Instituto Tecnológico de Buenos Aires

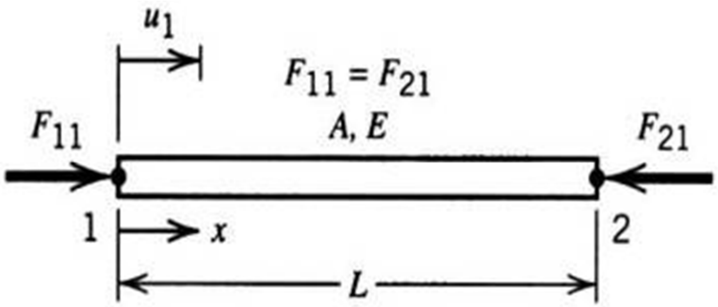
Elementos Finitos I

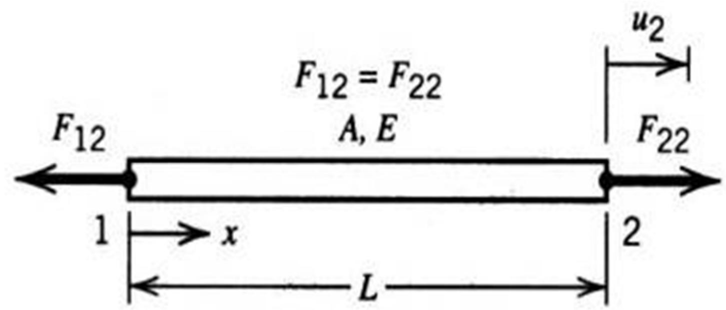
“Things should be described as simply as possible, but not simpler” – A. Einstein

Elementos Unidimensionales

Elemento Barra

Método Directo





$$\begin{bmatrix} k^I{}_{11} & k^I{}_{12} \\ k^I{}_{21} & k^I{}_{22} \end{bmatrix} \begin{Bmatrix} u^I{}_1 \\ u^I{}_2 \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \end{Bmatrix}$$

Desplazamientos Unitarios:

$$u^I{}_1 = 1 \; ; \; u^I{}_2 = 0$$
$$u^I{}_1 = 0 \; ; \; u^I{}_2 = 1$$

$$[K_e] \qquad \{D_e\} = \{R_e\}$$

$$\begin{bmatrix} k^I{}_{11} & k^I{}_{12} \\ k^I{}_{21} & k^I{}_{22} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} k^I{}_{11} \\ k^I{}_{21} \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \end{Bmatrix} = \frac{A_e E_e}{L_e} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$\begin{bmatrix} k^I{}_{11} & k^I{}_{12} \\ k^I{}_{21} & k^I{}_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} k^I{}_{12} \\ k^I{}_{22} \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \end{Bmatrix} = \frac{A_e E_e}{L_e} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

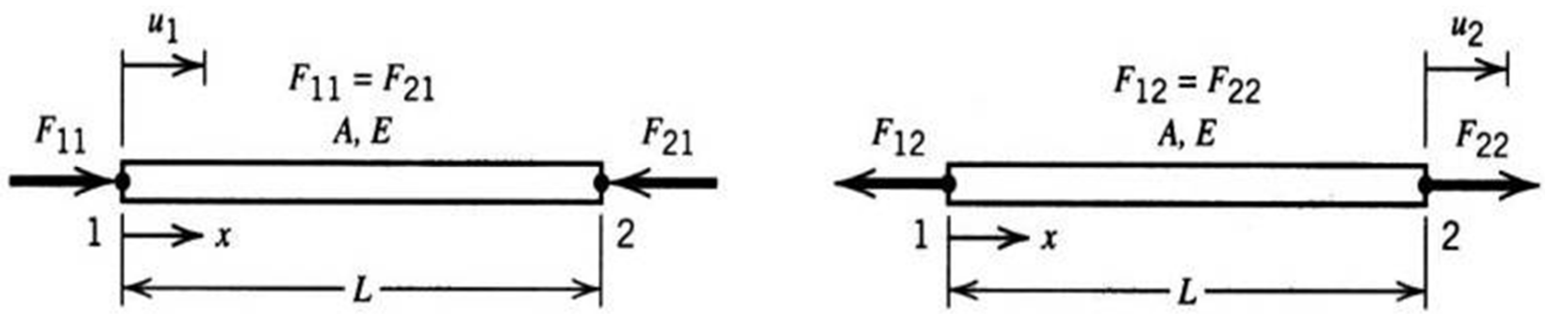
Matriz de rigidez:

$$[K_e] = \begin{bmatrix} X^I & -X^I \\ -X^I & X^I \end{bmatrix} ; X^I = \frac{A_e E_e}{L_e}$$

