Huggett (1993)

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July 10, 2020

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Huggett (1993)

$$v(z_0, a_0, 0) = \max_{\{c_t\}_{t \geq 0}} \mathbb{E} \int_0^\infty e^{-\rho t} u(c(t)) dt$$

$$\dot{a} = ra + z - c$$

$$a \geq \underline{a}$$

- z follows a two-state Poisson process with intensity λ

HJB equation

Rewrite the problem

$$v(z_t, a_t, t) = \max_{c_s} E\left(\int_t^{t+dt} e^{-\rho s} u(c_s) ds + v(z_{t+dt}, a_{t+dt}, t+dt)\right)$$

rearrange and divide by dt

$$0 = \max_{c_s} E\left(\frac{1}{dt}\int_t^{t+dt} e^{-\rho s} u(c_s) ds + \frac{1}{dt}\left(v(z_{t+dt}, a_{t+dt}, t+dt) - v(z_t, a_t, t)\right)\right)$$

using $v(x) \equiv v(x, 0)$, $\quad v(x,t) = e^{-\rho t} v(x) \quad v_t(x,t) = -\rho e^{-\rho t} v(x)$

if $z_t = z$, $E[v(z_{t+dt}, a_{t+dt})] = \lambda dt v(z', a_{t+dt}) + (1 - \lambda dt) v(z, a_{t+dt})$

$$0 = \max_{c_t} e^{-\rho t} u(c_t) + \frac{1}{dt}\left(e^{-\rho t + dt}\left(\lambda dt v(z', a_{t+dt}) + (1 - \lambda dt) v(z, a_{t+dt})\right) - e^{-\rho t} v(z, a_t)\right)$$

HJB equation

$$0 = \max_{c_t} e^{-\rho t} u(c_t) + \frac{1}{dt}\left(e^{-\rho t + dt}\lambda dt\Big(v(z', a_{t+dt}) - v(z, a_{t+dt})\Big) + e^{-\rho t + dt} v(z, a_{t+dt}) - e^{-\rho t} v(z, a_t)\right)$$

$$= \max_{c_t} e^{-\rho t} u(c_t) + e^{-\rho t + dt}\lambda\Big(v(z', a_{t+dt}) - v(z, a_{t+dt})\Big) + \frac{1}{dt}\left(e^{-\rho t + dt} v(z, a_{t+dt}) - e^{-\rho t} v(z, a_t)\right)$$

$dt \to 0$

$$= \max_{c_t} e^{-\rho t} u(c_t) + e^{-\rho t}\lambda\Big(v(z', a_t) - v(z, a_t)\Big) - \rho e^{-\rho t} v(z, a_t) + e^{-\rho t} v(z, a_t)\dot{a}_t$$

drop t, $e^{-\rho t}$ and rearrange

$$\rho v(z, a) = \max_{c} u(c) + v_a(z, a)\dot{a} + \lambda\Big(v(z', a) - v(z, a)\Big)$$

Numerical solution

Similar to the solution for the stochastic growth model, but simpler

- Set the **Z** grid
  - For example, assume agents draw z from a distribution when they receive a shock, and choose a support for z
  - How to choose λ value? $\lambda = \frac{1}{\text{mean duration}}$
    e.g. If an income shock arrives every 30 years, $\lambda = \frac{1}{30}$
  - This is because a holding times has an exponential distribution, and the expected value of an exponentially distributed random variables with rate parameter λ is $\frac{1}{\lambda}$
    Holding times
- Set the **A** grid

Numerical solution

Discretized the HJB equation. $v_{i,j} = v(z_i, a_j)$

$$\rho v_{i,j} = \max_c u(c_{i,j}) + v_a(z_i, a_j)(ra_j + z_i - c_{i,j}) + \lambda\big(v_{-i,j} - v_{i,j}\big)$$

From FOC, $u'(c_{i,j}) - v_a(z_i, a_j) = 0 \implies c_{i,j} = u^{-1}\big(v_a(z_i, a_j)\big)$

