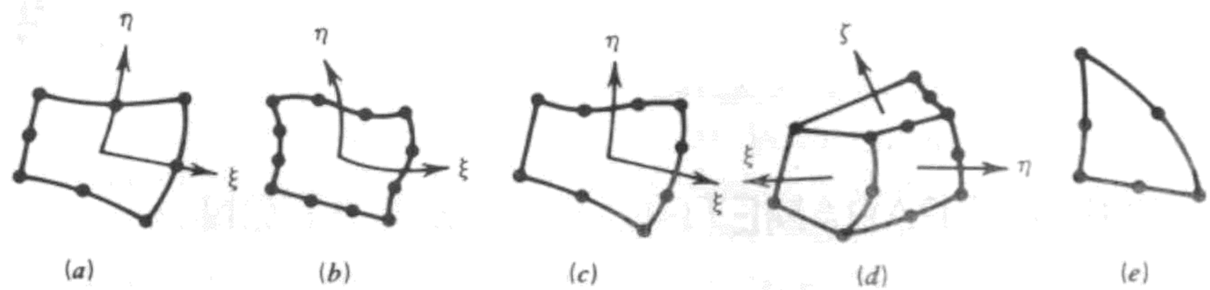
Formulación Isoparamétrica



Cambio de coordenadas

Estructurales X,Y,Z $\Rightarrow$ Naturales ξ,η,ζ

- Geometría Irregular $\Rightarrow$ Regular
- Dimensiones variables $\Rightarrow$ Fijas

Deformación — Geometría

$$\{u \quad v \quad w\} = [N(x,y,z)]\{d\} \Rightarrow \{x \quad y \quad z\} = [N(\xi,\eta,\zeta)]\{c\}$$

Mapeamos con las funciones de forma las posiciones de los elementos

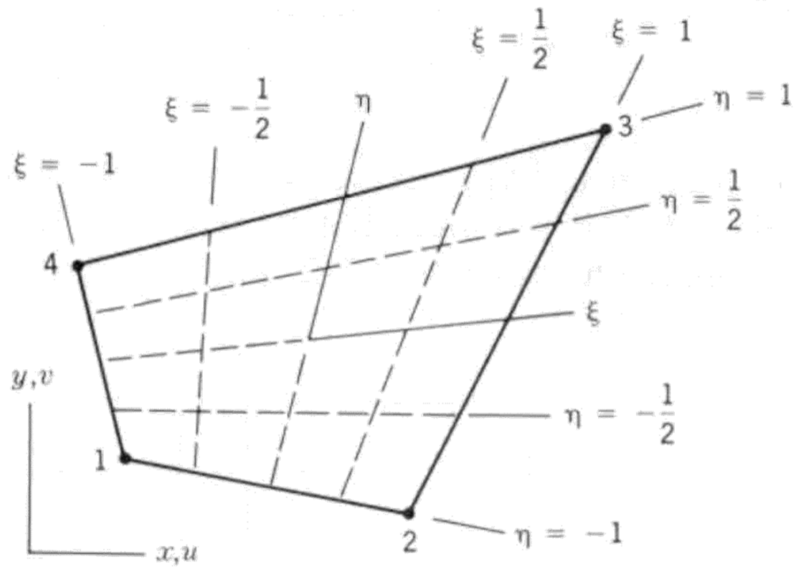
Funciones de forma

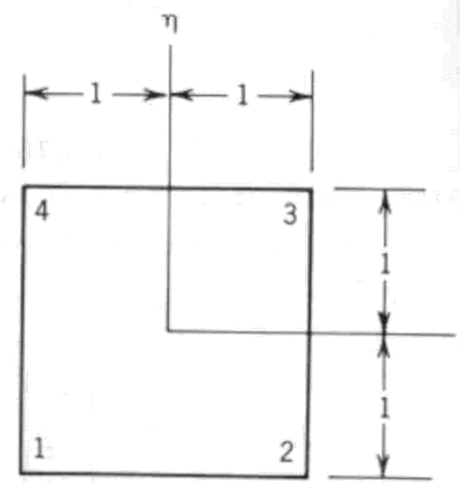
$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$





Discretización Geometría $\quad x(\xi,\eta) = \sum_i N(\xi,\eta)_i x_i \qquad y(\xi,\eta) = \sum_i N(\xi,\eta)_i y_i$

