Stochastic Growth Model

Soyoung Lee

July 10, 2020

Ohio State University

Stochastic Growth Model

$$v(z_0, k_0, 0) = \max_{\{c_t\}_{t\geq 0}} \mathbb{E} \int_0^{\infty} e^{-\rho t} u(c(t)) dt$$

$$\begin{aligned} \dot{k} &= (zf(k) - \delta k - c) \\ dz_t &= \tilde{\mu}_t dt + \tilde{\sigma}_t dw_t, \quad k_0 \text{ and } z_0 \text{ are given} \end{aligned}$$

- w_t follows standard Brownian motion

HJB equation

The problem has two state variables. We can use bivariate version of Ito's lemma where

$$dx_{1,t} = a_1 dt + \sigma_{11} dz_{1,t} + \sigma_{12} dz_{2,t} : dk_t = (zf(k) - \delta k - c)dt$$
$$dx_{2,t} = a_2 dt + \sigma_{21} dz_{1,t} + \sigma_{22} dz_{2,t} : dz_t = \tilde{\mu}_t dt + \tilde{\sigma}_t dw_t$$

Notice $\sigma_{11} = \sigma_{12} = \sigma_{21} = 0$, $a_1 = (zf(k) - \delta k - c)$, $a_2 = \tilde{\mu}_t$, and $\sigma_{22} = \tilde{\sigma}_t$
According to Ito's lemma,

$$\begin{aligned} df(t, x_1, x_2) =& \Big(f_t + f_1 a_1 + f_2 a_2 + \frac{1}{2}\Big(f_{x_1 x_1}(\sigma_{11}^2 + \sigma_{12}^2 + \rho_{12}\sigma_{11}\sigma_{12}) + f_{x_2 x_2}(\sigma_{21}^2 + \sigma_{22}^2 + \rho_{12}\sigma_{21}\sigma_{22}) \Big) \\ & + f_{x_1 x_2}(\sigma_{11}\sigma_{21} + \rho_{12}\sigma_{11}\sigma_{22} + \rho_{12}\sigma_{12}\sigma_{21} + \sigma_{12}\sigma_{22}) \Big) dt \\ & + (f_1\sigma_{11} + f_2\sigma_{21}) dz_{1,t} + (f_1\sigma_{12} + f_2\sigma_{22}) dz_{2,t} \\ \implies dv(t, z, k) =& \Big(v_t + v_k(zf(k) - \delta k - c) + v_z\tilde{\mu}_t + \frac{1}{2} v_{zz}\tilde{\sigma}_t^2 \Big) dt + (v_z\tilde{\sigma}_t) dw \end{aligned}$$

HJB equation

Rewrite the problem

$$v(z_t, k_t, t) = \max_{c_s} E\left(\int_t^{t+dt} e^{-\rho s} u(c_s) ds + v(z_{t+dt}, k_{t+dt}, t+dt) \right)$$

rearrange and divide by dt

$$0 = \max_{c_s} E\left(\frac{1}{dt} \int_t^{t+dt} e^{-\rho s} u(c_s) ds + \frac{1}{dt}\big(v(z_{t+dt}, k_{t+dt}, t+dt) - v(z_t, k_t, t) \big) \right)$$

using $v(x) \equiv v(x, 0)$, $v(x,t) = e^{-\rho t} v(x)$ $v_t(x,t) = -\rho e^{-\rho t} v(x)$ and apply Ito's lemma,

$$0 = \max_{c_t} e^{-\rho t} u(c_t) +$$

$$\frac{1}{dt} E\left(\left(-\rho e^{-\rho t} v + e^{-\rho t} v_k (z f(k) - \delta k - c) + e^{-\rho t} v_z \tilde{\mu}(z) + e^{-\rho t} \frac{1}{2} v_{zz} \tilde{\sigma}(z)^2 \right) dt + e^{-\rho t} v_z \tilde{\sigma}(z) dw \right)$$

divide by $e^{-\rho t}$, $E(dw) = 0$

$$\implies \rho v(z,k) = \max_c u(c) + v_k(z,k)(z f(k) - \delta k - c) + v_z(z,k) \tilde{\mu}(z) + \frac{1}{2} v_{zz}(z,k) \tilde{\sigma}(z)^2$$

