Viscosity Solution

Soyoung Lee

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Ohio State University

Why is viscosity solution useful

From Fleming and Soner (2006)

In general however, the value function is not smooth enough to satisfy the dynamic programming equations in the classical or usual sense. ... Indeed the lack of smoothness of the value function is more of a rule than the exception.

Therefore often we cannot find classical solutions. Instead, we can find viscosity solutions.

- Viscosity solution is a generalization of the classical concept of a solution to PDEs
- It has been known that the viscosity solution is the natural solution concept to use in many applications of PDEs, including HJB equations
- Under the viscosity solution concept, a solution does not need to be everywhere differentiable.

Computations with bounded domain

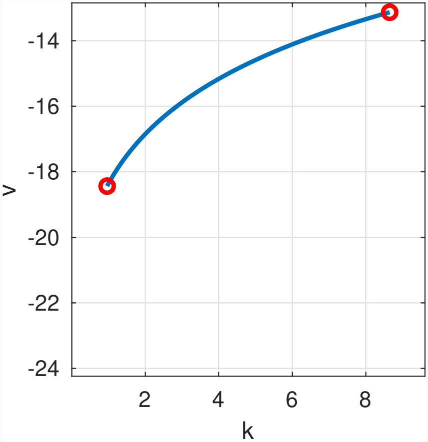
Recall the deterministic growth model

$$\rho v(k) = \max_c u(c) + v_k(k)(f(k) - \delta k - c)$$

We solve the problem on k grid.

At k_1 and k_{nk}, we don't know v is *differentiable*
(= left derivative equals right derivative)

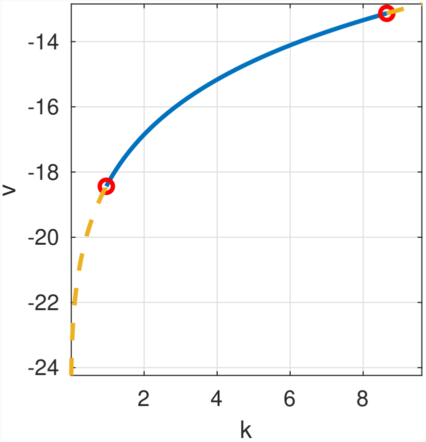
If v is not differentiable at k_1 or k_{nk}, v is not a classical solution



Computations with bounded domain

It is possible that v is differentiable at k_1 and k_{nk}, but we solve on a constrained grid. We imposed boundary conditions so that it is indeed the case.

Set low enough k_1 so that backward difference is never be selected at k_1 and vice versa.



Borrowing constraints

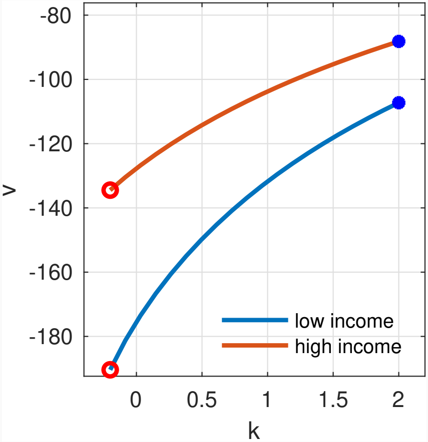
Huggett (1993)

$$\rho v(z,a) = \max_c u(c) + v_a(z,a)\dot{a} + \lambda\Big(v(z',a) - v(z,a)\Big)$$

$$\dot{a} = ra + z - c, \quad a \geq \underline{a}$$

Unlike the deterministic growth model, the borrowing constraint may bind

$\implies$ v is not differential at $\underline{a}$, v is not a *classical* solution