Formulación Isoparamétrica

Interpolación 2D

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \sum_i N(\xi,\eta)_i x_i \\ \sum_i N(\xi,\eta)_i y_i \end{Bmatrix} = [N(\xi,\eta)]\{c\} \quad ; \quad \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} \sum_i N(x,y)_i u_i \\ \sum_i N(x,y)_i v_i \end{Bmatrix} = [N(x,y)]\{d\}$$

Matriz de Rigidez Isoparamétrica

$$[K]_i = \int_{A_e} [B(x,y)]^T [E][B(x,y)] t\, dA_e = \int_{-1}^{1}\int_{-1}^{1} [B(\xi,\eta)]^T [E][B(\xi,\eta)] |J| t\, d\xi\, d\eta$$

$$B(x,y) = B(x(\xi,\eta), y(\xi,\eta)) = B(\xi,\eta) = [\partial(x,y)][N(x,y)] = [\partial(x,y)][N(\xi(x,y),\eta(x,y))]$$

Derivación

$$\frac{\partial N_i(\xi(x,y),\eta(x,y))}{\partial x} = \frac{\partial N_i(\xi(x,y),\eta(x,y))}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_i(\xi(x,y),\eta(x,y))}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial N_i(\xi(x,y),\eta(x,y))}{\partial y} = \frac{\partial N_i(\xi(x,y),\eta(x,y))}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_i(\xi(x,y),\eta(x,y))}{\partial \eta}\frac{\partial \eta}{\partial y}$$

Formulación Isoparamétrica

Jacobiano $J = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \sum_k N_k(\xi,\eta) x_k}{\partial \xi} & \dfrac{\partial \sum_k N_k(\xi,\eta) y_k}{\partial \xi} \\ \dfrac{\partial \sum_k N_k(\xi,\eta) x_k}{\partial \eta} & \dfrac{\partial \sum_k N_k(\xi,\eta) y_k}{\partial \eta} \end{bmatrix}$; $J^{-1} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix}$

Matriz Desplazamiento Deformación

$$\mathrm{B}(x,y) = \mathrm{B}(x(\xi,\eta), y(\xi,\eta)) = \mathrm{B}(\xi,\eta) = [\partial(x,y)][\mathrm{N}(x,y)] = [\partial(x,y)][\mathrm{N}(\xi(x,y), \eta(x,y))]$$

$$\varepsilon(x,y) = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} u,_x \\ v,_y \\ u,_y + v,_x \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u,_x \\ u,_y \\ v,_x \\ v,_y \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} [J^{-1}] & 0 \\ 0 & [J^{-1}] \end{bmatrix} \begin{Bmatrix} u,_\xi \\ u,_\eta \\ v,_\xi \\ v,_\eta \end{Bmatrix}$$

$$\begin{Bmatrix} u,_x \\ u,_y \end{Bmatrix} = \begin{Bmatrix} u,_\xi \dfrac{\partial \xi}{\partial x} + u,_\eta \dfrac{\partial \eta}{\partial x} \\ u,_\xi \dfrac{\partial \xi}{\partial y} + u,_\eta \dfrac{\partial \eta}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} u,_\xi \\ u,_\eta \end{Bmatrix} = [J^{-1}] \begin{Bmatrix} u,_\xi \\ u,_\eta \end{Bmatrix}$$

Formulación Isoparamétrica

$$\begin{Bmatrix} u,_x \\ u,_y \end{Bmatrix} = \left[J^{-1}\right]\begin{Bmatrix} u,_\xi \\ u,_\eta \end{Bmatrix} = \left[J^{-1}\right]\begin{Bmatrix} \dfrac{\partial \sum_k N_k(\xi,\eta) u_k}{\partial \xi} \\ \dfrac{\partial \sum_k N_k(\xi,\eta) u_k}{\partial \eta} \end{Bmatrix} = \left[J^{-1}\right]\begin{bmatrix} N_{1,\xi} & \cdots & N_{n,\xi} \\ N_{1,\eta} & \cdots & N_{n,\eta} \end{bmatrix}\begin{Bmatrix} u_1 \\ \vdots \\ u_n \end{Bmatrix} = \begin{bmatrix} N_{1,x} & \cdots & N_{n,x} \\ N_{1,y} & \cdots & N_{n,y} \end{bmatrix}\begin{Bmatrix} u_1 \\ \vdots \\ u_n \end{Bmatrix}$$

