Stochastic Process and Stochastic HJB Equation

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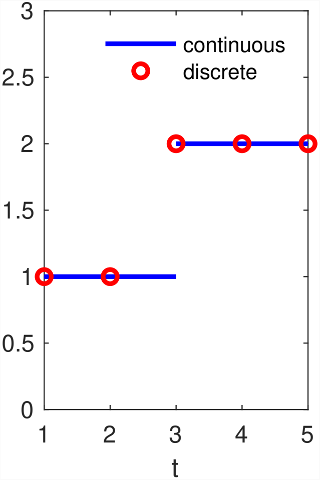
Stochastic process

A stochastic process is a collection of **random variables** defined on a **probability space**. Definitions

We think of the random variables are indexed with t, a time parameter. Let a set T as the set of all possible time points. $T = \mathbb{Z}_+ = \{0, 1, ...\}$ in discrete-time, and $T = \mathbb{R}_+ = [0, \infty)$ in continuous-time. A discrete-time process can always be viewed as a continuous-time process which is constant on the intervals $[n-1, n)$ for all $n \in N$.



Overview

Stochastic process

- Standard Brownian Motion
- The differential dz
  - Brownian Motion with a drift
  - Geometric Brownian Motion
  - Ornstein-Uhlenbeck Process
- Poisson process

Ito's Lemma

Stochastic HJB equation

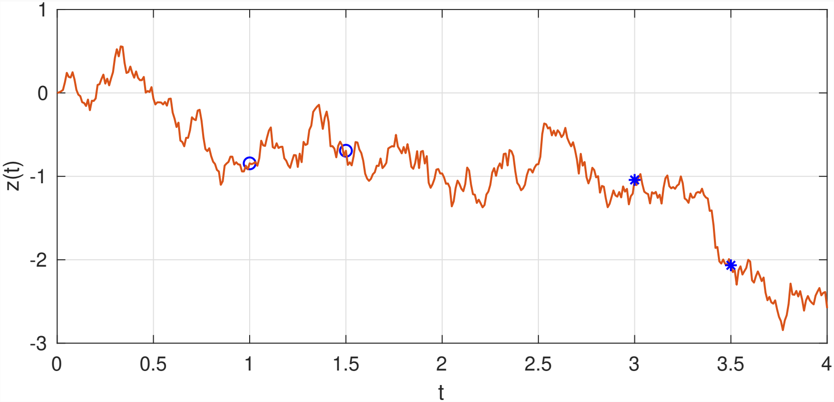
Standard Brownian Motion

Continuous-time analogue of the random walk

There exists a probability distribution over the set of continuous functions **z** $: \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfying the following conditions:

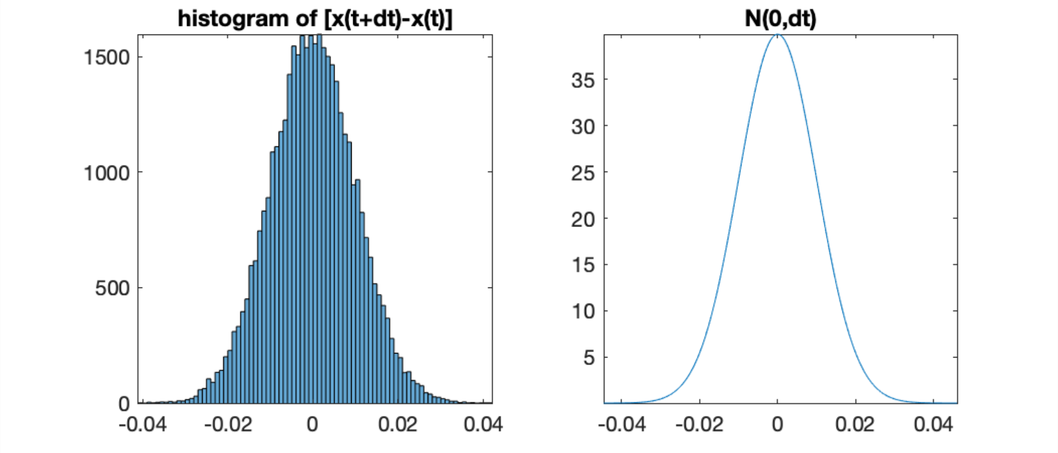
1. $z_0 = 0$
2. **(stationary)** for all $0 \leq s < t$, the distribution of $z(t) - z(s)$ is the normal distribution with mean 0 and variance $t - s$
3. **(independent increment)** the random variables $z(t) - z(s)$ are mutually independent if the intervals $[s, t]$ are non-overlapping

