Deterministic HJB equation

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Overview of the course

1. Deterministic HJB equation
   - Derivation of deterministic HJB equation
   - Numerical solution with Finite Difference Method
2. Stochastic processes and stochastic HJB equation
   - Continuous-time stochastic processes
   - Ito's lemma
   - Derivation of stochastic HJB equation
3. Stochastic growth model
4. Viscosity solution
5. Huggett (1993)
   - Kolmogorov forward equation

Hamilton–Jacobi–Bellman equation

- **Continuous time counterpart of the Bellman equation**
- A result of the theory of dynamic programming which was pioneered in the 1950s by Richard Bellman and coworkers

These slides contain derivations of HJB equations

- Finite horizon
- Infinite horizon with discount
- Examples: Neoclassical growth model

Finite Horizon

Consider an optimal control problem over time period $[0, T]$

$$v(x_0, 0) \equiv \max_{\alpha(s)} \int_0^T r(s, x(s), \alpha(s))ds + g(x(T))$$

subject to

$$\dot{x}(s) = f(s, x(s), \alpha(s)), \quad x_0 \text{ is given}$$

- $x \in X \subseteq \mathbb{R}^n$: state vector
- $\alpha \in A \subseteq \mathbb{R}^m$: control vector
- $r : X \times A \to \mathbb{R}$: instantaneous return
- g: terminal value

Example: Life-cycle model

- wealth (a)
- consumption (c)
- $r(t, a(t), c(t)) = e^{-\rho t} u(c(t))$
- $g > 0$ or $g = 0$
- $\dot{a}(t) = r(t)a(t) + w(t) - c(t)$

Finite Horizon

Suppose we look at the problem at t

$$v(x(t),t) = \max_{\alpha(s)} \int_t^T r(s, x(s), \alpha(s))ds + g(x(T))$$

subject to

$$\dot{x}(s) = f(s, x(s), \alpha(s)), \quad x(t) \text{ is given}$$

$v(x(t),t)$

- a fraction of the value that starts at t
- from the point of view of time 0

Finite Horizon

Now focus on the interval between t and $t+h$

$$v(x(t),t) = \max_{\alpha(s)} \int_t^{t+h} r(s,x(s),\alpha(s))ds + v(x(t+h),t+h)$$

subject to

$$\dot{x}(s) = f(s,x(s),\alpha(s)), \;\; x(t) \text{ is given}$$

$$v(x(T),T) = g(x(T))$$

Principle of Optimality: the continuation of an optimal plan is optimal.

Finite Horizon

Rearrange

$$0 = \max_{\alpha(s)} \int_t^{t+h} r(s, x(s), \alpha(s))ds + v(x(t+h), t+h) - v(x(t), t)$$

Divide by h

$$0 = \max_{\alpha(s)} \frac{1}{h} \int_t^{t+h} r(s, x(s), \alpha(s))ds + \frac{1}{h}(v(x(t+h), t+h) - v(x(t), t))$$

Take a limit as $h \to 0$ (Assuming v is differentible)

$$0 = \max_{\alpha(t)} r(t, x(t), \alpha(t)) + v_x(x(t), t)\dot{x}(t) + v_t(x(t), t)$$

$$\dot{x}(t) = f(t, x(t), \alpha(t))$$

Finite Horizon

Rewrite

$$0 = \max_{\alpha} r(t, x, \alpha) + v_x(x, t) f(t, x, \alpha) + v_t(x, t)$$

This is a **HJB equation**

- The HJB equation is a Differential Equation
- v is the solution to the HJB
- Optimal plans are derived just like in the discrete case: $\alpha^*(x, t)$ is the maximizer of the value

Infinite Horizon with Discount

$$v(x_0, 0) \equiv \max_{\alpha(s)} \int_0^\infty e^{-\rho s} r(x(s), \alpha(s)) ds$$

subject to

$$\dot{x}(s) = f(x(s), \alpha(s))$$

$$\lim_{s \to \infty} b(s) x(s) \geq 0$$

for some exogenous b and x_0 is given

- Assume that r and f are not functions of t

Infinite Horizon with Discount

At t

$$v(x,t) = \max_{\alpha(s)} \int_t^{\infty} e^{-\rho s} r(x(s), \alpha(s)) ds$$

subject to

$$\dot{x}(s) = f(x(s), \alpha(s))$$

$$\lim_{s \to \infty} b(s) x(s) \geq 0$$

for some exogenous b and x is given

Infinite Horizon with Discount

For problems where r and f do not depend on t, it is useful to have a value function that is not a function of time