Choose between $v^F_{a,i,j}$ and $v^B_{a,i,j}$ following upwind scheme

Boundaries

- **A**: At a_1, set $v^B_a(z_i, a_1) = u'(ra_1 + z_i)$. Choose high enough a_{na}.
- **Z**: No need to consider

Numerical Solution

Let $\dot{x}^+ = max(\dot{x}, 0)$, $\dot{x}^- = min(\dot{x}, 0)$

$$\rho v_{i,j} = \max_c u(c_{i,j}) + \frac{v_{i,j+1} - v_{i,j}}{\Delta a}\dot{a}^+_{F,i,j} + \frac{v_{i,j} - v_{i,j-1}}{\Delta a}\dot{a}^-_{B,i,j} + \lambda\big(v_{-i,j} - v_{i,j}\big)$$

Collecting coefficients of $v_{-i,j}$, $v_{i,j}$, $v_{i,j-1}$, $v_{i,j+1}$

$v_{-i,j}$: λ

$v_{i,j}$: $\underbrace{-\frac{\dot{a}^+_{F,i,j}}{\Delta a} + \frac{\dot{a}^-_{B,i,j}}{\Delta a}}_{d^{a0}_{ij}} - \lambda = d^{a0}_{ij} - \lambda = d^0_{ij}$

$v_{i,j-1}$: $-\frac{\dot{a}^-_{B,i,j}}{\Delta a} = d^{a1}_{ij}$, $\qquad v_{i,j+1}$: $\frac{\dot{a}^+_{F,i,j}}{\Delta a} = d^{a2}_{ij}$

Matrix representation

For example, $n_z = 2$ and $n_k = 4$

$$\rho \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \end{bmatrix} = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{34} \end{bmatrix} + \left[\begin{array}{cccc|cccc} d_{11}^{0} & d_{11}^{a1} & 0 & 0 & \lambda & 0 & 0 & 0 \\ d_{12}^{a1} & d_{12}^{0} & d_{12}^{a2} & 0 & 0 & \lambda & 0 & 0 \\ 0 & d_{13}^{a1} & d_{13}^{0} & d_{13}^{a2} & 0 & 0 & \lambda & 0 \\ 0 & 0 & d_{14}^{a1} & d_{14}^{0} & 0 & 0 & 0 & \lambda \\ \hline \lambda & 0 & 0 & 0 & d_{21}^{0} & d_{21}^{a2} & 0 & 0 \\ 0 & \lambda & 0 & 0 & d_{22}^{a1} & d_{22}^{0} & d_{22}^{a2} & 0 \\ 0 & 0 & \lambda & 0 & 0 & d_{23}^{a1} & d_{23}^{0} & d_{23}^{a2} \\ 0 & 0 & 0 & \lambda & 0 & 0 & d_{24}^{a1} & d_{24}^{0} \end{array}\right] \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \end{bmatrix}$$

$\rho \mathbf{v} = \mathbf{u} + \mathbf{A}\mathbf{v}$

- Solve the system of equations using explicit or implicit update

$nz > 2$

Suppose $nz = 3$. We can assume a process such as

- A shock arrives with intensity λ_i, $i = 1, 2, 3$
- People draw z from a distribution $[z_1, z_2, z_3]$ with probability $[p_1, p_2, p_3]$

$$\rho v(z_i, a) = \max_c u(c) + v_a(z_i, a)\dot{a} + \lambda_i \Big(\sum_{j=1}^{3} p_j v(z_j, a) - v(z_i, a) \Big)$$

Az $=$

$$\left[\begin{array}{c|c|c} \lambda_1(p_1 - 1)\mathbf{I} & \lambda_1 p_2 \mathbf{I} & \lambda_1 p_3 \mathbf{I} \\ \hline \lambda_2 p_1 \mathbf{I} & \lambda_2(p_2 - 1)\mathbf{I} & \lambda_2 p_3 \mathbf{I} \\ \hline \lambda_3 p_1 \mathbf{I} & \lambda_3 p_2 \mathbf{I} & \lambda_1(p_3 - 1)\mathbf{I} \end{array}\right]$$

- **I** is $na \times na$ identity matrix
- Diagonal terms are negative, others are positive
- `sum(Az,2) = 0`