Standard Brownian Motion



- $(z_{t+0.5} - z_t) \sim N(0, 0.5)$
- $(z_{1.5} - z_1)$ and $(z_{3.5} - z_3)$ are independent

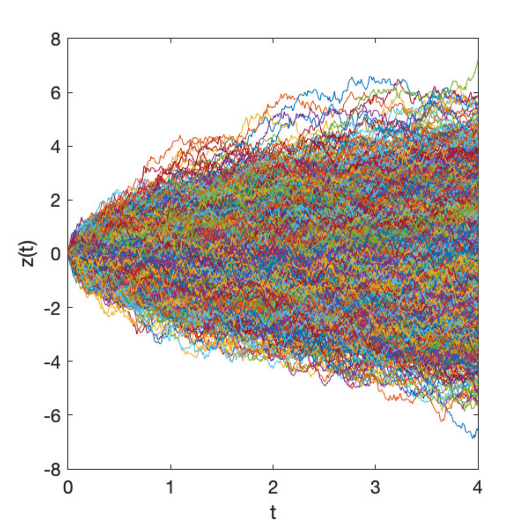
Standard Brownian Motion



- dt = 0.0001. $[-3\sqrt{dt}, 3\sqrt{dt}] = [-0.03, 0.03]$

Standard Brownian Motion

- At t, z_t is likely to be within $[-3\sqrt{t}, 3\sqrt{t}]$
- **A Brownian trajectory is nowhere differentiable.** proof (proposition 2.4)
  $\Longrightarrow$ cannot use calculus to analyze it



The differential dz

We can write Standard Brownian Motion as

$$z_{t+\Delta} = z_t + \varepsilon_t \sqrt{\Delta}, \quad \varepsilon_t \sim N(0,1), \quad z(0) = 0$$

Let's define

$$dz_t = \lim_{\Delta \to 0} (z_{t+\Delta} - z_t)$$

- dz_t is a fundamental building block of diffusion processes
    - $E(dz_t) = 0 \qquad E(z_{t+\Delta} - z_t) = E(\varepsilon_t \sqrt{\Delta}) = 0$
    - $var(dz_t) = dt \qquad var(z_{t+\Delta} - z_t) = var(\varepsilon_t \sqrt{\Delta}) = \Delta \equiv dt$
    - $cov(dz_t, dz_s) = 0$ where $s \neq t$

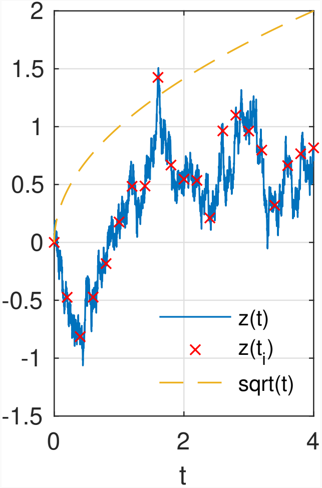
Quadratic variation

For a partition $\pi = t_0, t_1, ..., t_j$ of an interval $[0, T]$, let $|\pi| = \max_i (t_{i+1} - t_i)$. A Brownian motion z_t satisfies the following equation with probability 1:

$$\lim_{|\pi| \to 0} \left(\sum_i (z_{t_{i+1}} - z_{t_i})^2 \right) = T$$

proof) Let's assume uniform partition, $\frac{T}{n+1}$. $(z_{t_{i+1}} - z_{t_i})$, $\forall i$ are independent random variables that are drawn from $N(0, T/n)$.
Rewrite

$$\lim_{n \to \infty} \left(\sum_{i=1}^{n} (z_{i+1} - z_i)^2 \right)$$



The differential dz

Notice $E\left(z_{i+1} - z_i\right)^2$ is a variance of $\left(z_{i+1} - z_i\right)$.
As $n \to \infty$, $E\left(z_{i+1} - z_i\right)^2$ converges to T/n by Weak Law of Large Numbers.

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} (z_{i+1} - z_i)^2 \right) = \sum_{i=1}^{n} \frac{T}{n} = T$$

Quadratic variation implies

- $(dz_t)^2 = dt$ $\quad$ $dz_t = \sqrt{dt}$ $\quad$???

