ON THE LAST KERVAIRE INVARIANT PROBLEM

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Dedicated to Mark Mahowald

ABSTRACT. We prove that the element h_6^2 is a permanent cycle in the Adams spectral sequence. As a result, we establish the existence of smooth framed manifolds with Kervaire invariant one in dimension 126, thereby resolving the final case of the Kervaire invariant problem.

Combining this result with the theorems of Browder, Mahowald–Tangora, Barratt–Jones–Mahowald, and Hill–Hopkins–Ravenel, we conclude that smooth framed manifolds with Kervaire invariant one exist in and only in dimensions 2, 6, 14, 30, 62, and 126.

1. Introduction

For a smooth framed manifold in dimension 4k+2, the Kervaire invariant, taking values in \mathbb{F}_2 , determines whether the manifold could be converted into a homotopy sphere via surgery – it takes value 0 if the manifold can be converted to a homotopy sphere and 1 otherwise. Formally speaking, the Kervaire invariant is defined as the Arf invariant of the quadratic refinement of the intersection pairing in the cohomology of the manifold with \mathbb{F}_2 coefficients. Using this invariant, Kervaire [33] discovered a PL-manifold in dimension 10 that does not admit any smooth structure.

The Kervaire invariant problem seeks to identify the dimensions in which there exist framed smooth manifolds with Kervaire invariant one. In these dimensions, exactly half of the cobordism classes of framed manifolds have Kervaire invariant one, and the other half have Kervaire invariant 0. The Kervaire invariant problem is closely related to many problems in differential topology, especially Kervaire–Milnor's classification theorem [34] on exotic smooth structures on spheres.

In this paper we prove the following Theorem 1.1.

Theorem 1.1. There exist framed manifolds of Kervaire invariant one in dimension 126.

Together with previous results by Browder [10], Mahowald–Tangora [44], Barratt–Jones–Mahowald [4], and Hill–Hopkins–Ravenel [23], this is the last case of the Kervaire invariant problem.

Corollary 1.2. The dimensions that there exist framed manifolds of Kervaire invariant one are 2, 6, 14, 30, 62, and 126.

In dimensions 2, 6, and 14, the product of spheres $S^1 \times S^1$, $S^3 \times S^3$, $S^7 \times S^7$ can be framed to have Kervarie invariant one. In dimension 30, an explicit framed manifold of Kervarie invariant one was constructed by J.Jones in [32]. For dimensions 62 and 126, we would like to comment that no explicit manifold of Kervarie invariant one

was known, although 50% of all framed manifolds in these two dimensions have Kervarie invariant one.

Our Theorem 1.1 is a consequence of Browder's theorem (Theorem 1.3) and the following main theorem of this paper, Theorem 1.4.

Theorem 1.3 (Browder [10]). There exist framed manifolds in dimension n of Kervaire invariant one if and only if

- (1) $n=2^{j+1}-2$, and (2) the element h_j^2 in the Adams spectral sequence survives to the E_{∞} -page.

The Adams spectral sequence [1] converges to the stable homotopy groups of spheres, which, by Pontryagin's theorem, correspond to framed manifolds up to framed cobordism. When the element h_i^2 survives (see below for an introduction to the Adams spectral sequence and the element h_i^2 , the homotopy classes it detects are denoted by $\theta_i \in \pi_{2^{j+1}-2}$. Browder showed that these classes correspond to framed manifolds with Kervaire invariant one.

Theorem 1.4 (Theorem 7.1). The element h_6^2 survives to the E_{∞} -page in the Adams spectral sequence.

In addition to Theorem 1.1, further implications of Theorem 1.4, particularly in manifold topology and unstable homotopy theory, can be found in [47, 42, 23, 3], among others.

Recall that the 2-primary Adams spectral sequence has the following form.

$$E_2^{s,t} = \operatorname{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s} S^0,$$
$$d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}.$$

Here the E_2 -page is the cohomology of the mod 2 Steenrod algebra A, S^0 is the 2completed sphere spectrum, and the spectral sequence converges to the 2-completed stable homotopy groups of spheres. In general, for a spectrum X, its HF₂-Adams spectral sequence is denoted by $E_r^{*,*}(X)$, converging to the 2-completed homotopy groups of X.

For the sphere, Adams [1] computed the Adams 1-line:

$$\operatorname{Ext}_{A}^{1,t}(\mathbb{F}_{2},\mathbb{F}_{2}) = \begin{cases} \mathbb{F}_{2}, & \text{if } t = 2^{j} \text{ for some } j \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Denote by h_j the generator of \mathbb{F}_2 in the bidegree $(s,t)=(1,2^j)$. Adams [2] proved that the element h_i survives in the Adams spectral sequence if and only if $j \geq 3$, resolving the famous Hopf invariant problem. In fact, Adams proved that

$$d_2(h_j) = h_0 h_{j-1}^2 \neq 0$$
, for $j \geq 4$.

The Kervaire invariant element h_i^2 lives on the Adams 2-line, which is generated by elements of the form $\{h_ih_j\}$ that are subject to the relations $h_ih_{i+1}=0$. Notably, h_6^2 was the last unknown element on the Adams 2-line whose survival in the Adams spectral sequence remained uncertain prior to our work. As a corollary of Theorem 1.4, we have:

Corollary 1.5. On the Adams 2-line, the only non-trivial elements that survive in the Adams spectral sequence are:

$$h_0h_2$$
, h_0h_3 , h_2h_4 , h_1h_j $(j \ge 3)$, h_j^2 $(j \le 6)$.

Proof. From Adams's Hopf invariant one differentials, Lin's computations of the Adams 4-line [40], and the Leibniz rule, we know that the only elements that survive to the Adams E_3 -page are

$$h_0h_2$$
, h_0h_3 , h_2h_4 , h_2h_5 , h_3h_5 , h_3h_6 , h_1h_j $(j \ge 3)$, h_j^2 $(j \le 6)$.

May [45] proved that the first three elements and h_j^2 for $j \leq 3$ survive. Mahowald–Tangora [44] proved that h_2h_5 supports a nonzero d_3 -differential, h_3h_5 supports a nonzero d_4 -differential, and h_4^2 survives. Isaksen–Wang–Xu [29] proved that h_3h_6 supports a nonzero d_3 -differential. Mahowald [43] proved that h_1h_j for $j \geq 3$ survive. Barratt–Jones–Mahowald [4] proved that h_5^2 survives (see [56, 29] for alternative proofs). Hill–Hopkins–Ravenel [23] proved that h_j^2 supports nonzero differentials for $j \geq 7$ (the targets of these differentials are still unknown). Finally, by Theorem 1.4, h_6^2 survives.

Before Hill–Hopkins–Ravenel's proof on the non-existence of θ_j for $j \geq 7$, Barratt–Jones–Mahowald [3] had an inductive argument trying to establish the existence of all θ_j : They proved that if there exists a θ_j satisfying $2 \cdot \theta_j = 0$, $\theta_j^2 = 0$, then there exists a θ_{j+1} satisfying $2 \cdot \theta_{j+1} = 0$. In particular, the problem of whether a θ_{j+1} of order 2 exists is known as the strong Kervaire invariant problem. It was known that $2 \cdot \theta_j = 0$ for $j \leq 5$ (see [44] for j = 4 and [56, 29] for j = 5). Although we prove that θ_6 exists in this paper, the following questions are still open:

Question 1.6. Does there exist a θ_6 that has order 2?

Question 1.7. Does there exist a θ_5 such that $\theta_5^2 = 0$?

By Barratt–Jones–Mahowald's theorem [3], if the answer to Question 1.7 is positive, then it would imply our Theorem 1.4 and a positive answer to Question 1.6. We plan to study these questions in a future project.

As suggested by the proof of Corollary 1.5, computation of differentials in the Adams spectral sequence has a long history. See Section 2 of [53] for a brief summary at all primes and computations at the prime 2 up to around 60-stem. For more recent computations up to around 90-stem, see [29, 28, 55, 30].

Many methods were introduced to compute differentials in the Adams spectral sequence. We highlight some of the major methods as follows.

(1) Multiplicative structure of the Adams E_2 -page and the Leibniz rule.

By computing the Adams E_2 -page with its multiplicative structure via the May spectral sequence, and comparing with Toda's unstable computations, May [45] computed all differentials up to the 28-stem.

- (2) Higher product structure interactions between Massey products and Toda brackets through Moss's theorem [48].
- (3) The Mahowald trick [44], which translates between differentials and extension problems.

By using both Moss's theorem and the Mahowald trick, Barratt–Mahowald–Tangora [44, 5] computed all but one differentials up to the 45-stem; The

remaining one was computed by Bruner [11] using power operations.

(4) Comparison with the motivic Adams spectral sequence via the Betti realization functor [20].

By comparison via the Betti realization functor, Isaksen [26] gave rigorous arguments for all but one differentials in both the motivic and *classical* Adams spectral sequence up to the 59-stem; The remaining one was computed by Xu [31] using the higher Leibniz rule for motivic Massey products.

(5) Wang–Xu's $\mathbb{R}P^{\infty}$ -method [53].

Using Lin's algebraic Kahn–Priddy theorem [39], Wang–Xu introduced a technique to prove Adams differentials inductively using differentials in subquotients of $\mathbb{R}P^{\infty}$, and completed computations of Adams differentials up to the 61-stem. (See also [54] for application of this method on extension problems.)

There is a nice geometric application of these differentials in stems 60 and 61. The sphere S^{61} has a unique smooth structure and it is the last odd dimensional case: The only ones are S^1 , S^3 , S^5 and S^{61} .

(6) Gheorghe-Isaksen-Wang-Xu's motivic cofiber of tau method [22, 29].

By identifying the algebraicity of the special fiber of a motivic deformation, Gheorghe–Wang–Xu [22] proved that the motivic Adams spectral sequence of the cofiber of tau is isomorphic to the algebraic Novikov spectral sequence for BP_{*}, which can be completely computed in a large range by Wang's program [52]. Then differentials in the classical Adams spectral sequence follow from naturality.

Using this method, together with some others, Isaksen–Wang–Xu [29] computed Adams differentials up to the 90-stem, with only a few exceptions.

(7) HF₂-synthetic/filtered spectra method [50, 13, 18, 17].

The homotopy ring of the $H\mathbb{F}_2$ -synthetic sphere [50, 16] can be viewed as a tool to encode homotopical information of the classical Adams spectral sequence, in analog to the relation between the homotopy ring of the \mathbb{C} -motivic sphere [26] and the Adams–Novikov spectral sequence.

By studying the HF_2 -synthetic sphere, Burklund–Isaksen–Xu [17] proved a few Adams differentials up to the 90-stem that were left out by Isaksen–Wang–Xu [29].

Our strategy for proving our main Theorem 1.4 that h_6^2 survives is three-fold.

• Lin's Program.

By establishing the theory of noncommutative Gröbner bases for Steenrod algebras [36], Lin made computer programs that compute Ext groups in a very effective way. We use Lin's program to compute the Adams E_2 -pages for a vast collection of finite spectra and the maps between them. For a detailed account of the range of Adams E_2 data computed, see [38].

Building on the theory of secondary Steenrod algebras [6, 49, 19], we also leverage Lin's program to compute Adams d_2 -for certain finite spectra. Details on the d_2 -differentials we have computed can also be found in [38].

Additionally, Lin's program facilitates the propagation of differentials and extensions by employing the Leibniz rule and the naturality of Adams spectral sequences and extension spectral sequences (Definition 2.1).

• The Generalized Leibniz Rule and the Generalized Mahowald Trick.

The use of HF_2 -synthetic/filtered spectra allows us to discuss elements on the Adams E_n -page as classes in the homotopy groups of certain synthetic spectra [50, 18]. In particular, naturality on synthetic homotopy groups allows us to discuss jump-of-filtration phenomena on any Adams E_n -page in a rigorous way.

Using HF_2 -synthetic spectra, we develop the Generalized Leibniz Rule (Theorem 6.1) and the Generalized Mahowald Trick (Theorem 6.12), which can be viewed as generalizations of the classical Mahowald's trick and geometric boundary theorem studies by Ravenel, Behrens and others [51, 8, 41, 14]. We then use them to further propagate differentials from the known ones.

With the Generalized Leibniz Rule and the Generalized Mahowald Trick, Lin's program becomes even more powerful, enabling the derivation of additional differentials. The program rigorously verifies the conditions of the relevant theorems and generates human-readable proofs [37] for all differentials computed by the machine. In the Appendix, readers will find tables of Adams differentials that are used in the proof of the main Theorem 1.4.

• The inductive approach and adhoc arguments near stem 126.

By studying Barratt–Jones–Mahowald's inductive approach [3] in the HF₂-synthetic context, Burklund–Xu (Proposition 7.19 of [18]) proved that h_6^2 is a permanent cycle in the classical Adams spectral sequence if and only if $\lambda\eta\theta_5^2=0$ in the homotopy group of the synthetic sphere. Here λ is our notation for the synthetic deformation parameter, and θ_5 is any synthetic homotopy class that is detected by h_5^2 on the Adams E_2 -page.

There are 105 additive generators in stem 125 of the classical Adams E_2 page – these are the potential targets that could be hit by a nonzero differential from h_6^2 . Using Lin's program and the inductive approach, we can

rule out 101 out of the 105 elements.

By further applying the Generalized Leibniz Rule and the Generalized Mahowald Trick, we can reduce the possibility of a nonzero differential supported by h_6^2 to only one case: A nonzero d_{12} hits a certain element in stem 125, filtration 14 (see Proposition 7.8(2)). A necessary condition for this nonzero d_{12} is an η -extension from stem 124, filtration 10 to stem 125, filtration 14 (see Proposition 7.8(5)).

By a careful inspection of the homotopy groups of the synthetic sphere (and on certain Adams E_n -pages) near stem 126, we use adhoc arguments to prove that the η -extension in the previous paragraph cannot happen (see Proposition 7.9) and conclude that h_6^2 survives in the Adams spectral sequence.

Organization. In Section 2, we setup and discuss properties of an extension spectral sequence for a map $f: X \to Y$ between two spectra, whose E_0 -page is isomorphic to the direct sum of E_∞ -pages of the Adams spectral sequences of X and Y, and differentials correspond to $f_*: \pi_*X \to \pi_*Y$, filtered by the Adams filtration. This setup allows us to discuss the jump-of-filtration phenomena rigorously. In Section 3, we recall and discuss certain properties of $H\mathbb{F}_2$ -synthetic spectra. In Section 4, we discuss the extension spectral sequence from Section 2 in the context of $H\mathbb{F}_2$ -synthetic spectra. In Section 5, we discuss extensions on a classical Adams E_n -page, defined in terms of $H\mathbb{F}_2$ -synthetic spectra. In Section 6, using the language of Sections 4 and 5, we develop the Generalized Leibniz Rule and the Generalized Mahowald Trick. In Section 7, we use the inductive approach, and ad hoc arguments prove that h_6^2 survives in the Adams spectral sequence. Necessary information of the classical Adams spectral sequence for the final ad hoc arguments is included in the Appendix.

Acknowledgement. The second author is partially supported by grants NSFC-12325102, NSFC-12226002, the New Cornerstone Science Foundation, and Shanghai Pilot Program for Basic Research–Fudan University 21TQ1400100 (21TQ002). The third author is partially supported by NSF Grant DMS 2105462 and the AMS Centennial Research Fellowship.

The authors express their deepest gratitude to Mark Behrens and Peter May for their invaluable guidance, insightful advice, and continuing support throughout this journey.

We also extend our heartfelt thanks to Soren Galatius, Lars Hesselholt, Mike Hill, Mike Hopkins, Dan Isaksen, Ciprian Manolescu, Haynes Miller, Yi Ni, Doug Ravenel, Gang Tian, Chenyang Xu, and Weiping Zhang for their encouragement and support of this project. The authors also thank Yuchen Wu and Shangjie Zhang for helpful comments on the early drafts of this paper.

This work is dedicated to the memory of Mark Mahowald, whose groundbreaking contributions to algebraic topology continue to inspire generations of mathematicians. His deep insights and unwavering passion for the field remain a guiding light in our endeavors.

2. A Spectral Sequence for Extensions

In this paper, we assume that all symbols X, Y, Z, W, \ldots for spectra represent 2-completed connective spectra of finite type. In particular, their HF₂-Adams spectral sequences converge strongly.

Consider a map $f:X\to Y$ between two spectra and we can construct the following chain complex

$$0 \to \pi_* X \xrightarrow{\pi_* f} \pi_* Y \to 0$$

whose homology is

$$\ker(\pi_* f) \oplus \operatorname{coker}(\pi_* f).$$

We encounter f-extension problems when analyzing the homotopy groups through the E_{∞} -pages of the HF₂-based Adams spectral sequences. To address these extensions, we introduce the following spectral sequence.

Definition 2.1. For a map $f: X \to Y$, we define an f-extension spectral sequence (denoted f-ESS) as the spectral sequence derived from filtering the above chain complex according to the Adams filtration on π_* . The E_0 -page of this spectral sequence is isomorphic to the direct sum of E_∞ -pages of the Adams spectral sequences of X and Y:

$${}^fE_0^{s,t} \cong E_\infty^{s,t}(X) \oplus E_\infty^{s,t}(Y) \Longrightarrow \ker(\pi_* f) \oplus \operatorname{coker}(\pi_* f)$$

where the d_0 -differential is induced by f on the E_{∞} -pages.

Differentials in this spectral sequence has the form

$$d_n^f: {}^fE_n^{s,t} \to {}^fE_n^{s+n,t+n}.$$

To prevent confusion between differentials d_n in the Adams spectral sequence of a spectrum X and those in the f-extension spectral sequence, we denote the latter with a superscript, writing them as d_n^f .

In the remainder of the paper, we denote by AF(f) the Adams filtration of a map f. Recall that $AF(f) \geq k$ if and only if there exist k composable maps f_1, \ldots, f_k , each inducing a trivial map in $H\mathbb{F}_2$ -homology, such that f is homotopic to the composition $f_k \circ \cdots \circ f_1$ (see [51, Theorem 2.2.14] for example).

Notation 2.2. Let ${}^fZ_n^{s,t}(X) \subset E_\infty^{s,t}(X)$ be the subgroup consisting of elements for which the f-extension differentials d_0^f, \ldots, d_n^f vanish. Let ${}^fB_n^{s,t}(Y) \subset E_\infty^{s,t}(Y)$ be the subgroup generated by the sum of images of the f-extension differentials d_0^f, \ldots, d_n^f . We define ${}^fB_{-1}^{s,t}(Y) = 0$ and ${}^fZ_{-1}^{s,t}(X) = E_\infty^{s,t}(X)$.

By definition we have

$${}^{f}E_{n}^{s,t} \cong {}^{f}Z_{n-1}^{s,t}(X) \oplus \left(E_{\infty}^{s,t}(Y)/{}^{f}B_{n-1}^{s,t}(Y)\right).$$

A nonzero d_n^f differential takes the form

$$d_n^f: {}^fZ_{n-1}^{s,t}(X) \to E_{\infty}^{s+n,t+n}(Y)/\,{}^fB_{n-1}^{s+n,t+n}(Y).$$

When we write

$$d_n^f(x) = y$$

for $x \in E^{s,t}_{\infty}(X)$ and $y \in E^{s+n,t+n}_{\infty}(Y)$, it is understood that x belongs to ${}^fZ^{s,t}_{n-1}(X)$, and

$$d_n^f(x) = y + {}^f B_{n-1}^{s+n,t+n}(Y).$$

Furthermore, we may sometimes write

$$d_n^f(x) \equiv y \mod M$$

for some subgroup $M \subset E_{\infty}^{s+n,t+n}(Y)$. This means that

$$d_n^f(x) = y + m + {}^f B_{n-1}^{s+n,t+n}(Y)$$

for some $m \in M$.

Definition 2.3. We say there is an f-extension from $x \in E_{\infty}^{s,t}(X)$ to $y \in E_{\infty}^{s+n,t+n}(Y)$ if $d_n^f(x) = y$ in the f-ESS. We say this f-extension is **essential** if y is nontrivial in the E_n -page of the f-ESS, or equivalently

$$y \notin {}^fB_{n-1}^{s+n,t+n}(Y).$$

Otherwise we refer to it as inessential.

Notation 2.4. For $x \in E_{\infty}^{s,t}(X)$, we denote by $\{x\}$ the set of all classes in $\pi_{t-s}X$ that are detected by x. We use [x] to refer to a specific class or a general class in $\{x\}$, with the choice understood from the context.

Proposition 2.5. Consider $f: X \to Y$, $x \in E^{s,t}_{\infty}(X)$, $y \in E^{s+n,t+n}_{\infty}(Y)$ and $y' \in E^{s+m,t+m}_{\infty}(Y)$ for $m, n \geq 0$.

- (1) There is an f-extension from x to y, i.e., $d_n^f(x) = y$ in the f-ESS if and only if there is a class $[x] \in \{x\}$ such that $f[x] \in \{y\}$.
- (2) An f-extension from x to y is inessential, i.e., y is trivial in the E_n -page of the f-ESS if and only if there exists an element $x' \in E_{\infty}^{s+a,t+a}(X)$ for $0 < a \le n$ such that we have an essential differential $d_{n-a}^f(x') = y$. Equivalently, there exists a class $[x'] \in \{x'\} \subset \pi_* X$ with AF(x') > AF(x), such that $f[x'] \in \{y\}$.
- (3) Suppose we have an f-extension from x to both y and y'.
 - (a) If m = n, then $y y' \in {}^fB_{n-1}^{s+n,t+n}(Y)$ and there exists an element $x' \in E_{\infty}^{s+a,t+a}(X)$ for $0 < a \le n$ such that we have an essential differential

$$d_{n-a}^f(x') = y - y'$$

in f-ESS. Equivalently, there exists a class $[x'] \in \{x'\} \subset \pi_* X$ with AF(x') > AF(x), such that $f[x'] \in \{y - y'\}$.

(b) If m > n, then the f-extension from x to y is inessential.

Proof. Part (1) is straightforward from the setup of the extension spectral sequence and Definition 2.3. Parts (2) and (3) follow from Part (1).

Corollary 2.6. If the Adams filtration of f is k, then $d_i^f = 0$ for i < k.

Proof. When k > 0, it is clear that $d_0^f = 0$, which means that for any s and $[x] \in F_s\pi_*(X)$, we have $f([x]) \in F_{s+1}\pi_*(Y)$. Since AF(f) = k, we know that there exist k maps

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} X_k = Y$$

such that each f_i induces trivial map in HF₂-homology (hence AF(f_i) > 0) and f is homotopic to the composition $f_k \circ \cdots \circ f_1$. Then for any s and $[x] \in F_s\pi_*(X)$, inductively we have

$$f_1([x]) \in F_{s+1}\pi_*(X_1)$$

 $f_2 \circ f_1([x]) \in F_{s+2}\pi_*(X_2)$

. .

$$f([x]) = f_k \circ \cdots \circ f_1([x]) \in F_{s+k}\pi_*(X_k) = F_{s+k}\pi_*(Y).$$

By Proposition 2.5.(1) this means that $d_i^f x = 0$ for any $x \in E_{\infty}^{*,*}(X)$ and i < k. \square

Definition 2.7. (1) For $x \in E_{\infty}^{s,t}(X)$, we say that the an f-extension $d_n^f(x) = y$ has a crossing that hits Adams filtration p for some $p \leq s + n$, if there exists an element $x' \in E_{\infty}^{s+a,t+a}(X)$ with a > 0 and an f-extension $d_m^f(x') = y'$ for $0 \neq y' \in E_{\infty}^{s+a+m,t+a+m}(Y)$ such that

$$p = AF(y') = s + a + m \le AF(y) = s + n.$$

(2) We say that an f-extension $d_n^f(x) = y$ has no crossing that hits the range AF $\geq p$ if there does not exist an element $x' \in E_{\infty}^{s+a,t+a}(X)$ with a>0 and an f-extension $d_m^f(x') = y'$ for $0 \neq y' \in E_{\infty}^{s+a+m,t+a+m}(Y)$ such that

$$p \le AF(y') = s + a + m \le AF(y) = s + n.$$

(3) We say that an f-extension $d_n^f(x) = y$ has no crossing if it has no crossing that hits the range $AF \ge AF(x) + 1$.

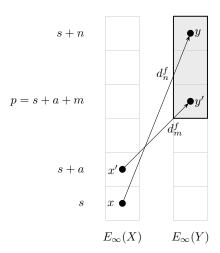


FIGURE 1. A crossing of d_n^f that hits Adams filtration p

Remark 2.8. The following statements are equivalent.

- An f-extension $d_n^f(x) = y$ has no crossing that hits the range AF $\geq p$.
- If for any a > 0 there is an f-extension from $x' \in E^{s+a,t+a}_{\infty}(X)$ to some nontrivial y', then AF(y') < p or AF(y') > AF(y).

Remark 2.9. We do not require the crossings in Definition 2.7 (1)(2) to be essential. A crossing implies the existence of an essential crossing, possibly a shorter one.

Proposition 2.10. An f-extension from x to y has no crossing in the range $AF \ge p$ if and only if for all $[x] \in \{x\}$ such that $AF(f[x]) \ge p$ we have $f[x] \in \{y\}$. In particular, we have

(1) An f-extension from x to y has no crossing if and only if for all $[x] \in \{x\}$ we have $f[x] \in \{y\}$.

(2) An f-extension $d_n^f(x) = 0$ has no crossing if and only if for all $[x] \in \{x\}$,

$$AF(f[x]) > AF(x) + n.$$

(3) If AF(f) = n, then all d_n^f -differentials have no crossing in the f-ESS.

Proof. First we prove the only if part. Assume by contradiction that there exists $[x] \in \{x\}$ such that $f[x] = [y'] \notin \{y\}$ is detected by $y' \in E_{\infty}(Y)$ with $AF(y') \ge p$. Then there is an f extension from x to y'. By Proposition 2.5 (3), y' (if AF(y) > AF(y')) or y - y' (if AF(y) = AF(y')) or y (if AF(y) < AF(y')) should be hit by a shorter d^f differential. This shorter differential is a crossing of the f-extension from x to y that hits $AF \ge p$, which is a contradiction. Now we prove the if part. Assume that the f-extension has a crossing from x' to y' with $p \le AF(y') \le AF(y)$. Then there exists $[x'] \in \{x'\}$ such that $f([x']) \in \{y'\}$. Note that $[x] + [x'] \in \{x\}$ since AF(x') > AF(x), we have

$$f([x] + [x']) = f([x]) + f([x']) = [y] + [y'] \in \begin{cases} \{y'\} & \text{if } AF(y') < AF(y) \\ \{y + y'\} & \text{if } AF(y') = AF(y). \end{cases}$$

However, by assumption we should have $f([x] + [x']) \in \{y\}$ and it is contradictory to the equation above in either case.