Numerical Solution: Z

Solve on discretized grids, **Z** $= \{z_1, ..., z_{n_z}\}$, **K** $= \{k_1, ..., k_{n_k}\}$ using FDM

First, need to discretize **Z**

- Assume that $log(z)$ follows Ornstein–Uhlenbeck process.
  - Let $x = log(z)$, $dx_t = \theta(\mu - x_t)dt + \sigma dw_t$
- Approximately, $x_t \sim N(\mu, \frac{\sigma^2}{2\theta})$. $[\mu - 2\sqrt{\frac{\sigma^2}{2\theta}}, \mu + 2\sqrt{\frac{\sigma^2}{2\theta}}]$ covers 95% of x, $[e^{\mu}/(e^{\sqrt{\frac{\sigma^2}{2\theta}}})^2, e^{\mu} * (e^{\sqrt{\frac{\sigma^2}{2\theta}}})^2]$ covers 95% of z. Choose z_{min} and z_{max} accordingly
- We assumed that z_t is a solution of $dz_t = \tilde{\mu}_t dt + \tilde{\sigma}_t dw_t$. $\tilde{\mu}_t$ and $\tilde{\sigma}_t$ shows up in HJB equation and we need to infer these terms from the Ornstein-Uhlenbeck Process

Numerical Solution: Z

First, need to discretize **Z** (continued)

- $x = log(z)$, $dx_t = \theta(\mu - x_t)dt + \sigma dw_t$
  - We want to know drift and diffusion of z_t, not those of $log(z_t)$.
    $\rightarrow$ Let $f(x_t) = e^{x_t}$. Since $z_t = e^{x_t} = f(x_t)$, computing drift and diffusion of $df(x_t)$ is computing those of dz_t
  - From Ito's Lemma,

$$df(t, x_t) = \Big(\frac{\partial f(t, x_t)}{\partial t} + \frac{\partial f(t, x_t)}{\partial x_t}\mu_t + \frac{1}{2}\frac{\partial^2 f(t, x_t)}{\partial x_t^2}{\sigma_t}^2\Big)dt + \frac{\partial f(t, x_t)}{\partial x_t}\sigma_t dz_t$$

$$= \underbrace{(e^{x_t}\theta(\mu - x_t) + \frac{1}{2}e^{x_t}\sigma_t^2)}_{\tilde{\mu}(z)} dt + \underbrace{e^{x_t}\sigma_t}_{\tilde{\sigma}(z)} dt$$

- $\tilde{\mu}(z) = (\theta(\mu - log(z)) + \frac{\sigma^2}{2})z, \quad \tilde{\sigma}(z) = \sigma z$

Numerical Solution

- Discretize **K**
- The number of grid points $n_z \times n_k$. Our method will involve solving decision rules at each $(z_i, k_j), i = 1, ..., n_z$ and $j = 1, ..., n_k$.
- Shorthand notation: $v_{i,j} = v(z_i, k_j)$
- Approximate

$$v_z(z_i, k_j) \approx \frac{v_{i+1,j} - v_{i,j}}{\Delta z} \text{ or } \frac{v_{i,j} - v_{i-1,j}}{\Delta z}, \quad v_k(z_i, k_j) \approx \frac{v_{i,j+1} - v_{i,j}}{\Delta k} \text{ or } \frac{v_{i,j} - v_{i,j-1}}{\Delta k} \quad \text{(F or B)}$$

$$v_{zz}(z_i, k_j) \approx \frac{\frac{v_{i+1,j} - v_{i,j}}{\Delta z} - \frac{v_{i,j} - v_{i-1,j}}{\Delta z}}{\Delta z} = \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{(\Delta z)^2} \quad \text{(central)}$$

Why use ‘central’ for v_{zz}?