$$\begin{Bmatrix} v,_x \\ v,_y \end{Bmatrix} = \left[J^{-1}\right]\begin{Bmatrix} v,_\xi \\ v,_\eta \end{Bmatrix} = \left[J^{-1}\right]\begin{Bmatrix} \dfrac{\partial \sum_k N_k(\xi,\eta) v_k}{\partial \xi} \\ \dfrac{\partial \sum_k N_k(\xi,\eta) v_k}{\partial \eta} \end{Bmatrix} = \left[J^{-1}\right]\begin{bmatrix} N_{1,\xi} & \cdots & N_{n,\xi} \\ N_{1,\eta} & \cdots & N_{n,\eta} \end{bmatrix}\begin{Bmatrix} v_1 \\ \vdots \\ v_n \end{Bmatrix} = \begin{bmatrix} N_{1,x} & \cdots & N_{n,x} \\ N_{1,y} & \cdots & N_{n,y} \end{bmatrix}\begin{Bmatrix} v_1 \\ \vdots \\ v_n \end{Bmatrix}$$

Matriz Desplazamiento Deformación

$$B(\xi,\eta) = \begin{bmatrix} N_{1,x} & 0 & & N_{n,x} & 0 \\ 0 & N_{1,y} & \cdots & 0 & N_{n,y} \\ N_{1,y} & N_{1,x} & & N_{n,y} & N_{n,x} \end{bmatrix}$$

$$\varepsilon(\xi,\eta) = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = B(\xi,\eta)\begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ u_n \\ v_n \end{Bmatrix}$$

Formulación Isoparamétrica

Matriz Desplazamiento Deformación

$$B(\xi,\eta)=\begin{bmatrix} \frac{\partial N_1}{\partial \xi}J^{-1}{}_{11}+\frac{\partial N_1}{\partial \eta}J^{-1}{}_{12} & 0 & & \frac{\partial N_4}{\partial \xi}J^{-1}{}_{11}+\frac{\partial N_4}{\partial \eta}J^{-1}{}_{12} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi}J^{-1}{}_{21}+\frac{\partial N_1}{\partial \eta}J^{-1}{}_{22} & \cdots & 0 & \frac{\partial N_4}{\partial \xi}J^{-1}{}_{21}+\frac{\partial N_4}{\partial \eta}J^{-1}{}_{22} \\ \frac{\partial N_1}{\partial \xi}J^{-1}{}_{21}+\frac{\partial N_1}{\partial \eta}J^{-1}{}_{22} & \frac{\partial N_1}{\partial \xi}J^{-1}{}_{11}+\frac{\partial N_1}{\partial \eta}J^{-1}{}_{12} & & \frac{\partial N_4}{\partial \xi}J^{-1}{}_{21}+\frac{\partial N_4}{\partial \eta}J^{-1}{}_{22} & \frac{\partial N_4}{\partial \xi}J^{-1}{}_{11}+\frac{\partial N_4}{\partial \eta}J^{-1}{}_{12} \end{bmatrix}$$

Matriz de Rigidez Integrada en coordenadas Isoparamétricas

$$\boxed{[K]=\int_{-1}^{1}\int_{-1}^{1}[B(\xi,\eta)]^T[E][B(\xi,\eta)]\left|J\right|t\,d\xi\,d\eta}$$