Elementos Unidimensionales

Elemento Barra

Método Directo

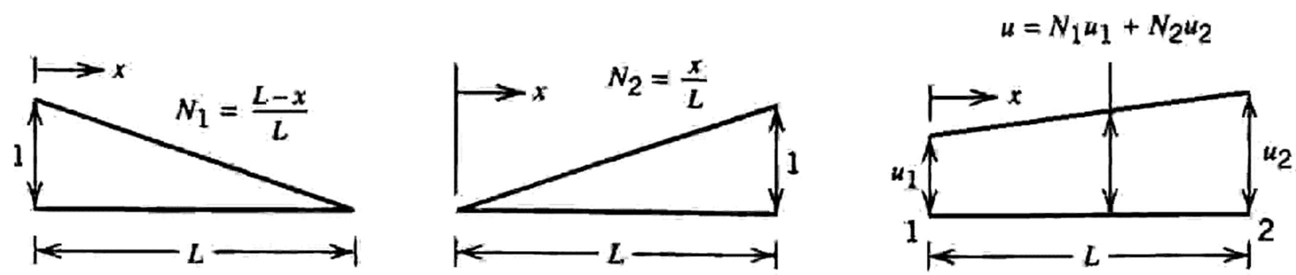


$$[K_e]\{D_e\} = \{R_e\}; \qquad \begin{bmatrix} X^I & -X^I \\ -X^I & X^I \end{bmatrix} \begin{Bmatrix} u^I{}_1 \\ u^I{}_2 \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \end{Bmatrix}$$

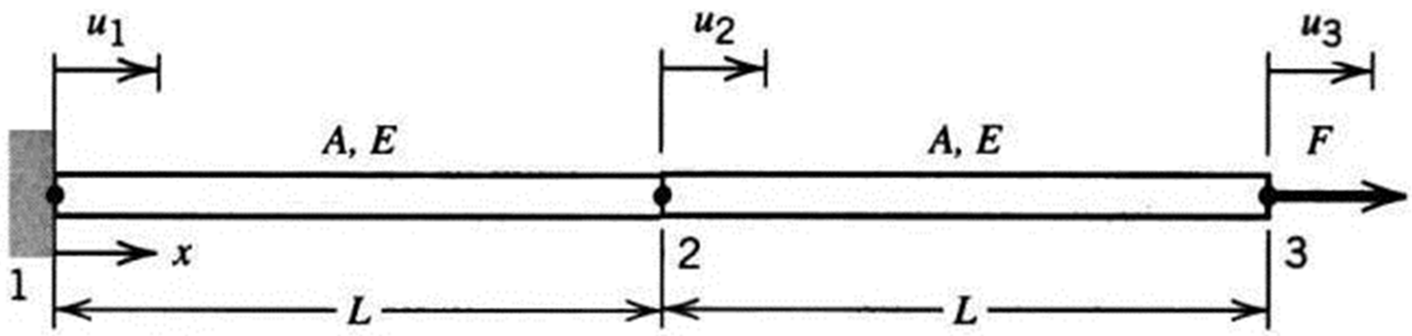
Desplazamiento

Funciones de Forma

$$u(x) = u_1\left(\frac{L-x}{L}\right) + u_2\left(\frac{x}{L}\right) = [N(x)]\{D_e\} = \begin{bmatrix} \dfrac{L-x}{L} & \dfrac{x}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