Hence the main statement is true. Part (1) (2) are special cases of the main statement and Part (3) holds for degree reasons.

Example 2.11. Consider the Hopf map $\eta: S^1 \to S^0$ between 2-completed spheres. It is known that it induces a surjective map on π_{46} (see [IWX] for example).

- In the η -ESS, we have the following nonzero differentials these are the essential η -extensions:
 - (1) $d_1^{\eta}(h_5d_0) = h_1h_5d_0$ in AF= 6,
 - (2) $d_2^{\eta}(\Delta h_1 g) = d_0 l$ in AF= 11,
 - (3) $d_3^{\eta}(h_1g_2) = \Delta h_2c_1$ in AF= 8,
 - (4) $d_4^{\eta}(h_3^2h_5) = Mh_1$ in AF= 7.
- The η -extension $d_2^{\eta}(h_0h_3^2h_5) = h_1h_5d_0$ is inessential.

This is a zero differential on the E_2 -page of the η -ESS due to the nonzero d_1^{η} -differential that hits $h_1h_5d_0$. Note that although this η -extension is inessential, we still have a class $[h_0h_3^2h_5] \in \{h_0h_3^2h_5\}$ such that $\eta \cdot [h_0h_3^2h_5] \in \{h_1h_5d_0\}$, due to the presence of an essential η -extension from h_5d_0 to $h_1h_5d_0$.

• The η -extension $d_4^{\eta}(h_3^2h_5) = Mh_1$ has a crossing that hits AF= 6.

In fact, both the essential extension $d_1^{\eta}(h_5d_0) = h_1h_5d_0$ and the inessential extension $d_2^{\eta}(h_0h_3^2h_5) = h_1h_5d_0$ are such crossings. Due to the existence of a crossing, the following statement is NOT true:

For all
$$[h_3^2h_5] \in \{h_3^2h_5\}$$
, we have $\eta \cdot [h_3^2h_5] \in \{Mh_1\}$.

In fact, there exists a class $[h_3^2h_5] \in \{h_3^2h_5\}$ such that $\eta \cdot [h_3^2h_5] \in \{h_1h_5d_0\}$ due to the crossing η -extension from h_5d_0 to $h_1h_5d_0$.

• The η -extension $d_1^{\eta}(h_5d_0) = h_1h_5d_0$ has no crossing by part (3).

Theorem 2.12. Consider a homotopy commutative diagram of spectra

$$X \xrightarrow{f} Y$$

$$\downarrow q$$

$$Z \xrightarrow{q} W$$

Suppose $m, n, l \ge 0, 0 < k \le m + l - n,$

$$\begin{array}{ll} x \in E^{s,t}_{\infty}(X) & y \in E^{s+n,t+n}_{\infty}(Y), \\ z \in E^{s+m,t+m}_{\infty}(Z) & w \in E^{s+m+l,t+m+l}_{\infty}(W), \end{array}$$

and

- $(1) d_n^f(x) = y,$
- (2) $d_m^p(x) = z$,
- (3) the differential in (1) or the differential in (2) has no crossing,
- (4) $d_l^g(z) = w$, and has no crossing that hits the range $AF \ge s + n + k$,
- (5) $d_{k-1}^q y = 0$, and has no crossing.

Then $d_{m+l-n}^q(y) = w$. (See Figure 2.)

Proof. First we find a representative [x] of x such that

$$(2.13) p[x] \in \{z\}$$

and

$$(2.14) f[x] \in \{y\}.$$

If (1) has no crossing, by (2) we can pick $[x] \in \{x\}$ such that (2.13) holds; If (2) has no crossing, by (1) we can pick $[x] \in \{x\}$ such that (2.14) holds. In both cases the other equation also holds because of the no crossing condition.

By (5), (2.14) and Proposition 2.10 (2) we know that qf[x] has $AF \ge s + n + k$. Since the diagram on the Adams E_{∞} -pages commutes, we have

$$AF(gp[x]) = AF(qf[x]) \ge s + n + k.$$

Combining with (4), (2.13) and Proposition 2.10 we have

$$gp[x] \in \{w\}.$$

Therefore, $qf[x] \in \{w\}$ and by (2.14) there is a q-extension from y to w.

Corollary 2.15. Consider a homotopy commutative diagram of spectra

$$X \xrightarrow{f} Y$$

$$\downarrow q$$

$$Z \xrightarrow{g} W$$

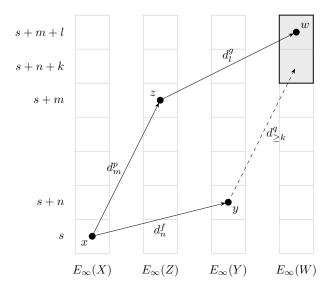


Figure 2. A demonstration of Theorem 2.12

Suppose $m, n, l \geq 0$,

$$\begin{array}{ll} x \in E^{s,t}_{\infty}(X) & y \in E^{s+n,t+n}_{\infty}(Y), \\ z \in E^{s+m,t+m}_{\infty}(Z) & w \in E^{s+m+l,t+m+l}_{\infty}(W), \end{array}$$

and

- (1) $d_n^f(x) = y$,
- $(2) \ d_m^p(x) = z,$
- (3) the differential in (1) or the differential in (2) has no crossing,
- (4) $d_l^g(z) = w$, and has no crossing.

Then $d_{m+l-n}^q(y) = w$.

Corollary 2.16. Consider a homotopy commutative diagram of spectra



Suppose $n, m \geq 0$,

$$x \in E^{s,t}_{\infty}(X), \ y \in E^{s+n,t+n}_{\infty}(Y), \ z \in E^{s+m,t+m}_{\infty}(Z)$$

and

- (1) $d_n^f(x) = y$, (2) $d_m^p(x) = z$, (3) the differential in (1) or the differential in (2) has no crossing.

Then $d_{m-n}^q(y) = z$.

Proof. This is a special case of Corollary 2.15 when Z = W and $g = id_Z$.

Corollary 2.17. Consider a homotopy commutative diagram of spectra

$$\begin{array}{c}
X \\
p \\
Z \xrightarrow{g} W
\end{array}$$

Suppose $n, m \geq 0$,

$$x\in E^{s,t}_\infty(X),\ z\in E^{s+m,t+m}_\infty(Y),\ w\in E^{s+m+l,t+m+l}_\infty(W)$$

and

- $(1) \ d_m^p(x) = z,$
- (2) $d_l^g(z) = w$, and has no crossing.

Then $d_{m+l}^q(x) = w$.

Proof. This is a special case of Corollary 2.15 when X = Y and $f = id_X$.

Corollary 2.18. Consider a homotopy commutative diagram of spectra

$$\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow q \\
Z \xrightarrow{g} W
\end{array}$$

Assume ${}^{p}E_{0}^{*,*} = {}^{p}E_{r}^{*,*}$ and ${}^{q}E_{0}^{*,*} = {}^{q}E_{r}^{*,*}$ for some $r \geq 0$. Then

$$(d_r^p, d_r^q): E_{\infty}^{*,*}(X) \oplus E_{\infty}^{*,*}(Y) \to E_{\infty}^{*,*}(Z) \oplus E_{\infty}^{*,*}(W)$$

induces a map from the f-ESS to the g-ESS.

Proof. Since ${}^pE_0^{*,*} = {}^pE_r^{*,*}$ and ${}^qE_0^{*,*} = {}^qE_r^{*,*}$, we know that d_r^p , d_r^q have no crossings. By Corollary 2.15, we have $d_0^g \circ d_r^p = d_r^q \circ d_0^f$ and therefore (d_r^p, d_r^q) induces a map from ${}^fE_1^{*,*}$ to ${}^gE_1^{*,*}$. Inductively we can apply Corollary 2.15 again and show that (d_r^p, d_r^q) induces a map from ${}^fE_n^{*,*}$ to ${}^gE_n^{*,*}$ and $d_n^g \circ d_r^p = d_r^q \circ d_n^f$ for all n.

Corollary 2.19. If the composition $g \circ f$ of two maps $f: X \to Y$, $g: Y \to Z$ is trivial, and $d_n^f(x) = y$ for $x \in E_{\infty}^{s,t}(X)$ and $y \in E_{\infty}^{s+n,t+n}(Y)$, then y is a permanent cycle in the g-ESS, i.e., we have $d_m^g(y) = 0$ for all $m \ge 0$.

Proof. This follows immediately from the following commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow g \\
0 & \longrightarrow & Z
\end{array}$$

and Corollary 2.15.

There is a second short proof using homotopy groups. By Proposition 2.5, we know that there exists $[x] \in \{x\}$ such that $f([x]) \in \{y\}$. Let $[y] = f([x]) \in \{y\}$ and we have g([y]) = g(f([x])) = 0. Hence y is a permanent d^g cycle.

Proposition 2.20. Suppose that the sequence of spectra

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

induces a sequence of homotopy groups

$$\pi_* X \xrightarrow{\pi_* f} \pi_* Y \xrightarrow{\pi_* g} \pi_* Z$$

that is exact in π_*Y . Then all permanent cycles in $E_{\infty}^{*,*}(Y)$ of the g-ESS are boundaries in the f-ESS.

Proof. If we consider the following sequence

$$(2.21) 0 \to \pi_* X \xrightarrow{\pi_* f} \pi_* Y \xrightarrow{\pi_* g} \pi_* Z \to 0$$

and treat it as a chain complex filtered by the Adams filtration, we obtain an extension spectral sequence comprising d^f and d^g differentials. The abutment of this spectral sequence, when projected onto the Y component, is zero. Therefor, all permanent d^g -cycles are killed by d^f -differentials.

Again we offer another proof via homotopy groups. Assume that $y \in E_{\infty}^{*,*}(Y)$ is a permanent d^g cycle. By the convergence of ESS we know that it detects an g-torsion $[y] \in \{y\}$ with g([y]) = 0. Hence there exists $[x] \in \pi_*X$ such that f([x]) = [y] by exactness. Assume that [x] is detected by $x \in E_{\infty}^{*,*}(X)$. Then there exists an f extension from x to y.

Corollary 2.19 and Proposition 2.20 are particularly useful because they provide numerous potential applications whenever we have a cofiber sequence of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Cf \xrightarrow{h} \Sigma X$$

These results can also be implemented in a computer program to facilitate their application.

3. $\mathrm{H}\mathbb{F}_2$ -SYNTHETIC SPECTRA

In Section 2, we explored classical extension problems in the context of commutative diagrams and cofiber sequences of spectra, framed in terms of the extension spectral sequence. The power of the extension spectral sequence can be further enhanced by generalizing and applying it to the category of $\mathrm{H}\mathbb{F}_2$ -synthetic spectra, a topic we will discuss in Section 4.

In this section, we recall some recent work of Pstragowski [50] and Appendix A of Burklund–Hahn–Senger [16] on E-synthetic spectra, which is strongly motivated by the work of Hu–Kriz–Ormsby [25], Isaksen [26], Gheorghe–Wang–Xu [22], and Gheorghe–Isaksen–Krause–Ricka [21] on the \mathbb{C} -motivic spectra. We will specialize some of these results to $H\mathbb{F}_2$ -synthetic spectra, refining and highlighting specific properties.

The stable, presentable, symmetric monoidal ∞ -category $\operatorname{Syn}_{\operatorname{H}\mathbb{F}_2}$ of $\operatorname{H}\mathbb{F}_2$ -synthetic spectra is constructed in [50], along with a functor

$$\nu: \mathrm{Sp} \to \mathrm{Syn}_{\mathrm{HF}_2},$$

from the ∞ -category of spectra Sp. This functor is lax symmetric monoidal and preserves filtered colimits. However, ν does not generally preserve cofiber sequences. Instead, the following holds.

Proposition 3.1 ([50, Lemma 4.23]). Suppose that

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a cofiber sequence of spectra. Then

$$\nu X \xrightarrow{\nu f} \nu Y \xrightarrow{\nu g} \nu Z$$

is a cofiber sequence of $H\mathbb{F}_2$ -synthetic spectra if and only if

$$0 \to \mathrm{H}\mathbb{F}_{2*}X \xrightarrow{\mathrm{H}\mathbb{F}_{2*}f} \mathrm{H}\mathbb{F}_{2*}Y \xrightarrow{\mathrm{H}\mathbb{F}_{2*}g} \mathrm{H}\mathbb{F}_{2*}Z \to 0$$

is a short exact sequence.

Definition 3.2 ([50, Definition 4.6]). The bigraded spheres $S^{s,w}$ are the synthetic spectra defined by $\Sigma^{s-w}\nu(S^w)$.

Definition 3.3 ([50, Definition 4.27]). The natural comparison map

$$S^{0,-1} = \Sigma \nu(S^{-1}) \to \nu(\Sigma S^{-1}) = S^{0,0}$$

in the category of HF₂-synthetic spectra is denoted by $\lambda \in \pi_{0,-1}S^{0,0}$. We denote by $S^{0,0}/\lambda$ the cofiber of λ . In general for synthetic X the following

$$\lambda \wedge id_X : \Sigma^{0,-1}X \to X.$$

is a natural transformation and by abuse of notation this map is also denoted by λ . We denote by X/λ the cofiber of λ which is equivalent to $(S^{0,0}/\lambda) \wedge X$.

Remark 3.4. For every $n \geq 1$, $S^{0,0}/\lambda^n$ is an E_{∞} -object in $\operatorname{Syn}_{H\mathbb{F}_2}$ (see [15, Appendix B, C] for example).

Remark 3.5. The element λ can be viewed as a parameter of a categorical deformation on $\operatorname{Syn}_{H\mathbb{F}_2}$ (see [50]).

$$\operatorname{Sp}_{2}^{\wedge} \xrightarrow{\lambda^{-1}} \operatorname{Syn}_{\operatorname{H}\mathbb{F}_{2}} \xrightarrow{\operatorname{mod } \lambda} \mathcal{D}(A_{*}\operatorname{-Comod})$$

Inverting λ , the generic fiber is equivalent to the classical category of 2-completed spectra $\operatorname{Sp}_2^{\wedge}$. Modding out by λ , the special fiber is equivalent to Hovey's derived category [24] of comodules over the mod 2 dual Steenrod algebra $\mathcal{D}(A_*\operatorname{-Comod})$. Historically, this synthetic deformation is motivated by, and analogous to, the motivic deformation conjectured by Isaksen and proven by Gheorghe–Wang–Xu [22]: Specifically, there exists a map

$$\tau:\widehat{S^{0,-1}}\to\widehat{S^{0,0}}$$

between p-completed motivic spheres. This element τ can be viewed as a parameter of a categorical deformation on the category of cellular objects over $\widehat{S^{0,0}}$.

$$\operatorname{Sp}_p^\wedge \xleftarrow{\tau^{-1}} \widehat{\operatorname{S}^{0,0}}\operatorname{-Mod} \xrightarrow{\mod \tau} \mathcal{D}(BP_*BP\operatorname{-Comod})$$

Inverting τ , the generic fiber is equivalent to the classical category of p-completed spectra $\operatorname{Sp}_p^{\wedge}$. Modding out by τ , the special fiber is equivalent to Hovey's derived category of comodules over the Hopf algebroid BP_*BP .

Theorem 3.6 ([16, Theorem A.8]). The synthetic Adams spectral sequence for νX

$$^{\text{syn}}E_2^{s,t,w}(\nu X) \Longrightarrow \pi_{t-s,w}(\nu X)$$

has an E_2 -page

$$^{\mathrm{syn}}E_{2}^{*,*,*}(\nu X) \cong E_{2}^{*,*}(X) \otimes \mathbb{F}_{2}[\lambda]$$

with differentials

$$d_r: {}^{\operatorname{syn}}E^{s,t,w}_r \to {}^{\operatorname{syn}}E^{s+r,t+r-1,w}_r.$$

Here $E_2^{*,*}(X) \cong \operatorname{Ext}_A^{*,*}(X)$ is the classical Adams E_2 -page of X. An element in $E_2^{s,t}(X)$ is viewed as having tridegree (s,t,t) in $\sup_{x \in S_2^{s,t}(x)} E_2^{s,t,t}(\nu X)$, and λ is in tridegree (0,0,-1). Given a classical Adams differential $d_r^{\operatorname{cl}}(x) = y$, the corresponding synthetic differential is $d_r(x) = \lambda^{r-1}y$, which is λ -linear, and all synthetic Adams differentials arise in this way.

Remark 3.7. Here, we use a third grading of the synthetic Adams E_2 -page that differs from [16] but more compatible with [17]. Specifically, we view elements in $E_2^{s,t}(X)$ as having tridegree (s,t,t), instead of (s,t,s) as in [16]. This choice ensures that differentials and maps preserve the third degree, which proves convenient for the extension spectral sequence in the \mathbb{HF}_2 -synthetic category in Section 4.

Theorem 3.8 ([16, Theorem A.1]). The synthetic Adams spectral sequence for νX is isomorphic to the λ -Bockstein spectral sequence (with no sign difference because we are working over \mathbb{F}_2 here). More specifically, we have

- (1) $E_2^{s,t}(X) \cong \pi_{t-s,t}(\nu X/\lambda)$. (2) If we have a classical Adams differential $d_r x = y$ for

$$x \in E_2^{s,t}(X) \cong \pi_{t-s,t}(\nu X/\lambda),$$

then x admits a lift to $\pi_{t-s,t}(\nu X/\lambda^{r-1})$ whose image under the τ -Bockstein

$$\nu X/\lambda^{r-1} \to \Sigma^{1,-r+1} \nu X/\lambda$$

is equal to $d_r(x)$.

Remark 3.9. The isomorphism with the λ -Bockstein spectral sequence in Theorem 3.8 and the correspondence of differentials between the classical and synthetic Adams spectral sequences in Theorem 3.6 can be interpreted as manifestations of the rigidity of the synthetic Adams spectral sequence. This type of rigidity was first observed by Hu-Kriz-Ormsby [25] and Isaksen [26] in the context of the motivic Adams-Novikov spectral sequence, and it was utilized extensively in the work of Gheorghe-Wang-Xu [22]. Subsequently, Burklund-Xu [18] uncovered the rigidity of the motivic Cartan-Eilenberg spectral sequence and performed computations based on this property. See also [27], [13] for applications of this type of rigidity.

Together with Remark 3.5, $H\mathbb{F}_2$ -synthetic spectra can be seen as a categorification of the classical Adams spectral sequence, just as cellular motivic spectra can be viewed a categorification of the classical Adams-Novikov spectral sequence.

Notation 3.10. Consider the classical Adams spectral sequence of X. Let $B_r^{s,t}(X)$ denote the subgroups of $E_2^{s,t}(X)$ generated by the sum of images of Adams differentials d_2, \ldots, d_r . Let $Z_r^{s,t}(X)$ denote the subgroup of $E_2^{s,t}(X)$ consisting of elements on which d_2, \ldots, d_r vanish. For r=1, we define $Z_1^{s,t}(X)=E_2^{s,t}(X)$ and $B_1^{s,t}(X)=0$. For $r=\infty$, let $Z_\infty^{s,t}(X)$ be the intersection of all $Z_r^{s,t}(X)$, and $B_\infty^{s,t}(X)$ be the union of all $B_r^{s,t}(X)$. We then have the following inclusions

$$0 = B_1 \subset B_2 \subset B_3 \subset \cdots \subset B_\infty \subset Z_\infty \subset \cdots \subset Z_3 \subset Z_2 \subset Z_1 = E_2.$$

In this article, we interpret a classical Adams differential d_r as the map

$$d_r: Z_{r-1}^{s,t}(X) \to E_2^{s+r,t+r-1}(X)/B_{r-1}^{s+r,t+r-1}(X),$$

and follow the pattern in Notation 2.2 for writing differentials.

The following two propositions are taken from [16, Corollary A.9, A.11]. Note that our third grading, w, differs from that used in the reference.

Proposition 3.11. The E_{∞} -page of the synthetic Adams spectral sequence for νX is given by

$$E^{s,t,w}_{\infty}(\nu X) \cong \begin{cases} Z^{s,t}_{\infty}(X)/B^{s,t}_{1+t-w}(X) & \text{if } t \geq w, \\ 0 & \text{otherwise} \ . \end{cases}$$

Proposition 3.12. The E_{∞} -page of the synthetic Adams spectral sequence for $\nu X/\lambda^r$ for $r \geq 2$ is given by

$$E_{\infty}^{s,t,w}(\nu X/\lambda^r) \cong \begin{cases} Z_{r-t+w}^{s,t}(X)/B_{1+t-w}^{s,t}(X) & \text{if } 0 \leq t-w < r, \\ 0 & \text{otherwise.} \end{cases}$$

Notation 3.13. For convenience, whenever $n \leq 0$ or $m \leq 0$, we will define $Z_n(X)/B_m(X)$ as a trivial group. This is a bit counterintuitive to the definition of Z_n and B_m but it allows us to consistently express the following

$$E^{s,t,w}_{\infty}(\nu X)\cong Z^{s,t}_{\infty}(X)/B^{s,t}_{1+t-w}(X)$$

and

$$E^{s,t,w}_{\infty}(\nu X/\lambda^r) \cong Z^{s,t}_{r-t+w}(X)/B^{s,t}_{1+t-w}(X).$$

Remark 3.14. The isomorphisms in Propositions 3.11 and 3.12 are compatible with the synthetic maps $\lambda^{r'-r}: \Sigma^{-r'+r}\nu X/\lambda^r \to \nu X/\lambda^{r'}$ and $\rho: \nu X/\lambda^r \to \nu X/\lambda^{r''}$ for $r' \geq r \geq r''$. The induced map

$$\lambda^{r'-r}: E^{s,t,w}_{\infty}(\nu X/\lambda^r) \to E^{s,t,w-r'+r}_{\infty}(\nu X/\lambda^{r'})$$

corresponds to the quotient map (when all subscripts are positive):

$$\lambda^{r'-r}: Z^{s,t}_{r-t+w}(X)/B^{s,t}_{1+t-w}(X) \to Z^{s,t}_{r-t+w}(X)/B^{s,t}_{1+t-w+r'-r}(X)$$

Similarly, the induced map

$$\rho: E^{s,t,w}_{\infty}(\nu X/\lambda^r) \to E^{s,t,w}_{\infty}(\nu X/\lambda^{r^{\prime\prime}})$$

corresponds to the embedding map (when all subscripts are positive):

$$\rho: Z_{r-t+w}^{s,t}(X)/B_{1+t-w}^{s,t}(X) \to Z_{r''-t+w}^{s,t}(X)/B_{1+t-w}^{s,t}(X).$$

These data do not extend beyond classical information. However, when we consider a map $f: X \to Y$, we can obtain interesting extensions, as will be discussed in Section 4.

Example 3.15. Consider the first few nonzero differentials in the classical Adams spectral sequence from the 15-stem to the 14-stem:

$$d_2(h_4) = h_0 h_3^2$$
, $d_3(h_0 h_4) = h_0 d_0$, $d_3(h_0^2 h_4) = h_0^2 d_0$,

where

$$h_0 \in \operatorname{Ext}_A^{1,1}, \ h_3 \in \operatorname{Ext}_A^{1,8}, \ h_4 \in \operatorname{Ext}_A^{1,16}, \ d_0 \in \operatorname{Ext}_A^{4,18}.$$

The E_2 -page of the synthetic Adams spectral sequence for $S^{0,0}$ has the form

$$^{\mathrm{syn}}E_2^{*,*,*}(S^{0,0}) \cong \mathrm{Ext}_A^{*,*} \otimes \mathbb{F}_2[\lambda],$$

and we have

$$h_0 \in {}^{\mathrm{syn}}E_2^{1,1,1}, \ h_3 \in {}^{\mathrm{syn}}E_2^{1,8,8}, \ h_4 \in {}^{\mathrm{syn}}E_2^{1,16,16}, \ d_0 \in {}^{\mathrm{syn}}E_2^{4,18,18}.$$

The corresponding nonzero synthetic Adams differentials for $S^{0,0}$ are

$$d_2(h_4) = \lambda h_0 h_3^2$$
, $d_3(h_0 h_4) = \lambda^2 h_0 d_0$, $d_3(h_0^2 h_4) = \lambda^2 h_0^2 d_0$,

along with their λ -multiples. As a result, the 14-stem of the synthetic E_{∞} -page is generated by λ -free elements h_3^2 and d_0 , along with the following λ -torsion elements:

$$h_0h_3^2$$
, h_0d_0 , λh_0d_0 , $h_0^2d_0$, $\lambda h_0^2d_0$.

On the other hand, since $h_0h_3^2 \in \operatorname{Ext}_A^{3,17}$ is a d_2 -cycle, we have

$$0 = B_1^{3,17} \subset B_2^{3,17} = Z_{\infty}^{3,17} = E_2^{3,17} \cong \mathbb{F}_2\{h_0 h_3^2\},\,$$

which matches the description:

$$E_{\infty}^{3,17,w}(S^{0,0}) \cong \begin{cases} Z_{\infty}^{3,17}/B_{18-w}^{3,17} \cong \mathbb{F}_2\{h_0h_3^2\} & \text{if } w = 17, \\ Z_{\infty}^{3,17}/B_{18-w}^{3,17} = 0 & \text{if } w \leq 16, \\ 0 & \text{if } w \geq 18. \end{cases}$$

The element $h_0h_3^2$ detects homotopy classes of AF = 3 in $\pi_{14,17}$. Due to the presence of the λ -free element d_0 in the synthetic E_{∞} -page, the element λd_0 also detects homotopy classes in $\pi_{14,17}$, but with AF = 4. This illustrates why it is more convenient to describe the E_{∞} -page as in Proposition 3.11, rather than in terms of synthetic homotopy groups.

As a comparison, the only nonzero synthetic Adams differential in the 14-stem for $S^{0,0}/\lambda^2$ is

$$d_2(h_4) = \lambda h_0 h_3^2,$$

and the elements λh_4 , $h_0 h_4$, $h_0^2 h_4$ are all permanent cycles.

For $S^{0,0}/\lambda^3$, the nonzero synthetic Adams differentials in the 14-stem are

$$d_2(h_4) = \lambda h_0 h_3^2, \ d_2(\lambda h_4) = \lambda^2 h_0 h_3^2,$$

$$d_3(h_0 h_4) = \lambda^2 h_0 d_0, \ d_3(h_0^2 h_4) = \lambda^2 h_0^2 d_0,$$

and the elements $\lambda^2 h_4$, $\lambda h_0 h_4$, $\lambda^2 h_0 h_4$, $\lambda h_0^2 h_4$, $\lambda^2 h_0^2 h_4$ are all permanent cycles. These results can be compared with the statements in Proposition 3.12.