Numerical Solution: Boundary conditions

Discretized the HJB equation

$$\rho v_{i,j} = \max_c u(c_{i,j}) + v_k(z_i, k_j)(z_i f(k_j) - \delta k_j - c_{i,j}) + v_z(z_i, k_j)\tilde{\mu}_i + \frac{1}{2} v_{zz}(z_i, k_j)\tilde{\sigma}_i^2$$

From FOC, $u'(c_{i,j}) - v_k(z_i, k_j) = 0 \implies c_{i,j} = u^{-1}\big(v_k(z_i, k_j)\big)$

Choose between $v^F_{k,i,j}$ and $v^B_{k,i,j}$ and between $v^F_{z,i,j}$ and $v^B_{z,i,j}$ following upwind scheme

Boundaries

- **K**: At k_1, there is no backward difference. Set low enough k_1 so that backward difference never be selected at k_1
- **Z**: Assume $v^B_z(z_1, k) = 0$ and $v^F_z(z_{n_z}, k) = 0$ for all k

Numerical Solution

Let $\dot{x}^+ = max(\dot{x}, 0)$, $\dot{x}^- = min(\dot{x}, 0)$

$$\rho v_{i,j} = \max_c u(c_{i,j}) + \frac{v_{i,j+1} - v_{i,j}}{\Delta k}\dot{k}^+_{F,i,j} + \frac{v_{i,j} - v_{i,j-1}}{\Delta k}\dot{k}^-_{B,i,j} + \frac{v_{i+1,j} - v_{i,j}}{\Delta z}\tilde{\mu}^+_i + \frac{v_{i,j} - v_{i-1,j}}{\Delta z}\tilde{\mu}^-_i + \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{2(\Delta z)^2}\tilde{\sigma}^2_i$$

Collecting coefficients of $v_{i-1,j}$, $v_{i,j}$, $v_{i+1,j}$, $v_{i,j-1}$, $v_{i,j+1}$

$v_{i-1,j}$: $-\frac{\tilde{\mu}^-_i}{\Delta z} + \frac{\tilde{\sigma}^2_i}{2(\Delta z)^2} = d^{z1}_{ij}$, $v_{i+1,j}$: $\frac{\tilde{\mu}^+_i}{\Delta z} + \frac{\tilde{\sigma}^2_i}{2(\Delta z)^2} = d^{z2}_{ij}$

$$v_{i,j}: \underbrace{-\frac{\dot{k}^+_{F,i,j}}{\Delta k} + \frac{\dot{k}^-_{B,i,j}}{\Delta k}}_{d^{k0}_{ij}} \underbrace{-\frac{\tilde{\mu}^+_i}{\Delta z} + \frac{\tilde{\mu}^-_i}{\Delta z} - \frac{2\tilde{\sigma}^2_i}{2(\Delta z)^2}}_{d^{z0}_{ij}} = d^{k0}_{ij} + d^{z0}_{ij} = d^0_{ij}$$

$v_{i,j-1}$: $-\frac{\dot{k}^-_{B,i,j}}{\Delta k} = d^{k1}_{ij}$, $v_{i,j+1}$: $\frac{\dot{k}^+_{F,i,j}}{\Delta k} = d^{k2}_{ij}$ Barles-Souganidis

Numerical Solution

Due to **Z** boundary condition, $v_z^B(z_1,k) = \frac{v_1 - v_0}{\Delta z} = 0$ and $v_z^F(z_{n_z},k) = \frac{v_{n_z+1} - v_{n_z}}{\Delta z} = 0 \; \forall \; k$