Estructura - Ensamblado



Dof Locales y Globales

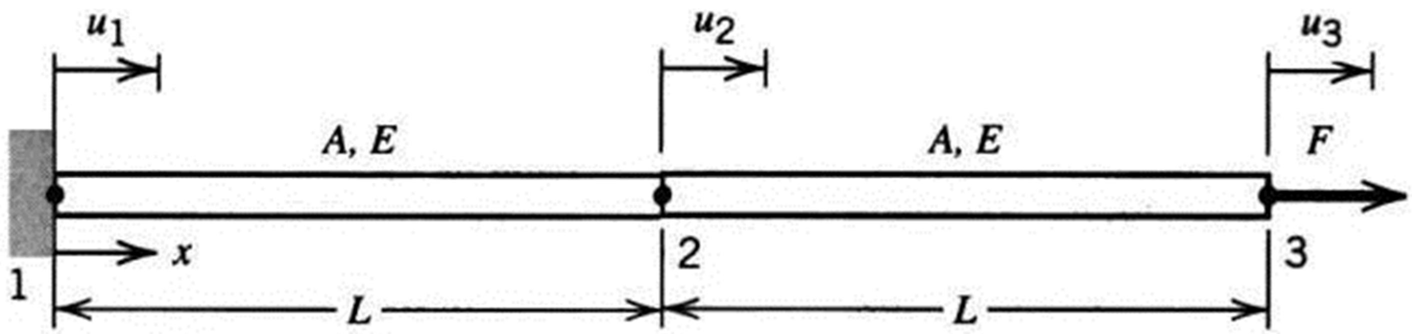
$$\begin{cases} u_1 = u^{I}{}_{1} \\ u_2 = u^{I}{}_{2} = u^{II}{}_{1} \\ u_3 = u^{II}{}_{2} \end{cases}$$

$$\begin{bmatrix} X^I & -X^I \\ -X^I & X^I \end{bmatrix} \begin{Bmatrix} u^I{}_1 \\ u^I{}_2 \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \end{Bmatrix} \quad \rightarrow \quad \begin{bmatrix} X^I & -X^I & 0 \\ -X^I & X^I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 \\ 0 \end{Bmatrix}$$

$$\underbrace{\begin{bmatrix} X^{II} & -X^{II} \\ -X^{II} & X^{II} \end{bmatrix}}_{[K_e]} \underbrace{\begin{Bmatrix} u^{II}{}_1 \\ u^{II}{}_2 \end{Bmatrix}}_{\{D_e\}} = \underbrace{\begin{Bmatrix} F^{II}{}_1 \\ F^{II}{}_2 \end{Bmatrix}}_{\{R_e\}} \quad \rightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^{II} & -X^{II} \\ 0 & -X^{II} & X^{II} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F^{II}{}_1 \\ F^{II}{}_2 \end{Bmatrix}$$

$$\left(\begin{bmatrix} X^I & -X^I & 0 \\ -X^I & X^I & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^{II} & -X^{II} \\ 0 & -X^{II} & X^{II} \end{bmatrix} \right) \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F^I{}_1 \\ F^I{}_2 + F^{II}{}_1 \\ F^{II}{}_2 \end{Bmatrix} \rightarrow \begin{bmatrix} X^I & -X^I & 0 \\ -X^I & X^I + X^{II} & -X^{II} \\ 0 & -X^{II} & X^{II} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Estructura - Ensamblado

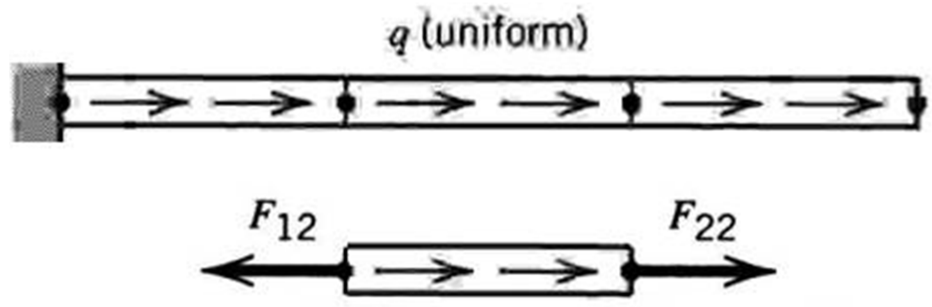


$= 0$

$$\underbrace{\begin{bmatrix} X^{I} & -X^{I} & 0 \\ -X^{I} & X^{I}+X^{II} & -X^{II} \\ 0 & -X^{II} & X^{II} \end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}}_{\{D\}} = \underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}}_{\{R\}} \quad ; \quad \textit{Condiciones de Borde} \qquad \begin{cases} u_1 = 0 \\ F_2 = 0 \end{cases}$$