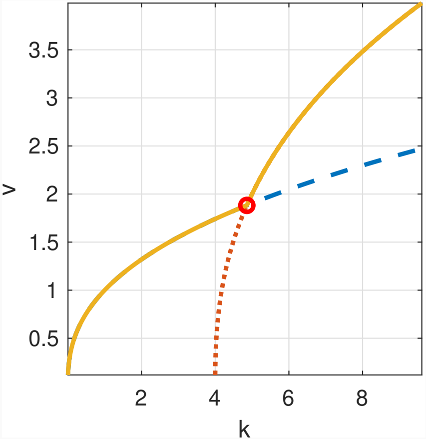


Kinks

Consider a model

$$\rho v(k) = \max_c u(c) + v_k(k)(f(k) - \delta k - c)$$
$$f(k) = max\{fl(k), fh(k)\},$$
$$fl(k) = zlk^{\alpha},$$
$$fh(k) = zh(max((k-\kappa),0)^{\alpha}, \kappa > 0, zh > zl$$

v is not differential at the kinked point



Definition

Consider a PDE

$$F(x, v(x), v'(x), v''(x)) = 0$$

$$\rho v(x) = \max_{\alpha} r(x,\alpha) + v_x(x) f(x,\alpha)$$

where F is a continuous mapping and satisfy the condition

$$F(x, r, p, X) \le F(x, s, p, Y) \text{ whenever } r \le s \text{ and } Y \le X$$

$F(\cdot)$ is non-decreasing in v and non-increasing in v''

$$\rho v(x) - \max_{\alpha} r(x,\alpha) - v_x(x) f(x,\alpha) = 0$$

Definition

A continuous function v is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution

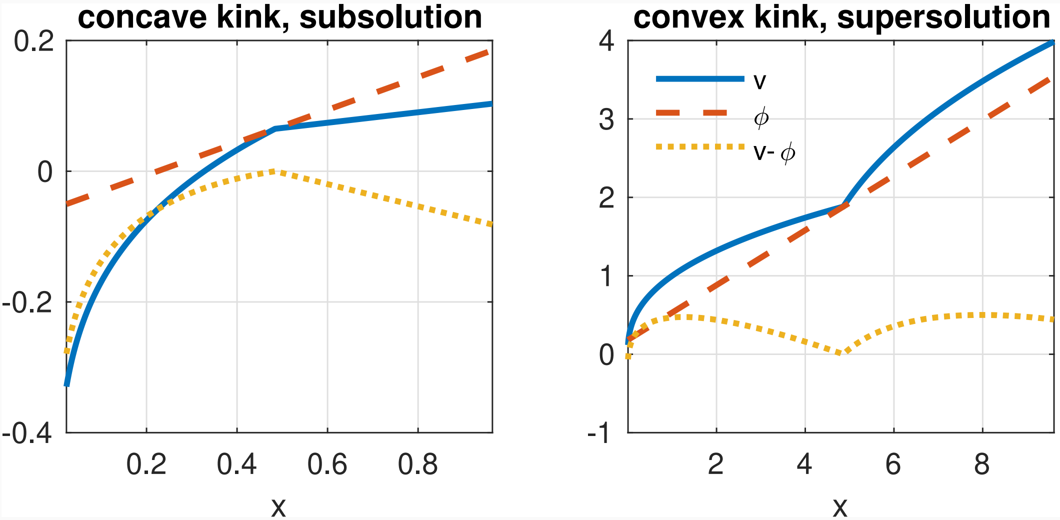
- **(Subsolution)** If ϕ is any smooth function and if $v - \phi$ has a local maximum at point x^*, then

$$F(x^*, \phi(x^*), \phi'(x^*), \phi''(x^*)) \leq 0$$
$$\rho v(x^*) \leq \max_{\alpha} r(x^*, \alpha) + \phi'(x^*) f(x^*, \alpha)$$

- **(Supersolution)** If ϕ is any smooth function and if $v - \phi$ has a local minimum at point x^*, then

$$F(x^*, \phi(x^*), \phi'(x^*), \phi''(x^*)) \geq 0$$
$$\rho v(x^*) \geq \max_{\alpha} r(x^*, \alpha) + \phi'(x^*) f(x^*, \alpha)$$

Subsolution and supersolution



Example

From Wikipedia. Consider a problem

$|u'(x)| = 1$, with boundary conditions $u(-1) = u(1) = 0$. Find u.