Proposition 3.16 ([16, Lemma 9.15]). If a map $f: X \to Y$ has Adams filtration AF(f) = k, then there exists a factorization

$$\begin{array}{ccc}
& \Sigma^{0,-k}\nu Y \\
& \Sigma^{0,-k}\tilde{f} & \nearrow & \downarrow \lambda^{k} \\
\nu X & & \nu Y
\end{array}$$

where $\tilde{f}: \Sigma^{0,k} \nu X \to \nu Y$ is called a synthetic lift of f.

In Proposition 3.16 the map νf can be also factorized as follows:

$$\begin{array}{c|c}
\nu X & \xrightarrow{\nu f} \nu Y \\
\downarrow^{\lambda^k} & & \uparrow \\
\Sigma^{0,k} \nu X
\end{array}$$

Proposition 3.17. Suppose that we have a distinguished triangle of spectra

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

with AF(h) > 0, and consequently a short exact sequence on $H\mathbb{F}_2$ -homology

$$0 \to H_*X \xrightarrow{H_*f} H_*Y \xrightarrow{H_*g} H_*Z \to 0.$$

Then there exists a distinguished triangle of synthetic spectra

$$\nu X \xrightarrow{\nu f} \nu Y \xrightarrow{\nu g} \nu Z \xrightarrow{\Sigma^{0,-1} \hat{h}} \Sigma^{0,-1} \nu \Sigma X = \Sigma^{1,0} \nu X$$

such that $\nu h = \lambda \hat{h}$.

Proof. The proof is contained in the proof of [16, Lemma 9.15].

Remark 3.18. For h in Proposition 3.17, let \tilde{h} be a synthetic map such that $\nu h = \lambda^k \tilde{h}$, then \hat{h} is equal to $\lambda^{k-1} \tilde{h}$ up to some λ -torsion.

Notation 3.19. For a map $f: X \to Y$ which is part of a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Cf \to \Sigma X$$

we define

$$e(f) = \begin{cases} 0 & \text{if } AF(f) = 0, \\ 1 & \text{if } AF(f) > 0. \end{cases}$$

When AF(f) = 0, we also denote νf by \hat{f} . In both cases, we have

$$\hat{f}: \Sigma^{0,e(f)} \nu X \to \nu Y$$

and $\nu f = \lambda^{e(f)} \hat{f}$. Further more, $C\hat{f} \simeq \Sigma^{0,-e(g)} \nu Cf$ or equivalently,

- if f induces a trivial map or an injection on $H\mathbb{F}_2$ -homology, then $C\hat{f} \simeq \nu Cf$;
- if f induces a surjection on HF₂-homology, then $C\hat{f} \simeq \Sigma^{0,-1} \nu C f$.

With the notation above, we can rewrite Proposition 3.17 into the following form which will be used in Theorem 6.12 and help us unify different cases.

Proposition 3.20. Suppose that we have a distinguished triangle of spectra

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

such that e(f) + e(g) + e(h) = 1. Then we have a distinguished triangle of synthetic spectra

$$\nu X \xrightarrow{\hat{f}} \Sigma^{0,-e(f)} \nu Y \xrightarrow{\hat{g}} \Sigma^{0,-e(f)-e(g)} \nu Z \xrightarrow{\hat{h}} \Sigma^{0,-1} \nu \Sigma X$$

(For simplicity we often omit the suspensions before maps but keep the suspensions before spectra.)

Remark 3.21. When AF(f) > 1, it is often more advantageous to work with $C\tilde{f}$ rather than $C\hat{f}$, as the λ -Bockstein spectral sequence for $C\tilde{f}$ corresponds to a modified Adams spectral sequence, which differs from the classical one but may provide additional information. On the other hand, given the equivalence $C\hat{f} \simeq \nu Cf$ from Notation 3.19, the rigidity Theorems 3.6 and 3.8 imply that the synthetic Adams spectral sequence for $C\hat{f}$ is isomorphic to the λ -Bockstein spectral sequence, and aligns with the classical Adams spectral sequence for Cf. Since the computational data used in this work is based on the classical Adams spectral sequence, we defer the exploration of $C\tilde{f}$ to future work.

To illustrate this in more detail, consider the example of the synthetic lift $\tilde{\eta}^3 \in \pi_{3,6}$ of $\eta^3 \in \pi_3$, which has AF = 3. We can examine several λ -multiples of $\tilde{\eta}^3$:

$$\lambda \tilde{\eta^3} \in \pi_{3,5}, \quad \hat{\eta^3} = \lambda^2 \tilde{\eta^3} \in \pi_{3,4}, \quad \nu(\eta^3) = \lambda^3 \tilde{\eta^3} \in \pi_{3,3}.$$

Now, consider the synthetic 2-cell complexes formed by taking the cofibers of these maps:

$$C(\tilde{\eta^3}), \ C(\lambda \tilde{\eta^3}), \ C(\lambda^2 \tilde{\eta^3}) \simeq C(\hat{\eta^3}) \simeq \nu(C\eta^3), \ C(\lambda^3 \tilde{\eta^3}) \simeq C\nu(\eta^3).$$

The rigidity Theorems 3.6 and 3.8 apply only to $C(\hat{\eta}^3)$, as it is equivalent to $\nu(C\eta^3)$. In this case, its synthetic Adams E_2 -page is λ -free over $\operatorname{Ext}_A^{*,*}(C\eta^3)$; in particular the identity element from the top cell has tridegree (s,t,w)=(0,4,4). There is a correspondence of differentials between the classical and synthetic Adams spectral sequence, and a synthetic differential generates λ -torsion corresponding to its length.

For $C(\tilde{\eta}^3)$, $C(\lambda \tilde{\eta}^3)$ and $C(\lambda^3 \tilde{\eta}^3)$, their synthetic Adams E_2 -pages remain λ -free, but the identity element from the top cell has tridegree

$$(s, t, w) = (0, 4, 6), (0, 4, 5), (0, 4, 7)$$

respectively. As a result, each supports a nonzero Adams d_3 -differential killing h_1^3 from the bottom cell when multiplied by $1, \lambda, \lambda^3$, respectively. This leaves no λ -torsion, λ^1 -torsion and λ^3 -torsion in $\pi_{4,6}$, respectively.

On the other hand, the λ -Bockstein spectral sequences for $C(\tilde{\eta}^3)$ and $C(\lambda \tilde{\eta}^3)$ correspond to a modified Adams spectral sequence in the sense of [7, Section 3], with the AF of η^3 "modified" to AF = 1 and AF = 2, respectively. In particular, the E_2 -page of the λ -Bockstein spectral sequence for $C(\tilde{\eta}^3)$, i.e., $\pi_{*,*}C(\tilde{\eta}^3)/\lambda$, may contain more multiplicative information than $\operatorname{Ext}_4^{**}(C\eta^3)$.

4. Synthetic Extensions

The extension spectral sequence introduced in Section 2 can be generalized for $H\mathbb{F}_2$ -synthetic spectra. Consider a map $f:X\to Y$ between two connective finite $H\mathbb{F}_2$ -synthetic spectra¹. By filtering

$$0 \to \pi_{*,*} X \xrightarrow{\pi_{*,*} f} \pi_{*,*} Y \to 0$$

using the Adams filtration, we obtain an f-extension spectral sequence:

$${}^fE_0^{s,t,w} \cong E_\infty^{s,t,w}(X) \oplus E_\infty^{s,t,w}(Y) \Longrightarrow \ker(\pi_{*,*}f) \oplus \operatorname{coker}(\pi_{*,*}f)$$

¹In our solution to the Kervaire invariant problem, we only consider finite synthetic spectra. However, the arguments in this section apply to any synthetic spectra for which the synthetic Adams spectral sequence strongly converges.

with differential

$$d_n^f: {}^f\!Z^{s,t,w}_{n-1}(X) \to E^{s+n,t+n,w}_{\infty}(Y)/{}^f\!B^{s+n,t+n,w}_{n-1}(Y)$$

where ${}^fZ_*^{*,*,*}$ and ${}^fB_*^{*,*,*}$ are analogous to the notations in Notation 2.2, with the addition of an extra degree.

In fact all the definitions and results in Section 2 remain valid in this synthetic setting, provided we include the extra weight degree in addition to the topological degree and ensure that all maps also preserve the weight degree. For convenience, we will not restate these results here but will occasionally refer to them and apply their synthetic versions when needed.

Notation 4.1. There are certain special maps between synthetic spectra that we want to consider. For any $n < m \le \infty$ and a spectrum X, we have the following distinguished triangles of HF_2 -synthetic spectra

$$\Sigma^{0,-n}\nu X/\lambda^{m-n}\xrightarrow{\lambda^n}\nu X/\lambda^m\xrightarrow{\rho_{n,m}}\nu X/\lambda^n\xrightarrow{\delta_{n,m}}\Sigma^{1,-n}\nu X/\lambda^{m-n}.$$

We simply write $\rho = \rho_{n,m}$, $\delta = \delta_{n,m}$ by abuse of notation if n,m is understood in the context. When $m = \infty$, this sequence is interpreted as

$$\Sigma^{0,-n}\nu X \xrightarrow{\lambda^n} \nu X \xrightarrow{\rho} \nu X/\lambda^n \xrightarrow{\delta} \Sigma^{1,-n}\nu X.$$

Many arguments for finite m implies the corresponding statement when $m=\infty$ because of

$$\nu X = \varprojlim_m \nu X / \lambda^m$$

as shown in [16, Proposition A.13].

We also note that our grading for the triangulation translation functor is smashing with $S^{1,0}$, which is consistent with [16, Appendix A] but is different from [18, Section 7].

Proposition 4.2. The only nonzero differentials in the extension spectral sequences for the maps λ^n and ρ from Notation 4.1 are the d_0 's:

$$d_0^{\lambda^n} = \lambda^n \text{ and } d_0^{\rho} = \rho.$$

As a result, these d_0 's have no crossings.

Proof. First, we show that the following sequence is exact in the middle:

$$(4.3) E_{\infty}^{s,t,w}(\Sigma^{0,-n}\nu X/\lambda^{m-n}) \xrightarrow{\lambda^n} E_{\infty}^{s,t,w}(\nu X/\lambda^m) \xrightarrow{\rho} E_{\infty}^{s,t,w}(\nu X/\lambda^n).$$

We apply Proposition 3.11 and prove this case by case.

When t-w<0 or $t-w\geq m$, the middle group $E^{s,t,w}_{\infty}(\nu X/\lambda^m)=0$, so the sequence is exact in the middle.

When $0 \le t - w < n$, the sequence is isomorphic to

$$0 \to Z^{s,t}_{m-t+w}(X)/B^{s,t}_{1+t-w}(X) \to Z^{s,t}_{n-t+w}(X)/B^{s,t}_{1+t-w}(X)$$

which is exact in the middle because the second map is clearly injective.

When $n \leq t - w < m$, the sequence is isomorphic to

$$Z^{s,t}_{m-t+w}(X)/B^{s,t}_{1+t-w-n}(X) \to Z^{s,t}_{m-t+w}(X)/B^{s,t}_{1+t-w}(X) \to 0$$

which is exact in the middle because the first map is clearly surjective.

Thus, we have shown that (4.3) is always exact in the middle. Combined with Corollary 2.19 we conclude that all $d_i^{\lambda^n}$, d_i^{ρ} are trivial for i > 0.

Remark 4.4. From the proof above, we see that in (4.3), λ^n is surjective or trivial, while ρ is injective or trivial.

However, the δ -extension spectral sequence for

$$\delta: \nu X/\lambda^n \to \Sigma^{1,-n} \nu X/\lambda^{m-n}$$

is more complicated, as it encodes classical Adams differentials d_2 through d_m .

Remark 4.5. For the convenience of readers to check gradings, whenever we write

$$d_n^f(x) = \lambda^k y$$

for $f: \Sigma^{m,w}\nu X \to \nu Y$, $x \in E_{\infty}^{s_1,t_1,w_1}(\nu X)$ and $y \in E_{\infty}^{s_2,t_2,w_2}(\nu Y)$, the following conditions must hold:

$$s_2 = s_1 + n$$
, $t_2 - s_2 = t_1 - s_1 + m$, and $w_2 = w_1 + w$.

Proposition 4.6. Suppose in the classical Adams spectral sequence of X we have $d_r(x) = y$, where $x \in Z_{r-1}^{s,t}(X)$ and $y \in Z_{\infty}^{s+r,t+r-1}(X)$. Consider the map

$$\delta: \nu X/\lambda^n \to \Sigma^{1,-n} \nu X/\lambda^{m-n}$$
.

(1) If $r \ge n+1$, then we view x as an element of

$$E^{s,t,t}_{\infty}(\nu X/\lambda^n) \cong Z^{s,t}_n(X),$$

and $\lambda^{r-n-1}y$ as an element of

$$E_{\infty}^{s+r,t+r-1,t+n}(\nu X/\lambda^{m-n}) \cong Z_{m-r+1}^{s+r,t+r-1}(X)/B_{r-n}^{s+r,t+r-1}(X).$$

We then have

$$d_r^{\delta}(x) = \lambda^{r-n-1} y,$$

which is trivial if r > m.

(2) If r < n+1, then we view $\lambda^{n+1-r}x$ as an element of

$$E^{s,t,t-n-1+r}_{\infty}(\nu X/\lambda^n) \cong Z^{s,t}_{r-1}(X)/B^{s,t}_{n+2-r}(X),$$

and y as an element of

$$E^{s+r,t+r-1,t+r-1}_{\infty}(\nu X) \cong Z^{s+r,t+r-1}_{\infty}(X).$$

In this case, we have

$$d_r^{\delta}(\lambda^{n+1-r}x) = y.$$

Proof. Since $\delta = \delta_{n,m}$ is the composition of ρ and $\delta_{n,\infty}$ as the following

$$\begin{array}{c}
\nu X/\lambda^n \\
\delta_{n,\infty} \downarrow \\
\Sigma^{1,-n} \nu X \xrightarrow{\rho} \Sigma^{1,-n} \nu X/\lambda^{m-n}
\end{array}$$

it suffices to prove the case when $m = \infty$ by Corollary 2.17. In the rest of the proof we will write $\delta_n = \delta_{n,\infty}$.

First, we prove by induction on n that

$$(4.7) d_r^{\delta_n}(x) \equiv \lambda^{r-n-1} y \mod B_{r-1}^{s+r,t+r-1}(X)$$

when $r \ge n + 1$, and

$$(4.8) d_r^{\delta_n}(\lambda^{n+1-r}x) \equiv y \mod B_{r-1}^{s+r,t+r-1}(X)$$

when r < n + 1. (These two expressions coincide when r = n + 1.) For n = 1, the claim holds since the τ -Bockstein spectral sequence is isomorphic to the classical Adams spectral sequence. Now, assume $n \geq 2$ and the claim holds for n-1.

Consider the following commutative diagram.

$$\Sigma^{0,-1}\nu X/\lambda^{n-1} \xrightarrow{\lambda} \nu X/\lambda^n$$

$$\downarrow^{\delta_n}$$

$$\Sigma^{1,-n}\nu X$$

By Corollary 2.16, if $r \leq n$, we have

$$d_r^{\delta_n}(\lambda^{n+1-r}x) = d_r^{\delta_{n-1}}(\lambda^{n-r}x) \equiv y \mod B_{r-1}^{s+r,t+r-1}(X).$$

If $r \ge n+1$, x can be viewed as an element of the E_{∞} -pages of $\nu X/\lambda^{n-1}$ or $\nu X/\lambda^n$. We then have

$$\lambda d_r^{\delta_n}(x) = d_r^{\delta_n}(\lambda x) = d_r^{\delta_{n-1}}(x) \equiv \lambda^{r-n} y \mod B_{r-1}^{s+r,t+r-1}(X)$$

which implies

$$d_r^{\delta_n}(x) \equiv \lambda^{r-n-1} y \mod B_{r-1}^{s+r,t+r-1}(X),$$

since $r-n+1 \le r-1$ and hence the indeterminacy B_{r-n+1} introduced by dividing

 λ is contained in B_{r-1} . The induction for (4.7) and (4.8) is now complete. We will show that $B_{r-1}^{s+r,t+r-1}(X)$ in both equations are actually equal to the sum of images of $d_0^{\delta_n}$ through $d_{r-1}^{\delta_n}$. Consider any $y' \in B_{r-1}^{s+r,t+r-1}(X)$ and assume that $d_{r'}x' = y'$ is an essential classical Adams differential for $2 \le r' \le r-1$. If $r \ge n + 1$, using (4.7), we have

$$d_{r'}^{\delta_n}(\lambda^{r-r'}x) \equiv \lambda^{r-n-1}y' \mod B_{r'-1}^{s+r,t+r-1}(X),$$

and if r < r' + 1, using (4.8), we have

$$d_{r'}^{\delta_n}(\lambda^{n+1-r'}x') \equiv y' \mod B_{r'-1}^{s+r,t+r-1}(X).$$

Notice that the extra indeterminacy here is $B_{r'-1}$ instead of B_{r-1} . By induction on r, this shows that $B_{r-1}^{s+r,t+r-1}(X)$ equals the sum of images of $d_0^{\delta_n}$ through $d_{r-1}^{\delta_n}$. Therefore, we can omit $B_{r-1}^{s+r,t+r-1}(X)$ and simply write

$$d_r^{\delta_n}(x) = \lambda^{r-n-1} y$$

when $r \geq n + 1$, and

$$d_r^{\delta_n}(\lambda^{n+1-r}x) = y$$

when $r \leq n + 1$.

Remark 4.9. The $d_r^{\delta}(x)$ we calculated in Proposition 4.6 may be inessential.

Corollary 4.10. For x, y, δ in Proposition 4.6 we always have

$$(4.11) d_r^{\delta}(\lambda^a x) = \lambda^{a+r-n-1} y$$

if $0 \le a \le n$ and $0 \le a + r - n - 1 < m - n$ (the differential is trivial if a exceeds this range).

Remark 4.12. As indicated in the proof, the right-hand side of equation (4.11) (considered as a subset of $Z_{\infty}^{s+r,t+r-1}(X)$) is a coset of

$$B_{r-1}^{s+r,t+r-1}(X)$$

which is the same as the value of the classical Adams differential $d_r(x) = y$. This implies that the equation (4.11) holds if and only if $d_r(x) = y$. Therefore the δ -ESS encodes the same information as the classical Adams spectral sequence.

Example 4.13. We continue the discussion from Example 3.15 regarding the implications of the classical Adams differentials in the 14-stem:

$$d_2(h_4) = h_0 h_3^2$$
, $d_3(h_0 h_4) = h_0 d_0$.

The reader is advised to begin by identifying the elements on the E_{∞} -pages of the synthetic Adams spectral sequences for $S^{0,0}$ and $S^{0,0}/\lambda^k$ for k=1,2,3 in the relevant tridegrees, as illustrated in Example 3.15.

For $\delta_1: S^{0,0}/\lambda \to S^{1,-1}$, we have

$$d_2^{\delta_1}(h_4) = h_0 h_3^2, \ d_3^{\delta_1}(h_0 h_4) = \lambda h_0 d_0.$$

For $\delta_2: S^{0,0}/\lambda^2 \to S^{1,-2}$, we have

$$d_2^{\delta_2}(\lambda h_4) = h_0 h_3^2, \ d_3^{\delta_2}(h_0 h_4) = h_0 d_0, \ d_3^{\delta_2}(\lambda h_0 h_4) = \lambda h_0 d_0$$

For $\delta_3: S^{0,0}/\lambda^3 \to S^{1,-3}$, we have

$$d_2^{\delta_3}(\lambda^2 h_4) = h_0 h_3^2, \ d_3^{\delta_3}(\lambda h_0 h_4) = h_0 d_0, \ d_3^{\delta_3}(\lambda^2 h_0 h_4) = \lambda h_0 d_0.$$

Definition 4.14. Suppose $r \ge n+1$ and $d_r(x) = y$, where

$$x \in E_2^{s,t}(X), \quad y \in E_2^{s+r,t+r-1}(X).$$

A crossing of $d_r(x) = y$ on the E_{n+1} -page refers to an essential Adams differential

$$d_{r-a-b}(x') = y',$$

where

$$x' \in E_2^{s+a,t+a}(X), \quad y' \in E_2^{s+r-b,t+r-b-1}(X),$$

with $0 < a \le n-1$ and $0 \le b \le r-n-1$. See Figure 3.

Remark 4.15. The crossing defined here is opposite to crossings in Moss's theorem.

The emphasis on a crossing occurring "on the E_{n+1} -page" in Definition 4.14 may seem counter-intuitive. However, this is clarified in Proposition 4.16 and Example 4.18 that follow.

Proposition 4.16. The Adams differential $d_r(x) = y$ has a crossing on the E_{n+1} page if and only if the corresponding δ_n -extension

$$d_r^{\delta_n}(x) = \lambda^{r-n-1} y$$

for

$$\delta_n: \nu X/\lambda^n \to \Sigma^{1,-n} \nu X$$

has a crossing.

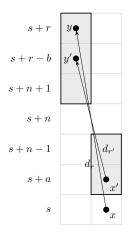


Figure 3. A crossing of $d_r(x) = y$ on the E_{n+1} -page

Proof. By Propositions 3.11, 3.12 and 4.6, a crossing of $d_r^{\delta_n}(x) = \lambda^{r-n-1}y$ takes the form

$$d_{r-a-b}^{\delta_n}(\lambda^a x') = \lambda^{r-n-1-b} y',$$

where $0 < a \le n - 1, 0 \le b \le r - n - 1,$

$$\lambda^a x' \in E^{s+a,t+a,t}_{\infty}(\nu X/\lambda^n) \cong Z^{s+a,t+a}_{n-a}(X)/B^{s+a,t+a}_{1+a}(X),$$

and

$$y' \in E_{\infty}^{s+r-b,t+r-b-1}(\nu X).$$

By Corollary 4.10, we see that this crossing corresponds to the classical Adams differential

$$d_{r-a-b}(x') = y'.$$

Remark 4.17. From Definition 4.14, it immediately follows that there are no crossings of any differential on the E_2 -page, as this would require $0 < a \le n-1 = 0$. According to Proposition 4.16, this reflects the fact that δ_1 -extensions have no crossings for degree reasons.

Example 4.18. We examine the following classical Adams differentials from stem 38 to stem 37:

$$d_3(e_1) = h_1 t$$
, $d_4(h_0 h_3 h_5) = h_0^2 x$,

where both targets, h_1t and h_0^2x , reside within the same (s,t)-bidegree in $\operatorname{Ext}_A^{7,44}$.

According to Definition 4.14, this nonzero d_3 -differential is a crossing of the d_4 -differential on both the E_3 -page and the E_4 -page, but not on the E_2 -page or any E_r -page for $r \geq 5$.

This corresponds to the following facts:

(1) For the map $\delta_2: S^{0,0}/\lambda^2 \to S^{1,-2}$, the differential

$$d_3^{\delta_2}(\lambda e_1) = \lambda h_1 t$$

is a crossing for the differential

$$d_4^{\delta_2}(h_0h_3h_5) = \lambda h_0^2 x.$$

(2) For the map $\delta_3: S^{0,0}/\lambda^3 \to S^{1,-3}$, the differential

$$d_3^{\delta_3}(\lambda e_1) = h_1 t$$

is a crossing for the differential

$$d_4^{\delta_3}(h_0h_3h_5) = h_0^2x.$$

In both cases (1) and (2), the target elements have the same weight.

(3) In comparison, for the map $\delta_1: S^{0,0}/\lambda \to S^{1,-1}$, we have the differential

$$d_4^{\delta_1}(h_0h_3h_5) = \lambda^2 h_0^2 x,$$

which does not have any crossing differential. The only differential whose target resides in the same (s, t)-bidegree is

$$d_3^{\delta_1}(e_1) = \lambda h_1 t,$$

but its target has a different weight.

(4) For later pages, specifically for $r \geq 4$, and considering the synthetic spectrum $S^{0,0}/\lambda^r$, there is still a crossing for the differential

$$d_4^{\delta_r}(\lambda^{r-3}h_0h_3h_5) = h_0^2x.$$

However, the element $h_0h_3h_5$ itself does not survive to the synthetic Adams E_{∞} -page, as it supports a nonzero Adams differential:

$$d_4(h_0h_3h_5) = \lambda^3 h_0^2 x.$$

5. Extensions on a classical E_r -page

To state the Generalized Leibniz Rule in terms of the classical Adams spectral sequence, we need to define f-extensions not only on homotopy groups but on the E_r -page as well.

Notation 5.1. For a map between classical spectra $f: X \to Y$, consider the associated synthetic map from Notation 3.19

$$\hat{f}: \Sigma^{0,e(f)} \nu X \to \nu Y$$
.

For any $2 \le r \le \infty$, we denote the following mod λ^{r-1} reduction maps by \hat{f}_{r-1} :

$$\hat{f}_{r-1}: \Sigma^{0,e(f)} \nu X/\lambda^{r-1} \to \nu Y/\lambda^{r-1}$$

The E_0 -page of the \hat{f}_{r-1} -ESS is isomorphic to

$$\begin{split} \hat{f}_{r-1} E_0^{s,t,t-k+e(f)} &\cong & E_{\infty}^{s,t,t-k}(\nu X/\lambda^{r-1}) \oplus E_{\infty}^{s,t,t-k+e(f)}(\nu Y/\lambda^{r-1}) \\ &\cong & \left(Z_{r-1-k}^{s,t}(X)/B_{1+k}^{s,t}(X) \right) \oplus \left(Z_{r-1-k+e(f)}^{s,t}(Y)/B_{1+k-e(f)}^{s,t}(Y) \right). \end{split}$$

A nontrivial $d_n^{\hat{f}_{r-1}}$ differential can be interpreted as a map from the subgroup

(5.2)
$$\hat{f}_{r-1}Z_{n-1}^{s,t,t-k}(X) \subset Z_{r-1-k}^{s,t}(X)/B_{1+k}^{s,t}(X)$$

to the quotient group

$$(5.3) \qquad (Z_{r-1-k-n+e(f)}^{s+n,t+n}(Y)/B_{1+k+n-e(f)}^{s+n,t+n}(Y))/\hat{f}_{r-1}B_{n-1}^{s+n,t+n,t-k}(Y).$$

The differential $d_n^{\hat{f}_{r-1}}$ is trivial for degree reasons when

$$n < AF(f) \text{ or } n > r - 2 - k + e(f).$$

Definition 5.4. Let $x \in Z^{s,t}_{r-1}(X)$ and $y \in Z^{s+n,t+n}_{r-1-n+e(f)}(Y)$ for some

$$e(f) \le n \le r - 2 + e(f).$$

We say that there is an (f, E_r) -extension from x to y, denoted by

$$d_n^{f,E_r}(x) = y$$

if there exists a synthetic \hat{f}_{r-1} -extension

(5.6)
$$d_n^{\hat{f}_{r-1}}(x) = \lambda^{n-e(f)}y.$$

where x is viewed as an element of the subgroup (5.2) with k=0, and $\lambda^{n-e(f)}y$ is viewed as an element of the quotient group (5.3) with k = 0.

