A Generalized Endogenous Grid Method for Models with the Option to Default*

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Abstract

We develop an endogenous grid method for models with the option to default in which price schedules are endogenously determined in equilibrium and depend on individuals' states. The algorithm has noticeable computational benefits in efficiency and accuracy. We obtain these computational benefits by combining Fella's (2014) identification for non-concave regions with our algorithm that numerically searches for risky borrowing limits. These two procedures identify the region of solution sets to which Carroll's (2006) endogenous grid method is applicable. To demonstrate the method, we apply our method to Nakajima and Ríos-Rull's (2014) model. In terms of computation time, this method is seven to twenty-seven times faster than the conventional grid search method. Moreover, various types of accuracy tests indicate that our method yields more accurate results than the grid search method.

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1 Introduction

Dynamic models which allow agents to default on their debts are often difficult to solve accurately and efficiently because the features of these models hinder the use of efficient algorithms, such as Carroll's (2006) endogenous grid method (EGM) and Arellano, Maliar, Maliar, and Tsyrennikov's (2016) envelope condition method. These algorithms require a feasible set for the solution to be defined before being implemented through an exogenous borrowing constraint or a collateral constraint. However, in models with the option to default, it is hard to define the feasible set before solving the models because it differs across individual states and is endogenously determined in equilibrium.[1] Additionally, because the choice of default is discrete, value functions become non-concave and non-differentiable. Therefore, it is difficult to use efficient algorithms that exploit the necessity and sufficiency of the first-order condition for an interior optimum.

In this paper, we propose a solution method which addresses these computational issues. First, we handle issues from the non-concavity and non-differentiability by employing Fella's (2014) algorithm. Fella (2014) suggests a generalized EGM that can handle non-concavity and non-differentiability of value functions. However, Fella's (2014) EGM cannot be directly applied to models with the option to default because it works only when the feasible set for the solution is predetermined through an exogenous borrowing constraint or a collateral constraint. Second, to address this issue, we define the feasible set for the solution of asset holdings by adding a step that numerically calculates its lower bound, which is called the risky borrowing limit. This numerical procedure is based on a theoretical result in Arellano (2008) and Clausen and Strub (2019). They show that for every optimal debt contract, the size of debt, defined as the product of the price and the quantity of debt, increases with the quantity of debt. For each state, we numerically compute a level of asset holdings above which this theoretical finding always holds. Then, we take it as the risky borrowing limit, which acts as the lower bound of the feasible set of asset holdings. These

[1] In contrast, when the option to default is unavailable, this issue does not appear because the feasible set of the solution is irrelevant to its equilibrium. It is predetermined through an exogenous borrowing constraint or a collateral constraint.

procedures allow us to exploit the computational benefits of the EGM in solving models with the option to default.

To demonstrate our algorithm, we apply our method to the model in Nakajima and Ríos-Rull (2014), which studies the effects of access to credit along with the nature of business cycles. In the model, besides deciding whether to repay or default on their debt, households determine saving and labor supply and are exposed to idiosyncratic risks on their labor productivity as well as aggregate shocks. To find equilibrium in this model, using the conventional grid search method is inefficient. The value function iterations must be repeated multiple times because it is in general equilibrium and uses the algorithm of Krusell and Smith (1998) to address aggregate shocks in the model. By applying our algorithm to this model, we examine how helpful our algorithm is in solving rich models that require significant computational time.

Our algorithm has noticeable computational benefits in accuracy and efficiency. It converges seven to twenty-seven times faster and yields more accurate results than the grid search method based on various accuracy tests, such as Bellman equation errors, Den Haans forecasting test, and R^2 of the forecasting rules. We attribute this improvement in accuracy to the first-order conditions (FOCs) in the EGM. Several components in our algorithm contribute to improvements in efficiency. First, our algorithm defines the regions of solution sets to which Carroll's (2006) EGM is applicable. This step reduces the use of a forward-looking nonlinear equation solver, which is one of the most time-consuming procedures in computation. Second, our algorithm simultaneously updates risky borrowing limits, the loan price schedule, and value functions, instead of iterating separately on them.[2] Finally, our algorithm solves the choice of labor supply by combining the first-order condition of the leisure-consumption choice with the EGM following Khan (2016).[3]

[2]These three objects are inter-winded in the equilibrium. When the option to default is available, a risk-neutral intermediary needs to price loans so that it can operate at a zero profit for each type of borrower. Therefore, on the one hand, the financial intermediary takes into account that each type of individual makes their default decision by comparing default and non-default values. On the other hand, these individuals take into consideration that the financial intermediary charges loan prices and sets up the risky borrowing limit based on their decision. Since the risky borrowing limit, loan price schedule, and value functions interact in this way, it is not clear how to find fixed points of these objects stably. Our algorithm updates these objects at the same time, which contributes to improvements in efficiency.

[3]Note that solving the choice of labor supply is costly with the EGM. For example, Barillas and Fernández-Villaverde (2007) add a step of searching the grid into the EGM to solve the choice of labor supply.

This paper belongs to the stream of the literature of EGM, originally developed by Carroll (2006). Barillas and Fernández-Villaverde (2007) extend it to solve models with endogenous labor supply. Fella (2014) develops a generalized EGM to solve models with discrete choices and exogenous borrowing constraints by addressing the non-concavity and non-differentiability of value functions. Iskhakov, Jørgensen, Rust, and Schjerning (2017) propose another type of EGM that can handle discrete choices and show that introducing taste shocks improves its efficiency. Hintermaier and Koeniger (2010) develop an EGM to solve multidimensional models with smooth and concave value functions. Druedahl and Jørgensen (2017) extend it to solve multidimensional models with non-convexity and constraints. However, none of these EGMs can be applied to models with the option to default because they require the feasible set for the solution to be defined before they can be implemented. Our method extends these EGMs to be applicable to models with the option to default by adding a numerical procedure of identifying the feasible set for the solution through the risky borrowing limit.

This paper is also related to the literature on other types of computational methods to solve models with default risks. Arellano, Maliar, Maliar, and Tsyrennikov (2016) develop an envelope condition method applicable to default models. However, as mentioned in Arellano, Maliar, Maliar, and Tsyrennikov (2016), it is difficult to achieve the convergence of the envelope condition method in models with endogenous default rules. This paper contributes to this margin by suggesting a computational method that attains stable convergence with endogenous default rules without loss of efficiency and accuracy.

The organization of this paper is as follows. Section 2 describes the Nakajima and Ríos-Rull (2014) model, to which we apply our method. Section 3 describes the details of the algorithm. Section 4 presents the results. Section 5 concludes.

