Restricciones - Constraints
Clasificación

- Punto simple:

$u_4 = 0$	Lineal, homogénea
$v_1 = 0.2$	Lineal, no homogénea

- Multi punto

$u_2 = 2\ u_3$	Lineal, homogénea
$u_2 = 2\ u_3 - v_2 + 0.1$	Lineal, no homogénea
$(u_1 - v_1)^2 + u_5 . u_6 = 0$	No líneal, homogénea.

Restricciones - Constraints

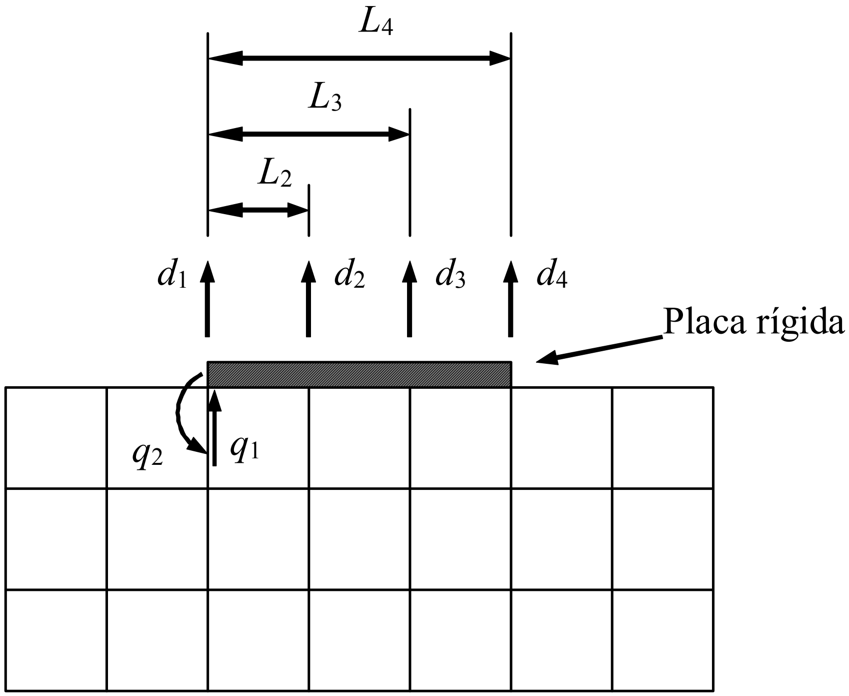
Ejemplo: Placa rígida

$d_1 = q_1$

$d_2 = q_1 + q_2 L_2$

$d_3 = q_1 + q_2 L_3$

$d_4 = q_1 + q_2 L_4$



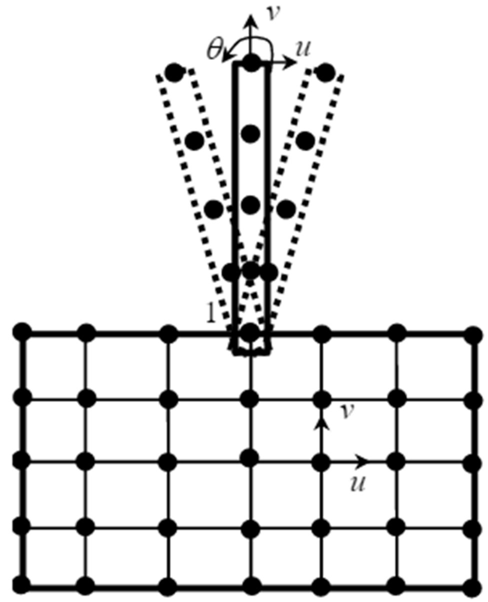
Eliminando q1 y q2:

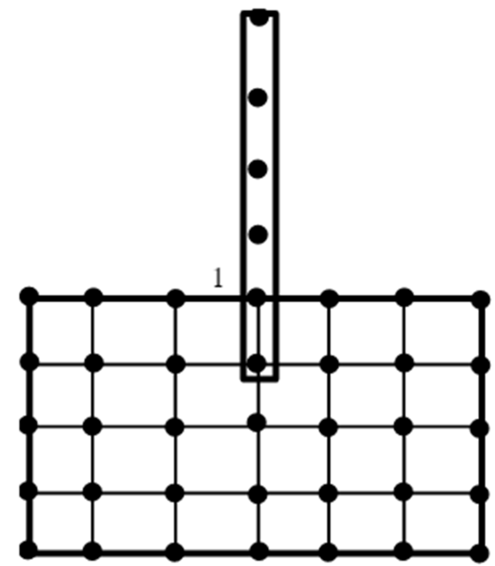
$(L_3 / L_2 - 1)\, d_1 - (L_3 / L_2)\, d_2 + d_3 = 0$

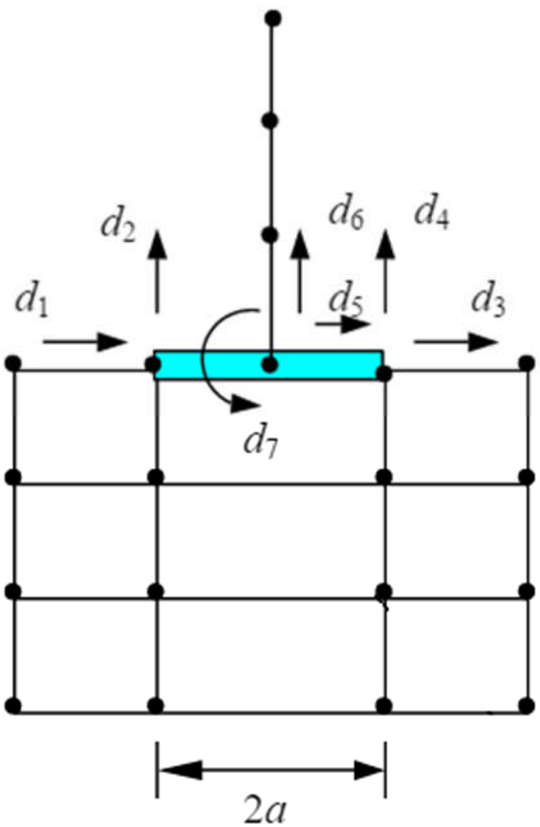
$(L_4 / L_2 - 1)\, d_1 - (L_4 / L_2)\, d_2 + d_4 = 0$