We say that this (f, E_r) -extension in (5.5) is essential if the corresponding synthetic \hat{f}_{r-1} -extension in (5.6) is an essential differential in the \hat{f}_{r-1} -ESS.

For $r = \infty$, we similar define an (f, E_{∞}) -extension using the corresponding synthetic \hat{f} -extension.

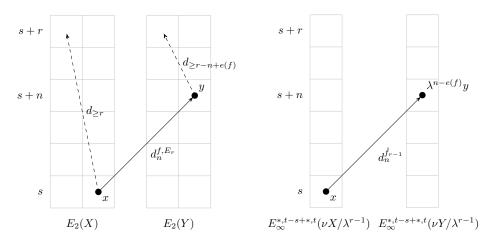


FIGURE 4. (f, E_r) -extension

Remark 5.7. Consider an (f, E_r) -extension $d_n^{f, E_r}(x) = y$ in (5.5). The element y should be interpreted as a coset with indeterminacy given by $B_{1+n-e(f)}^{s+n,t+n}(Y)$, plus the sum of images of $d_{\leq n}^{\hat{f}_{r-1}}$ differentials. This (f, E_r) -extension is essential if this coset does not contain 0.

Example 5.8. We present the following examples of (f, E_r) -extensions.

(1) Consider $f = \nu : S^3 \to S^0$. We have an (f, E_2) -extension:

$$d_1^{f,E_2}(h_5) = h_2 h_5,$$

where $h_5 \in \operatorname{Ext}_A^{1,32}$ and $h_2h_5 \in \operatorname{Ext}_A^{2,36}$. In fact, the classical homotopy class $\nu \in \pi_3$ is detected by h_2 with AF = 1 in Ext, and this (f, E_2) -extension simply states that, in Ext, the product of h_5 by h_2 is h_2h_5 .

(2) For the same map f as in (1), we have an (f, E_3) -extension:

$$d_2^{f,E_3}(h_0h_4^2) = h_0p,$$

where $h_0 h_4^2 \in \operatorname{Ext}_A^{3,33}$ and $h_0 p \in \operatorname{Ext}_A^{5,38}$. By definition, $\hat{f} = [h_2]$, detected by h_2 in tridegree (s, t, w) = (3, 4, 4)of the synthetic Adams E_2 -page. This (f, E_3) -extension states that for the synthetic map

$$\hat{f}_2 = [h_2]/\lambda^2 : S^{3,4}/\lambda^2 \to S^{0,0}/\lambda^2,$$

there exists a synthetic \hat{f}_2 -extension

$$d_2^{\hat{f}_2}(h_0 h_4^2) = \lambda h_0 p.$$

This is equivalent to the existence of a synthetic homotopy class $[h_0h_4^2]$ in $\pi_{30+3,33+4}S^{3,4}/\lambda^2$, detected by $h_0h_4^2$ on the synthetic Adams E_{∞} -page, that maps to the unique class $[\lambda h_0 p]$ in $\pi_{33,37} S^{0,0}/\lambda^2$, detected by $\lambda h_0 p$ on the synthetic Adams E_{∞} -page. In other words, we have

$$[h_0 h_4^2] \cdot [h_2] = [\lambda h_0 p] \text{ in } \pi_{33,37} S^{0,0} / \lambda^2.$$

We will justify this (f, E_3) -extension later on in Example 6.19 using the Generalized Mahowald Trick, Theorem 6.12.

(3) For the same map f as in (1), we also have an (f, E_{∞}) -extension:

$$d_2^{f,E_{\infty}}(h_0h_4^2) = h_0p.$$

This is equivalent of saying that the relation in $\pi_{33,37}S^{0,0}/\lambda^2$ from (2), can be lifted to a relation

$$[h_0 h_4^2] \cdot [h_2] = [\lambda h_0 p] \text{ in } \pi_{33,37} S^{0,0},$$

for some classes $[h_0h_4^2]$ in $\pi_{30+3,33+4}S^{3,4}$ and $[\lambda h_0p]$ in $\pi_{33,37}S^{0,0}$.

We will justify this (f, E_{∞}) -extension later on in Example 6.24 using Corollary 6.23, which allows us to stretch extensions across pages.

(4) Consider $f = 2: S^0 \to S^0$. We have an (f, E_∞) -extension:

$$d_2^{f,E_{\infty}}(h_0h_3^2) = 0.$$

By definition, $\hat{f} = [h_0]$, detected by h_0 in tridegree (s, t, w) = (1, 1, 1) of the synthetic Adams E_2 -page. We remark that there is a relation $\lambda \cdot [h_0] =$ 2 in $\pi_{0,0}S^{0,0}$. For the synthetic map $[h_0]: S^{0,1} \to S^{0,0}$, this (f, E_{∞}) extension is equivalent to the existence of a synthetic homotopy class $[h_0h_3^2]$ in $\pi_{14,17}S^{0,0}$, such that

$$[h_0 h_3^2] \cdot [h_0] = 0 \text{ in } \pi_{14,18} S^{0,0}.$$

In fact, by inspection, we learn that $\pi_{14.18}S^{0.0} \cong \mathbb{Z}/2$, generated by $[\lambda h_0 d_0]$, which is an $[h_0]$ -multiple of a class $[\lambda d_0]$ in a higher Adams filtration than $h_0h_3^2$. Therefore, there exists a class $[h_0h_3^2]$ whose $[h_0]$ -multiple is zero.

In other words, we have an essential (f, E_{∞}) -extension:

$$d_1^{f, E_{\infty}}(d_0) = h_0 d_0,$$

which makes the following alternative (f, E_{∞}) -extension inessential:

$$d_2^{f,E_{\infty}}(h_0h_3^2) = h_0d_0.$$

Definition 5.9. A crossing of the (f, E_r) -extension $d_n^{f, E_r}(x) = y$ in (5.5) is defined as an essential (f, E_{r-a}) -extension from some $x' \in Z_{r-1-a}^{s+a,t+a}(X)$ to

$$y' \in Z^{s+n-b,t+n-b}_{r-1-n+b+e(f)}(Y) \backslash B^{s+n-b,t+n-b}_{1+n-b-e(f)}(Y)$$

(where \ denotes the difference of sets and it means that y' should survive to the classical Adams $E_{r-n+b+e(f)}$ page while it should not be hit by an Adams differential of length at most 1+n-b-e(f)) for $0 < a \le r-2$ and $0 \le b \le n-a-e(f)$. See Figure 5.

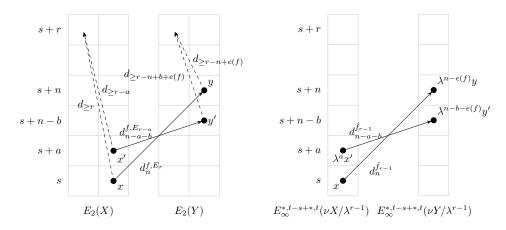


FIGURE 5. A crossing of an (f, E_r) -extension

Proposition 5.10. An (f, E_r) -extension $d_n^{f, E_r}(x) = y$ (5.5) has a crossing if and only if the synthetic \hat{f}_{r-1} -extension $d_n^{\hat{f}_{r-1}}(x) = \lambda^{n-e(f)}y$ (5.6) has a crossing.

Proof. By definition a crossing of the synthetic extension (5.6) has form

(5.11)
$$d_{n-a-b}^{\hat{f}_r}(\lambda^a x') = \lambda^{n-b-e(f)} y'$$

for some $x' \in Z^{s+a,t+a}_{r-1-a}(X)$ and $y' \in Z^{s+n-b,t+n-b}_{r-1-n+b+e(f)}(Y)$. Consider the following commutative diagram

$$\begin{array}{ccc} \Sigma^{0,e(f)}\nu X/\lambda^{r-1-a} & \xrightarrow{\hat{f}_{r-1-a}} \nu Y/\lambda^{r-1-a} \\ & & & \downarrow \lambda^a \\ & & & \downarrow \lambda^a \\ \Sigma^{0,e(f)+a}\nu X/\lambda^{r-1} & \xrightarrow{\hat{f}_{r-1}} \Sigma^{0,a}\nu Y/\lambda^{r-1} \end{array}$$

We see that (5.11) lifts to

(5.12)
$$d_{n-a-b}^{\hat{f}_{r-1-a}}(x') = \lambda^{n-a-b-e(f)}y''$$

for some $y'' \in Z^{s+n-b,t+n-b}_{r-1-n+b+e(f)}(Y)$ such that

$$y'' \equiv y' \mod B_{1+n-b-e(f)}^{s+n-b,t+n-b}(Y)$$

where the indeterminacy $B_{1+n-b-e(f)}^{s+n-b,t+n-b}(Y)$ is introduced by dividing λ^a . Therefore, the differential (5.11) is essential if and only if the differential (5.12) is essential and $y'' \notin B_{1+n-b-e(f)}^{s+n-b,t+n-b}(Y)$. This completes the proof.

The above Definition 5.9, and Proposition 5.10 can be similarly defined and stated for the case $r = \infty$.

Example 5.13. For the four (f, E_r) -extensions in Example 5.8, we consider their crossings.

(1) For $f = \nu : S^3 \to S^0$, with an (f, E_2) -extension:

$$d_1^{f,E_2}(h_5) = h_2 h_5,$$

there are no crossings for degree reasons. In fact, since the map $\hat{f}_1 = [h_2]/\lambda$ has AF = 1, $d_1^{\hat{f}_1}$ -extensions have no crossings. (2) For $f = \nu : S^3 \to S^0$, with an (f, E_3) -extension:

$$d_2^{f,E_3}(h_0h_4^2) = h_0p,$$

we can verify that it has no crossings from two perspectives.

First, by Definition 5.9, a crossing of this (f, E_3) -extension would have n=2, e(f)=1, which implies a=1,b=0. This corresponds to an essential (f, E_2) -extension from some

$$x' \in Z_1^{4,37}(S^3) \cong \operatorname{Ext}_A^{4,37}(S^3) \cong \mathbb{F}_2,$$

generated by $h_0^2 h_4^2$, to some

$$y' \in Z_1^{5,38}(S^0) \cong \operatorname{Ext}_A^{5,38}(S^0) \cong \mathbb{F}_2,$$

generated by h_0p . However, since in Ext we have

$$h_0^2 h_4^2 \cdot h_2 = 0 \neq h_0 p,$$

this (f, E_3) -extension has no crossing.

Alternatively, as seen in Example 5.8(2), this is equivalent to a synthetic f_2 -extension

$$d_2^{\hat{f}_2}(h_0 h_4^2) = \lambda h_0 p,$$

where

$$\hat{f}_2 = [h_2]/\lambda^2 : S^{3,4}/\lambda^2 \to S^{0,0}/\lambda^2.$$

By Proposition 5.10, this (f, E_3) -extension has a crossing if and only if the synthetic f_2 -extension has a crossing, which is a non-trivial differential of the form

$$d_1^{\hat{f}_2}(x') = y'.$$

For degree reasons, the only possible candidates are

$$x' = \lambda h_0^2 h_4^2, \ y' = \lambda h_0 p.$$

This again contradicts the relation in Ext, confirming that there is no cross-

(3) For $f = \nu : S^3 \to S^0$, with an (f, E_∞) -extension:

$$d_2^{f,E_{\infty}}(h_0 h_4^2) = h_0 p,$$

we can similarly confirm, using both perspectives outlined in (2), that it has no crossings.

(4) For $f = 2: S^0 \to S^0$, with an (f, E_∞) -extension:

$$d_2^{f,E_{\infty}}(h_0h_3^2) = 0,$$

the discussion in Example 5.8(4) can be rephrased to indicate a crossing of the form

$$d_1^{f, E_{\infty}}(d_0) = h_0 d_0.$$

We comment that this d_1 -differential is also a crossing of the inessential (f, E_{∞}) -extension:

$$d_2^{f,E_{\infty}}(h_0h_3^2) = h_0d_0.$$

Proposition 5.14. Consider $f: X \to Y$, $x \in Z^{s,t}_{\infty}(X)$ and $y \in Z^{s+n,t+n}_{\infty}(Y)$. Then

$$d_n^{f,E_\infty}(x) = y$$

implies that

$$d_n^f(x + B_{\infty}^{s,t}(X)) = y + B_{\infty}^{s,t}(Y).$$

(The implied differential could be inessential.)

Proof. This follows directly from inverting λ , which induces a map of spectral sequences from the f-ESS to the f-ESS. The induced map on the E_0 -pages are of the following form:

$$Z^{s,t}_{\infty}(X)/B^{s,t}_{1+t-w}(X) \xrightarrow{d^{\hat{f}}_{0}} Z^{s,t}_{\infty}(Y)/B^{s,t}_{1+t-w-e(f)}(Y)$$

$$\downarrow^{\lambda^{-1}} \qquad \qquad \downarrow^{\lambda^{-1}}$$

$$E^{s,t}_{\infty}(X) \xrightarrow{d^{f}_{0}} E^{s,t}_{\infty}(Y)$$

Remark 5.15. Since the synthetic Adams E_{∞} -page contains more information than the classical Adams E_{∞} -page, the (f, E_{∞}) -extensions similarly provide more information compared to the classical f-extensions.

6. The Generalized Leibniz Rule and Generalized Mahowald Trick

Now, we introduce the theorem of the Generalized Leibniz Rule, a valuable tool for computing new Adams differentials.

Theorem 6.1 (Generalized Leibniz Rule). Let $f: X \to Y$ be a map between two classical spectra. Suppose that $2 \le n \le r$, $e(f) \le m \le n - 2 + e(f)$, $l \ge e(f)$, and we have

$$x \in Z^{s,t}_{r-1}(X), \quad y \in Z^{s+m,t+m}_{r-1-m+e(f)}(Y)$$

$$x_{\infty} \in Z^{s+r,t+r-1}_{\infty}(X), \quad y_{\infty} \in Z^{s+r+l,t+r+l-1}_{\infty}(Y)$$

and the following conditions hold:

- $\begin{array}{ll} (1) \ d_r(x) = x_{\infty}, \\ (2) \ d_m^{f,E_n}(x) = y, \\ (3) \ d_l^{f,E_{\infty}}(x_{\infty}) = y_{\infty}, \end{array}$
- (4) the differential in (1) has no crossing on the E_n -page or (2) has no crossing.
- (5) the differential in (3) has no crossing.

Then we have an Adams differential

$$d_{r+l-m}(y) = y_{\infty}.$$

Proof. Consider the commutative diagram of synthetic spectra

$$\begin{array}{ccc}
\Sigma^{0,e(f)}\nu X/\lambda^{n-1} & \xrightarrow{\hat{f}_{n-1}} & \nu Y/\lambda^{n-1} \\
\delta_X \downarrow & \delta_Y \downarrow \\
\Sigma^{1,-n+1+e(f)}\nu X & \xrightarrow{\hat{f}} & \Sigma^{1,-n+1}\nu Y.
\end{array}$$

By condition (1) and Proposition 4.6, we have

$$(6.2) d_r^{\delta_X}(x) = \lambda^{r-n} x_{\infty}.$$

By conditions (2) and (3), and Definition 5.4, we have

(6.3)
$$d_m^{\hat{f}_{n-1}}(x) = \lambda^{m-e(f)}y$$

and

$$d_l^{\hat{f}}(x_{\infty}) = \lambda^{l - e(f)} y_{\infty},$$

which implies

(6.4)
$$d_l^{\hat{f}}(\lambda^{r-n}x_{\infty}) = \lambda^{r+l-n-e(f)}y_{\infty}.$$

Applying Proposition 4.16 and Proposition 5.10 to conditions (4) and (5), we know that the differential in (6.2) or (6.3) has no crossing, and that the differential in (6.4) has no crossing.

Therefore, we can apply Corollary 2.15 and obtain

$$d_{r+l-m}^{\delta_Y}(\lambda^{m-e(f)}y) = \lambda^{r+l-n-e(f)}y_{\infty}.$$

By Remark 4.12, this is equivalent to

$$d_{r+l-m}(y) = y_{\infty}.$$

Remark 6.5. We can further generalize Theorem 6.1 by using the conditions in Theorem 2.12 rather than those Corollary 2.15. We leave this generalization to the reader.

We emphasize that the no-crossing conditions in Theorem 6.1, the Generalized Leibniz Rule, are crucial. We first demonstrate the strength of the Generalized Leibniz Rule with Example 6.7, which uses a d_2 -differential to prove a d_3 -differential in the classical Adams spectral sequence. Additionally, Example 6.8 shows that, without the no-crossing condition, the conclusion could be false.

Remark 6.6. A version of the Generalized Leibniz Rule without the no-crossing conditions is presented in the synthetic setting in Chua's work [19, Theorem 12.9]. However, there is no doubt that this version is incorrect. Indeed, Example 6.8 is a counter-example to Chua's theorem. For further details, see Remark 6.10.

Example 6.7. We show that the Generalized Leibniz Rule can be used to prove the following classical Adams differential

$$d_3(h_2h_5) = h_0p,$$

using the classical Hopf invariant one differential

$$d_2(h_5) = h_0 h_4^2,$$

and the (f, E_3) -extension

$$d_2^{f,E_3}(h_0h_4^2) = h_0p,$$

for the map $f = \nu : S^3 \to S^0$ from Example 5.8. Specifically, we have:

$$n=2, r=2, s=1, t=35, m=1, l=2, e(f)=1,$$

and

$$x = h_5 \in \operatorname{Ext}_A^{1,32} \cong Z_1^{1,35}(S^3), \ y = h_2 h_5 \in \operatorname{Ext}_A^{2,36} \cong Z_1^{2,36}(S^0),$$

 $x_{\infty} = h_0 h_4^2 \in \operatorname{Ext}_A^{3,33} \cong Z_{\infty}^{3,36}(S^3), \ y_{\infty} = h_0 p \in \operatorname{Ext}_A^{5,38} \cong Z_{\infty}^{5,38}(S^0).$

In terms of the conditions in Theorem 6.1, we have:

- (1) $d_2(h_5) = h_0 h_4^2$,
- (2) $d_1^{f,E_2}(h_5) = h_2 h_5$, from Example 5.8(1),
- (3) $d_2^{f,E_{\infty}}(h_0h_4^2) = h_0p$, from Example 5.8(3),
- (4) (a) the differential in (1) has no crossing, from Remark 4.17, and
 - (b) the differential in (2) has no crossing, from Example 5.13(1),
- (5) the differential in (3) has no crossing, from Example 5.13(3).

Since all conditions of Theorem 6.1 are satisfied, we conclude that there is a classical Adams differential

$$d_3(h_2h_5) = h_0p.$$

Example 6.8. We show that, without the no crossing conditions, the Generalized Leibniz Rule could lead to incorrect conclusions.

Consider the map $f=2:S^0\to S^0$, and the classical Hopf invariant one differential

$$d_2(h_4) = h_0 h_3^2$$
.

Set

$$n=2, r=2, s=1, t=16, m=1, l=2, e(f)=1,$$

and

$$x = h_4 \in \operatorname{Ext}_A^{1,16} \cong Z_1^{1,16}(S^0), \ y = h_0 h_4 \in \operatorname{Ext}_A^{2,17} \cong Z_1^{2,17}(S^0),$$
$$x_{\infty} = h_0 h_3^2 \in \operatorname{Ext}_A^{3,17} \cong Z_{\infty}^{3,17}(S^0), \ y_{\infty} = 0 \in \operatorname{Ext}_A^{5,19} \cong Z_{\infty}^{5,19}(S^0).$$

In terms of the conditions in Theorem 6.1, we have:

- (1) $d_2(h_4) = h_0 h_3^2$,
- (2) $d_1^{f,E_2}(h_4) = h_0 h_4$, since the product of h_4 by h_0 is $h_0 h_4$ in Ext, (3) $d_2^{f,E_\infty}(h_0 h_3^2) = 0$, from Example 5.8(4),
- (4) (a) the differential in (1) has no crossing, from Remark 4.17, and
 - (b) the differential in (2) has no crossing, for degree reasons,
- (5) the differential in (3) has a crossing:

$$d_1^{f,E_\infty}(d_0) = h_0 d_0,$$

from Example 5.13(4).

For condition (5), if there were no crossing for the differential in (3), then the Generalized Leibniz Rule would imply a classical Adams differential

$$d_3(h_0h_4) = 0,$$

which is incorrect. This example illustrates that the no-crossing conditions are essential when applying the Generalized Leibniz Rule.

Remark 6.9. As noted in Example 5.8(4) and Example 5.13(4), the essential (f, E_{∞}) -extension:

$$d_1^{f,E_\infty}(d_0) = h_0 d_0,$$

is a crossing for both the (f, E_{∞}) -extension,

$$d_2^{f,E_{\infty}}(h_0h_3^2) = 0,$$

and the inessential (f, E_{∞}) -extension:

$$d_2^{f,E_{\infty}}(h_0h_3^2) = h_0d_0.$$

This indicates that, in Example 6.8, if we were to disregard the no-crossing condition (5), and apply the Generalized Leibniz Rule to these two cases of the (f, E_{∞}) -extensions, we would arrive at two conflicting classical statements:

$$d_3(h_0h_4) = 0$$
, $d_3(h_0h_4) = h_0d_0$.

Remark 6.10. In Chua's work [19, Theorem 12.9], it is stated that for a synthetic map $\alpha: X \to Y$, and an element $x \in \pi_{*,*}X/\lambda$, there exists a differential from a maximal α -extension of x to a maximal α -extension of $d_r(x)$. According to [19, Definition 12.5], a maximal α -extension of x' is defined as $\alpha[x']$, where [x'] represents a lift of x' to the $E_{r'}$ -page, chosen such that $\alpha[x']$ is the most λ -divisible among all such lifts.

In the context of Example 6.8, let $r = 2, r' = \infty$,

$$\alpha = [h_0]: S^{0,1} \to S^{0,0}$$

and consider $x = h_4$ in Ext, with $x' = d_2(x) = h_0 h_3^2$.

As noted in Example 5.8(4), the (f, E_{∞}) -extension:

$$d_2^{f,E_{\infty}}(h_0 h_3^2) = 0$$

is equivalent to the existence of a synthetic homotopy class $[h_0h_3^2]$ in $\pi_{14,17}S^{0,0}$, such that

$$[h_0 h_3^2] \cdot [h_0] = 0 \text{ in } \pi_{14,18} S^{0,0}.$$

Similarly, the inessential (f, E_{∞}) -extension:

$$d_2^{f,E_{\infty}}(h_0h_3^2) = h_0d_0$$

implies the existence of another synthetic homotopy class $[h_0h_3^2]$ in $\pi_{14,17}S^{0,0}$, such that

$$[h_0 h_3^2] \cdot [h_0] = [h_0 d_0]$$
 in $\pi_{14,18} S^{0,0}$.

Between these two lifts of $h_0h_3^2$, the first $[h_0h_3^2]$ is clearly the maximal $[h_0]$ -extension according to Chua's definition [19, Definition 12.5]. Consequently, the incorrect version of the Generalized Leibniz Rule in [19, Theorem 12.9], would lead to an incorrect conclusion:

$$d_3(h_0h_4) = 0.$$

Next, we discuss the Generalized Mahowald Trick.

In order to apply the Generalized Leibniz Rule, we need to provide a method for computing extensions on specific Adams E_k -pages, such as the (f, E_3) -extension $d_2^{f,E_3}(h_0h_4^2) = h_0p$ in Example 6.7. This is provided by Theorem 6.12 (the Generalized Mahowald Trick). The crux of the proof of the Generalized Mahowald Trick lies in the following lemma.

Lemma 6.11 (May [46]). Let $X \to Y \to Z \to \Sigma X$ and $X' \to Y' \to Z' \to \Sigma X'$ be distinguished triangles of (synthetic) spectra. By smashing these distinguished triangles together, we obtain the following commutative diagram of cofiber sequences:

If $a \in \pi_n(X \wedge Z')$ and $b \in \pi_n(Y \wedge Y')$ map to the same element in $\pi_n(Y \wedge Z')$, then there exists $c \in \pi_n(Z \wedge X')$ such that

- (1) b and c map to the same element in $\pi_n(Z \wedge Y')$, and
- (2) a and c map to the same element via boundary maps in $\pi_{n-1}(X \wedge X')$.

Proof. The proof follows directly from [46, Lemma 4.6].

In fact, consider V in Axiom TC3 of [46, Section 4] and the corresponding commutative diagram. By [46, Lemma 4.6] we know that V is the pull back of the following diagram.

$$\begin{array}{c}
V \longrightarrow Y \wedge Y' \\
\downarrow \qquad \qquad \downarrow \\
X \wedge Z' \longrightarrow Y \wedge Z'
\end{array}$$

Since a and b map to the same element in $\pi_n(Y \wedge Z')$, we know that we can find $v \in V$ such that v maps to a in $\pi_n(X \wedge Z')$ and b in $\pi_n(Y \wedge Y')$. Then we let $c = j_3(v) \in \pi_n(Z \wedge X')$, where $j_3 : V \to Z \wedge X'$ is the map in the commutative diagram in Axiom TC3 of [46, Section 4]. This lemma follows from the commutativity of the diagram.

