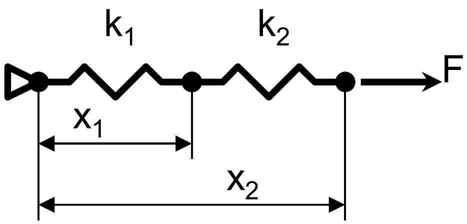
Principio de los Trabajos Virtuales vs Mínima energía potencial total



$\pi = U + V \; ; \; \delta(\pi) = 0$

$$W_{\text{int}} = \delta x_1 (x_1 k_1) + \delta (x_2 - x_1)((x_2 - x_1) k_2) \qquad ; \qquad W_{ext} = F \delta x_2$$

$$\delta x_1 (x_1 k_1) + (\delta x_2 - \delta x_1)((x_2 - x_1) k_2) = F \delta x_2$$

$$\delta x_1 (x_1 k_1 - (x_2 - x_1) k_2) + \delta x_2 ((x_2 - x_1) k_2 - F) = 0 \;\; \forall \delta x_1 \;\; y \;\; \forall \delta x_2$$

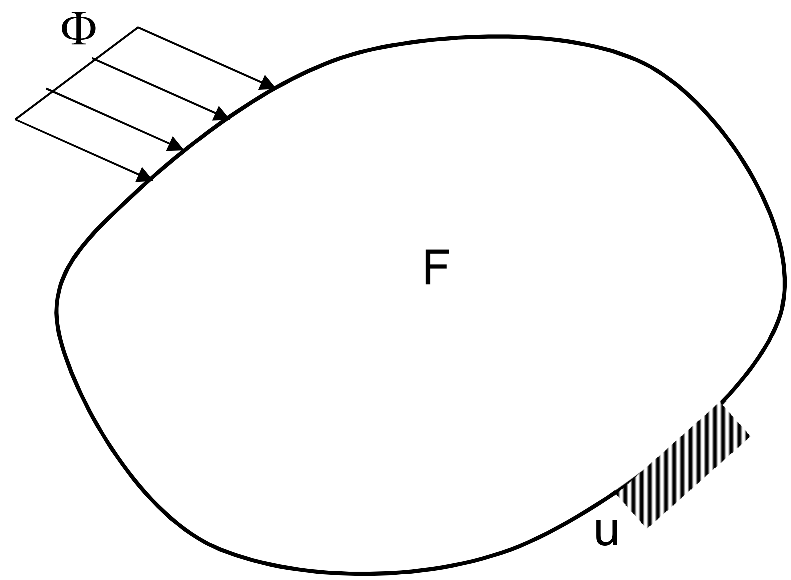
$x_1 k_1 - (x_2 - x_1) k_2 = 0$ Equilibrio en x_1

$(x_2 - x_1) k_2 - F = 0$ Equilibrio en x_2

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

Principio de los trabajos virtuales ¿Qué resuelve?

Al trabajar sólo con fuerzas conservativas, el trabajo externo es igual al trabajo interno.



1 contra 1 (como un producto escalar pero de tensores)

$$W_{\text{int}} = \int_v \underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}}\, dv \qquad W_{ext} = \int_v \underline{F} \cdot \delta\underline{u}\, dv + \int_S \underline{\Phi} \cdot \delta\underline{u}\, ds$$

$$W_{\text{int}} = W_{ext} \quad \Rightarrow \quad \int_v \underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}}\, dv = \int_v \underline{F} \cdot \delta\underline{u}\, dv + \int_S \underline{\Phi} \cdot \delta\underline{u}\, ds$$

$$\Leftrightarrow \begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{F} = \underline{0} \\ \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right) = \underline{\underline{\varepsilon}} \end{cases}$$

Pequños desplazamientos

Principio de los trabajos virtuales ¿Qué resuelve?