Restricciones - Constraints

Ejemplo: Unión viga / elemento plano







$$d_1 = d_5$$

$$d_2 = d_6 - a d_7$$

$$d_3 = d_5$$

$$d_4 = d_6 + a d_7$$

http://mashayekhi.iut.ac.ir/sites/mashayekhi.iut.ac.ir/files/files_course/lesson_16_1.pdf

Restricciones - Constraints

Método directo

$$[C]\{D\}-\{Q\}=0$$

C y Q matrices constantes que imponen restricciones.

$$[C_r C_c]\{D\}-\{Q\}=\{0\}$$

r: “restringidos”

c: “condensados”

$$\{D\}=\begin{Bmatrix} D_r \\ D_c \end{Bmatrix}$$

$$\begin{Bmatrix} D_r \\ D_c \end{Bmatrix}=\begin{bmatrix} I \\ -C_c^{-1}C_r \end{bmatrix}\{D_r\}+\begin{Bmatrix} 0 \\ C_c^{-1}Q \end{Bmatrix}\rightarrow\{D\}=[T]\{D_r\}+\{Q_0\}$$

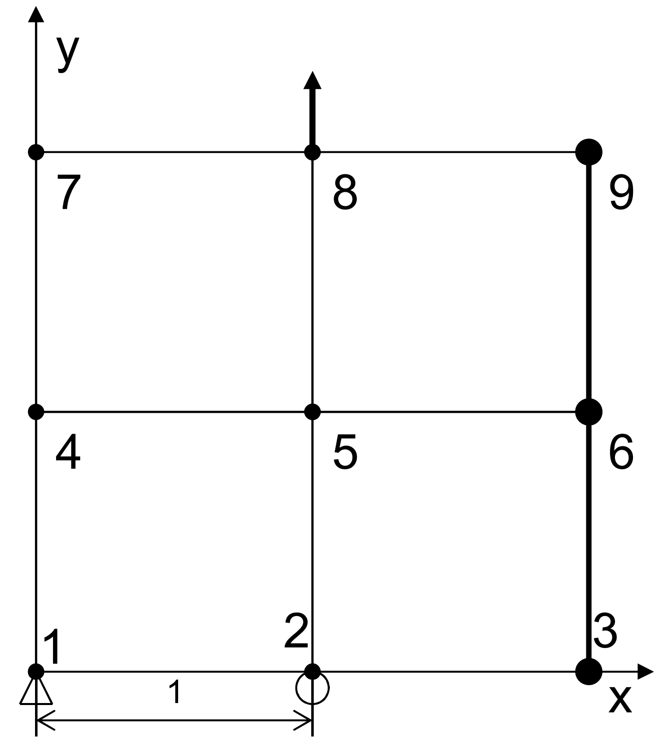
$$[K]\{D\}=\{R\}\rightarrow[K_r]\{D_r\}=\{R_r\}$$

$$[K_r]=[T]^T[K][T]$$

$$\{R_r\}=[T]^T(\{R\}-[K]\{Q_0\})$$

Restricciones - Constraints

Ejemplo: $u_3=u_6=u_9$



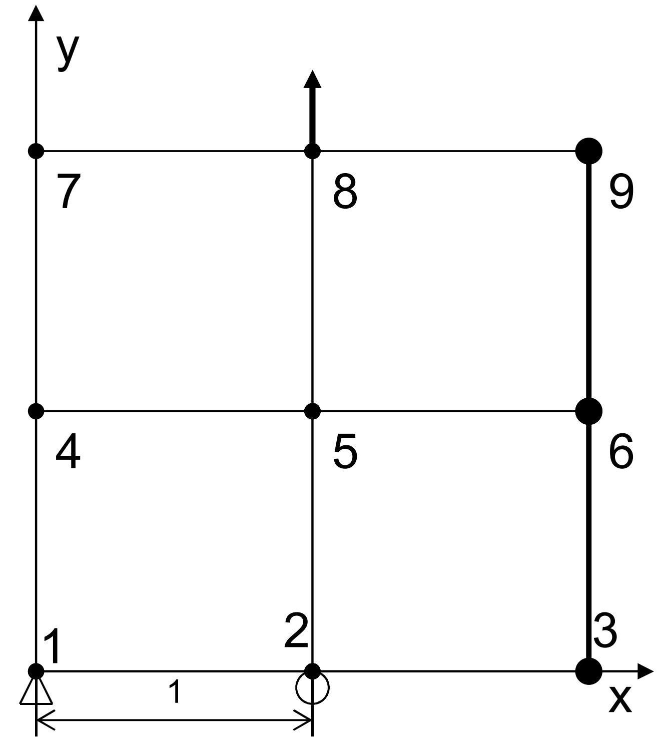
$$\underbrace{\begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 & \cdots & -1 & 0 \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & -1 & 0 \end{bmatrix}}_{[C]} \cdot \begin{Bmatrix} u_1 \\ \vdots \\ u_3 \\ \vdots \\ u_6 \\ \vdots \\ u_9 \\ v_9 \end{Bmatrix} = \underbrace{\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}_{\{Q\}}$$

$$\left[-C_c{}^{-1}C_r\right] = -\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}}_{cxc} \underbrace{\begin{bmatrix} 0 & \cdots & 0 & -1 & 0 \\ 0 & \cdots & 0 & -1 & 0 \end{bmatrix}}_{cxr} = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}}_{cxr} \quad ; \quad \{Q\} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{cx1}$$

