Chapter 4. Canonical correlation analysis. ①

4.1 Multivariate population and sample CCA.

$\underset{p\times 1}{\underset{\sim}{X}} \in \mathbb{R}^p$ and $\underset{q\times 1}{\underset{\sim}{Y}} \in \mathbb{R}^q$ (random vectors)

set

$\underset{p\times 1}{\underset{\sim}{a}} \in \mathbb{R}^p$ and $\underset{q\times 1}{\underset{\sim}{b}} \in \mathbb{R}^q$ (deterministic vectors)

- Define r.v's $\underset{1\times 1}{A} = \underset{\sim}{a}^T \underset{\sim}{X}$ and $\underset{1\times 1}{B} = \underset{\sim}{b}^T \underset{\sim}{Y}$.

* Goal: to find $\underset{\sim}{a}$ and $\underset{\sim}{b}$ that maximize $\mathrm{Corr}(A, B) = \dfrac{\mathrm{Cov}(A, B)}{\sqrt{\mathrm{Var}(A)\,\mathrm{Var}(B)}}$.

Goal

NOTE: Assume $\underset{\sim}{a}$ and $\underset{\sim}{b}$ maximize $\mathrm{Corr}(A, B)$.

Then $\underset{\sim}{a}^* = c\underset{\sim}{a}$ and $\underset{\sim}{b}^* = d\underset{\sim}{b}$, $c, d > 0$ maximize $\mathrm{Corr}(\underset{\sim}{a}^{*T}\underset{\sim}{X}, \underset{\sim}{b}^{*T}\underset{\sim}{Y})$,

because $\mathrm{Corr}(\underset{\sim}{a}^{*T}\underset{\sim}{X}, \underset{\sim}{b}^{*T}\underset{\sim}{Y}) = \mathrm{Corr}(c\underset{\sim}{a}^T\underset{\sim}{X}, d\underset{\sim}{b}^T\underset{\sim}{Y}) = \mathrm{corr}(\underset{\sim}{a}^T\underset{\sim}{X}, \underset{\sim}{b}^T\underset{\sim}{Y})$.

Multiple Solutions! Remedy → impose normalizing condition.

$\mathrm{Var}(A) = 1$ and $\mathrm{Var}(B) = 1$.

$\Leftrightarrow$ $\mathrm{Corr}(A, B) = \mathrm{Cov}(A, B)$.

* Let $\underset{\sim}{a} = \underset{\sim}{a}_1$ and $\underset{\sim}{b} = \underset{\sim}{b}_1$ maximize $\mathrm{Corr}(A_1, B_1) = \mathrm{Corr}(\underset{\sim}{a}_1^T\underset{\sim}{X}, \underset{\sim}{b}_1^T\underset{\sim}{Y})$.

Terminology

(A_1, B_1) : "first pair of canonical variables".

$\rho_1 = \mathrm{Cov}(A_1, B_1) = \max\{\mathrm{Cov}(\underset{\sim}{a}^T\underset{\sim}{X}, \underset{\sim}{b}^T\underset{\sim}{Y}) : \forall\, \underset{\sim}{a}, \underset{\sim}{b}$ satisfy $\mathrm{Var}(\underset{\sim}{a}^T\underset{\sim}{X}) = \mathrm{Var}(\underset{\sim}{b}^T\underset{\sim}{Y}) =$

"first cannonical correlation"

* Second pair $(\underset{\sim}{a}, \underset{\sim}{b})$ that maximize $Corr(A, B)$ subject to $Var(A) = Var(B) = 1$, ②

AND additional condition, $Cov(A, A_1) = Cov(A, B_1) = Cov(B, B_1) = cov(B, A_1) = 0$.

(analogous to Orthogonality in fpca
not correlated to the first pair).

$\Rightarrow$ denote the second pair $(\underset{\sim}{a}_2, \underset{\sim}{b}_2)$,

the second canonical variables $(A_2, B_2) = (\underset{\sim}{a}_2^T \underset{\sim}{X}, \underset{\sim}{b}_2^T \underset{\sim}{Y})$,

the resulting correlation $\rho_2 = cov(A_2, B_2)$.

NOTE: $\rho_2 \le \rho_1$ (b/c max in a smaller subspace)

* Continue for subsequent pairs. $(\underset{\sim}{a}_k, \underset{\sim}{b}_k, \rho_k, A_k, B_k)$.

* How to obtain the canonical components?

(i) Assume $\mathbb{E}(\underset{\sim}{X}) = 0$ and $\mathbb{E}(\underset{\sim}{Y}) = 0$ for simplicity.