Integración Numérica

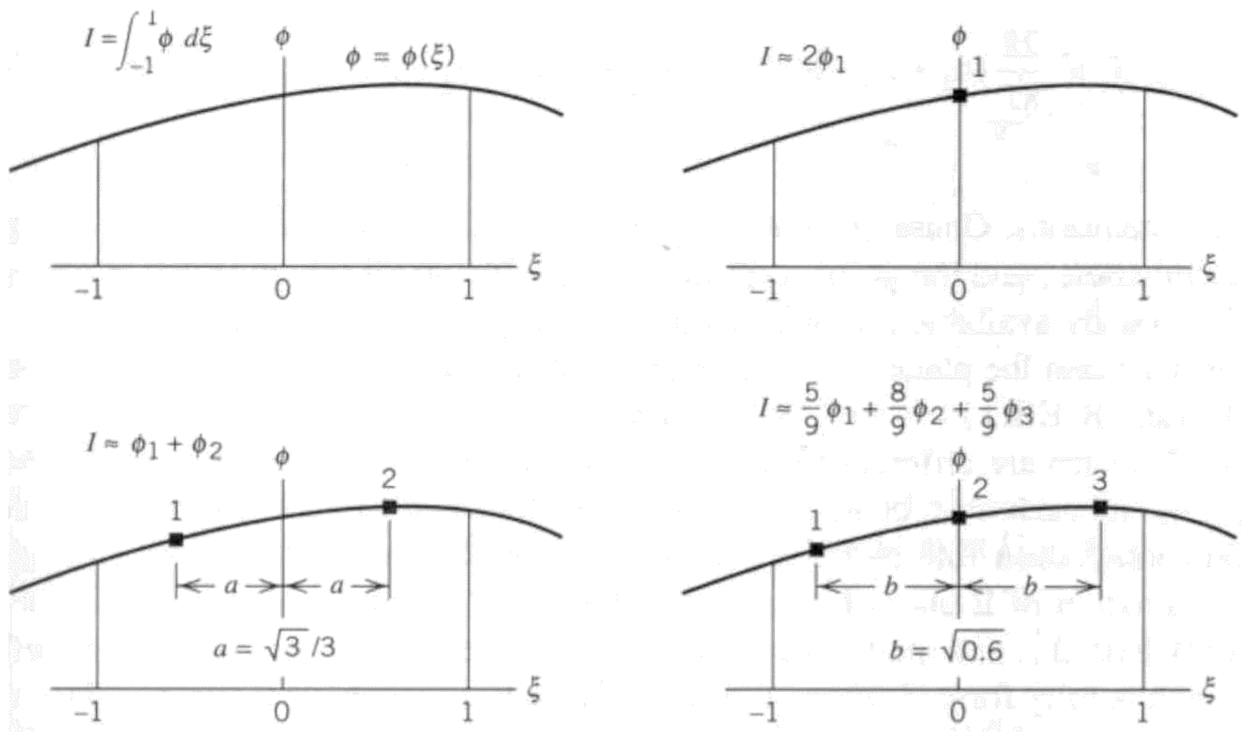
$$I = \int_{-1}^{1} \phi(\xi) d\xi$$

W$_i$: Pesos

$$\downarrow$$

$$I \cong \sum_{i=1}^{n} W_i \cdot \phi(\xi_i)$$

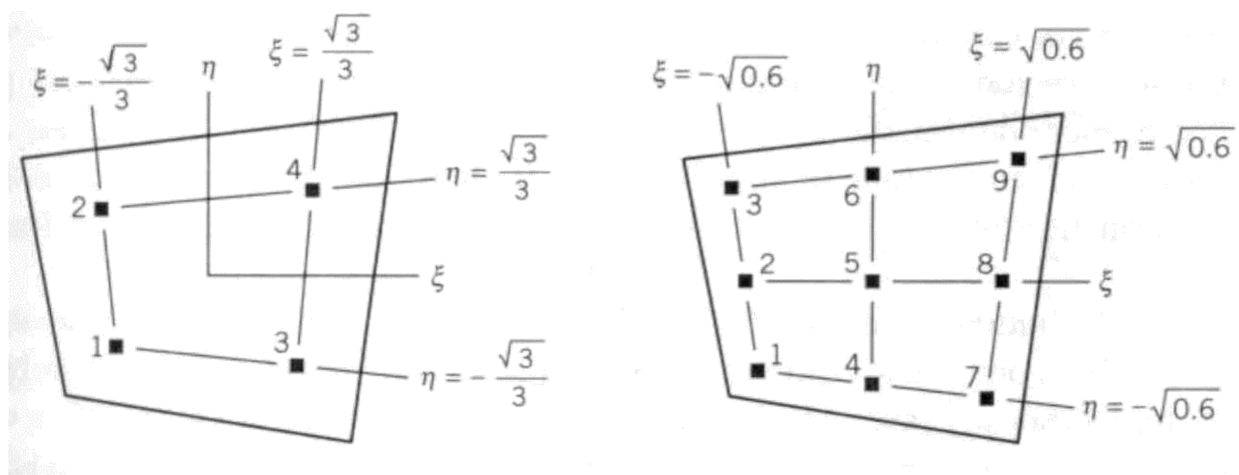
Presición de Gauss: Gr=2n-1



$$I = \int_{-1}^{1}\int_{-1}^{1} \phi(\xi,\eta) d\xi d\eta$$

$$\downarrow$$

$$I \cong \sum_{j=1}^{m}\sum_{i=1}^{n} W_i W_j \phi(\xi_i,\eta_j)$$



(a) (b)