$$Ext)\quad W_{ext} = \int_v \underline{F}\cdot\delta\underline{u}\,dv + \int_S \underline{\Phi}\cdot\delta\underline{u}\,ds = \int_v \underline{F}\cdot\delta\underline{u}\,dv + \int_S \underline{n}\cdot\underline{\underline{\sigma}}\cdot\delta\underline{u}\,ds = \int_v \underline{F}\cdot\delta\underline{u}\,dv + \int_v \nabla\cdot\left(\underline{\underline{\sigma}}\cdot\delta\underline{u}\right)dv$$

$$\nabla\cdot\left(\underline{\underline{\sigma}}\cdot\delta\underline{u}\right) \qquad = \qquad \left(\nabla\cdot\underline{\underline{\sigma}}\cdot\delta\underline{u}\right) + \left(\underline{\underline{\sigma}} : \nabla\delta\underline{u}\right)$$

$$\downarrow \qquad\qquad\qquad\qquad \uparrow$$

$$\frac{\partial}{\partial x_l}\underline{e}_l\cdot\left(\sigma_{ij}\,\underline{e}_i\,\underline{e}_j\cdot\delta u_k\,\underline{e}_k\right) = \frac{\partial}{\partial x_l}\left(\sigma_{lj}\cdot\delta u_j\right) = \sigma_{lj,l}\cdot\delta u_j + \sigma_{lj} : \delta u_{j.l}$$

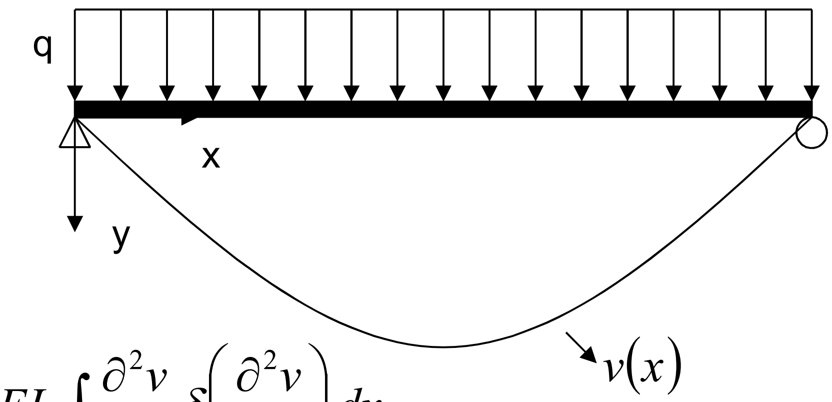
$$W_{ext} = \int_v \underline{F}\cdot\delta\underline{u} + \nabla\cdot\underline{\underline{\sigma}}\cdot\delta\underline{u} + \underline{\underline{\sigma}} : \nabla\delta\underline{u}\,dv = \int_v \left(\nabla\cdot\underline{\underline{\sigma}} + \underline{F}\right)\cdot\delta\underline{u} + \underline{\underline{\sigma}} : \nabla\delta\underline{u}\,dv = \int_v \underline{\underline{\sigma}} : \nabla\delta\underline{u}\,dv$$

$$Int)\quad W_{\text{int}} = \int_v \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}\,dv = \int_v \underline{\underline{\sigma}} : \tfrac{1}{2}\left(\nabla\delta\underline{u} + \nabla\delta\underline{u}^T\right)dv = \int_v \tfrac{1}{2}\underline{\underline{\sigma}} : \nabla\delta\underline{u} + \tfrac{1}{2}\underline{\underline{\sigma}} : \nabla\delta\underline{u}^T\,dv = \int_v \underline{\underline{\sigma}} : \nabla\delta\underline{u}\,dv$$

Sigma es simetrico

$$W_{ext} = W_{\text{int}}$$

Principio de los Trabajos Virtuales



$$W_{ext} = \int_L q\,\delta v\,dx$$

$$W_{\text{int}} = \int_V \sigma\,\delta\varepsilon\,dv = \int_V E\varepsilon\,\delta\varepsilon\,dv = E\int_L\int_A \left(-y\frac{\partial^2 v}{\partial x^2}\right)\delta\left(-y\frac{\partial^2 v}{\partial x^2}\right)dA\,dx = EI_y\int_L \frac{\partial^2 v}{\partial x^2}\,\delta\left(\frac{\partial^2 v}{\partial x^2}\right)dx$$

La escuación de fuerte de vigas de de 4to orden, sin embargo, la debil es de 2do orden. Para barras la fuerte es de 2do y la debil de 1er.