$u(x) = 1 - |x|$ satisfies the problem except $x = 0$. Thus not a classical solution
But $1 - |x|$ can be a viscosity solution, and this is the unique viscosity solution.
Let $F(u'(x)) = |u'(x)| - 1 = 0$

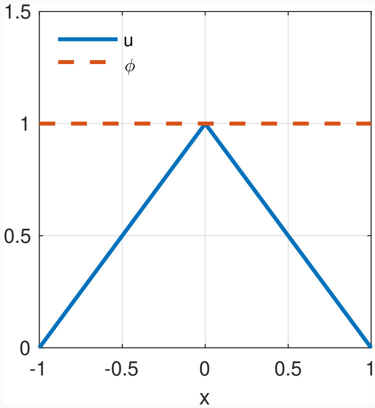
Example: subsolution

We can find a smooth function ϕ that is $u - \phi$ has a local maximum at point $x = 0$. Need to check

$$F(\phi'(0)) = |\phi'(0)| - 1 \leq 0$$

$$\begin{aligned}
\phi(x) - u(x) &\geq 0 \\
\phi(x) + u(0) - u(0) - u(x) = \phi(x) + u(0) - \phi(0) - u(x) &\geq 0 \\
\phi(x) - \phi(0) &\geq u(x) - u(0) \\
&= 1 - |x| - 1 = -|x|
\end{aligned}$$

$$\implies \phi(x) - \phi(0) \geq -|x|$$



Example: subsolution

$$\phi(x) - \phi(0) \geq -|x|$$

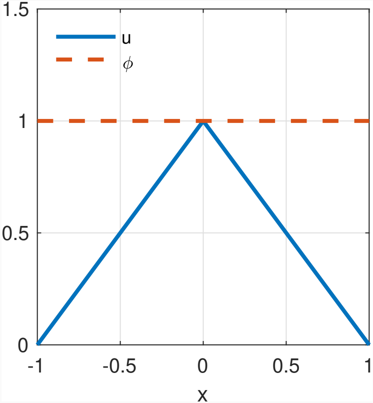
For positive x, this inequality implies

$$\lim_{x \to 0^+} \frac{\phi(x) - \phi(0)}{x} \geq -1$$

For negative x, this inequality implies

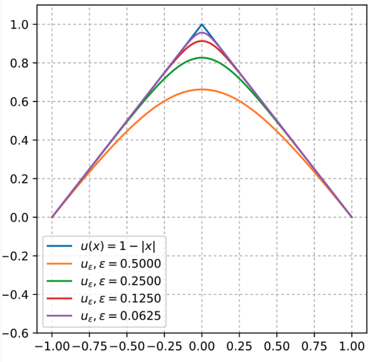
$$\lim_{x \to 0^-} \frac{\phi(x) - \phi(0)}{x} \leq 1$$

Because ϕ is differentiable, the left and right limits agree and are equal to $\phi'(0)$.
Therefore $|\phi'(0)| \leq 1 \Leftrightarrow F(\phi'(0)) \leq 0$.



Example: supersolution

There is no smooth function ϕ that $u - \phi$ has a local minimum at point x^*, therefore u is a supersolution.



Viscosity solution

- If v is differentiable at x^*, then v is a classical solution
- Under some conditions*, **HJB equation has unique viscosity solution**
  Crandall and Lions (1986)
  * viscosity solutions are bounded and uniformly continuous
- There is a theory for numerical solutions using finite difference method
  - **Finite difference scheme converges to unique viscosity solution** under three conditions (monotonicity, consistency, stability)
    Barles and Souganidis (1991)

Viscosity solution

Back to HJB equations,

1. Computations with bounded domain $\Longrightarrow$ boundary inequalities
2. Borrowing constraints $\Longrightarrow$ constrained viscosity solution
3. Kinks $\Longrightarrow$ viscosity solution
    - Show v is a supersolution
    - No concave kink in a miximization problem; v is a subsolution

Computations with bounded domain: boundary inequalities

Recall the deterministic growth model

$$\rho v(k) = \max_c u(c) + v_k(k)(f(k) - \delta k - c)$$

- For numerical solution, want to impose $k_{min} \leq k \leq k_{max}$
- To do so, impose two boundary inequalities
  1. $v_k(k_{min}) \geq u'(f(k_{min}) - \delta k_{min})$
  2. $v_k(k_{max}) \leq u'(f(k_{max}) - \delta k_{max})$

