A Generalized Endogenous Grid Method for Default Risk Models*

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Abstract

Researchers often use the grid search method to solve models with default risk because the complexity of the problem prevents the use of more efficient but less general tools. In this paper, we propose an extension of the endogenous grid method for default risk models in which price schedules are dependent on individuals' state variables and endogenously determined in equilibrium. Our method combines Fella's (2014) identification for non-concave regions and our algorithm that numerically searches for the risky borrowing limit, a limit that is a theoretical lower bound of the feasible set of asset holdings. The demarcation of this combination enables us to exploit the endogenous grid method by identifying the region of solution sets. Our method is faster and more accurate than the conventional grid search method while preserving the stability; these computational gains are amplified in richer models. With higher accuracy, our method is approximately eight to nine times faster with a simple canonical model of Arellano (2008); approximately 19 to 27 times faster with the richer model of Nakajima and Ríos-Rull (2014). Finally, we show that our method is applicable to a broad class of default risk models by characterizing sufficient conditions for the application. This method may contribute to facilitating the use of model environments with default choices, thereby allowing us to explore further questions of credit and financial frictions.

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1 Introduction

In any financial market, defaults have widely been observed. World capital markets often have experienced sovereign defaults on a large scale, and consumer bankruptcy is one of the important social insurances in many countries. Researchers have employed default risk models to better understand the sources of defaults and the implications of default-related policies.[1] Default risk models have tended to become quite intricate over time as they are applied to more heterogeneous situations and financial markets become increasingly complex.

Despite the increased computational burden, we have relied on the stability of the grid search method to solve default risk models at the cost of efficiency because there are few robust and efficient solution methods for them. Arellano, Maliar, Maliar and Tsyrennikov (2016) developed an envelope condition method (ECM) for default risk models, but this method is not applicable to life-cycle models as it solves problems in a forward manner. Moreover, as mentioned in the paper, this ECM does not guarantee the convergence of value functions in default risk models. Another efficient algorithm is the endogenous grid method (EGM), which was originally developed by Carroll (2006). This EGM has progressed to solve a broader class of dynamics problems. For example, the EGM has been extended to solve models with endogenous labor supply (Barillas and Fernández-Villaverde, 2007), discrete choices (Fella, 2014; Iskhakov, Jørgensen, Rust and Schjerning, 2017), and multiple dimensional choices (Hintermaier and Koeniger, 2010; Druedahl and Jørgensen, 2017). Unfortunately, these extended EGMs cannot be applied to default risk models.

These EGMs do not work with default risk models because they have not addressed the following features of default risk models in a comprehensive manner. First, the value functions are non-concave and not differentiable across their whole domain because defaulting is a discrete choice variable which causes kinks in value functions. Several extended EGMs have addressed this problem (Fella, 2014; Iskhakov et al., 2017; Druedahl and Jørgensen, 2017), but none of them have accurately handled another issue in default risk models: the feasible set of asset holdings is not predetermined and differs across individual states.[2] This set must be established before solving the models to implement these solution methods. Villemot (2012) is an exception using the EGM to solve a default risk model in Arellano (2008) by introducing a heuristic algorithm that updates the lower bound of the feasible set. This algorithm, however, is not guaranteed to find the correct

[1] For example, there have been studies for the episodes of sovereign default (Aguiar and Gopinath, 2006; Arellano, 2008; Yue, 2010), the implications of consumer bankruptcy reforms (Chatterjee et al., 2007; Livshits et al., 2007; Athreya, 2008; Athreya et al., 2009; Livshits et al., 2010; Chatterjee and Gordon, 2012; Nakajima, 2017), and the interactions between household default and business cycles (Nakajima and Ríos-Rull, 2014; Gordon, 2015).

[2] In contrast, when an option to default is unavailable, this issue does not appear because the feasible set of the solution is irrelevant to its equilibrium. It is predetermined through an exogenous borrowing constraint or a collateral constraint.

feasible set. Further, it is uncertain how generally this algorithm works with other default risk models because Villemot (2012) did not consider the formalization of the class of models for the application.

In this paper, we propose an extension of the EGM that is faster, more accurate, and applicable to a broader class of default risk models than previous methods in the literature. Our method comprehensively handles the computational issues above. First, we address issues from the non-concavity and non-differentiability by employing Fella's (2014) algorithm. But, as mentioned previously, Fella's (2014) EGM cannot directly be applied to endogenous default risk models because it works only when the feasible set for the solution is predetermined through an exogenous borrowing constraint or a collateral constraint. To overcome this issue, we introduce a numerical procedure to identify the feasible set for the solution. Arellano (2008) and Clausen and Strub (2020) show that in every optimal debt contract, the size of debt, which is the product of the price and the quantity of debt, increases with the quantity of debt. This condition will only be satisfied on a closed interval of debt holdings, which we can find numerically for each state. We can then take the minimum of this interval as the risky borrowing limit, which serves as the lower bound of the feasible set of debt holdings. This combination enables us to exploit the computational benefits of the EGM in solving default risk models.

We illustrate the detailed procedures of our method with a canonical model of Arellano (2008).[3] Our EGM has noticeable computational benefits in accuracy and efficiency. Our method converges approximately eight to nine times faster and yields more accurate results than the grid search method according to Bellman equation errors. These computational benefits are amplified when we apply our method to solve the richer model of Nakajima and Ríos-Rull (2014). We show that our method is approximately 19 to 27 times faster than the grid search method.

Finally, we characterize sufficient conditions for our method to be applicable to a model. The sufficient conditions imply that our EGM can be applied to a broad class of default risk models: both finite- and infinite-horizon models, along with other discrete choices (e.g., housing, durable goods, health insurance, and retirement) and multiple types of defaults (e.g., Chapter 7 vs. Chapter 13 in consumer bankruptcy).

The organization of this paper is as follows. Section 2 describes the model of Arellano (2008), to which we apply our method. In Section 3, the detailed procedures of our algorithm are demonstrated, and the results are reported in Section 4. In Section 5, we provide and discuss sufficient conditions for the application of our method. Finally, Section 6 concludes this paper.

[3]This model is well suited as a pedagogical example because it contains all the necessary components in a relatively simple model.

2 Model

In this section, we lay out Arellano's (2008) model. In the model, the government starts each period with assets a and endowment S. Endowment S follows an AR(1) process with a persistence of ρ_S:

$$S^{'} = \rho_S S + \epsilon^{'},\ \epsilon^{'} \sim N(0, \sigma_S). \tag{1}$$

The AR(1) process is approximated in a Markov chain $\pi_{S,S'}$. The government has an option to default on its debt $a < 0$. Given the option to default, the government solves the following problem:

$$V(S,a) = \max\left\{V^c(S,a), V^d(S)\right\} \tag{2}$$

where $V(S,a)$ is the value of the government, $V^c(S,a)$ is the value associated with not defaulting and $V^d(S)$ is the value associated with defaulting.

The value associated with not defaulting is as follows:

$$V^c(S,a) = \max_{\{a' \geq -Z\}} \left\{ u(S - q(S,a^{'})a^{'} + a) + \beta \sum_{S'} \pi_{S,S'} V(S^{'}, a^{'}) \right\} \tag{3}$$

where Z is a lower bound on debt to prevent Ponzi schemes but is otherwise not binding in the equilibrium. $u(\cdot)$ is the utility function that is differentiable, and $q(\cdot,\cdot)$ is the loan price schedule over the current endowment S and the next period asset $a^{'}$.

The value associated with defaulting is as follows:

$$V^d(S) = u(h(S)) + \beta \sum_{S'} \pi_{S,S'} \left[\theta V(S^{'},0) + (1-\theta)V^d(S^{'})\right] \tag{4}$$

$$h(S) = \begin{cases} \lambda & \text{if } S > \lambda \\ S & \text{if } S \leq \lambda, \end{cases} \tag{5}$$

where θ is the probability that the economy will regain access to the international credit markets. The above value function implies that default causes two kinds of penalties. The first type of penalty is exclusion costs. This is the opportunity cost of not having access to the credit market in the following period with probability 1-θ. The other type of penalty is output costs. The government must pay $S - \lambda$ when S is greater than λ under a bad credit history.

The financial market is competitive with risk neutral lenders whose expected profit is zero.

With these lenders, the loan price schedule, $q(S, a^{'})$, satisfies

$$q(S, a^{'}) = \frac{1 - \delta(S, a^{'})}{1 + r} \tag{6}$$

where $\delta(S, a^{'})$ is the probability of default associated with S and $a^{'}$ and r is the risk-free interest rate.

To set the loan price, we need to characterize the default probability $\delta(S, a^{'})$. To do so, let us define $D(a)$ as

$$D(a) = \{S : V^c(S, a) < V^d(S)\}. \tag{7}$$

The probability of default with endowment S and assets in the next period $a^{'}$ is

$$\delta(S, a^{'}) = \sum_{\{S^{'} \in D(a^{'})\}} \pi_{S,S^{'}} = \sum_{\{S^{'}: V^c(S^{'}, a^{'}) < V^d(S^{'})\}} \pi_{S,S^{'}} \tag{8}$$

If $D(a^{'}) = \emptyset$, the equilibrium default probability becomes zero, and the bond price is equal to that of risk-free bond.

3 Algorithm

Let us begin with notations to explain the algorithm. Let n be the number of iterations for the value function and loan price schedule. Let $EV^n(S, a^{'})$ be the expected value function, $\beta \sum_{S^{'}} \pi_{S,S^{'}} V^n(S^{'}, a^{'})$. We will denote $G_{a^{'}} = \{a^{'}_1, \ldots, a^{'}_{N_{a^{'}}}\}$ as the grid for assets, $a^{'}$, in the next period. In addition, we define $D_{a^{'}} EV^n(S, a^{'})$ as the derivative of the expected value function with respect to the next period asset holdings, $a^{'}$. We compute the numerical derivative of the expected value function in the following way:

$$D_{a^{'}} EV^n(S, a^{'}_k) = \begin{cases} \frac{EV^n(S, a^{'}_{k+1}) - EV^n(S, a^{'}_k)}{a^{'}_{k+1} - a^{'}_k}, \text{ for } k < N_{a^{'}} \\ \frac{EV^n(S, a^{'}_{N_{a^{'}}}) - EV^n(S, a^{'}_{N_{a^{'}}-1})}{a^{'}_{N_{a^{'}}} - a^{'}_{N_{a^{'}}-1}}, \text{ for } k = N_{a^{'}} \end{cases} \tag{9}$$

where $N_{a^{'}}$ is the number of grid for a. The numerical derivative of the discount loan rate with respect to $a^{'}$, $D_{a^{'}} q^n(S, a^{'})$, is computed in the same way.