Stationary distribution

To find an equilibrium price, we need to solve a **stationary distribution**
Recall the Huggett (1993) with equilibrium conditions

$$\rho v(z,a) = \max_c u(c) + v_a(z,a)s(z,a) + \lambda(v(z',a) - v(z,a))$$

$$s(z,a) = ra + z - c, \quad a \geq \underline{a}$$

$$0 = \sum_{i=1}^{nz} \int_{\underline{a}}^{\infty} a g(z_i, a) da \quad \textbf{(bond market clears)}$$

$$0 = -\frac{d}{da}[s(z,a)g(z,a;t)] + \lambda\Big(g(z',a;t) - g(z,a;t)\Big) \quad \textbf{(invariant distribution)}$$

Stationary distribution

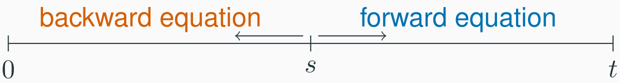
$$\frac{\partial g(z,a;t)}{\partial t} = -\frac{d}{da}[s(z,a)g(z,a;t)] + \lambda\Big(g(z',a;t) - g(z,a;t)\Big)$$

is **Kolmogorov forward equation**, and 'stationary' implies

$$\frac{\partial g(z,a;t)}{\partial t} = 0$$

Kolmogorov equation

Kolmogorov equation is a PDE that describes the time evolution of a function under the influence of drag forces and random forces, as in Brownian motion.



- Kolmogorov forward equations are obtained by letting s approach t from below and we use it to describe the evolution of a density function
- Kolmogorov backward equations are obtained by letting s approach 0 from above
  - When a terminal value is known
  - We can use it for e.g. bond pricing of default models, option pricing

Derivation of Kolmogorov forward equation

Assume the intensity of income shock depends on the level of income. $\lambda_1 \neq \lambda_2$
G is a CDF, $G(z, a; t) = Pr(\tilde{z}_t = z, \tilde{a}_t \leq a)$

Over a time period of Δ, G evolves as

$$\begin{aligned} Pr(\tilde{z}_{t+\Delta} = z_i, \tilde{a}_{t+\Delta} \leq a) = (1 - \Delta\lambda_i) Pr(\tilde{z}_t = z_i, \tilde{a}_t \leq a - \Delta s(a, z_i)) \\ + \Delta\lambda_{-i} Pr(\tilde{z}_t = z_{-i}, \tilde{a}_t \leq a - \Delta s(a, z_{-i})) \end{aligned}$$

- **(Decision)** $\tilde{a}_{t+\Delta} = \tilde{a}_t + \Delta s(\tilde{a}_t, z)$
  If one has $a - \Delta s(a, z)$ at t, will have a at $t + \Delta$
- **(Shock)** Income switches from z_i to z_{-i} with an intensity $\Delta\lambda_i$
  $1 - \Delta\lambda_i$: Those who with z_i at t and don't get income shocks over Δ
  $\Delta\lambda_{-i}$: Those who with z_{-i} at t and get income shocks over Δ, flow into z_i

Derivation of Kolmogorov forward equation

$$\begin{aligned} G(z_i, a; t+\Delta) &= Pr(\tilde{z}_{t+\Delta} = z_i, \tilde{a}_{t+\Delta} \leq a) \\ &= (1 - \Delta\lambda_i) Pr(\tilde{z}_t = z_i, \tilde{a}_t \leq a - \Delta s_i) + \Delta\lambda_{-i} Pr(\tilde{z}_t = z_{-i}, \tilde{a}_t \leq a - \Delta s_{-i}) \\ &= (1 - \Delta\lambda_i) G(z_i, a - \Delta s_i; t) + \Delta\lambda_{-i} G(z_{-i}, a - \Delta s_{-i}; t) \\ &= G(z_i, a - \Delta s_i; t) - \Delta\lambda_i G(z_i, a - \Delta s_i; t) + \Delta\lambda_{-i} G(z_{-i}, a - \Delta s_{-i}; t) \end{aligned}$$