Theorem 6.12 (Generalized Mahowald Trick). Consider a distinguished triangle of spectra

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

with e(f) + e(g) + e(h) = 1. Suppose that r = n + m + l, $n_1 = n - e(f) \ge 1$, $m_1 = m - e(g) \ge 0$, $l_1 = l - e(h) \ge 0$, and

$$\begin{array}{ll} x \in Z^{s+l,t+l-1}_{n_1}(X), & y \in Z^{s+n+l,t+n+l-1}_{m_1+1}(Y), \\ \bar{x} \in Z^{s,t}_{r-1}(Z), & \bar{y} \in Z^{s+r,t+r-1}_{\infty}(Z), \end{array}$$

such that

(1)
$$d_l^{h,E_{r'}}\bar{x} = x$$
, where $r' = r - m_1 = n_1 + l_1 + 1$,

(2)
$$d_r \bar{x} = \bar{y}$$
,

(3) the $(h, E_{r'})$ -extension in (1) has no crossing, or the Adams differential (2) has no crossing on the $E_{r'}$ -page, (4) $d_m^{g,E_{m_1+2}}y=\bar{y}$.

$$(4) \ d_m^{g, E_{m_1+2}} y = \bar{y}$$

Then we have $x \in Z_{n+m+e(h)}^{s+l,t+l-1}(X)$ and

$$d_n^{f,E_{n+m+1+e(h)}}x\equiv y\mod B_{r'}^{s+n+l,t+n+l-1}(Y).$$

(See Figure 6.)

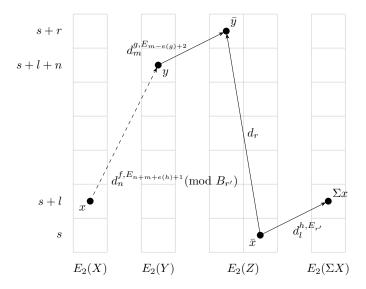


Figure 6. A demonstration of Theorem 6.12

FIGURE 7. Elements in the homotopy groups of the smash products

Proof. Consider the following two distinguished triangles of synthetic spectra

$$\nu X \xrightarrow{\hat{f}} \Sigma^{0,-e(f)} \nu Y \xrightarrow{\hat{g}} \Sigma^{0,-e(f)-e(g)} \nu Z \xrightarrow{\hat{h}} \Sigma^{0,-1} \nu \Sigma X = \Sigma^{1,0} \nu X$$
$$S^{0,0} / \lambda^r \xrightarrow{\rho} S^{0,0} / \lambda^{r'-1} \xrightarrow{\delta_{r'-1}} S^{1,-r'+1} / \lambda^{m_1+1} \xrightarrow{\lambda^{r'-1}} S^{1,0} / \lambda^r$$

and their smash product. See Figure 7.

By condition (1), we have

(6.13)
$$d_l^{\hat{h}_{r'-1}}(\bar{x}) = \lambda^{l_1} x$$

where $\hat{h}_{r'-1}$ is the map $\Sigma^{0,e(h)}\nu Z/\lambda^{r'-1} \to \nu X/\lambda^{r'-1}$ induced by \hat{h} . By condition (2), we have

(6.14)
$$d_r^{\delta_{r'-1}}(\bar{x}) = \lambda^{m_1} \bar{y}.$$

Applying Proposition 4.16 and Proposition 5.10 to condition (3), we know that the differential in (6.13) or the differential in (6.14) has no crossing. Hence, there exists

$$[\bar{x}] \in {\{\bar{x}\}} \subset \pi_{t-s,t}(\nu Z/\lambda^{r'-1})$$

such that

$$\hat{h}_{r'-1}([\bar{x}]) \in {\lambda^{l_1} x} \subset \pi_{t-s-1, t-1+e(h)}(\nu X/\lambda^{r'-1})$$

and

$$\delta_{r'-1}([\bar{x}]) \in {\lambda^{m_1}\bar{y}} \subset \pi_{t-s-1,t+r'-1}(\nu Z/\lambda^{m_1+1}).$$

By condition (4), we have

$$d_m^{\hat{g}_{m_1+1}}(y) = \lambda^{m_1} \bar{y}.$$

This implies that there exists

$$[y] \in \{y\} \subset \pi_{t-s-1,t+n+l-1}(\nu Y/\lambda^{m_1+1})$$

such that

$$\hat{g}_{m_1+1}([y]) \in {\lambda^{m_1}\bar{y}} \subset \pi_{t-s-1,t+r'-1}(\nu Z/\lambda^{m_1+1}),$$

where \hat{g}_{m_1+1} is the map $\Sigma^{0,e(g)}\nu Y/\lambda^{m_1+1}\to \nu Z/\lambda^{m_1+1}$ induced by \hat{g} .

Due to degree reasons (Proposition 3.12), $\lambda^{m_1}\bar{y}$ detects a unique element in homotopy,

$$[\lambda^{m_1}\bar{y}] \in \pi_{t-s-1,t+r'-1}(\nu Z)/\lambda^{m_1+1}.$$

Therefore, we have

$$\delta_{r'-1}([\bar{x}]) = \hat{g}_{m_1+1}([y]) = [\lambda^{m_1}\bar{y}],$$

By Lemma 6.11, there exists

$$[\lambda^{l_1}x] \in \pi_{t-s-1,t-1+e(h)}(\nu X/\lambda^r),$$

such that

(6.15)
$$\rho([\lambda^{l_1} x]) = \hat{h}_{r'-1}([\bar{x}]),$$

(6.16)
$$\hat{f}_r([\lambda^{l_1}x]) = \lambda^{r'-1}[y],$$

where ρ is the map $\nu X/\lambda^r \to \nu X/\lambda^{r'-1}$, and \hat{f}_r is the map $\Sigma^{0,e(f)}\nu X/\lambda^r \to \nu Y/\lambda^r$ induced by \hat{f} .

The equality in (6.15) indicates that $[\lambda^{l_1}x]$ can be lifted to the E_{∞} -page of $\nu X/\lambda^r$, so we have

$$x \in Z_{n+m+e(h)}^{s+l,t+l-1}(X).$$

The equation in (6.16) indicates that

$$d_n^{\hat{f}_r}(\lambda^{l_1}x) = \lambda^{r'-1}y.$$

Therefore, by dividing λ^{l_1} , we have

$$d_n^{\hat{f}_{n+m+e(h)}}(x) \equiv \lambda^{n_1} y \mod B^{s+n+l,t+n+l-1}_{r'}(Y)$$

for x in the E_{∞} page of $\nu X/\lambda^{r-l_1} = \nu X/\lambda^{n+m+e(h)}$. By definition, this is equivalent to

$$d_n^{f,E_{n+m+1+e(h)}}x\equiv y\mod B^{s+n+l,t+n+l-1}_{r'}(Y).$$

We refer to Theorem 6.12 as the Generalized Mahowald Trick, as this approach was first utilized by Mahowald and his collaborators in various works (see, for example, [44]), particularly in the case where $m_1 = l_1 = 0$. The synthetic setting advances this method by allowing for the consideration of cases where $m_1 > 0, l_1 > 0$ as well.

Remark 6.17. Several versions of classical generalizations of Mahowald's original trick appear in the literature and are often referred to as geometric boundary theorems. Notable examples include the works of Behrens [8], and Ma [41].

Remark 6.18. A synthetic generalization of the Mahowald Trick, again lacking any no-crossing conditions, is also presented in Chua's work [19, Theorem 12.11]. This version is also incorrect for similar reasons.

Example 6.19. We apply the Generalized Mahowald Trick to prove the claim in Example 5.8(2) regarding an (f, E_3) -extension:

$$d_2^{f,E_3}(h_0h_4^2) = h_0p,$$

for the map $f = \nu : S^3 \to S^0$.

Specifically, let $X = S^3, Y = S^0, Z = S^0/\nu$, where Z is the cofiber of $\nu \in \pi_3$, we have a distinguished triangle

$$S^3 \xrightarrow{\quad \nu \quad} S^0 \xrightarrow{\quad i \quad} S^0/\nu \xrightarrow{\quad q \quad} S^4.$$

We set

$$f = \nu$$
, $g = i$, $h = q$, $e(f) = 1$, $e(g) = 0$, $e(h) = 0$.

The short exact sequence on $H\mathbb{F}_2$ -homology

$$0 \longrightarrow \mathrm{H}\mathbb{F}_{2*}S^0 \xrightarrow{i_*} \mathrm{H}\mathbb{F}_{2*}S^0/\nu \xrightarrow{q_*} \mathrm{H}\mathbb{F}_{2*}S^4 \longrightarrow 0,$$

induces a long exact sequence on Ext-groups

$$\cdots \xrightarrow{\cdot h_2} \operatorname{Ext}_A^{*,*}(S^0) \xrightarrow{i_*} \operatorname{Ext}_A^{*,*}(S^0/\nu) \xrightarrow{q_*} \operatorname{Ext}_A^{*,*}(S^4) \xrightarrow{\cdot h_2} \cdots.$$

We use Atiyah–Hirzebruch notation (as in [53, 54, Notation 3.3]) to denote elements in $\operatorname{Ext}_A^{*,*}(S^0/\nu)$. Specifically, if an element x in $\operatorname{Ext}_A^{*,*}(S^0/\nu)$ satisfies

$$q_*(x) = a \neq 0 \in \operatorname{Ext}_A^{*,*}(S^4),$$

we denote x by a[4]; otherwise, due to exactness,

$$x = i_*(b)$$
 for some $b \in \operatorname{Ext}_A^{*,*}(S^0)$.

and in this case, we denote x by b[0].

We have

$$m = 0, l = 0, n = 2, r = 2, r' = 2, n_1 = 1, m_1 = 0, l_1 = 0,$$

and

$$x = h_0 h_4^2 \qquad \qquad \in \operatorname{Ext}_A^{3,33} \qquad \cong Z_1^{3,36}(S^3),$$

$$\bar{x} = h_0 h_4^2[4] \qquad \qquad \in \operatorname{Ext}_A^{3,37}(S^0/\nu) \qquad \cong Z_1^{3,37}(S^0/\nu),$$

$$y = h_0 p \qquad \qquad \in \operatorname{Ext}_A^{5,38} \qquad \cong Z_1^{5,38}(S^0),$$

$$\bar{y} = h_0 p[0] \qquad \qquad \in \operatorname{Ext}_A^{5,38}(S^0/\nu) \qquad \cong Z_\infty^{5,38}(S^0/\nu).$$

For conditions in Theorem 6.12, we have:

- (1) $d_0^{h,E_2}(h_0h_4^2[4]) = h_0h_4^2$, since in Ext $h_2 \cdot h_0h_4^2 = 0$. (2) $d_2(h_0h_4^2[4]) = h_0p[0]$. This is a classical Adams d_2 -differential for S^0/ν , and can be computed using Lin's computer program via the method of the secondary Steenrod algebra [6, 49, 19].
- (3) (a) The differential in (1) has no crossing, as it is a d_0 -differential, and (b) the differential in (2) has no crossing, from Remark 4.17.
- (4) $d_0^{g,E_2}(h_0p) = h_0p[0]$, as h_0p is not divisible by h_2 in Ext.

Since all conditions of Theorem 6.12 are satisfied, we conclude that $x = h_0 h_4^2$ survives to the E_3 -page, and there is an (f, E_3) -extension:

$$d_2^{f,E_3}(h_0h_4^2) = h_0p,$$

as promised.

The outcome of the Generalized Mahowald Trick, as stated in Theorem 6.12, is an f-extension on a specific Adams page, such as the (f, E_3) -extension $d_2^{f, E_3}(h_0 h_4^2) =$ h_0p in Example 5.8(2) and Example 6.19. In practice, the source of an (f, E_r) extension may survive to later Adams pages, prompting interest in the $(f, E_{>r})$ extensions, such as the (f, E_{∞}) -extension $d_2^{f, E_{\infty}}(h_0 h_4^2) = h_0 p$ in Example 5.8(3), which is applied in Example 6.7. The following Propositions 6.20 and 6.23 describe the relationships between extensions across different pages.

Proposition 6.20. Suppose that we have an (f, E_r) -extension $d_n^{f, E_r}(x) = y$, where $x \in Z_{r-1}^{s,t}(X)$ and $y \in Z_{r-1-n+e(f)}^{s+n,t+n}(Y)$. Then for all $2 \le r' < r$, we also have

(6.21)
$$d_n^{f,E_{r'}}(x) = y.$$

Furthermore, if $d_n^{f,E_r}(x) = y$ is essential and $n \leq r' - 2 + e(f)$, then (6.21) is inessential if and only if there exists some $0 < a' \le n - e(f)$ and an element

$$x' \in Z^{s+a',t+a'}_{r'-1-a'}(X) \backslash Z^{s+a',t+a'}_{r-1-a'}(X)$$

such that

$$d_{n-a'-b}^{\hat{f}_{r'-1}}(\lambda^{a'}x') = \lambda^{n-b-e(f)}y'$$

for some b > 0.

Proof. Consider the following commutative diagram:

$$\Sigma^{0,e(f)} \nu X/\lambda^{r-1} \xrightarrow{\hat{f}_{r-1}} \nu Y/\lambda^{r-1}$$

$$\rho_X \downarrow \qquad \qquad \rho_Y \downarrow$$

$$\Sigma^{0,e(f)} \nu X/\lambda^{r'-1} \xrightarrow{\hat{f}_{r'-1}} \nu Y/\lambda^{r'-1}$$

By Corollary 2.18, (ρ_X, ρ_Y) induces a map from the \hat{f}_{r-1} -ESS to the $\hat{f}_{r'-1}$ -ESS. Therefore, by naturality,

$$d_n^{\hat{f}_{r-1}}(x) = \lambda^{n-e(f)}y \text{ implies } d_n^{\hat{f}_{r'-1}}(x) = \lambda^{n-e(f)}y,$$

which, by definition, is

$$d_n^{f,E_{r'}}(x) = y.$$

Next, we prove the second part of the proposition. By Definition 5.4, the (f, E_r) -extension (6.21) is inessential if and only if there exists $0 < a \le n - e(f)$ and

$$\lambda^a x' \in E^{s+a,t+a,t}_{\infty}(\nu X/\lambda^{r'-1}) \cong Z^{s+a,t+a}_{r'-1-a}(X)/B^{s+a,t+a}_{1+a}(X),$$

such that

(6.22)
$$d_{n-a}^{\hat{f}_{r'-1}}(\lambda^a x') = \lambda^{n-e(f)}y,$$

and this differential is not induced by (ρ_X, ρ_Y) , as we assume that $d_n^{f, E_r}(x) = y$ is essential.

There are two scenarios where this differential is not induced by (ρ_X, ρ_Y) . The first case occurs when $\lambda^a x'$ is not in the image of ρ_X at all, which is equivalent to $x' \notin Z_{r-1-a}^{s+a,t+a}(X)$. (See Figure 8.)

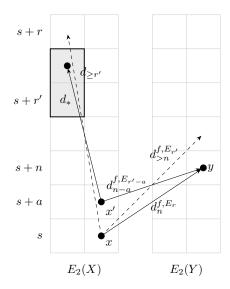


FIGURE 8. Extensions across pages

The second case occurs when $\lambda^a x'$ is in the image of ρ_X , but it supports an essential \hat{f}_{r-1} -extension that is strictly shorter than the differential (6.22).

To further explore this scenario, we replace x with $\lambda^a x'$ and analyze successive essential extensions. This process is repeated iteratively until the first case is reached. Ultimately, this leads to the existence of some $0 < a' \le n - e(f)$ and an element

$$x'' \in Z_{r'-1-a'}^{s+a',t+a'}(X) \setminus Z_{r-1-a'}^{s+a',t+a'}(X)$$

such that

$$d_{n-a'-b}^{\hat{f}_{r'-1}}(\lambda^{a'}x'') = \lambda^{n-b-e(f)}y'$$

for some b>0. This represents a crossing of $d_n^{\hat{f}_{r'-1}}(x)=\lambda^{n-e(f)}y$.

The following Corollary 6.23 is the contrapositive statement of the second part of Proposition 6.20.

Corollary 6.23. Suppose $r \geq r'$ and there exists an $(f, E_{r'})$ -extension

$$d_n^{f,E_{r'}}(x) = y$$

for $x \in Z^{s,t}_{r-1}(X)$ and $y \in Z^{s+n,t+n}_{r'-1-n+e(f)}(Y)$. Assume that this extension has no crossing of the form $d^{f,E_{r'-a}}_{n-a-b}(x')=y'$ for any b>0, $0< a \le n-e(f)$, and

$$x' \in Z_{r'-1-a}^{s+a,t+a}(X) \setminus Z_{r-1-a}^{s+a,t+a}(X).$$

Under these conditions, we also have an (f, E_r) -extension

$$d_n^{f,E_r}(x) = y.$$

Example 6.24. In Example 6.19, we use the Generalized Mahowald Trick to establish the (f, E_3) -extension:

$$d_2^{f,E_3}(h_0h_4^2) = h_0p,$$

for the map $f = \nu: S^3 \to S^0$, as discussed in Example 5.8(2). However, to apply the Generalized Leibniz Rule in proving the classical Adams differential

$$d_3(h_2h_5) = h_0p$$

in Example 6.7, we need the E_{∞} -page version of the (f, E_3) -extension:

$$d_2^{f,E_{\infty}}(h_0h_4^2) = h_0p.$$

We apply Corollary 6.23 to confirm this.

As checked in Example 5.13(2), the (f, E_3) -extension has no crossing. Consequently, by Corollary 6.23, we obtain the required (f, E_{∞}) -extension:

$$d_2^{f,E_{\infty}}(h_0h_4^2) = h_0p.$$

7. Proof of the main theorem

In this section, we present the proof of our main Theorem 1.4.

Theorem 7.1 (Theorem 1.4). The element h_6^2 survives to the E_{∞} -page in the Adams spectral sequence.

We first recall the following theorem, known as the inductive method, originally by Barratt–Jones–Mahowald [3] and later extended to the $H\mathbb{F}_2$ -synthetic setting by Burklund–Xu [18, Proposition 7.19].

Notation 7.2. Let $\theta_5 = [h_5^2]$ represent any synthetic homotopy class in $\pi_{62,62+2}S^{0,0}$ detected by h_5^2 in the Adams E_2 -page. For convenience, we use the same notation, θ_5 , to denote its image in $\pi_{62,62+2}S^{0,0}/\lambda^r$ via the map $S^{0,0} \to S^{0,0}/\lambda^r$ for all $r \ge 1$. Similarly, let $\eta = [h_1] \in \pi_{1,1+1}S^{0,0}$.

Theorem 7.3 (Barratt-Jones-Mahowald, Burklund-Xu).

(1) The element h_6^2 survives to the E_{r+3} -page of the classical Adams spectral sequence if and only if for some θ_5 ,

$$\lambda \eta \theta_5^2 = 0 \text{ in } \pi_{125,125+4} S^{0,0} / \lambda^{r+1}.$$

(2) In particular, h_6^2 is a permanent cycle in the classical Adams spectral sequence if and only if for some θ_5 ,

$$\lambda \eta \theta_5^2 = 0 \text{ in } \pi_{125,125+4} S^{0,0}$$

Remark 7.4. The statement in Theorem 7.3(1) was originally stated as

$$\eta\theta_5^2 = 0 \text{ in } \pi_{125,125+5}S^{0,0}/\lambda^r$$

in [18, Proposition 7.19]. Upon inspection, $\pi_{125,125+5}S^{0,0}$ doesn't contain any λ -torsion classes, so this is equivalent to the version we stated, which is more consistent with the statement in part (2).

Remark 7.5. Theorem 7.3 is proved using the quadratic construction on a map from the mod 2 Moore spectrum to the sphere spectrum, where the restriction on the bottom cell is θ_5 . Therefore, it is necessary to use a θ_5 of order 2.

Notably, [56, 29] confirms that all classical θ_5 's indeed have order 2. Furthermore, from Proposition 3.11 and an analysis of the differentials in the classical Adams spectral sequence, we find that $\pi_{62,62+2}S^{0,0}$ doesn't contain any λ -torsion. Consequently, all synthetic θ_5 's also have order 2, making it valid to apply Theorem 7.3 to any θ_5 .

Additionally, since the proof of Theorem 7.3 shows that the expression $\lambda \eta \theta_5^2$ corresponds to the total differential $\delta_1: S^{0,0}/\lambda \to S^{1,-1}$ on h_6^2 , the value of the expression $\lambda \eta \theta_5^2$ is consistent for every choice of θ_5 . (Note that our grading for the triangulation translation functor is smashing with $S^{1,0}$, which is consistent with [16, Appendix A] but is different from [18, Section 7], so the target of δ_1 is $S^{1,-1}$.)

We pay special attention to the following three elements on the classical Adams E_2 -page (see Figure 9 and Tables 7, 9, 5, 6 in the Appendix):

$$h_1 h_4 x_{109,12} \in \operatorname{Ext}_A^{14,125+14},$$

$$x_{126,8,4} + x_{126,8} \in \operatorname{Ext}_A^{8,126+8},$$

$$h_0^2 x_{124,8} \in \operatorname{Ext}_A^{10,124+10},$$

$$g^4 \Delta h_1 g \in \operatorname{Ext}_A^{25,125+25}.$$

For the right side of Figure 9, we use dashed differentials to indicate the shortest possible nonzero differentials that these elements could support.

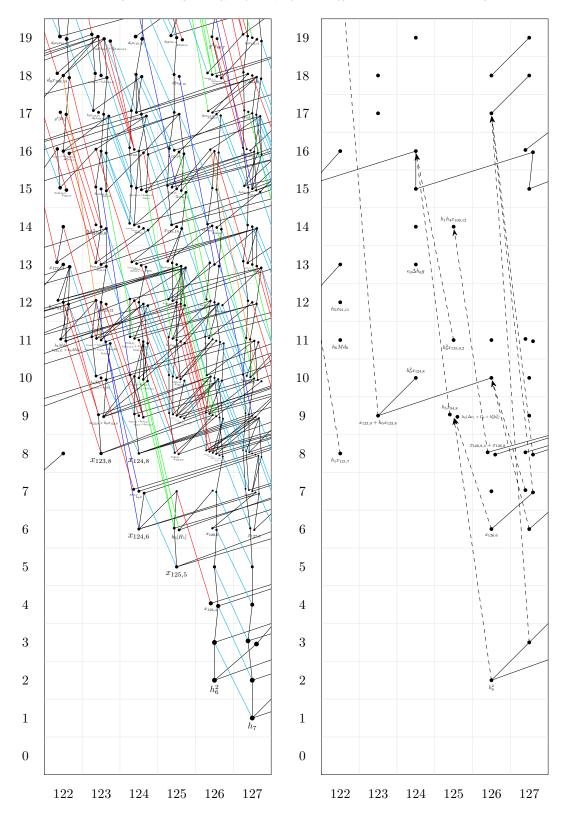


FIGURE 9. The Adams E_2 and E_∞ -pages of S^0 near h_6^2

From Figure 9 and Tables 5, 9, 7, 6 in the Appendix, we know that

Fact 7.6. (1) $x_{126,8,4} + x_{126,8}$ survives to the E_6 -page.

(2) $h_1h_4x_{109,12}$ is a permanent cycle, and can only be killed by

$$d_6(x_{126,8,4} + x_{126,8})$$
 or $d_{12}(h_6^2)$.

- (3) $h_0^2 x_{124,8}$ survives to the E_{∞} -page. (4) In $\operatorname{Ext}_A^{25,125+25}$, the element $g^4 \Delta h_1 g$ is the only one that survives to the classical E_5 -page.

Remark 7.7. From Table 9 in the Appendix, we have

$$d_3(x_{126.6}) = h_5 x_{94.8}$$
, or $h_5 x_{94.8} + h_6(\Delta e_1 + C_0 + h_0^6 h_5^2) \neq 0$.

Therefore, the element $x_{126,6} \in \operatorname{Ext}_{4}^{6,126+6}$ cannot kill $h_1 h_4 x_{109,12}$.

We will apply Theorem 7.3 to prove the following Proposition 7.8.

Proposition 7.8. Exactly one of the following two statements is true:

- (1) The element h_6^2 survives to the E_{∞} -page.
- (2) There is a nonzero classical Adams differential

$$d_{12}(h_6^2) = h_1 h_4 x_{109,12}.$$

Furthermore, statement (2) is true if and only if the following three statements are all true:

- $d_6(x_{126,8,4} + x_{126,8}) = 0.$
- (4) There exists a θ_5 such that θ_5^2 is detected by $\lambda^6 h_0^2 x_{124.8}$. In particular,

$$\theta_5^2 = \lambda^6 [h_0^2 x_{124,8}] \neq 0 \in \pi_{124,124+4} S^{0,0}$$

for some $[h_0^2x_{124,8}]$.

(5) There exists a homotopy class $[h_0^2x_{124,8}]$ such that $\lambda^3\eta[h_0^2x_{124,8}]$ is detected by $\lambda^6 h_1 h_4 x_{109,12}$. In particular, we have

$$\lambda^3 \eta[h_0^2 x_{124,8}] = \lambda^6 [h_1 h_4 x_{109,12}] \in \pi_{125,125+8} S^{0,0}$$

for some $[h_1h_4x_{109.12}]$.

By further analyzing classical Adams differentials, we reduce the η -extension in statement (5) of Proposition 7.8 to a specific 2-extension in stem 125 (Corollary 7.18), and then compare it with a particular ν -extension in stem 125 (Lemma 7.20) to demonstrate that the 2-extension cannot hold. This ultimately leads to the proof of Proposition 7.9.

Proposition 7.9. If statement (3) is true, then statement (5) in Proposition 7.8 must be false.

Proof of Theorem 7.1. From Proposition 7.9, at least one of statements (3) or (5) in Proposition 7.8 is false. Consequently, statement (2) is also false, which confirms that statement (1) in Proposition 7.8 is true.

In the rest of this section, we prove Propositions 7.8 and 7.9.

To prove Proposition 7.8, we first establish Lemmas 7.10 and 7.11, which demonstrate that the existential statements (4) and (5) in Proposition 7.8 are equivalent to their corresponding universal statements (4') and (5').

Lemma 7.10. The statement (4) in Proposition 7.8 is equivalent to the following statement (4'):

(4') For every θ_5 , we have θ_5^2 is detected by $\lambda^6 h_0^2 x_{124,8}$. In particular,

$$\theta_5^2 = \lambda^6 [h_0^2 x_{124,8}] \neq 0 \in \pi_{124,124+4} S^{0,0}$$

for some $[h_0^2x_{124,8}]$.

Proof. We only need to show that statement (4) implies statement (4'). According to [29],

$$\pi_{62} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$$
,

and is generated by θ_5 and classes of AF = 6, 8, 10. Therefore, the indeterminacy of the classical θ_5 , or the difference of any two choices of classical θ_5 , lies in AF \geq 6.

As explained in Remark 7.5, $\pi_{62,62+2}S^{0,0}$ contains no λ -torsion, and therefore, the indeterminacy of the synthetic θ_5 also belongs to AF ≥ 6 . Since every θ_5 has order 2, the indeterminacy of the synthetic θ_5^2 lies in AF ≥ 12 .