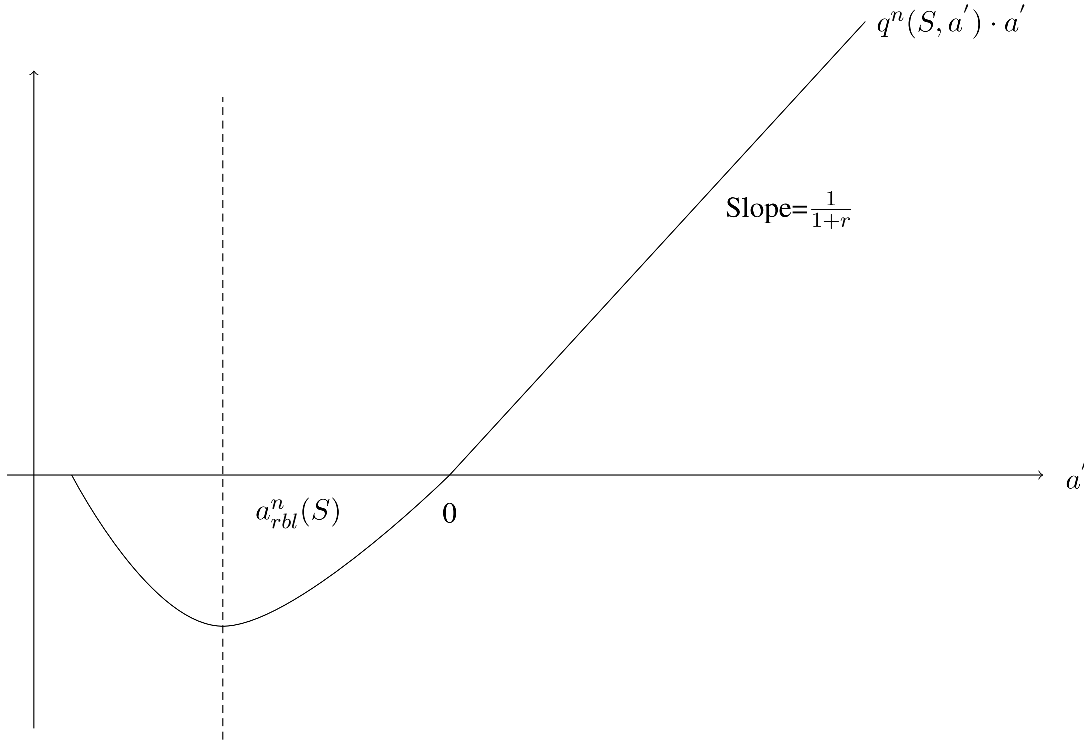

Figure 1: Risky Borrowing Limit

3.1 Calculating Risky Borrowing Limit

We set up the feasible sets of the solution through the risky borrowing limit (credit limit), which is studied in Arellano (2008) and Clausen and Strub (2020). They show that, for each state S, the size of loan $q(S, a^{'})a^{'}$ increases with $a^{'}$ in every optimal debt contract. If the size of loan $q(S, a^{'})a^{'}$ decreases in $a^{'}$, households can increase their consumption by reducing debts (increasing $a^{'}$), which implies that it cannot be an optimal debt contract. Arellano (2008) defines the risky borrowing limit to be the lower bound of the set for optimal contract. Figure 1 illustrates the risky borrowing limit, $a^n_{rbl}(S)$.

Using this theoretical finding, we numerically compute the risky borrowing limit for each state S using the following definition:

Definition 3.1.1. For each n and S, $a^n_{rbl}(S)$ is the risky borrowing limit if

$$\forall a^{'} > a^n_{rbl}(S),\ D_{a^{'}} \left(q^n(S, a^{'}) \cdot a^{'}\right) = D_{a^{'}} q^n(S, a^{'}) \cdot a^{'} + q^n(S, a^{'}) > 0. \tag{10}$$

Going forward, when we compute the endogenous grid, we will only use grid points above the risky borrowing limit.[4]

[4]We argue that the risky borrowing limit might be a general feature of default risk models. More details will be

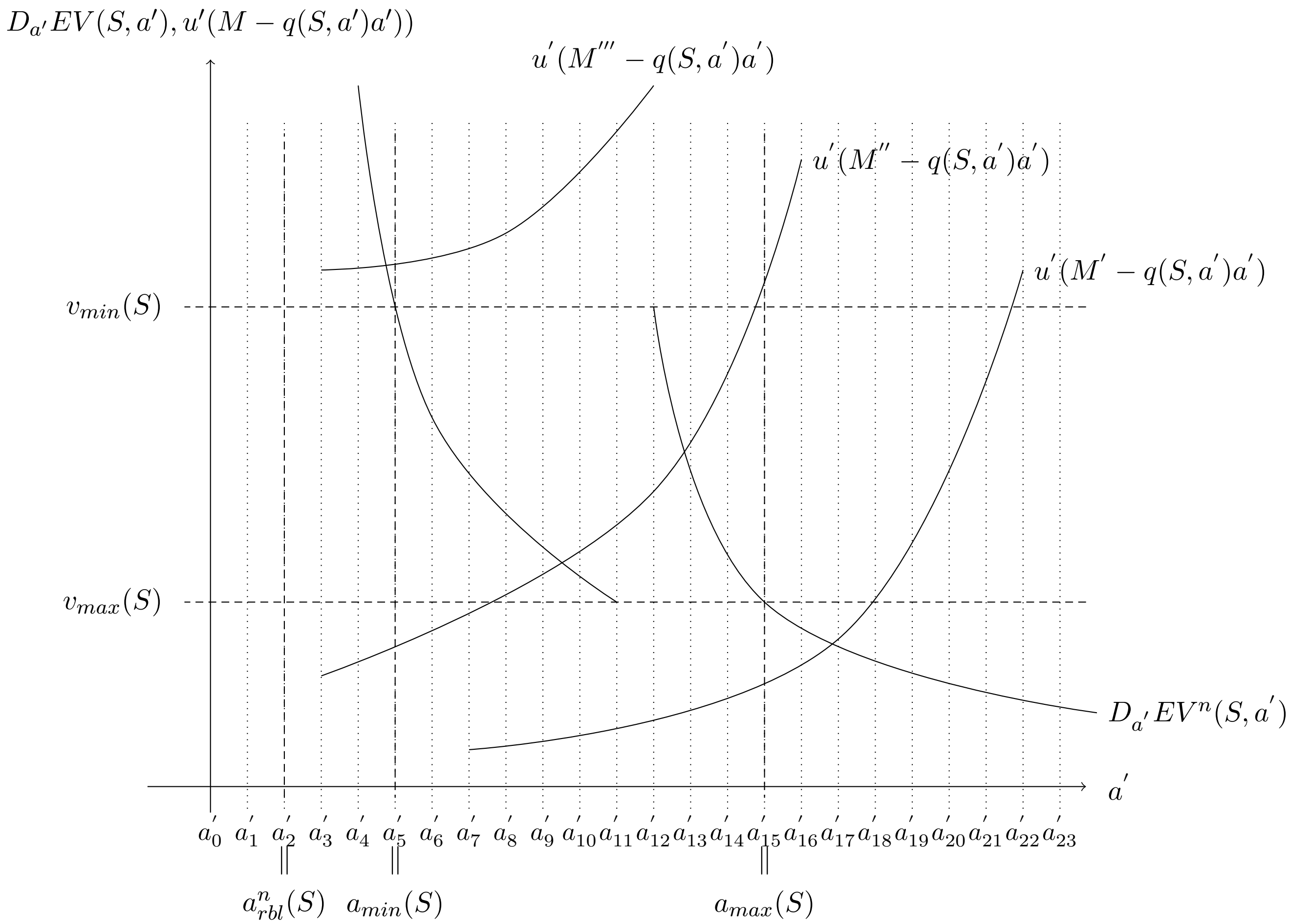

Figure 2: Illustrating the Algorithm

3.2 Identifying the (Non-) Concave Region

Fella (2014) presented an algorithm that divides the space of assets in the next period $a^{'}$ into concave and non-concave regions. In the concave region, the FOC is sufficient and necessary for the global solutions, whereas in the non-concave region, the FOC is only a necessary condition. This algorithm uses information on the curvature of the expected value function. The details are as follows.

As in Fella (2014), we use Figure 2 to understand how his algorithm works. The vertical axis represents, for a given S, the values for the derivative of the expected value function, $D_{a^{'}}EV^n(S,\cdot)$, and the marginal utility of consumption, $u^{'}(\cdot)$. The horizontal axis is the value of asset holdings in the next period, $a^{'}$. Given a level of cash on hand M, the marginal utility of present consumption increases with asset holdings in the next period, $a^{'}$. Let $M^{'''} < M^{''} < M^{'}$ be three arbitrary levels of cash on hand. We can then see from Figure 2 that marginal utility of present consumption is decreasing in cash on hand. The non-monotonic and discontinuous line is the derivative of the expected value function, $D_{a^{'}}EV^n(S,\cdot)$. The curve is discontinuous at those values of $a^{'}$ for which the default probability jumps discretely as $a^{'}$ changes.[5] The risky borrowing limit, $a^n_{rbl}(S)$, is represented at $a^{'}_2$. Let us define $G_{a^n_{rbl}(S)}$ as the set of all grid points for assets above the risky borrowing limit $a^n_{rbl}(S)$. Here, $G_{a^n_{rbl}(S)} = \{a^{'}_2, \ldots, a^{'}_{23}\}$.

This algorithm identifies the concave region by using information related to the First Order Condition (FOC). In Figure 2, the intersection points between $u^{'}(\cdot)$ and $D_{a^{'}}EV^n(S,\cdot)$ are where the FOC holds. A non-concavity causes a jump in $D_{a^{'}}EV(S,\cdot)$, which can lead to multiple crossing points. (e.g. $u^{'}(M^{''} - q(S,a^{'})a^{'})$ intersects $D_{a^{'}}EV(S,\cdot)$ twice; between $a^{'}_9$ and $a^{'}_{10}$, between $a^{'}_{12}$ and $a^{'}_{13}$). The multiple crossing points means that the FOC is not a sufficient condition but a necessary condition. In contrast, if there is only one crossing point then necessity of the FOC combined with uniqueness means that it is also sufficient for a global solution.[6] As in Fella (2014), we identify the concave region where the two curves are single-crossed by using the following criterion.

$a^{'}_i \in G_{a^n_{rbl}(S)}$ is on the concave region either if

$$\begin{aligned} &\forall\ a^{'}_j \in G_{a^n_{rbl}(S)} \text{ with } a^{'}_j < a^{'}_i,\ D_{a^{'}}EV^n(S,a^{'}_i) < D_{a^{'}}EV^n(S,a^{'}_j) \text{ or} \\ &\forall\ a^{'}_j \in G_{a^n_{rbl}(S)} \text{ with } a^{'}_j > a^{'}_i,\ D_{a^{'}}EV^n(S,a^{'}_i) > D_{a^{'}}EV^n(S,a^{'}_j). \end{aligned} \tag{11}$$

This condition implies that the derivative of the expected value function, $D_{a^{'}}EV^n(S,a^{'})$, is strictly

addressed in Section 5.4.

[5] Whereas the decision on default is the only discrete choice in the model, other types of discrete choices can be addressed along with default options. More details will be discussed in Section 5.

[6] In Figure 2 this happens at the point where $u^{'}(M^{'''} - q(S,a^{'})a^{'})$ intersects $D_{a^{'}}EV(S,\cdot)$; between $a^{'}_4$ and $a^{'}_5$.

decreasing on the concave region. In Figure 2, $a^{'}_5$ and $a^{'}_{15}$ are the two thresholds of this condition; we denote them as $a_{min}(S)$ and $a_{max}(S)$, respectively. As a result, in Figure 2, the concave region is $\{a^{'}_2, a^{'}_3, a^{'}_4\} \cup \{a^{'}_{16}, \cdots a^{'}_{23}\}$. The remaining region becomes the non-concave region, $\{a_{min}(S) = a^{'}_5, a^{'}_6, \cdots, a^{'}_{14}, a^{'}_{15} = a_{max}(S)\}$. $v_{max}(S)$ and $v_{min}(S)$ are the corresponding values of $D_{a^{'}}EV(\cdot)$ at $a_{max}(S)$ and $a_{min}(S)$, respectively.

Note that identifying the concave region is equivalent to finding $a_{min}(S)$ and $a_{max}(S)$. To find the thresholds, we take the following steps. First, we check the discontinuous points of the derivative of the expected value function, $D_{a^{'}}EV^n(S, a^{'}_i)$ with respect to $a^{'}_i$. In Figure 2, the discontinuous points arise at $a^{'}_{11}$ and $a^{'}_{12}$. Second, over the discontinuous points, we find the minimum value of $D_{a^{'}}EV^n(S, \cdot)$ and denote it as $v_{max}(S)$. In Figure 2, $v_{max} = D_{a^{'}}EV^n(S, a^{'}_{11})$. Next, we search for a set of $a^{'}_i$ that satisfies $D_{a^{'}}EV^n(S, a^{'}_i) \leq v_{max}(S)$, which is $\{a^{'}_{15}, a^{'}_{16}, \cdots, a^{'}_{23}\}$ in Figure 2. We choose the minimum among this set and define it as $a_{max}(S)$. In Figure 2, $a_{max}(S) = a^{'}_{15}$. v_{min} and a_{min} are computed analogously.