2 Model

In this section, we demonstrate our algorithm using a simplified version of Nakajima and Ríos-Rull's (2014) model.[4] In the model, A continuum of households exists with a stochastic lifespan and a risk neutral credit intermediary. They can access a one-period non-contingent debt and can default on it. They face two types of idiosyncratic shocks: preference shocks γ, and labor productivity shocks i. Labor productivity shocks, i, takes the following form:

$$\log i = \log e + \log p + \log t \tag{1}$$

where e is the permanent shock, p is the persistent shock, and t is the transitory shock. The permanent shock and the transitory shock are drawn from $N(0, (\eta\sigma_e)^2)$ and $N(0, (\eta\sigma_t)^2)$, respectively. The persistent shock, p, follows an AR(1) process:

$$p' = \rho_p p + \epsilon_p \tag{2}$$

Let us denote $x = (\gamma, e, p, t)$. Then the individual state variables are $\{x, h, a\}$, where $h \in \{0, 1\}$ is a households credit history, and a is a current asset position of the household. The aggregate state variables are $\{z, K, m\}$, where z is an aggregate shock to productivity, K is the aggregate capital stock, and $m(x, h, a)$ is a distribution of households.[5]

A household with a good credit history, $h = 0$, solves the following problem:

$$V(z, K, m, x, 0, a) = \max\{V_0(z, K, m, x, 0, a), V_1(z, K, m, x, 0, a)\} \tag{3}$$

where $V(z, K, m, x, 0, a)$ is the value of a household with a good credit history, $V_0(z, K, m, x, 0, a)$ is the value associated with not defaulting, and $V_1(z, K, m, x, 0, a)$ is the value associated with

[4] Their model includes uncertainty shocks on labor productivity. We do not consider them while preserving idiosyncratic shocks on labor productivity and TFP shocks.

[5] As the authors pointed out, aggregate capital, K, is a state variable along with the distribution m, since m is not a sufficient statistic for the aggregate capital in the current period.

defaulting.

A non-defaulting household with a good credit history solves the following problem:

$$V_0(z,K,m,x,0,a)=\max_{c,l,a'}\left\{\frac{(c^{\alpha}(1-l)^{1-\alpha})^{1-\sigma}}{1-\sigma}+\beta\pi\gamma\sum_{z'x'}\Gamma^z_{z,z'}\Gamma^x_{x,x'}V(z',K',m',x',0,a')\right\}, \tag{4}$$

such that

$$c+a'\pi q(z,K,m,x,a')=a[1+r(z,K,m)\mathbb{1}_{a\geq 0}]+eptw(z,K,m)\cdot l$$

$$m'=\phi_m(z,z',K,m)$$

$$K'=\phi_K(z,K,m)$$

where c is the current consumption, l is the hours worked, α is the parameter of weight on consumption over the utility function, σ is the coefficient of relative risk aversion, and a' is the asset holdings in the next period. Households die with a probability of $(1-\pi)$. $q(\cdot)$ is the discount rate of debt, and $w(\cdot)$ is the wage. For savers, $q(\cdot)=1$, because there is no discount due to default risks. Borrowers have an option to default on their debt, which implies that $q(\cdot)$ includes a default premium as well as the inverse of the expected interest rate. $\mathbb{1}_{a\geq 0}$ is the indicator function of holding positive assets. $\phi_m(\cdot)$ is the law of motion for the distribution, m, and $\phi_K(\cdot)$ is the law of motion for the aggregate capital, K.

A household that files for bankruptcy with a good credit history solves the following problem:

$$V_1(z,K,m,x,0,a)=\max_{c,l}\left\{\frac{(c^{\alpha}(1-l)^{1-\alpha})^{1-\sigma}}{1-\sigma}+\beta\pi\gamma\sum_{z'x'}\Gamma^z_{z,z'}\Gamma^x_{x,x'}V(z',K',m',x',1,0)\right\} \tag{5}$$

such that

$$c=eptlw(z,K,m)(1-\xi)$$

Those who file for bankruptcy cannot save during the current period. They pay a fraction of ξ out of their labor income for a wage garnishment to creditors. In the next period, their credit status changes to a bad credit history, $h' = 1$, and they start with zero assets, $a^{'} = 0$.

A household with a bad credit history $(h = 1)$ solves the following problem:

$$V(z, K, m, x, 1, a) = \max_{c,l,a' \geq 0} \left\{ \frac{(c^{\alpha}(1-l)^{1-\alpha})^{1-\sigma}}{1-\sigma} + \beta\pi\gamma \sum_{z'x',h'} \Gamma^{z}_{z,z'} \Gamma^{x}_{x,x'} \Gamma^{h}_{h'} V(z', K', m', x', h', a^{'}) \right\}, \tag{6}$$

such that

$$c + a^{'}\pi = a[1 + r(z, K, m)] + eptlw(z, K, m)$$

Households with a bad credit history cannot borrow. Their credit status, h, reverts to be good according to a stochastic transitional process $\Gamma^{h}_{h'}$ in the next period.

Given an aggregate state (z, K, m), the probability of defaulting for a household of type $x = (\gamma, e, p, t)$ with a good credit history, $h = 0$, and amount of debt, $a^{'}$, is

$$d(z, K, m, x, a^{'}) = \sum_{z^{'},x^{'}} \Gamma^{z}_{z,z'} \Gamma^{x}_{x,x'} \mathbb{1}_{g^h=1}(z', \phi_K(z, K, m), \phi_m(z, z^{'}, K, m), x^{'}, a) \tag{7}$$

where $\Gamma^{z}_{z,z'}$ is the transitional probability function of $z^{'}$ conditional on z, $\Gamma^{x}_{x,x'}$ is the transitional probability function of $x^{'}$ conditional on x, and $\mathbb{1}_{g^h}$ is the indicator function of the default decision rule, $g^h = 1$.

The following condition needs to hold for intermediaries in the credit industries who operate at expected zero profit:

$$[1 + r(z, K, m)]q(z, K, m, x, a^{'})(-a^{'}) = \sum_{z^{'},x^{'}} \Gamma_{z,z'} \Gamma_{x,x'|z'} [\mathbb{1}_{g^{h'}=1} \xi e^{'} p^{'} t^{'} g^{l'} w(z', K', m') + \mathbb{1}_{g^{h'}=0}(-a^{'})]. \tag{8}$$

They follow a constant return to scale production technology, F, where the price of factors is determined from the standard marginal conditions:

$$w(z,K,L,m) = zF_L(K,L) \tag{9}$$

$$r_K(z,K,L,m) = zF_K(K,L) - \delta \tag{10}$$

where L is total labor input in efficiency unit, and K is aggregate capital in the current period. Due to the default option, r_K is different from the return on savings for households. There is a representative mutual fund and that all savers hold their wealth in the fund. The return on the mutual fund is

$$r(z,K,L,m) = \frac{K}{K+D} r_K(z,K,L,m) + \frac{D}{K+D} r_D(z,K,L,m), \tag{11}$$

$$r_D(z,K,L,m) = \frac{\int \mathbf{1}_{a<0}[\mathbf{1}_{g^h=1}\xi eptl g^l w(z,K,L,m) + \mathbf{1}_{g^h=0}(-a)]dm}{D} - 1, \tag{12}$$

where D is the aggregate amount of loans today, g^l is a working hours decision, and g^h is a default decision. I follow all the other details, such as the market clearing condition and recursive equilibrium, in Nakajima and Ríos-Rull (2014).

3 Algorithm

We focus on demonstrating how our algorithm is applied to solve the problem of non-defaulting households with a good credit history because that with a bad credit history can be solved by the EGM of Fella (2014) owing to its exogenous borrowing constraint.