Define $v(x) \equiv v(x,0)$
Then, $v(x,t) = e^{-\rho t} v(x)$
$v_t(x,t) = -\rho e^{-\rho t} v(x)$

Plug these into a HJB equation,

$$
\begin{aligned}
0 &= \max_{\alpha} r(t,x,\alpha) + v_x(x,t) f(t,x,\alpha) + v_t(x,t) \\
&= \max_{\alpha} e^{-\rho t} r(x,\alpha) + e^{-\rho t} v_x(x) f(x,\alpha) - \rho e^{-\rho t} v(x) \\
&= \max_{\alpha} r(x,\alpha) + v_x(x) f(x,\alpha) - \rho v(x) \\
\rho v(x) &= \max_{\alpha} r(x,\alpha) + v_x(x) f(x,\alpha)
\end{aligned}
$$

Example: Neoclassical growth model

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\dot{k}_t = f(k_t) - \delta k_t - c_t, \;\; k_0 \text{ is given}$$

- k: capital, $\dot{k}$: saving, δ: depreciation rate
- c: consumption, $u(c)$: utility, ρ: discount rate
- Let $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $f(k) = k^\alpha$

Solve the model: 1. Hamiltonian

$$H(t,k,c,\lambda) \equiv e^{-\rho t}u(c) + \lambda(f(k) - c - \delta k)$$

Optimality conditions are

$$\frac{\partial H}{\partial c} = e^{-\rho t}u'(c) - \lambda = 0, \quad \frac{\partial H}{\partial k} = -\lambda(f'(k) - \delta) = \dot{\lambda}$$

Rearrange. The equilibrium path is a solution to

$$\frac{\dot{c}}{c} = \frac{1}{\sigma(c)}(f'(k) - \rho - \delta)$$

$$\dot{k} = f(k) - \delta k - c$$

with k_0 is given and $\lim_{T\to\infty} e^{-\rho T}u'(c_T)k_T = 0$

* $\sigma(c) \equiv -\frac{u''(c)c}{u'(c)}$: coefficient of relative risk aversion

Steady State

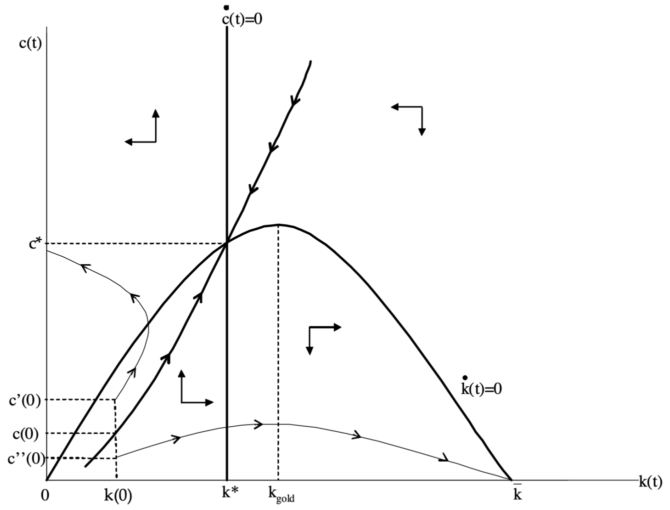
Steady states is where $\dot{k} = 0$ and $\dot{c} = 0$

$$f'(k^*) = \rho + \delta$$

$$c^* = f(k^*) - \delta k^*$$

$f(k) = k^\alpha, \alpha < 1,\ k^* = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}$

Dynamics: Phase Diagram



Dynamics: Numerical solution

Recall the two equations. They are differential equations.

$$\frac{\dot{c}}{c} = \frac{1}{\sigma(c)}(f'(k) - \rho - \delta)$$

$$\dot{k} = f(k) - \delta k - c$$

Finite difference methods are methods of solving differential equations

- FDMs discretize problems by approximating the derivatives with finite differences
- FDMs convert ordinary differential equations (ODE) or partial differential equations (PDE) into a system of equations that can be solved by matrix algebra techniques