$$\underbrace{\begin{matrix} \begin{bmatrix} -X^{I} & 0 \end{bmatrix} \\ \begin{bmatrix} X^{I}+X^{II} & -X^{II} \\ -X^{II} & X^{II} \end{bmatrix} \end{matrix}}_{[K_r]} \underbrace{\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}}_{\{D_r\}} = \underbrace{\begin{matrix} \{F_1\} \\ \begin{Bmatrix} 0 \\ F_3 \end{Bmatrix} \end{matrix}}_{\{R_r\}} \quad \therefore \quad \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{X} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ F_3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{LF_3}{AE} \\ \dfrac{2LF_3}{AE} \end{Bmatrix}$$

Carga Distribuida



$$\frac{A_eE_e}{L_e}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$F^I{}_1 = F^I{}_2 = \frac{qL_e}{2} \qquad \rightarrow \qquad \frac{A_eE_e}{L_e}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} R \\ qL \\ qL \\ 0.5qL \end{Bmatrix}$$

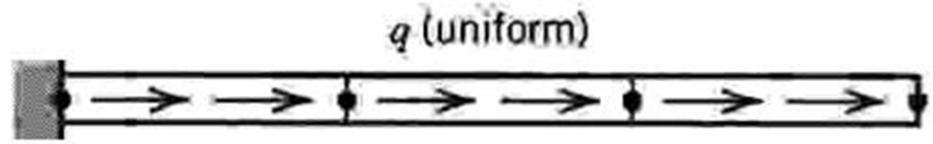
$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{L_e}{A_eE_e}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} qL_e \\ qL_e \\ 0.5qL_e \end{Bmatrix} = \frac{qL_e^2}{A_eE_e}\begin{Bmatrix} 2.5 \\ 4 \\ 4.5 \end{Bmatrix}$$

$$\{R\} = \frac{A_eE_e}{L_e}\begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \frac{qL_e^2}{A_eE_e}\begin{Bmatrix} 2.5 \\ 4 \\ 4.5 \end{Bmatrix} = -2.5qL_e$$