Let $S = (z,K,m,x,h=0)$. Then we can write the value function of households with a good credit history, $V_0(z,K,m,x,h=0,a) = V_0(S,a)$, and the loan discount rate, $q(z,K,m,x,h=0,a^{'}) = q(S,a^{'})$. n is the number of iterations for the value function and loan price schedule. $EV^n(S,a^{'}) = \beta\pi\gamma\sum_{z'x'}\Gamma^z_{z,z'}\Gamma^x_{x,x'}V^n(z',K',m',x',0,a^{'})$ is the expected value function. We will denote $G_{a'} = \{a^{'}_1,\ldots,a^{'}_{N_{a'}}\}$ as the grid for assets, $a^{'}$, in the next period. In addition, we define

$D_{a'}EV^n(S,a')$ as the derivative of the expected value function with respect to asset holdings, a', in the next period. We compute the numerical derivative of the expected value function in the following way:

$$D_{a'}EV^n(S,a'_k) = \begin{cases} \frac{EV^n(S,a'_{k+1})-EV^n(S,a'_k)}{a'_{k+1}-a'_k}, \text{ for } k < N_{a'} \\ \frac{EV^n(S,a'_{N_{a'}})-EV^n(S,a'_{N_{a'}-1})}{a'_{N_{a'}}-a'_{N_{a'}-1}}, \text{ for } k = N_{a'}. \end{cases} \tag{13}$$

The numerical derivative of the discount loan rate with respect to a', $D_{a'}q^n(S,a')$, is computed in the same way.

Before dipping into the details, we provide a road map of our algorithm.

1. For each S, define the feasible set of the solution for asset holdings by calculating the risky borrowing limit and save it.
2. Identify the (non-) concave region of asset holdings a' by using the algorithm of Fella (2014).
3. For each S, for each grid point of future asset holdings, a', whose value is larger than the borrowing limit, compute the endogenously-determined cash on hand by solving the FOC. Save these pairs of the endogenously-determined cash on hand and the grid for future asset holdings a'.
4. Compute the value function for non-defaulting over the endogenous grid for cash on hand.
5. Identify the global solution over the endogenous grid for cash on hand.
   - If a' is on the concave region, save the pair of the endogenous grid for cash on had and asset holdings a'.
   - If a' is on the non-concave region, verify whether the pair implies the maximum by solving the value function. If this is the maximum, save the pair. Otherwise, discard it.
6. For the saved pairs of the endogenous cash on hand and a', compute the corresponding endogenous grid for the current assets, $a(S,a')$. Save the pairs of the endogenous grid for the current assets $a(S,a')$ and a'.
7. Evaluate the value function for non-defaulting over the endogenous grid for the current assets

into the exogenous grid for the current assets.

- Compute the value of the endogenous grid for the current assets $a(S, a^{'} = 0)$ corresponding to $a^{'} = 0$
  - If $a_i \geq a(S, a^{'} = 0)$, use a linear interpolation.
  - If $a_i < a(S, a^{'} = 0)$, solve the value function $V_0(S, a_i)$ by searching for the grid between the risky borrowing limit and zero assets.

8. Compute the value function for defaulting.
9. Update the value function and loan price schedules.
10. Start a new iteration until the updated value function is close enough to the current value function.

In the following subsections, we describe each step of the algorithm with more details.

3.1 Calculating the Risky Borrowing Limit

We set up the feasible sets of the solution through the risky borrowing limit, which is studied in Arellano (2008) and Clausen and Strub (2019). They show that , for each state S, the size of loan $q(S, a^{'})a^{'}$ increases with $a^{'}$ for any optimal debt contract. If the size of loan $q(S, a^{'})a^{'}$ decreases in $a^{'}$, households can increase their consumption by increasing debts, which cannot an optimal debt contract. Arellano (2008) (Clausen and Strub (2019)) defines the risky borrowing limit to be the lower bound of the set for optimal contract. Figure 1 illustrates the risky borrowing limit, $a^n_{rbl}(S)$.

Based on this theoretical finding, for each state S, we numerically compute the risky borrowing limit as follows.

Definition 3.1.1. For each n and S, $a^n_{rbl}(S)$ is the risky borrowing limit if

$$\forall a^{'} > a^n_{rbl}(S),\ D_{a^{'}} q^n(S, a^{'}) \cdot a^{'} > 0. \tag{14}$$

Going forward, when we compute the endogenous grid, we will only use grid points above the

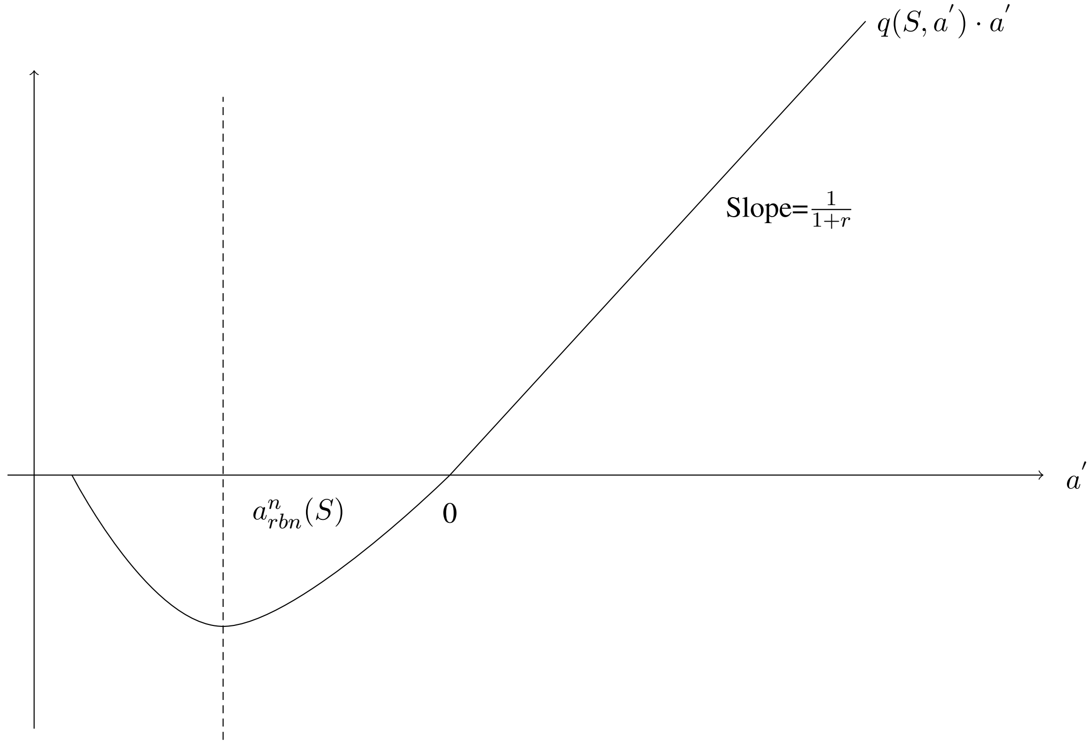

Figure 1: Risky Borrowing Limit

risky borrowing limit.