Therefore, if for some θ_5 , θ_5^2 is detected by this specific element $\lambda^6 h_0^2 x_{124,8}$, which is nonzero in AF = 10, then for any θ_5 , θ_5^2 is nonzero and detected by $\lambda^6 h_0^2 x_{124,8}$.

Lemma 7.11. The statement (5) in Proposition 7.8 is equivalent to the following statement (5'):

(5') For every homotopy class $[h_0^2x_{124,8}]$, we have $\lambda^3\eta[h_0^2x_{124,8}]$ is detected by $\lambda^6h_1h_4x_{109,12}$. In particular, we have

$$\lambda^3 \eta[h_0^2 x_{124,8}] = \lambda^6 [h_1 h_4 x_{109,12}] \in \pi_{125,125+8} S^{0,0}$$

for some $[h_1h_4x_{109,12}]$.

Proof. We only need to show that statement (5) implies statement (5'), which is sufficient to show that the indeterminacy of $[h_0^2x_{124,8}]$, or the difference between any two homotopy classes $[h_0^2x_{124,8}]$, when multiplied by $\lambda^3\eta$, belongs to AF \geq 15, given that $[h_1h_4x_{109,12}]$ has AF = 14.

Since $[h_0^2x_{124,8}]$ has AF = 10, the indeterminacy is generated by cycles in AF \geq 11 of stem 124. We observe that classical cycles in AF = 11, 12 of stem 124 are all killed by Adams d_2 or d_3 -differentials, and cycles in AF = 13 are all annihilated by h_1 in Ext. Therefore, we conclude that the indeterminacy, when multiplied by $\lambda^3 \eta$, belongs to AF \geq 15.

Remark 7.12. From Fact 7.6(2) and the rigidity Theorems 3.6 and 3.8 for the synthetic Adams spectral sequence for $S^{0,0}$, the right side of the equation in statement (5') is nonzero if and only if statement (3) is true.

By Lemmas 7.10 and 7.11, we will freely interchange statements (4) and (4'), as well as (5) and (5'), depending on the context.

Now we prove Proposition 7.8.

Proof of Proposition 7.8. From the proof of Theorem 7.3 [18] we know that

$$\delta_1(h_6^2) = \lambda \eta \theta_5^2,$$

where δ_1 is the map $S^{0,0}/\lambda \to S^{1,-1}$. Suppose that $\eta \theta_5^2$ is detected by $\lambda^{n-5}T_n$ in AF = n of the synthetic Adams spectral sequence for some $T_n \in \operatorname{Ext}_A^{n,125+n}$. This implies a differential in the δ_1 -ESS:

$$d_{n-2}^{\delta_1}(h_6^2) = \lambda^{n-2}T_n.$$

By Theorems 3.6, 3.8 and Corollary 4.10, this is equivalent to a synthetic Adams differential

$$d_{n-2}(h_6^2) = \lambda^{n-3} T_n$$

and a classical Adams differential

$$d_{n-2}(h_6^2) = T_n.$$

This differential is nonzero if and only if T_n is nonzero on the classical E_{n-2} -page, i.e., not the image of a differential $d_{\leq n-3}$. In particular, when $\lambda \eta \theta_5^2 = 0$, we conclude that h_6^2 is a permanent cycle.

We first show that statements (3), (4') and (5') together imply statement (2). By statements (4') and (5'), we have that for any θ_5 ,

$$\lambda \eta \theta_5^2 = \lambda \eta \cdot \lambda^6 [h_0^2 x_{124,8}] = \lambda^4 \cdot \lambda^3 \eta [h_0^2 x_{124,8}] = \lambda^{10} [h_1 h_4 x_{109,12}].$$

Recall from Fact 7.6(2) that $h_1h_4x_{109,12}$ is a permanent cycle, and can only be killed classically by

$$d_6(x_{126,8,4} + x_{126,8})$$
 or $d_{12}(h_6^2)$.

Therefore, statement (3) implies that the expression $\lambda^{10}[h_1h_4x_{109,12}]$ is nonzero in synthetic homotopy, leading to a nonzero classical d_{12} -differential:

$$d_{12}(h_6^2) = h_1 h_4 x_{109,12}.$$

We will complete the proof of Proposition 7.8 by showing that if one the statements (3), (4) or (5) is false, then statement (2) is also false, and in this case,

$$\eta\theta_5^2 = 0 \in \pi_{125,125+5}S^{0,0},$$

making statement (1) true. This conclusion is reached by estimating the Adams filtration of θ_5^2 and subsequently that of the expression $\eta\theta_5^2$.

We begin by estimating the Adams filtration of θ_5^2 in $\pi_{124,124+4}S^{0,0}$. First, we observe that this group $\pi_{124,124+4}S^{0,0}$ does not contain any λ -torsion classes. This is because, in the 125-stem, the group (from Table 7 in the Appendix)

$$\operatorname{Ext}_A^{i,125+i} = 0 \text{ for } i \le 4,$$

and thus, by the rigidity Theorems 3.6 and 3.8 for the synthetic Adams spectral sequence for $S^{0,0}$, in the 124-stem, λ -torsion classes can only appear in $\pi_{124,124+j}S^{0,0}$ for $j \geq 7$.

Additionally, by analyzing classical Adams differentials, the Adams filtration of θ_5^2 is at least 10. In the case where it is of Adams filtration 10, it is detected by the element $h_0^2 x_{124,8}$, which, according to Fact 7.6(3), is a permanent cycle and cannot be killed.

Upon further inspection of the differentials associated with elements in stem 124 and filtration between 10 and 13, we are left with three possibilities:

- (a) $\theta_5^2 = \lambda^6 \cdot [h_0^2 x_{124,8}]$, which is statement (4), or
- (b) $\theta_5^2 = \lambda^9 \cdot [e_0 \Delta h_6 g]$, where $e_0 \Delta h_6 g$ is permanent cycle in AF = 13, or
- (c) θ_5^2 is a λ^{10} -multiple.

Since in Ext, we have $h_1 \cdot e_0 \Delta h_6 g = 0$, we deduce that in either possibilities (b) or (c), $\eta \theta_5^2$ is a λ^{10} -multiple. In other words, $\eta \theta_5^2$ has AF ≥ 15 . In the group $\pi_{125,125+5}S^{0,0}$, the only class that is a λ^{10} -multiple is actually λ -free: By Fact 7.6(4), it is $\lambda^{20}g^4\Delta h_1g$ in AF = 25 and is detected by tmf (see [9] for the Hurewicz image of tmf). Since θ_5 maps to zero in tmf, we have that $\eta \theta_5^2$ maps to zero in tmf and therefore must be zero in this case.

This shows that if statement (4) is false, then $\eta\theta_5^2=0$, and therefore h_6^2 is a permanent cycle.

Hence, we focus on the remaining possibility (a), assuming statement (4) holds:

$$\theta_5^2 = \lambda^6 \cdot [h_0^2 x_{124.8}].$$

By the proof of Lemma 7.11, the only nonzero possibility for $\lambda^3 \eta[h_0^2 x_{124,8}]$ is $\lambda^6[h_1 h_4 x_{109,12}]$. Therefore, if statement (5) is false, then using the same tmf detection argument, we conclude that $\eta\theta_5^2=0$, and consequently, h_6^2 is a permanent cycle.

Finally, if statement (3) is false, an inspection reveals the only alternative classical differential:

$$d_6(x_{126,8,4} + x_{126,8}) = h_1 h_4 x_{109,12}.$$

This, combined with the tmf detection argument, would again imply $\eta\theta_5^2=0$, and consequently, h_6^2 is a permanent cycle. This completes the proof.

Before we prove Proposition 7.9, we first state and prove a few lemmas. For Lemma 7.14, we draw attention to the following element in Ext:

$$x_{123,9} + h_0 x_{123,8} \in \text{Ext}_A^{9,123+9}$$
.

From Tables 3, 7 in the Appendix, we have

- **Fact 7.13.** (1) $x_{123,9} + h_0 x_{123,8}$ survives to the Adams E_{12} -page, and is not killed by any classical differential.
 - (2) We have a classical nonzero differential

$$d_2(x_{125,8}) = h_1(x_{123,9} + h_0x_{123,8}) + h_0^2x_{124,8}.$$

Lemma 7.14. There exists a homotopy class

$$\alpha_1 = [x_{123,9} + h_0 x_{123,8}] \in \pi_{123,123+9} S^{0,0} / \lambda^9$$

with the following properties:

(1) For any homotopy class $[h_0^2x_{124.8}]$, there exist homotopy classes

$$\alpha_2 \in \pi_{124,124+13} S^{0,0}/\lambda^9, \ \alpha_3 \in \pi_{125,125+15} S^{0,0}/\lambda^9,$$

such that

$$\lambda^{3} \eta \cdot \alpha_{1} = \lambda^{3} [h_{0}^{2} x_{124,8}] + \lambda^{6} \alpha_{2} \qquad \in \pi_{124,124+7} S^{0,0} / \lambda^{9},$$

$$\eta \cdot \alpha_{2} = \lambda \cdot \alpha_{3} \qquad \in \pi_{125,125+14} S^{0,0} / \lambda^{9},$$

$$(2) \qquad \lambda^{3} \cdot \alpha_{1} \cdot [h_{0}] = 0 \in \pi_{123,123+7} S^{0,0} / \lambda^{9}.$$

Proof. From Fact 7.13(1), the element $x_{123,9} + h_0 x_{123,8}$ survives to the Adams E_{12} -page, and is not killed by any classical differential.

Let $\alpha_1 = [x_{123,9} + h_0 x_{123,8}] \in \pi_{123,123+9} S^{0,0}/\lambda^{11}$ denote any homotopy class detected by $x_{123,9} + h_0 x_{123,8}$. For simplicity, we also use α_1 to denote its images in $\pi_{123,123+9} S^{0,0}/\lambda^r$ for $1 \le r \le 10$, under the following sequence of maps:

$$S^{0,0}/\lambda^{11} \longrightarrow S^{0,0}/\lambda^{10} \longrightarrow \cdots \longrightarrow S^{0,0}/\lambda$$

$$\alpha_1 = [x_{123,9} + h_0 x_{123,8}] \longmapsto \alpha_1 \longmapsto \cdots \longmapsto x_{123,9} + h_0 x_{123,8}$$

From the nonzero d_2 -differential in Fact 7.13, we have

$$h_1 \cdot (x_{123.9} + h_0 x_{123.8}) = h_0^2 x_{124.8},$$

on the classical E_3 -page. This implies synthetically, for $2 \le r \le 11$,

$$\lambda \eta \cdot \alpha_1 + \lambda [h_0^2 x_{124.8}] \in \pi_{124.124+9} S^{0,0} / \lambda^r$$

lies in AF ≥ 11 for any $[h_0^2 x_{124,8}]$.

By inspection, as in the proof of Lemma 7.11,

$$\lambda^3 \eta \cdot \alpha_1 + \lambda^3 [h_0^2 x_{124,8}] \in \pi_{124,124+7} S^{0,0} / \lambda^9$$

has AF ≥ 13 . The only possibility for it to have AF = 13 is that it is detected by the element $\lambda^6 e_0 \Delta h_6 g$. (Note that the class $[\lambda^6 h_4 x_{109,12}]$ is irrelevant due to the nonzero differential $d_3(\lambda^6 h_4 x_{109,12}) = \lambda^8 h_1 x_{122,15,2}$.) Since in Ext we have

$$h_1 \cdot e_0 \Delta h_6 g = 0,$$

we may choose $\alpha_2 \in \pi_{124,124+13} S^{0,0}/\lambda^9$ to be any class detected by $\lambda^6 e_0 \Delta h_6 g$, with the property that $\eta \alpha_2$ is λ -divisible. Thus, there exists an α_3 such that $\eta \alpha_2 = \lambda \alpha_3$. This proves the required property (1).

For the relation in property (2), we will first prove it in $\pi_{123,123+7}S^{0,0}/\lambda^{11}$ and then map it to $\pi_{123,123+7}S^{0,0}/\lambda^9$.

By Proposition 3.12 for the synthetic Adams spectral sequence for $S^{0,0}/\lambda^{11}$, the expression

$$\lambda^3 \cdot \alpha_1 \cdot [h_0] \in \pi_{123,123+7} S^{0,0} / \lambda^{11}$$

has AF \geq 17. In particular, the values

$$\lambda^4(x_{123,11,2} + x_{123,11} + h_0 h_6 B_4)$$
 in AF = 11,

$$\lambda^8 h_0^2 x_{123,13,2}$$
 in AF = 15

can be ruled out due to the nonzero Adams differentials

$$d_7(\lambda^4 \cdot (x_{123,11,2} + x_{123,11} + h_0 h_6 B_4)) = \lambda^{10} h_1 x_{121,17},$$

$$d_3(\lambda^8 \cdot h_0^2 x_{123,13,2}) = \lambda^{10} h_0^2 x_{122,16},$$

in the synthetic Adams spectral sequence for $S^{0,0}/\lambda^{11}$, which are zero in the spectral sequence for $S^{0,0}/\lambda^{9}$.

Therefore, the expression $\lambda^3 \cdot \alpha_1 \cdot [h_0]$ is λ^{10} -divisible in $\pi_{123,123+7}S^{0,0}/\lambda^{11}$. Mapping it further to $\pi_{123,123+7}S^{0,0}/\lambda^9$, we conclude that it is zero.

For Lemma 7.16, we draw attention to the following element in Ext:

$$h_0^2 x_{125,9,2} \in \operatorname{Ext}_A^{11,125+11}$$
.

From Table 7 in the Appendix, we have

Fact 7.15. The element $h_0^2x_{125,9,2}$ survives to the Adams E_5 -page, and is not killed by any classical differential.

Lemma 7.16. Assuming that both statements (3) and (5') in Proposition 7.8 are true, the synthetic Toda bracket

$$\langle \lambda^3 \alpha_1, [h_0], \eta \rangle \subset \pi_{125, 125 + 7} S^{0,0} / \lambda^9$$

does not contain zero, and is detected by $\lambda^4 h_0^2 x_{125,9,2}$. Here $\alpha_1 = [x_{123,9} + h_0 x_{123,8}]$ refers to the homotopy class described in Lemma 7.14.

Note that the synthetic Toda bracket in Lemma 7.16 is well defined, as the homotopy class α_1 in Lemma 7.14 satisfies the relation $\lambda^3 \alpha_1 \cdot [h_0] = 0$.

Remark 7.17. According to Fact 7.15, it is not yet known whether the element $h_0^2x_{125,9,2}$ supports a nonzero d_5 -differential. Assuming that both statements (3) and (5') in Proposition 7.8 are true, Lemma 7.16 specifically implies that $\lambda^4h_0^2x_{125,9,2}$ detects a nonzero homotopy class in $\pi_{125,125+7}S^{0,0}/\lambda^9$. Therefore, under these assumptions, we would have $d_5(h_0^2x_{125,9,2}) = 0$.

Proof of Lemma 7.16. We assume both statements (3) and (5') in Proposition 7.8 are true. From statement (5'), we have

$$\lambda^3 \eta[h_0^2 x_{124,8}] = \lambda^6 [h_1 h_4 x_{109,12}] \in \pi_{125,125+8} S^{0,0}.$$

Mapping this relation to $S^{0,0}/\lambda^9$, and applying the following relations from Lemma 7.14

$$\lambda^{3} \eta \cdot \alpha_{1} = \lambda^{3} [h_{0}^{2} x_{124,8}] + \lambda^{6} \alpha_{2} \qquad \in \pi_{124,124+7} S^{0,0} / \lambda^{9},$$

$$\eta \cdot \alpha_{2} = \lambda \cdot \alpha_{3} \qquad \in \pi_{125,125+14} S^{0,0} / \lambda^{9},$$

we have

$$\eta \cdot \lambda^3 \eta \alpha_1 = \eta \cdot \lambda^3 [h_0^2 x_{124,8}] + \eta \cdot \lambda^6 \alpha_2$$

= $\lambda^6 [h_1 h_4 x_{109,12}] + \lambda^7 \alpha_3 \in \pi_{125,125+8} S^{0,0} / \lambda^9$,

which, from statement (3) and Remark 7.17, is nonzero and detected by $\lambda^6 h_1 h_4 x_{109,12}$ in AF = 14.

On the other hand, since $\eta^2 = \langle [h_0], \eta, [h_0] \rangle$, we have

$$\eta \cdot \lambda^3 \eta \alpha_1 = \lambda^3 \alpha_1 \cdot \langle [h_0], \eta, [h_0] \rangle = \langle \lambda^3 \alpha_1, [h_0], \eta \rangle \cdot [h_0].$$

Therefore, the synthetic Toda bracket $\langle \lambda^3 \alpha_1, [h_0], \eta \rangle$ does not contain zero, and its $[h_0]$ -multiple is detected by $\lambda^6 h_1 h_4 x_{109,12}$ in AF = 14. Since $h_1 h_4 x_{109,12}$ is not h_0 -divisible in Ext, the synthetic Toda bracket is detected by an element in AF ≤ 12 .

This synthetic Toda bracket $\langle \lambda^3 \alpha_1, [h_0], \eta \rangle$ lies in $\pi_{125,125+7} S^{0,0}/\lambda^9$, whose AF \leq 12 part is generated by

$$[h_0^2 x_{125,5}] \text{ in AF} = 7,$$

$$\lambda^2 [h_6 (\Delta e_1 + C_0 + h_0^6 h_5^2)] \text{ in AF} = 9,$$

$$[\lambda^4 h_0^2 x_{125,9,2}] \text{ in AF} = 11.$$

From Table 9 in the Appendix, we have

$$0 \neq d_3(x_{126,6}) = \lambda^2 h_5 x_{94,8} + \text{ possibly } \lambda^2 h_6(\Delta e_1 + C_0 + h_0^6 h_5^2).$$

In both scenarios, $\lambda^2 [h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)]$ remains and is in AF = 9.

For the rest of the proof, we only need to rule out the cases $[h_0^2 x_{125,5}]$ in AF = 7 and $\lambda^2 [h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)]$ in AF = 9.

For $[h_0^2 x_{125,5}]$ in AF = 7, due to the nonzero d_3 -differential

$$d_3(x_{126,4}) = \lambda^2 h_0^2 x_{125,5},$$

it can be chosen to be annihilated by λ^2 .

However, from statement (3) and Remark 7.17, $\lambda^8[h_1h_4x_{109,12}]$ remains nonzero in the homotopy of $S^{0,0}/\lambda^9$. Therefore, the synthetic Toda bracket is not annihilated by λ^2 . As discussed earlier, its $[h_0]$ -multiple is detected by $\lambda^6h_1h_4x_{109,12}$, and thus, this case of $[h_0^2x_{125,5}]$ can be ruled out.

For $\lambda^2[h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)]$ in AF = 9, we first consider a classical Toda bracket in stem 125:

$$\langle \theta_5, 2, [\Delta e_1 + C_0 + h_0^6 h_5^2] \rangle.$$

From [56, 29], $\pi_{62} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, so in particular both θ_5 and $[\Delta e_1 + C_0 + h_0^6 h_5^2]$ have order 2, and this classical Toda bracket is well defined.

From classical d_2 -differentials:

$$d_2(h_6) = h_0 h_5^2$$
, $d_2(h_0^6 h_6) = h_0(\Delta e_1 + C_0 + h_0^6 h_5^2)$,

we obtain the following Massey product on the E_3 -page

$$h_6(\Delta e_1 + C_0 + h_0^6 h_5^2) = \langle h_5^2, h_0, \Delta e_1 + C_0 + h_0^6 h_5^2 \rangle,$$

and we check that it has zero indeterminacy. Further analysis shows that there are no crossing differentials, as per the criteria of Moss's theorem [48, Theorem 1.2] (noting that crossing differentials in Moss's theorem have a different meaning from our definition). Therefore, we have a classical Toda bracket

$$[h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)] \in \langle \theta_5, 2, [\Delta e_1 + C_0 + h_0^6 h_5^2] \rangle.$$

Synthetically, by inspection, we also have $2\theta_5 = 0$ and $2[\Delta e_1 + C_0 + h_0^6 h_5^2] = 0$. It follows that there is a corresponding synthetic Toda bracket

$$[h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)] \in \langle \theta_5, 2, [\Delta e_1 + C_0 + h_0^6 h_5^2] \rangle.$$

Multiplying by $\lambda^2[h_0]$, we get:

$$\lambda^{2}[h_{0}] \cdot [h_{6}(\Delta e_{1} + C_{0} + h_{0}^{6}h_{5}^{2})] = \lambda^{2}[h_{0}] \cdot \langle \theta_{5}, 2, [\Delta e_{1} + C_{0} + h_{0}^{6}h_{5}^{2}] \rangle$$

$$= \lambda \cdot \langle 2, \theta_{5}, 2 \rangle [\Delta e_{1} + C_{0} + h_{0}^{6}h_{5}^{2}]$$

$$= \lambda^{3}\eta\theta_{5}[\Delta e_{1} + C_{0} + h_{0}^{6}h_{5}^{2}].$$

Note that all expressions in the above equation have zero indeterminacy.

If the synthetic Toda bracket $\langle \lambda^3 \alpha_1, [h_0], \eta \rangle$ were $\lambda^2 [h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)]$, mapping the above equation to $S^{0,0}/\lambda^9$, we would then have a nonzero equation in $\pi_{125,125+8}S^{0,0}/\lambda^9$.

$$\lambda^{3} \eta[h_{0}^{2} x_{124,8}] = \lambda^{6} [h_{1} h_{4} x_{109,12}] = \langle \lambda^{3} \alpha_{1}, [h_{0}], \eta \rangle [h_{0}] + \lambda^{6} \eta \alpha_{2}$$

$$= \lambda^{2} [h_{6} (\Delta e_{1} + C_{0} + h_{0}^{6} h_{5}^{2})] [h_{0}] + \lambda^{6} \eta \alpha_{2}$$

$$= \lambda^{3} \eta \theta_{5} [\Delta e_{1} + C_{0} + h_{0}^{6} h_{5}^{2}] + \lambda^{6} \eta \alpha_{2}.$$

For this equation to be nonzero, we must have a nonzero equation

$$\lambda^3 [h_0^2 x_{124,8}] = \lambda^3 \theta_5 [\Delta e_1 + C_0 + h_0^6 h_5^2] + \lambda^6 \alpha_2$$
 in $\pi_{124,124+7} S^{0,0} / \lambda^9$.

However, in Ext, we have

$$h_5^2(\Delta e_1 + C_0 + h_0^6 h_5^2) = 0 \neq h_0^2 x_{124,8} \in \text{Ext}_A^{10,124+10},$$

so this equation is not possible.

We have ruled out the possibility that the synthetic Toda bracket $\langle \lambda^3 \alpha_1, [h_0], \eta \rangle$ is detected by $\lambda^2 [h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)]$ or $[h_0^2 x_{125,5}]$. Therefore, we conclude that it must be detected by $[\lambda^4 h_0^2 x_{125,9,2}]$.

From the proof of Lemma 7.16, we have the following $[h_0]$ -extension.

Corollary 7.18. Assuming that both statements (3) and (5') in Proposition 7.8 are true, we have a relation: For any homotopy class $[\lambda^4 h_0^2 x_{125,9,2}]$,

$$[\lambda^4 h_0^2 x_{125,9,2}] \cdot [h_0] = \lambda^6 [h_1 h_4 x_{109,12}] \neq 0 \in \pi_{125,125+8} S^{0,0} / \lambda^9,$$

for some $[h_1h_4x_{109,12}]$.

Proof. From the proof of Lemma 7.16, there exists a homotopy class $[\lambda^4 h_0^2 x_{125,9,2}]$ contained in the synthetic Toda bracket $\langle \lambda^3 \alpha_1, [h_0], \eta \rangle$, and we have

$$[\lambda^4 h_0^2 x_{125,9,2}] \cdot [h_0] = \langle \lambda^3 \alpha_1, [h_0], \eta \rangle \cdot [h_0] = \lambda^6 [h_1 h_4 x_{109,12}] + \lambda^7 \alpha_3.$$

Since $\lambda^7 \alpha_3$ is in a strictly higher filtration than $\lambda^6[h_1h_4x_{109,12}]$, we may choose another class $[h_1h_4x_{109,12}]$ such that the right-hand side of the above equation is simply $\lambda^6[h_1h_4x_{109,12}]$. From the discussion of this synthetic Toda bracket in the proof of Lemma 7.16, we know that the difference of any two classes detected by $\lambda^4h_0^2x_{125,9,2}$, when multiplied by $[h_0]$, is not detected by $\lambda^6[h_1h_4x_{109,12}]$. Therefore, the corollary holds for any choice of $[\lambda^4h_0^2x_{125,9,2}]$.

We have one more Lemma 7.20 before we prove Proposition 7.9.

We draw attention to the following element in Ext:

$$h_1 x_{121,7} \in \operatorname{Ext}_A^{8,122+8}$$

From Table 2 in the Appendix, we have

Fact 7.19. The element $h_1x_{121,7}$ survives to the Adams E_6 -page, and is not killed by any classical differential.

Lemma 7.20. There exists a homotopy class $[\lambda^4 h_1 x_{121,7}] \in \pi_{122,122+4} S^{0,0}/\lambda^9$, such that

$$[\lambda^4 h_1 x_{121,7}] \cdot [h_2] = \lambda [\lambda^5 h_0^2 x_{125,9,2}] \in \pi_{125,125+5} S^{0,0} / \lambda^9.$$

Proof. From Fact 7.19, we know that the element $h_1x_{121,7}$ may only support a nonzero d_r -differential for $r \geq 6$. By Proposition 4.6, $\lambda^4 h_1 x_{121,7}$ detects nonzero homotopy classes in $\pi_{122,122+4} S^{0,0}/\lambda^9$, and hence the required existence of such a homotopy class.

For the desired relation, we follow Example 6.19 and apply the Generalized Mahowald Trick Theorem 6.12. Consider the distinguished triangle

$$S^3 \xrightarrow{\quad \nu \quad} S^0 \xrightarrow{\quad i \quad} S^0/\nu \xrightarrow{\quad q \quad} S^4.$$

The short exact sequence on $H\mathbb{F}_2$ -homology

$$0 \longrightarrow \mathrm{H}\mathbb{F}_{2*}S^0 \xrightarrow{\ i_* \ } \mathrm{H}\mathbb{F}_{2*}S^0/\nu \xrightarrow{\ q_* \ } \mathrm{H}\mathbb{F}_{2*}S^4 \longrightarrow 0,$$

induces a long exact sequence on Ext-groups

$$\cdots \xrightarrow{\cdot h_2} \operatorname{Ext}_A^{*,*}(S^0) \xrightarrow{i_*} \operatorname{Ext}_A^{*,*}(S^0/\nu) \xrightarrow{q_*} \operatorname{Ext}_A^{*,*}(S^4) \xrightarrow{\cdot h_2} \cdots.$$

Also recall for notations, if an element x in $\operatorname{Ext}_A^{*,*}(S^0/\nu)$ satisfies

$$q_*(x) = a \neq 0 \in \operatorname{Ext}_A^{*,*}(S^4),$$

we denote x by a[4]; otherwise, due to exactness,

$$x = i_*(b)$$
 for some $b \in \operatorname{Ext}_A^{*,*}(S^0)$,

and in this case, we denote x by b[0].

