Interpolación:

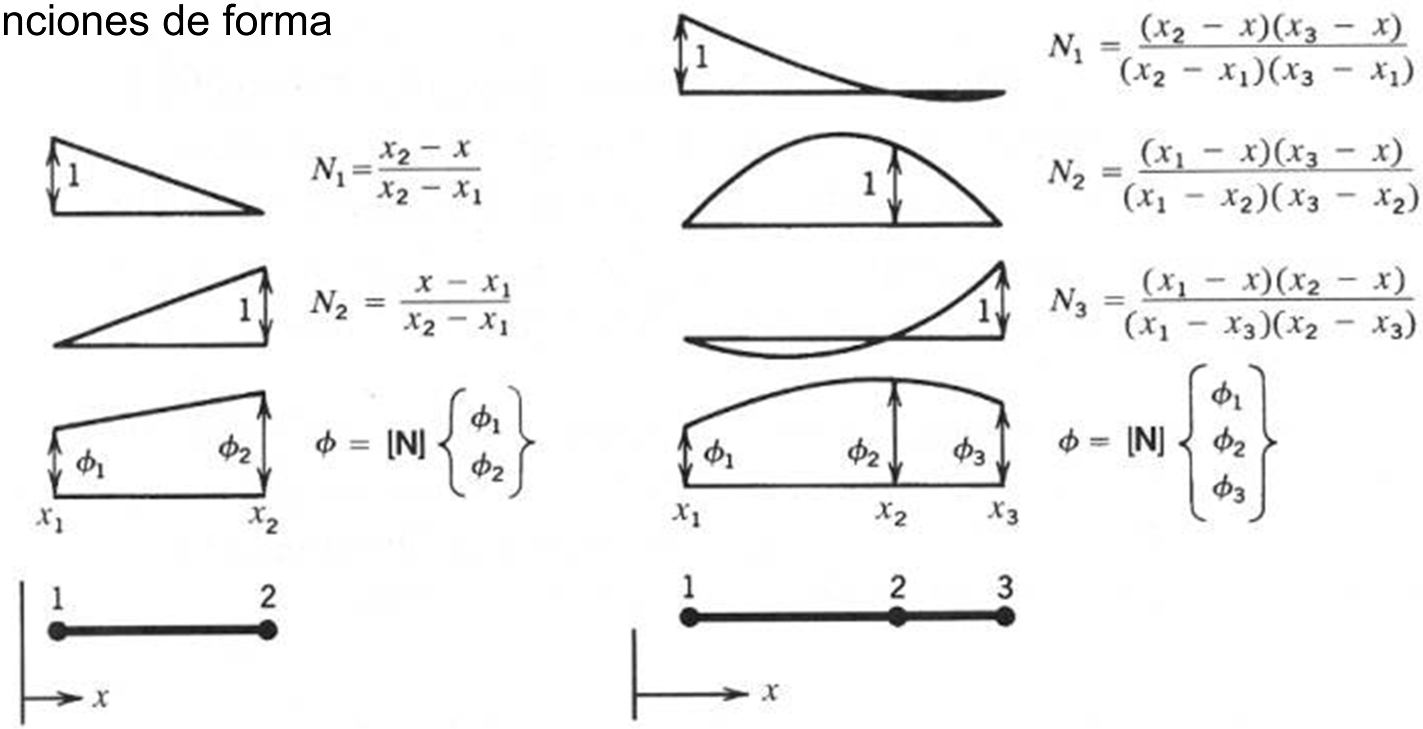
$$\phi(x) \cong \tilde{\phi}(x) = \sum_{i=0}^{n} a_i x^i; \quad \tilde{\phi}(x) = \underbrace{\{1 \quad x \quad \cdots \quad x^n\}}_{\{X(x)\}} \begin{Bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{Bmatrix}; \quad \underbrace{\begin{Bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_n \end{Bmatrix}}_{\{\phi_e\}} = \underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^n \\ 1 & x_1 & x_1^2 & x_1^n \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & x_n^n \end{bmatrix}}_{[A]} \begin{Bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{Bmatrix}; \quad \{a\} = [A]^{-1}\{\phi_e\}$$

$$\tilde{\phi}(x) = \{X(x)\}\{a\} = \underbrace{\{X\}[A]^{-1}}_{[N]}\{\phi_e\} = [N(x)]\{\phi_e\}$$