Restricciones - Constraints

Ejemplo: $u_3=u_6=u_9$

$$
\begin{Bmatrix} u_1 \\ \vdots \\ v_2 \\ v_3 \\ \vdots \\ v_5 \\ v_6 \\ \vdots \\ v_9 \\ u_3 \\ u_6 \end{Bmatrix}
=
\underbrace{\begin{bmatrix}
1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 1 & 0 & \vdots & 0 & 0 & \vdots & 0 \\
0 & \cdots & 0 & 1 & \vdots & 0 & 0 & \vdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \vdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \vdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & 0
\end{bmatrix}}_{[T](c+r)xr}
\begin{Bmatrix} u_1 \\ \vdots \\ v_2 \\ v_3 \\ \vdots \\ v_5 \\ v_6 \\ \vdots \\ v_9 \end{Bmatrix}
+
\underbrace{\begin{Bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}}_{Q_0}
$$



Sistema Global

$$
\underbrace{[T]}_{rx(c+r)}{}^{T}\ \underbrace{[K]}_{(c+r)x(c+r)}\ \underbrace{[T]}_{(c+r)xr}\ \{D_r\} = \underbrace{[T]}_{rx(c+r)}{}^{T}\left(\underbrace{\{R\}}_{(c+r)} - \underbrace{[K]}_{(c+r)x(c+r)}\ \underbrace{\{Q_0\}}_{(c+r)}\right) \rightarrow [K_r]\{D_r\} = \{R_r\}
$$

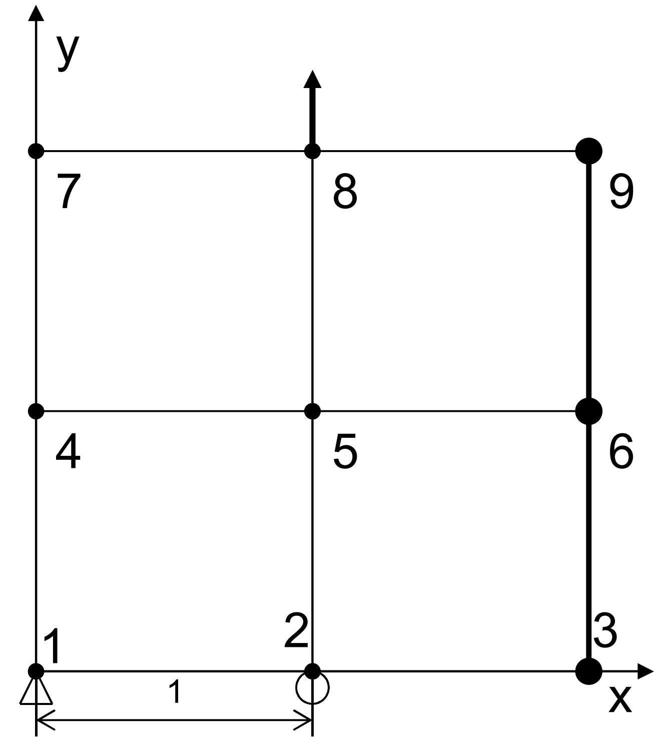
Restricciones Constraints

Multiplicadores de Lagrange

$$\Pi_p = \tfrac{1}{2}\{D\}^T[K]\{D\} - \{D\}^T\{R\} + \{\lambda\}^T\left([C]\{D\} - \{Q\}\right)$$

$$\downarrow$$

$$\frac{\partial}{\partial D_i};\frac{\partial}{\partial \lambda_i} \quad \rightarrow \begin{bmatrix} K & C^T \\ C & 0 \end{bmatrix}\begin{Bmatrix} D \\ \lambda \end{Bmatrix} = \begin{Bmatrix} R \\ Q \end{Bmatrix}$$

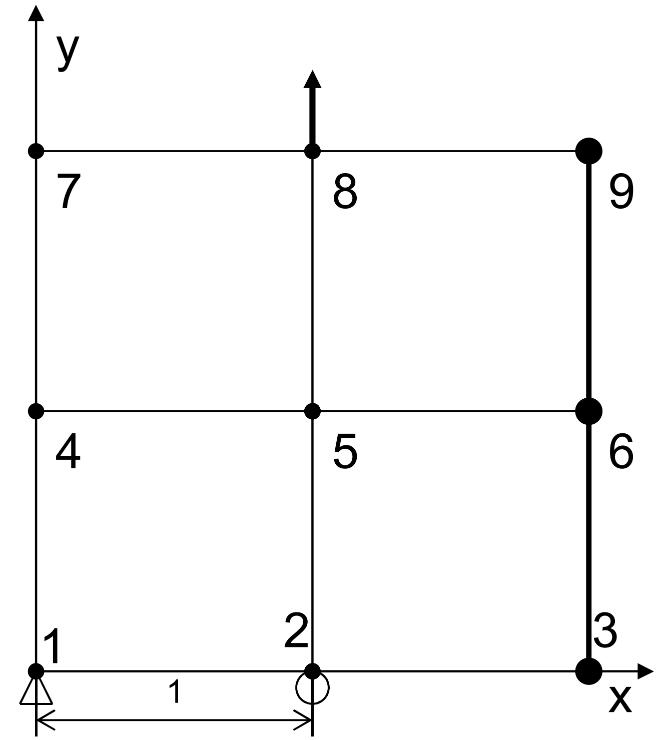


Ejemplo

$$\underbrace{\begin{bmatrix} K_{11} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & K_{1n} & 0 & 0 \\ \vdots & \ddots & & & & & & \vdots & \vdots & \vdots \\ \vdots & & \ddots & & & & & \vdots & 1 & 0 \\ \vdots & & & \ddots & & & & \vdots & \vdots & \vdots \\ \vdots & & & & \ddots & & & \vdots & 0 & 1 \\ \vdots & & & & & \ddots & & \vdots & \vdots & \vdots \\ \vdots & & & & & & \ddots & \vdots & -1 & -1 \\ K_{n1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & K_{nn} & \vdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & -1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & -1 & 0 & 0 & 0 \end{bmatrix}}_{[K](2c+r)x(2c+r)} \begin{Bmatrix} u_1 \\ \vdots \\ u_3 \\ \vdots \\ u_6 \\ \vdots \\ u_9 \\ v_9 \\ \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} Ru_1 \\ \vdots \\ Ru_3 \\ \vdots \\ Ru_6 \\ \vdots \\ Ru_9 \\ Rv_9 \\ Q_1 \\ Q_2 \end{Bmatrix}$$