Proponemos

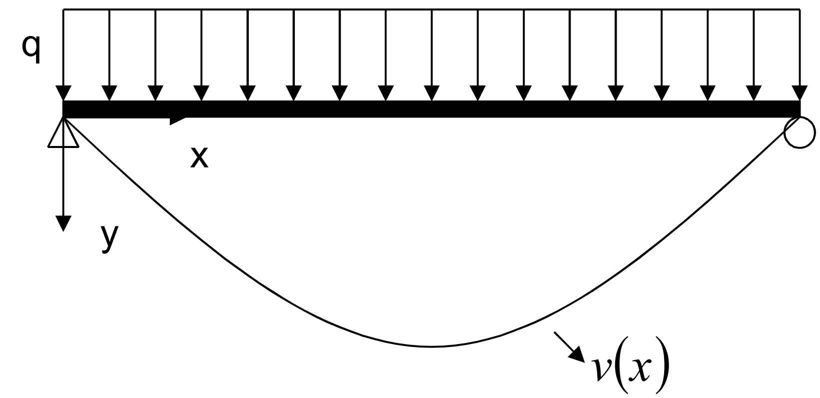
$$v = x(x-L)\left(a_1 + a_2 x + a_3 x^2\right) \quad ; \quad \delta v = x(x-L)\left(\delta a_1 + \delta a_2 x + \delta a_3 x^2\right)$$

$$W_{ext} = q\int_L x(x-L)\left(\delta a_1 + \delta a_2 x + \delta a_3 x^2\right)dx = q\left(\delta a_1\frac{L^3}{6} + \delta a_2\frac{L^4}{12} + \delta a_3\frac{L^5}{25}\right)$$

$$\frac{\partial^2 v}{\partial x^2} = 2a_1 + a_2(6x-2L) + a_3\left(12x^2 - 6Lx\right) \quad ; \quad \delta\frac{\partial^2 v}{\partial x^2} = 2\delta a_1 + \delta a_2(6x-2L) + \delta a_3\left(12x^2 - 6Lx\right)$$

$$W_{\text{int}} = EI_y\int_L \left(2a_1 + a_2(6x-2L) + a_3\left(12x^2-6Lx\right)\right)\left(2\delta a_1 + \delta a_2(6x-2L) + \delta a_3\left(12x^2-6Lx\right)\right)dx$$

Principio de los Trabajos Virtuales



$$W_{\text{int}} - W_{ext} = \delta a_1\left(2EIL\left(a_3L^2 + a_2L + 2a_1\right) - \frac{qL^3}{6}\right)$$
$$+\delta a_2\left(2EIL\left(2a_3L^3 + 2a_2L^2 + a_1L\right) - \frac{qL^4}{12}\right)$$
$$+\delta a_3\left(2EIL\left(\frac{12}{5}a_3L^4 + 2a_2L^3 + a_1L^2\right) - \frac{qL^5}{20}\right)$$

$$W_{\text{int}} - W_{ext} = 0 \qquad \forall\, \delta a_1, \delta a_2, \delta a_3$$

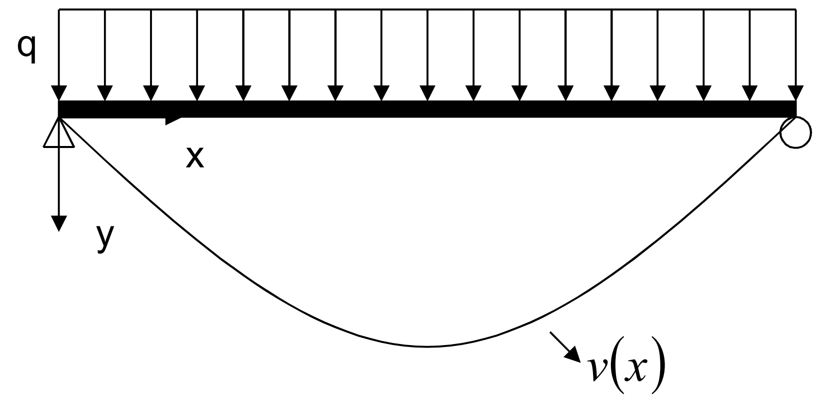
$$\left.\begin{array}{r} 12EIL\left(a_3L^2 + a_2L + 2a_1\right) - qL^3 = 0 \\ 24EIL\left(2a_3L^3 + 2a_2L^2 + a_1L\right) - qL^4 = 0 \\ 8EIL\left(12a_3L^4 + 10a_2L^3 + 5a_1L^2\right) - qL^5 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a_1 = -\dfrac{qL^2}{24EI} \\ a_2 = -\dfrac{qL}{24EI} \\ a_3 = \dfrac{q}{24EI} \end{array}\right.$$