Subtract $G(z_i, a; t)$ from both sides and divide by Δ

$$\frac{G(z_i, a; t+\Delta) - G(z_i, a; t)}{\Delta} = \frac{G(z_i, a - \Delta s_i; t) - G(z_i, a; t)}{\Delta} - \lambda_i G(z_i, a - \Delta s_i; t) + \lambda_{-i} G(z_{-i}, a - \Delta s_{-i}; t)$$

$^* s_i = s(a, z_i)$

Derivation of Kolmogorov forward equation

$$\frac{G(z_i, a; t+\Delta) - G(z_i, a; t)}{\Delta} = \frac{G(z_i, a - \Delta s_i; t) - G(z_i, a; t)}{\Delta} - \lambda_i G(z_i, a - \Delta s_i; t) + \lambda_{-i} G(z_{-i}, a - \Delta s_{-i}; t)$$

Take the limit as $\Delta \to 0$

$$\partial_t G(z_i, a; t) = -s_i \partial_a G(z_i, a; t) - \lambda_i G(z_i, a; t) + \lambda_{-i} G(z_{-i}, a; t)$$

Take the derivative wrt a gives a PDE

$$\partial_t g(z_i, a; t) = \partial_a[-s_i g(z_i, a; t)] - \lambda_i g(z_i, a; t) + \lambda_{-i} g(z_{-i}, a; t)$$

This is a Kolmogorov forward equation and g is a density function

* $g(z, a; t) = \partial_a G(z, a; t)$

Solving Kolmogorov forward equation

To solve the KFE, approximate $\frac{d}{da}g$ with numerical derivatives and apply upwind scheme.

$$0 = -\frac{d}{da}[s(z_i, a)g(z_i, a)] - \lambda_i g(z_i, a) + \lambda_{-i} g(z_{-i}, a)$$

The most convenient/correct approximation is as follows: Deep (?) underlying reason

$$0 = -\frac{s^+_{F,i,j} g_{i,j} - s^+_{F,i,j-1} g_{i,j-1}}{\Delta a} - \frac{s^-_{B,i,j+1} g_{i,j+1} - s^-_{B,i,j} g_{i,j}}{\Delta a} - \lambda_i g_{i,j} + \lambda_{-i} g_{-i,j}$$

$$g_{-i,j}\text{:}\ \lambda_{-i}$$

$$g_{i,j}\text{:}\ -\frac{\dot{a}^+_{F,i,j}}{\Delta a} + \frac{\dot{a}^-_{B,i,j}}{\Delta a} - \lambda_i$$

$$g_{i,j-1}\text{:}\ -\frac{\dot{a}^+_{F,i,j-1}}{\Delta a}, \qquad g_{i,j+1} : \frac{\dot{a}^-_{B,i,j+1}}{\Delta a}$$

$$v_{-i,j}\text{:}\ \lambda_i$$

$$v_{i,j}\text{:}\ -\frac{\dot{a}^+_{F,i,j}}{\Delta a} + \frac{\dot{a}^-_{B,i,j}}{\Delta a} - \lambda_i$$

$$v_{i,j-1}\text{:}\ -\frac{\dot{a}^-_{B,i,j}}{\Delta a}, \qquad v_{i,j+1} : \frac{\dot{a}^+_{F,i,j}}{\Delta a}$$

Solving Kolmogorov forward equation

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
=
\left[\begin{array}{cccc|cccc}
d^0_{11} & s^-_{B,12} & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\
s^+_{F,11} & d^0_{12} & s^-_{B,13} & 0 & 0 & \lambda_2 & 0 & 0 \\
0 & s^+_{F,12} & d^0_{13} & s^-_{B,14} & 0 & 0 & \lambda_2 & 0 \\
0 & 0 & s^+_{F,13} & d^0_{14} & 0 & 0 & 0 & \lambda_2 \\
\hline
\lambda_1 & 0 & 0 & 0 & d^0_{21} & s^-_{B,22} & 0 & 0 \\
0 & \lambda_1 & 0 & 0 & s^+_{F,21} & d^0_{22} & s^-_{B,23} & 0 \\
0 & 0 & \lambda_1 & 0 & 0 & s^+_{F,22} & d^0_{23} & s^-_{B,24} \\
0 & 0 & 0 & \lambda_1 & 0 & 0 & s^+_{F,23} & d^0_{24}
\end{array}\right]
\begin{bmatrix} g_{11} \\ g_{12} \\ g_{13} \\ g_{14} \\ g_{21} \\ g_{22} \\ g_{23} \\ g_{24} \end{bmatrix}
$$