$+ 0{,}5qL$

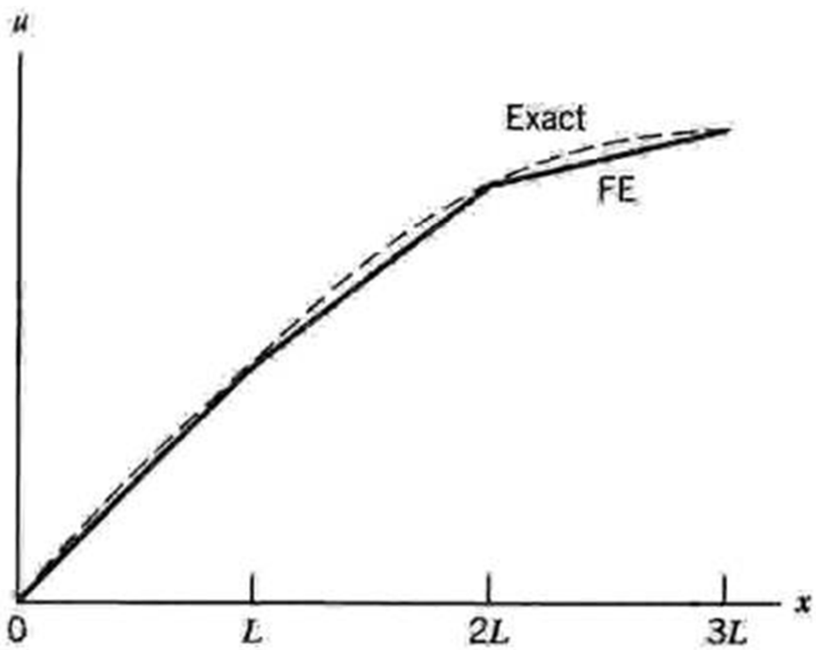
Carga Distribuida

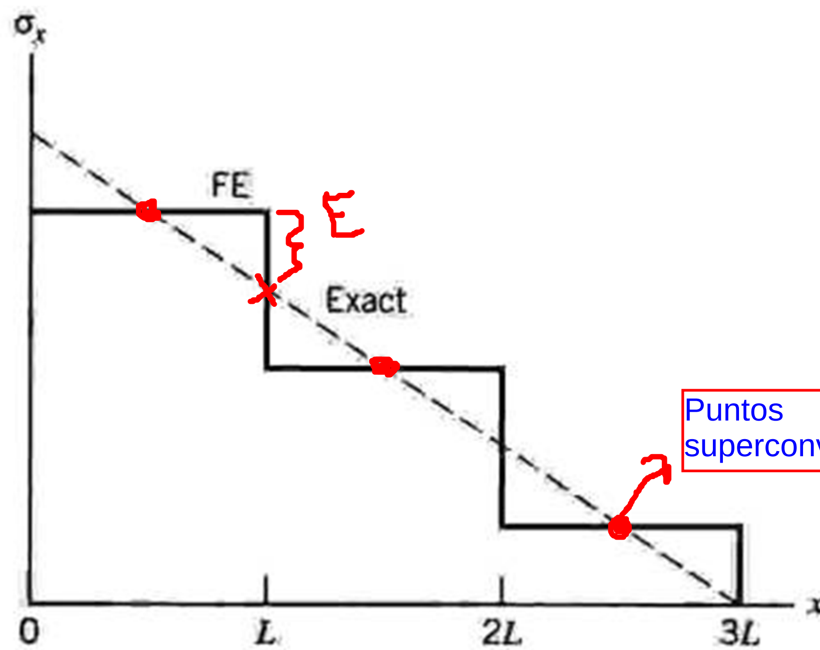
Tensiones



$$u(x)=[N(x)]\{D_e\}=\begin{bmatrix}\frac{L_e-x}{L_e} & \frac{x}{L_e}\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} \quad ; \quad \varepsilon_x=\frac{du(x)}{dx}=\overbrace{\frac{d}{dx}}^{[\partial]}[N(x)]\{d_e\}=\underbrace{[\partial][N(x)]}_{[B(x)]}\{d_e\}=[B(x)]\{D_e\}$$

$$[B(x)]=\begin{bmatrix}-\frac{1}{L_e} & \frac{1}{L_e}\end{bmatrix} \quad ; \quad \sigma_x=E_e\varepsilon_x=E_e\begin{bmatrix}-\frac{1}{L_e} & \frac{1}{L_e}\end{bmatrix}\begin{Bmatrix}4\\4.5\end{Bmatrix}=\frac{qL_e}{2A_e}$$





Transformaciones – DOF no alineados - Barras

Matriz de transformación T

Nota: $\dfrac{V_{12}}{|V_{12}|}=\{l_1, m_1\}$; $\underbrace{\begin{bmatrix} u_1' \\ v_1' \end{bmatrix}}_{\{d'\}}=\underbrace{\begin{bmatrix} l_1 & m_1 \\ -m_1 & l_1 \end{bmatrix}}_{[T]}\underbrace{\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}}_{\{d\}}$; $[T][T]^T=\begin{bmatrix} l_1 & m_1 \\ -m_1 & l_1 \end{bmatrix}\begin{bmatrix} l_1 & -m_1 \\ m_1 & l_1 \end{bmatrix}$; $[T]^T=[T]^{-1}$