Computations with bounded domain: boundary inequalities

1. $v_k(k_{min}) \geq u'(f(k_{min}) - \delta k_{min})$
    - If v is differentiable, the FOC holds: $v_k(k_{min}) = u'(c(k_{min}))$
    - If $\dot{k} < 0$ at k_{min}, it's possible that $k_t < k_{min}$ at some point; $\dot{k} = f(k_{min}) - \delta k_{min} - c(k_{min}) \geq 0$
    - Since u is concave, $v_k(k_{min}) = u'(c(k_{min})) \geq u'(f(k_{min}) - \delta k_{min})$
2. $v_k(k_{max}) \leq u'(f(k_{max}) - \delta k_{max})$
    - same logic applies

In practice, choose low enough k_{min} and high enough k_{max} so that $\dot{k}_{min} \geq 0$ and $\dot{k}_{max} \leq 0$

Borrowing constraints: constrained viscosity solution

$$\text{Huggett (1993)} \quad \rho v(z,a) = \max_c u(c) + v_a(z,a)\dot{a} + \lambda\Big(v(z',a) - v(z,a)\Big)$$
$$\dot{a} = ra + z - c, \quad a \geq \underline{a}$$

Borrowing constraints may bind at $\underline{a}$ and v may not be differentiable at $\underline{a}$

Constrained viscosity solution

A constrained viscosity solution of HJB is a continuous function v such that

1. v is a viscosity solution for all $a > \underline{a}$
2. v is a subsolution at $a = \underline{a}$: if ϕ is any smooth function and if $v - \phi$ has a local maximum at point $\underline{a}$, then $\rho v(z,\underline{a}) \leq \max_c u(c) + \phi_a(z,\underline{a})\dot{a} + \lambda\Big(\phi(z',a) - \phi(z,a)\Big)$

Why subsolution?

From the derivation of HJB,

$$0 = \max_{c_s} E\left(\frac{1}{dt}\int_t^{t+dt} u(c_s)ds + \frac{1}{dt}\big(v(z_{t+dt}, a_{t+dt}, t+dt) - v(z_t, a_t, t)\big)\right)$$

Let $dt \to 0$. We can compute $\frac{\partial v(z,a,t)}{\partial t}$ because $v(z,a,t) = e^{-\rho t}v(z,a)$, and it's differentiable w.r.t. t. Let $a_t = \underline{a}$

$$\rho v(z, a_t) = \max_c u(c) + \lim_{dt\to 0}\frac{1}{dt}\big(v(z, a_{t+dt}) - v(z, a_t)\big) + \lambda\Big(v(z', a_t) - v(z, a_t)\Big)$$

Consider a smooth function ϕ such that $v - \phi$ has a local maximum at point a_t

$$\leq \max_c u(c) + \lim_{dt\to 0}\frac{1}{dt}\big(\phi(z, a_{t+dt}) - \phi(z, a_t)\big) + \lambda\Big(\phi(z', a_t) - \phi(z, a_t)\Big)$$

$$\leq \max_c u(c) + \phi_a(z, a_t)\dot{a_t} + \lambda\Big(\phi(z', a_t) - \phi(z, a_t)\Big)$$

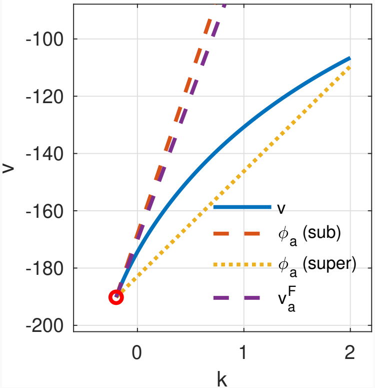
Why not supersolution?

v is a supersolution if ϕ any smooth function and if $v - \phi$ has a local minimum at a_{min}, and $\rho v(z, \underline{a}) \geq \max_c u(c) + \phi_a(z, \underline{a})\dot{\underline{a}} + \lambda\Big(v(z', \underline{a}) - v(z, \underline{a})\Big)$

$v \geq \phi$ near $\underline{a}$

$$\begin{aligned}\rho v(z, a_t) &= \max_c u(c) + \lim_{dt \to 0} \frac{1}{dt}\left(v(z, a_{t+dt}) - v(z, a_t)\right) \\ &\quad + \lambda\Big(v(z', a_t) - v(z, a_t)\Big) \\ &\geq \max_c u(c) + \phi_a(z, a_t)\dot{a}_t + \lambda\Big(\phi(z', a_t) - \phi(z, a_t)\Big)\end{aligned}$$

When the constraint binds, v_a^F implies negative saving.
Therefore ϕ_a implies $\dot{a}_t < 0$
If v_a exists, it must be $\phi_a > v_a$ to satisfy the condition.