3.3 Computing the Endogenous Grid for the Cash on Hand

Given n, for each S and $a^{'}_i \in G_{a^n_{rbl}(S)}$, we compute the endogenously-determined cash on hand, $M(S, a^{'}_i)$. To retrieve this endogenously-determined cash on hand, $M(S, a^{'}_i)$, we need to obtain the endogenously-determined consumption by using the following FOC:

$$\text{For each } n,\ S \text{ and } a^{'}_i \in G_{a^n_{rbl}(S)}, \quad u^{'}(c(S, a^{'}_i)) = \frac{D_{a^{'}}EV^n(S, a^{'}_i)}{D_{a^{'}}q^n(S, a^{'}_i) \cdot a^{'}_i + q^n(S, a^{'}_i)} \tag{12}$$

where $c(S, a^{'}_i)$ is the endogenously-determined consumption. The FOC (12) is locally well-defined and easy to compute.[7,8] Recall that the derivative of the expected value function, $D_{a^{'}}EV^n(S, a^{'}_i)$, and the loan price schedules, $D_{a^{'}}q^n(S, a^{'}_i)$, are computed using equation (9). Given $D_{a^{'}}EV^n(S, a^{'}_i)$ and $D_{a^{'}}q^n(S, a^{'}_i)$, $c(S, a^{'}_i) = u^{'-1}\left(\frac{D_{a^{'}}EV^n(S, a^{'}_i)}{D_{a^{'}}q^n(S, a^{'}_i) \cdot a^{'}_i + q^n(S, a^{'}_i)}\right)$.

Given $c(S, a^{'}_i)$, we retrieve the endogenously-determined cash on hand $M(S, a^{'}_i)$ as follows:

$$M(S, a^{'}_i) = c(S, a^{'}_i) + q^n(S, a^{'}_i)a^{'}_i. \tag{13}$$

For each S and $a^{'}_i \in G_{a^n_{rbl}(S)}$, we save the pairs of $(M(S, a^{'}_i), a^{'})$.

[7] Clausen and Strub (2020) proved the local differentiability of the expected value function and the loan price schedules and showed the existence of the FOC (12). The proof in Clausen and Strub (2020) was for the case of iid shocks on earnings; yet as they mentioned, the inclusion of AR-1 shocks does not make a huge difference in the proof.

[8] For each $a^{'}_i \in G_{a^{'}}$ with $a^{'}_i > a_{rbl}(S)$, the derivative of the size of the loan, $D_{a^{'}}q^n(S, a^{'}_i)a^{'}_i + q^n(S, a^{'}_i)$, is always positive by the definition of the risky borrowing limit, $a^n_{rbl}(S)$. We assume that the utility function $u(\cdot)$ is differentiable with respect to c. The derivative of the expected value function and price function can be obtained numerically using equation (9).

3.4 Storing the No-Default Value Function on the Endogenous Grid for Cash on Hand

Given n, for each S and $a_i^{'} \in G_{a_{rbl}^n(S)}$, we compute the value function over the endogenous grid for cash on hand, $M(S, a_i^{'})$ as follows:

$$V^{c,n+1}(S, M(S, a_i^{'})) = u(M(S, a_i^{'}) - q^n(S, a_i^{'}) \cdot a_i^{'}) + EV^n(S, a_i^{'}) \tag{14}$$

It is worth noting two things in this step. First, the value function is computed without any max-operator, which contributes to efficiency. Second, the value functions are defined on the endogenous grid of $M(S, a_i^{'})$, not on its exogenous grid.

3.5 Identifying the Global Solution over the Endogenous Grid for Cash on Hand

Given a specific level of cash on hand, the corresponding $a^{'}$ may not be a global solution as illustrated in Figure 2. In this step, we identify a set of the global solutions and save the corresponding pairs of $(M(S, a_i^{'}), a_i^{'})$. Given n and S, $a_i^{'} \in G_{a_{rbl}^n(S)}$ is either on the concave region or on the non-concave region. When $a_i^{'}$ is on the concave region, as $a_3^{'}$ in Figure 2, the pair of $(M(S, a_i^{'}), a_i^{'})$ implies a global solution because the FOC (12) is a sufficient and necessary condition. We save all of the pairs $(M(S, a_i^{'}), a_i^{'})$ on the concave region.

When $a_i^{'}$ is on the non-concave region – e.g., $a_i^{'} = a_9^{'}$ in Figure 2, the pair of $(M(S, a_i^{'}), a_i^{'})$ does not guarantee a global maximum because the FOC (12), while necessary, is not sufficient. As in Fella (2014), for each S and $a_i^{'}$ on the non-concave region, we verify whether this $a_i^{'}$ is the global solution by solving the following problem:

$$a_g^{'} = \underset{\{a_k^{'} \in \{a_{min}(S), \cdots, a_{max}(S)\}\}}{\operatorname{argmax}} \Big[u(M(S, a_i^{'}) - q^n(S, a_k^{'}) \cdot a_k^{'}) + EV^n(S, a_k^{'})\Big] \tag{15}$$

where $\{a_{min}(S), \cdots, a_{max}(S)\}$ is the non-concave region. If $a_i^{'} = a_g^{'}$, this implies that the pair of $(M(S, a_i^{'}), a_i^{'})$ corresponds to a global solution, thus we save this pair. If $a_i^{'} \neq a_g^{'}$, we discard this pair. This step does not add the computational intensity much since it only searches over the non-concave region.

3.6 Computing the Endogenous Grid for the Current Assets Related to the Global Solutions

For the saved pairs of $(M(S, a_i^{'}), a_i^{'})$ for the global solutions, we store the corresponding pairs of $(a(S, a_i^{'}), a_i^{'})$. For each saved $a_i^{'}$, we compute the endogenous grid for the current assets, $a(S, a_i^{'})$, as follows:

$$a(S, a_i^{'}) = M(S, a_i^{'}) - S. \tag{16}$$

Note that when a set of $a_i^{'}$ corresponds to the global solutions, $a(S, a_i^{'})$ monotonically increases with $a_i^{'}$, thereby allowing for a one-to-one mapping from a to $a^{'}$. This mapping enables us to use splines to evaluate the policy function over the exogenous grid in the following step.

3.7 Evaluating the Policy Function and the No-Default Value Function on the Exogenous Grid for the Current Assets

Using the one-to-one mapping between $a(S, \cdot)$ and $a^{'}$, we compute the policy function over the exogenous grid of the current assets G_a as follows.

First, for each state S, we find the value of the endogenous grid for the current assets corresponding to $a^{'} = 0$, $a(S, a^{'} = 0)$. Then, for each S and for each $a_i \in G_a$, we compare the value of a_i to that of $a(S, a^{'} = 0)$. If $a_i \geq a(S, a^{'} = 0)$, we use a linear interpolation to evaluate the policy function of asset holdings, $a^{'} = g_a(S, a_i)$ and evaluate the corresponding value $V^{c,n+1}(S, a_i) = u(a_i + S - q^n(S, g_a(S, a_i)) \cdot g_a(S, a_i)) + EV^n(S, g_a(S, a_i))$. If $a_i < a(S, a^{'} = 0)$, we compute $V^{c,n+1}(S, a_i)$ and $g_a(S, a_i)$ by solving the following problem:

$$V^{c,n+1}(S, a_i) = \max_{\{a_{rbl}^n(S) < a_j^{'} < 0\}} u(a_i + S - q^n(S, a_j^{'}) \cdot a_j^{'}) + EV^n(S, a_j^{'}). \tag{17}$$

Note that this inclusion of the grid search does not cause a huge loss in efficiency because this problem searches the grid just between the risky borrowing limit, $a_{rbl}^n(S)$, and zero assets, $a^{'} = 0$.

We restrict the usage of interpolation to non-negative asset holdings, $a^{'} \geq 0$, due to computational issues. As Hatchondo et al. (2010) pointed out, the computational accuracy is sensitive to the numerical method of computing the derivative of the loan rate schedule, $D_{a^{'}} q^n(S, a^{'})$. We also find that the convergence of the value function, $V(S, \cdot)$, and loan rate schedules, $q(S, \cdot)$, is sensitive to how to compute their derivative on the borrowing region, $a^{'} < 0$. For these reasons, we employ the grid search method for the borrowing region, $a^{'} < 0$.

3.8 Computing the Value of Defaulting

We solve the value function with a bad credit history:

$$V^{d,n+1}(S) = u(y(S)) + \beta \sum_{S'} \pi_{S,S'} \left[\theta V^{c,n+1}(S',0) + (1-\theta)V^{d,n}(S'),\right]. \tag{18}$$

Since the value function of defaulting is not related to any continuous endogenous state, it is not costly to compute it.

3.9 Updating the Value Function and Loan Price Schedules

We update the value function, $V^{n+1}(S,a)$ and the price function, $q^{n+1}(S,a')$ in the following way:

$$V^{n+1}(S,a) = \max\left\{V^{c,n+1}(S,a), V^{d,n+1}(S)\right\} \tag{19}$$

$$q^{n+1}(S,a') = \frac{1-\delta(S,a')}{1+r}$$

where

$$\delta(S,a') = \sum_{\{S':V^{c,n+1}(S',a')<V^{d,n+1}(S')\}} \pi_{S,S'}.$$

We compute $EV^{n+1}(S,a)$. If $||EV^{n+1}(S,a) - EV^{n}(S,a)||_\infty > 10^{-5}$ where $||\cdot||_\infty$ is the sup norm over $\mathbb{S}X\mathbb{A}$, start a new iteration with $n = n+1$.

3.10 Summary of the Algorithm

To sum up, given an iteration number, n, and the expected value function $EV^n(S,a)$, the algorithm is as follows:

1. For each S, calculate the risky borrowing limit, $a^n_{rbl}(S)$, and save it.
2. Identify the (non-) concave region of asset holdings a' by using the algorithm of Fella (2014).
3. Given $(S, a^n_{rbl(S)})$, compute the endogenously-determined cash on hand, $M(S,a_i')$, by solving the FOC (12). Save these pairs of $(M(S,a_i'), a_i')$.
4. Compute the value function for non-defaulting over the endogenous grid for cash on hand, $(V^{c,n+1}(S, M(S,a_i'))$.
5. Identify the global solution over the endogenous grid for cash on hand.
    - If a_i' is on the concave region, save the pair of $(M(S,a_i'), a_i')$

- If $a_i^{'}$ is on the non-concave region, verify whether the candidate $(M(S, a_i^{'}), a_i^{'})$ implies the maximum by solving the value function. If this is the maximum, save the pair of $(M(S, a_i^{'}), a_i^{'})$. Otherwise, discard it.