Formulación Isoparamétrica

En cada punto de Gauss

$$J_{ij} = \begin{bmatrix} \dfrac{\partial \sum_k N_k(\xi_i,\eta_j) x_k}{\partial \xi} & \dfrac{\partial \sum_k N_k(\xi_i,\eta_j) y_k}{\partial \xi} \\ \dfrac{\partial \sum_k N_k(\xi_i,\eta_j) x_k}{\partial \eta} & \dfrac{\partial \sum_k N_k(\xi_i,\eta_j) y_k}{\partial \eta} \end{bmatrix} \quad ; \quad J_{ij}^{-1} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix}$$

Matriz elemental

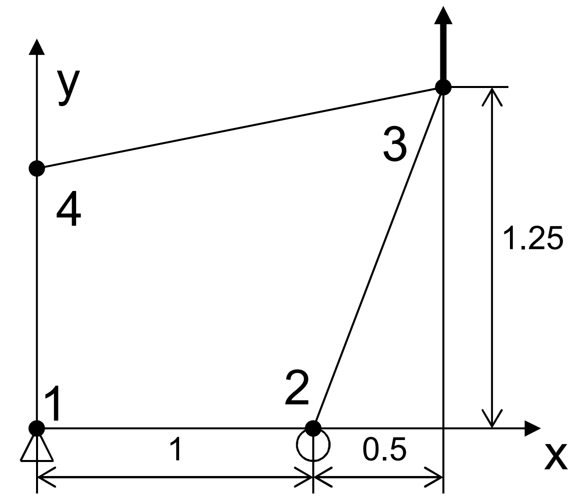
$$\begin{bmatrix} N_{1,\xi}(\xi_i,\eta_j) & \cdots & N_{n,\xi}(\xi_i,\eta_j) \\ N_{1,\eta}(\xi_i,\eta_j) & \cdots & N_{n,\eta}(\xi_i,\eta_j) \end{bmatrix} \rightarrow \left[J^{-1}(\xi_i,\eta_j)\right] \begin{bmatrix} N_{1,\xi}(\xi_i,\eta_j) & \cdots & N_{n,\xi}(\xi_i,\eta_j) \\ N_{1,\eta}(\xi_i,\eta_j) & \cdots & N_{n,\eta}(\xi_i,\eta_j) \end{bmatrix} = \begin{bmatrix} N_{1,x} & \cdots & N_{n,x} \\ N_{1,y} & \cdots & N_{n,y} \end{bmatrix}$$

$$B(\xi_i,\eta_j) = \begin{bmatrix} N_{1,x} & 0 & & N_{n,x} & 0 \\ 0 & N_{1,y} & \cdots & 0 & N_{n,y} \\ N_{1,y} & N_{1,x} & & N_{n,y} & N_{n,x} \end{bmatrix}$$

Matriz de Rigidez Integrada numéricamente en coordenadas isoparamétricas

$$[\mathrm{K}] = \sum_i \sum_j \mathrm{w}_i \mathrm{w}_j \left[\mathrm{B}(\xi_i,\eta_j)\right]^{\mathrm{T}} [\mathrm{E}] \left[\mathrm{B}(\xi_i,\eta_j)\right] \left|\mathrm{J}_{ij}\right| \, \mathrm{t}$$

Problema Plano – Tensión Plana



Matriz Constitutiva (E=1, ν=0.33):

$$C := \begin{pmatrix} 1.125 & 0.375 & 0 \\ 0.375 & 1.125 & 0 \\ 0 & 0 & 0.375 \end{pmatrix}$$

Funciones de Forma:

$$N(\xi,\eta) := \frac{1}{4}\cdot\begin{bmatrix} (1-\xi)\cdot(1-\eta) \\ (1+\xi)\cdot(1-\eta) \\ (1+\xi)\cdot(1+\eta) \\ (1-\xi)\cdot(1+\eta) \end{bmatrix} \qquad N(\xi,\eta) = \begin{pmatrix} N1(\xi,\eta) & 0 & N2(\xi,\eta) & 0 & N3(\xi,\eta) & 0 & N4(\xi,\eta) & 0 \\ 0 & N1(\xi,\eta) & 0 & N2(\xi,\eta) & 0 & N3(\xi,\eta) & 0 & N4(\xi,\eta) \end{pmatrix}$$