Dynamics: Numerical solution

$$\frac{\dot{c}}{c} = \frac{1}{\sigma(c)}(f'(k) - \rho - \delta)$$
$$\dot{k} = f(k) - \delta k - c$$

- Approximate k_t and c_t over time dimension, $t = 1, ..., n$. Let the distance between points by Δt
- Approximate the derivative, $\dot{k}_t \approx \frac{k_{t+\Delta t} - k_t}{\Delta t}$

Using the approximation, can rewrite the above equations

$$\frac{c_{t+\Delta t} - c_t}{\Delta t}\frac{1}{c_t} = \frac{1}{\sigma(c_t)}(f'(k_t) - \rho - \delta)$$
$$\frac{k_{t+\Delta t} - k_t}{\Delta t} = f(k_t) - \delta k_t - c_t$$

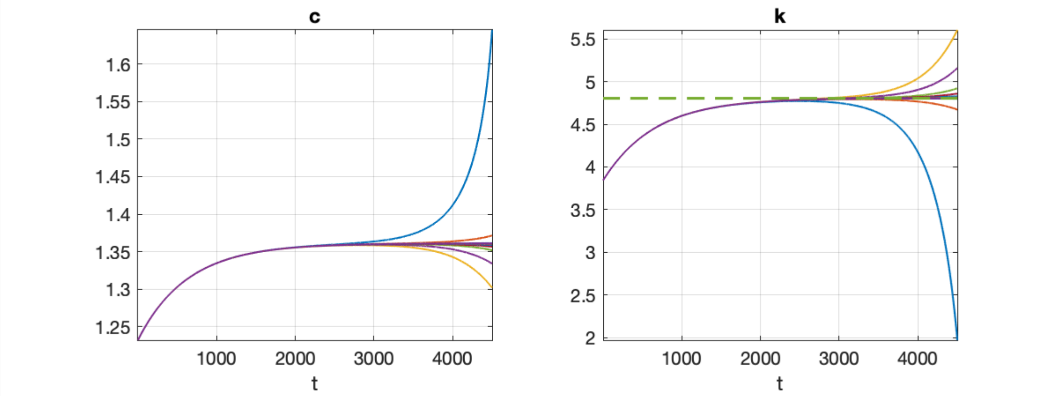
Shooting algorithm

$$\frac{c_{t+1} - c_t}{\Delta t} \frac{1}{c_t} = \frac{1}{\sigma(c_t)} (f'(k_t) - \rho - \delta)$$

$$\frac{k_{t+1} - k_t}{\Delta t} = f(k_t) - \delta k_t - c_t$$

- Guess c_0
- Compute $\{c_t, k_t\}, t = 1, ..., n$
- If the sequence converges to (c^*, k^*), then exit. If not, back to the first step and try different c_0.

Shooting algorithm



Solve the model: 2. HJB equation

$$v(k_0) = \max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\dot{k}_t = f(k_t) - \delta k_t - c_t$$

and k_0 is given

HJB equation is

$$\rho v(k) = \max_c u(c) + v_k(k)(f(k) - \delta k - c)$$

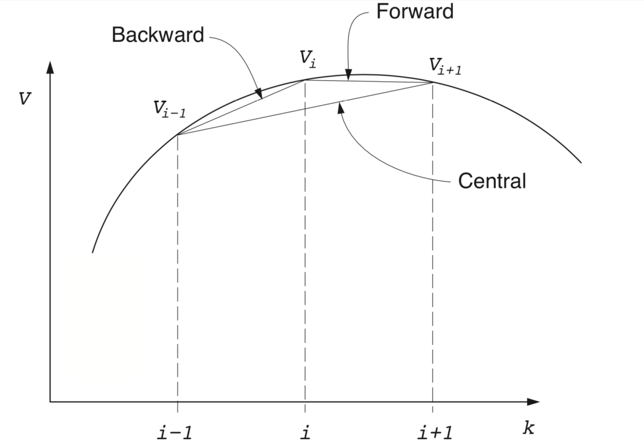
$$\dot{k} = f(k) - \delta k - c$$

Numerical Solution with Finite Difference Method

We will solve on discretized grids: $\mathbf{K} = \{k_1, ..., k_{n_k}\}$

- Grid on state space has n_k points. Our method will involve solving decision rules at each $k_i, i = 1, ..., n_k$
- Guess a value funtion over k: $v_0(k) = \frac{(k^\alpha)^{1-\sigma}}{1-\sigma}\frac{1}{\rho}$ $\quad \rho v_0(k) = u(f(k))$
- Shorthand notation: $v_i = v(k_i)$
- FDMs discretize the problem by approximating the derivatives with finite differences. Approximate