3.2 Identifying the (Non-) Concave Region

We use the algorithm of Fella (2014) to identify the concave region of the expected value function. Figure 2 helps us understand how his algorithm works. The vertical axis represents, given S, the values for the derivative of the expected value function, $D_{a'}V^n(S, \cdot)$, and the marginal utility of consumption, $D_c u(\cdot, 1 - l(S))$. The horizontal axis implies the value of asset holdings in the next period, a'. The upward-sloping curve indicates that, given a level of cash on hand M, the marginal utility of present consumption increases with asset holdings in the next period, a'. The shift of marginal utility of consumption to the right means that the marginal utility of consumption declines with cash on hand, M, which implies $M''' < M'' < M'$. The non-monotonic and discontinuous graph is the derivative of the expected value function, $D_{a'}EV^n(S, \cdot)$. The curve is discontinuous at those values of a' for which the default probability discontinuously varies along

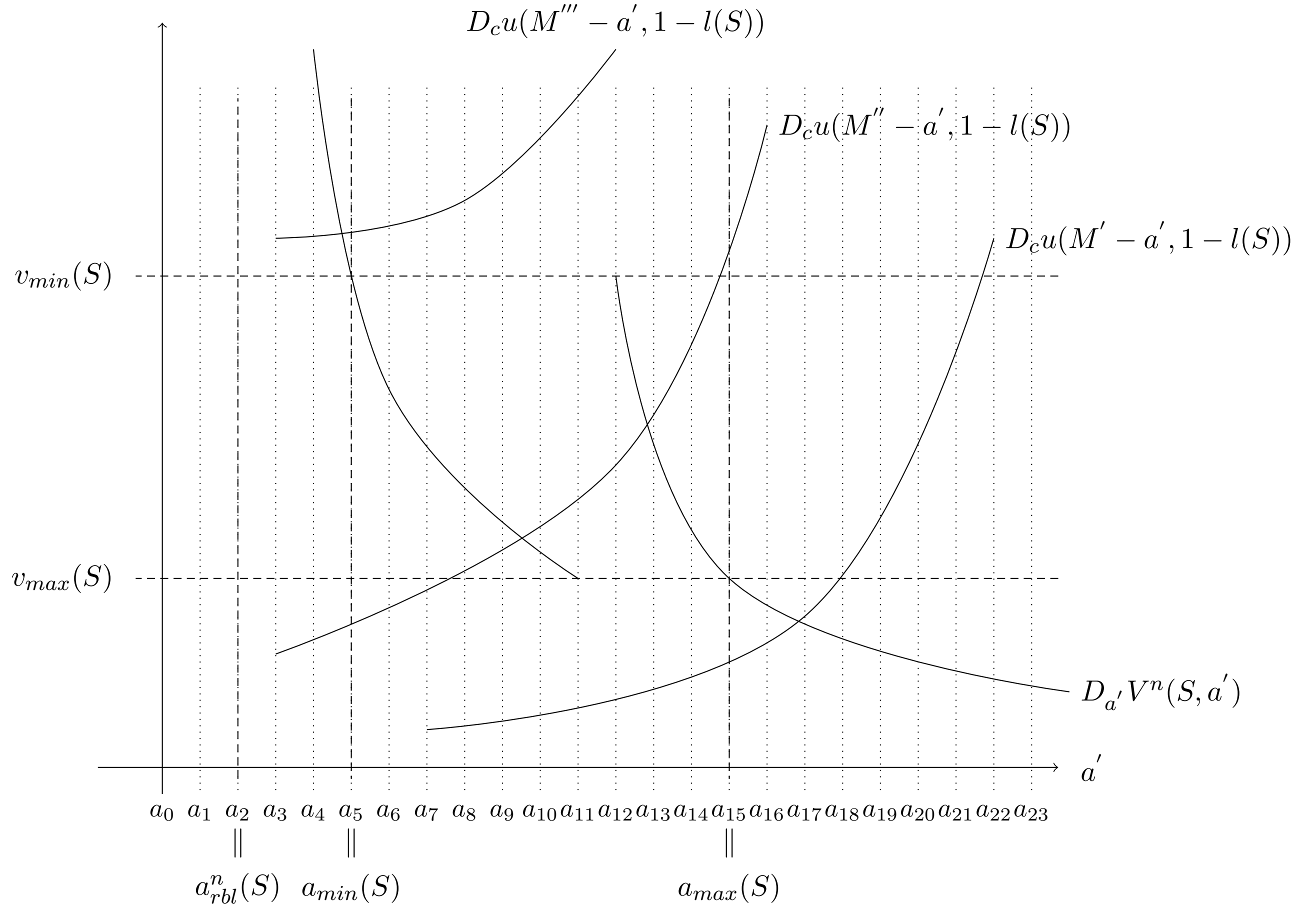

Figure 2: Illustrating the Algorithm.

changes in $a^{'}$.[6] The First Order Condition (FOC) is satisfied whenever the two curves intersect. The multiple crossing points mean that the FOC, while necessary, is not a sufficient condition for equilibrium.

Recall that, for each state S, we calculated the risky borrowing limit, $a^n_{rbl}(S)$, which plays a role in the lower bound of feasible solutions for asset holdings, $a^{'}$. In Figure 2, the risky borrowing limit, $a^n_{rbl}(S)$, is represented at a_2. Let's define $G^{rbl(S)}_{a^{'}}$ as the set of all grid points for assets above the risky borrowing limit $a^n_{rbl}(S)$. Here, $G^{rbl}_{a^{'}} = \{a_2, \dots, a_{23}\}$. Following Fella (2014), we identify regions in which the two curves are single-crossed, which means the FOC is a sufficient and necessary condition for the global solution of asset holdings in the next period, $a^{'}$. For each state S, Let us denote $G^c_a(S)$ ($G^{nc}_{a^{'}}(S)$) as the concave (non-concave) subset of $G^{rbl(S)}_{a^{'}}$. $a^{'}_i \in G^{rbl(S)}_{a^{'}}$ is on the concave region either if for any $a^{'}_j \in G^{rbl(S)}_{a^{'}}$ with $a^{'}_j < a^{'}_i$, $D_{a^{'}}V^n(S, a^{'}_i) < D_{a^{'}}V^n(S, a^{'}_j)$ or if for any $a^{'}_j \in G^{rbl(S)}_{a^{'}}$ with $a^{'}_j > a^{'}_i$, $D_{a^{'}}V^n(S, a^{'}_i) > D_{a^{'}}V^n(S, a^{'}_j)$. This condition implies that the derivative of the expected value function with regard to future assets, $D_{a^{'}}V^n(S, a^{'})$, monotonically decreases on the concave region. In Figure 2, a_5 and a_{15} are the two thresholds of this condition. Thus, in Figure 2, the concave region is $G^c_{a^{'}}(S) = \{a^{'}_2, a^{'}_3, a^{'}_4\} \cup \{a^{'}_{16}, \cdots a_{N^{'}_{a^{'}}}\}$, and the non-concave region is $G^{nc}_{a^{'}}(S) = \{a^{'}_{min} = a^{'}_5, \cdots, a^{'}_{15} = a^{'}_{max}\}$. $v_{max}(S)$ ($v_{min}(S)$) is the corresponding value of $a^{'}_{max}(S)$ ($a^{'}_{min}(S)$).