Borrowing constraints: numerical solution

At $a_1 = \underline{a}$

- Simply set $v_a^B(a_1, z_j) = u'(ra_1 + z_j) \Leftrightarrow \dot{a}_{1,j}^B = ra_1 + z_j - (ra_1 + z_j) = 0$
- That is, at a_1, set backward difference appx as if an agent saves zero
  - If $\dot{a}_{1,j}^F > 0$, $v_{1,j}^B$ won't be used. $\dot{a}_{1,j}^F > 0$ means an agent with current wealth a_1 choose to save a positive amount and the borrowing constraint does not bind
  - If $\dot{a}_{1,j}^F < 0$, the constraint binds. Force her to save 0 $\Leftrightarrow v_{1,j}^B$ will be selected and we made $\dot{a}_{1,j}^B = 0$

At a_{na}

- If the grid is set properly, all agents will dis-save at a_{na}
- Backward difference will be selected

Kinks

Consider a model

$$\rho v(k) = \max_c u(c) + v_k(k)(f(k) - \delta k - c)$$
$$f(k) = max\{fl(k), fh(k)\},$$
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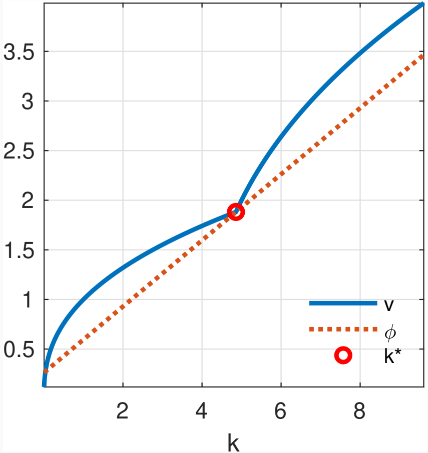
v is not differential at the kinked point



Kinks: supersolution

Let $k_t = k^*$

$$v(k_t, t) = \max_{c_s} \int_t^{t+dt} u(c_s)ds + v(k_{t+dt}, t + dt)$$

Consider a smooth function ϕ such that $v - \phi$ has a local minimum at point k_t

$$\geq \max_{c_s} \int_t^{t+dt} u(c_s)ds + \phi(k_{t+dt}, t + dt)$$

$$\implies 0 \geq \max_{c_s} \frac{1}{dt}\Big(\int_t^{t+dt} u(c_s)ds + \big(\phi(k_{t+dt}, t + dt) - \phi(k_t, t)\big)\Big)$$

$$\implies \rho\phi(k^*) \geq \max_{c} u(c) + \phi_k(k^*)\dot{k}^*$$

$$v(k_t) = \phi(k_t)$$

$$\implies \rho v(k^*) \geq \max_{c} u(c) + \phi_k(k^*)\dot{k}^*$$

Kinks: subsolution

The viscosity solution of the maximization problem only admits convex (downward) kinks, but not concave (upward) kinks. proof (p.17-19)
$\implies$ There is no smooth ϕ such that $v - \phi$ has a local maximum at a kinked point

Kinks: numerical solution

If v is smooth and concave,
$v_k^B > v_k^F$ and $\dot{k}^B < \dot{k}^F$

If there are kinks, there are cases
$v_k^B < v_k^F$ and $\dot{k}^B < 0 \;\; \& \;\; \dot{k}^F > 0$

If there is a such point, choose

- $\dot{k}^F$ if $u(c^B) + v_k^B \dot{k}^B < u(c^F) + v_k^F \dot{k}^F$
- $\dot{k}^B$ if $u(c^B) + v_k^B \dot{k}^B \geq u(c^F) + v_k^F \dot{k}^F$