  $v_k(k_i) \approx \frac{v_{i+1}-v_i}{\Delta k}$ or $\frac{v_i - v_{i-1}}{\Delta k}$ (forward or backward)



- $v_{k,i}$ (forward) $< v_{k,i}$ (backward)
- $c = (v_k(k))^{-\frac{1}{\sigma}}$, c (forward) $> c$ (backward)
- $\dot{k} = f(k) - \delta k - c$, $\dot{k}$ (forward) $< \dot{k}$ (backward)

Numerical Solution with Finite Difference Method

Discretized the HJB

$$\rho v_i = \max_c u(c_i) + v_k(k_i)(f(k_i) - \delta k_i - c_i)$$

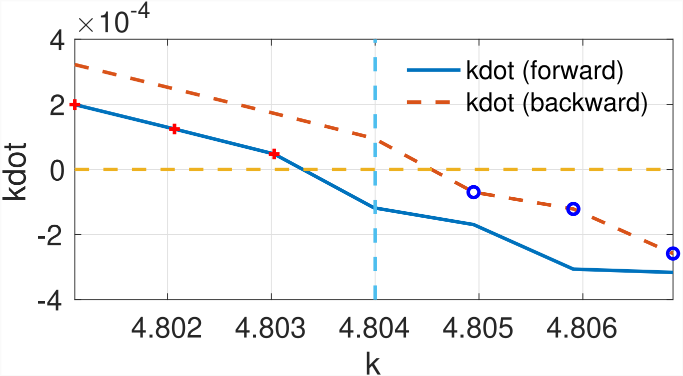
From FOC, $c = (v_k(k))^{-\frac{1}{\sigma}}$
There are two apprx. $v_k(k_i) \approx \frac{v_{i+1} - v_i}{\Delta k} (= v_{k,i}^F)$ and $\frac{v_i - v_{i-1}}{\Delta k} (= v_{k,i}^B)$.

Which one to use? **Upwind scheme**

- forward difference whenever drift of state variable positive, $\dot{k}_{F,i} = f(k_i) - \delta k_i - c_i = f(k_i) - \delta k_i - (v_{k,i}^F)^{\frac{-1}{\sigma}}$ when $\dot{k}_{F,i} > 0$
- backward difference whenever drift of state variable negative, $\dot{k}_{B,i} = f(k_i) - \delta k_i - c_i = f(k_i) - \delta k_i - (v_{k,i}^B)^{\frac{-1}{\sigma}}$ when $\dot{k}_{B,i} < 0$

At k_1, there is no backward difference and there is no forward difference at k_{n_k}. Set low enough k_1 so that backward difference is never be selected at k_1 and vice versa.

Upwind Scheme



- 1st - 3rd point: use $\dot{k}_F$. Positive saving
- 5th - 7th point: use $\dot{k}_B$. Negative saving
- 4th point: $\dot{k}_F < 0$ and $\dot{k}_B > 0 \implies$ steady state. $\dot{k} = 0$

Numerical Solution with Finite Difference Method

After selecting which approximation to use, we can write a discretized HJB for each k_i.

$$\begin{aligned}\rho v_i &= \max_c u(c_i) + v_k(k_i)(f(k_i) - \delta k_i - c_i) \\ &= \max_c u(c_i) + \frac{v_{i+1} - v_i}{\Delta k}\dot{k}^+_{F,i} + \frac{v_i - v_{i-1}}{\Delta k}\dot{k}^-_{B,i}\end{aligned}$$

where $\dot{k}^+ = max(\dot{k}, 0)$, $\dot{k}^- = min(\dot{k}, 0)$, $c_i = f(k_i) - \delta k_i - \dot{k}^+_{F,i} - \dot{k}^-_{B,i}$

