Krusell and Smith (1998)

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Krusell and Smith (1998)

$$v(\varepsilon_0, a_0; z_0, g_0) = \max_{\{c_t\}_{t\geq 0}} \mathbb{E} \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\dot{a}_t = r_t a_t + w_t \varepsilon_t - c_t, \;\; a_t \geq \underline{a}$$

- Labor productivity ε follows a two-state Poisson process with λ_ε
- TFP z follows a two-state Poisson process with λ_z
- g is a density function

Derivation of HJB

Look at the problem at t, and focus on an interval between t and $t + dt$

Drop t to ease the notation, and let $x' = x_{t+dt}$

$\forall i, j$

$$e^{-\rho t} v(\varepsilon_i, a; z_j, g) = \max_c \mathbb{E} \int_t^{t+dt} e^{-\rho t} u(c_t) dt + \mathbb{E} e^{-\rho t + dt} v(\varepsilon', a'; z', g')$$

$$0 = \max_c \mathbb{E} \Big(\frac{1}{dt} \int_t^{t+dt} e^{-\rho t} u(c_t) dt + \frac{1}{dt} \big(e^{-\rho t + dt} v(\varepsilon', a'; z', g') - e^{-\rho t} v(\varepsilon_i, a; z, g) \big) \Big)$$

Thanks to the property of Poisson process, we can write

$$\mathbb{E} v(\varepsilon', a', z', g') = dt\lambda_z \big[dt\lambda_\varepsilon v(\varepsilon_{-i}, a', z_{-j}, g') + (1 - dt\lambda_\varepsilon) v(\varepsilon_i, a', z_{-j}, g') \big]$$
$$+ (1 - dt\lambda_z) \big[dt\lambda_\varepsilon v(\varepsilon_{-i}, a', z_j, g') + (1 - dt\lambda_\varepsilon) v(\varepsilon_i, a', z_j, g') \big]$$

Derivation of HJB

$$
\begin{aligned}
&\frac{1}{dt}\mathbb{E}\Big(e^{-\rho t+dt}v(\varepsilon', a'; z', g') - e^{-\rho t}v(\varepsilon_i, a; z, g))\Big) \\
&= \frac{1}{dt}\Big\{e^{-\rho t+dt}dt\lambda_z\big[dt\lambda_\varepsilon v(\varepsilon_{-i}, a', z_{-j}, g') + (1-dt\lambda_\varepsilon)v(\varepsilon_i, a', z_{-j}, g')\big] \\
&+ e^{-\rho t+dt}(1-dt\lambda_z)\big[dt\lambda_\varepsilon v(\varepsilon_{-i}, a', z_j, g') + (1-dt\lambda_\varepsilon)v(\varepsilon_i, a', z_j, g')\big] - e^{-\rho t}v(\varepsilon_i, a; z_j, g)\Big\} \\
&= \frac{1}{dt}\Big\{e^{-\rho t+dt}dt\lambda_z\big[dt\lambda_\varepsilon(v(\varepsilon_{-i}, a', z_{-j}, g') - v(\varepsilon_i, a', z_{-j}, g'))\big] \\
&+ e^{-\rho t+dt}(1-dt\lambda_z)\big[dt\lambda_\varepsilon(v(\varepsilon_{-i}, a', z_j, g') - v(\varepsilon_i, a', z_j, g'))\big] \\
&+ e^{-\rho t+dt}dt\lambda_z(v(\varepsilon_i, a', z_{-j}, g') - v(\varepsilon_i, a', z_j, g')) \\
&+ e^{-\rho t+dt}v(\varepsilon_i, a'; z_j, g') - e^{-\rho t}v(\varepsilon_i, a; z_j, g)\Big\}
\end{aligned}
$$