Define $C_{11} = \mathbb{E}[\underset{\sim}{X}\underset{\sim}{X}^T]$, $C_{22} = \mathbb{E}[\underset{\sim}{Y}\underset{\sim}{Y}^T]$, $C_{12} = \mathbb{E}[\underset{\sim}{X}\underset{\sim}{Y}^T]$, $C_{21} = \mathbb{E}[\underset{\sim}{Y}\underset{\sim}{X}^T]$.

(ii) Assume that C_{11} and C_{22} are nonsingular (invertible).

Define Correlation matrices as

$$R = C_{11}^{-1/2} C_{12} C_{22}^{-1/2} \quad \text{and} \quad R^T = C_{22}^{-1/2} C_{21} C_{11}^{-1/2}$$

(iii) Then first $m = \min(p, q)$ eigenvalues of

$$M_X = RR^T = \left(C_{11}^{-1/2} C_{12} C_{22}^{-1/2}\right)\left(C_{22}^{-1/2} C_{12} C_{11}^{-1/2}\right) \text{ and}$$

$$M_Y = R^T R = \left(C_{22}^{-1/2} C_{21} C_{11}^{-1/2}\right)\left(C_{11}^{-1/2} C_{12} C_{22}^{-1/2}\right)$$

Matrix w/ rank m

m eigenvalues, eigenvectors
$\Rightarrow$ m CCA components exist.

are the <u>same</u> and they are $\underbrace{\rho_1^2 \ge \rho_2^2 \ge \cdots \ge \rho_m^2}_{m \text{ correlations}} > 0$

NOTE: $$\rho = \frac{Cov(A, B)}{\sqrt{Var(A)}\sqrt{Var(B)}} = \frac{\underset{\sim}{a}^T Cov(\underset{\sim}{X}, \underset{\sim}{Y}) \underset{\sim}{b}}{\sqrt{\underset{\sim}{a}^T Var(\underset{\sim}{X}) \underset{\sim}{a}}\sqrt{\underset{\sim}{b}^T Var(\underset{\sim}{Y}) \underset{\sim}{b}}} = \frac{\underset{\sim}{a}^T C_{12} \underset{\sim}{b}}{\sqrt{\underset{\sim}{a}^T C_{11} \underset{\sim}{a}}\sqrt{\underset{\sim}{b}^T C_{22} \underset{\sim}{b}}}$$

let $\underset{\sim}{c} = C_{11}^{1/2} \underset{\sim}{a}$, $\underset{\sim}{d} = C_{22}^{1/2} \underset{\sim}{b}$.

by Cauchy Schwartz.

$$= \frac{\underset{\sim}{c}^T \boxed{C_{11}^{-1/2} C_{12} C_{22}^{-1/2}} \underset{\sim}{d}}{\sqrt{\underset{\sim}{c}^T \underset{\sim}{c}}\sqrt{\underset{\sim}{d}^T \underset{\sim}{d}}} = \frac{(\underset{\sim}{c}^T R) \underset{\sim}{d}}{\sqrt{c^T c}\sqrt{d^T d}} \le \frac{1}{\sqrt{c^T c}\,\cancel{\sqrt{d^T d}}} (c^T R R^T c)^{1/2} \cancel{(\underset{\sim}{d}^T \underset{\sim}{d})^{1/2}}$$

$$\Leftrightarrow \rho \le \left(\frac{c^T R R^T c}{c^T c}\right)^{1/2} \Leftrightarrow \rho^2 \le \frac{c^T \overset{= M_X}{RR^T} c}{c^T c}$$ follows Rayleigh quotient form.

$\le$ 1st eigenvalue of M_X. (bounded)

which is obtained when c = 1st eigenvector of M_X.

NOTE : (cont'd)

$$\rho = \frac{c^T(Rd)}{\sqrt{c^Tc}\sqrt{d^Td}} \overset{C.S}{\leq} \left(\frac{d^TR^TRd}{d^Td}\right)^{1/2}$$

$$\Leftrightarrow \quad \rho^2 \leq \frac{d^T(\overset{= M_Y}{R^TR})d}{d^Td} \leq \text{1st eigenvalue of } M_Y$$

d = 1st eigenvector of M_Y.

(iv) The corresponding eigenvectors are e_k and f_k s.t. $M_X e_k = \rho_k^2 e_k$ and $M_Y f_k = \rho_k^2 f_k$ for $k = 1, \ldots, m$.