At z_1,

$$\rho v_{1,j} = \max_c u(c_{1,j}) + \frac{v_{1,j+1} - v_{1,j}}{\Delta k}\dot{k}^+_{F,1,j} + \frac{v_{1,j} - v_{1,j-1}}{\Delta k}\dot{k}^-_{B,1,j} + \frac{v_{2,j} - v_{1,j}}{\Delta z}\tilde{\mu}_1^+ + \frac{v_{2,j} - v_{1,j}}{2(\Delta z)^2}\tilde{\sigma}_1^2$$

$$d^0_{1j} = -\frac{\dot{k}^+_{F,1,j}}{\Delta k} + \frac{\dot{k}^-_{B,1,j}}{\Delta k} - \frac{\tilde{\mu}_1^+}{\Delta z} - \frac{\tilde{\sigma}_1^2}{2(\Delta z)^2}$$

At z_{nz},

$$\rho v_{nz,j} = \max_c u(c_{nz,j}) + \frac{v_{nz,j+1} - v_{nz,j}}{\Delta k}\dot{k}^+_{F,nz,j} + \frac{v_{nz,j} - v_{nz,j-1}}{\Delta k}\dot{k}^-_{B,nz,j} + \frac{v_{nz,j} - v_{nz-1,j}}{\Delta z}\tilde{\mu}^-_{nz} - \frac{v_{nz,j} - v_{nz-1,j}}{2(\Delta z)^2}\tilde{\sigma}^2_{nz}$$

$$d^0_{nz,j} = -\frac{\dot{k}^+_{F,nz,j}}{\Delta k} + \frac{\dot{k}^-_{B,nz,j}}{\Delta k} + \frac{\tilde{\mu}^-_{nz}}{\Delta z} - \frac{\tilde{\sigma}^2_{nz}}{2(\Delta z)^2}$$

For example, $n_z = 3$ and $n_k = 4$

$$
\rho \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \\ v_{31} \\ v_{32} \\ v_{33} \\ v_{34} \end{bmatrix}
= \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{34} \\ u_{31} \\ u_{32} \\ u_{33} \\ u_{34} \end{bmatrix}
+ \left[\begin{array}{cccc|cccc|cccc}
d_{11}^{0} & d_{11}^{k1} & 0 & 0 & d_{11}^{z2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_{12}^{k1} & d_{12}^{0} & d_{12}^{k2} & 0 & 0 & d_{12}^{z2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_{13}^{k1} & d_{13}^{0} & d_{13}^{k2} & 0 & 0 & d_{13}^{z2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{14}^{k1} & d_{14}^{0} & 0 & 0 & 0 & d_{14}^{z2} & 0 & 0 & 0 & 0 \\
\hline
d_{21}^{z1} & 0 & 0 & 0 & d_{21}^{0} & d_{21}^{k2} & 0 & 0 & d_{21}^{z2} & 0 & 0 & 0 \\
0 & d_{22}^{z1} & 0 & 0 & d_{22}^{k1} & d_{22}^{0} & d_{22}^{k2} & 0 & 0 & d_{22}^{z2} & 0 & 0 \\
0 & 0 & d_{23}^{z1} & 0 & 0 & d_{23}^{k1} & d_{23}^{0} & d_{23}^{k2} & 0 & 0 & d_{23}^{z2} & 0 \\
0 & 0 & 0 & d_{24}^{z1} & 0 & 0 & d_{24}^{k1} & d_{24}^{0} & 0 & 0 & 0 & d_{24}^{z2} \\
\hline
0 & 0 & 0 & 0 & d_{31}^{z1} & 0 & 0 & 0 & d_{31}^{0} & d_{31}^{k2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_{32}^{z1} & 0 & 0 & d_{32}^{z1} & d_{32}^{0} & d_{32}^{k2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_{33}^{z1} & 0 & 0 & d_{33}^{k1} & d_{33}^{0} & d_{33}^{k2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{34}^{z1} & 0 & 0 & d_{34}^{k1} & d_{34}^{0}
\end{array}\right]
\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \\ v_{31} \\ v_{32} \\ v_{33} \\ v_{34} \end{bmatrix}
$$