Derivation of HJB

crossed out dt where it's possible

$$
\begin{aligned}
&e^{-\rho t+dt}dt\lambda_z\left[\lambda_\varepsilon(v(\varepsilon_{-i},a',z_{-j},g')-v(\varepsilon_i,a',z_{-j},g'))\right]\\
&+e^{-\rho t+dt}(1-dt\lambda_z)\left[\lambda_\varepsilon(v(\varepsilon_{-i},a',z_j,g')-v(\varepsilon_i,a',z_j,g'))\right]\\
&+e^{-\rho t+dt}\lambda_z(v(\varepsilon_i,a',z_{-j},g')-v(\varepsilon_i,a',z_j,g'))\\
&+\frac{e^{-\rho t+dt}v(\varepsilon_i,a';z_j,g')-e^{-\rho t}v(\varepsilon_i,a;z_j,g)}{dt}
\end{aligned}
$$

$dt \to 0$

$$
\begin{aligned}
&\Longrightarrow 0\\
&\Longrightarrow e^{-\rho t}\lambda_\varepsilon[v(\varepsilon_{-i},a,z_j,g)-v(\varepsilon_i,a,z_j,g)]\\
&\Longrightarrow e^{-\rho t}\lambda_z[v(\varepsilon_i,a,z_{-j},g)-v(\varepsilon_i,a,z_j,g)]\\
&\Longrightarrow e^{-\rho t}[-\rho v(\varepsilon_i,a;z_j,g)+v_a(\varepsilon_i,a;z_j,g)\dot{a}\\
&\qquad +v_g(\varepsilon_i,a;z_j,g)\dot{g}]
\end{aligned}
$$

Derivation of HJB

$$0 = \max_c \mathbb{E}\Big(\frac{1}{dt}\int_t^{t+dt} e^{-\rho t}u(c_t)dt + \frac{1}{dt}\big(e^{-\rho t+dt}v(\varepsilon', a'; z', g') - e^{-\rho t}v(\varepsilon_i, a; z, g)\big)\Big)$$

$dt \to 0$

$$= \max_c e^{-\rho t}u(c) + e^{-\rho t}\Big(\lambda_\varepsilon[v(\varepsilon_{-i}, a, z_j, g) - v(\varepsilon_i, a, z_j, g)] + \lambda_z[v(\varepsilon_i, a, z_{-j}, g) - v(\varepsilon_i, a, z_j, g)]$$

$$- \rho v(\varepsilon_i, a; z_j, g) + v_a(\varepsilon_i, a; z_j, g)\dot{a} + v_g(\varepsilon_i, a; z_j, g)\dot{g}\Big)$$

rearrange

$$\rho v(\varepsilon_i, a; z_j, g) = \max_c u(c) + v_a(\varepsilon_i, a; z_j, g)\dot{a} + \lambda_\varepsilon[v(\varepsilon_{-i}, a, z_j, g) - v(\varepsilon_i, a, z_j, g)]$$

$$+ \lambda_z[v(\varepsilon_i, a, z_{-j}, g) - v(\varepsilon_i, a, z_j, g)] + v_g(\varepsilon_i, a; z_j, g)\dot{g}$$

HJB equation

$$\rho v(\varepsilon_i, a; z_j, g) = \max_c u(c) + v_a(\varepsilon_i, a; z_j, g)\dot{s}(\cdot) + \lambda_\varepsilon[v(\varepsilon_{-i}, a, z_j, g) - v(\varepsilon_i, a, z_j, g)]$$
$$+ \lambda_z[v(\varepsilon_i, a, z_{-j}, g) - v(\varepsilon_i, a, z_j, g)] + \sum_{i=1}^{n\varepsilon} \int \frac{\partial v(\varepsilon_i, a; z_j, g)}{\partial g(\varepsilon_i, a)} \mathcal{K}_z g(\varepsilon_i, a) da$$

$$s(\varepsilon, a; z, g) = r(z, g)a + w(z, g)\varepsilon - c, \quad a \geq \underline{a}$$
$$\mathcal{K}_z g(\varepsilon, a) := \frac{d}{da}[s(\varepsilon, a; z, g)g(\varepsilon, a)] - \lambda_\varepsilon g(\varepsilon, a) + \lambda_\varepsilon g(\varepsilon', a)$$