(v) Finally the weights of the kth pair of canonical variables are

$$a_k = C_{11}^{-1/2} e_k \quad \text{and} \quad b_k = C_{22}^{-1/2} f_k .$$

NOTE : e_k and f_k are related via $e_k = \rho_k^{-1} R f_k$ and $f_k = \rho_k^{-1} R^T e_k$.

because

$$\rho_k = \mathrm{Cov}(A_k, B_k) = \mathrm{Cov}(a_k^T X, b_k^T Y) = a_k^T C_{12} b_k$$

$$= e_k \underbrace{C_{11}^{-1/2} C_{12} C_{22}^{-1/2}}_{R} f_k .$$

* Notation using inner product .

i) $\mathrm{Cov}(a^T X, b^T Y) = \mathbb{E}[a^T X b^T Y] = \mathbb{E}[a^T X Y^T b] = a^T \mathbb{E}[XY^T] b = \langle a, C_{12} b \rangle$

ii) $\mathrm{Var}(a^T X) = \langle a, C_{11} a \rangle$

iii) $\mathrm{Var}(b^T Y) = \langle b, C_{22} b \rangle$

iv) $\rho_k = \langle a_k, C_{12} b_k \rangle = \max \{ \langle a, C_{12} b \rangle : a \in \mathbb{R}^p, b \in \mathbb{R}^q, \langle a, C_{11} a \rangle = \langle b, C_{22} b \rangle = 1,$ subj. to $\langle A_k, A_j \rangle = \langle A_k, B_j \rangle = \langle B_k, B_j \rangle = \langle B_k, A_j \rangle = 0, \ \forall j < k$.

v) $A_j = \langle a_j, X \rangle$, $B_j = \langle b_j, Y \rangle$.

* Sample : $(x_1, y_1) \cdots (x_N, y_N)$.

$$\hat{A} = (\hat{a}^T x_1, \ldots, \hat{a}^T x_N)^T$$

$$\hat{B} = (\hat{b}^T y_1, \ldots, \hat{b}^T y_N)^T$$

Find $\hat{a}$ & $\hat{b}$ s.t $\mathrm{Corr}(\hat{A}, \hat{B})$ is maximum.

$(\hat{A}_1, \hat{B}_1)$: first pair of sample canonical variates.

4.2 Functional Canonical Components

$X(\cdot) \in \mathcal{H}_1 = L^2(T_1)$

$Y(\cdot) \in \mathcal{H}_2 = L^2(T_2)$

Assume $\mathbb{E}(X(t)) = \mathbb{E}(Y(s)) = 0, \ \forall t, s.$

* Covariance Operators :

$\mathbb{E}\int X(s)\,x(s)\,X(t)\,ds.$

$C_{11} : \mathcal{H}_1 \to \mathcal{H}_1 \qquad C_{11}(x)(t) = \int_{T_1} C_{11}(t,s)\,x(s)\,ds = \mathbb{E}[\langle X, x\rangle X(t)], \quad C_{11}(t,s) = \mathbb{E}(X(t)X($

$C_{12} : \mathcal{H}_2 \to \mathcal{H}_1 \qquad C_{12}(y)(t) = \int C_{12}(t,s)\,y(s)\,ds = \mathbb{E}[\langle Y, y\rangle X(t)], \quad C_{12}(t,s) = \mathbb{E}(X(t)Y(s$

$C_{21} : \mathcal{H}_1 \to \mathcal{H}_2 \qquad C_{21}(x)(t) = \int C_{21}(t,s)\,x(s)\,ds = \mathbb{E}[\langle X, x\rangle Y(t)], \quad C_{21}(t,s) = \mathbb{E}(Y(t)X($

$C_{22} : \mathcal{H}_2 \to \mathcal{H}_2 \qquad C_{22}(y)(t) = \int C_{22}(t,s)\,y(s)\,ds = \mathbb{E}[\langle Y, y\rangle Y(t)], \quad C_{22}(t,s) = \mathbb{E}(Y(t)Y(s$

- C_{11}, C_{22} : covariance operator (symmetric, p.d, Hilbert-Schmidt) (see ch2).

* Hilbert-Schmidt operator to the space $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ of bounded operators from $\mathcal{H}_2 \to \mathcal{H}_1$.

(section 4.5, pg 58)

suppose $\{v_i\}$ is a basis in $\mathcal{H}_1$ and $\{u_i\}$ is a basis in $\mathcal{H}_2$.

If $A \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$, then

$$A(u_i) = \sum_{j=1}^{\infty} \langle A(u_i), v_j\rangle v_j = \sum_{j=1}^{\infty} a_{ij} v_j \in \mathcal{H}_1.$$

(compare w/ covariance operator defined in $\mathcal{H}_1$ (4.15))

$$C(x) = \sum_{j=1}^{\infty} \lambda_j \langle x, v_j\rangle v_j \quad \text{where} \quad \|C\|_S^2 = \sum_{j\geq 1} \lambda_j^2 \quad \text{(ch2)}$$

* C_{12} is Hilbert-Schmidt.