$\rightarrow \mathbf{0} = \mathbf{Bg}$

This is the preferred because $\mathbf{B} = \mathbf{A}^T$ where $\mathbf{A}$ from the HJB equation, $\rho \mathbf{v} = \mathbf{u} + \mathbf{Av}$

Solving Kolmogorov forward equation

$\mathbf{0} = \mathbf{B}\mathbf{g}$

- If we solve above equation, $g_i = 0$, $i = 1, ..n$ is a soltuion.
- We have an implicit constraint $\sum_{i=1}^{nz} \int g(z_i, a)da = 1$
- **B** is singular so cannot be inverted
    - **A** is singular. e.g. `-sum(A(:,1:n-1),2) = A(:,n)`
        - But we invert $(\rho + \frac{1}{\Delta})\mathbf{I} - \mathbf{A}$ when we solve the value function
    - $\mathbf{A}^T$ is singular too. e.g. `-sum(A(1:n-1,:),1) = A(n,:)`

Adjusting B

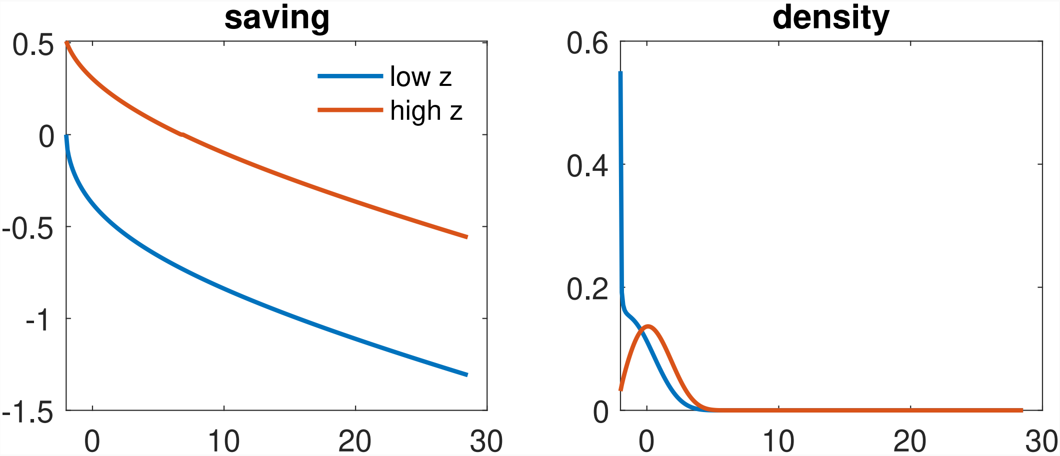
$$\begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{cccc|cccc} d^0_{11} & s^-_{B,12} & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s^+_{F,12} & d^0_{13} & s^-_{B,14} & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & s^+_{F,13} & d^0_{14} & 0 & 0 & 0 & \lambda_2 \\ \hline \lambda_1 & 0 & 0 & 0 & d^0_{21} & s^-_{B,22} & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & s^+_{F,21} & d^0_{22} & s^-_{B,23} & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & s^+_{F,22} & d^0_{23} & s^-_{B,24} \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & s^+_{F,23} & d^0_{24} \end{array}\right] \begin{bmatrix} g_{11} \\ g_{12} \\ g_{13} \\ g_{14} \\ g_{21} \\ g_{22} \\ g_{23} \\ g_{24} \end{bmatrix}$$

Let g_0 be a $na \times nz$ zero vector. Fix $g_{0,i} = 0.1$ (or any positive number)

Replace the i^{th} row of **B** with a row of zeros everywhere except for 1 on the diagonal

By setting g_0 and adjusting **B**, we fix $\tilde{g}_i = 0.1$ and find the rest of g_j values. $\tilde{g} = \mathbf{B}^{-1} g_0$

