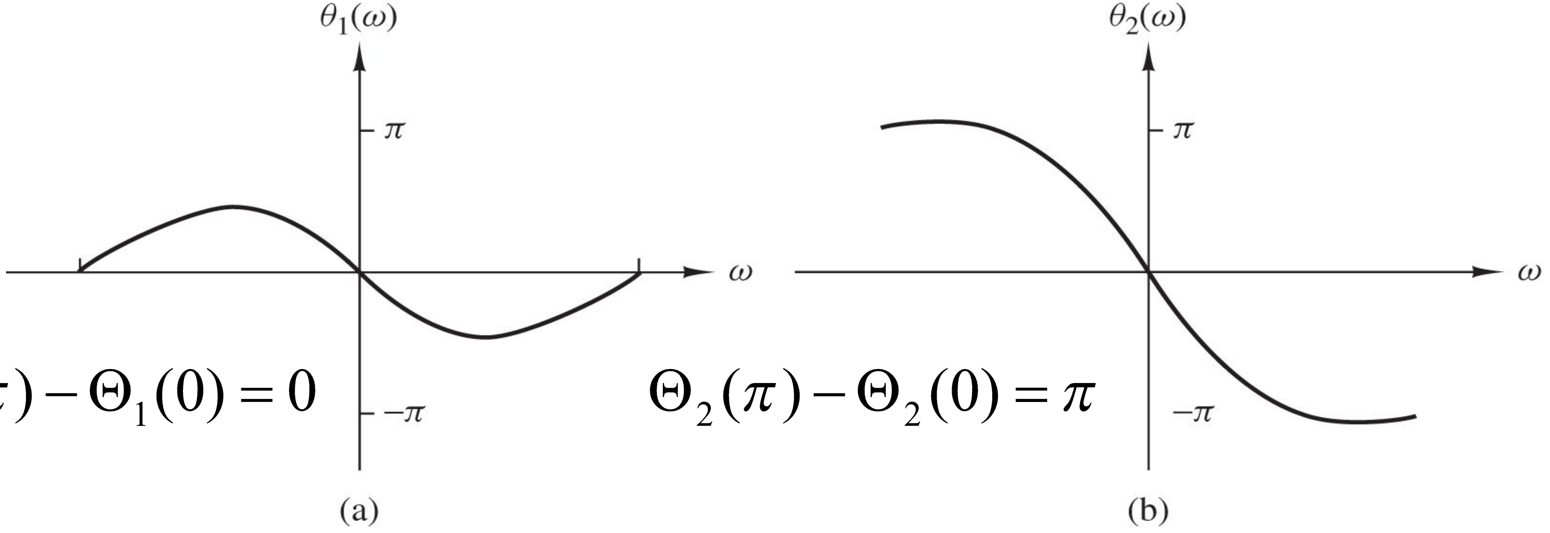
$$|H_1(\omega)| = |H_2(\omega)| = \sqrt{\frac{5}{4} + \cos\omega}$$

$$\Theta_1(\omega) = -\omega + \tan^{-1}\frac{\sin\omega}{0.5 + \cos\omega}$$

$$\Theta_2(\omega) = -\omega + \tan^{-1}\frac{\sin\omega}{2 + \cos\omega}$$

Example

$$\Theta_1(\omega) = -\omega + \tan^{-1}\frac{\sin\omega}{0.5+\cos\omega} \qquad \Theta_2(\omega) = -\omega + \tan^{-1}\frac{\sin\omega}{2+\cos\omega}$$



$$\Theta_1(\pi) - \Theta_1(0) = 0 \qquad \Theta_2(\pi) - \Theta_2(0) = \pi$$

Figure 5.5.3 Phase response characteristics for the systems in (5.5.10). and (5.5.11).

Filter Design via Pole-Zero Placement

1. Place a pole near the unit circle corresponding to frequencies to be emphasized
2. Place zeros near the frequencies to be deemphasized
3. All poles should be inside the unit circle for stability. Zeros can be placed anywhere in the z-plane
4. All complex zeros and poles must occur in complex conjugate pairs for real filter coefficients

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=0}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=0}^{M} \left(1 - z_k z^{-1}\right)}{\prod_{k=0}^{N} \left(1 - p_k z^{-1}\right)}$$