6. For the saved pairs of $(M(S, a_i^{'}), a_i^{'})$, compute the corresponding endogenous grid for the current assets, $a(S, a_i^{'})$. Save the pairs of $(a(S, a_i^{'}), a_i^{'})$.
7. Using the monotonicity between $a(S, a_i^{'})$ and $a_i^{'}$, compute the policy function of asset holdings and the value function for non-defaulting over the exogenous grid for the current assets.
   - Compute the value of the endogenous grid for the current assets $a(S, a^{'} = 0)$ corresponding to $a^{'} = 0$
     - If $a_i \geq a(S, a^{'} = 0)$, use a linear interpolation to compute the policy function of asset holdings in the next period, $g_a(S, a_i)$. With $g_a(S, a_i)$, compute $V^{c,n+1}(S, a_i)$.
     - If $a_i < a(S, a^{'} = 0)$, solve the value function $V^{c,n+1}(S, a_i)$ by searching for the grid between the risky borrowing limit, $a_{rbl}^n(S)$ and zero assets, $a^{'} = 0$.
8. Compute the value function for defaulting, $V^{d,n+1}(S)$.
9. Update the value function, $V^{n+1}(S, a) = \max\{V^{c,n+1}(S, a), V^{d,n+1}(S)\}$, and loan price schedules, $q^{n+1}(S, a^{'})$.
10. Compute $EV^{n+1}(S, a)$.
11. Start a new iteration with $n = n + 1$ if $||EV^{n+1}(S, a) - EV^n(S, a)||_\infty > 10^{-5}$. Otherwise, stop.

4 Results

4.1 Parameterization

Table 1: Parameters

r	1.7%	Risk-free interest rate
σ	2.0	Risk aversion
ρ_S	0.945	Endowment process
σ_S	0.025	Endowment process
β	0.953	Discount factor
θ	0.282	Probability of reentry
λ	$0.969E(S)$	Output cost

We follow Arellano's (2008) choice of parameter values. The utility function is

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

The Markov chain approximation to the process follows Tauchen (1986). The number of grid points for S is 21. We set lower and upper bound of a to -2.5 and 3.5. The grid points for a are log-spaced around zero, as default decisions are made and measures are located near zero. The model is solved a grid of 200, 500, 1000 and 2000 for the continuous state variable a. Table 1 shows the values of the chosen parameters.

Figure 3 shows the results of our computations. Our solutions resemble the solution of Arellano (2008), and the results from our EGM and those from grid search method are very close.

4.2 Computing Time and Accuracy

As mentioned above, we vary the size of the grid for a across computational exercises. In all computational exercises, we keep the number of the grid points for the other variables. We compute Bellman equation errors instead of computing Euler equation errors to measure the accuracy. To compute the Euler error, we must calculate the derivative of the loan price schedule, the value of which depends on types of numerical derivatives. As Hatchondo et al. (2010) point out, Euler equation errors are sensitive to how to calculate the derivative of the loan rate schedule, $D_{a'}q(S, a^{'})$. To avoid this issue, we compute Bellman equation errors. Recall the following notation: S is the state vector other than assets a. Then, the Bellman equation

$$V(S, a) = u(c(S, a)) + E_{S'}\left[V(S^{'}, a^{'}(S, a))\right] \tag{20}$$

should hold exactly for the true decision rules. Because the decision rules are numerically computed in our computational exercises, the Bellman equation (20) does not hold exactly with the numerically calculated decision rules. We define c^* as the solution for

$$u(c^*(S, a)) = V(S, a) - E_{S'}\left[V(S^{'}, \bar{a}^{'}(S, a))\right] \tag{21}$$

where bars indicate the numerically calculated decision rules. We define the Bellman equation error as

$$BE(s, a) = \left|1 - \frac{c^*(S, a)}{\bar{c}(S, a)}\right|. \tag{22}$$

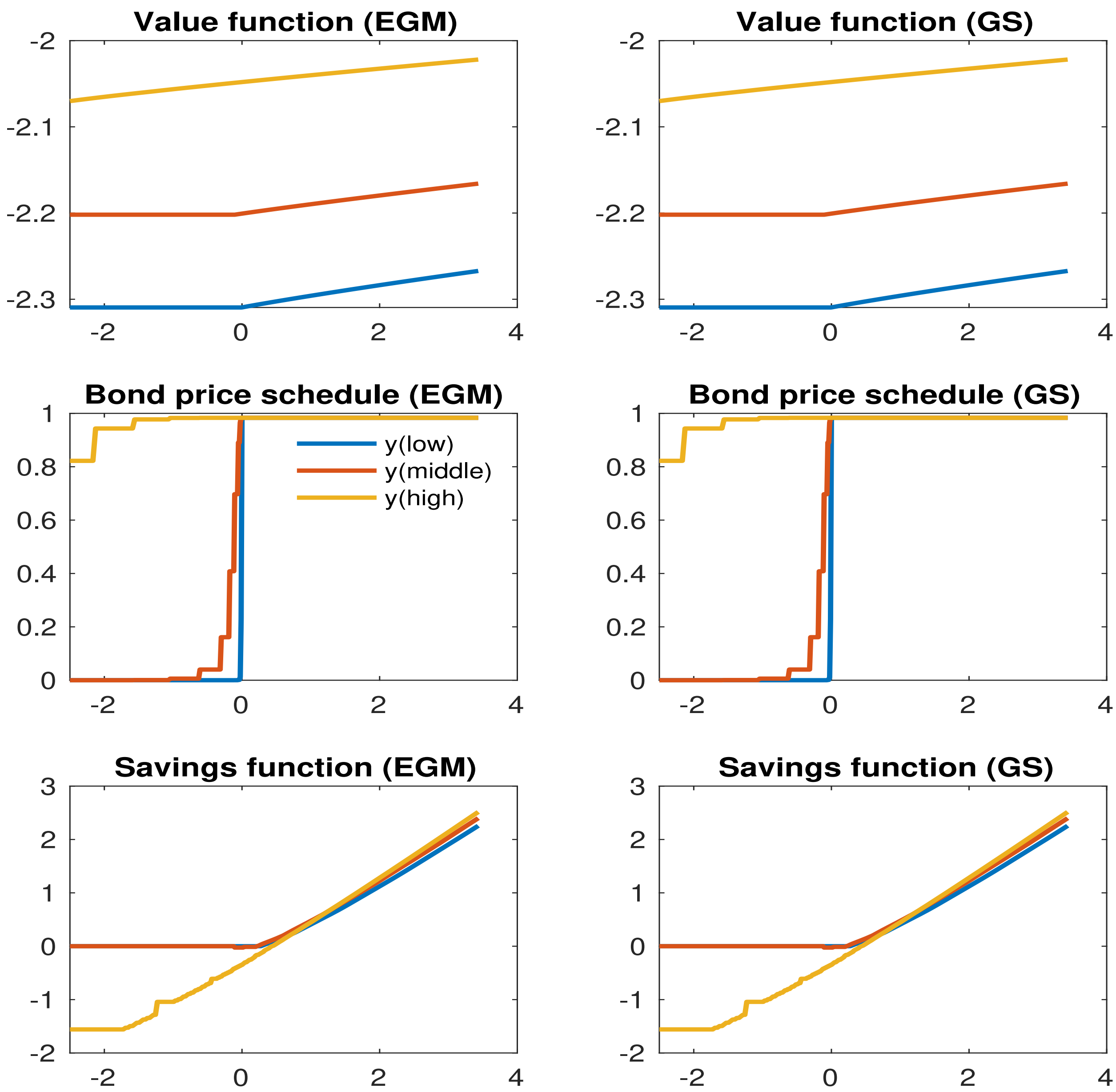

Figure 3: Value Function, Bond Price Schedule, and Savings Function

Following the literature, we report both the maximum and the average of Bellman equation errors. To do so, we compute the stationary distributions. While we vary the number of grid points for a when solve the value functions, we fix the number of grid points for a to 2,000 when computing the stationary distributions. Average errors are weighted sum of errors with weights being the mass on the stationary distribution. We only consider errors on points with positive mass. The programs were written in Fortran 95 and all computations were carried out on on a single core of an Intel i7-4770 processor.

Table 2: Computing Time (second)

# of Grid Points	200		500		1000		2000	
Computational Method	EGM	GS	EGM	GS	EGM	GS	EGM	GS
AVG CPU Time	0.438	3.375	2.219	20.375	8.234	80.125	35.016	320.984
AVG CPU Time per iteration	0.002	0.018	0.012	0.107	0.043	0.420	0.183	1.681

Table 4 shows that the EGM is approximately eight to nine times faster than the grid search method across all grid settings. Since the model we solve is simple, one might think that the efficiency gain seems small. Further, considering the additional time to implement the EGM algorithm, it might not be worth using the EGM to a simple model. However, if a model entails more complicated features and hence takes longer to solve, the EGM can reduce computing time with significance. For example, Nakajima and Ríos-Rull (2014) use a more complex income process and aggregate uncertainty. They use a method in Krusell and Smith (1998) to approximate the aggregate states of the model, thus solving it requires a long simulation (outer loop) after computing value functions and decision rules (inner loop). Also, finding equilibrium requires several iterations of inner loops and outer loops. We use our method to solve Nakajima and Ríos-Rull (2014) and find the EGM is from 18.5 to 27.3 times faster than the grid search method in the inner loops. In the outer loops, the EGM is approximately 7.5 times faster than the grid search method. The details about our implementation and numerical results can be found in Appendix.

Table 3: Computational Accuracy

# of Grid Points	200		500		1000		2000	
Computational Method	EGM	GS	EGM	GS	EGM	GS	EGM	GS
AVG Bellman EQ Error (%)	0.025	0.028	0.020	0.027	0.017	0.029	0.015	0.029
MAX Bellman EQ Error (%)	12.052	12.052	12.048	12.056	9.617	12.056	9.614	12.057

As mentioned above, Bellman equation error is our accuracy measure and Figure 5 shows the errors and stationary distributions. In both figures, there are low mass where errors are large. The differences are not noticeable in the figure, but Table 5 shows that the EGM produces more accurate outcomes than the grid search method. Both average and maximum Bellman equation errors in the EGM is same or lower than those in the grid search methods across all grid settings.

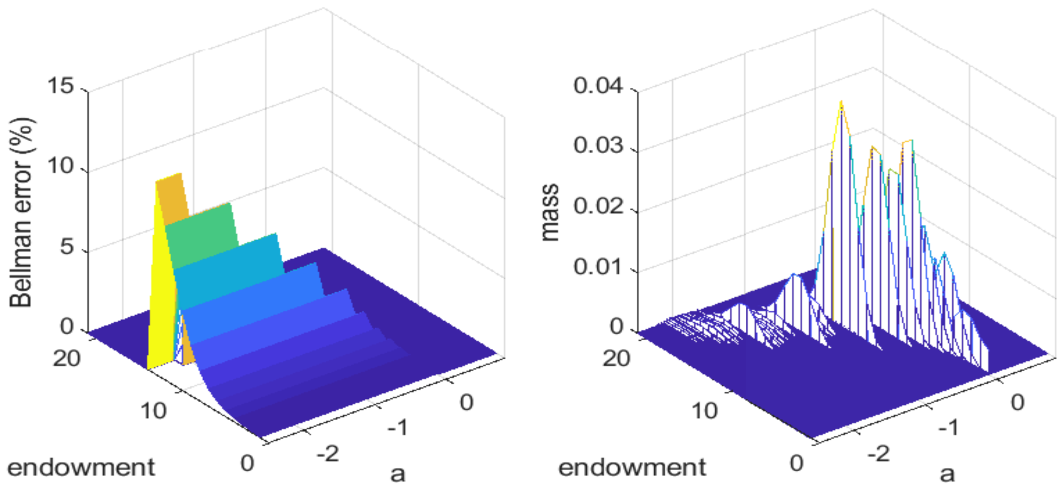

Figure 4: Bellman error and stationary distribution for the EGM

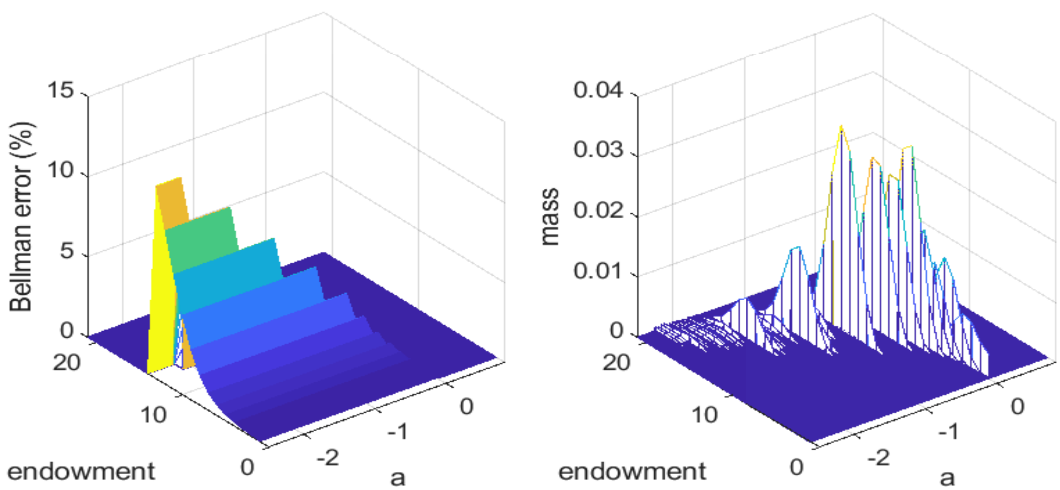

Figure 5: Bellman error and stationary distribution for the Grid Search Method

5 Generalization of the EGM

In Section 2 and 3, we chose Arellano's (2008) model to demonstrate our EGM because it is applicable to this model. However, the EGM does not work in all models. For example, the EGM may not be useful when asymmetric information exists between lenders and borrowers, nor when the risky borrowing limit does not exist. This section formalizes the method in a theoretical framework and provides its sufficient conditions in order to understand the class of models to which our EGM is applicable. We modify theorems in White (2015), in accordance with the circumstances of default risk models.