Funciones de forma

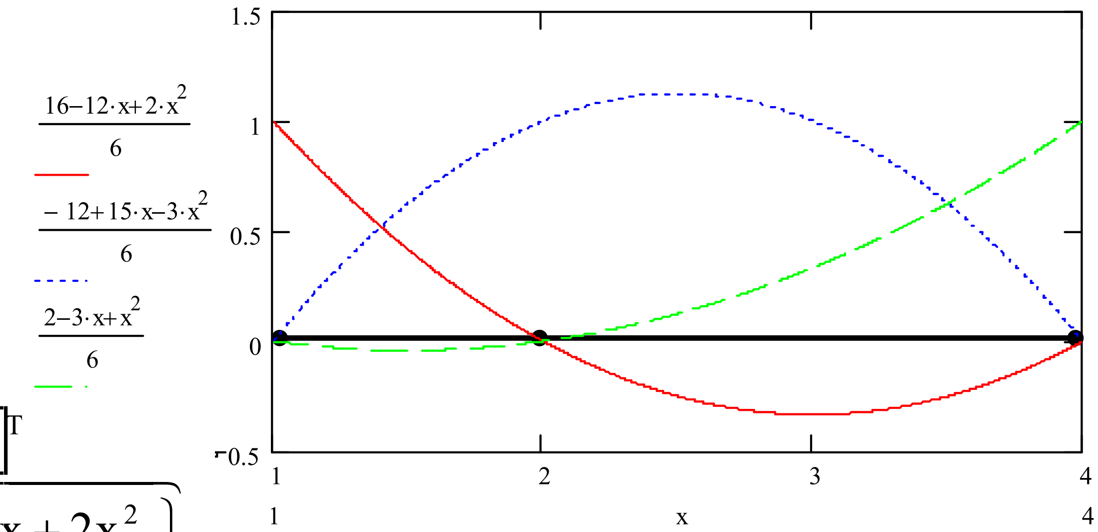
Condiciones

$$\begin{cases} [N_i(x_i)] = 1 \wedge [N_i(x_j)] = 0 \\ \qquad i \neq j \\ \sum_{i=0}^{n} N_i = 1 \end{cases}$$



Elemento Barra 3 nodos:

$$\tilde{\phi}(x)=\begin{Bmatrix}1 & x & x^2\end{Bmatrix}\begin{Bmatrix}a_0\\a_1\\a_2\end{Bmatrix};\begin{Bmatrix}\phi_0\\\phi_1\\\phi_2\end{Bmatrix}=\begin{bmatrix}1 & 1 & 1\\1 & 2 & 4\\1 & 4 & 16\end{bmatrix}\begin{Bmatrix}a_0\\a_1\\a_2\end{Bmatrix};$$



$$\tilde{\phi}(x)=\begin{Bmatrix}1\\x\\x^2\end{Bmatrix}^T\frac{1}{6}\begin{bmatrix}16 & -12 & 2\\-12 & 2 & -3\\2 & -3 & 1\end{bmatrix}\{\phi_e\}=\overbrace{\frac{1}{6}\begin{Bmatrix}16-12x+2x^2\\-12+15x-3x^2\\2-3x+x^2\end{Bmatrix}}^{[N]^T}\{\phi_e\}$$

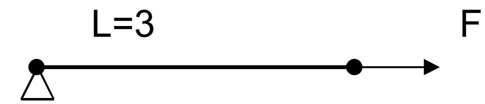
Matriz de Rigidez:

$$\varepsilon_x=\underbrace{\{\partial\}[N(x)]}_{[B(x)]}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix}=\left[\frac{1}{6}\begin{Bmatrix}-12+4x\\15-6x\\-3+2x\end{Bmatrix}\right]^T\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix};\qquad [K]=\int_{L_e}[B]^T E[B]A\,dL=\frac{AE}{36}\int_{L_e}\begin{Bmatrix}-12+4x\\15-6x\\-3+2x\end{Bmatrix}\begin{Bmatrix}-12+4x\\15-6x\\-3+2x\end{Bmatrix}^T dL$$