↳ def. Let $\Psi(x)(t) = \int \psi(t,s)\,x(s)\,ds$.

(P.22) Ψ is Hilbert-Schmidt operator iff $\iint \psi(t,s)^2\,dt\,ds < \infty$.

NOTE : want to show $\iint C_{12}(t,s)^2\,dt\,ds < \infty$.

$$\iint C_{12}(t,s)^2\,dt = \iint \mathbb{E}(X(t)Y(s))^2\,dt\,ds$$

$$\leq \iint \mathbb{E}(X(t)^2)\,\mathbb{E}(Y(s)^2)\,dt\,ds = \int \mathbb{E}(X(t)^2)\,dt \int \mathbb{E}(Y(t)^2)\,dt$$

$$= \mathbb{E}(\|X\|^2)\,\mathbb{E}(\|Y\|^2) < \infty$$

b/c X, Y are square-integrable.

* Then defining cannonical components are identical to Multivariate case.
(Only diff is that inner product is in Hilbert space that $\langle x, y\rangle = \int x(t)y(t)\,dt$.

4.5. Square root of the covariance operator

* Definition of Square root of the covariance operator, C.

i) Spectral decomposition of the operator C.

$$C(x) = \sum_{j=1}^{\infty} \lambda_j \langle x, v_j\rangle v_j, \quad x \in L^2$$

(λ_j: nonnegative, v_j: orthonormal.)

ii) An operator R is called square root of C if $RR = C$. (~ cholesky decomp)

All covariance operator has the corresponding unique, positive definite square root

$$C^{1/2}(x) = \sum_{j=1}^{\infty} \lambda_j^{1/2} \langle x, v_j\rangle v_j, \quad x \in L^2$$

Symmetric and positive definite

NOTE : ① Symmetric. $\Leftrightarrow$ want to show $\langle C^{1/2}(x), y\rangle = \langle x, C^{1/2}(y)\rangle$.

$$\begin{aligned}
\langle C^{1/2}(x), y\rangle &= \left\langle \sum_{j=1}^{\infty} \lambda_j^{1/2} \langle x, v_j\rangle v_j, y\right\rangle \\
&= \int \sum_{j=1}^{\infty} \lambda_j^{1/2} \int x(t) v_j(t)\,dt\, v_j(s)\, y(s)\,ds \\
&= \int x(t) \sum_{j=1}^{\infty} \lambda_j^{1/2} \int v_j(s) y(s)\,ds\, v_j(t)\,dt \\
&= \int x(t) \sum_{j=1}^{\infty} \lambda_j^{1/2} \langle v_j, y\rangle v_j(t)\,dt \\
&= \langle x, C^{1/2}(y)\rangle. \quad \blacksquare
\end{aligned}$$

② positive-definite. $\Leftrightarrow$ Want to show $\langle C^{1/2}(x), x\rangle \geq 0$, $x \in H$.

$$\begin{aligned}
\langle C^{1/2}(x), x\rangle &= \sum_{j=1}^{\infty} \lambda_j^{1/2} \langle x, v_j\rangle \langle v_j, x\rangle \\
&= \sum_{j=1}^{\infty} \lambda_j^{1/2} \langle x, v_j\rangle^2 \geq 0. \quad \blacksquare
\end{aligned}$$

non-negative

* Def : Inverse of operator.

A defined on a subspace $\mathcal{D}(A)$ with range $\mathcal{R}(A)$.
B defined on a subspace $\mathcal{R}(A)$. is called the inverse of A

if $B(A(x)) = x$ for all $x \in \mathcal{D}(A)$
AND $A(B(y)) = y$ for all $y \in \mathcal{R}(A)$.

i) if A has an inverse, it is <u>unique</u> and is denoted A^{-1}.

ii) A is invertible iff $A(x) = 0$ implies $x = 0$.

* C and $C^{1/2}$ are invertible iff $\lambda_j > 0$ for each $j \geq 1$.