5.1 Formalization of the EGM

Consider a set of agents facing a dynamic maximization problem with a number of decision variables each of which can be either continuous or discrete.[9] Time t begins at period 0 and ends after period $T \in \mathbb{N}$.[10] We denote $s_t \in S_t \subset \mathbb{R}^p$ as the exogenous state where $p \in \mathbb{N}$. Let $a_t \in A_t \subset \mathbb{R}$ be the continuous state. Let us denote $d_t \in D_t = \{1, \cdots, N_d\}$ as the endogenous default state. State variables (s_t, a_t, d_t) correspond to (S, a, d) in Section 3. We define $b_t \in B_t = \{1, \cdots, N_b\}$ as the endogenous discrete state other than the default state, which does not exist in Arellano's (2008) model. Note that changes in the current endogenous state variables (a_t, d_t, b_t) affect $(a_{t+1}, d_{t+1}, b_{t+1})$, but not s_{t+1}.

Agents make a choice of y_t from the closed and convex feasible set $\Gamma_t(s_t, a_t, b_t, d_t)$, where the constraint correspondence, $\Gamma_t(\cdot)$, represents the budget set for the current state: $\Gamma_t : S_t \times A_t \times B_t \times D_t \longrightarrow Y_t \subset \mathbb{R}$. We denote the feasible set $Y_t = \cup_{\{s_t, a_t, b_t, d_t\} \in S_t \times A_t \times D_t \times B_t} \Gamma_t(s_t, a_t, b_t, d_t) \subset \mathbb{R}$. In Section 3, the choice y_t corresponds to consumption c, and the constraint correspondence $\Gamma_t(\cdot)$ corresponds to the budget constraints, $\Gamma(S, d = 0, a) = \{c : 0 < c \leq S + a - q(S, -Z)(-Z)\}$ and $\Gamma(S, d = 1, a) = \{c : 0 < c \leq h(s)\}$. Agents obtain a flow of utility from their choices and states through the utility function $U_t : S_t \times B_t \times D_t \times Y_t \longrightarrow \mathbb{R}$. U_t is continuous, strictly monotonic, strictly concave, and twice differentiable to y_t on its interior domain. In addition, the discount factor β_t is between 0 and 1.

Once agents determine y_t, their state changes from (s_t, a_t, b_t, d_t) to the interim period state, $(s_{t+1}, a_{t+1}, b_{t+1}, d_t) \in S_{t+1} \times A_{t+1} \times B_{t+1} \times D_t$ according to the transition function $\Delta_t(\cdot)$. Note that this interim timing indicates the time before making decision on default in period $t+1$ but after deciding y_t, and thereby s_t, a_t, and b_t change to s_{t+1}, a_{t+1}, and b_{t+1}. The transition $\Delta_t(\cdot)$ depends on random shock $\epsilon_{t+1} \in E_{t+1} \subset \mathbb{R}^l$, drawn from the CDF of $F_{t+1}(\epsilon_{t+1})$. ϵ_{t+1} corresponds to ϵ' in Arellano's (2008) model. Agents do not know the exact value of ϵ_{t+1} when making the decision on y_t, but know the distribution of $F_{t+1}(\epsilon_{t+1})$. The transition function Δ_t is formally defined as

$$\begin{aligned} \Delta_t \colon S_t \times A_t \times B_t \times D_t \times Y_t \times E_{t+1} &\longrightarrow S_{t+1} \times A_{t+1} \times B_{t+1} \times D_t \\ (s_t, a_t, b_t, d_t, y_t, \epsilon_{t+1}) &\longmapsto (s_{t+1}, a_{t+1}, b_{t+1}, d_t). \end{aligned} \tag{23}$$

Since d_t does not change in the interim period, we can express the transition function Δ_t conditional on d_t. Further, we regard the future endogenous state b_{t+1} as given state $\bar{b}_{t+1}$ to use the algorithm of Fella (2014) afterward. We define the transition function conditional on

[9] For the sake of easy exposition, we describe the problem in terms analagous to Section 3. However, while we refer to the discrete decision as 'defaulting', our algorithm can be applied in the context of other problems with discrete choices such as housing.

[10] As in White (2015), it can be applied to infinite horizon models, with the time subscripts skipped.

$(d_t \times \bar{b}_{t+1}) \in D_t \times B_{t+1}$, $\Delta_{d_t,\bar{b}_{t+1}}$, as follows:

$$\begin{aligned} \Delta_{d_t,\bar{b}_{t+1}} : S_t \times A_t \times B_t \times Y_t \times E_{t+1} &\longrightarrow S_{t+1} \times A_{t+1} \\ (s_t, a_t, b_t, y_t, \epsilon_{t+1}) &\longmapsto (s_{t+1}, a_{t+1}). \end{aligned} \tag{24}$$

Recall that y_t affects the transition from a_t to a_{t+1}, but not that from s_t to s_{t+1}. Thus, we can decompose the conditional transition function $\Delta_{d_t,\bar{b}_{t+1}}$ into the portion $\Delta^S_{d_t,\bar{b}_{t+1}}$ independent of y_t and the portion $\Delta^A_{d_t,\bar{b}_{t+1}}$ dependent on y_t. This decomposition implies that $\partial\Delta^S_{d_t,\bar{b}_{t+1}}/\partial y_t = 0|_{p\times 1}$, $\partial\Delta^S_{d_t,\bar{b}_{t+1}}/\partial a_t = 0|_{p\times 1}$, $\partial\Delta^S_{d_t,\bar{b}_{t+1}}/\partial b_t = 0|_{p\times N_b}$, and $\partial\Delta^A_{d_t,\bar{b}_{t+1}}/\partial y_t \neq 0$. Since $\Delta^S_{d_t,\bar{b}_{t+1}}$ is irrelevant to (a_t, b_t, y_t), we represent $\Delta^S_{d_t,\bar{b}_{t+1}}$ as a function of (s_t, ϵ_{t+1}). For example, in Section 2, $\Delta^S_{d_t,\bar{b}_{t+1}}$ corresponds to the first-order Markov chain $\pi_{S,S'}$ which was independent of consumption c. $\Delta^A_{d_t,\bar{b}_{t+1}}$ corresponds to the transition of the current assets a to the next period assets a' that depends not only on $\pi_{S,S'}$ but also on consumption c.

We recursively represent the agent's problem. At the beginning of each period, agents solve

$$V_t(s_t, a_t, b_t) = \max_{d_t \in \{0,1,\cdots N_d\}} \{v_t(s_t, a_t, b_t, d_t))\} \tag{25}$$

where $V_t(s_t, a_t, b_t)$ is the value after default decision and $v_t(s_t, a_t, b_t, d_t)$ is the value before default decision. Before agents decide whether to default or not, for each $(d_t \times \bar{b}_{t+1}) \in D_t \times B_{t+1}$, they solve

$$\begin{aligned} v_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1}) &= \max_{y_t \in \Gamma_t(s_t,a_t,b_t,d_t;\bar{b}_{t+1})} U_t(s_t, b_t, d_t, y_t) + \beta_t E\left[\{V_{t+1}(s_{t+1}, a_{t+1}, \bar{b}_{t+1})\}\right] \\ &= \max_{y_t \in \Gamma_t(s_t,a_t,b_t,d_t;\bar{b}_{t+1})} U_t(s_t, b_t, d_t, y_t) \\ &+ \beta_t \int V_{t+1}(\Delta_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t, \epsilon_{t+1}))dF_{t+1}(\epsilon_{t+1}). \\ &= \max_{y_t \in \Gamma_t(s_t,a_t,b_t,d_t;\bar{b}_{t+1})} U_t(s_t, b_t, d_t, y_t) \\ &+ \beta_t \int V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t, \epsilon_{t+1}), \Delta^A_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t, \epsilon_{t+1}))dF_{t+1}(\epsilon_{t+1}). \end{aligned} \tag{26}$$

The terminal value is defined as:

$$v_t(s_T, a_T, b_T, d_T) = \max_{y_T \in \Gamma_T(s_T,a_T,b_T,d_T)} U_t(s_T, b_T, d_T, y_T). \tag{27}$$

We define the policy function $\Psi_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1})$ as follows:

$$\begin{aligned} &\Psi_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1}) \\ &= \underset{y_t \in \Gamma_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1})}{\operatorname{argmax}} \quad U_t(s_t, b_t, d_t, y_t) \\ &+\beta_t \int V_{t+1}(\Delta^S_{d_t, \bar{b}_{t+1}}(s_t, \epsilon_{t+1}), \Delta^A_{d_t, \bar{b}_{t+1}}(s_t, a_t, b_t, y_t, \epsilon_{t+1}))dF_{t+1}(\epsilon_{t+1}). \end{aligned} \tag{28}$$

If a model is included in a subset of the general class of problems described above, then the EGM is applicable when the model satisfies the conditions described in the next subsection.

5.2 Conditions for the EGM

The EGM works only when the FOC exists and is well-defined as a necessary condition for the global solutions. This holds when condition C1 is satisfied.

- **C1** : $E[V_{t+1}(s_{t+1}, a_{t+1}, \bar{b}_{t+1})]$ is locally differentiable with respect to the optimal choice of a_{t+1}. In addition, $a_{t+1} = \Delta^A_{d_t, \bar{b}_{t+1}}(s_t, a_t, b_t, y_t, \epsilon_{t+1})$ is locally differentiable with respect to the optimal choice of y_t.

Researchers can check whether this condition holds in a specific problem by using the "Reverse Calculus" method from Clausen and Strub (2020). They provide useful instruments for checking the local differentiability applicable to models with discrete choices and default options.

Assuming C1 is satisfied, the following FOC will then be a necessary but not sufficient condition for optimal consumption.

$$\begin{aligned} &\frac{\partial U_t(s_t, b_t, d_t, y_t)}{\partial y_t} = \\ &-\beta_t \int \left[\frac{\partial V_{t+1}(\Delta^S_{d_t, \bar{b}_{t+1}}(s_t, \epsilon_{t+1}), \Delta^A_{d_t, \bar{b}_{t+1}}(s_t, a_t, b_t, y_t, \epsilon_{t+1}))}{\partial a_{t+1}} \cdot \frac{\partial \Delta^A_{d_t, \bar{b}_{t+1}}(s_t, a_t, b_t, y_t, \epsilon_{t+1}))}{\partial y_t} \right] dF_{t+1}(\epsilon_{t+1}). \end{aligned} \tag{29}$$

Since U_t is differentiable, $\frac{\partial U_t(s_t, b_t, d_t, y_t)}{\partial y_t}$ is well-defined. Note that the FOC (29) is not sufficient but necessary for the global solution of a_{t+1} because EV_{t+1} might not be strictly concave due to the default options and other discrete choices.