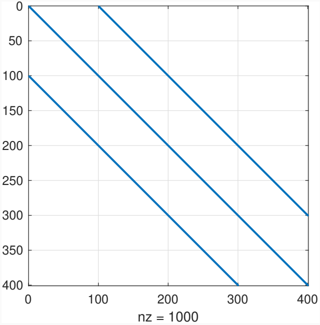
- Compare to the deterministic case, there are additional non-zero terms: d_{ij}^{z1} and d_{ij}^{z2} due to z process

Sparse matrix Az: z process

- Terms that are related to drift and diffusion of dz ($d_{ij}^{z0}, d_{ij}^{z1}, d_{ij}^{z2}$) do not change over iterations
- $nz = 4$, $nk = 100$

Making `sparse` matrix in Matlab

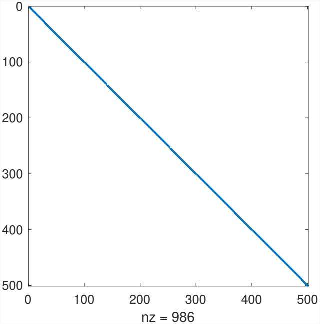
```
Az = sparse(nz*nk,nz*nk);

% when iz = 1
i1 = 1; i2 = nk;
Az(i1:i2,i1:i2) = spdiags(ones(nk,1).*dz0(1),0,nk,nk);

i3 = nk+1; i4 = 2*nk;
Az(i1:i2,i3:i4) = spdiags(ones(nk,1).*dz2(1),0,nk,nk);
```



Sparse matrix Ak: $\dot{k}$

- Terms that are related to drift of k ($d_{ij}^{k0}, d_{ij}^{k1}, d_{ij}^{k2}$) change over iterations

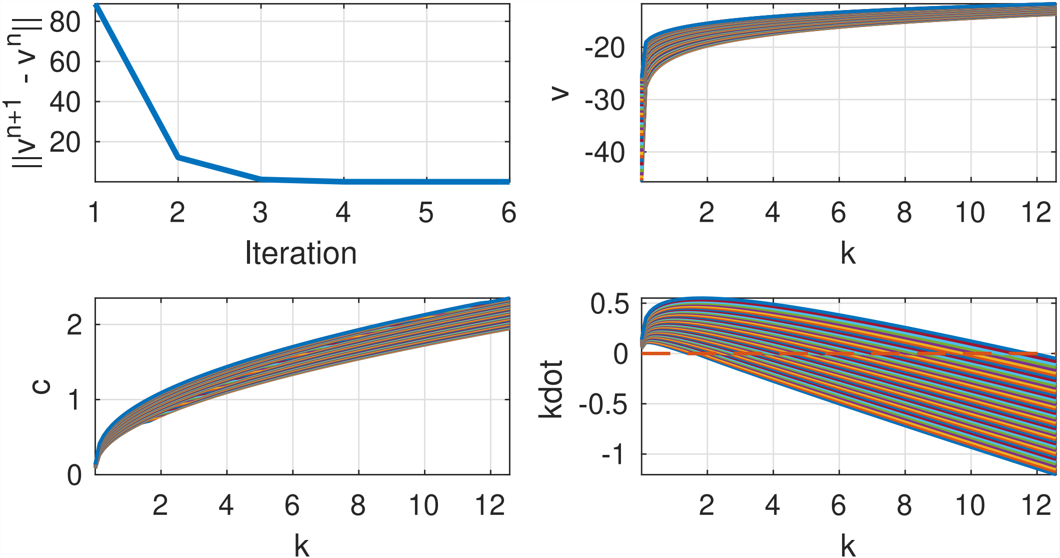
```
Ak = sparse(nz*nk,nz*nk);

for i = 1:nz

i1 = (i-1)*nk + 1; i2 = i*nk;
Ak(i1:i2,i1:i2) = spdiags(dk0(:,i),0,nk,nk)
+spdiags([0;dk1(1:nk-1,i)],1,nk,nk)
+spdiags([dk2(2:nk,i);0],-1,nk,nk);

end
```