$$[K]=\frac{AE}{36}\int_{L_e}\begin{bmatrix}(-12+4x)^2 & (-12+4x)(15-6x) & (-12+4x)(-3+2x)\\(-12+4x)(15-6x) & (15-6x)^2 & (15-6x)(-3+2x)\\(-12+4x)(-3+2x) & (-3+2x)(15-6x) & (-3+2x)^2\end{bmatrix}dL=\frac{AE}{36}\begin{bmatrix}48 & -54 & 6\\-54 & 81 & -27\\6 & -27 & 21\end{bmatrix}$$

Solución Exacta:

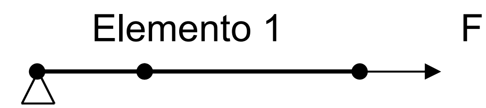
$$u_3 = 3\frac{F}{AE}$$



$$[K]\{D\} = \{R\}$$

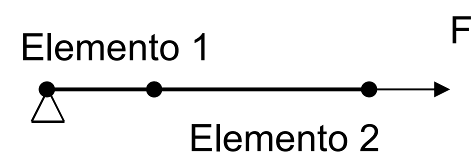
1 Elemento Barra de 3 nodos

$$\frac{AE}{12}\begin{bmatrix} 24 & -18 & 2 \\ -18 & 27 & -9 \\ 2 & -9 & 7 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ F \end{Bmatrix}$$



$$\frac{1}{9AE}\begin{bmatrix} 7 & 9 \\ 9 & 27 \end{bmatrix}\begin{Bmatrix} 0 \\ F \end{Bmatrix} = \frac{1}{AE}\begin{Bmatrix} F \\ 3F \end{Bmatrix}$$

2 Elementos Barra de 2 nodos



$$AE\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+\tfrac{1}{2} & -\tfrac{1}{2} \\ 0 & -\tfrac{1}{2} & \tfrac{1}{2} \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ F \end{Bmatrix}$$

$$\frac{1}{AE}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ F \end{Bmatrix} = \frac{1}{AE}\begin{Bmatrix} F \\ 3F \end{Bmatrix}$$

Elemento Viga:

$$\varepsilon_x(x,y)=\frac{\partial u(x,y)}{\partial x}=\frac{\partial\left(-y\dfrac{\partial v(x)}{\partial x}\right)}{\partial x}=-y\frac{\partial^2 v(x)}{\partial x^2}=-y\kappa$$

$$W_{int}=\int_V \cancel{\partial}\varepsilon_x^T E\varepsilon_x dv=\int_0^L\int_{\frac{-b}{2}}^{\frac{b}{2}}\int_{\frac{-h}{2}}^{\frac{h}{2}} y^2\cancel{\partial}\kappa^T E\kappa\, dz\, dy\, dx=\int_0^L \cancel{\partial}\kappa^T E\overbrace{\frac{h^3b}{12}}^{I_z}\kappa\ dx$$

$$\kappa=\frac{\partial^2 v(x)}{\partial x^2}=\frac{\partial^2\{N(x)\}}{\partial x^2}\{d\}=\begin{Bmatrix}-\dfrac{6}{L^2}+\dfrac{12x}{L^3}\\ -\dfrac{4}{L}+\dfrac{6x}{L^2}\\ \dfrac{6}{L^2}-\dfrac{12x}{L^3}\\ -\dfrac{2}{L}+\dfrac{6x}{L^2}\end{Bmatrix}^T\begin{Bmatrix}v_1\\ \theta_{z1}\\ v_2\\ \theta_{z2}\end{Bmatrix}$$