Restricciones - Constraints

Penalización



$$[C]\{D\} - \{Q\} = \{t\} \Rightarrow \{t\} = \{0\}; \qquad [\alpha] = \begin{bmatrix} \alpha_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{nn} \end{bmatrix}$$

$$\Pi_p = \tfrac{1}{2}\{D\}^T[K]\{D\} - \{D\}^T\{R\} + \tfrac{1}{2}\cdot\{t\}^T[\alpha]\{t\}$$

$$\frac{\partial \Pi_p}{\partial D_i} = \{D\}^T[K] - \{R\} + \{t\}^T[\alpha][C] = 0$$

$$\{D\}^T[K] - \{R\} + \{D\}^T[C]^T[\alpha][C] - \{Q\}^T[\alpha][C] = 0 \;\rightarrow\; \left([K] + [C]^T[\alpha][C]\right)\{D\} = \{R\} + [C]^T[\alpha]\{Q\}$$

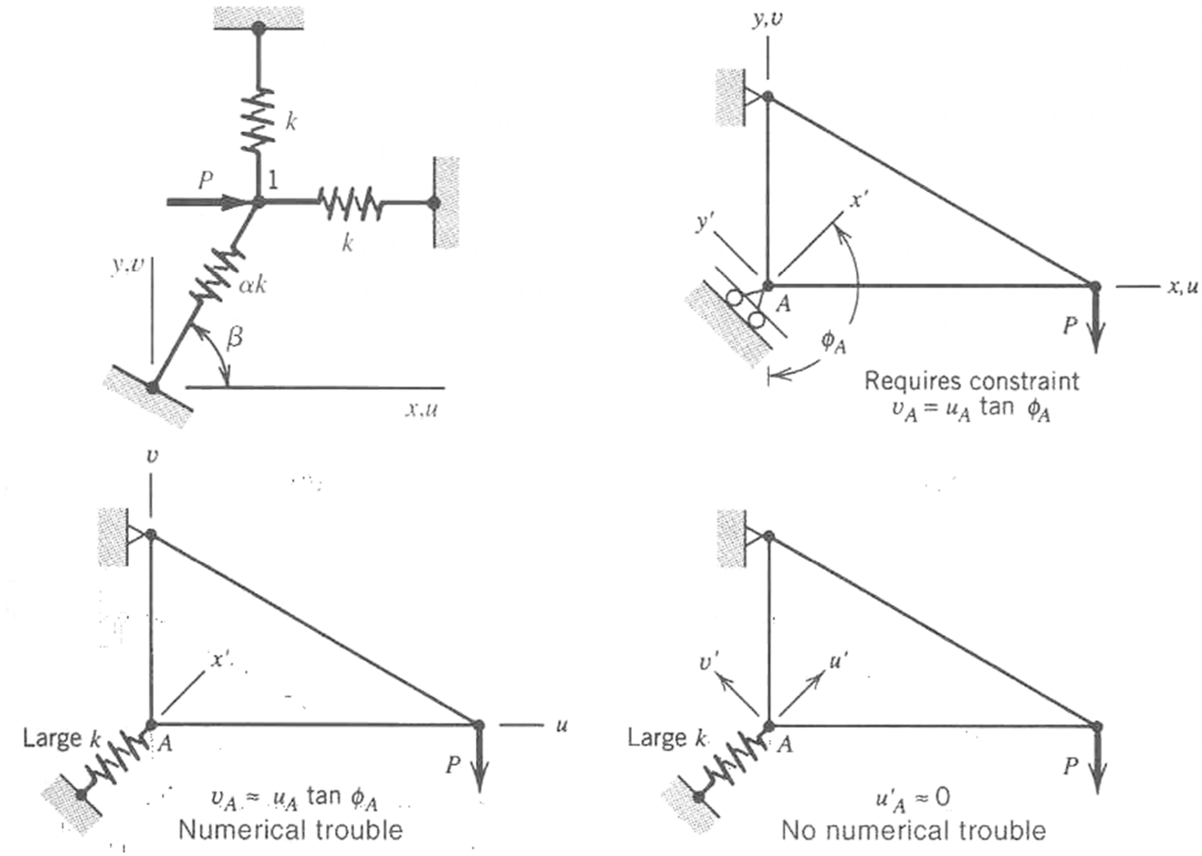
Ejemplo

$$\left([K] + [C]^T \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 & \cdots & -1 & 0 \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & -1 & 0 \end{bmatrix}\right)\{D\} = \{R\} + [C]^T \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Restricciones - Dof no alineados (Skew)

$$\begin{Bmatrix} u'_A \\ v'_A \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\phi_A & \sin\phi_A \\ -\sin\phi_A & \cos\phi_A \end{bmatrix}}_{\left[T_A\right]} \begin{Bmatrix} u_A \\ v_A \end{Bmatrix}$$

$$\left[T\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\phi_A & \sin\phi_A & 0 & 0 \\ 0 & 0 & -\sin\phi_A & \cos\phi_A & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\left[K\right]\left\{D\right\} = \left\{R\right\}$$

$$\left\{D\right\} = \begin{Bmatrix} u_A \\ v_A \\ u_B \\ \vdots \\ v_C \end{Bmatrix} \rightarrow \left\{D'\right\} = \begin{Bmatrix} u'_A \\ v'_A \\ u_B \\ \vdots \\ v_C \end{Bmatrix} \Rightarrow \left\{D\right\} = \left[T\right]^T \left\{D'\right\}$$