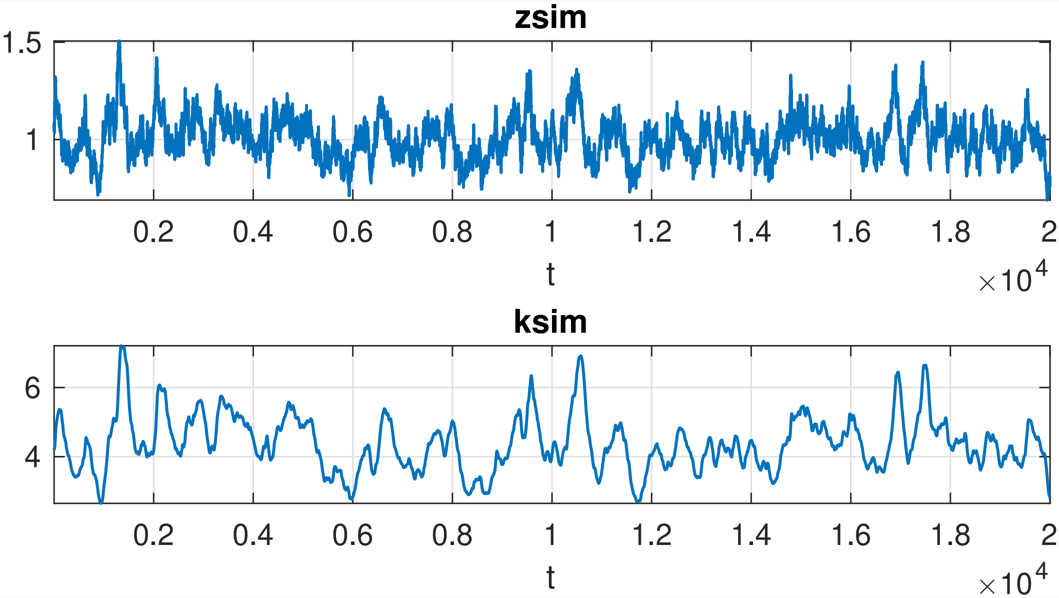
Numerical Solution

Once construct **A = Az + Ak** matrix, solve value function using explicit/implicit updating

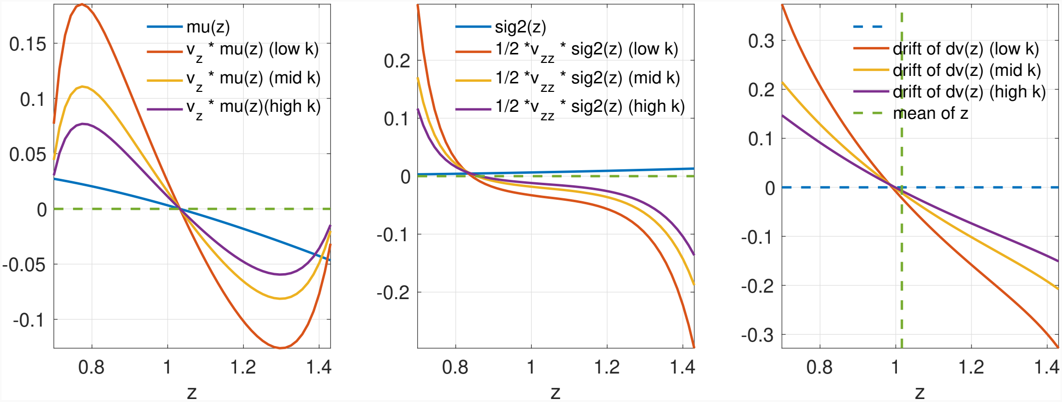
- **Explicit** $\frac{\mathbf{v}^{n+1}-\mathbf{v}^{n}}{\Delta} + \rho \mathbf{v}^{n} = \mathbf{u}^{n} + \mathbf{A}^{n}\mathbf{v}^{n}$
- **Implicit** $\frac{\mathbf{v}^{n+1}-\mathbf{v}^{n}}{\Delta} + \rho \mathbf{v}^{n+1} = \mathbf{u}^{n} + \mathbf{A}^{n}\mathbf{v}^{n+1}$



||v^{n+1} - v^n||
80
60
40
20
1
2
3
4
5
6
Iteration
v
-20
-30
-40
2
4
6
8
10
12
k
c
2
1
0
2
4
6
8
10
12
k
kdot
0.5
0
-0.5
-1
2
4
6
8
10
12
k


zsim
1.5
1
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
t
$\times 10^4$
ksim
6
4
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
t
$\times 10^4$