With dz_t, we can build more complex processes

Brownian Motion with drift

Discrete $x_{t+1} - x_t = \mu + \sigma\varepsilon_t$

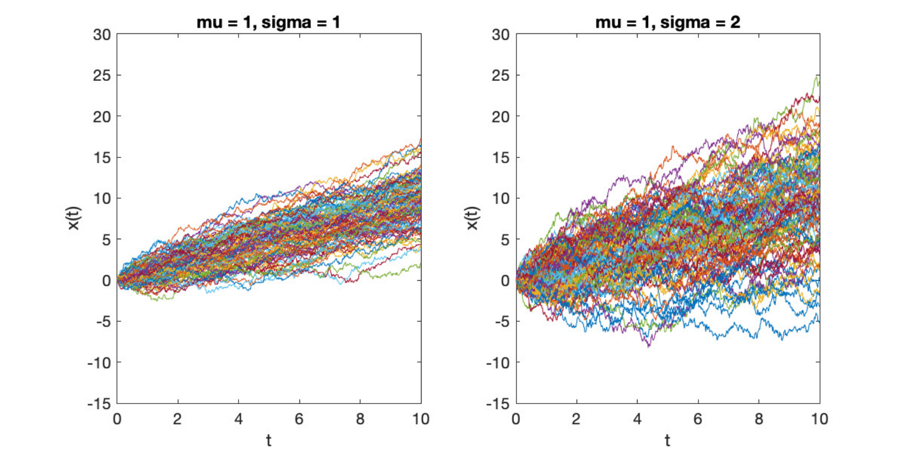
Continuous $dx_t = \mu dt + \sigma dz_t$

- x_t has a trend
- $E(dx_t) = \mu dt + \sigma E dz_t = \mu dt$
- $var(dx_t) = E[(dx_t - E(dx_t))^2] = E[(dx_t - \mu dt)^2] = \sigma^2 dt$

$$\begin{aligned} E[(dx_t - \mu dt)^2] &= E(dx_t^2 - 2dx_t\mu dt + \mu^2 dt^2) \\ &= E(\mu^2 dt^2 + \sigma^2 dz_t^2 + 2\mu dt\sigma dz_t - 2\mu^2 dt^2 - 2\sigma dz_t \mu dt + \mu^2 dt^2) \end{aligned}$$

Keep terms of order dt, and ignore terms of order higher than dt; $dt^{3/2}$, dt^2 and so on, because they get arbitrarily small as the time interval shrinks.

Brownian Motion with drift



- μ determines trends and σ determines dispersion

Geometric Brownian Motion with drift

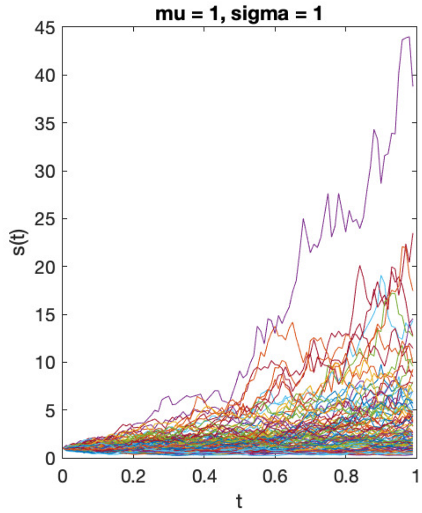
$$s_t = s_0 e^{x_t}, \quad s_0 > 0, \quad dx_t = \mu dt + \sigma dz_t$$

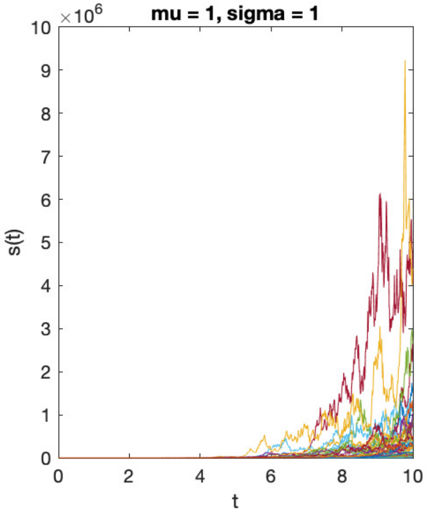
It is known that derivation

$$\frac{ds_t}{s_t} = \mu_s dt + \sigma dz_t, \quad \mu_s = \mu + \frac{\sigma^2}{2}$$

- Percentile differences are normally distributed
- Given $s_0 > 0$, $s_t > 0$
- Commonly used to analyze stock prices