- Problem: g is infinite-dimensional object

g approximation

- Krusell and Smith (1998) approximate g with finite set of moments of a distribution
  $\Longrightarrow$ replace the infinite g with the first moment, $k = \sum_{\varepsilon} \int_{\underline{a}}^{\infty} a g(\varepsilon, a) da$
- Same idea applies here

$$\rho v(\varepsilon_i, a; z_j, k) = \max_c u(c) + v_a(\varepsilon_i, a; z_j, k)\dot{s}(\cdot) + \lambda_\varepsilon[v(\varepsilon_{-i}, a, z_j, k) - v(\varepsilon_i, a, z_j, k)]$$
$$+ \lambda_z[v(\varepsilon_i, a, z_{-j}, k) - v(\varepsilon_i, a, z_j, k)] + v_k(\varepsilon_i, a, z_j, k)\dot{k}$$

$$s(\varepsilon, a; z, g) = r(z, g)a + w(z, g)\varepsilon - c, \quad a \geq \underline{a}$$
$$\dot{k} = h(k, z)$$

Numerical solution

Solve on $\mathcal{E} \times \mathbf{A} \times \mathbf{Z} \times \mathbf{K}$
Guess $h(k, z)$, compute $\dot{k}(k, z)$

Discretized the HJB equation. $v_{i,j,m,n} = v(\varepsilon_i, a_j; z_m, k_n)$

$$\rho v_{i,j,m,n} = \max_c u(c_{i,j,m,n}) + v_{a,i,j,m,n}(r_{m,n}a_j + w_{m,n}\varepsilon_i - c_{i,j,m,n}) + \lambda_\varepsilon\big(v_{-i,j,m,n} - v_{i,j,m,n}\big)$$
$$+ \lambda_z\big(v_{i,j,-m,n} - v_{i,j,m,n}\big) + v_{k,i,j,m,n}\dot{k}_{m,n}$$

- From FOC, $u'(c_{i,j,m,n}) - v_{a,i,j,m,n} = 0 \implies c_{i,j,m,n} = u^{-1}\big(v_{a,i,j,m,n}\big)$
- Choose between $v^F_{a,i,j,m,n}$ and $v^B_{a,i,j,m,n}$ following upwind scheme
- Choose between $v^F_{k,i,j,m,n}$ and $v^B_{k,i,j,m,n}$ following upwind scheme

Numerical solution

Boundaries

- **A**: At a_1, set $v_a^B(a_1, \cdot) = u'(ra_1 + w\varepsilon_i)$. Choose high enough a_{na}.
- **K**: Choose low enough k_1 and high enough k_{nk}.
- **Z**, $\mathcal{E}$: No need to consider

Numerical Solution

Let $\dot{x}^+ = max(\dot{x}, 0)$, $\dot{x}^- = min(\dot{x}, 0)$

$$\rho v_{i,j,m,n} = \max_c u(c_{i,j,m,n}) + \frac{v_{i,j+1,m,n} - v_{i,j,m,n}}{\Delta a}\dot{a}^+_{F,i,j,m,n} + \frac{v_{i,j,m,n} - v_{i,j-1,m,n}}{\Delta a}\dot{a}^-_{B,i,j,m,n}$$
$$+ \lambda_\varepsilon \left(v_{-i,j,m,n} - v_{i,j,m,n}\right) + \lambda_z \left(v_{i,j,-m,n} - v_{i,j,m,n}\right)$$
$$+ \frac{v_{i,j,m,n+1} - v_{i,j,m,n}}{\Delta k}\dot{k}^+_{m,n} + \frac{v_{i,j,m,n} - v_{i,j,m,n-1}}{\Delta k}\dot{k}^-_{m,n}$$

Collecting coefficients
at (z_m, k_n)

$v_{-i,j}$: λ_ε

$v_{i,j}$: $-\frac{\dot{a}^+_{F,i,j}}{\Delta a} + \frac{\dot{a}^-_{B,i,j}}{\Delta a} - \lambda_\varepsilon = d^0_{ij}$

$v_{i,j-1}$: $-\frac{\dot{a}^-_{B,i,j}}{\Delta a} = d^{a1}_{ij}$

$v_{i,j+1}$: $\frac{\dot{a}^+_{F,i,j}}{\Delta a} = d^{a2}_{ij}$

$v_{i,j,-m,n}$: λ_z

$v_{i,j,m,n}$: $-\frac{\dot{k}^+_{m,n}}{\Delta k} + \frac{\dot{k}^-_{m,n}}{\Delta k} - \lambda_z = x^{k0}_{mn} - \lambda_z$