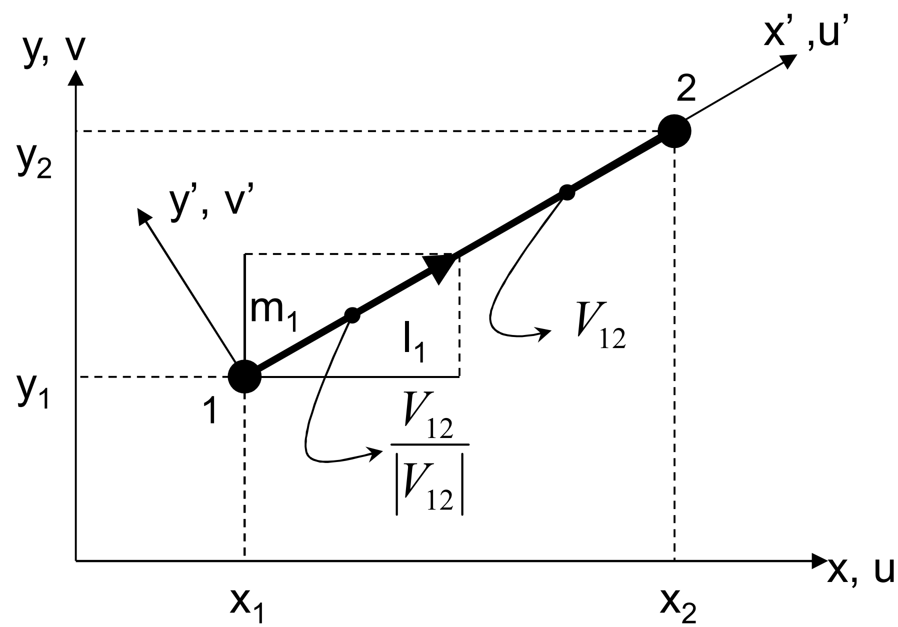
$$\{d'\}=[T]\{d\} \rightarrow \underbrace{\begin{bmatrix} u_1' \\ u_2' \end{bmatrix}}_{\{d'\}}=\underbrace{\begin{bmatrix} l_1 & m_1 & 0 & 0 \\ 0 & 0 & l_1 & m_1 \end{bmatrix}}_{[T]}\underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}}_{\{d\}}$$

$$[k']\{d'\}=\{r'\} \quad \rightarrow \quad [k'][T]\{d\}=\{r'\}$$

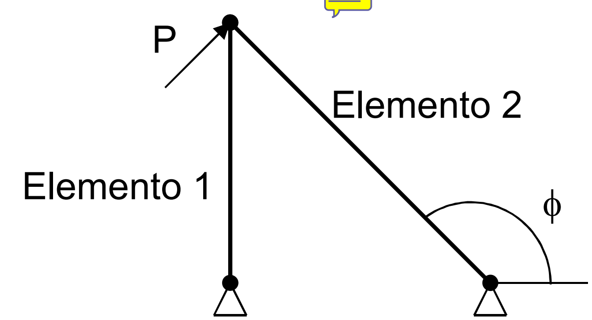
$$[T]^T[k'][T]\{d\}=[T]^T\{r'\}$$

$$\{r'\}=[T]\{r\} \quad ; \quad [T]^T\{r'\}=[T]^T[T]\{r\}$$

$$[T]^T[k'][T]\{d\}=\{r\} \rightarrow [k]=[T]^T[k'][T]$$



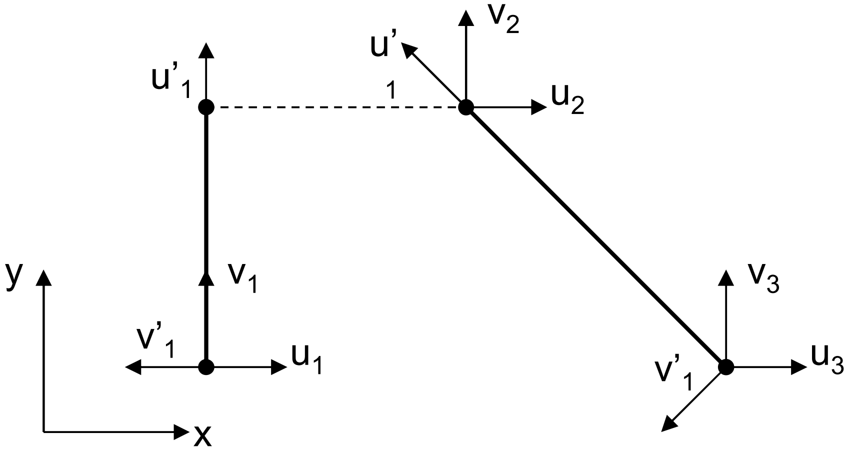
Transformaciones – DOF no alineados- Barras



Matrices de rigidez elementales

$$[K'_1] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \quad \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} \quad k_1 = \frac{A_1 E_1}{L_1}$$

$$[K'_2] = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \quad \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} \quad k_2 = \frac{A_2 E_2}{L_2}$$