Let us define $Z_t \subset \mathbb{R}$ as the set of post-decision endogenous state. $z_t \in Z_t$ is the intermediate state after agents have chosen and executed their choices y_t but before the transition shock ϵ_{t+1} arises. For example, in Section 3, the post-decision endogenous state z_t is the size of debt $q(S, a') \cdot a'$ because it is determined after consumption c is chosen, but before the transition shock ϵ' is realized. This post-decision endogenous state enables us to decompose the endogenous state transition function $\Delta^A_{d_t,\bar{b}_{t+1}}(\cdot)$ into intra- and inter-period components. This decomposition is important in employing the EGM because it requires to use the FOC (29) represented by the post-decision endogenous state z_t. Formally, we decompose the the endogenous state transition function $\Delta^A_{d_t,\bar{b}_{t+1}}(\cdot)$ as follows:

- **C2** : For each d_t and for each $\bar{b}_{t+1}$, there exist functions $\Xi_{d_t,\bar{b}_{t+1}} : S_t \times A_t \times B_t \times Y_t \longrightarrow \mathbb{R}$ and $\chi_{d_t,\bar{b}_{t+1}} : S_t \times Z_t \times B_t \times E_{t+1} \longrightarrow A_{t+1}$ such that $\Delta^A_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t, \epsilon_{t+1}) = \chi_{d_t,\bar{b}_{t+1}}(s_t, \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t), b_t, \epsilon_{t+1})$ for all $(s_t, a_t, b_t, y_t) \in W_t \times A_t \times B_t \times Y_t$ when $y_t \in \Gamma_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1})$.

This condition implies that for each d_t and $\bar{b}_{t+1}$, through the intra-period transition function $\Xi_{d_t,\bar{b}_{t+1}}$, the states (s_t, a_t, b_t) and choice y_t generate a post-decision state z_t. In Section 3, recall that $z_t = q(S, a^{'})a^{'}$ and $a_{t+1} = a^{'}$. Our EGM requires z_t to act as a sufficient statistic for the endogenous state a_{t+1}. This implies that there must exist a unique one-to-one mapping between $z_t = q(S, a^{'})a^{'}$ and $a_{t+1} = a^{'}$.

To ensure that this mapping exists, we first need to ensure that $q(S, a^{'})a^{'}$ should increase with $a^{'}$ (so that the mapping is one-to-one). Second, $z_t = q(S, a^{'}) \cdot a^{'}$ needs to be independent of the value of $\epsilon^{'}$, so that the decomposition of $a^{'}$ out of $z_t = q(S, a^{'}) \cdot a^{'}$ is unique.[11] We generalize these conditions as follows:

- **C3** : For each s_t, d_t and $\bar{b}_{t+1}$, a_{t+1} is independent of ϵ_{t+1}, but dependent on the conditional expectation of the default decision $E_t(D(\epsilon_{t+1}))$ that is formed in the current period t. i.e, $a_{t+1} = \chi_{d_t,\bar{b}_{t+1}}(s_t, z_t, b_t, E_t(D(\epsilon_{t+1})))$.

- **C4** : For each s_t, d_t and $\bar{b}_{t+1}$, $a_{t+1} = \chi_{d_t,\bar{b}_{t+1}}(s_t, z_t, b_t, E_t(D(\epsilon_{t+1})))$ monotonically increases with z_t; therefore, there exists a function $\chi^{-1}_{d_t,\bar{b}_{t+1}}(\cdot)$ where $z_t = \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t(D(\epsilon_{t+1})))$ and $\chi^{-1}_{d_t,\bar{b}_{t+1}}(\cdot)$ is increasing in in a_{t+1}.

C3 implies that $\chi_{d_t,\bar{b}_{t+1}}(s_t, z_t, b_t, E_t(D(\epsilon_{t+1})))$ is affected not by a future value of ϵ_{t+1} but by $E_t(D(\epsilon_{t+}1))$ that is formed in the current period t. **C4** is a generalized version of the risky borrowing limit. In Section 3, on the region of assets greater than the risky borrowing limit, $z_t = q(S, a^{'})a^{'}$ is monotonic increasing in $a_{t+1} = a^{'}$. Note that $z_t = \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t(D(\epsilon_{t+1})))$ is independent of the current continuous state a_t. This feature allows us to predetermine the lower bound of the feasible set for the solution of a_{t+1} at the initial step, thereby insulating the interactions between

[11]In Section 3, $q(S, a^{'})$ is not a function of shocks $\epsilon^{'}$ (or $S^{'}$) but a function of the conditional mean of defaulting, $\delta(S, a^{'}) = \sum_{\{S^{'}:V^c(S^{'},a^{'})<V^d(S^{'})\}} \pi_{S,S^{'}}$. Note that the default set $\{S^{'} : V^c(S^{'}, a^{'}) < V^d(S^{'})\}$ can be different across default risk models because it depends on their default rules. What matters is that $q(S, a^{'})$ is affected not by a specific value of $S^{'}$ but by a function of its conditional expectation $\sum_{\{S^{'}:V^c(S^{'},a^{'})<V^d(S^{'})\}} \pi_{S,S^{'}}$ that is formed in the current period.

the step of searching for the lower bound and that of computing the endogenously-determined current state a_t in our EGM.

$\mathbf{C2}, \mathbf{C3}$ and $\mathbf{C4}$ do not guarantee that the solution to the problem will be contained in the feasibility set implied by Γ_t. To make the post-decision consistent with the feasible set such as budget constraint, we need the following condition:

- $\mathbf{C5}$: Let d_t and $\bar{b}_{t+1}$ be given. For all $y_t \in Y_t$ and $(s_t, a_t, b_t) \in S_t \times A_t \times B_t$, $y_t \in \Gamma_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1})$ if and only if $\Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t) \in Z_t$ where $Z_t = \cup_{(s_t,a_t,b_t)\in S_t\times A_t\times B_t} \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, \Gamma_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1}))$.

$\mathbf{C5}$ means that the post-decision endogenous state z_t is a sufficient statistic for evaluating the feasibility of choice y_t. Additionally, this condition implies that Z_t spans the entire space of possible outcome. In Section 3, this condition implies that given S and a, knowing $z = q(S, a^{'})a^{'}$ is equivalent to knowing c because they satisfy the budget constraint, $c + z = c + q(S, a^{'})a^{'} = S + a$.

Unlike the transitional functions in White (2015), the transitional functions might not be differentiable because of the inclusion of discrete decision variables. We need a condition to ensure the local differentiability of these functions.

- $\mathbf{C6}$: For each d_t and $\bar{b}_{t+1}$, (1) $z_t = \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t)$ is locally differentiable with respect to the optimal choices of y_t, and (2) $a_{t+1} = \chi_{d_t,\bar{b}_{t+1}}(s_t, z_t, b_t, E_t[D(\epsilon_{t+1})])$ is locally differentiable with respect to the optimal post-decision state of z_t.

In Arellano's (2008) model, (1) of $\mathbf{C6}$ implies that $z_t = q(S, a^{'}) \cdot a^{'}$ is differentiable with respect to c. This condition is satisfied for any S and $a^{'}$ as $\frac{\partial q(S,a^{'})\cdot a^{'}}{\partial c} = -1$. (2) of $\mathbf{C6}$ implies that $a^{'}$ is differentiable with respect to $z = q(S, a^{'})a^{'}$. The differentiability of optimal next-period asset holdings $a^{'}$ with regard to z implies that $q(S, a^{'})$ is differentiable with regard to $a^{'}$ because $\frac{\partial a^{'}}{\partial z} = [\frac{\partial z}{\partial a^{'}}]^{-1} = 1/(D_{a^{'}} q(S, a^{'})a^{'} + q(S, a^{'}))$. As with $\mathbf{C1}$, a researcher can check whether their setting has this property using Clausen and Strub's (2020) reverse calculus technique.

With $\mathbf{C1}, \mathbf{C2}, \mathbf{C3}, \mathbf{C4}, \mathbf{C5}$ and $\mathbf{C6}$, for each $(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}$, the FOC (29) can be rewritten as follows:

$$\frac{\partial U_t(s_t, b_t, d_t, y_t)}{\partial y_t}$$
$$= -\beta_t \int \left[\frac{\partial V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t, \epsilon_{t+1}), \chi_{d_t,\bar{b}_{t+1}}(s_t, \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t), b_t, E_t[D(\epsilon_{t+1})]))}{\partial a_{t+1}} \right. \tag{30}$$
$$\left. \cdot \frac{\partial \chi_{d_t,\bar{b}_{t+1}}(s_t, \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t), b_t, E_t[D(\epsilon_{t+1})]))}{\partial z_t} \cdot \frac{\partial \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t)}{\partial y_t} \right] dF_{t+1}(\epsilon_{t+1})$$

$$= -\beta_t \int \left[\frac{\partial V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t, \epsilon_{t+1}), a_{t+1})}{\partial a_{t+1}} \right] dF_{t+1}(\epsilon_{t+1}) \cdot \frac{\partial \chi_{d_t,\bar{b}_{t+1}}(s_t, z_t, b_t, E_t[D(\epsilon_{t+1})]))}{\partial z_t}$$
$$\cdot \frac{\partial \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t)}{\partial y_t}.$$

where $\chi_{d_t,\bar{b}_{t+1}}(s_t, \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t), b_t, E_t[D(\epsilon_{t+1})]) = a_{t+1}$ and $\Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t) = z_t$. Note that $\frac{\partial a_{t+1}}{\partial z_t} = \frac{\partial \chi_{d_t,\bar{b}_{t+1}}(s_t,z_t,b_t,E_t[D(\epsilon_{t+1})])}{\partial z_t}$ and $\frac{\partial z_t}{\partial y_t} = \frac{\partial \Xi_{d_t,\bar{b}_{t+1}}(s_t,a_t,b_t,y_t)}{\partial y_t}$ can be pulled out of the integral because they are independent of ϵ_{t+1}. Recall that **C4** and **C6** imply $\frac{\partial a_{t+1}}{\partial z_t} > 0$. Furthermore, we assume the following condition:

- **C7** : For each $(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}$, $\frac{\partial z_t}{\partial y_t} = \frac{\partial \Xi_{d_t,\bar{b}_{t+1}}(s_t,a_t,b_t,y_t)}{\partial y_t} = K \in \mathbb{R} \setminus \{0\}$, and $sgn(\frac{\partial \Xi_{d_t,\bar{b}_{t+1}}(s_t,a_t,b_t,y_t)}{\partial y_t}) = -sgn(\frac{\partial U_t(s_t,a_t,b_t,d_t,y_t)}{\partial y_t})$.

C7 implies that given a budget set, a change in y_t must be accompanied by a change in z_t of the opposite sign. Furthermore, this change is a constant proportion such that the tradeoff between y_t and z_t is linear. This condition is a general feature of dynamic problems in economics. For example, in the model of Arellano (2008), **C7** implies that $\frac{\partial q(S,a^{'})\cdot a^{'}}{\partial c} = -1 < 0$; and thus, $sgn(\frac{\partial q(S,a^{'})\cdot a^{'}}{\partial c}) = -sgn(u'(c)) < 0$. In the neoclassical growth model, this condition means $\frac{\partial k_{t+1}}{\partial c_t} = -1$ in the budget constraint $c_t + k_{t+1} = (1 + r_t)k_t + w_t l_t$.