$v_{i,j,m,n-1}$: $-\frac{\dot{k}^-_{m,n}}{\Delta k} = x^{k1}_{mn}$

$v_{i,j,m,n+1}$: $\frac{\dot{k}^+_{mn}}{\Delta k} = x^{k2}_{mn}$

Matrix representation: saving and income shock

For example, $n_\varepsilon = 2$ and $n_a = 4$. Given (z_m, k_n),

$$\mathbf{Aa}_{mn} = \left[\begin{array}{cccc|cccc}
d_{11}^{0} & d_{11}^{a2} & 0 & 0 & \lambda_\varepsilon & 0 & 0 & 0 \\
d_{12}^{a1} & d_{12}^{0} & d_{12}^{a2} & 0 & 0 & \lambda_\varepsilon & 0 & 0 \\
0 & d_{13}^{a1} & d_{13}^{0} & d_{13}^{a2} & 0 & 0 & \lambda_\varepsilon & 0 \\
0 & 0 & d_{14}^{a1} & d_{14}^{0} & 0 & 0 & 0 & \lambda_\varepsilon \\
\hline
\lambda_\varepsilon & 0 & 0 & 0 & d_{21}^{0} & d_{21}^{a2} & 0 & 0 \\
0 & \lambda_\varepsilon & 0 & 0 & d_{22}^{a1} & d_{22}^{0} & d_{22}^{a2} & 0 \\
0 & 0 & \lambda_\varepsilon & 0 & 0 & d_{23}^{a1} & d_{23}^{0} & d_{23}^{a2} \\
0 & 0 & 0 & \lambda_\varepsilon & 0 & 0 & d_{24}^{a1} & d_{24}^{0}
\end{array}\right]$$

Matrix representation: saving and income shock

Suppose $n_z = 2$ and $n_k = 3$

$$\mathbf{Aa} = \left[\begin{array}{c|c|c|c|c|c} \mathbf{Aa}_{11} & & & & & \\ \hline & \mathbf{Aa}_{12} & & & & \\ \hline & & \mathbf{Aa}_{13} & & & \\ \hline & & & \mathbf{Aa}_{21} & & \\ \hline & & & & \mathbf{Aa}_{22} & \\ \hline & & & & & \mathbf{Aa}_{23} \end{array}\right]$$

- $\mathbf{Aa}_{mn}$ is $n_\varepsilon n_a \times n_\varepsilon n_a$
- I ordered k first then z

Matrix representation: z shock

$$
\mathbf{Az} = \left[\begin{array}{c|c|c|c|c|c}
-\lambda_z \mathbf{I} & & & \lambda_z \mathbf{I} & & \\ \hline
 & -\lambda_z \mathbf{I} & & & \lambda_z \mathbf{I} & \\ \hline
 & & -\lambda_z \mathbf{I} & & & \lambda_z \mathbf{I} \\ \hline
\lambda_z \mathbf{I} & & & -\lambda_z \mathbf{I} & & \\ \hline
 & \lambda_z \mathbf{I} & & & -\lambda_z \mathbf{I} & \\ \hline
 & & \lambda_z \mathbf{I} & & & -\lambda_z \mathbf{I}
\end{array}\right]
$$

- $\mathbf{I}$ is $n_\varepsilon n_a \times n_\varepsilon n_a$ identity matrix

Matrix representation: $\dot{k}$

$$
\mathbf{Ak} = \left[\begin{array}{c|c|c|c|c|c}
x_{11}^{k0}\mathbf{I} & x_{11}^{k2}\mathbf{I} & & & & \\
\hline
x_{12}^{k1}\mathbf{I} & x_{12}^{k0}\mathbf{I} & x_{12}^{k2}\mathbf{I} & & & \\
\hline
 & x_{13}^{k1}\mathbf{I} & x_{13}^{k0}\mathbf{I} & & & \\
\hline
 & & & x_{21}^{k0}\mathbf{I} & x_{21}^{k2}\mathbf{I} & \\
\hline
 & & & x_{22}^{k1}\mathbf{I} & x_{22}^{k0}\mathbf{I} & x_{22}^{k2}\mathbf{I} \\
\hline
 & & & & x_{23}^{k1}\mathbf{I} & x_{23}^{k0}\mathbf{I}
\end{array}\right]
$$