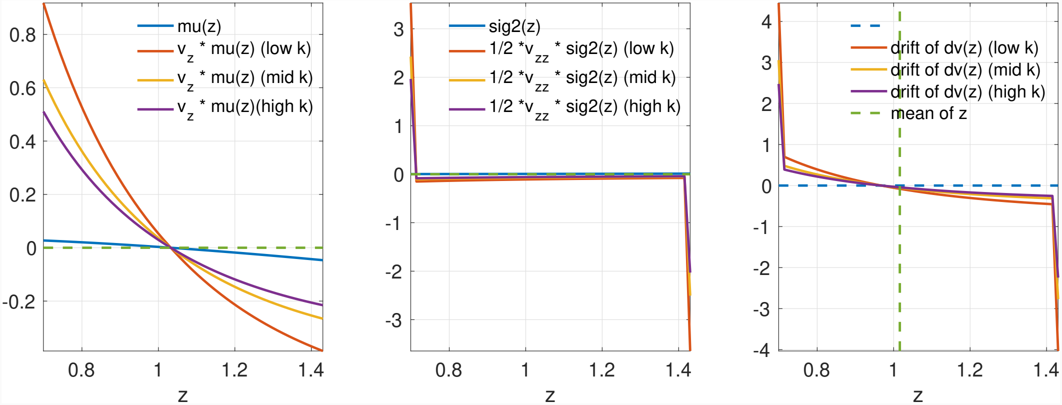
$\tilde{\mu}(z)$ and $\tilde{\sigma}(z)$



$$\rho v(z,k) = \max_c u(c) + v_k(z,k)(zf(k) - \delta k - c) + v_z(z,k)\tilde{\mu}(z) + \frac{1}{2}v_{zz}(z,k)\tilde{\sigma}(z)^2$$

drift with initial guess of value

$\tilde{\mu}(z)$ and $\tilde{\sigma}(z)$ using initial guess of value



$$\rho v(z,k) = \max_c u(c) + v_k(z,k)(zf(k) - \delta k - c) + v_z(z,k)\tilde{\mu}(z) + \frac{1}{2}v_{zz}(z,k)\tilde{\sigma}(z)^2$$

Back

Barles-Souganidis

We use unwind scheme to satisfy one of the condition that is required to apply Barles-Souganidis Theorem
('Convergence of Approximation Schemes For Fully Nonlinear Second Order Equation', Barles and Souganidis, 1990)

Barles-Souganidis

If the scheme satisfies the monotonicity, consistency and stability conditions, then as $\Delta k \to 0$ its solution $v_i, i = 1, ..., n$ converges locally uniformly to the unique viscosity solution of G

Back

Barles-Souganidis

Explain with the deterministic growth model, but can be generalized

`https://www.fields.utoronto.ca/programs/scientific/09-10/finance/courses/tourin.pdf`

- Can write any HJB equation with one state variable as

$$0 = G(k, v(k), v_k(k)) = \rho v(k) - \max_c u(c) - v_k(k)(f(k) - \delta k - c)$$

- Corresponding FD scheme

$$\begin{aligned} 0 &= S(\Delta k, k_i, v_i; v_{i-1}, v_{i+1}) \\ &= \rho v(k_i) - u(c_i) - \frac{v(k_{i+1}) - v(k_i)}{\Delta k}(f(k_i) - \delta k_i - c_i)^+ - \frac{v(k_i) - v(k_{i-1})}{\Delta k}(f(k_i) - \delta k_i - c_i)^- \end{aligned}$$