Verificación

$$v = \frac{qx}{24EI}\left(L^3 - 2Lx^2 + x^3\right)$$
$$\theta = \frac{\partial v}{\partial x} = \frac{q}{24EI}\left(L^3 - 6Lx^2 + 4x^3\right)$$
$$M = EI\,\frac{\partial \theta}{\partial x} = \frac{q}{24}\left(-12Lx + 12x^2\right)$$
$$V = -\frac{\partial M}{\partial x} = -\frac{q}{24}\left(-12L + 24x\right)$$
$$Q = \frac{\partial V}{\partial x} = -q$$

Solución
Exacta

Principio de los Trabajos Virtuales



Proponemos un orden menos

$$v = x(x-L)(a_1 + a_2 x) \quad ; \qquad \delta v = x(x-L)(\delta a_1 + \delta a_2 x)$$

$$W_{\text{int}} - W_{ext} = \delta a_1\left(2EIL\left(a_2 L + 2a_1\right) - \frac{qL^3}{6}\right) + \delta a_2\left(2EIL\left(2a_2 L^2 + a_1 L\right) - \frac{qL^4}{12}\right)$$

$$W_{\text{int}} - W_{ext} = 0 \qquad \forall\, \delta a_1, \delta a_2$$

$$\left.\begin{array}{l} 12EIL\left(a_2 L + 2a_1\right) - qL^3 = 0 \\ 24EIL\left(2a_2 L^2 + a_1 L\right) - qL^4 = 0 \end{array}\right\} \Rightarrow \begin{cases} a_1 = -\dfrac{qL^2}{24EI} \\ a_2 = 0 \end{cases}$$

Solución
Aproximada

Verificación

$$v = \frac{qL^2 x(x-L)}{24EI}$$

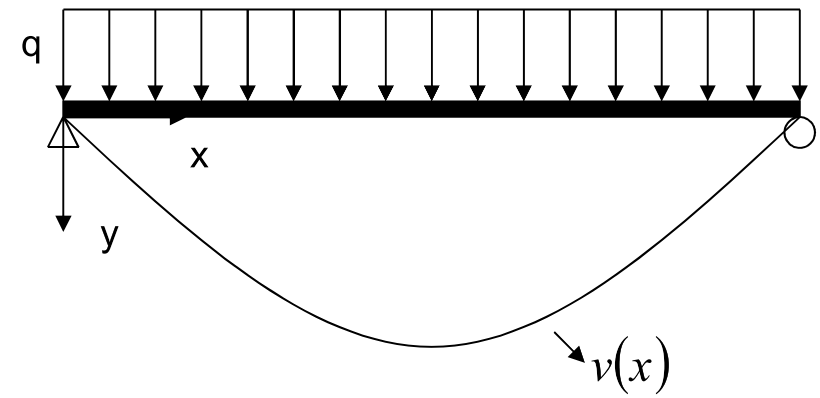
$$\theta = \frac{\partial v}{\partial x} = \frac{L^2 q}{24EI}(2x - L)$$

$$M = EI\frac{\partial \theta}{\partial x} = \frac{L^2 q}{12}$$

$$V = -\frac{\partial M}{\partial x} = 0$$

$$Q = \frac{\partial V}{\partial x} = 0$$

Principio de los Trabajos Virtuales



Proponemos un orden más

$$v = x(x-L)\left(a_1 + a_2 x + a_3 x^2 + a_4 x^3\right)$$

$$\delta v = x(x-L)\left(\delta a_1 + \delta a_2 x + \delta a_3 x^2 + \delta a_4 x^3\right)$$

$$W_{\text{int}} - W_{ext} = 0 \qquad \forall\, \delta a_1, \delta a_2, \delta a_3, \delta a_4$$

$$\begin{cases} a_1 = -\dfrac{qL^2}{24EI} \\ a_2 = -\dfrac{qL}{24EI} \\ a_3 = \dfrac{q}{24EI} \\ a_4 = 0 \end{cases}$$

Solución Exacta

Problema Elástico

Equilibrio

3 Ec.

6 Incógnitas

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f} = \rho \frac{\partial \underline{v}}{\partial t} \quad ; \quad \sigma_{ji,j} + f_i = \rho \frac{\partial v_i}{\partial t}$$