Geometric Brownian Motion with drift





Ornstein-Uhlenbeck process

Discrete

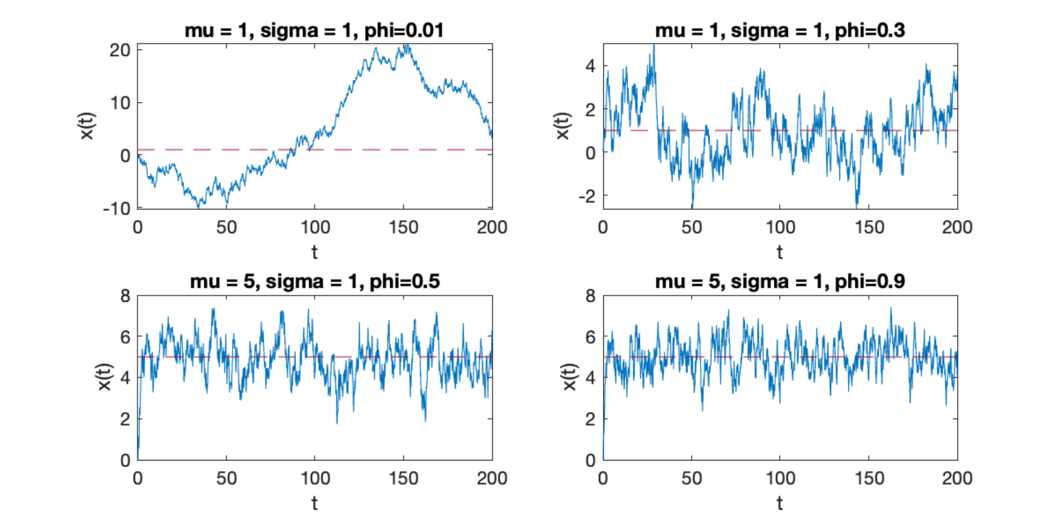
$$
\begin{aligned}
x_{t+1} &= (1-\rho)\mu + \rho x_t + \sigma\varepsilon_t \\
x_{t+1} - x_t &= -(1-\rho)(x_t - \mu) + \sigma\varepsilon_t
\end{aligned}
$$

Continuous

$$dx_t = -\phi(x_t - \mu)dt + \sigma dz_t$$

- Continuous-time analogue of discrete-time AR(1)
- Stationary, mean-reverting process
- Approximately, $x_t \sim N(\mu, \frac{\sigma^2}{2\phi})$
- ϕ: how strongly it reacts to perturbations. lower ϕ implies more persistent

Ornstein-Uhlenbeck process



Poisson process

Continuous-time version of Bernoulli process (A sequence of binary random variables)

λ: arrival rate. Expected number of arrival per unit time.
the probability of k arrivals during time interval τ is:

$$p(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}$$

For very small time interval dt, the probability of k arrival is:

$$p(k,dt) \approx \begin{cases} 1-\lambda dt & \text{if } k=0 \\ \lambda dt & \text{if } k=1 \\ 0 & \text{if } k>1 \end{cases}$$

Ito's Lemma

From the derivation of deterministic HJB:

$$0 = \max_{\alpha(s)} \frac{1}{h} \int_t^{t+h} r(s, x(s), \alpha(s)) ds + \frac{1}{h}(v(x(t+h), t+h) - v(x(t), t))$$

$$h \to 0, \quad 0 = \max_{\alpha(t)} r(t, x(t), \alpha(t)) + v_x(x(t), t) \underbrace{\dot{x}(t)}_{\frac{\partial x(t)}{\partial t}} + v_t(x(t), t)$$

Let x_t be a solution of $dx_t = \mu dt + \sigma dz_t$

We know that x_t is nowhere differentiable, so we cannot use the classical calculus.

Then how do we think about $\lim_{h \to 0} \frac{1}{h}(v(x(t+h), t+h) - v(x(t), t))$?

Ito's Lemma

Although x_t is not differentiable, we understand properties of dx_t. We can also try to compute change of v over a short time interval dt.
If $v(x(t), t)$ is differentiable, can we write

$$v(x(t+dt), t+dt) - v(x(t), t) = v_t(x(t), t)dt + v_x(x(t), t)dx_t?$$

Above equation is incorrect.