With **C1** − **C7**, we can rearrange the previous FOC as follows:

$$\frac{\partial U_t(s_t, b_t, d_t, y_t)}{\partial y_t} \cdot \frac{1}{K} \tag{31}$$
$$= -\left[\frac{\partial \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t[D(\epsilon_{t+1})]))}{\partial a_{t+1}} \right]^{-1} \cdot \beta_t \int \left[\frac{\partial V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t, \epsilon_{t+1}), a_{t+1})}{\partial a_{t+1}} \right] dF_{t+1}(\epsilon_{t+1}).$$

where $\chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t[D(\epsilon_{t+1})])) = z_t$. Note that $\left[\frac{\partial \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t,a_{t+1},b_t,E_t[D(\epsilon_{t+1})]))}{\partial a_{t+1}} \right]^{-1}$ is well-defined because of **C4** and (2) of **C6**. Now, a_t and z_t disappear in the state vector.

We need a monotonicity of a_{t+1} with respect to a_t to use a spline to approximate the decision rule of a_{t+1} over the exogenous grid. Condition **C8** guarantees this monotonicity.

- **C8** : For each $(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}$, at the policy function $\Psi_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1})$, $z_t = \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, \Psi_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1}))$ is non-decreasing in a_t.

In Arellano's (2008) model, **C8** implies that $z_t = q(s, a^{'})a^{'} = q(s, g_a(S, a)) \cdot g_a(S, a)$ is non-decreasing in a. Since **C4** implies that $q(s, a^{'})a^{'}$ is non-decreasing in $a^{'}$, these two conditions indicate that $a^{'} = g_a(S, a)$ is weakly monotonic increasing in a. This monotonicity must be required to use splines.

Let us denote $\hat{g}_{d_t,\bar{b}_{t+1}}(s_t, b_t, y_t)$ and $\hat{V}_{d_t,\bar{b}_{t+1}}(s_t, b_t, a_{t+1})$ as follows:

$$\hat{g}_{d_t,\bar{b}_{t+1}}(s_t, b_t, y_t) = \frac{\partial U_t(s_t, b_t, d_t, y_t)}{\partial y_t} \cdot \frac{1}{K} \tag{32}$$

$$\begin{aligned} &\hat{V}_{d_t,\bar{b}_{t+1}}(s_t, b_t, a_{t+1}) \\ &= -\left[\frac{\partial \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t[D(\epsilon_{t+1})]))}{\partial a_{t+1}}\right]^{-1} \cdot \beta_t \int \left[\frac{\partial V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t, \epsilon_{t+1}), a_{t+1})}{\partial a_{t+1}}\right] dF_{t+1}(\epsilon_{t+1}). \end{aligned} \tag{33}$$

In Arellano's (2008) model, $K = \left[\frac{\partial \Xi_{d_t,\bar{b}_{t+1}}(s_t,a_t,b_t,y_t)}{\partial y_t}\right]^{-1} = -1$ and $[\frac{\partial z_t}{\partial a_{t+1}}]^{-1} = \frac{1}{D_{a^{'}} q(S,a^{'})a^{'} + q(S,a^{'})}$. Therefore, $\hat{g}_{d_t,\bar{b}_{t+1}}(s_t, b_t, y_t) = \hat{V}_{d_t,\bar{b}_{t+1}}(s_t, b_t, a_{t+1})$ is consistent with the FOC (12) in Arellano's (2008) model.

Note that for each $(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}$, the EGM retrieves the endogenously-driven choice variable y_t from the given decision states (s_t, b_t, a_{t+1}). Next, the EGM finds the endogenously-driven current state a_t by using the retrieved $y_t(s_t, b_t, a_{t+1})$ with information on the budget set $\Gamma_t(\cdot)$. As a result, for each $(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}$ and given (s_t, b_t, a_{t+1}), the EGM solves the following system:

$$\hat{g}_{d_t,\bar{b}_{t+1}}(s_t, b_t, y_t(s_t, b_t, a_{t+1})) = \hat{V}_{d_t,\bar{b}_{t+1}}(s_t, b_t, a_{t+1}) \tag{34}$$

where

$$\hat{g}_{d_t,\bar{b}_{t+1}}(s_t, b_t, y_t(s_t, b_t, a_{t+1})) = \left.\frac{\partial U_t(s_t, b_t, d_t, y_t)}{\partial y_t}\right|_{y_t=y_t(s_t,b_t,a_{t+1})} \cdot \frac{1}{K}$$

$$\hat{V}_{d_t,\bar{b}_{t+1}}(s_t, b_t, a_{t+1})$$
$$= -\left[\frac{\partial \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t[D(\epsilon_{t+1})]))}{\partial a_{t+1}}\right]^{-1} \cdot \int \left[\frac{\partial V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t, \epsilon_{t+1}), a_{t+1})}{\partial a_{t+1}}\right] dF_{t+1}(\epsilon_{t+1}).$$

It is worth noting that there is no max operator, thereby leading to sufficient improvements in computational efficiency.

Let us explain notations for the procedure of the implementation. G_{a_t} is the exogenous grid of a_t and $G_{a_{t+1}}$ is the exogenous grid of a_{t+1}.

5.3 Implementation

The EGM is applicable if a problem satisfies $\mathbf{C1} - \mathbf{C8}$. The method follows a seven-step procedure. Initially, let us begin with $t = T$ and $E[V_{T+1}] = 0$.

1. In each period t, let the expected value function $E_t[V_{t+1}(s_{t+1}, a_{t+1}, \bar{b}_{t+1})]$ and the conditional mean of defaulting $E_t[D(\epsilon_{t+1})]$ be given. For each s_t, b_t, d_t, and $\bar{b}_{t+1}$, characterize an interval of $a_{t+1} \in G_{a_{t+1}}$ satisfying **C4**; find the minimum; and save it as $a^{rbl}_{d_t,\bar{b}_{t+1}}(s_t, b_t, E_t[D(\epsilon_{t+1})])$ (generalized risky borrowing limit). Going forward, take the states $(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}$ and $(s_t, b_t) \in S_t \times B_t$ as given to make the notations simple.
2. For each $a_{t+1} \in G_{a_{t+1}}$ with $a_{t+1} > a^{rbl}_{d_t,\bar{b}_{t+1}}(s_t, b_t, E_t[D(\epsilon_{t+1})])$, compute the endogenously-driven $y_t(s_t, b_t, a_{t+1})$ by solving $\hat{g}_{d_t,\bar{b}_{t+1}}(s_t, b_t, y_t(s_t, b_t, a_{t+1})) = \hat{V}_{d_t,\bar{b}_{t+1}}(s_t, b_t, a_{t+1})$ (Equation (34)). Then, save the pairs of $(y_t(s_t, b_t, a_{t+1}), b_t, a_{t+1})$.
3. For each $a_{t+1} \in G_{a_{t+1}}$ with $a_{t+1} > a^{rbl}_{d_t,\bar{b}_{t+1}}(s_t, b_t, E_t[D(\epsilon_{t+1})])$, use the algorithm of Fella (2014) to refine the global solution out of the candidates from the previous step. Save the refined pairs of $(y_t(s_t, b_t, a_{t+1}), b_t, a_{t+1})$ for the global solutions.
4. For the refined pairs of $(y_t(s_t, b_t, a_{t+1}), b_t, a_{t+1})$, retrieve the corresponding endogenously-driven current state $a_t(s_t, b_t, a_{t+1})$. To do so, first, compute $z_t = \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t[D(\epsilon_{t+1})]))$. Then, use z_t, $y_t(s_t, b_t, a_{t+1})$, and the budget

set Γ_t to find $a_t(s_t, b_t, a_{t+1})$ that satisfies $\Gamma_t(s_t, a_t(s_t, b_t, a_{t+1}), b_t, d_t; b_{t+1})$. This search is possible due to $\mathbf{C5}$. Now, $a_t(s_t, b_t, a_{t+1})$ lies in the endogenously-determined grid points of a_t. Save the pairs of $(a_t(s_t, b_t, a_{t+1}), b_t, a_{t+1})$. Note that these pairs correspond to the global solutions.

5. For the saved pairs of $(a_t(s_t, b_t, a_{t+1}), b_t, a_{t+1})$, approximate the decision rule $a_{t+1} = g_a(s_t, \cdot, b_t, d_t, b_{t+1})$ over the exogenous grid of a_t, G_{a_t}. We can do this using a spline due to the one-to-one mapping between a_{t+1} and a_t ($\mathbf{C8}$). Then, using the approximated decision rule $a_{t+1} = g_a(s_t, a_t, b_t, d_t, b_{t+1})$, compute the value function $v_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1})$ over G_{a_t}.
6. Solve $\max_{\{(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}\}} v_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1})$ and update $V_t(s_t, a_t, b_t)$.
7. Compute $E_{t-1}[V_t(s_t, a_t, b_t)]$ and $E_{t-1}[D(\epsilon_t)]$. Take t=t-1; go back to Step 1 until $t = 0$.

5.4 Discussion

It is worth discussing how restrictive the sufficient conditions are. If these conditions were too restrictive to address many types of default risk models, our method might not be more useful than the conventional approach. Since $\mathbf{C5} - \mathbf{C8}$ are widely shared features in general dynamic problems in macroeconomics, we focus on $\mathbf{C1} - \mathbf{C4}$.

$\mathbf{C1}$ is about the local differentiability of the expected value function $E_t[V_{t+1}]$ with respect to the optimal decision of a_{t+1} and that of the decision rule a_{t+1} with respect to the optimal choice of y_t. It might be hard to claim that all default risk models satisfy $\mathbf{C1}$; however, Clausen and Strub (2020) have shown that this feature is prevalent in many types of dynamic problems. More importantly, we can, at least, certainly check whether this condition holds to a specific problem with default risks by using the "Reverse Calculus" in Clausen and Strub (2020).

$\mathbf{C2}$, $\mathbf{C3}$ and $\mathbf{C4}$ imply that there exists a post-decision state z_t that is a sufficient statistic for the future endogenous state a_{t+1}. According to White (2015), this property of sufficient statistics must be required to use EGMs. To do so, in addition to $\mathbf{C2}$, our EGM requires more conditions than the EGM of White (2015) because in default risk models, the post-decision state z_t is non-linearly related to the future endogenous state a_{t+1} (i.e., $z = q(s, a^{'})a^{'}$). This additional feature brings $\mathbf{C3}$ and $\mathbf{C4}$. $\mathbf{C3}$ implies that the EGM might not work in default risk models with shocks on assets or investment, in which z_t may not uniquely identify a_{t+1} because the realization of shocks ϵ_{t+1} may affect assets a_{t+1} (i.e., Glover et al. (2010)). Shocks on defaultable assets, however, are not a commonly used assumption in the literature.

$\mathbf{C4}$ implies that the EGM might not be applicable to default risk models where the generalized risky borrowing limit is not well-defined or depends on endogenous states other than $(s_t, z_t, b_t, E_t(D(\epsilon_{t+1})))$. We argue that the first issue might not be problematic, but the second issue does impose some limitations on our method. The first issue implies that either the borrowing constraint is unbounded, or utility is decreasing in the amount of debt any time a_{t+1} is negative. An unbounded borrowing limit allows for Ponzi schemes which most researchers prefer to rule out. For the case of utility decreasing in debt for the whole borrowing region, the generalized borrowing limit is well-defined at zero-assets, $a_{t+1} = 0$, if the return on savings $(a_{t+1} > 0)$ is independent of the individual choice of a_{t+1}. For example, in Arellano (2008), $a^{'} = 0$ is the upper bound of the feasible set for the risk-borrowing limit because $\partial[q(S, a^{'}) \cdot a^{'}]/\partial a^{'} = q(S, a^{'}) = 1/(1 + r_f) > 0$ if $a^{'} \geq 0$. This assumption is quite common in the literature.