For each state S, we compute the threshold pair of $(a_{max}(S),\ v_{max}(S))$ in the following way. First, we check the discontinuous points of the derivative function, $D_{a^{'}}V^n(S, \cdot)$. Next, among the discontinuous points, we find the minimum value, which is $v_{max}(S)$. To compute $a_{max}(S)$, we search for the maximum of $a^{'}_i$ satisfying $D_{a^{'}}V^n(S, a^{'}_i) \leq v_{max}(S)$, which is defined as $a_{max}(S)$. $v_{min}(S)$ is similarly computed. We check the discontinuous points of the derivative function, $D_{a^{'}}V^n(S, \cdot)$. Next, among the discontinuous points, we find the maximum value, $v_{min}(S)$. Then, we search for the minimum of $a^{'}_i$ satisfying $D_{a^{'}}V^n(S, a^{'}_i) \geq v_{min}(S)$, which is defined as $a_{min}(S)$.

[6] Whereas the decision on default is the only discrete choice in the model, other types of discrete choices can be addressed along with the option to default.

3.3 Computing the Endogenous Grid for the Cash on Hand

For each S and $a_i^{'} \in G_{a'}$ with $a_i^{'} > a_{rbl}^n(S)$, we compute the endogenously-determined cash on hand, $M(S, a_i^{'})$. To retrieve this endogenously-determined cash on hand, $M(S, a_i^{'})$, we need to obtain the endogenously-determined consumption by using the following FOC:

$$\text{For each } n,\ S \text{ and } a_i^{'} \in G_{a'} \text{ with } a_i^{'} > a_{rbl}(S),$$
$$D_c u(c(S, a_i^{'}), 1 - l(S, a_i^{'})) = \frac{D_{a'} EV^n(S, a_i^{'})}{\pi(D_{a'} q^n(S, a_i^{'}) \cdot a_i^{'} + q^n(S, a_i^{'}))}. \tag{15}$$

where $c(S, a_i^{'})$ and $l(S, a_i^{'})$ are the endogenously-determined consumption and hours worked, respectively. The derivative of the expected value function, $D_{a'} EV^n(S, a_i^{'})$, and the loan price schedules, $D_{a'} q^n(S, a_i^{'})$, are computed by using the equation (13). Given $D_{a'} EV^n(S, a_i^{'})$ and $D_{a'} q^n(S, a_i^{'})$, $c(S, a_i^{'})$ and $l(S, a_i^{'})$ are obtained by using the consumption-leisure optimal condition. Appendix A describes the details as to how to compute them.

Given $c(S, a_i^{'})$, we retrieve the endogenously-determined cash on hand $M(S, a_i^{'})$ as follows:

$$M(S, a_i^{'}) = c(S, a_i^{'}) + \pi q^n(S, a_i^{'}) a_i^{'}. \tag{16}$$

For each S and $a_i^{'} \in G_{a'}$ with $a_i^{'} > a_{rbl}(S)$, we save this corresponding cash on hand, $M(S, a_i^{'})$.

Note that the right hand side of the FOC (15) is numerically computable.[7] Moreover, the FOC (15) is well-defined. Clausen and Strub (2019) prove the differentiability of the expected value function and the loan price schedules and show the existence of the FOC (15).[8]

[7] For each $a_i^{'} \in G_{a'}$ with $a_i^{'} > a_{rbl}(S)$, the derivative of the size of the loan, $D_{a'} q^n(S, a_i^{'}) a_i^{'} + q^n(S, a_i^{'})$, is always positive by the definition of the risky borrowing limit, $a_{rbl}(S)$. We assume that the utility function $u(c, 1 - l)$ is differentiable with respect to c and l. Moreover, the derivative of the expected value function and price function can be obtained numerically using equation (13).

[8] The proof in Clausen and Strub (2019) is based on the case of iid shocks on earnings; yet as they mentioned, the inclusion of AR-1 shocks does not make a huge difference in the proof.

3.4 Storing the Value Function for Non-Defaulting over the Endogenous Grid for Cash on Hand

Given n, for each S and $a_i^{'} \in G_{a'}$ with $a_i^{'} > a_{rbl}^n(S)$, we compute the value function over the endogenous grid for cash on hand, $M(S, a_i^{'})$ as follows:

$$V_0(S, M(S, a_i^{'})) = u(M(S, a_i^{'}) - \pi q^n(S, a_i^{'}) \cdot a_i^{'},\ 1 - l(S, a_i^{'})) + EV^n(S, a_i^{'}) \tag{17}$$

It is worth noting two things at this step. First, the value function is computed without any max-operator, which contributes to efficiency. We use the endogenously-driven cash on hand, $M(S, a_i^{'})$. Second, the value functions are defined on the endogenous grid of $M(S, a_i^{'})$, not on its exogenous grid.

3.5 Identifying the Global Solution over the Endogenous Grid for Cash on Hand

We identify a set of the global solutions and save the corresponding pairs of $(M(S, a_i^{'}), a_i^{'})$. Given n and S, $a_i^{'} \in G_{a'}$ with $a_i^{'} > a_{rbl}^n(S)$ is either on the concave region, $G_{a'}^c(S)$, or on the non-concave region $G_{a'}^{nc}(S)$. When $a_i^{'} \in G_{a'}^c(S)$, as a_3 in Figure 2, the pair of $(M(S, a_i^{'}), a_i^{'})$ implies a global solution because the FOC (15) is a sufficient and necessary condition for equilibrium when the value function is concave in $a^{'}$. We save all of the pairs $(M(S, a_i^{'}), a_i^{'})$ on the concave region.

When $a_i^{'} \in G_{a'(S)}^{nc}$ – e.g., $a_i^{'} = a_9$ in Figure 2, the pair of $(M(S, a_i^{'}), a_i^{'})$ does not guarantee a global maximum because the FOC (15) is not a sufficient condition. As in Fella (2014), for each S and $a_i \in G_{a'}^{nc}(S)$, we verify whether this a_i is the global solution by solving the following problem:

$$a_g^{'} = \underset{\{a_k^{'} \in G_{a'}^{nc}(S)\}}{\operatorname{argmax}} \Big[u(M(S, a_i^{'}) - \pi \cdot q(S, a_k^{'}) \cdot a_k^{'},\ 1 - l(S, a_k^{'})) + EV^n(S, a_k^{'})\Big]. \tag{18}$$

If $a_i^{'} = a_g^{'}$, this implies that the pair of $(M(S, a_i^{'}), a_i^{'})$ implies a global solution. Thus, we save this pair. If $a_i^{'} \neq a_g^{'}$, we discard this pair. Note that we search the grid only on the non-concave region $G_{a'}^{nc}(S)$, which reduces computational losses in efficiency. we compute the decision rule for labor supply, $l(S, a_k^{'})$, by using the consumption-leisure optimal condition in Appendix A.

Note that the stored pairs are corresponding to the global solutions. Thus, when $a_i^{'}$ corresponds to the global solutions, $V_0(S, \cdot)$ and $M(S, \cdot)$ monotonically increases with $a_i^{'}$, which allows us to use splines in the following steps.