Adjust the level of g_js to make the total mass 1. $g = \frac{\tilde{g}}{\sum_{i=1}^{nz} \int \tilde{g}(z_i, a) da}$


saving
0.5
0
-0.5
-1
-1.5
0
10
20
30
low z
high z
density
0.6
0.4
0.2
0
0
10
20
30

Derivation of Kolmogorov forward equation: Diffusion

Let x_t be a solution of

$$dx_t = \mu(x_t)dt + \sigma(x_t)dw_t$$

Let $f(x)$ be a smooth function. Applying Ito's lemmas

$$\begin{aligned} df(x) &= f_x(x)\mu(x)dt + f_{xx}(x)\frac{\sigma(x)^2}{2}dt + f_x(x)\sigma(x)dw_t \\ E\big[df(x)\big] &= E\big[f_x(x)\mu(x)dt + f_{xx}(x)\frac{\sigma(x)^2}{2}dt + f_x(x)\sigma(x)dw_t\big] \\ &= E\big[f_x(x)\mu(x)dt + f_{xx}(x)\frac{\sigma(x)^2}{2}dt\big] \qquad E(dw_t) = 0 \end{aligned}$$

by Dominated convergence theorem,

$$\frac{d}{dt}E\big[f(x)\big] = E\big[f_x(x)\mu(x) + f_{xx}(x)\frac{\sigma(x)^2}{2}\big]$$

Derivation of Kolmogorov forward equation: Diffusion

$$\frac{d}{dt}E[f(x)] = E[f_x(x)\mu(x) + f_{xx}(x)\frac{\sigma(x)^2}{2}]$$

Let $g(x,t)$ be a density function.

$$\frac{d}{dt}\int_{-\infty}^{\infty} f(x)g(x,t)dx = \int_{-\infty}^{\infty}\Big(f_x(x)\mu(x) + f_{xx}(x)\frac{\sigma(x)^2}{2}\Big)g(x,t)dx$$

$$\int_{-\infty}^{\infty} f(x)\Big(\frac{\partial g(x,t)}{\partial t}\Big)dx = \int_{-\infty}^{\infty}\Big(g(x,t)\mu(x)f_x(x) + g(x,t)\frac{\sigma(x)^2}{2}f_{xx}(x)\Big)dx$$

Integration by parts:

$$\int_{-\infty}^{\infty} u(x)v'(x)dx = \alpha - \int_{-\infty}^{\infty} u'(x)v(x)dx,\ \int_{-\infty}^{\infty} y(x)v''(x)dx = \beta + \int_{-\infty}^{\infty} y''(x)v(x)dx$$

Let $v(x) = f(x)$, $\quad u(x) = g(x,t)\mu(x)$, $\quad y(x) = g(x,t)\frac{\sigma(x)^2}{2}$

$\alpha = u(\infty)v(\infty) - u(-\infty)v(-\infty)$. Let $\alpha = \beta = 0$

Derivation of Kolmogorov forward equation: Diffusion

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\left(\frac{\partial g(x,t)}{\partial t}\right)dx &= \int_{-\infty}^{\infty}\left(g(x,t)\mu(x)f_x(x) + g(x,t)\frac{\sigma(x)^2}{2}f_{xx}(x)\right)dx \\ &= \int_{-\infty}^{\infty}\left(-\frac{d}{dx}\big(g(x,t)\mu(x)\big)f(x) + \frac{d^2}{dx^2}\Big(g(x,t)\frac{\sigma(x)^2}{2}\Big)f(x)\right)dx \\ &= \int_{-\infty}^{\infty} f(x)\left(-\frac{d}{dx}\big(g(x,t)\mu(x)\big) + \frac{d^2}{dx^2}\Big(g(x,t)\frac{\sigma(x)^2}{2}\Big)\right)dx\end{aligned}$$

implies

$$\frac{\partial g(x,t)}{\partial t} = -\frac{d}{dx}\Big(g(x,t)\mu(x)\Big) + \frac{d^2}{dx^2}\Big(g(x,t)\frac{\sigma(x)^2}{2}\Big)$$

This is Kolmogorov forward equation.