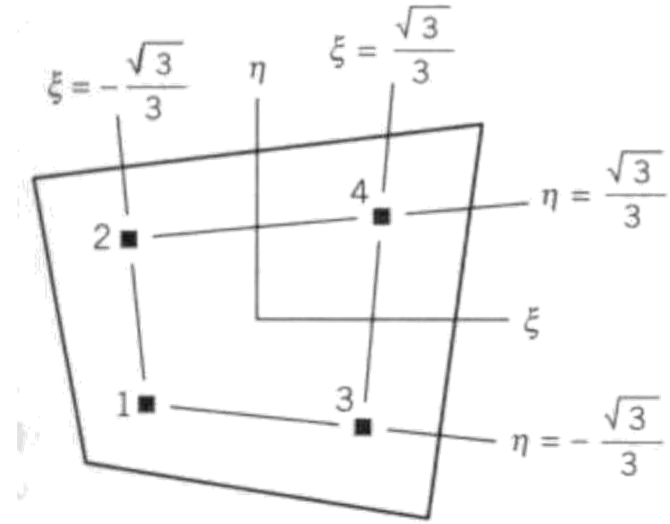
Jacobiano:

$$J(\xi,\eta) = \begin{pmatrix} \frac{d}{d\xi}N(\xi,\eta)_1 & \frac{d}{d\xi}N(\xi,\eta)_2 & \frac{d}{d\xi}N(\xi,\eta)_3 & \frac{d}{d\xi}N(\xi,\eta)_4 \\ \frac{d}{d\eta}N(\xi,\eta)_1 & \frac{d}{d\eta}N(\xi,\eta)_2 & \frac{d}{d\eta}N(\xi,\eta)_3 & \frac{d}{d\eta}N(\xi,\eta)_4 \end{pmatrix}\cdot\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1.5 & 1.25 \\ 0 & 1 \end{pmatrix}$$

$$J(\xi,\eta) = \frac{1}{4}\begin{pmatrix} -1+\eta & 1-\eta & 1+\eta & -1-\eta \\ -1+\xi & -1-\xi & 1+\xi & 1-\xi \end{pmatrix}\cdot\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1.5 & 1.25 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} .625+.125\cdot\eta & .625+.625\cdot\eta \\ .125+.125\cdot\xi & .5625+.0625\cdot\xi \end{pmatrix}$$

Problema Plano – Tensión Plana

$$\mathrm{J}(\xi,\eta)=\begin{bmatrix}\frac{5}{8}+\frac{1}{8}\eta & \frac{1}{16}+\frac{1}{16}\eta\\ \frac{1}{8}+\frac{1}{8}\xi & \frac{9}{16}+\frac{1}{16}\xi\end{bmatrix}\qquad\left|\mathrm{J}(\xi,\eta)\right|=\frac{1}{32}(11+\xi+2\eta)$$



$$\mathrm{J}\left(\tfrac{1}{\sqrt{3}},\tfrac{1}{\sqrt{3}}\right)=\begin{bmatrix}0.69717 & 0.09858\\ 0.19717 & 0.59858\end{bmatrix}\qquad\left|\mathrm{J}\left(\tfrac{1}{\sqrt{3}},\tfrac{1}{\sqrt{3}}\right)\right|=0.39788$$

$$\begin{bmatrix}N_{1,\xi}(\xi_i,\eta_j) & \cdots & N_{n,\xi}(\xi_i,\eta_j)\\ N_{1,\eta}(\xi_i,\eta_j) & \cdots & N_{n,\eta}(\xi_i,\eta_j)\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-1+\eta & 1-\eta & 1+\eta & -1-\eta\\ -1+\xi & -1-\xi & 1+\xi & 1-\xi\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-1+\eta_j & 1-\eta_j & 1+\eta_j & -1-\eta_j\\ -1+\xi_i & -1-\xi_i & 1+\xi_i & 1-\xi_i\end{bmatrix}$$