3.6 Computing the Endogenous Grid for the Current Assets

Until the previous step, the value function V_0 is evaluated on the endogenous grid for cash on hand M. For each n, S and $a_i^{'} \in G_{a'}$ with $a_i^{'} > a_{rbl}^{n}(S)$, we compute the endogenous grid for the current assets, $a(S, a_i^{'})$ as follows:

$$a(S, a_i^{'}) = \frac{M(S, a_i^{'}) - eptw(S)l(S, a_i^{'})}{(1 + r(S) \cdot \mathbb{1}_{a \geq 0})}. \tag{19}$$

Note that when a set of $a_i^{'}$ involves the stored pair $(M(s, a_i^{'}), a_i^{'})$ in the previous step, $a(S, \cdot)$ monotonically increases with $a_i^{'}$. As mentioned previously, for the saved pairs of $(M(S, a_i^{'}), a_i^{'})$, $M(S, \cdot)$ is monotonically increasing with $a_i^{'}$. In addition, Equation (19) presents that $a(S, \cdot)$ increases with $M(S, \cdot)$, which implies that $a(S, \cdot)$ is monotonic in $a^{'}$. We save these pairs of $(a(S, a_i^{'}), a_i^{'})$. In the same logic, for these saved pairs of $(a(S, a_i^{'}), a_i^{'})$, $V_0(S, \cdot)$ monotonically increases with $a(S, a_i^{'})$.

3.7 Evaluating the Value Function for Non-Defaulting and the Policy Function on the Exogenous Grid for the Current Assets

Due to the monotonicity of $V_0(S, \cdot)$ in $a(S, \cdot)$ and that of $a(S, \cdot)$ in $a^{'}$, we can use an interpolation to evaluate them on the exogenous grid of the current asset, G_a; yet we restrict the usage of interpolation to non-negative asset holdings, $a^{'} \geq 0$, due to computational issues. As Hatchondo et al. (2010) point out, the computational accuracy is sensitive to how the derivative of the loan rate schedule, $D_{a'}q(S, a^{'})$, is calculated. We also find that the convergence of the value function, $V(S, \cdot)$, and loan rate schedules, $q(S, \cdot)$, are sensitive to the method used to compute their derivatives in the borrowing region where $a^{'} < 0$. For these reasons, we employ the grid search method for the borrowing region, $a^{'} < 0$.

Because this endogenous grid method produces a mapping from asset holdings $a^{'}$ in the next period into the current assets a, it is not very costly to find the threshold of borrowing over the current assets a. For each S, we find $a(S, a^{'} = 0)$, which is the value of the endogenous grid for the current assets corresponding to $a^{'} = 0$. When a grid point of the current assets a_i is greater than the threshold of borrowing $a(S, a^{'} = 0)$, we use a linear interpolation to evaluate the value, $V_0(S, a_i)$, and the policy function of asset holdings, $a^{'}(S, a_i)$. When $a_i < a(S, a^{'} = 0)$, we compute $V_0(S, a_i)$ and $a^{'}(S, a_i)$ by solving the following problem:

$$V_0(S, a_i) = \tag{20}$$

$$\max_{\{a_{rbl}(S) < a_j^{'} < 0\}} u((1 + r(s) \cdot \mathbb{1}_{a_i \geq 0})a_i + eptw(S) \cdot l(S, a_j^{'}) - \pi q(S, a_j^{'})a_j^{'},\ 1 - l(S, a_j^{'})) + EV^n(S, a_j^{'}).$$

Note that this inclusion of the grid search does not bring about a huge loss in efficiency because this problem searches the grid just between the risky borrowing limit, $a_{rbl}(S)$, and zero assets, $a^{'} = 0$. We compute the decision rule for labor supply, $l(S, a_j^{'})$, by using the consumption-leisure optimal condition in Appendix A.

3.8 Computing the Value Function for Defaulting

We solve the value function of defaulting with a good credit history:

$$V_1^{n+1}(S, a) = u(eptlw(z, K, m)(1 - \xi)) + \beta\pi\gamma \sum_{z'x'} \Gamma^z_{z,z'} \Gamma^x_{x,x'} V^n(z', K', m', x', 1, 0) \tag{21}$$

Since the value function of defaulting is not related to any continuous endogenous state, it is not costly to compute it.

3.9 Updating the Value Function and Loan Price Schedules

We update the value function, $V^{n+1}(S,a)$ and the price function, $q^{n+1}(S,a^{'})$ in the following way:

$$V^{n+1}(S,a) = \max\{V_0^{n+1}(S,a), V_1^{n+1}(S,a)\} \tag{22}$$

$$q^{n+1}(S,a^{'}) = \sum_{z^{'},x^{'}} \Gamma^z_{z,z^{'}} \Gamma^x_{x,x^{'}} \frac{\mathbb{1}_{g^{'h}=1}\xi e^{'}p^{'}t^{'}g^{'l}w(z^{'},K^{'},m^{'}) + \mathbb{1}_{g^{'h}=0}(-a^{'})}{[1+r(z^{'},K^{'},m^{'})](-a^{'})}$$

where

$$d(z,K,m,x,a^{'}) = \sum_{z^{'},x^{'}} \Gamma^{'}_{z,z^{'}} \Gamma^x_{x,x^{'}|z^{'}} \mathbb{1}_{g^h(z^{\prime},\phi_K(z,K,m),\phi_m(z,z^{'},K,m),x^{'},a^{'})=1}.$$

If $||V^{n+1}(S,a) - V^n(S,a)||_\infty > 10^{-5}$ with $||\cdot||_\infty$ the sup norm over $\mathbb{S}X\mathbb{A}$, start a new iteration.

3.10 Summary of the Algorithm

To sum up, given an iteration number, n, and the expected value function $EV^n(S,a)$, the algorithm is as follows:

1. For each S, calculate the risky borrowing limit, $a^n_{rbl}(S)$, and save it.
2. Identify the (non-) concave region of asset holdings $a^{'}$ by using the algorithm of Fella (2014).
3. Given $(S, a^{'}_{rbl(S)})$, compute the endogenously-determined cash on hand, $M(S,a^{'})$, by solving the FOC (15). Save these pairs of $(M(S,a^{'}), a^{'})$.
4. Compute the value function for non-defaulting over the endogenous grid for cash on hand, $(V^n_0(S, M(S,a^{'}))$.
5. Identify the global solution over the endogenous grid for cash on hand.
   - If $a^{'}_i$ is on the concave region, save the pair of $(M(S,a^{'}_i), a^{'}_i)$
   - If $a^{'}_i$ is on the non-concave region, verify whether the candidate $(M(S,a^{'}_i), a^{'}_i)$ implies

the maximum by solving the value function. If this is the maximum, save the pair of $(M(S, a_i^{'}), a_i^{'})$. Otherwise, discard it.