- **I** is $n_\varepsilon n_a \times n_\varepsilon n_a$ identity matrix

Matrix representation

- After constructing **Aa**,**Az** and **Ak**, we can compactly write the system of equations

$$\mathbf{AA} = \mathbf{Aa} + \mathbf{Az} + \mathbf{Ak}$$
$$\rho \mathbf{v} = \mathbf{u} + \mathbf{AAv}$$

- Solve the system of equations using explicit/implicit updating
- Save $s(\varepsilon, a; z, k)$ for simulation

Update $h(k, z)$

Once solve value function, simulate the model and update the forecasting function $h(k, z)$

- Guess an initial density g_0
- Select simulation period t and time interval dt. (simulation length is t/dt)
- Simulate a z series
  - Find a transition probability matrix. If a shock intensity is λ, a duration of not getting a shock is $\frac{1}{\lambda dt}$ (in terms of simulation time interval); a probability of switching state is λdt.
    $$\mathbf{P} = \begin{bmatrix} 1-\lambda dt & \lambda dt \\ \lambda dt & 1-\lambda dt \end{bmatrix}$$

Update $h(k, z)$

Once solve value function, simulate the model and update the forecasting function $h(k, z)$

- Simulate k series by solving KFE:
  (g is over $\mathcal{E} \times$ **A**. (z, k) affects $s(\varepsilon, a; t)$)

$$\frac{\partial g(\varepsilon, a; t)}{\partial t} = -\frac{d}{da}[s(\varepsilon, a; t)g(\varepsilon, a; t)] + \lambda(g(\varepsilon', a; t) - g(\varepsilon, a; t))$$

- Given g_0, can compute $k(1) = \sum_\varepsilon \int_{\underline{a}}^{\infty} ag(\varepsilon, a; 1)da$
- Given (z_1, k_1), compute $s(\varepsilon, a; 1)$ using interpolation. Construct $\mathbf{Aa}_1$ with $s(\varepsilon, a; 1)$
  - k_1 may not be on **K**

Update $h(k, z)$

- Simulate a k series by solving KFE:

$$\frac{\partial g(\varepsilon, a; t)}{\partial t} = -\frac{d}{da}[s(\varepsilon, a; t)g(\varepsilon, a; t)] + \lambda(g(\varepsilon', a; t) - g(\varepsilon, a; t))$$

- Approximate $\frac{\partial g}{\partial t}$, $\frac{d}{da}g$ with numerical derivatives and apply upwind scheme. Given (z_t, k_t),

$$\frac{g_{i,j}(t+\Delta t) - g_{i,j}(t)}{\Delta t} = -\frac{s^+_{F,i,j}g_{i,j} - s^+_{F,i,j-1}g_{i,j-1}}{\Delta a} - \frac{s^-_{B,i,j+1}g_{i,j+1} - s^-_{B,i,j}g_{i,j}}{\Delta a} - \lambda_i g_{i,j} + \lambda_{-i}g_{-i,j}$$

$$\implies \frac{\mathbf{g}_{t+\Delta t} - \mathbf{g}_t}{\Delta t} = \mathbf{Aa}_t^T \mathbf{g}_t$$