Derivation of Kolmogorov forward equation: Diffusion

If we assume a diffusion process for z, we can replace $f(x)$ to $f(z,a)$ where

$$da = s(z,a)dt, \quad dz = \mu(z)dt + \sigma(z)dw$$

$$\frac{\partial g(z,a;t)}{\partial t} = -\frac{d}{da}\Big(g(z,a;t)s(z,a)\Big) - \frac{d}{dz}\Big(g(z,a;t)\mu(z)\Big) + \frac{d^2}{dz^2}\Big(g(z,a;t)\frac{\sigma(z)^2}{2}\Big)$$

Derivation of Kolmogorov forward equation: Diffusion

The most convenient/correct approximation is as follows:

$$
\begin{aligned}
0 = & -\frac{s^{+}_{F,i,j} g_{i,j} - s^{+}_{F,i,j-1} g_{i,j-1}}{\Delta a} - \frac{s^{-}_{B,i,j+1} g_{i,j+1} - s^{-}_{B,i,j} g_{i,j}}{\Delta a} \\
& - \frac{\mu^{+}_{i} g_{i,j} - \mu^{+}_{i-1} g_{i-1,j}}{\Delta z} - \frac{\mu^{-}_{i+1} g_{i+1,j} - \mu^{-}_{i} g_{i,j}}{\Delta z} \\
& + \frac{\sigma^2_{i+1} g_{i+1,j} - 2\sigma^2_{i} g_{i,j} + \sigma^2_{i-1} g_{i-1,j}}{2(\Delta z)^2}
\end{aligned}
$$

$g_{i-1,j}$: $\frac{\mu^{+}_{i-1}}{\Delta z} + \frac{\sigma^2_{i-1}}{2(\Delta z)^2}$

$g_{i+1,j}$: $-\frac{\mu^{-}_{i+1}}{\Delta z} + \frac{\sigma^2_{i+1}}{2(\Delta z)^2}$

$g_{i,j-1}$: $-\frac{\dot{a}^{+}_{F,i,j-1}}{\Delta a}$

$g_{i,j+1}$: $\frac{\dot{a}^{-}_{B,i,j+1}}{\Delta a}$

$g_{i,j}$: $-\frac{s^{+}_{F,i,j}}{\Delta a} + \frac{s^{-}_{B,i,j}}{\Delta a} - \frac{\mu^{+}_{i}}{\Delta z} + \frac{\mu^{-}_{i}}{\Delta z} + \frac{-2\sigma^2_{i}}{2(\Delta z)^2}$

Due to boundary conditions,

$g_{1,j}$: $-\frac{s^{+}_{F,i,j}}{\Delta a} + \frac{s^{-}_{B,i,j}}{\Delta a} - \frac{\mu^{+}_{i}}{\Delta z} + \frac{-\sigma^2_{i}}{2(\Delta z)^2}$

$g_{nz,j}$: $-\frac{s^{+}_{F,i,j}}{\Delta a} + \frac{s^{-}_{B,i,j}}{\Delta a} + \frac{\mu^{-}_{i}}{\Delta z} + \frac{-\sigma^2_{i}}{2(\Delta z)^2}$

Holding times

Let x_t be a continuous time Markov chain. Define s_x to be the holding time at x:

$$x_0 = x, s_x = inf\{t \geq 0 : x_t \neq x\}$$

s_x has exponential distribution.

Memoryless property

Let T be a positive random variable. T has memoryless property:

$$p[T > t + s | T > s] = p[T > t]$$

if and only if T has exponential distribution.

Back

Solving Kolmogorov forward equation

Achdou et al. (2020) p33.

The deep underlying reason for this choice of discretization is that the KF equation actually is the "transpose" problem of the HJB equation. More precisely, the differential operator in the KF equation is the adjoint of the operator in the HJB equation, the "infinitesimal generator." Our transpose discretization of the KF equation is not only well-founded mathematically; it is also extremely convenient: having solved the HJB equation, the solution of the Kolmogorov Forward equation is essentially "for free."

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