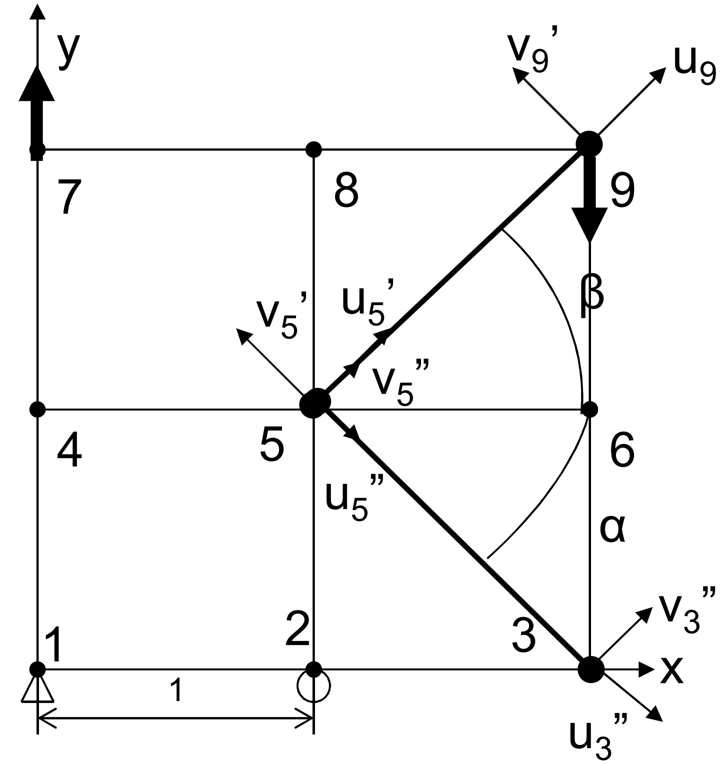
$$\underbrace{\left[T\right]\left[K\right]\left[T\right]^T}_{\left[K'\right]} \left\{D'\right\} = \underbrace{\left[T\right]\left\{R\right\}}_{\left[R'\right]}$$

Restricciones – Rigid Links

$$\begin{Bmatrix} u_5' \\ v_5' \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}}_{[T_\beta]} \begin{Bmatrix} u_5 \\ v_5 \end{Bmatrix}$$

$$\begin{Bmatrix} u_9' \\ v_9' \end{Bmatrix} = [T_\beta] \begin{Bmatrix} u_9 \\ v_9 \end{Bmatrix}$$

$$u_5' = u_9' \Rightarrow u_5 \cos\beta + v_5 \sin\beta - u_9 \cos\beta - v_9 \sin\beta = 0$$

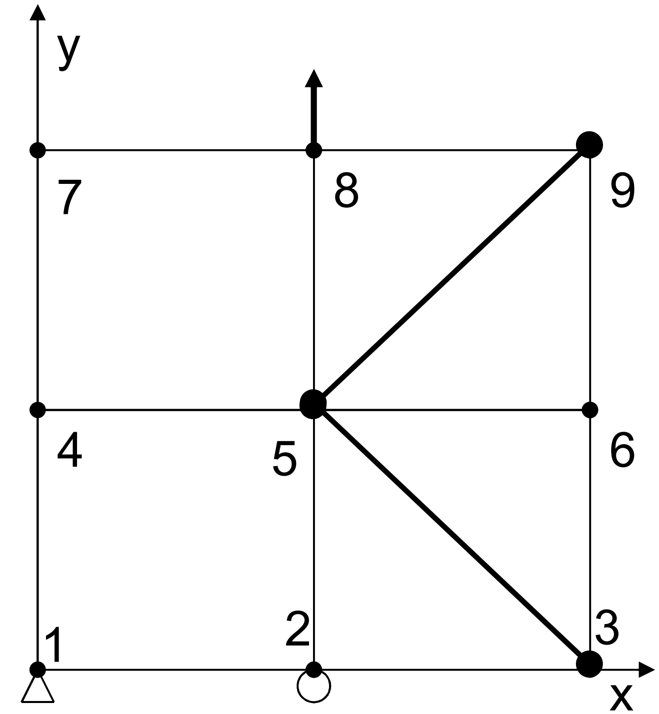


$$\begin{Bmatrix} u_5'' \\ v_5'' \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}} \begin{Bmatrix} u_5 \\ v_5 \end{Bmatrix}$$

$$\begin{Bmatrix} u_3'' \\ v_3'' \end{Bmatrix} = [T_\alpha] \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix}$$

$$u_5'' = u_3'' \Rightarrow u_5 \cos\alpha + v_5 \sin\alpha - u_3 \cos\alpha - v_3 \sin\alpha = 0$$

Restricciones – Rigid Links



$$u_5' = u_9' \Rightarrow \begin{array}{c} u_5 \cos\beta + v_5 \sin\beta - u_9 \cos\beta - v_9 \sin\beta = 0 \\ u_9 = v_5 \tan\beta + u_5 - v_9 \tan\beta \end{array}$$

$$u_5'' = u_3'' \Rightarrow \begin{array}{c} u_5 \cos\alpha + v_5 \sin\alpha - u_9 \cos\alpha - v_9 \sin\alpha = 0 \\ u_3 = v_5 \tan\alpha + u_5 - v_3 \tan\alpha \end{array}$$

$$\begin{Bmatrix} u_3 \\ v_3 \\ u_5 \\ v_5 \\ u_9 \\ v_9 \end{Bmatrix} = \underbrace{\begin{bmatrix} -\tan\alpha & 1 & \tan\alpha & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \tan\beta & -\tan\beta \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{[T]} \begin{Bmatrix} v_3 \\ u_5 \\ v_5 \\ v_9 \end{Bmatrix}$$

$$\{D\} = \begin{Bmatrix} u_1 \\ \vdots \\ u_3 \\ \vdots \\ u_9 \\ v_9 \end{Bmatrix} \rightarrow \{\widetilde{D}\} = \begin{Bmatrix} u_1 \\ \vdots \\ v_9 \end{Bmatrix} \Rightarrow \{D\} = [T]\{\widetilde{D}\}$$