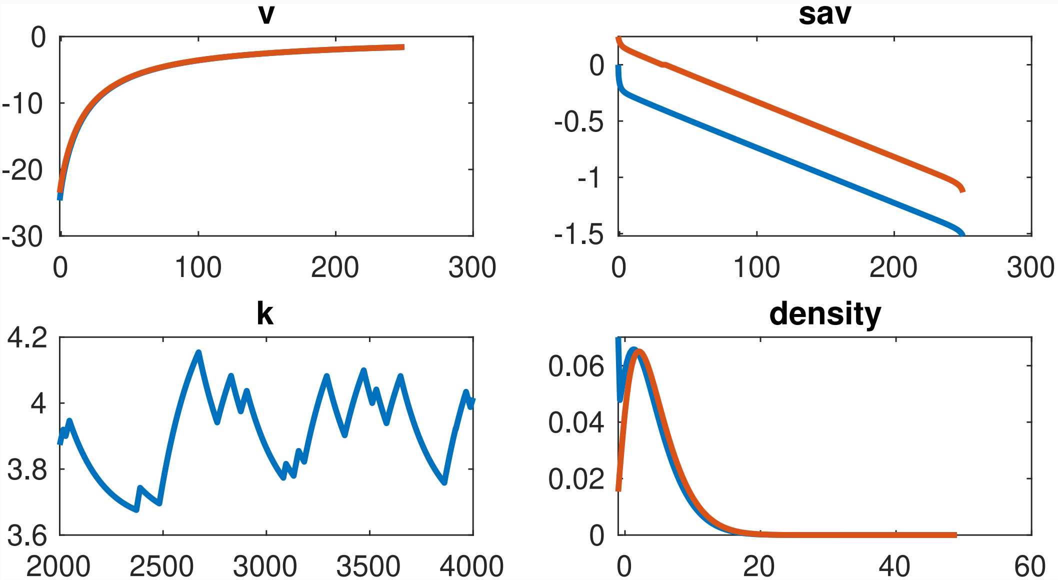
$$\implies \mathbf{g}_{t+\Delta t} = (\mathbf{I} + \Delta t \mathbf{Aa}_t^T)\mathbf{g}_t$$

Update $h(k, z)$

- Once compute $\mathbf{g}_{t+\Delta t}$, can compute $k(t+\Delta)$ and can simulate a k series by repeatedly constructing $\mathbf{Aa}_t$ and solving KFE
- Assume functional form and update $h(k, z)$ using the simluated k

$$log(k_{t+dt}) = \beta_0^z + \beta_1^z log(k_t) + e_t$$

$$h(k, z) = \frac{k_{t+dt} - k_t}{dt}$$


v
0
-10
-20
-30
0
100
200
300
sav
0
-0.5
-1
-1.5
0
100
200
300
k
4.2
4
3.8
3.6
2000
2500
3000
3500
4000
density
0.06
0.04
0.02
0
0
20
40
60

Mean Field Game

- Games with a very large number of agents interacting in a mean field manner
  - Each agent has a very small impact on the outcome
  - The term mean-field refers to the fact that the strategy of each player is affected only by the average density (mean-field) of the other players
- As a result, the game can be analyzed in the limit of an infinite number of agents
- One approaches to the formulation and the analysis of MFGs is based on the solution of a coupled system of PDEs: HJB and Fokker-Planck

Stationary MFG

For example, Huggett (1993)

$$\rho v(z,a) = \max_c u(c) + v_a(z,a)s(z,a) + \lambda(v(z',a) - v(z,a)) \qquad \text{(HJB)}$$

$$s(z,a) = ra + z - c, \quad a \geq \underline{a}$$

$$0 = -\frac{d}{da}[s(z,a)g(z,a;t)] + \lambda(g(z',a;t) - g(z,a;t)) \qquad \text{(Fokker-Planck)}$$

$$0 = \sum_{i=1}^{nz} \int_{\underline{a}}^{\infty} a g(z_i,a)da, \qquad 1 = \sum_{i=1}^{nz} \int_{\underline{a}}^{\infty} g(z_i,a)da \qquad \text{(EQ. condition)}$$

- g is transported by the $s(z,a)$ which is determined by solving the HJB
- v depends on g through the price r
- There exist unique solution to stationary MFG

Time-dependent MFG

In other words, MFGs with common noise (= aggregate uncertainty)

From ‘Partial differential equation models in macroeconomics’, Achdou et al. (2014)
These things are unknown:

1. Existence and uniqueness of solutions
2. A theoretical understanding of the behaviour of g.
   - For example, are there certain regions of the space of density functions S in which g lives ‘most of the time’?
3. Development of efficient and robust approximation schemes and results regarding their convergence