We consider

$$x = h_1 x_{121,7} \qquad \in \operatorname{Ext}_A^{8,130},$$

$$\bar{x} = h_1 x_{121,7} [4] + x_{126,8} [0] + x_{126,8,2} [0] \qquad \in \operatorname{Ext}_A^{8,134} (S^0 / \nu),$$

$$y = h_0^2 x_{125,9,2} \qquad \in \operatorname{Ext}_A^{11,136},$$

$$\bar{y} = h_0^2 x_{125,9,2} [0] \qquad \in \operatorname{Ext}_A^{11,136} (S^0 / \nu).$$

For conditions in Theorem 6.12, we have:

- (1) $d_0^{q,E_2}(h_1x_{121,7}[4] + x_{126,8}[0] + x_{126,8,2}[0]) = h_1x_{121,7}.$
- (2) $d_3(h_1x_{121,7}[4] + x_{126,8}[0] + x_{126,8,2}[0]) = h_0^2x_{125,9,2}[0]$. This is a classical Adams d_3 -differential for S^0/ν , and is obtained from our computations.
- (3) (a) The differential in (1) has no crossing, as it is a d_0 -differential, and (b) Upon inspection, the differential in (2) has no crossing.
- (4) $d_0^{i, E_2}(h_0^2 x_{125,9,2}) = h_0^2 x_{125,9,2}[0]$, as $h_0^2 x_{125,9,2}$ is not divisible by h_2 in Ext.

Since all conditions of Theorem 6.12 are satisfied, we conclude that there is an $([h_2], E_4)$ -extension:

$$d_3^{[h_2],E_4}(h_1x_{121,7}) = h_0^2x_{125,9,2}.$$

In other words, we have the following relation:

$$[h_1 x_{121,7}] \cdot [h_2] = \lambda^2 [h_0^2 x_{125,9,2}] \in \pi_{125,125+9} S^{0,0} / \lambda^3.$$

Using the map $\rho: S^{0,0}/\lambda^5 \to S^{0,0}/\lambda^3$ from Notation 4.1, we lift the above relation and obtain:

$$[h_1 x_{121,7}] \cdot [h_2] = [\lambda^2 h_0^2 x_{125,9,2}] \in \pi_{125,125+9} S^{0,0} / \lambda^5.$$

In fact, since $h_1x_{121,7} \cdot h_2 = 0$ in AF = 9 of Ext, we might obtain a relation of the form:

$$[h_1 x_{121,7}] \cdot [h_2] = [\lambda x] + [\lambda^2 h_0^2 x_{125,9,2}] \in \pi_{125,125+9} S^{0,0} / \lambda^5,$$

for some element x in AF = 10. However, upon inspection, for all $x \in \operatorname{Ext}_A^{10,125+10}$, the homotopy class $[\lambda x]$ either does not exist or can be chosen to be zero.

Using the map $\lambda^4: \Sigma^{0,-4}S^{0,0}/\lambda^5 \to S^{0,0}/\lambda^9$ from Notation 4.1, we further push the above relation and obtain the following relation:

$$[\lambda^4 h_1 x_{121,7}] \cdot [h_2] = \lambda [\lambda^5 h_0^2 x_{125,9,2}] \in \pi_{125,125+5} S^{0,0} / \lambda^9$$

This completes the proof.

Now we prove Proposition 7.9.

We draw attention to the following elements in Ext:

$$h_6 M d_0 \in \operatorname{Ext}_A^{11,122+11},$$

 $h_5 x_{91,11} \in \operatorname{Ext}_A^{12,122+12}.$

From Table 2 in the Appendix, we have

Fact 7.21.

- (1) The element h_6Md_0 survives to the Adams E_{∞} -page.
- (2) The element $h_5x_{91,11}$ survives to the Adams E_{∞} -page.

Proof of Proposition 7.9. We assume that statement (3) in Proposition 7.8 is true. For the sake of a contradiction, we also assume statement (5') is true.

From Lemma 7.20 and Corollary 7.18, there exists a homotopy class $[\lambda^4 h_1 x_{121,7}] \in \pi_{122,122+4} S^{0,0}/\lambda^9$, such that

$$[\lambda^4 h_1 x_{121,7}] \cdot [h_2] \cdot [h_0] = \lambda [\lambda^5 h_0^2 x_{125,9,2}] \cdot [h_0]$$
$$= \lambda^8 [h_1 h_4 x_{109,12}] \neq 0 \in \pi_{125,125+6} S^{0,0} / \lambda^9,$$

for some $[h_1h_4x_{109,12}]$. Note that statement (3) in Proposition 7.8, Fact 7.6(2), and Proposition 4.6 imply that the expression $\lambda^8[h_1h_4x_{109,12}]$ is nonzero in the homotopy of $S^{0,0}/\lambda^9$.

Since the element detecting $h_1h_4x_{109,12}$ in not an h_2 -multiple in Ext, we must have the expression

$$[\lambda^4 h_1 x_{121,7}] \cdot [h_0]$$

in $\pi_{122,122+5}S^{0,0}/\lambda^9$ be nonzero, and have AF ≤ 12 . Upon inspection, the only possibilities are:

$$\lambda^{6}[h_{6}Md_{0}]$$
 in AF = 11, $\lambda^{7}[h_{5}x_{91,11}]$ in AF = 12.

From Fact 7.21 both h_6Md_0 and $h_5x_{91,11}$ are nonzero permanent cycles in the classical Adams spectral sequence, so either possibility would lift to a relation in the homotopy groups of $S^{0,0}$.

For the case of $\lambda^6[h_6Md_0]$, consider the expression

$$\lambda^2 [h_6 M d_0] \cdot [h_2] \in \pi_{125,125+10} S^{0,0}.$$

Since $h_6Md_0 \cdot h_2$ in AF = 12 is killed by a classical d_2 -differential, we know that

$$\lambda[h_6Md_0] \cdot [h_2] \in \pi_{125,125+11}S^{0,0}$$

is in AF \geq 13. Upon inspection, the permanent cycles in $\operatorname{Ext}_A^{13,125+13}$ are all killed by d_2 or d_4 -differentials, and thus,

$$\lambda^2[h_6Md_0] \cdot [h_2] \in \pi_{125,125+10}S^{0,0}$$

is in AF \geq 14. Therefore, for the above relation in the homotopy of $S^{0,0}/\lambda^9$ to hold, we must have

$$\lambda^{2}[h_{6}Md_{0}] \cdot [h_{2}] = \lambda^{4}[h_{1}h_{4}x_{109,12}] \in \pi_{125,125+10}S^{0,0}.$$

For the case of $\lambda^7[h_5x_{91,11}]$, consider the expression

$$\lambda[h_5x_{91,11}] \cdot [h_2] \in \pi_{125,125+12}S^{0,0}.$$

Since $h_5 x_{91,11} \cdot h_2$ in AF = 13 is killed by a classical d_2 -differential, we know that

$$\lambda[h_5x_{91.11}] \cdot [h_2] \in \pi_{125.125+12}S^{0,0}$$

is in AF \geq 14. Therefore, for the above relation in the homotopy of $S^{0,0}/\lambda^9$ to hold, we must have

$$\lambda[h_5x_{91.11}] \cdot [h_2] = \lambda^2[h_1h_4x_{109.12}] \in \pi_{125.125+12}S^{0,0}.$$

In both cases, $\lambda^4[h_1h_4x_{109,12}]$ is a $\lambda[h_2]$ -multiple in the homotopy of $S^{0,0}$. Since the classical $\nu \in \pi_3$ has AF = 1, we have

$$S^{0,0}/(\lambda[h_2]) \simeq \nu(S^0/\nu).$$

By the rigidity Theorem 3.6 of the synthetic Adams spectral sequence for $S^{0,0}/(\lambda[h_2])$, we know that the element $\lambda^4 h_1 h_4 x_{109,12}[0]$ must be killed by a synthetic Adams differential, which corresponds to to a statement that in the classical Adams spectral sequence of S^0/ν , the element $h_1 h_4 x_{109,12}[0]$ must be killed by a d_r -differential for $r \leq 5$.

	Diamanta	1	1
s	Elements	d_r	value
	$h_0^{12}h_6^2[0]$	d_2^{-1}	$h_0^{11}h_7[0]$
14	$x_{126,14}[0]$	d_3^{-1}	$(((x_{123,11,2}) + (x_{123,11}) + h_0h_6[B_4])[4])$
1.1	$Q_2D_2(h_3[4]) + D_2x_{68,8}[0]$	d_3^{-1}	$(((x_{123,11,2}) + h_5(x_{92,10}))[4])$
	$D_2x_{68,8}[0]$	d_2	$h_0Q_2x_{68,8}[0]$
	$h_0^{11}h_6^2[0]$	d_2^{-1}	$h_0^{10}h_7[0]$
	$h_1 x_{120,11}(h_1[4])$	d_{12}	?
13	$h_1 x_{125,12,2}[0]$	d_5	$d_0^2 x_{97,10}[0]$
10	$h_6 x_{56,10} (h_0 h_2[4])$	d_3	$h_1 x_{124,15}[0]$
	$(((x_{122,13}) + h_1^2(x_{120,11}) + h_0^2h_6(Md_0))[4])$	d_2	$x_{125,15}[0] + h_0^3 x_{125,12}[0]$
	$h_0 h_3 x_{119,11}[0]$	d_2	$h_0^3 x_{125,12}[0]$
	$ d_1 x_{94,8}[0] h_0^{10} h_6^{2}[0] $	d_2^{-1}	$x_{127,10}[0]$
12	$h_0^{10}h_6^2[0]$	d_2^{-1}	$h_0^9 h_7[0]$
12	$((h_5(x_{91,11}) + h_0(x_{122,11}))[4])$	d_3	$Q_2x_{68,8}[0]$
	$h_3x_{119,11}[0]$	d_2	$h_0^2 x_{125,12}[0]$
	$h_0^9 h_6^2[0]$	d_2^{-1}	$h_0^8 h_7[0]$
11	$x_{126,11}[0]$	d_3^{-1}	$x_{127,8}[0]$
11	$h_1x_{125,10,2}[0] + h_1x_{125,10}[0]$		Permanent
	$h_1 x_{125,10}[0]$	d_{14}	?
	$h_0^2 x_{126,8}[0]$	d_2^{-1}	$h_0 x_{127,7}[0]$
10	$h_0^8 h_6^2 [0]$	d_2^{-1} d_2^{-1}	$h_0^7 h_7[0]$
	$x_{126,10}[0]$	d_3	$nx_{94,8}[0]$
	$h_0x_{126,8}[0]$	d_2^{-1}	$x_{127,7}[0]$
	$h_0^{7}h_0^{2}[0]$	d_2^{-1}	$h_0^{6}h_7[0]$
9	$h_1x_{125,8}[0]$	d_{16}	?
	$h_0 x_{126,8,3}[0]$	d_4	$h_0 x_{125,12}[0]$
	$x_{126,9}[0]$	d_3	$h_0^4 x_{125,8}[0]$

Table 1. The classical Adams spectral sequence of S^0/ν for $9 \le s \le 14$ in stem 126

However, from Table 1 (obtained from [38] [37], and can be visualized from [35]), in the classical Adams spectral sequence of S^0/ν , the element $h_1h_4x_{109,12}[0]$ is not killed by any d_r for $r \leq 5$ (from the range of $9 \leq s \leq 14$ in stem 126). Therefore, we arrive at a contradiction.

8. Appendix: The classical Adams spectral sequence in the range 122 < t-s < 127, s < 25

We provide a brief overview of Lin's computer program for computing Adams differentials and extensions. The program's functionality for propagating differentials and extensions relies on the following data:

- The Adams E_2 -pages of a collection of CW spectra.
- Maps between these E_2 -pages.
- \bullet Adams d_2 -differentials for certain CW spectra.
- There are three manually added differentials: $d_5h_0^{24}h_6 = h_0^2P^6d_0$ and $d_6h_0^{55}h_7 = h_0^2x_{126,60}$ in the Adams spectral sequence of S^0 (from the image of J), and $d_3v_2^{16} = \beta^5g$ in the Adams spectral sequence of tmf, derived from power operations (Bruner–Rognes [12]).

Detailed descriptions of these data are available in [37]. Lin's program extends these results by computing additional Adams differentials using tools such as the Leibniz rule, naturality, the Generalized Leibniz Rule, and the Generalized Mahowald Trick. All computed differentials and extensions are accessible via interactive plots [35].

Moreover, the proofs provided in [37] offer more information than the interactive plots. These proofs include numerous disproofs of potential differentials, even for cases where the differentials remain unresolved. For example, consider

$$x_{126,21} \in E_2^{21,126+21}(S^0).$$

The spectral sequence plot for S^0 shows that $x_{126,21}$ survives to the E_4 -page, but the value of $d_4(x_{126,21})$ undetermined. By analyzing the proofs in [37], we observe that many potential values for $d_4(x_{126,21})$ have been ruled out. Consequently, we conclude:

$$d_4(x_{126,21}) = x_{125,25,2} + x_{125,25} + g^4 \Delta h_1 g + \text{possibly } d_0^2 e_0 g B_4.$$

Tables 2–12 present results from Lin's program for the classical Adams spectral sequence of the sphere in the range $122 \le t-s \le 127, s \le 25$.

s	Elements	d_r	value
05	$e_0 g^3 \Delta h_1 g$	d_2^{-1}	$g^2\Delta^2m$
25	$d_0^3 g[B_4]$	$\overline{d_4}$	$d_0^4 x_{65,13}$
24	$d_0g\Delta^2g^2$	d_2	$d_0e_0g^3\Delta h_2^2$
23	$Ph_1x_{113,18,2}$	d_3^{-1}	$x_{123,20}$
23	$h_0^2 d_0 x_{108,17}$	d_3	$d_0^4 Mg$
	e_0g^2Mg	d_3^{-1}	$h_0^2 h_3 x_{116,16}$
22	$x_{122,22}$	d_3	$h_0^2 d_0 x_{107,19}$
	$h_0d_0x_{108,17}$	d_2	$h_0^2 d_0 x_{107,18}$
21	$h_0^2 d_0 e_0 x_{91,11}$		Permanent
21	$d_0x_{108,17}$	d_2	$d_0 e_0 \Delta h_2^2 [B_4] + h_0 d_0 x_{107,18}$
	$h_0^4 x_{122,16}$	d_2^{-1}	$h_0h_3x_{116,16}$
20	$g^3(C_0 + h_0^6 h_5^2)$	d_3	$g^3\Delta h_2^2n$
	$h_0 d_0 e_0 x_{91,11}$	d_2	$h_0^6 x_{121,16}$
19	$h_0^3 x_{122,16}$	d_2^{-1}	$h_3x_{116,16}$
10	$d_0e_0x_{91,11}$	d_2	$h_0 d_0^2 x_{93,12}$
	$h_0^2 x_{122,16}$	d_3^{-1}	$h_0^2 x_{123,13,2}$
18	$h_1 x_{121,17}$	d_7^{-1}	$x_{123,11,2} + x_{123,11} + h_0 h_6 [B_4]$
	$d_0x_{108,14}$	d_3	$h_0 d_0^2 x_{93,12} + h_0^5 x_{121,16}$
17	$h_0 x_{122,16}$	d_3^{-1} d_3^{-1} d_3^{-1}	$h_0 x_{123,13,2}$
	$g^{3}[H_{1}]$	d_3^{-1}	$\Delta h_2^2 x_{93,8}$
	$x_{122,16} + h_0 x_{122,15,2}$	d_3^{-1} d_3^{-1}	$x_{123,13,2}$
16	$\Delta h_2^2 x_{92,10}$	d_3^{-1}	$x_{123,13}$
	$h_0 x_{122,15,2}$		Permanent
15	$x_{122,15}$	d_3	g^3A
10	$x_{122,15,2}$	d_2	$h_0^2 h_4 x_{106,14}$
14	$h_0x_{122,13}$	d_3^{-1}	$x_{123,11,2}$
	$h_0^2 h_6 M d_0$	d_2^{-1}	$x_{123,11}$
13	$h_1^2 x_{120,11}$		Permanent
	$x_{122,13}$	d_3	$h_0h_4x_{106,14}$
	$h_0h_6Md_0$	d_2^{-1} d_3^{-1}	$x_{123,10}$
12	$h_0x_{122,11}$	d_3^{-1}	$h_0 x_{123,8}$
	$h_5 x_{91,11}$		Permanent
11	$x_{122,11} + h_6 M d_0$	d_3^{-1}	$x_{123,8}$
	h_6Md_0		Permanent
9-10			
8	$h_1x_{121,7}$	d_6	?
0-7			

Table 2. The classical Adams spectral sequence of S^0 for $s \leq 25$ in stem 122

s	Elements	d_r	value
25			
	$h_1Ph_1x_{113,18,2}$	d^{-1}	m
24	$\frac{h_1 \Gamma h_1 x_{113,18,2}}{d_0^2 \Delta h_2^2 M g}$	d_2^{-1} d_2^{-1}	$x_{124,22}$
24	$\frac{d_0\Delta h_2Mg}{d_0MPx_{56,10}}$	$\frac{a_2}{d_4}$	$\begin{array}{c} d_0 x_{110,18} \\ MP x_{69,18} \end{array}$
02	$\frac{a_0MLx_{56,10}}{g^2\Delta^2m}$		
23	$g^-\Delta^-m$	d_2	$e_0g^3\Delta h_1g$
21-22			
20	$x_{123,20}$	d_3	$Ph_1x_{113,18,2}$
	$e_0 g^2 x_{66,7} + h_0^6 x_{123,13,2}$	d_2^{-1}	$x_{124,17}$
19	$h_0^6 x_{123,13,2}$	$\frac{d_2}{d_2^{-1}}$	$h_0^3 x_{124,14,2}$
10	$h_0e_0x_{106,14} + h_0^2h_3x_{116,16}$	d_3	$e_0x_{107,12}$
	$h_0^2 h_3 x_{116,16}$	d_3	e_0g^2Mg
	$h_0^5 x_{123,13,2}$	d_2^{-1}	$h_0^2 x_{124,14,2}$
18	$e_0 x_{106,14} + h_0 h_3 x_{116,16}$		Permanent
	$h_0h_3x_{116,16}$	d_2	$h_0^4 x_{122,16}$
	$h_0^2 x_{123,15} + h_0^4 x_{123,13,2}$	d_2^{-1} d_2^{-1}	$h_0x_{124,14}$
17	$h_0^4 x_{123,13,2}$	d_2^{-1}	$h_0 x_{124,14,2} + h_0 x_{124,14}$
11	$d_0x_{109,13}$	_	Permanent
	$h_3x_{116,16}$	d_2	$h_0^3 x_{122,16}$
	$h_0 x_{123,15} + h_0^3 x_{123,13,2}$	d_2^{-1}	$x_{124,14}$
16	$h_0^3 x_{123,13,2}$	d_2^{-1}	$x_{124,14,2} + x_{124,14}$
	$h_1x_{122,15,2}$	d_2^{-1} d_3^{-1}	$h_4x_{109,12}$
	$x_{123,15}$	d_4^{-1}	$x_{124,11,2} + x_{124,11}$
15	$h_4x_{108,14}$	d_5^{-1}	$h_0 x_{124,9}$
	$h_0^2 x_{123,13,2}$	d_3	$h_0^2 x_{122,16}$
	$h_0^3 x_{123,11}$	d_2^{-1}	$h_0x_{124,11}$
14	$h_0 x_{123,13,2}$	d_3	$h_0x_{122,16}$
	$\Delta h_2^2 x_{93,8}$	d_3	$g^3[H_1]$
	$h_0^2 x_{123,11}$	d_2^{-1}	$x_{124,11}$
13	$x_{123,13,2}$	d_3	$x_{122,16} + h_0 x_{122,15,2}$
	$x_{123.13}$	d_3	$\Delta h_2^2 x_{92,10}$
	$h_0 x_{123,11} + h_0^2 h_6 [B_4]$		$x_{124,9,2} + h_0 x_{124,8}$
12	$x_{123,12}$	d_3^{-1} d_3^{-1}	$x_{124,9} + h_0 x_{124,8}$
	$h_0^2 h_6 [B_4]$	d_5^{-1}	h_6A
	$h_0^2 x_{123,9}$	d_2^{-1}	-
	1	d_5^{-1}	$x_{124,9}$ $x_{124,6}$
11	$\frac{h_5 x_{92,10}}{x_{123,11,2} + x_{123,11} + h_0 h_6[B_4]}$	d_{7}	$h_1x_{121,17}$
	$x_{123,11} + h_0 h_6 [B_4]$	d_3	$h_0 x_{122,13}$
	$h_0h_6[B_4]$	d_2	$h_0^2 h_6 M d_0$
	$h_0 x_{123,9}$	d_2^{-1}	$x_{124,8}$
10	$\frac{x_{123,10} + h_6[B_4]}{x_{123,10} + h_6[B_4]}$	d_3^{-1}	$x_{124,8}$ $x_{124,7}$
	$h_6[B_4]$	d_2	$h_0 h_6 M d_0$
	$x_{123,9} + h_0 x_{123,8}$	d_{12}	7
9	$h_0 x_{123,8} + h_0 x_{123,8}$	d_3	$h_0 x_{122,11} + h_0 h_6 M d_0$
8		d_3	$x_{122,11} + h_0 M d_0$
0-7	$x_{123,8}$	a_3	$\omega_{122,11} + n_6 m \omega_0$
0-7			

Table 3. The classical Adams spectral sequence of S^0 for $s \leq 25$ in stem 123

s	Elements	d_r	value
		d_2^{-1}	$h_0^9 x_{125,14}$
25	$\begin{array}{c} h_0^{11} x_{124,14,2} \\ i x_{101,18} \end{array}$	d_2	$\frac{h_0 x_{125,14}}{d_0^2 \Delta h_2^2 x_{65,13} + h_0 P d_0 x_{101,18}}$
20	$\frac{du_{101,18}}{d_0e_0\Delta^3h_1g}$	d_2	$\frac{d_0^2 g^3 m}{d_0^2 g^3 m}$
	$h_{10}^{10}r_{10}$		$h_0^8 x_{125,14}$
24	$\begin{array}{c} h_0^{10} x_{124,14,2} \\ h_0^2 d_0 x_{110,18} \end{array}$	d_2^{-1} d_2^{-1}	$h_0 a_{125,14} $ $h_0 e_0 x_{108,17}$
	h^9x_1	d^{-1}	$h_0^7 x_{125,14}$
	$h_0^9 x_{124,14,2} d_0 g \Delta h_2^2 [B_4] + h_0 d_0 x_{110,18}$	d_2^{-1} d_2^{-1} d_3^{-1}	$n_0 x_{125,14}$ $e_0 x_{108,17}$
23	$\frac{h_0 g \Delta h_2 [D4] + h_0 a_0 x_{110,18}}{h_0 d_0 x_{110,18}}$	$\frac{u_2}{d^{-1}}$,
	$\frac{d_0x_{110,18}}{d_0x_{110,19}}$	u_3	$x_{125,20}$ Permanent
		<i>d</i> −1	
	$\frac{h_0^8 x_{124,14,2}}{g^2 \Delta^2 t}$	d_2^{-1} d_3^{-1}	$h_0^6 x_{125,14}$
22	$\frac{y \Delta t}{x_{124,22}}$	d_3	$gx_{105,15}$
	$\frac{x_{124,22}}{d_0 x_{110,18}}$		$h_1 P h_1 x_{113,18,2}$ $d_0^2 \Delta h_2^2 M g$
	1.7	d_2	$a_0 \Delta n_2 m g$
21	$\begin{array}{c} h_0^7 x_{124,14,2} \\ h_0 d_0^2 [\Delta \Delta_1 g] \end{array}$	a_2	$ \frac{d_0^2 \Delta h_2^2 M g}{h_0^5 x_{125,14}} \\ d_0 g x_{91,11} $
	$n_0 a_{\bar{0}} [\Delta \Delta_1 g]$	a_2	$a_0 g x_{91,11}$
20	$h_0^6 x_{124,14,2}$	d_2^{-1} d_5^{-1}	$h_0^4 x_{125,14}$
	$d_0^2[\Delta\Delta_1g]$	d_5	$h_1x_{124,14}$
19	$h_0^5 x_{124,14,2}$	d_2^{-1}	$h_0^3 x_{125,14}$
10	$d_0x_{110,15}$		Permanent
18	$h_0 x_{124,17} + h_0^4 x_{124,14,2}$	d_2^{-1}	$x_{125,16}$
10	$h_0^4 x_{124,14,2}$	d_2^{-1} d_2^{-1}	$h_0^2 x_{125,14}$
	$h_0^3 x_{124,14}$	$ \begin{array}{c c} d_2^{-1} \\ d_2^{-1} \\ d_2 \end{array} $	$x_{125,15}$
17	$h_0^2 x_{124,15}$	d_2^{-1}	$h_0x_{125,14}$
11	$x_{124,17}$	d_2	$e_0g^2x_{66,7} + h_0^6x_{123,13,2}$
	$h_0^3 x_{124,14,2}$	d_2	$h_0^6 x_{123,13,2}$
	$h_0x_{124,15}$	d_2^{-1}	$x_{125,14}$
	$h_1x_{123,15}$	d_3^{-1}	$h_3x_{118,12}$
16	$h_0^2 x_{124,14}$		Permanent
	$e_0x_{107,12}$	d_3	$e_0 g^2 x_{66,7} + h_0 e_0 x_{106,14} + h_0^2 h_3 x_{116,16}$
	$h_0^2 x_{124,14,2}$	d_2	$h_0^5 x_{123,13,2}$
	$x_{124,15}$	d_4^{-1}	$h_6x_{62,10}$
15	$h_3^2 x_{110,13} + h_0 x_{124,14}$		Permanent
	$h_0 x_{124,14}$	d_2	$h_0^2 x_{123,15} + h_0^4 x_{123,13,2}$
<u></u>	$h_0 x_{124,14,2}$	d_2	$h_0^2 x_{123,15}$
	$h_1x_{123,13}$	d_2^{-1}	$x_{125,12}$
	$h_1x_{123,13,2}$	d_2^{-1}	$x_{125,12,2}$
14	$\Delta h_2^2 x_{94,8}$		Permanent
	$x_{124,14}$	d_2	$h_0 x_{123,15} + h_0^3 x_{123,13,2}$
igsquare	$x_{124,14,2}$	d_2	$h_0 x_{123,15}$
	$h_0^5 x_{124,8}$	d_2^{-1}	$h_0^3 x_{125,8}$
13	$[H_1](\Delta e_1 + C_0 + h_0^6 h_5^2)$	d_3^{-1}	$x_{125,10,2}$
	$e_0\Delta h_6 g$		Permanent
	$h_4x_{109,12}$	d_3	$h_1 x_{122,15,2}$