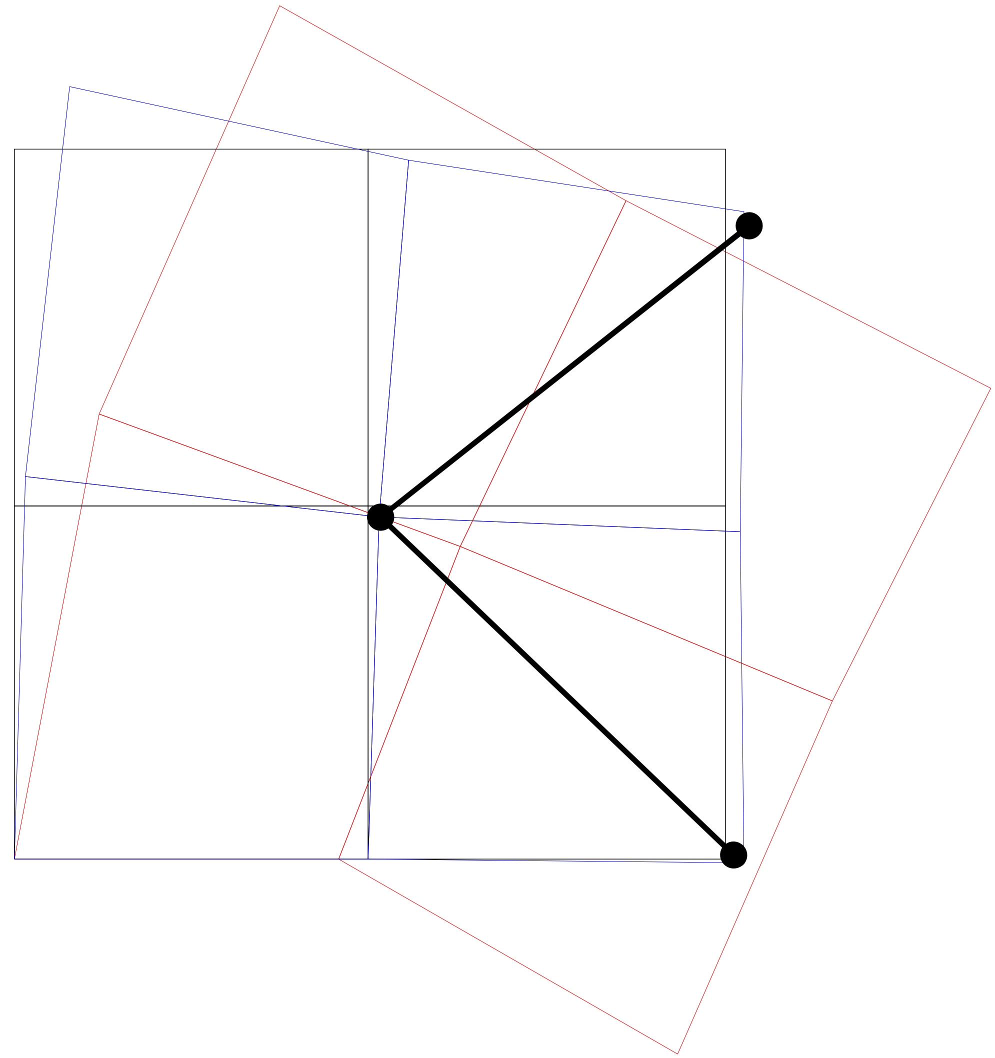
Restricciones – Rigid Links

$$[K]\{D\}=\{R\}$$

$$[K][T]\{\widetilde{D}\}=\{R\}$$

$$[T]^T[K][T]\{\widetilde{D}\}=[T]^T\{R\}$$

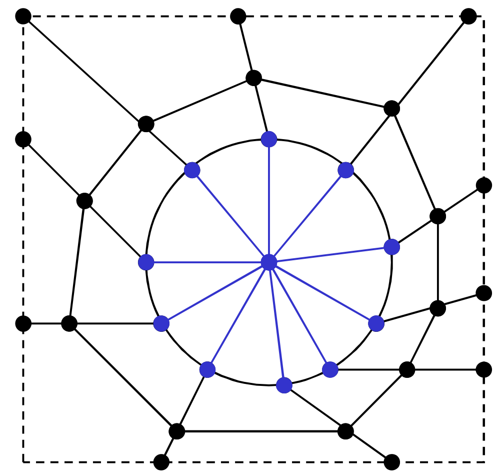
$$\underbrace{[T]^T[K][T]}_{[\widetilde{K}]}\{D'\}=\underbrace{[T]^T\{R\}}_{[\widetilde{R}]}$$



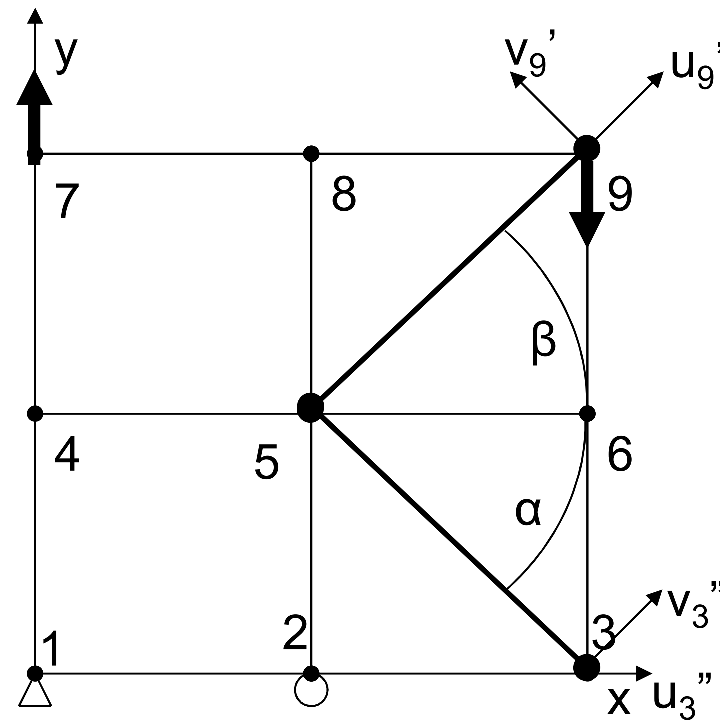
Restricciones – Rigid Links – Giros

$$v_9' = -u_9 \sin\beta + v_9 \cos\beta$$
$$v_3'' = -u_3 \sin\alpha + v_3 \cos\alpha$$