Assumption to ensure C, $C^{1/2}$ are invertible

NOTE : want to show that $C(x) = 0 \Rightarrow x = 0$

Suppose $C(x) = 0$

$$C(x) = \sum_{j=1}^{\infty} \lambda_j \langle x, v_j \rangle v_j$$
$$= \sum_{j=1}^{\infty} \underbrace{\lambda_j}_{>0} \underbrace{\int x(t) v_j(t) v_j(s)\, dt} = 0 \Rightarrow \text{each term} = 0.$$

* Range of $C^{1/2}$

$$\mathcal{R}(C^{1/2}) = \{ y \in L^2 : \sum_{j=1}^{\infty} \lambda_j^{-1} \langle y, v_j \rangle^2 < \infty \}.$$

NOTE $(\Rightarrow)$ WTS if $y \in \mathcal{R}(C^{1/2})$, then $\sum_{j=1}^{\infty} \lambda_j^{-1} \langle y, v_j \rangle^2 < \infty$

$$y \in \mathcal{R}(C^{1/2}) \Rightarrow y = C^{1/2} x \quad \text{for } x \in L^2$$
$$= \sum_{i=1}^{\infty} \lambda_i^{1/2} \langle x, v_i \rangle v_i$$

and $\sum_{j=1}^{\infty} \lambda_j^{-1} \langle y, v_j \rangle^2 = \sum_{j=1}^{\infty} \lambda_j^{-1} \langle \sum_{i=1}^{\infty} \lambda_i^{1/2} \langle x, v_i \rangle v_i, v_j \rangle^2$

$$= \sum_{j=1}^{\infty} \langle x, v_j \rangle^2 = \|x\|^2 < \infty$$

Parseval's eq — square integrable.

$(\Leftarrow)$ WTS if $\sum_{j=1}^{\infty} \lambda_j^{-1} \langle y, v_j \rangle^2 < \infty$, then $x = \sum_{j=1}^{\infty} \lambda_j^{-1/2} \langle y, v_j \rangle v_j$ is well-defined element of L^2 and we can show that $C^{1/2}(x) = y$.

$$\sum_{j=1}^{\infty} \left(\lambda_j^{-1/2} \langle y, v_j \rangle \right)^2 = \sum_{j=1}^{\infty} \langle x, v_j \rangle^2$$

$$\Rightarrow \lambda_j^{-1/2} \langle y, v_j \rangle = \langle x, v_j \rangle \text{ for each } j \Rightarrow x = \sum_{j=1}^{\infty} \lambda^{-1/2} \langle y, v_j \rangle v_j \in L^2$$

(cont'd) $C^{1/2}(x) = \sum_{i=1}^{\infty} \lambda_i^{1/2} \langle \sum_{j=1}^{\infty} \lambda^{-1/2} \langle y, v_j \rangle v_j, v_i \rangle v_i = y.$

* Inverse of $C^{1/2}$.

$C^{1/2}(x) = y$. is defined on a subspace $\mathcal{D}(C^{1/2})$ with range $\underbrace{R(C^{1/2})}_{\text{given in (4.17)}}$

want to define $C^{-1/2}$ on domain $R(C^{1/2})$ s.t. $C^{-1/2}(C^{1/2}(x)) = x$.

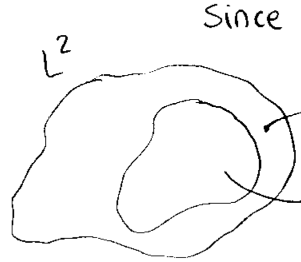
given in (4.18) $C^{-1/2}(y) = \sum_{j=1}^{\infty} \lambda_j^{-1/2} \langle y, v_j \rangle v_j, \quad y \in R(C^{1/2})$

* Note that each v_k in $C^{1/2}(x) = \sum_{j=1}^{\infty} \lambda_j^{1/2} \langle x, v_j \rangle v_j$ is in $R(C^{1/2})$.

NOTE: Set $y = v_k$ in (4.17)

$\sum_{j=1}^{\infty} \lambda_j^{-1} \langle v_k, v_j \rangle^2 = \lambda_j^{-1} < \infty \Rightarrow v_k \in R(C^{1/2})$

Since $R(C^{1/2})$ is a linear subspace, all finite linear comb of v_k's are in $R(C^{1/2})$



$\to y = \sum_{k=1}^{\infty} \lambda_k^{1/2} v_k \in L^2$ because

$\int y(t)^2 dt = \sum \lambda_k < \infty$.

$\to R(C^{1/2})$

but $y \notin R(C^{1/2})$ because

$\sum_{j=1}^{\infty} \lambda_j^{-1} \langle \sum_{k=1}^{\infty} \lambda_k^{1/2} v_k, v_j \rangle^2 = \sum_{j=1}^{\infty} \lambda_j^{-1} \lambda_j = \infty$.

Def:

* Operator A defined in $\mathcal{L}(H_1, H_2)$.