Ito's Lemma shows how to compute the differential of a function $f(t, x_t)$ of a diffusion process x_t, $df(t, x_t) = f(t+dt, x_{t+dt}) - f(t, x_t)$

Ito process

First, let's define Ito processes

An Ito process is of the form

$$dx_t = \mu_t dt + \sigma_t dz_t$$

- μ_t is called the instantaneous drift, and σ_t is called the instantaneous variance/diffusion coefficient
- This is a Brownian motion with an instantaneous drift μ_t and variance σ_t^2

Ito's Lemma

Let x_t be an Ito process.

$$dx_t = \mu_t dt + \sigma_t dz_t$$

Suppose f is twice continuously differentiable. **Ito's Lemma states that $f(t, x_t)$ is an Ito process as well and shows how to compute the drift and diffusion coefficient of $df(\cdot)$**

$$df(t, x_t) = \Big(\frac{\partial f(t, x_t)}{\partial t} + \frac{\partial f(t, x_t)}{\partial x_t}\mu_t + \frac{1}{2}\frac{\partial^2 f(t, x_t)}{\partial x_t^2}\sigma_t^2\Big)dt + \frac{\partial f(t, x_t)}{\partial x_t}\sigma_t dz_t$$

Ito's Lemma

Ito's Lemma is, heuristically, a second-order Taylor expansion in t and x_t using the rules: $(dz_t)^2 = dt$, $dtdz_t = dt^{3/2} = o(dt)$, $dt^2 = o(dt)$

$$
\begin{aligned}
df(t,x_t) &\approx \frac{\partial f(t,x_t)}{\partial t}dt + \frac{\partial f(t,x_t)}{\partial x_t}dx_t + \frac{1}{2}\frac{\partial^2 f(t,x_t)}{\partial t^2}dt^2 + \frac{1}{2}\frac{\partial^2 f(t,x_t)}{\partial x_t^2}(dx_t)^2 + \frac{\partial f(t,x_t)}{\partial t \partial x_t}dtdx_t \\
&= \frac{\partial f(t,x_t)}{\partial t}dt + \frac{\partial f(t,x_t)}{\partial x_t}dx_t + \frac{1}{2}\frac{\partial^2 f(t,x_t)}{\partial x_t^2}(dx_t)^2 + o(dt) \\
&= \frac{\partial f(t,x_t)}{\partial t}dt + \frac{\partial f(t,x_t)}{\partial x_t}(\mu_t dt + \sigma_t dz_t) + \frac{1}{2}\frac{\partial^2 f(t,x_t)}{\partial x_t^2}(\mu^2 dt^2 + \sigma^2 dz_t^2 + 2\mu dt\sigma dz_t) \\
&= \left(\frac{\partial f(t,x_t)}{\partial t} + \frac{\partial f(t,x_t)}{\partial x_t}\mu_t + \frac{1}{2}\frac{\partial^2 f(t,x_t)}{\partial x_t^2}\sigma_t^2\right)dt + \frac{\partial f(t,x_t)}{\partial x_t}\sigma_t dz_t + o(dt)
\end{aligned}
$$

* $\frac{\partial f(t,x_t)}{\partial t\partial x_t}dtdx_t = \frac{\partial f(t,x_t)}{\partial t\partial x_t}dt(\mu_t dt + \sigma_t dz_t) = \frac{\partial f(t,x_t)}{\partial t\partial x_t}\mu_t dt^2 + \frac{\partial f(t,x_t)}{\partial t\partial x_t}\sigma_t dz_t dt = o(dt)$

Example

Compute $df(t, x_t)$

$$f(t,x) = x^2, \;\; dx_t = \mu dt + \sigma dz_t$$

$$\begin{aligned} df(t,x_t) &= \Big(\frac{\partial f(t,x_t)}{\partial t} + \frac{\partial f(t,x_t)}{\partial x_t}\mu_t + \frac{1}{2}\frac{\partial^2 f(t,x_t)}{\partial x_t^2}\sigma_t^2\Big)dt + \frac{\partial f(t,x_t)}{\partial x_t}\sigma_t dz_t \\ &= (2x_t\mu + \sigma^2)dt + 2x_t\sigma dz_t \end{aligned}$$

Example: Geometric Brownian Motion

Compute ds_t.