$$v_9' = v_3'' \Rightarrow -u_9 \sin\beta + v_9 \cos\beta + u_3 \sin\alpha - v_3 \cos\alpha = 0$$

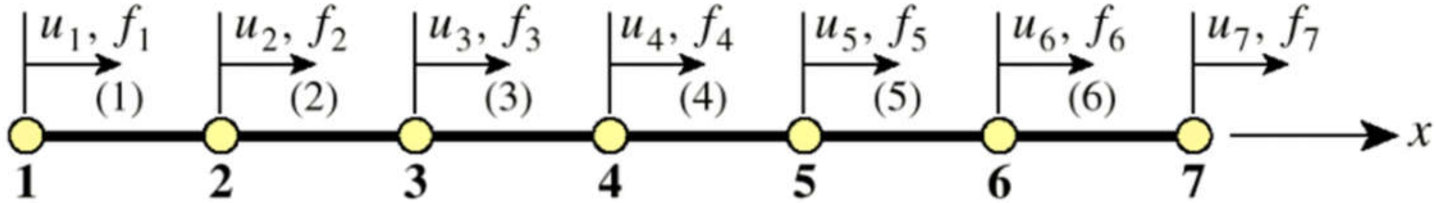


$$-u_9 \sin\beta + v_9 \cos\beta + u_3 \sin\alpha - v_3 \cos\alpha = 0$$
$$u_5 \cos\beta + v_5 \sin\beta - u_9 \cos\beta - v_9 \sin\beta = 0$$
$$u_5 \cos\alpha + v_5 \sin\alpha - u_9 \cos\alpha - v_9 \sin\alpha = 0$$



Restricciones - Constraints

Ejemplo de aplicación*: Método directo



$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

Imponemos un vínculo rígido: $u_2 = u_6 \rightarrow u_2 - u_6 = 0$

Tomando u_2 como Master:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix}$$

$$\mathbf{u} = \mathbf{T}\hat{\mathbf{u}}.$$

*http://www.colorado.edu/engineering/CAS/courses.d/IFEM.d

Restricciones - Constraints

Ejemplo de aplicación: Método directo

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$
$$\mathbf{u} = \mathbf{T}\hat{\mathbf{u}}$$

$$\hat{\mathbf{K}} = \mathbf{T}^T\mathbf{K}\mathbf{T}$$
$$\hat{\mathbf{f}} = \mathbf{T}^T\mathbf{f}$$

$$\hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}}$$

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 \\ K_{12} & K_{22}+K_{66} & K_{23} & 0 & K_{56} & K_{67} \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 \\ 0 & K_{56} & 0 & K_{45} & K_{55} & 0 \\ 0 & K_{67} & 0 & 0 & 0 & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2+f_6 \\ f_3 \\ f_4 \\ f_5 \\ f_7 \end{bmatrix}$$

Tomando u_6 como Master:

$$\begin{bmatrix} K_{11} & 0 & 0 & 0 & K_{12} & 0 \\ 0 & K_{33} & K_{34} & 0 & K_{23} & 0 \\ 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ K_{12} & K_{23} & 0 & K_{56} & K_{22}+K_{66} & K_{67} \\ 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_3 \\ f_4 \\ f_5 \\ f_2+f_6 \\ f_7 \end{bmatrix}$$

Múltiples restricciones:

$$\begin{aligned} 2u_3 + u_4 + u_5 &= 0 \\ u_2 - u_6 &= 0 \\ u_1 + 4u_4 &= 0 \end{aligned}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_5 \\ u_7 \end{bmatrix}$$

Restricciones - Constraints

Ejemplo de aplicación: Método directo

Caso no homogéneo

$$u_2 - u_6 = 0.2$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.2 \\ 0 \end{bmatrix}$$

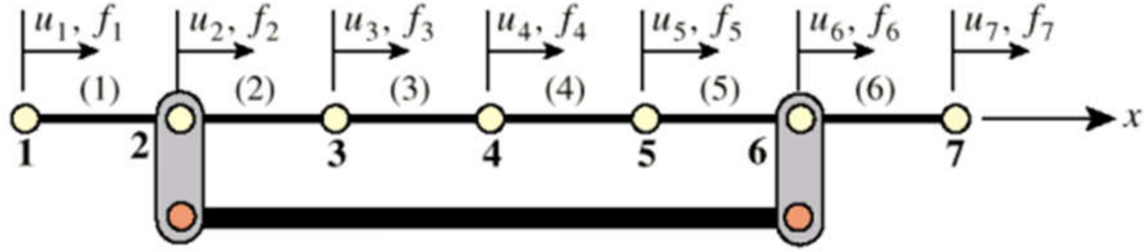
$$\mathbf{u} = \mathbf{T}\hat{\mathbf{u}} - \mathbf{g}$$

$$\mathbf{T}^T\mathbf{K}\mathbf{T}\hat{\mathbf{u}} = \hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{f}} = \mathbf{T}^T\mathbf{f} + \mathbf{T}^T\mathbf{K}\mathbf{g}$$