Matriz de Rigidez

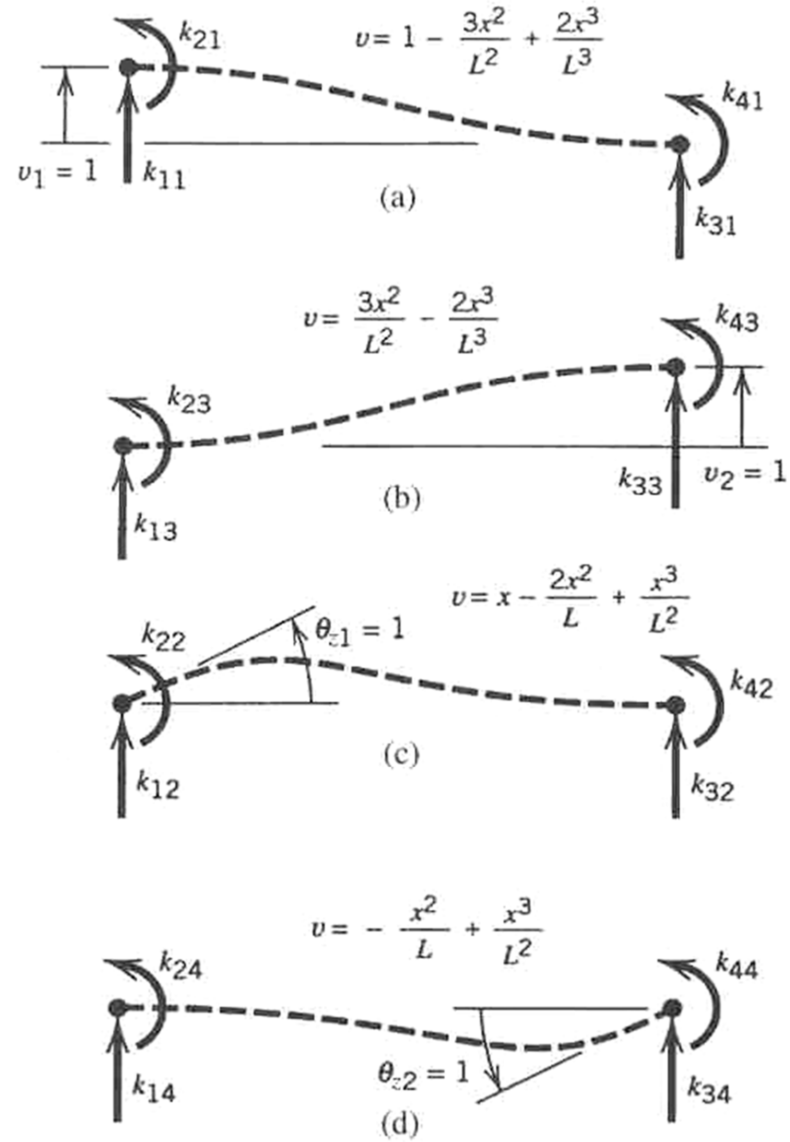
$$[K]=\int_{L_e}[B]^T EI_z[B]dx;$$

Elemento Viga:

$$v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \overbrace{\{1 \quad x \quad x^2 \quad x^3\}}^{\{X\}} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix} = \begin{Bmatrix} a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ a_1 + 2a_2 x + 3a_3 x^2 \\ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ a_1 + 2a_2 x + 3a_3 x^2 \end{Bmatrix} = \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix}}^{[A]} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

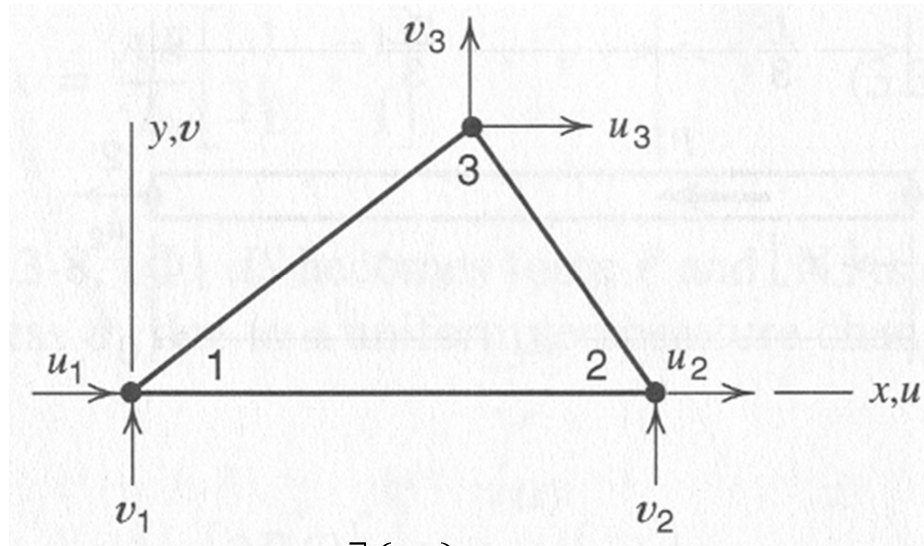
$$[N]\begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix} = \{X\}[A]^{-1}\begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix} = \begin{bmatrix} 1 - \dfrac{3x^2}{L^2} + \dfrac{2x^3}{L^3} \\ x - \dfrac{2x^2}{L} + \dfrac{x^3}{L^2} \\ -\dfrac{2x^3}{L^3} + \dfrac{3x^2}{L^2} \\ -\dfrac{x^2}{L} + \dfrac{x^3}{L^2} \end{bmatrix}^T \begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix}$$