Consistencia

6 Ecs.

9 Incógnitas

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{I}}\right) \begin{cases} \varepsilon_{kl} = \frac{1}{2}\left(u_{l,k} + u_{k,l} + u_{i,k}u_{i,l}\right) \\ \downarrow \\ \varepsilon_{kl} \approx \frac{1}{2}\left(u_{l,k} + u_{k,l}\right) \end{cases}$$

Pequeñas deformaciones $\rightarrow$ Lineal

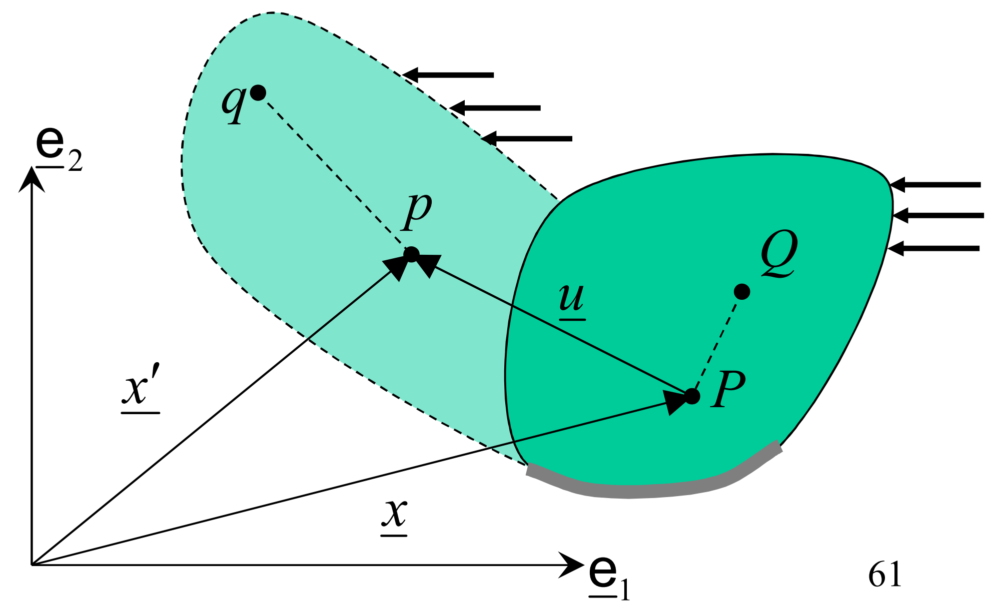
Constitutivas

6 Ecs.

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}}\,\underline{\underline{\varepsilon}} \quad ; \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

Condiciones de Borde $u_i = \tilde{u}_i$

$$^{(\underline{n})}\underline{\Phi} = \underline{\underline{\sigma}} \cdot \underline{n} \quad ; \quad ^{(\underline{n})}t_i = \sigma_{ij} n_j$$



Relación Constitutiva General

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

Consistencia: Operador

Desplazamiento-Deformación

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Casos Planos – Relaciones Constitutivas

Esfuerzo Membranal $\sigma_{x,z} = \sigma_{y,z} = 0$

Tensión Plana

Condiciones

- $\sigma_z = \tau_{xz} = \tau_{zy} = 0$
- $\varepsilon_z \neq 0$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{\left(1-\nu^2\right)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Deformación Plana

Condiciones

- $w=0$
- $\varepsilon_z = \gamma_{xz} = \gamma_{zy} = 0$
- $\sigma_z \neq 0$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\nu)} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Operador