Note that, however, when the generalized risky borrowing limit is a function of endogenous states other than $(s_t, z_t, b_t, E_t(D(\epsilon_{t+1})))$, it is uncertain whether our method works. For example, when there exists asymmetric information between lenders and borrowers, the price function might depend on the distribution over agents (i.e., Athreya et al. (2012)). In this type of models, we cannot make sure whether the generalized risky borrowing limit can be predetermined with the states, $(s_t, z_t, b_t, E_t(D(\epsilon_{t+1})))$. In this case, the EGM might not solve the problem. To address issue, one might need to include the additional endogenous states into the generalized risky borrowing limit. This inclusion, however, might dampen the efficiency gain of our method.

Nonetheless, the sufficient conditions clearly imply that our EGM can cover a broad class of default risk models. Since the EGM solves the problem in a backward direction, this method can be used for life-cycle consumer bankruptcy models (i.e., Athreya (2008); Athreya et al. (2009); Livshits et al. (2007, 2010); Gordon (2015)). These life-cycle models cannot solved by the envelope condition method of Arellano et al. (2016) because it is a forward-solving algorithm. Additionally, the EGM can address multiple options to default (i.e., Chapter 7 and Chapter 13 in Consumer bankruptcy) with other discrete choices (i.e., housing, durable goods, health insurance, and retirement). This versatility might be useful in investigating the interaction between default and other types of policies related to these discrete choices. Jang (2020), for example, used the EGM to solve a life-cycle model that examines the role of consumer bankruptcy in designing optimal health insurance policies.

6 Conclusion

We presented an extension of the endogenous grid method for default risk models. This method combines Fella's (2014) endogenous grid method by newly introducing a numerical step to search for the risky borrowing limit, which is the lower bound of the feasible set for the solution of asset holdings. By using the algorithm of Fella (2014) and our step for the risky borrowing limit, we identified the region of solution sets to which Carroll's (2006) endogenous grid method is applicable. Compared to the conventional grid search method, the method brings substantial improvements in computational efficiency and accuracy. We further showed that our EGM is applicable to a broad class of default risk models by providing sufficient conditions for the application. We hope that this method opens up possibilities for researchers to investigate topics with default options that have previously been left unexplored due to computational complexity.

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A Additional Results

We apply our algorithm to solve Nakajima and Ríos-Rull (2014), which is computationally heavier than Arellano (2008). We compare the computing time and accuracy of our EGM with those of the grid search method as well. For details of the model and parameterization, see Nakajima and Ríos-Rull (2014). As in Nakajima and Ríos-Rull (2014), we use Krusell and Smith's (1998) method to handle the aggregate uncertainty. Note that this method approximates aggregate states using a few moments and agents expect next period states using parameterized functional forms of those moments. The method achieves high accuracy, but it requires a long simulation to update forecasting rules and may take many trials to find a proper functional form.

A.1 Specification of Krusell and Smith's (1998) Method

Nakajima and Ríos-Rull (2014) approximated $(z, K; m)$ with (z, K, O), where z is total factor productivity, K is aggregate capital, m is household distribution, and O is average individual labor productivity. They use forecasting rules for K', L, r, and O' where L is aggregate labor and r is risk-free rate. Here, we abstract from the counter-cyclical earnings risk and approximate aggregate states $(z, K; m)$ to (z, K). Additionally, instead of forecasting L, which is necessary to calculate the wage w, we forecast the wage directly. We specify the forecasting functions for K', r, and w as the following log-linear forms:

$$log\, K' = \phi_{k1}(z, K) + \phi_{k2}(z, K) \cdot log\, K$$

$$log\, r = \phi_{r1}(z, K) + \phi_{r2}(z, K) \cdot log\, K$$

$$log\, w = \phi_{w1}(z, K) + \phi_{w2}(z, K) \cdot log\, K$$

A.2 Computing Time and Accuracy

We vary the size of the grid for assets across computational exercises. In all computational exercises, we keep the number of the grid points for the other variables as follows. The size of the grid for the permanent labor productivity shock is 2, that for the persistent shock is 15, and that for the transitory shock is 3. The number of the grid for the TFP shock is 3, and that for K is 5. Because we use Krusell and Smith's (1998) method, we must go through the inner and outer loops several times until the forecasting rules converge. We compute the average CPU time per iteration in the inner loop and outer loop, respectively. We simulated the model for 2,000 periods with Krusell and Smith's (1998) method, and all computations were carried out on on a single core of an Intel i7-4770 processor. The programs were written in Fortran 95.

Table 4: Computing Time

# of GRD. PTS. for INR. - OTR.	200-500		300-500		400-500		500-600	
Computational Method	EGM	GS	EGM	GS	EGM	GS	EGM	GS
AVG CPU Time in INR. per ITER.*	0.68	12.54	1.25	29.27	1.99	54.39	2.99	79.65
AVG CPU Time in OTR. per ITER.*	29.49	173.49	27.05	185.62	24.48	182.79	38.66	286.97

.$*$: Unit = minute.
'# of GRD. PTS. for INR. - OTR.' refers to the number of grid points for assets in the inner loops and in the outer loops, respectively.

Table 4 indicates that the EGM is faster than the grid search method both in the inner loop and in the other loop. In the inner loops, the EGM is from 18.5 to 27.3 times faster than the grid search method. In the outer loop, the EGM is approximately 7.5 times faster than the grid search method. The gap differs across the size of the asset grid, but the EGM is much more efficient than the grid search method across all grid settings.

To measure accuracy, we use three criteria in the literature. First, we compute Bellman equation errors (BE error) which is defined the same way in Section 4. Second, we take Den Hann's forecasting test (DH error) described in Algan et al. (2014). It is the difference between expected capital K_e' by the forecasting rules and realized capital K_r' from the simulations: $|logK_r' - logK_e'|$. Finally, we reports the R^2 of the forecasting rules in the simulation step.

Figure 6 shows that with the EGM, the price dynamics in the simulation are very close to those generated by the forecasting rules. Since they are very close to one another, it is hard to observe blue lines in the dynamics of the risk-free interest rate and wage. Figure 7 shows that, with the grid search method, there are differences between the simulated-dynamics of these prices and those generated by the forecasting rules. Den Hann error measures those differences. Overall, Den Han errors from the EGM are smaller than those from the grid search method.

Table 5: Computational Accuracy

# of GRD. PTS. for INR. - OTR.	200-500		300-500		400-500		500-600	
Computational Method	EGM	GS	EGM	GS	EGM	GS	EGM	GS
Average of BE Error*	0.11%	0.36%	0.06%	0.16%	0.03%	0.09%	0.02%	0.06%
Max of BE Error*	10.77%	15.53%	11.34%	11.76%	11.57%	17.49%	11.71%	15.46%
R^2 of K' function	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
R^2 of r function	0.9987	0.9911	0.9971	0.9882	0.9975	0.9963	0.9977	0.9940
R^2 of w function	0.9999	0.9997	0.9999	0.9997	0.9997	0.9997	0.9997	0.9996
Mean of DH error	0.004%	0.01%	0.005%	0.012%	0.009%	0.01%	0.007%	0.01%
Max of DH error	0.029%	0.06%	0.038%	0.081%	0.047%	0.073%	0.04%	0.05%

.$*$: The Bellman equation errors are computed in stationary equilibrium. The number of grid points refers to the number of points for asset grid

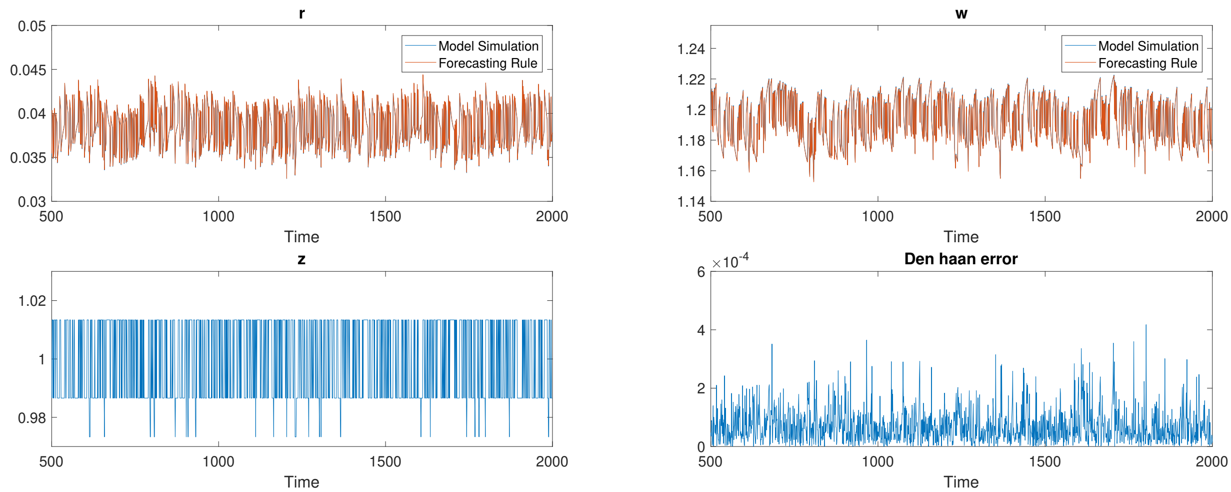

Figure 6: Simulation Results for the EGM with the 500-600 grid

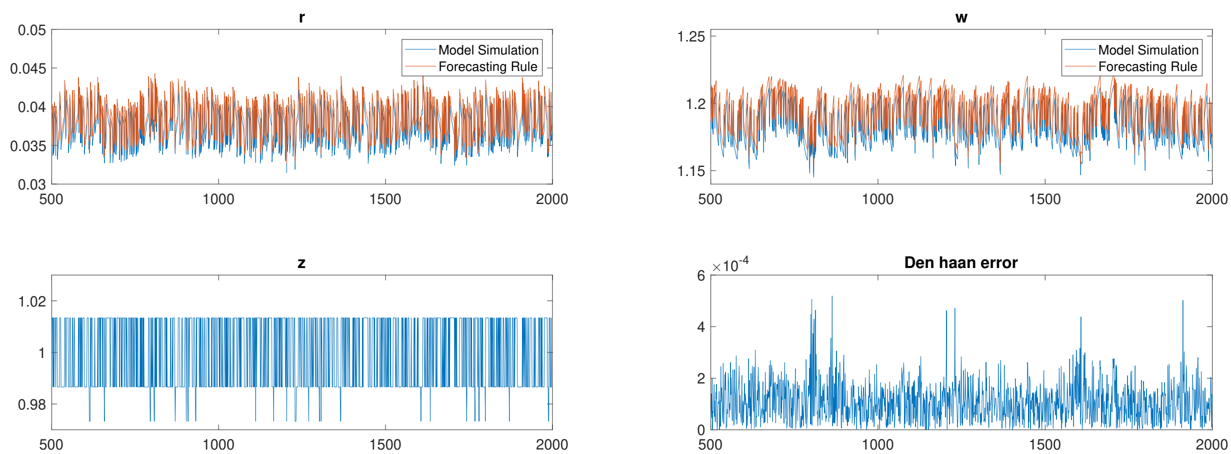

Figure 7: Simulation Results for the Grid Search Method with the 500-600 grid

Table 5 shows that the EGM produces more accurate outcomes than the grid search method. Regarding the Bellman equation errors, the average Bellman equation errors in the EGM are approximately three times lower than that in the grid search methods. Although the gaps in the maximum Bellman errors are smaller than that in the average Bellman error, the EGM generates smaller values of the maximum Bellman errors than the grid search method. These smaller gap appear because our EGM also uses the grid search method for the borrowing region. Additionally, the EGM produces higher R^2s of the forecasting functions than the grid search method. Lastly, the average of Den Hann errors and maximum of Den Hann errors from the EGM are lower than that from the grid search method across all grid settings.