$$s_t = s_0 e^{x_t}, \quad dx_t = \mu dt + \sigma dz_t$$

Let $f(t, x_t) = s_t = s_0 e^{x_t}$

$$\begin{aligned}
ds_t = df(t, x_t) &= \Big(\frac{\partial f(t,x_t)}{\partial t} + \frac{\partial f(t,x_t)}{\partial x_t}\mu_t + \frac{1}{2}\frac{\partial^2 f(t,x_t)}{\partial x_t^2}\sigma_t^2\Big)dt + \frac{\partial f(t,x_t)}{\partial x_t}\sigma_t dz_t \\
&= (s_0 e^{x_t}\mu + s_0 e^{x_t}\frac{\sigma^2}{2})dt + s_0 e^{x_t}\sigma dz_t \\
&= (s_t\mu + s_t\frac{\sigma^2}{2})dt + s_t\sigma dz_t \\
\implies \frac{ds_t}{s_t} &= (\mu + \frac{\sigma^2}{2})dt + \sigma dz_t
\end{aligned}$$

Back

Ito's Lemma: Multivariate

Let $z_1, z_2, ..., z_n$ be independent Brownian motion and $\mathbf{x} \equiv (x_1, x_2, ...x_m)$ be a vector process. Suppose $f : \mathbb{R}^m \to \mathbb{R}$ is twice continuously differentiable and x_i is an Ito process with $dx_i = a_i dt + \sum_{j=1}^{n} b_{ij} dz_j$. Then $df(\mathbf{x})$ is an Ito process with the differential,

$$df(\mathbf{x}) = \sum_{i=1}^{m} f_i(\mathbf{x}) d\mathbf{x}_i + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} f_{ij}(\mathbf{x}) d\mathbf{x}_i d\mathbf{x}_j$$

where $f_i = \frac{\partial f}{\partial x_i}$, $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$

Example: Bivariate

Compute $df(t, x_{1,t}, x_{2,t})$ where

$$dx_{1,t} = a_1 dt + \sigma_{11} dz_{1,t} + \sigma_{12} dz_{2,t}$$
$$dx_{2,t} = a_2 dt + \sigma_{21} dz_{1,t} + \sigma_{22} dz_{2,t}$$

$$df(t, x_1, x_2) = f_t dt + f_1 dx_1 + f_2 dx_2 + \frac{1}{2}\left(f_{x_1 x_1}(dx_1)^2 + f_{x_2 x_2}(dx_2)^2 + 2 f_{x_1 x_2} dx_1 dx_2\right)$$

$\times$	dt	dz_i
dt	$o(dt)$	$o(dt)$
dz_j	$o(dt)$	$\rho_{ij} dt$

$\rho_{ii} = 1$, $\rho_{ij} = 0$ if z_i and z_j are independent

Example: Bivariate

$$
\begin{aligned}
df(t, x_1, x_2) &= f_t dt + f_1(a_1 dt + \sigma_{11} dz_{1,t} + \sigma_{12} dz_{2,t}) + f_2(a_2 dt + \sigma_{21} dz_{1,t} + \sigma_{22} dz_{2,t}) \\
&+ \frac{1}{2}\Big(f_{x_1 x_1}(\sigma_{11}^2 + \sigma_{12}^2 + \rho_{12}\sigma_{11}\sigma_{12}) + f_{x_2 x_2}(\sigma_{21}^2 + \sigma_{22}^2 + \rho_{12}\sigma_{21}\sigma_{22})\Big) dt \\
&+ f_{x_1 x_2}(\sigma_{11}\sigma_{21} + \rho_{12}\sigma_{11}\sigma_{22} + \rho_{12}\sigma_{12}\sigma_{21} + \sigma_{12}\sigma_{22}) dt \\
&= \Big(f_t + f_1 a_1 + f_2 a_2 + \frac{1}{2}\Big(f_{x_1 x_1}(\sigma_{11}^2 + \sigma_{12}^2 + \rho_{12}\sigma_{11}\sigma_{12}) + f_{x_2 x_2}(\sigma_{21}^2 + \sigma_{22}^2 + \rho_{12}\sigma_{21}\sigma_{22})\Big) \\
&+ f_{x_1 x_2}(\sigma_{11}\sigma_{21} + \rho_{12}\sigma_{11}\sigma_{22} + \rho_{12}\sigma_{12}\sigma_{21} + \sigma_{12}\sigma_{22})\Big) dt \\
&+ (f_1\sigma_{11} + f_2\sigma_{21}) dz_{1,t} + (f_1\sigma_{12} + f_2\sigma_{22}) dz_{2,t}
\end{aligned}
$$

Stochastic HJB equation

A generic problem

$$v(x_0, 0) = \max_{\alpha_s} E \int_0^\infty e^{-\rho s} r(x_s, \alpha_s) ds$$

$$dx_t = \mu dt + \sigma dz_t, \quad x_0 \text{ is given}$$

- In general, x and α can be vectors
- Next four slides derive stochastic HJB equation when x is scalar