Elementos Triangulares

Deformación Constante - CST

Problema Plano



La matriz no depende de la posición (mas facil de integrar)

$$u(x,y) = a_1 + a_2 x + a_3 y = \{1 \quad x \quad y\}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \;;\; \begin{cases} u(x_1,y_1) = u_1 \\ u(x_2,y_2) = u_2 \\ u(x_3,y_3) = u_3 \end{cases}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}_{[A]}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u(x,y) = \underbrace{\{1 \quad x \quad y\}[A]^{-1}}_{\underbrace{[N]}_{1x3}} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \{N_1 \quad N_2 \quad N_3\} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = a_1 + a_2 x + a_3 y$$

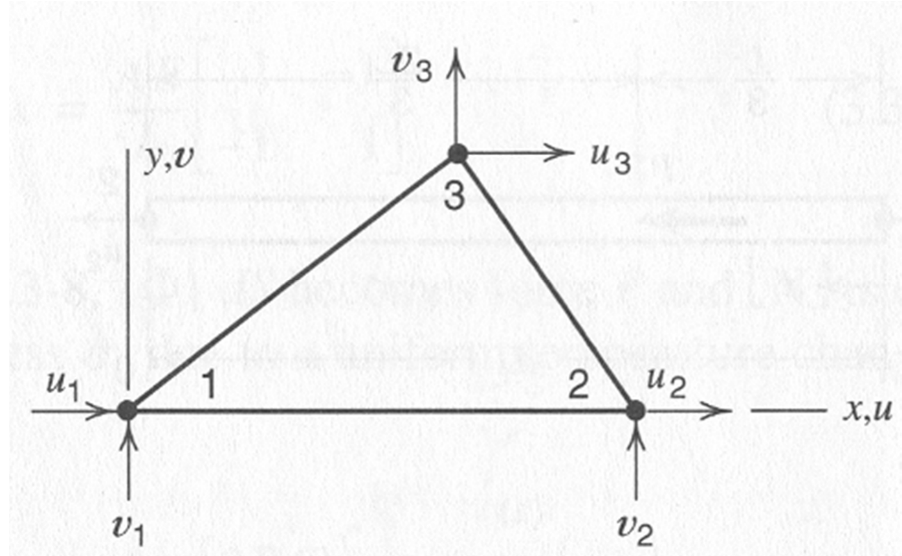
$$v(x,y) = \underbrace{\{1 \quad x \quad y\}[A]^{-1}}_{[N]} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \{N_1 \quad N_2 \quad N_3\} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = a_4 + a_5 x + a_6 y$$

Elementos Triangulares

Deformación Constante - CST

Deformaciones

$$N_1 = \det(A)^{-1}((x_2y_3 - x_3y_2) + x(y_2 - y_3) + y(x_3 - x_2))$$
$$N_2 = \det(A)^{-1}((x_3y_1 - x_1y_3) + x(y_3 - y_1) + y(x_1 - x_3))$$
$$N_3 = \det(A)^{-1}((x_1y_2 - x_2y_1) + x(y_1 - y_2) + y(x_2 - x_1))$$