Suppose $\{v_i\}$ is a basis in $\mathcal{H}_1$ and $\{u_j\}$ is a basis in $\mathcal{H}_2$.

then $A(v_i) = \sum_{j=1}^{\infty} \langle A(v_i), u_j \rangle u_j = \sum_{j=1}^{\infty} a_{ji} u_j$.

* A is Hilbert Schmidt norm iff $\sum_{j=1}^{\infty} a_{ji}^2 < \infty$.

and we denote the space of HS operators in $\mathcal{L}(H_1, H_2)$ with $S(\mathcal{H}_1, \mathcal{H}_2)$.

Suppose $A_1 \in S(\mathcal{H}_1, \mathcal{H}_2)$, $A_2 \in S(\mathcal{H}_2, \mathcal{H}_3)$.

i) $A_2 A_1 \in S(\mathcal{H}_1, \mathcal{H}_3)$

ii) $\|A_2 A_1\|_S \le \|A_1\|_S \|A_2\|_S$. C.S ineq. holds.

* if A is an integrable operator following $A(x)(t) = \int_{T_1} a(t,s)\, x(s)\, ds$, $x \in \mathcal{H}_1$,

then A is Hilbert-Schmidt iff $\iint a^2(t,s)\, dt\, ds < \infty$.

NOTE: $\|A\|_S^2 = \sum_{i=1}^{\infty} \langle A(e_i), A(e_i) \rangle = \sum_{i=1}^{\infty} \int \left\{ \int a(t,s) e_i(s) ds \right\}^2 dt = \int \underbrace{\sum_{i=1}^{\infty} \langle a(t,\cdot), e_i(\cdot) \rangle^2}_{\|a(t,\cdot)\|^2} dt$

Main idea of Section 4.6 :

Spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are too big to define functional canonical components.

Only possible on smaller subspace..

- Want to construct operators analogous to the matrices M_X and M_Y.

→ Solution is given in Proposition 4.2.

the correlation operator

$$R = C_{11}^{-1/2} \; C_{12} \; C_{22}^{-1/2} \quad : \quad R_2 \to R_1$$

its adjoint $\quad R^* = C_{22}^{-1/2} \, C_{12} \, C_{11}^{-1/2} \quad : \quad R_1 \to R_2$

(~ transpose)

$$M_Y = R^* R \; : \; R_2 \to R_2$$

$$\Rightarrow \quad M_X = R R^* \; : \; R_1 \to R_1$$

$R^* : \langle R(y), x \rangle = \langle y, R^*(x) \rangle$

Eigenvectors of M_Y and M_X : $\; e_K = \rho_K^{-1} R(f_K)$ and $f_K = \rho_K^{-1} R^*(e_K)$

Weight functions : $\quad a_K = C_{11}^{-1/2}(e_K) \quad$ and $\quad b_K = C_{22}^{-1/2}(f_K)$.

$R_1 = R(C_{11})$ and $R_2 = R(C_{22})$

* Composition of operators.

① $\; R = \underbrace{C_{11}^{-1/2}}_{R(C_{11}) \to \mathcal{H}_1} \quad \underbrace{C_{12}}_{\mathcal{H}_2 \to R(C_{11})} \quad \underbrace{C_{22}^{-1/2}}_{R(C_{22}) \to \mathcal{H}_2} \; : R_2 \to \mathcal{H}_1$

⇒ This means, we need to make sure

$$C_{12}(\mathcal{H}_2) \subset R(C_{11}) = R_1$$

Analogously for R^*, $\; C_{21}(\mathcal{H}_1) \subset R_2$.

$R^* : \mathcal{H}_1 \to R_2$

both are ensured by proposition 4.1.

$\Rightarrow \quad M_Y = \underbrace{R^*}_{\mathcal{H}_1 \to R_2} \underbrace{R}_{R_2 \to \mathcal{H}_1} \; : R_2 \to R_2 \qquad$ and $\; M_X : R_1 \to R_1$

② To use decomposition, we need M_Y and M_X to be Hilbert-Schmidt.

R and R^* are H.S because $\sum r_{jk}^2 < \infty$ by proposition 4.1

Composition of H-S operators is also Hilbert Schmidt.

$R : R_2 \to R_1$

$R^* : R_1 \to R_2$

③ $\; a_K = C_{11}^{-1/2} e_K$

We need $R(e_K) \subset R(C_{11})$ ← this is ensured by Assumption 4.1.

(1)

[4.1]. $X = \sum_{i\geq 1} \xi_i \vartheta_i$ & $Y = \sum_{j\geq 1} \zeta_j u_j$ — ①

& ϑ_i & u_j are chosen in such a way so that $C_{11}\vartheta_i = \lambda_i \vartheta_i$

& $C_{22} u_j = \gamma_j u_j$. — ②

SVD:

$C_{11} = UDV^T$

$C_{11}V = UD$.