Table 4. The classical Adams spectral sequence of S^0 for $13 \leq s \leq 25$ in stem 124

	$h_0 x_{124,11,2} + h_0 x_{124,11}$	d_2^{-1}	$x_{125,10}$
	$h_0^2 x_{124,10,2} + h_0^4 x_{124,8}$	$ \begin{array}{c} d_2^{-1} \\ d_2^{-1} \\ d_2^{-1} \\ d_3^{-1} \\ d_2 \end{array} $	$h_0x_{125,9,2}$
12	$h_0^4 x_{124,8}$	d_2^{-1}	$h_0^2 x_{125,8}$
	$h_1 x_{123,11,2}$	d_3^{-1}	$x_{125,9}$
	$h_0x_{124,11}$		$h_0^3 x_{123,11}$
	$h_0 x_{124,10,2} + h_0^3 x_{124,8}$	$\begin{array}{c} d_2^{-1} \\ d_2^{-1} \\ d_3^{-1} \end{array}$	$x_{125,9,2}$
	$h_0^3 x_{124,8}$	d_2^{-1}	$h_0x_{125,8}$
11	$x_{124,11,3}$	d_3^{-1}	$x_{125,8,2}$
	$x_{124,11,2} + x_{124,11}$	d_4	$x_{123,15}$
	$x_{124,11}$	d_2	$h_0^2 x_{123,11}$
	$h_1 x_{123,9} + h_0^2 x_{124,8}$	$\begin{array}{c} d_2^{-1} \\ d_4^{-1} \\ d_4^{-1} \end{array}$	$x_{125,8}$
	$x_{124,10,2} + h_0 x_{124,9}$	d_4^{-1}	$h_6[H_1]$
10	$x_{124,10} + h_0^2 x_{124,8}$	d_4^{-1}	$h_6[H_1] + h_0 x_{125,5}$
	$h_0^2 x_{124,8}$		Permanent
	$h_0x_{124,9}$	d_5	$h_4x_{108,14}$
	$x_{124,9,2} + h_0 x_{124,8}$	d_3	$h_0 x_{123,11} + h_0^2 h_6 [B_4]$
9	$x_{124,9} + h_0 x_{124,8}$	d_3	$x_{123,12}$
	$h_0x_{124,8}$	d_2	$h_0^2 x_{123,9}$
8	$x_{124,8}$	d_2	$h_0 x_{123,9}$
	$h_0 x_{124,6}$	d_2^{-1}	$x_{125,5}$
7	h_6A	d_5	$h_0^2 h_6 [B_4]$
	$x_{124,7}$	d_3	$x_{123,10} + h_6[B_4]$
6	$x_{124,6}$	d_5	$h_5 x_{92,10}$
0-5		·	

Table 5. The classical Adams spectral sequence of S^0 for $s \leq 12$ in stem 124

s	Elements	d_r	value
	$d_0^2 e_0 g[B_4]$	d_4^{-1} d_4^{-1}	$h_1x_{125,20}$
25	$x_{125,25,2} + x_{125,25} + g^4 \Delta h_1 g$	d_4^{-1}	$x_{126,21} + \text{possibly } h_1 x_{125,20}$
20	$g^4\Delta h_1 g$		Permanent
	$x_{125,25}$	d_2	$h_0x_{124,26}$
24	$e_0g\Delta^2g^2$	d_2	$d_0g^4\Delta h_2^2$
23	$h_0^2 e_0 x_{108,17}$	d_3	$d_0^3 e_0 Mg$
20	$h_0^9 x_{125,14}$	d_2	$h_0^{11}x_{124,14,2}$
	g^3Mg	d_4^{-1}	$x_{126,18}$
22	$ix_{102,15} + h_0^8 x_{125,14}$	d_4^{-1}	$gx_{106,14} + e_0x_{109,14,2}$
22	$h_0^8 x_{125,14}$	d_2	$h_0^{10}x_{124,14,2}$
	$h_0e_0x_{108,17}$	d_2	$h_0^2 d_0 x_{110,18}$
	$x_{125,21}$	d_4^{-1}	$d_0x_{112,13}$
21	$h_0^7 x_{125,14}$	d_2	$h_0^9 x_{124,14,2}$
	$e_0 x_{108,17}$	d_2	$d_0g\Delta h_2^2[B_4] + h_0d_0x_{110,18}$
	$h_0 d_0 g x_{91,11}$	d_2^{-1}	$x_{126,18,2}$
20	$x_{125,20}$	d_3	$d_0g\Delta h_2^2[B_4]$
	$h_0^6 x_{125,14}$	d_2	$h_0^8 x_{124,14,2}$

Table 6. The classical Adams spectral sequence of S^0 for $20 \le s \le 25$ in stem 125

	$gx_{105,15}$	d_3	$q^2\Delta^2t$
19	$h_0^5 x_{125,14}$	d_2	$h_0^7 x_{124,14,2}$
10	$d_0gx_{91,11}$	d_2	$h_0 d_0^2 [\Delta \Delta_1 g]$
	$d_0^2 x_{97,10}$	d_5^{-1}	$h_1x_{125,12,2}$
18	$h_0^4 x_{125,14}$	d_2	$h_0^{14,125,12,2}$ $h_0^{6}x_{124,14,2}$
		d_2^{-1}	
17	$h_0^2 Q_2 x_{68,8}$		$h_0 D_2 x_{68,8}$
	$h_0^3 x_{125,14}$	d_2	$h_0^5 x_{124,14,2}$
	$h_0Q_2x_{68,8}$	d_2^{-1}	$D_2x_{68,8}$
16	$h_1x_{124,15}$	d_4^{-1}	$h_0^2 x_{126,10}$
	$x_{125,16}$	d_2	$h_0 x_{124,17} + h_0^4 x_{124,14,2}$
	$h_0^2 x_{125,14}$	d_2	$h_0^4x_{124,14,2}$
	$h_0^3 x_{125,12}$	d_2^{-1}	$h_0h_3x_{119,11}$
1.5	$Q_2x_{68,8}$	d_4^{-1}	$h_1x_{125,10}$
15	$h_1x_{124,14}$	d_5	$\frac{d_0^2[\Delta\Delta_1 g]}{d_0^2[\Delta\Delta_1 g]}$
	x _{125,15}	d_2	$h_0^3 x_{124,14}$
	$h_0x_{125,14}$	d_2	$h_0^2 x_{124,15}$
, ,	$h_0^2 x_{125,12}$	d_2^{-1}	$h_3x_{119,11}$
14	$h_1h_4x_{109,12}$,	Permanent
	$x_{125,14}$	d_2	$h_0 x_{124,15}$
	$h_0^4 x_{125,9,2}$	d_2^{-1}	$x_{126,11}$
	$h_0^5 x_{125,8}$	d_2^{-1}	$h_0x_{126,10}$
13	$nx_{94,8}$	d_4^{-1}	$h_1x_{125,8}$
	$h_0x_{125,12}$	d_4^{-1}	$h_1x_{125,8,2}$
	$h_3x_{118,12}$	d_3	$h_1x_{123,15}$
	$h_0h_6x_{62,10}$	d_2^{-1}	$x_{126,10}$
	$h_0^3 x_{125,9,2} + h_0^4 x_{125,8}$	d_3^{-1}	$x_{126,9}$
12	$h_0^4 x_{125,8}$	d_3^{-1}	$x_{126,9} + h_0 x_{126,8,3}$
	$x_{125,12}$	d_2	$h_1x_{123,13}$
	$x_{125,12,2}$	d_2	$h_1 x_{123,13,2}$
	$h_1x_{124,10,2}$	d_3^{-1}	$x_{126,8}$
	$h_1x_{124,10}$	d_3^{-1}	$x_{126,8,2}$
11	$h_0^2 x_{125,9,2}$	d_5	?
	$h_6x_{62,10}$	d_4	$x_{124,15}$
	$h_0^3 x_{125,8}$	d_2	$h_0^5 x_{124,8}$
	$h_0x_{125,9}$	d_2^{-1}	$x_{126,8,3}$
	$x_{125,10,2}$	$\overline{d_3}$	$[H_1](\Delta e_1 + C_0 + h_0^6 h_5^2)$
10	$x_{125,10}$	d_2	$h_0 x_{124,11,2} + h_0 x_{124,11}$
	$h_0x_{125,9,2}$	d_2	$h_0^2 x_{124,10,2} + h_0^4 x_{124,8}$
<u></u>	$h_0^2 x_{125,8}$	d_2	$h_0^4 x_{124,8}$
	$h_6(\Delta e_1 + C_0 + h_0^6 h_5^2)$		Permanent
	$h_5x_{94,8}$	d_7	?
9	$x_{125,9}$	d_3	$h_1x_{123,11,2}$
	$x_{125,9,2}$	d_2	$h_0 x_{124,10,2} + h_0^3 x_{124,8}$
<u> </u>	$h_0x_{125,8}$	d_2	$h_0^3 x_{124,8}$
8	$x_{125,8,2}$	d_3	$x_{124,11,3}$
S	$x_{125,8}$	d_2	$h_1 x_{123,9} + h_0^2 x_{124,8}$
7	$h_0^2 x_{125,5}$	d_3^{-1}	$x_{126,4}$
	$h_6[H_1]$	d_4	$x_{124,10,2} + h_0 x_{124,9}$
6	$h_0 x_{125,5}$	d_4	$x_{124,10,2} + x_{124,10} + h_0 x_{124,9} + h_0^2 x_{124,8}$
5		d_2	$\frac{124,10,2 + 124,10 + 104,124,9 + 104,124,8}{h_0 x_{124,6}}$
0-4	$x_{125,5}$	ω_2	100w124,0
0-4			

Table 7. The classical Adams spectral sequence of S^0 for $s \leq 19$ in stem 125

s	Elements	d_r	value
25	$h_0^7 x_{126,18}$	d_3^{-1}	$h_0^{21}h_7$
20	$d_0 e_0 \Delta h_2^2 Mg$		
	$\frac{h_0^6 x_{126,18}}{h_0^6 x_{126,18}}$	d_2^{-1}	$\frac{d_0 x_{113,18}}{h_0^{20} h_7}$
24	$g^4 \Delta h_2 c_1$	$\frac{u_3}{d^{-1}}$	$\frac{n_0}{g^3C''}$
	$\frac{g \Delta h_2 c_1}{d_0 P d_0 M^2}$	$\frac{a_3}{d^{-1}}$	$d_0e_0[\Delta\Delta_1g]$
-		$\begin{array}{c} d_3^{-1} \\ d_3^{-1} \\ d_4^{-1} \\ \hline d_4^{-1} \\ \hline d_4^{-1} \\ \hline d_4^{-1} \\ \end{array}$	
23	$h_0^5 x_{126,18}$	a_3	$h_0^{19}h_7$
	$x_{126,23}$	u_4	$e_0x_{110,15}$
22	$h_0 x_{126,21} + h_0^4 x_{126,18}$	d_3^{-1} d_3^{-1}	$h_1x_{126,18,2}$
	$h_0^4 x_{126,18}$	a_3	$h_0^{18}h_7$
0.1	$h_0^3 x_{126,18}$	$d_3^{-1} \\ d_4$	$h_0^{17}h_7$
21	$h_1x_{125,20}$	a_4	$d_0^2 e_0 g[B_4]$
	$x_{126,21}$	d_4	$x_{125,25,2} + x_{125,25} + g^4 \Delta h_1 g + \text{possibly } d_0^2 e_0 g B_4$
20	$h_0^2 x_{126,18}$	d_3^{-1} d_5^{-1}	$h_0^{16}h_7$
	$d_0x_{112,16}$	d_5	$x_{127,15}$
19	$h_0x_{126,18}$	d_3^{-1} d_3^{-1}	$h_0^{15}h_7$
10	$g^3x_{66,7}$	d_3^{-1}	$gx_{107,12}$
	$x_{126,18} + e_0 x_{109,14,2}$	d_7	?
18	$e_0x_{109,14,2}$	d_4	g^3Mg
10	$gx_{106,14}$	d_4	$ix_{102,15} + g^3Mg + h_0^8x_{125,14}$
	$x_{126,18,2}$	d_2	$h_0 d_0 g x_{91,11}$
	$h_0^{15}h_6^2$	d_2^{-1}	$h_0^{14}h_7$
17	$h_1^2 x_{124,15}$	$d_2^{-1} d_8$	$h_6x_{64,14}$
	$x_{126,17}$	d_8	?
	$d_0x_{112,13}$	d_4	$x_{125,21}$
	$h_0^{14}h_6^2$	d_2^{-1}	$h_0^{13}h_7$
16	$h_0^2 D_2 x_{68,8}$	d_3^{-1}	$x_{127,13}$
	$h_1^2 x_{124,14}$	d_6^{-1}	$h_2 x_{124,9} + h_0^2 x_{127,8}$
15	$h_0^{13}h_6^2$	d_2^{-1} d_2	$h_0^{12}h_7$
10	$h_0 D_2 x_{68,8}$	d_2	$h_0^2 Q_2 x_{68,8}$
	$h_0^{12}h_6^2$	d_2^{-1} d_4^{-1}	$h_0^{11}h_7$
14	$x_{126,14}$	d_4^{-1}	$h_0^2 x_{127,8}$
	$h_1h_3x_{118,12}$	d_5^{-1}	$h_1x_{126,8,2}$
	$D_2x_{68,8}$	d_2	$h_0Q_2x_{68,8}$
	$h_0^{11}h_6^2$	d_2^{-1}	$h_0^{10}h_7$
13	1 120,12,2	d_5	$d_0^2 x_{97,10}$
	$h_0h_3x_{119,11}$	d_2	$h_0^3 x_{125,12}$
	$d_1x_{94,8}$	d_2^{-1}	$x_{127,10}$
	$h_0x_{126,11}$	d_2^{-1} d_2^{-1}	$h_3x_{120,9}$
12	$h_0^{10}h_6^2$		$h_0^9 h_7$
	$h_0^2 x_{126,10}$	d_4	$h_1x_{124,15}$
	$h_3x_{119,11}$	d_2	$h_0^2 x_{125,12}$
	$h_0^2 x_{126,9}$	d_2^{-1} d_2^{-1}	$h_0x_{127,8}$
	$h_0^9 h_6^2$	d_2^{-1}	$h_0^8h_7$
11	$h_1 x_{125,10,2} + h_1 x_{125,10}$,	Permanent
	$h_1x_{125,10}$	d_4	$Q_2x_{68,8}$
	x _{126,11}	d_2	$h_0^4 x_{125,9,2}$
Щ	$h_0 x_{126,10}$	d_2	$h_0^5 x_{125,8}$

Table 8. The classical Adams spectral sequence of S^0 for $11 \le s \le 25$ in stem 126

	$h_0 x_{126,9}$	d_2^{-1} d_2^{-1}	$x_{127,8}$
	$h_0^2 x_{126,8}$	d_2^{-1}	$h_0 x_{127,7}$
10	$h_0^8 h_6^2$	d_2^{-1}	$h_0^7 h_7$
	$h_0^2 x_{126,8,3}$		Permanent
	$x_{126,10}$	d_2	$h_0 h_6 x_{62,10}$
	$h_0x_{126,8}$	d_2^{-1}	$x_{127,7}$
	$h_0^7 h_6^2$	d_2^{-1}	$h_0^6 h_7$
9	$h_1x_{125,8}$	d_4	$nx_{94,8}$
Э	$h_1x_{125,8,2}$	d_4	$h_0x_{125,12}$
	$x_{126,9}$	d_3	$h_0^3 x_{125,9,2} + h_0^4 x_{125,8}$
	$h_0x_{126,8,3}$	d_3	$h_0^3 x_{125,9,2}$
	$h_0^6 h_6^2$	d_2^{-1}	$h_0^5 h_7$
	$h_6(C'+X_2)$	d_{17}	?
8	$x_{126,8,4} + x_{126,8}$	d_6	?
0	$x_{126,8}$	d_3	$h_1x_{124,10,2}$
	$x_{126,8,2}$	d_3	$h_1 x_{124,10}$
	$x_{126,8,3}$	d_2	$h_0x_{125,9}$
7	$h_0^5 h_6^2$	d_2^{-1}	$h_0^4 h_7$
'	$h_1h_6[H_1]$	d_{18}	?
C	$h_0^4 h_6^2$	d_2^{-1}	$h_0^3 h_7$
6	$x_{126,6}$	d_3	$h_5 x_{94,8} + \text{possibly } h_6 (\Delta e_1 + C_0 + h_0^6 h_5^2)$
5	$h_0^3 h_6^2$	d_2^{-1}	$h_0^2 h_7$
4	$h_0^2 h_6^2$	d_2^{-1}	h_0h_7
4	$x_{126,4}$	d_3	$h_0^2 x_{125,5}$
3	$h_0 h_6^2$	d_2^{-1}	h_7
-			?
2	h_6^2	d_7	1
0-1	-		

Table 9. The classical Adams spectral sequence of S^0 for $s \leq 10$ in stem 126

s	Elements	d_r	value
	$h_0^{24}h_7$	d_3	$h_0^{10}x_{126,18}$
25	$ix_{104,18}$	d_2	$d_0^3 x_{84,15,2} + h_0 d_0 x_{112,22}$
	$d_0g\Delta^3h_1g$	d_2	$d_0e_0g^3m$
	$h_0^2 d_0 x_{113,18}$	d_2^{-1}	$h_0 g x_{108,17}$
24	$h_1x_{126,23}$	d_3^{-1}	$x_{128,21}$
	$h_0^{23}h_7$	d_3	$h_0^9 x_{126,18}$
	$e_0g\Delta h_2^2[B_4] + h_0d_0x_{113,18}$	d_2^{-1}	$gx_{108,17}$
23	$h_0d_0x_{113,18}$	d_4	$d_0^3 x_{84,15,2}$
20	$h_0^{22}h_7$	d_3	$h_0^8 x_{126,18}$
	$d_0Pd_0x_{91,11}$	d_3	$h_1x_{125,25}$
	$d_0x_{113,18,2}$	d_4^{-1}	$d_0e_0x_{97,10}$
22	$h_0^{21}h_7$	d_3	$h_0^7 x_{126,18}$
	$d_0x_{113,18}$	d_2	$d_0 e_0 \Delta h_2^2 Mg$
	$h_0^6 x_{127,15}$	d_2^{-1}	$h_0^5 x_{128,14}$
21	$x_{127,21} + g^3 C''$		Permanent
21	$h_0^{20}h_7$	d_3	$h_0^6 x_{126,18}$
	$g^{3}C''$	d_3	$g^4\Delta h_2 c_1$

Table 10. The classical Adams spectral sequence of S^0 for $21 \le s \le 25$ in stem 127

1,5	5	7-1	1.4
h_0	$5x_{127,15}$	d_2^{-1}	$h_0^4 x_{128,14}$
	$e_0[\Delta\Delta_1 g]$	d_4	$d_0Pd_0M^2$
h_0^1	$h_0^{19}h_7$	d_3	$h_0^5 x_{126,18}$
h_0^2	$^{1}_{0}x_{127,15}$	d_2^{-1}	$h_0^3 x_{128,14}$
h_1	$x_{126,18}$		Permanent
	$x_{110,15}$	d_4	$x_{126,23}$
	$x_{126,18,2}$	d_3	$h_0 x_{126,21} + h_0^4 x_{126,18}$
h_0^1	$h_{0}^{18}h_{7}$	d_3	$h_0^4 x_{126,18}$
	$x_{127,15}$	d_2^{-1}	$h_0^2 x_{128,14}$
h3	$^{3}_{0}h_{6}x_{64,14}$	$\frac{d_2}{d_2^{-1}}$	$h_0^2 h_6 x_{65,13}$
$18 \frac{n_0}{a^2}$	$^{2}\Delta h_{1}H_{1}$	$\frac{d_2}{d_3^{-1}}$	
$\frac{10}{h}$	$\frac{\Delta n_1 n_1}{1 x_{126,17}}$	u_3	$gx_{108,11}$ Permanent
h.]	$^{12}_{0}^{126,17}$	d_3	
$\overline{}$			$h_0^3 x_{126,18}$
h_0^2	$^{2}_{0}h_{2}x_{124,14}$	d_2^{-1}	$x_{128,15}$
	$_{0}^{2}x_{127,15}$	d_2^{-1}	$x_{128,15} + h_0 x_{128,14}$
17 h_0^2	$^{2}_{5}h_{6}x_{64,14}$	d_2^{-1}	$h_0 h_6 x_{65,13}$
g_3	$r_{107,13}$	d_4^{-1}	$h_0 h_3 x_{121,11}$
h_0^1	h_{7}^{16}	d_3	$h_0^2 x_{126,18}$
h_0	$0x_{127,15} + h_0h_2x_{124,14}$	d_2^{-1}	$x_{128,14}$
h_0	$h_6x_{64,14}$	d_2^{-1}	$h_6x_{65,13}$
h_{ϵ}	$h_2x_{124,14}$		Permanent
10	127,16		Permanent
	$h_{0}^{15}h_{7}$	d_3	$h_0x_{126,18}$
	$r_{107,12}$	d_3	$g^3x_{66,7}$
	$1x_{126,14}$	d_2^{-1}	$x_{128,13,2}$
	$2x_{124,14}$	2	Permanent
	127,15	d_5	
h_i	$h_{1}^{14}h_{7}$	d_2	$\frac{d_0 x_{112,16}}{h_0^{15} h_6^2}$
	$5x_{64,14}$	d_2	$h_1^2 x_{124,15}$
	$g \Delta h_6 g$	d_2^{-1}	
		$\frac{d_2}{d_2^{-1}}$	$x_{128,12,2}$
	$h_3 x_{120,12}$		$h_3x_{121,11}$
		d_2	$h_0^{14}h_6^2$
	$x_{127,10}$	d_2^{-1}	$h_0x_{128,10}$
	$\Delta h_6 g$	d_3^{-1}	$x_{128,10,2}$
$13 h_3$	$3x_{120,12}$	d_4^{-1}	$h_2x_{125,8,2}$
x_1	127,13	d_3	$h_0^2 D_2 x_{68,8}$
	$h_{0}^{12}h_{7}$	d_2	$h_0^{13}h_6^2$
h_0^2	$x_{127,10}^{2}$	d_2^{-1}	$x_{128,10}$
$12 h_1$	$x_{126,11}$	d_3^{-1}	$h_1x_{127,8}$
h_0^1	$h_{0}^{11}h_{7}$	d_2	$h_0^{12}h_6^2$
	$h_3x_{120,9}$	d_3^{-1}	$h_3D_2h_6$
h_c	$h_2 x_{124,9}$	3	Permanent
	$0x_{127,10}$		Permanent
h_{i}	$h_0^{10}h_7$	d_2	$h_0^{11}h_6^2$
	$x_{125,8}$	2	Permanent
11 ₁	$\frac{1}{2}x_{125,8}$ $\frac{1}{2}x_{124,9} + h_0^2x_{127,8}$	de	$h_1^2 x_{124,14}$
102	$\frac{2x_{124,9} + h_0 x_{127,8}}{2x_{127,8}}$	$\frac{d_6}{d_4}$	
) ⁴ 127,8	u_4	$x_{126,14}$
10		d-	d. r
x_1	127,10	$\frac{d_2}{d_2}$	$d_1x_{94,8}$
$\begin{vmatrix} x_1 \\ h_3 \end{vmatrix}$		d_2 d_2 d_2	$ \begin{array}{c} d_1 x_{94,8} \\ h_0 x_{126,11} \\ h_0^{10} h_6^2 \end{array} $

Table 11. The classical Adams spectral sequence of S^0 for $10 \leq s \leq 20$ in stem 127

	$h_0^2 x_{127,7}$	d_2^{-1}	$h_0x_{128,6}$
	$h_1x_{126,8}$		Permanent
9	$h_1x_{126,8,2}$	d_5	$h_1h_3x_{118,12}$
	$h_0 x_{127,8}$	d_2	$h_0^2 x_{126,9}$
	$h_0^8 h_7$	d_2	$h_0^2 x_{126,9}$ $h_0^9 h_6^2$
	$h_0 x_{127,7,2} + h_0 x_{127,7} + h_0^2 x_{127,6}$	d_2^{-1} d_2^{-1}	$x_{128,6}$
	$h_0^2 x_{127,6}$	d_2^{-1}	$h_0x_{128,5}$
	h_2h_6A		Permanent
8	$h_2x_{124,7}$	d_9	?
	$x_{127,8}$	d_2	$h_0 x_{126,9}$
	$h_0 x_{127,7}$	d_2	$h_0^2 x_{126,8}$ $h_0^8 h_6^2$
	$h_0^7 h_7$	d_2	$h_0^8 h_6^2$
	$h_0 x_{127,6}$	d_2^{-1}	$x_{128,5}$
	$h_1x_{126,6}$	d_{10}	?
7	$x_{127,7,2} + x_{127,7}$	d_3	?
	$x_{127,7}$	d_2	$h_0x_{126,8}$
	$h_0^6 h_7$	d_2	$h_0 x_{126,8}$ $h_0^7 h_6^2$
6	$x_{127,6}$	d_4	?
U	$x_{127,6} \ h_0^5 h_7$	d_2	$h_0^6 h_6^2$
5	$h_0^4 h_7$	d_2	$h_0^5 h_6^2$
4	$h_0^3 h_7$	d_2	$h_0^4 h_6^2$
3	$h_1 h_6^2$	d_{14}	?
3	$h_0^2 h_7$	d_2	$h_0^3 h_6^2$
2	h_0h_7	d_2	$h_0^2 h_6^2$
1	h_7	d_2	$h_0 h_6^2$
0			

Table 12. The classical Adams spectral sequence of S^0 for $s \leq 9$ in stem 127

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