$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix};$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = a_2$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = a_6$$
$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = a_3 + a_5$$

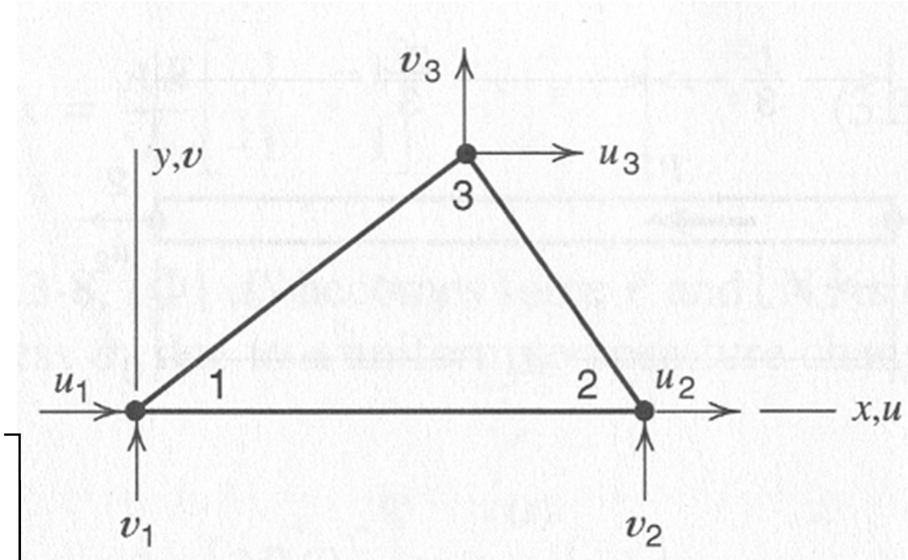
$$; \quad [B]\{d\} = \{\partial\} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & & 0 \\ 0 & \frac{\partial N_1}{\partial y} & \cdots & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & & \frac{\partial N_3}{\partial x} \end{bmatrix} = \det(A)^{-1} \begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix}$$

Elementos Triangulares

Deformación Constante - CST

Matriz de Rigidez



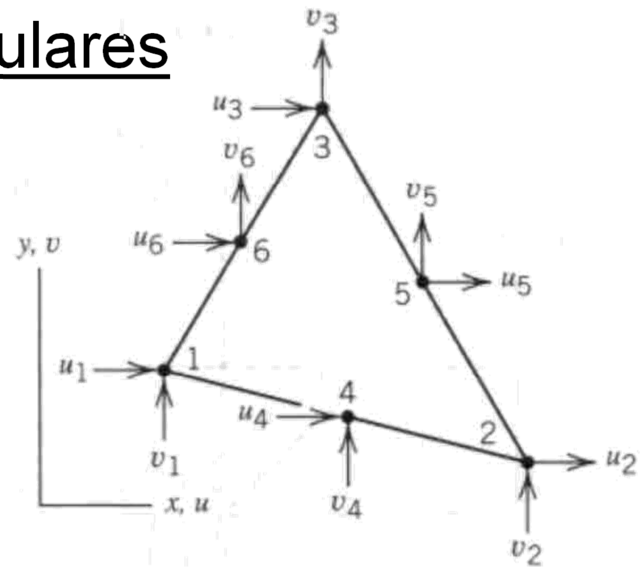
$$[B]=\frac{\begin{bmatrix} y_2-y_3 & 0 & y_3-y_1 & 0 & y_1-y_2 & 0 \\ 0 & x_3-x_2 & 0 & x_1-x_3 & 0 & x_2-x_1 \\ x_3-x_2 & y_2-y_3 & x_1-x_3 & y_3-y_1 & x_2-x_1 & y_1-y_2 \end{bmatrix}}{\det(A)}$$