$\gamma_j > 0$ & $\lambda_i > 0$ for $\rho_{ji} = \dfrac{E[\xi_i \zeta_j]}{\lambda_i^{1/2}\gamma_j^{1/2}}$ — ③ & $\sum_{i=j=1}^{\infty} \rho_{ji}^2 < \infty$ — ④

then for any x in $C_{12}(H_2) \subset R_1 = R(C_{11}^{1/2})$ & any y in $C_{21}(H_1) \subset R_2 = R(C_{22}^{1/2})$

Show → $\left\{ \sum_{i\geq 1} \lambda_i^{-1} \underline{\langle x, \vartheta_i\rangle^2} < \infty \equiv \langle x, \vartheta_i\rangle^2 < \infty \right\}$ & $\left\{ \sum_{j\geq 1} \gamma_j \langle y, u_j\rangle^2 < \infty \equiv \langle y, u_j\rangle^2 < \infty \right\}$

FACT:– $R(C^{1/2}) = \{ y \in L^2 : \underbrace{\sum \lambda_j^{-1} \langle y, \vartheta_j\rangle^2 < \infty}_{\text{need to show this holds to define range}} \}$

$X \in H_1 = L^2[T_1]$

$Y \in H_2 = L^2[T_2]$

$C_{11}: H_1 \to H_1$

$C_{12}: H_2 \to H_1$

Pf: 1. let, $x = C_{12}(y)$, $y \in H_2$

$x = C_{12}(y)(t) = \int_{T_2} C_{12}(t,s) y(s)\, ds \in H_1$

$= E[\langle Y, y\rangle X] = E[\langle \sum_{j\geq 1} \zeta_j u_j, y\rangle \sum_{k\geq 1} \xi_k \vartheta_k]$ (by ①)

$= \sum_{j,k} E[\xi_k \zeta_j] . \langle u_j, y\rangle \vartheta_k$ — ⑤

let, $\langle x, \vartheta_i\rangle = \langle \sum_{j,k} E[\xi_k \zeta_j] . \langle u_j, y\rangle \vartheta_k, \vartheta_i \rangle$ (if $k = i$)

$= \sum_{j} \left(\underbrace{E[\xi_i \zeta_j]} . \langle u_j, y\rangle \right)$

$= \sum_{j} \rho_{ji} \lambda_i^{1/2} \gamma_j^{1/2} \langle u_j, y\rangle$. [by ③].

then, $\langle x, \vartheta_i \rangle^2 \leq \left(\sum_j \rho_{ji}^2 \gamma_j\right)\underbrace{\left(\sum_j \langle \vartheta_j, y\rangle\right)^2}\cdot \lambda_i = \lambda_i \|y\|^2 \left(\sum_j \rho_{ji}^2 \gamma_j\right)$

(by Parseval's Inequality)

$$\left[\sum x_i y_i\right]^2 \leq \left(\sum x_i^2\right)\left(\sum y_j^2\right)$$

So, $\sum_i \lambda_i^{-1} \langle x, \vartheta_i\rangle^2 \leq \sum \|y\|^2 \left(\sum_{i,j=1}^{\infty} \rho_{ji}^2 \gamma_j\right) < \infty$.

So, $C_{12}(H_2) \subset R_1$ by (4).

Similarly, we can show $C_{21}(H_1) \subset R_2$

<u>NOTE:</u>

- $R = C_{11}^{-1/2} C_{12} C_{22}^{-1/2} = C_{11}^{-1/2} \underbrace{C_{12}(C_{22}^{-1/2})}_{\subset R_1} = C_{11}^{-1/2}(R_1) \in H_1$
- $R^* = C_{22}^{-1/2} C_{21} C_{11}^{-1/2} = C_{22}^{-1/2} \underbrace{C_{21}(C_{11}^{-1/2})}_{R_2} \in R_2$ /

so that $RR^* = M_y$.

- $R(C_{11}^{1/2}) \subset H_1$
- $R(C_{22}^{1/2}) \subset H_2$

$\boxed{\text{Lemma:- 4.1}}$ if $\sum_{i,j=1}^{\infty} \rho_{ji}^2 < \infty$ then $R : R_2 \to H_1$ & $R^* : H_1 \to R_2$

& $R(u_j) = \sum_{k\geq 1} \rho_{jk} \vartheta_k$ & $R^*(\vartheta_i) = \sum_{k \geq 1} \rho_{ki} u_k$.