Matrices transformadas

$$[K_1] = [T_1]^T [K'_1][T_1]$$

Con β=90°

$$[K_1] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K_1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_1 & 0 & -k_1 \\ 0 & 0 & 0 & 0 \\ 0 & -k_1 & 0 & k_1 \end{bmatrix} \quad \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

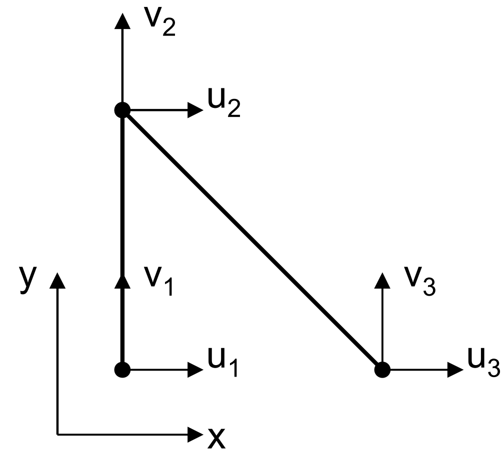
$$[K_2] = [T_2]^T [K'_2][T_2]$$

Con $\beta=\phi$

$$[K_2] = \begin{bmatrix} c\phi & 0 \\ s\phi & 0 \\ 0 & c\phi \\ 0 & s\phi \end{bmatrix} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} c\phi & s\phi & 0 & 0 \\ 0 & 0 & c\phi & s\phi \end{bmatrix}$$

$$[K_2] = \begin{bmatrix} k_2 c^2\phi & k_2 \cdot c\phi \cdot s\phi & -k_2 c^2\phi & -k_2 \cdot c\phi \cdot s\phi \\ k_2 \cdot c\phi \cdot s\phi & k_2 s^2\phi & -k_2 \cdot c\phi \cdot s\phi & -k_2 s^2\phi \\ -k_2 c^2\phi & -k_2 \cdot c\phi \cdot s\phi & k_2 c^2\phi & k_2 \cdot c\phi \cdot s\phi \\ -k_2 \cdot c\phi \cdot s\phi & -k_2 s^2\phi & k_2 \cdot c\phi \cdot s\phi & k_2 s^2\phi \end{bmatrix} \quad \begin{bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

Transformaciones – DOF no alineados- Barras



Matriz global

$$[K]=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & k_1 & 0 & -k_1 & 0 & 0\\ 0 & 0 & k_2c^2\phi & k_2\cdot c\phi\cdot s\phi & -k_2c^2\phi & -k_2\cdot c\phi\cdot s\phi\\ 0 & -k_1 & k_2\cdot c\phi\cdot s\phi & k_1+k_2s^2\phi & -k_2\cdot c\phi\cdot s\phi & -k_2s^2\phi\\ 0 & 0 & -k_2c^2\phi & -k_2\cdot c\phi\cdot s\phi & k_2c^2\phi & k_2\cdot c\phi\cdot s\phi\\ 0 & 0 & -k_2\cdot c\phi\cdot s\phi & -k_2s^2\phi & k_2\cdot c\phi\cdot s\phi & k_2s^2\phi\end{bmatrix}\quad\begin{bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\end{bmatrix}$$

Carga externas

$$\{R\}=\begin{Bmatrix}R_{x1}\\ R_{y1}\\ P_x\\ P_y\\ R_{x3}\\ R_{y3}\end{Bmatrix}$$

Matriz reducida

$$[K]=\begin{bmatrix}k_2c^2\phi & k_2\cdot c\phi\cdot s\phi\\ k_2\cdot c\phi\cdot s\phi & k_1+k_2s^2\phi\end{bmatrix}\quad\begin{bmatrix}u_2\\ v_2\end{bmatrix}$$

Desplazamientos ($\phi=45°$; $A_1=A_2=1$; $E_1=E_2=1$; $L_1=1$; $L_2=2^{0.5}$)

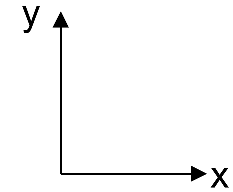
$$\{U\}=[K]^{1}\{R\}=\begin{bmatrix}\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4}\\ -\frac{\sqrt{2}}{4} & 1+\frac{\sqrt{2}}{4}\end{bmatrix}^{-1}\begin{Bmatrix}P\,\frac{\sqrt{2}}{2}\\ P\,\frac{\sqrt{2}}{2}\end{Bmatrix}=P\begin{Bmatrix}2+\sqrt{2}\\ \sqrt{2}\end{Bmatrix}$$