Stochastic HJB equation

As in the deterministic case, focus on the interval between t and $t + dt$. Also recall
$v(x) \equiv v(x, 0), \quad v(x,t) = e^{-\rho t} v(x), \quad v_t(x,t) = -\rho e^{-\rho t} v(x)$

$$v(x_t, t) = \max_{\alpha_s} E\left(\int_t^{t+dt} e^{-\rho s} r(x_s, \alpha_s) ds + v(x_{t+dt}, t + dt) \right)$$

rearrange and divide by dt

$$0 = \max_{\alpha_s} E\left(\frac{1}{dt} \int_t^{t+dt} e^{-\rho s} r(x_s, \alpha_s) ds + \frac{1}{dt} \big(v(x_{t+dt}, t + dt) - v(x_t, t) \big) \right)$$

$$dx_t = \mu dt + \sigma dz_t$$

Stochastic HJB equation

$dt \to 0$ and apply Ito's lemma,

$$0 = \max_{\alpha_t} e^{-\rho t} r(x_t, \alpha_t) + \frac{1}{dt} E\Bigg(\Big(\frac{\partial v(t,x_t)}{\partial t} + \frac{\partial v(t,x_t)}{\partial x_t}\mu + \frac{1}{2}\frac{\partial^2 v(t,x_t)}{\partial x_t^2}\sigma^2\Big)dt + \frac{\partial v(t,x_t)}{\partial x_t}\sigma dz_t\Bigg)$$

using $v(x,t) = e^{-\rho t} v(x), \quad v_t(x,t) = -\rho e^{-\rho t} v(x)$

$$= \max_{\alpha_t} e^{-\rho t} r(x_t, \alpha_t) + \frac{1}{dt} E\Bigg(\Big(-\rho e^{-\rho t} v(x_t) + e^{-\rho t} v_x(x_t)\mu + \frac{e^{-\rho t}}{2} v_{xx}(x_t)\sigma^2\Big)dt + e^{-\rho t} v_x(x_t)\sigma dz_t\Bigg)$$

$$= \max_{\alpha_t} r(x_t, \alpha_t) - \rho v(x_t) + v_x(x_t)\mu + \frac{1}{2} v_{xx}(x_t)\sigma^2 \qquad E(dz_t) = 0$$

drop t and rearrange

$$\rho v(x) = \max_{\alpha} r(x, \alpha) + v_x(x)\mu + \frac{1}{2} v_{xx}(x)\sigma^2$$

$$dx = \mu dt + \sigma dz$$

Stochastic HJB equation: Poisson Uncertainty

Consider the same problem but with a different stochastic process
Suppose $x \in [x_1, x_2]$. Shock arrival rate is λ. When shock arrives switch to x_1 with probability p_1 and switch to x_2 with p_2.

$$v(x_t, t) = \max_{\alpha_s} E\left(\int_t^{t+dt} e^{-\rho s} r(x_s, \alpha_s) ds + v(x_{t+dt}, t+dt)\right)$$

rearrange and divide by dt

$$0 = \max_{\alpha_s} E\left(\frac{1}{dt}\int_t^{t+dt} e^{-\rho s} r(x_s, \alpha_s) ds + \frac{1}{dt}(v(x_{t+dt}, t+dt) - v(x_t, t))\right)$$

For very small time interval dt, the probability of k arrival is:

$$p(k, dt) \approx \begin{cases} 1 - \lambda dt & \text{if } k = 0 \\ \lambda dt & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases}$$

Stochastic HJB equation: Poisson Uncertainty

$$E\Big(v(x_{t+dt}, t+dt)\Big) = \underbrace{\lambda dt\Big(p_1 v(x_1, t+dt) + p_2 v(x_2, t+dt)\Big)}_{\text{shock arrives}} + \underbrace{(1-\lambda dt)v(x_{t+dt}, t+dt)}_{\text{shock does not arrive, } x_{t+dt} = x_t}$$

rearrange, let $v(x') = p_1 v(x_1) + p_2 v(x_2)$, and $dt \to 0$

$$0 = \max_{\alpha_t} e^{-\rho t} r(x_t, \alpha_t) + \frac{1}{dt}\Big(e^{-\rho(t+dt)}\lambda dt v(x') + e^{-\rho(t+dt)}(1-\lambda dt)v(x_t) - e^{-\rho t}v(x_t)\Big)$$