$\boxed{\text{Pf}}$. $C_{22}(y) = \sum_{k\geq 1} \gamma_k \cdot \langle y, u_k\rangle u_k$ (by 4.15) (decomposition of Covariance Operator). (Chp 3)

$\Rightarrow C_{22}^{-1/2}(y) = \sum_{k\geq 1} \gamma_k^{-1/2} \langle y, u_k \rangle u_k$

$\Rightarrow$ ~~$C_{12}C_{22}(y) =$~~ $\sum \gamma_k^{-1/2}$

$$\Rightarrow C_{22}^{-1/2}(u_j) = \sum_{k \geq 1} \gamma_k^{-1/2} \langle u_j, u_k \rangle u_k = \gamma_j^{-1/2} u_j$$

$$\Rightarrow C_{12} C_{22}^{-1/2}(u_j) = \gamma_j^{-1/2} \underbrace{C_{12}(u_j)} = \gamma_j^{-1/2} \sum_{i,k} E[\xi_i \xi_k] \langle u_i, u_j \rangle \vartheta_k$$

$$= \gamma_j^{-1/2} \sum_k E[\xi_k \zeta_j] \vartheta_k \qquad (4.24)$$

So, $R(u_j) = \gamma_j^{-1/2} \sum_k E[\xi_k \zeta_j] \, C_{11}^{-1/2}(\vartheta_k)$

$$= \underbrace{\gamma_j^{-1/2} \sum_k E[\xi_k \zeta_j] \lambda_k^{-1/2}} \vartheta_k = \sum_k \rho_{jk} \vartheta_k.$$

$$\left(\because C_{11}^{-1/2}(\vartheta_k) = \sum_{j \geq 1} \lambda_j^{-1/2} \langle \vartheta_k, \vartheta_j \rangle \vartheta_j = \lambda_k^{-1} \vartheta_k \right)$$

Now, $\langle R(u_j), \vartheta_i \rangle = \left\langle \sum_k \rho_{jk} \vartheta_k, \vartheta_i \right\rangle = \rho_{ji} = \langle u_j, \underbrace{R^*(\vartheta_i)}_{\sum_k \rho_{ki} u_k} \rangle$

(4)

$\boxed{\text{Assump}^n\text{: 4.1}}$ • $\lambda_i > 0$ & $\gamma_j > 0$ $\forall i \& j$

• $\sum_{i,j=1} \lambda_i^{-1} \rho_{ji}^2 < \infty$ & $\sum_{i,j=1} \gamma_j^{-1} \rho_{ji}^2$

treat $f_k \equiv X(s,t)$ as a ~~sum~~ $\sum$

$X(s,t) = \sum_{k \geq 1} \xi_{ik}(t) \phi_k(s)$

$\xi_{ik}(\cdot) = \int_S X(\cdot,\cdot)\phi_k(\cdot)$
$= \langle X, \phi \rangle_S$

R is any operator

• $R(C^{1/2}) = \{ y \in L^2 : \sum \lambda_j^{-1} \langle y, \vartheta_j \rangle^2 < \infty \}$

T.S.T $y = R(f_k) \in R_1$

Pf. $\sum_{i \geq 1} \lambda_i^{-1} \langle y, \vartheta_i \rangle^2 = \sum_{i \geq 1} \lambda_i^{-1} \langle \overline{R(f_k)}, \vartheta_i \rangle^2$

$= \sum_i \lambda_i^{-1} \langle \sum_{j \geq 1} \langle f_k, u_j \rangle . R(u_j), \vartheta_i \rangle^2$

$= \sum \lambda_i^{-1} \langle \sum \langle f_k, u_j \rangle \sum_{k \geq 1} \rho_{jk} \vartheta_k, \vartheta_i \rangle^2$

$= \sum \lambda_i^{-1} \left(\sum_{j \geq 1} \langle f_k, u_j \rangle \rho_{ji} \right)^2$

$\leq \sum_{i \geq 1} \lambda_i^{-1} \underbrace{\sum_{j \geq 1} \langle f_k, u_j \rangle^2} \sum_{j \geq 1} \rho_{ji}^2$ (by CS)

$= \sum \lambda_i^{-1} \| f_k \|^2 . \sum_{j \geq 1} \rho_{ji}^2 = \left(\sum_{i,j=1} \lambda_i^{-1} \rho_{ji}^2 \right) \| f_k \|^2$

$< \infty$.

So, $R(f_k) \in R_1$.

Say, f_k be eigenfunctions. So $f_k \in R_2$

for any $y \in R_2 \Rightarrow R(y) \subset R_1$ ~~Range~~

So, the domain of R^* is R_1. (Range).