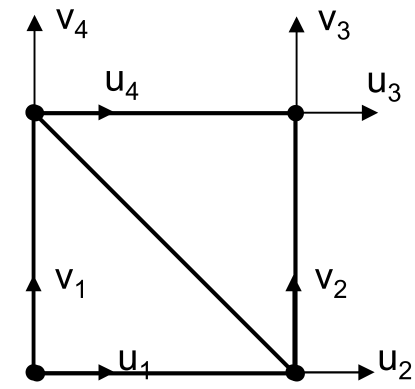
Reacciones Externas

$$\{R\}=[K]\{U\}=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ & 1 & 0 & -1 & 0 & 0\\ & & \sqrt{2}/4 & -\sqrt{2}/4 & -\sqrt{2}/4 & \sqrt{2}/4\\ & & & 1+\sqrt{2}/4 & \sqrt{2}/4 & -\sqrt{2}/4\\ & & & & \sqrt{2}/4 & -\sqrt{2}/4\\ & & & & & \sqrt{2}/4\end{bmatrix}P\begin{bmatrix}0\\ 0\\ 2+\sqrt{2}\\ \sqrt{2}\\ 0\\ 0\end{bmatrix}=P\begin{bmatrix}0\\ -\sqrt{2}\\ \sqrt{2}/2\\ \sqrt{2}/2\\ -\sqrt{2}/2\\ \sqrt{2}/2\end{bmatrix}$$

Ensamblado Matriz de Rigidez

$$
\begin{array}{c}
\begin{bmatrix}
\times & \_ & \times & \_ & \_ & \_ & \_ & \_ \\
\_ & \times & \_ & \_ & \_ & \_ & \_ & \times \\
\times & \_ & \times & \circ & \_ & \_ & \circ & \circ \\
\_ & \_ & \circ & \times & \_ & \times & \circ & \circ \\
\_ & \_ & \_ & \_ & \times & \_ & \times & \_ \\
\_ & \_ & \_ & \times & \_ & \times & \_ & \_ \\
\_ & \_ & \circ & \circ & \times & \_ & \times & \circ \\
\_ & \times & \circ & \circ & \_ & \_ & \circ & \times
\end{bmatrix}
\begin{array}{cc}
u_1 & d_1 \\
v_1 & d_2 \\
u_2 & d_3 \\
v_2 & d_4 \\
u_3 & d_5 \\
v_3 & d_6 \\
u_4 & d_7 \\
v_4 & d_8
\end{array}
\\
\begin{array}{cccccccc}
u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4
\end{array}
\end{array}
\quad \leftrightarrow
$$

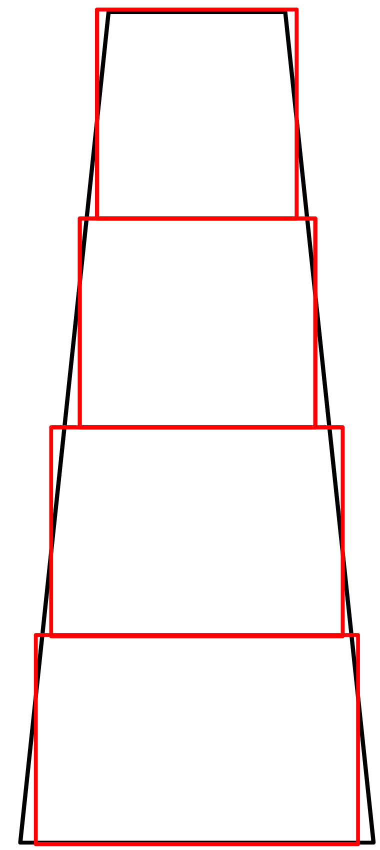




Numeración Local vs Numeración Global

Práctica Preliminar

$h = 4m$; $A_i = 100mm^2$; $A_s = 25mm^2$



Solución Exacta:

Solución Aproximada según área:

- Máxima
- Mínima
- Media