Barles-Souganidis

1. **Monotonicity**: the numerical scheme is monotone, that is S is non-increasing in both v_{i-1} and v_{i+1} (non-increasing in v_j terms where $j \neq i$)
2. **Consistency**: the numerical scheme is consistent, that is for every smooth function v with bounded derivatives $S(\Delta k, k_i, v_i; v_{i-1}, v_{i+1}) \to G(k, v(k), v_k(k))$ as $\Delta k \to 0$ and $k_i \to k$.
3. **Stability**: the numerical scheme is stable, that is for every $\Delta k > 0$, it has a solution $v_i, i = 1, .., n$ which is uniformly bounded independently of Δk.

Barles-Souganidis

Check monotonicity of the scheme. Recall the FD scheme

$$
\begin{aligned}
&S(\Delta k, k_i, v_i; v_{i-1}, v_{i+1}) \\
=&\rho v(k_i) - u(c_i) - \frac{v(k_{i+1}) - v(k_i)}{\Delta k}\dot{k}^+ - \frac{v(k_i) - v(k_{i-1})}{\Delta k}\dot{k}^- \\
=&\rho v(k_i) - u(c_i) - \frac{\dot{k}^+}{\Delta k}v(k_{i+1}) + \frac{\dot{k}^+}{\Delta k}v(k_i) - \frac{\dot{k}^-}{\Delta k}v(k_i) + \frac{\dot{k}^-}{\Delta k}v(k_{i-1})
\end{aligned}
$$

- $S(\Delta k, k_i, v_i; v_{i-1}, v_{i+1})$ is non-increasing in v_{i-1} and v_{i+1}

Barles-Souganidis

For the stochastic growth model, the FD scheme is Back

$$0 = \rho v_{i,j} - \max_c u(c_{i,j}) - \frac{v_{i,j+1} - v_{i,j}}{\Delta k}\dot{k}^+_{F,i,j} - \frac{v_{i,j} - v_{i,j-1}}{\Delta k}\dot{k}^-_{B,i,j}$$
$$- \frac{v_{i+1,j} - v_{i,j}}{\Delta z}\tilde{\mu}_i^+ - \frac{v_{i,j} - v_{i-1,j}}{\Delta z}\tilde{\mu}_i^- - \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{2(\Delta z)^2}\tilde{\sigma}_i^2$$

Collecting coefficients of $v_{i-1,j}, v_{i,j}, v_{i+1,j}, v_{i,j-1}, v_{i,j+1}$

$$v_{i-1,j}: \; -\left(-\frac{\tilde{\mu}_i^-}{\Delta z} + \frac{\tilde{\sigma}_i^2}{2(\Delta z)^2}\right) = d_{ij}^{z1}, \quad v_{i+1,j}: -\left(\frac{\tilde{\mu}_i^+}{\Delta z} + \frac{\tilde{\sigma}_i^2}{2(\Delta z)^2}\right) = d_{ij}^{z2}$$

$$v_{i,j}: \; -\left(-\frac{\dot{k}^+_{F,i,j}}{\Delta k} + \frac{\dot{k}^-_{B,i,j}}{\Delta k} - \frac{\tilde{\mu}_i^+}{\Delta z} + \frac{\tilde{\mu}_i^-}{\Delta z} - \frac{2\tilde{\sigma}_i^2}{2(\Delta z)^2}\right) = d_{ij}^0$$

$$v_{i,j-1}: \; -\left(-\frac{\dot{k}^-_{B,i,j}}{\Delta k} = d_{ij}^{k1}\right), \qquad v_{i,j+1}: -\left(\frac{\dot{k}^+_{F,i,j}}{\Delta k}\right) = d_{ij}^{k2}$$

Need to check; $d_{ij}^{z1} \leq 0$, $d_{ij}^{z2} \leq 0$, $d_{ij}^{k1} \leq 0$, and $d_{ij}^{k2} \leq 0$. Easy to see these hold.