6. For the saved pairs of $(M(S, a_i^{'}), a_i^{'})$, compute the corresponding endogenous grid for the current assets, $a(S, a_i^{'})$. Save the pairs of $(a(S, a_i^{'}), a_i^{'})$.
7. Evaluate the value function for non-defaulting over the endogenous grid for the current assets into the exogenous grid for the current assets.
   - Compute the value of the endogenous grid for the current assets $a(S, a^{'} = 0)$ corresponding to $a^{'} = 0$
     - If $a_i \geq a(S, a^{'} = 0)$, use a linear interpolation.
     - If $a_i < a(S, a^{'} = 0)$, solve the value function $V_0(S, a_i)$ by searching for the grid between the risky borrowing limit, $a^n_{rbl}(S)$ and zero assets, $a^{'} = 0$.
8. Compute the value function for defaulting, $V_1^{n+1}(S, a)$.
9. Update the value function, $V^{n+1}(S, a)$, and loan price schedules, $q^{n+1}(S, a^{'})$.
10. Start a new iteration if $||V^{n+1}(S, a) - V^n(S, a)||_\infty > 10^{-5}$.

4 Results

We compare the computing time and accuracy of our EGM with those of the grid search method. As in Nakajima and Ríos-Rull (2014), we use Krusell and Smith's (1998) method to handle the aggregate uncertainty. Note that this method approximates aggregate states using a few moments and agents expect next period states using parameterized functional forms of those moments. The method achieves high accuracy, but it requires a long simulation to update forecasting rules and may take many trials to find a proper functional form.

4.1 Parameterization

Table 1: Chosen Parameters

λ	π	σ	γ_1	θ	δ	σ_e	ρ_p
0.1000	0.9800	3.7167	1.0000	0.3600	0.0800	0.4400	0.9630
σ_p	σ_t	ξ	β	α	Γ_2^γ	γ_2	η
0.1300	0.3500	0.3395	1.0011	0.3681	0.0310	0.0000	0.7500
$\nu_1 = \nu_2$	ν_3	$\gamma_{1,1}^z$	$\gamma_{2,2}^z$	$\gamma_{3,3}^z$	γ_3^z		
0.0134	0.0267	0.6667	0.6667	0.3333	0.0200		

We follow Nakajima and Ríos-Rull's (2014) choice of parameter values. Table 1 shows the values of the chosen parameters.

4.2 Specification of Krusell and Smith's (1998) Method

Nakajima and Ríos-Rull (2014) approximated $(z, K; m)$ with (z, K, O), where O is average individual labor productivity and use forecasting rules for K', L, r, and O'. Here, we abstract from the counter-cyclical earnings risk and approximate aggregate states $(z, K; m)$ to (z, K). Additionally, instead of forecasting L, which is necessary to calculate the wage, we forecast the wage directly. We specify the forecasting functions for K', r, and w as the following log-linear forms:

$$log\, K' = \phi_{k1}(z, K) + \phi_{k2}(z, K) \cdot log\, K$$

$$log\, r = \phi_{r1}(z, K) + \phi_{r2}(z, K) \cdot log\, K$$

$$log\, w = \phi_{w1}(z, K) + \phi_{w2}(z, K) \cdot log\, K$$

4.3 Computing Time and Accuracy

We vary the size of the grid for assets across computational exercises. In all computational exercises, we keep the number of the grid points for the other variables as follows. The size of the grid for the permanent labor productivity shock is 2, $n_e = 2$, that for the persistent shock is 15, $n_p = 15$, and that for the transitory shock is 3, $n_t = 3$. The number of the grid for the TFP shock is 3, $n_z = 3$, and that for K is 5, $n_k = 5$. Because we use Krusell and Smith's (1998) method, we must go through the inner and outer loops several times until the forecasting rules are convergent. We compute the average CPU time per iteration in the inner loop and outer loop, respectively. We simulated the model for 2,000 periods with Krusell and Smith's (1998) method, and all computations were carried out on on a single core of an Intel i7-4770 processor. The programs were written in Fortran 95.

Table 2: Computing Time

# of GRD. PTS. for INR. - OTR.	200-500		300-500		400-500		500-600	
Computational Method	EGM	GS	EGM	GS	EGM	GS	EGM	GS
# of ITER	7	7	7	7	7	7	7	7
AVG CPU Time in INR. per ITER.*	0.68	12.54	1.25	29.27	1.99	54.39	2.99	79.65
AVG CPU Time in OTR. per ITER.*	29.49	173.49	27.05	185.62	24.48	182.79	38.66	286.97

.$*$: Unit = minute

Table 2 indicates that the EGM is faster than the grid search method both in the inner loop and in the other loop. In the inner loops, the EGM is from 18.5 to 27.3 times faster than the grid search method. In the outer loop, the EGM is approximately 7.5 times faster than the grid search method. The gap differs across the size of the asset grid, but the EGM is much more efficient than the grid search method across all grid settings.

To measure accuracy, we use three criteria in the literature. First, we compute not Euler equation errors but Bellman equation errors. To compute the Euler error, we must calculate the derivative of the loan price schedule, of which value depends on types of numerical derivatives. As Hatchondo et al. (2010) point out, Euler equation errors are sensitive to how to calculate the derivative of the loan rate schedule, $q(S, a^{'})$. To avoid this issue, we compute Bellman equation errors. Recall the following notation: S is the state vector other than assets a. Then, the Bellman equation

$$V(S,a) = u(c(S,a), l(S,a)) + E_{S'}\left[V(S^{'}, a^{'}(S,a))\right] \tag{23}$$

should hold exactly for the true decision rules. Because in our computational exercises, the decision rules are numerically computed, the Bellman equation (23) does not hold exactly with the numerically calculated decision rules. We define c^* as the solution for

$$u(c^*(S,a), \bar{l}(S,a)) = V(S,a) - E_{S'}\left[V(S^{'}, \bar{a}^{'}(S,a))\right] \tag{24}$$

where bars indicate the numerically calculated decision rules. We define the Bellman equation error as

$$BE(s,a) = \left|1 - \frac{c^*(S,a)}{\bar{c}(S,a)}\right|. \tag{25}$$

Following the literature, we report both the maximum and the average of Bellman equation errors. Second, we take Den Hann's forecasting test described in Algan et al. (2014). It is the difference between expected K_e' by the forecasting rules and realized K_r' from the simulations: $|logK_r' - logK_e'|$. Finally, we reports the R^2 of the forecasting rules in the simulation step.

Figure 3 implies that with the EGM, the price dynamics in the simulation are very close to those generated by the forecasting rules. Since they are very close to one another, it is hard to observe blue lines in the dynamics of the risk-free interest rate and wage. Figure 4 shows that, with the grid search method, there are differences between the simulated-dynamics of these prices and those generated by the forecasting rules. Den Hann error measures those differences. Overall, Den Han errors from the EGM are smaller than those from the grid search method.