Example

- Given a two-pole LPF, Find b_0 and p such that H(0) = 1 and |H(π/4)|2 = 0.5

$$H(z) = \frac{b_0}{\left(1 - pz^{-1}\right)^2}$$

Example

- $H(0) = 1$ and $|H(\pi/4)|^2 = 0.5$

$$H(z) = \frac{b_0}{\left(1 - pz^{-1}\right)^2}$$

$$H(0) = b_0 /(1\text{-}p)^2$$

$$1 = b_0 /(1\text{-}p)^2 \quad \textbf{(eqn. 1)}$$

$$|H(\pi/4)|^2 = (b_0 /(1\text{-}pe^{-0.25j\pi})^2)^2$$

$$0.5 \quad = (b_0 /(1\text{-}pe^{-0.25j\pi})^2)^2 \textbf{(eqn. 2)}$$

$$b_0 = 0.4624, \; p = 0.32$$

Example

- Design a two-pole BPF with $\omega_c = \pi/2$, has a zero at $\omega = 0$ and $\omega = \pi$, and a magnitude response of 0.707 at $\omega = 4\pi/9$

Example

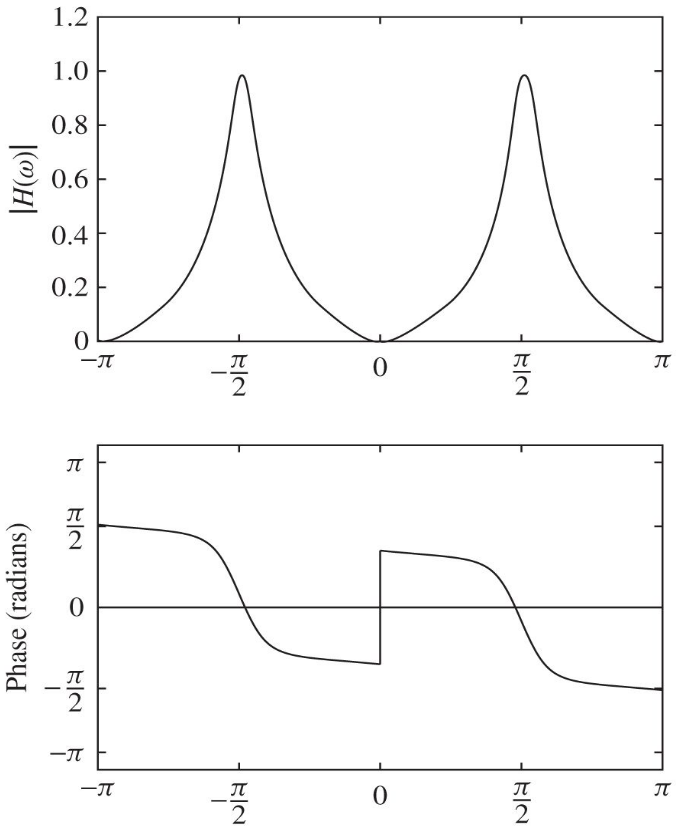
- Poles will be at p1,2 = $re^{\pm 0.5j\pi}$, zeros at z=1 and z=-1

$$H(z) = G\frac{(z-1)(z+1)}{(z-jr)(z+jr)}$$

$$H\left(\frac{\pi}{2}\right) = G\frac{2}{1-r^2} = 1 \quad \longrightarrow \quad G = \frac{1-r^2}{2}$$

$$\left|H\left(\frac{\pi}{4}\right)\right|^2 = \frac{1}{2} \quad \longrightarrow \quad r^2 = 0.707$$

Example



$$H(z) = 0.15 \frac{1 - z^{-2}}{1 + 0.7 z^{-2}}$$

Notes on the poles and zeros

- If a LPF DE is:

$$y(n) = -\sum_{k=1}^{N} a^k y(n-k) + \sum_{k=0}^{M} b^k x(n-k)$$

- Then we can transform the system into a HPF by:

$$H_{hp}(\omega) = H_{lp}(\omega - \pi) \qquad h_{hp}(n) = (e^{j\pi})^n h_{lp}(n) = (-1)^n h_{lp}(n)$$

$$y(n) = -\sum_{k=1}^{N} (-1)^k a^k y(n-k) + \sum_{k=0}^{M} (-1)^k b^k x(n-k)$$

Notes on the poles and zeros

- A system is **invertible** if there is a one-to-one correspondence between its input and output signals.

$$h(n) * h_I(n) = \delta(n)$$

$$H(z) = \frac{B(z)}{A(z)} \qquad H_I(z) = \frac{A(z)}{B(z)}$$

Summary

- DT systems are stable when the unit circle is included in the ROC of the z-transform
- Systems with poles along the unit circle are marginally stable
- The frequency response reveals the filter characteristics of a DT system. It can be easily obtained by setting $z = e^{j\omega}$ in the transfer function
- Pole-Zero placement method for filter design

For further reading...

- Chapters 4.4-4.6

  "Digital Signal Processing, 3rd ed. By Proakis, J. & Manolakis, D."

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