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 \\ K_{12} & K_{22}+K_{66} & K_{23} & 0 & K_{56} & K_{67} \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 \\ 0 & K_{56} & 0 & K_{45} & K_{55} & 0 \\ 0 & K_{67} & 0 & 0 & 0 & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + f_6 - 0.2K_{66} \\ f_3 \\ f_4 \\ f_5 - 0.2K_{56} \\ f_7 - 0.2K_{67} \end{bmatrix}$$

Restricciones - Constraints

Ejemplo de aplicación: Penalización



$$u_2 = u_6$$

$$\alpha \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_6 \end{bmatrix} = \begin{bmatrix} f_2^{(7)} \\ f_6^{(7)} \end{bmatrix}$$

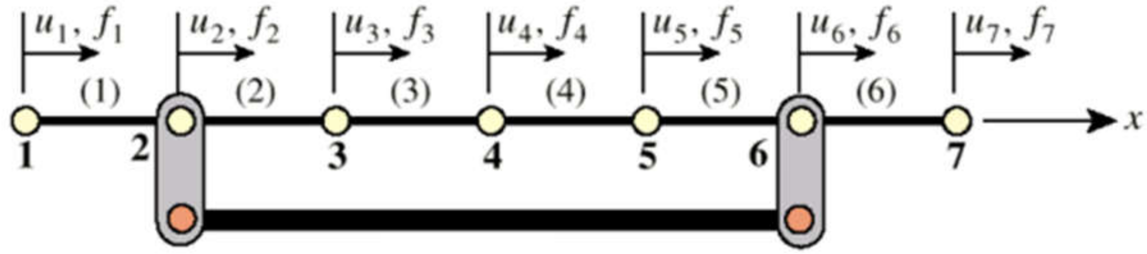
$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22}+\alpha & K_{23} & 0 & 0 & -\alpha & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & -\alpha & 0 & 0 & K_{56} & K_{66}+\alpha & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

Restricciones - Constraints

Ejemplo de aplicación: Penalización

Caso homogéneo

$$u_2 = u_6$$



$$\alpha \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_6 \end{bmatrix} = \begin{bmatrix} f_2^{(7)} \\ f_6^{(7)} \end{bmatrix}$$

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22}+\alpha & K_{23} & 0 & 0 & -\alpha & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 \\ 0 & -\alpha & 0 & 0 & K_{56} & K_{66}+\alpha & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

Caso no homogéneo

$$3u_3 + u_5 - 4u_6 = 1$$

$$\begin{bmatrix} 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_3 \\ u_5 \\ u_6 \end{bmatrix} = 1$$

$$\begin{bmatrix} 9 & 3 & -12 \\ 3 & 1 & -4 \\ -12 & -4 & 16 \end{bmatrix} \begin{bmatrix} u_3 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$$

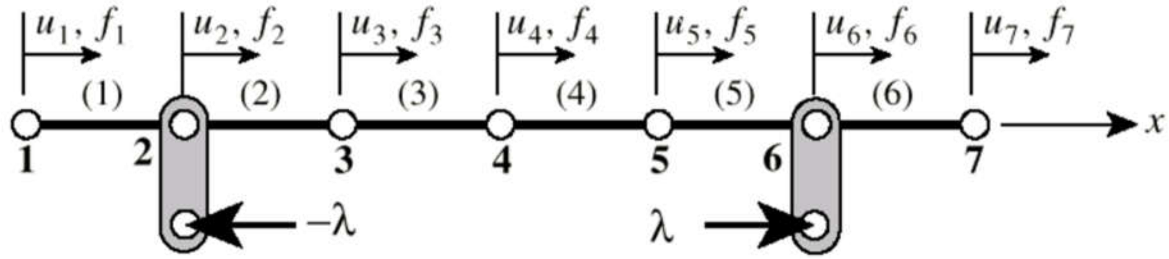
$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 \\ 0 & K_{23} & K_{33}+9\alpha & K_{34} & 3\alpha & -12\alpha & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 \\ 0 & 0 & 3\alpha & K_{45} & K_{55}+\alpha & K_{56}-4\alpha & 0 \\ 0 & 0 & -12\alpha & 0 & K_{56}-4\alpha & K_{66}+16\alpha & K_{67} \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3+3\alpha \\ f_4 \\ f_5+\alpha \\ f_6-4\alpha \\ f_7 \end{bmatrix}$$

Restricciones - Constraints

Ejemplo de aplicación: Multiplicadores de Lagrange

Caso homogéneo

$$u_2 = u_6$$



$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 & 1 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} & -1 \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ 0 \end{bmatrix}$$

Caso no homogéneo

$$\begin{aligned} 5u_2 - 8u_7 &= 3 \\ u_2 - u_6 &= 0 \\ 3u_3 + u_5 - 4u_6 &= 1 \end{aligned}$$

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & K_{23} & K_{33} & K_{34} & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & K_{34} & K_{44} & K_{45} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{45} & K_{55} & K_{56} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & K_{56} & K_{66} & K_{67} & -1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & K_{67} & K_{77} & 0 & -8 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & -4 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

Restricciones Constraints

Resumen métodos

- Método directo:
  Se eligen nodos “Master” y “Slave” y se eliminan estos últimos explícitamente.

- Multiplicadores de Lagrange
  Se agrega una incógnita para cada restricción. Físicamente representan las fuerzas necesarias que se deberían aplicar para lograr dicha restricción exactamente.

- Penalización
  Se introducen elementos elásticos ficticios que imponen aproximadamente el vínculo parametrizados por un peso. Se logra la restricción perfecta cuando el peso va a infinito. Se aumenta el modelo FEM con estos elementos de penalidad.

	Directo	Lagrange	Penalización
Generalidad	Aceptable	Excelente	Excelente
Implementación	Pobre - Aceptable	Sencilla	Fácil
Criterio del usuario	Alto	Casi ninguno	Alto
Precisión	Variable	Excelente	Mediocre
Sensitividad a la dependencia de las restricciones	Alta	Alta	Ninguna
[K] definida positiva	Sí	No	Sí
Modifica vector cargas (caso homog.)	Sí	Sí	No