Table 3 implies that the EGM produces more accurate outcomes than the grid search method. Regarding the Bellman equation error, the average Bellman equation error in the EGM is approx-

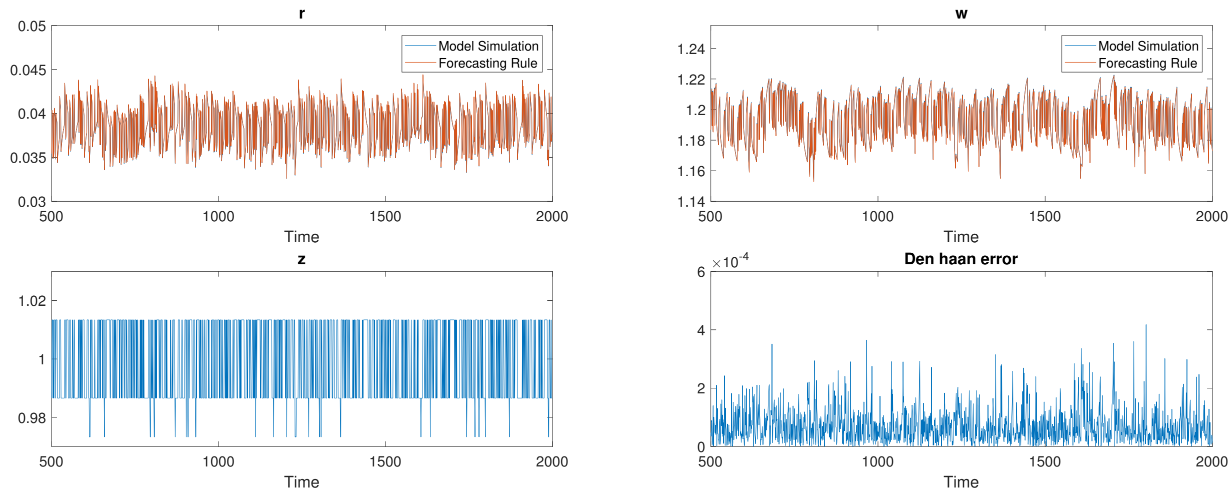

Figure 3: Simulation Results for the EGM with the 500-600 grid

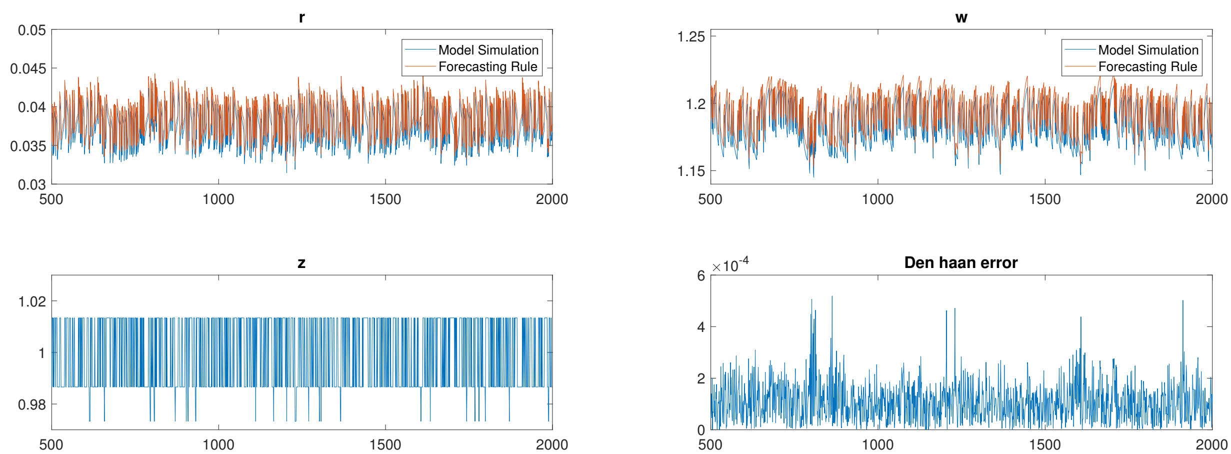

Figure 4: Simulation Results for the Grid Search Method with the 500-600 grid

imately three times lower than that in the grid search methods in various grid settings. Although the gap in the maximum Bellman error is smaller than that in the average Bellman error, the EGM generates a smaller value of the maximum Bellman error than the grid search method. This smaller gap appears because our EGM also uses the grid search method for the borrowing region, $a^{'} < 0$. Additionally, the EGM produces smaller values of R^2s than the grid search method. The gap in R^2 is well-observed for r. Moreover, the average Den Hann error from the EGM is lower than that from the grid search method across various grid settings. The EGM also generates smaller values of the maximum Den Hann error than the grid search method.

Table 3: Computational Accuracy

# of GRD. PTS. for INR. - OTR.	200-500		300-500		400-500		500-600	
Computational Method	EGM	GS	EGM	GS	EGM	GS	EGM	GS
AVG BE Error*	0.11%	0.36%	0.06%	0.16%	0.03%	0.09%	0.02%	0.06%
MAX BE Error*	10.77%	15.53%	11.34%	11.76%	11.57%	17.49%	11.71%	15.46%
R^2 of $K^{'}$	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
R^2 of r	0.9987	0.9911	0.9971	0.9882	0.9975	0.9963	0.9977	0.9940
R^2 of w	0.9999	0.9997	0.9999	0.9997	0.9997	0.9997	0.9997	0.9996
mean(DH)	0.004%	0.01%	0.005%	0.012%	0.009%	0.01%	0.007%	0.01%
max(DH)	0.029%	0.06%	0.038%	0.081%	0.047%	0.073%	0.04%	0.05%

.$*$: The Bellman equation errors are computed in stationary equilibrium.

5 Conclusion

We develop an endogenous grid method for models with the option to default. This method extends Fella's (2014) endogenous grid method by newly introducing a numerical step to search for the risky borrowing limit, which is the lower bound of the feasible set for the solution of asset holdings. By using the algorithm of Fella (2014) and our novel step for the risky borrowing limit, we identify the region of solution sets to which Carroll's (2006) endogenous grid method is applicable. Compared to the conventional grid search method, the method brings substantial improvements in computational efficiency and accuracy. We hope this method opens up possibilities for researchers to investigate topics with the default option that have not been explored due to computational complexities.

A Computing Labor Supply with the EGM

Following Khan (2016), we obtain the decision rule for labor supply by using the FOCs of consumption and labor. The FOCs of labor and consumption, respectively, are as follows.

$$\text{FOC:}[l] : [c^{\alpha}(1-l)^{1-\alpha}]^{-\sigma}(1-\alpha)c^{\alpha}(1-l)^{-\alpha} = \lambda eptw(z,K,m) \tag{26}$$

$$\text{FOC:}[c] : [c^{\alpha}(1-l)^{1-\alpha}]^{-\sigma}\alpha c^{\alpha-1}(1-l)^{1-\alpha} = \lambda. \tag{27}$$

where λ is the Lagrangian multiplier for the budget constraint. By rearranging the two FOCs, we can represent leisure $1-l$ in terms of consumption c as follows.

$$1-l = \frac{1-\alpha}{\alpha}\frac{c}{eptw(z,K,m)}. \tag{28}$$

We substitute (28) into the budget constraint, $c + a^{'}\pi q(z,K,m,x,a^{'}) = a[1+r(z,K,m)\mathbb{1}_{a\geq 0}] + eptw(z,K,m)\cdot l$. Then, we obtain

$$c = \alpha \cdot \Big(a[1+r(z,K,m)\mathbb{1}_{a\geq 0}] + eptw(z,K,m) - a^{'}\pi q(z,K,m,x,a^{'})\Big). \tag{29}$$

We substitute c in (29) into (28) again. If the implied l is greater than or equal to 0, we keep l and c from this step. Otherwise $(l < 0)$, we replace the value of l with 0 and compute c by using the FOC (15).

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