$$\left[J^{-1}\right]\begin{bmatrix}N_{1,\xi} & \cdots & N_{n,\xi}\\ N_{1,\eta} & \cdots & N_{n,\eta}\end{bmatrix}=\begin{bmatrix}N_{1,x} & \cdots & N_{n,x}\\ N_{1,y} & \cdots & N_{n,y}\end{bmatrix}=\frac{1}{(11+\xi+2\eta)}\begin{bmatrix}-4+5\eta-\xi & 5-4\eta+\xi & 4+4\eta & -5-5\eta\\ -4-2\eta+6\xi & -6-6\eta & 4+4\xi & 6+2\eta-4\xi\end{bmatrix}$$

$$\mathrm{B}(\xi,\eta)=\frac{\begin{bmatrix}\xi-4-5\eta & 0 & 5-4\eta+\xi & 0 & 4(1+\eta) & 0 & -5(1+\eta) & 0\\ 0 & 6\xi-4-2\eta & 0 & -6(1+\xi) & 0 & 4(1+\xi) & 0 & 6+2\eta-4\xi\\ 6\xi-4-2\eta & \xi-4-5\eta & -6(1+\xi) & 5-4\eta+\xi & 4(1+\xi) & 4(1+\eta) & 6+2\eta-4\xi & 5(1+\eta)\end{bmatrix}}{11+\xi+2\eta}$$

Problema Plano – Tensión Plana

$$B\left(\tfrac{1}{\sqrt{3}},\tfrac{1}{\sqrt{3}}\right)=\begin{bmatrix} -0.13278 & 0 & 0.25667 & 0 & 0.49555 & 0 & -0.61944 & 0 \\ 0 & -0.13278 & 0 & -0.74333 & 0 & 0.49555 & 0 & 0.38056 \\ -0.13278 & -0.13278 & -0.74333 & 0.25667 & 0.49555 & 0.49555 & 0.38056 & -0.61944 \end{bmatrix}$$

$$I(\xi,\eta)=[B(\xi,\eta)]^T[E][B(\xi,\eta)]|J|$$

$$I(\xi,\eta)=\frac{\begin{bmatrix} \xi-4-5\eta & 0 & 6\xi-4-2\eta \\ 0 & 6\xi-4-2\eta & \xi-4-5\eta \\ \vdots & \vdots & \vdots \end{bmatrix}}{11+\xi+2\eta}\begin{bmatrix} 1.125 & 0.375 & 0 \\ 0.375 & 1.125 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}\frac{\begin{bmatrix} \xi-4-5\eta & 0 & \cdots \\ 0 & 6\xi-4-2\eta & \cdots \\ 6\xi-4-2\eta & \xi-4-5\eta & \cdots \end{bmatrix}}{11+\xi+2\eta}\frac{11+\xi+2\eta}{32}$$