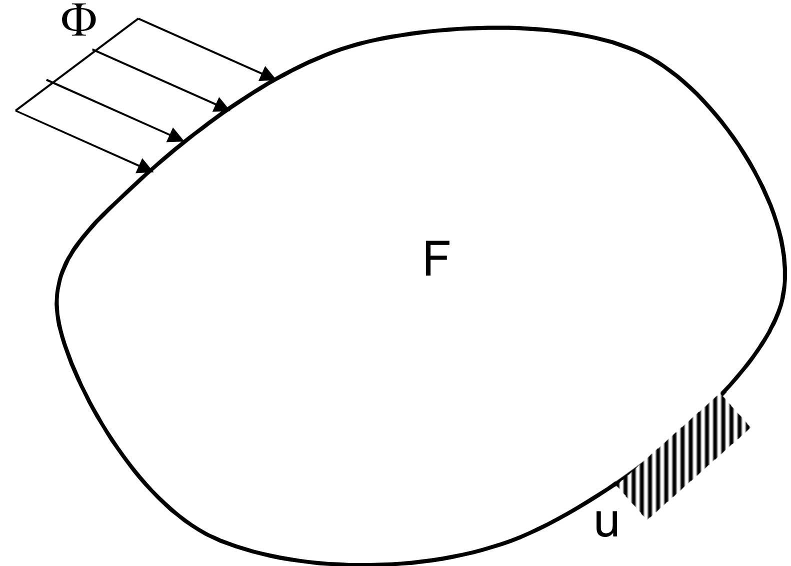
Desplazamiento-Deformación

$$2D) \quad \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Matriz B

Principio de los trabajos virtuales - Desplazamiento

$$\left.\begin{aligned} W_{\text{int}} &= \int_v \{\delta\varepsilon\}^T \{\sigma\} dv \\ W_{ext} &= \int_v \{\delta u\}^T \{F\} dv + \int_S \{\delta u\}^T \{\Phi\} ds + \{\delta d\}^T \{R\} \end{aligned}\right\}$$



$$\underbrace{W_{\text{int}} = W_{ext}}_{\Downarrow}$$

$$\int_v \{\delta\varepsilon\}^T \{\sigma\} dv = \int_v \{\delta u\}^T \{F\} dv + \int_S \{\delta u\}^T \{\Phi\} ds + \{\delta d\}^T \{R\}$$

<u>Discretización</u> $\{u(x,y,z)\} = [N(x,y,z)]\{d\}$

$$\{\varepsilon(x,y,z)\} = [\partial]\{u(x,y,z)\} = \underbrace{[\partial][N(x,y,z)]}_{[B(x,y,z)]}\{d\} \quad \{\varepsilon\}_T = \{\varepsilon\} - \{\varepsilon\}_0 = [B(x,y,z)]\{d\} - \{\varepsilon\}_0$$

$$\{\sigma(x,y,z)\} = [C]\{\varepsilon\}_T + \{\sigma_0\} = [C](\{\varepsilon(x,y,z)\} - \{\varepsilon_0(x,y,z)\}) + \{\sigma_0\}$$

<u>Variaciones</u> $\{\delta u\}^T = \{\delta d\}^T [N]^T$ $\qquad$ $\{\delta\varepsilon\}^T = \{\delta d\}^T [B]^T$

Principio de los trabajos virtuales - Desplazamiento

$$\int_v \{\delta\varepsilon\}^T [C]\{\varepsilon\}dv - \int_v \{\delta\varepsilon\}^T [C]\{\varepsilon_0\}dv + \int_v \{\delta\varepsilon\}^T \{\sigma_0\}dv = \int_v \{\delta u\}^T \{F\}dv + \int_S \{\delta u\}^T \{\Phi\}ds + \{\delta d\}^T \{R\}$$

$$\{\delta d\}^T \left(\int_v [B]^T [C][B]\{d\}dv - \int_v [B]^T [C]\{\varepsilon_0\}dv + \int_v [B]^T \{\sigma_0\}dv \right) = \{\delta d\}^T \left[\int_v [N]^T \{F\}dv + \int_S [N]^T \{\Phi\}ds + \{R\} \right]$$

$$\{\delta d\}^T \left(\int_v [B]^T [C][B]dv\{d\} - \int_v [B]^T [C]\{\varepsilon_0\}dv + \int_v [B]^T \{\sigma_0\}dv - \int_v [N]^T \{F\}dv - \int_S [N]^T \{\Phi\}ds - \{R\} \right) = 0 ; \forall \{\delta d\}^T$$

<u>Nota:</u> $\int_v [B]^T [C][B]dv = \sum_{elementos} \int_{v_e} [B_e]^T [C_e][B_e]dv_e$

$$\underbrace{\int_v [B]^T [C][B]dv}_{[K]} \{d\} = \underbrace{\int_v [N]^T \{F\}dv + \int_S [N]^T \{\Phi\}ds + \{R\} + \int_v [B]^T [C]\{\varepsilon_0\}dv - \int_v [B]^T \{\sigma_0\}dv}_{\{R\}}$$

$$[K]\{D\} = \{R\}$$