$$0 = \max_{\alpha_t} e^{-\rho t} r(x_t, \alpha_t) + \frac{1}{dt}\Big(e^{-\rho(t+dt)}\lambda dt(v(x') - v(x_t))\Big) + \underbrace{\frac{1}{dt}\Big(e^{-\rho(t+dt)}v(x_t) - e^{-\rho t}v(x_t)\Big)}_{\frac{\partial v(x,t)}{\partial t}}$$

$$0 = \max_{\alpha_t} e^{-\rho t} r(x_t, \alpha_t) + e^{-\rho t}\lambda\Big(v(x') - v(x_t)\Big) - \rho e^{-\rho t}v(x_t)$$

drop t and rearrange

$$\rho v(x) = \max_{\alpha} r(x, \alpha) + \lambda\Big(v(x') - v(x)\Big)$$

Stochastic processes

Back

Definition : Stochastic process

Let T be a set and $(E, \mathcal{E})$ a measurable space. A stochastic process indexed by T, taking values in $(E, \mathcal{E})$, is a collection $X = (X_t)_{t \in T}$ of measurable maps X_t from a probability space $(S, \mathcal{F}, \mu)$ to $(E, \mathcal{E})$. The space $(E, \mathcal{E})$ is called the state space of the process

Definition : Random Variable

If a real valued function $f : S \to \mathbb{R}$ is measurable in the probability space $(S, \mathcal{F}, \mu)$, then f is called a random variable.

Definition : Probability space

If $\mu(S) = 1$, then μ is a probability measure and $(S, \mathcal{F}, \mu)$ is called a probability space. Any measurable set $A \in \mathcal{F}$ is called an event, and $\mu(A)$ is called a probability event.

Stochastic processes

Back

Definition : Measure space

A measure space is triple $(S, \mathcal{F}, \mu)$, where S is a set, $\mathcal{F}$ is a σ-algebra of its subsets, and μ is a measure defined on $\mathcal{F}$.

Definition : σ-algebra

Let S be some set, and let $\mathcal{P}(S)$ its power set. Then a subset $\Sigma \subseteq \mathcal{P}(S)$ is called a σ-algebra if it satisfies the following three properties:

1. $\emptyset, S \in \Sigma$
2. Σ is closed under complementation: If A is in Σ, then so is its complement.
3. Σ is closed under countable unions: If $A_1, A_2, A_3, ... \in \Sigma$, then so is $A = A_1 \cup A_2 \cup A_3 \cup \dots$.

Stochastic processes

Back

Let $t \in [0, \infty) = \mathbb{R}_+$, $\mathcal{B}_+$ denote the Borel subsets of $\mathbb{R}_+$, and $\mathcal{F}_t$ be the set of events known at t.

Definition : Continuous-time stochastic process

Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space. A continuous-time stochastic process if a function $x : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ that is measurable with respect to $\mathcal{B}_+ \times \mathcal{F}_t$.

Definition : Filtration

A filtration is an increasing sequence of σ-algebras on a measurable space. That is, given a measurable space $(\Omega, \mathcal{F})$, a filtration is a sequence of σ-algebras $\{\mathcal{F}_t\}_{t \geq 0}$ with $\mathcal{F}_t \subseteq \mathcal{F}$ where each t, t is a non-negative real number and $t_1 \leq t_2 \implies \mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$.

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Definition : Filtered probability space

A filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$, is a probability space equipped with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of its σ-algebra $\mathcal{F}$. A filtered probability space is said to satisfy the usual conditions if it is complete (i.e., $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) and right-continuous (i.e. $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ for all times t.

The differential dz

figure in 7p, but t = [0,0.1]

$$
\begin{aligned}
(dz_t)^2 &= (z_{t+dt} - z_t)^2 \\
&= \left((z_{t+dt} - z_{t+dt\frac{n-1}{n}}) + \ldots + (z_{t+dt\frac{1}{n}} - z_t) \right)^2 \\
&= \left(dz_{t,n} + dz_{t,n-1} + \ldots + dz_{t,1} \right)^2 \\
&= \sum_{i=1}^{n} \left(dz_{t,i}^2 + dz_{t,i} \sum_{j=-i} dz_{t,j} \right)
\end{aligned}
$$

as $n \to \infty$, WLLN applies. $dz_{t,i} dz_{t,j} = 0$ in expectation.

$$
= \sum_{i=1}^{n} \frac{dt}{n} = dt
$$

$\Longrightarrow$ can find Brownian motion in any dz_t

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