We end up with a system of n_k equations

For example, let $i = 1$. $\dot{k}^+_{F,1} > 0$, $\dot{k}^-_{B,1} = 0$

$$\begin{aligned}\rho v_1 &= u(c_1) + \frac{v_2 - v_1}{\Delta k}\dot{k}^+_{F,1} \\ &= u(c_1) + \frac{\dot{k}^+_{F,1}}{\Delta k}v_2 - \frac{\dot{k}^+_{F,1}}{\Delta k}v_1 \\ \left(\text{Let } s = \frac{\dot{k}}{\Delta k}\right) \quad &= u(c_1) + s^+_{F1}v_2 - s^+_{F1}v_1\end{aligned}$$

Matrix representation

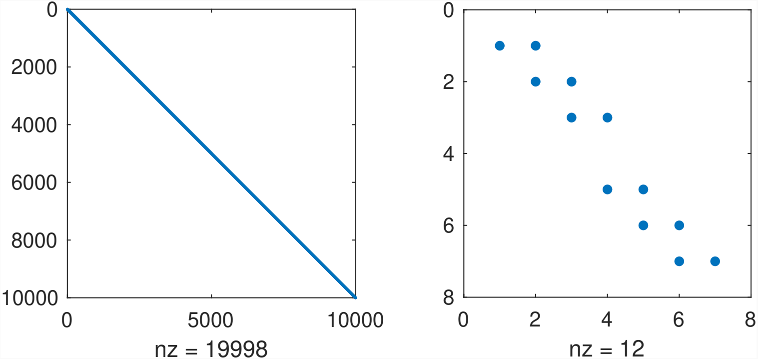
Suppose $n_k = 3$.

$$\rho \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -s^+_{F1} + s^-_{B1} & s^+_{F1} & 0 \\ -s^-_{B2} & -s^+_{F2} + s^-_{B2} & s^+_{F2} \\ 0 & -s^-_{B3} & -s^+_{F3} + s^-_{B3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{aligned} \rightarrow \rho \mathbf{v} &= \mathbf{u} + \mathbf{A}\mathbf{v} \\ (\rho \mathbf{I} - \mathbf{A})\mathbf{v} &= \mathbf{u} \\ \mathbf{v} &= (\rho \mathbf{I} - \mathbf{A})^{-1}\mathbf{u} \end{aligned}$$

- Set low enough k_1 so that backward difference is never be selected at k_1 and vice versa $\implies s^-_{B1} = 0$, $s^+_{F3} = 0$
- Notice `sum(A,2)` is always $n_k \times 1$ zero vector

Sparse Matrix A



- $n_k = 10000$
- nz: the number of non-zero elements
- left panel: **A**(:,:), right panel: **A**(i1:i1+6,i1:i1+6)

Value function updating: Explicit vs Implicit

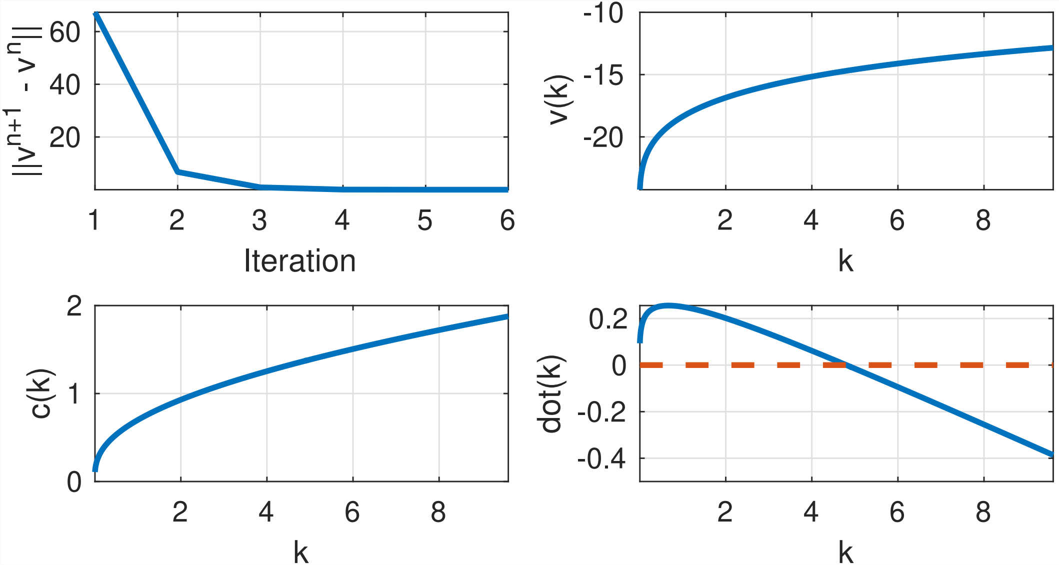
$$\mathbf{v} = (\rho \mathbf{I} - \mathbf{A})^{-1}\mathbf{u}$$