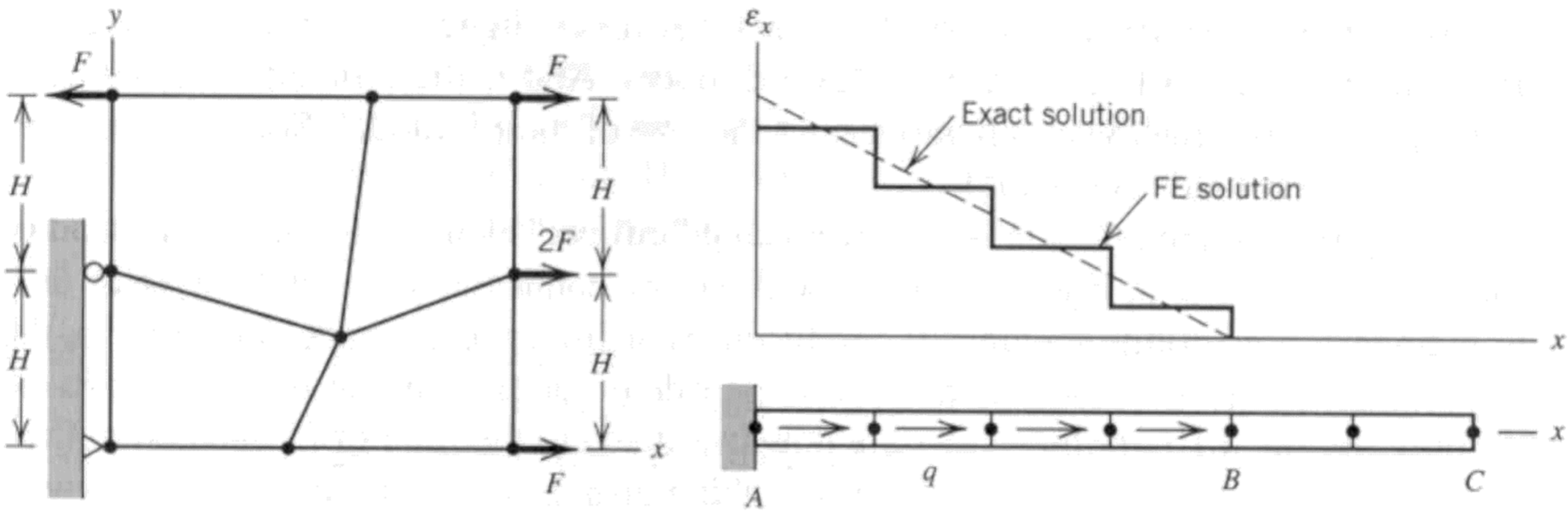
$$I(\xi,\eta)=\frac{1}{11+\xi+2\eta}\begin{bmatrix} \left(\frac{3}{4}-\frac{39}{32}\eta-\frac{9}{32}\xi+\frac{237}{256}\eta^2-\frac{81}{128}\xi\eta+\frac{117}{256}\xi^2\right) & \left(\frac{3}{8}-\frac{9}{32}\eta-\frac{15}{32}\xi-\frac{15}{64}\eta^2+\frac{3}{4}\xi\eta-\frac{9}{64}\xi^2\right) & \cdots \\ \left(\frac{3}{8}-\frac{9}{32}\eta-\frac{15}{32}\xi-\frac{15}{64}\eta^2+\frac{3}{4}\xi\eta-\frac{9}{64}\xi^2\right) & \left(\frac{3}{4}+\frac{3}{32}\eta-\frac{51}{32}\xi+\frac{111}{256}\eta^2-\frac{123}{128}\xi\eta+\frac{327}{256}\xi^2\right) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$I(\xi_i,\eta_j)=[B(\xi_i,\eta_j)]^T[E][B(\xi_i,\eta_j)]|J_{ij}| \qquad \rightarrow \qquad [K]=\sum_i\sum_j w_i w_j\ I(\xi_i,\eta_j)\,t$$

Matriz de Rigidez

Patch Test

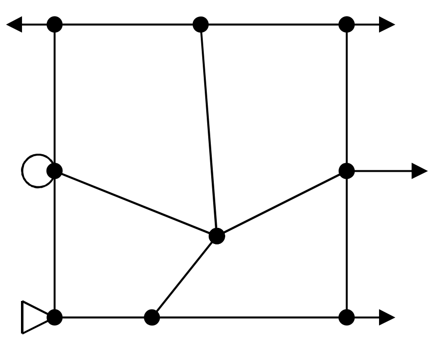
Se verifica con tracción. El elemento diseñad debe dar las tensiones reales
Tiene que poder reproducir un estao de tensión constante. SI hago esto puedo aproximar de manera convergente.
Es mejor si lo hacemos con un nodo en el medio



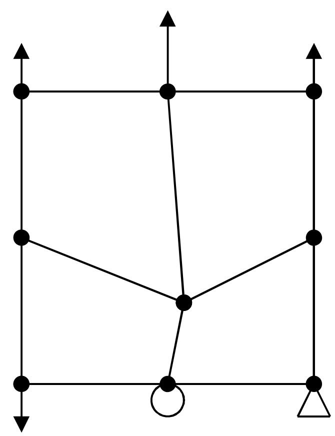
(a) (b)

El Q6 no pasa a menos que se le haga una correción.

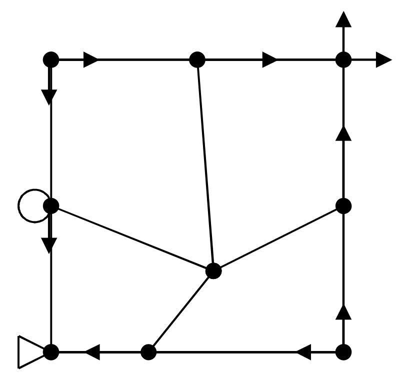
σ_x: cte



σ_y: cte



τ_{xy}: cte



Evidentemente estoy teniendo un problema con el programa porque no paso el patch test.... Alumno