After constructing **u** and **A**, can find a value function. But since the HJB equations are highly non-linear, so is the system of equations. Therefore it has to be solved using an iterative scheme.

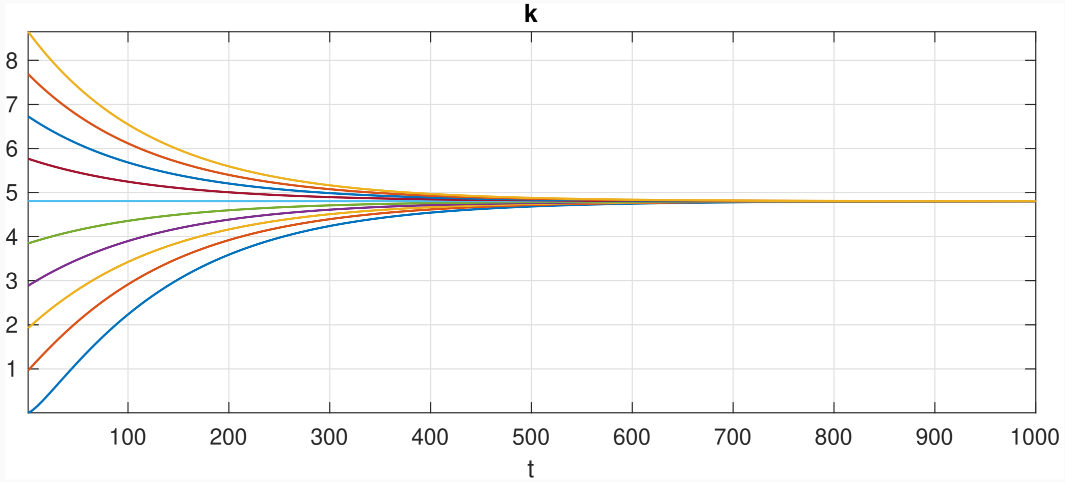
Let v^n be a value from n^{th} iteration

- **Explicit** $\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta} + \rho \mathbf{v}^n = \mathbf{u}^n + \mathbf{A}^n \mathbf{v}^n$, $\mathbf{v}^{n+1} = \left(\mathbf{u}^n + (\mathbf{A}^n + (\rho + \frac{1}{\Delta})\mathbf{I})\mathbf{v}^n\right)\Delta$
- **Implicit** $\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta} + \rho \mathbf{v}^{n+1} = \mathbf{u}^n + \mathbf{A}^n \mathbf{v}^{n+1}$, $\mathbf{v}^{n+1} = ((\rho + \frac{1}{\Delta})\mathbf{I} - \mathbf{A}^n)^{-1}(\mathbf{u}^n + \frac{\mathbf{v}^n}{\Delta})$

In general, the implicit method is the preferred approach because it is both more efficient and more stable



$||v^{n+1} - v^n||$
60
40
20
1
2
3
4
5
6
Iteration
-10
-15
-20
v(k)
2
4
6
8
k
2
1
0
c(k)
2
4
6
8
k
0.2
0
-0.2
-0.4
dot(k)
2
4
6
8
k


k
8
7
6
5
4
3
2
1
100
200
300
400
500
600
700
800
900
1000
t

Practice problem

$$v(k_0) = \max_{\{i_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t}\Big(k_t^\alpha - i_t - \frac{\theta}{2}\frac{(i-\delta k)^2}{k} - \delta k_t\Big)dt$$

$$\dot{k}_t = i_t - \delta k_t$$

and k_0 is given

- Derive HJB equation
- Solve the HJB equation
  $\rho = 0.05$, $\alpha = 0.3$, $\theta = 2$, $\delta = 0.06$