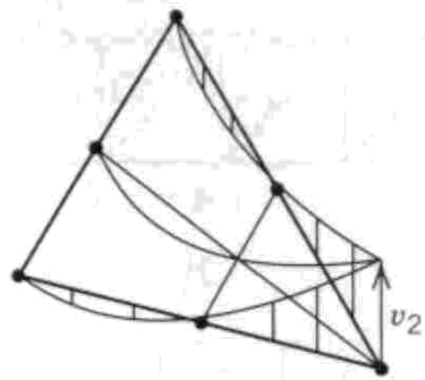
$$[K]=\int_S [B]^T[C][B]ds$$

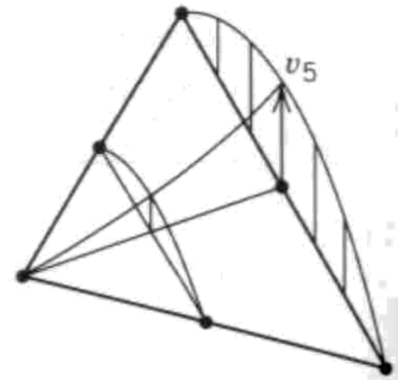
$$[K]=\frac{S}{\det(A)}\begin{bmatrix} y_2-y_3 & 0 & x_3-x_2 \\ 0 & x_3-x_2 & y_2-y_3 \\ y_3-y_1 & 0 & x_1-x_3 \\ 0 & x_1-x_3 & y_3-y_1 \\ y_1-y_2 & 0 & x_2-x_1 \\ 0 & x_2-x_1 & y_1-y_2 \end{bmatrix}\frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}[B]$$

Elementos Triangulares

Deformación Lineal - LST







$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

$$v = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2$$

La presición del poinomio depende de si completo un piso:

1
x y
x^2 xy y^2
x^3 x^2y xy^2 y^3`

El LST tiene el segundo piso completo

Deformaciones

$$\varepsilon_x = \frac{\partial u}{\partial x} = a_2 + 2a_4 x + a_5 y$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = a_9 + a_{11} x + 2a_{12} y$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = a_3 + a_8 + (a_5 + 2a_{10})x + (2a_6 + a_{11})y$$

Los términos a10 y a11 que aparecen por llenar el piso son los que imiden que aparezca el corte espúreo.

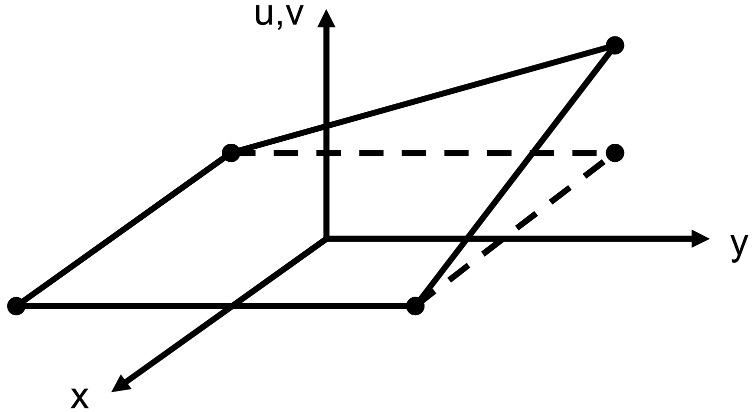
Elementos Rectangulares

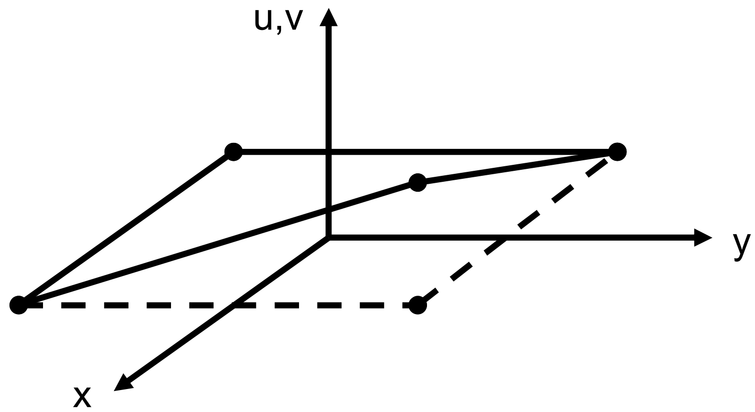
Deformación Constante - Q4

$$u = a_1 + a_2x + a_3y + a_4xy = N_1u_1 + \cdots + N_4u_4$$
$$v = a_5 + a_6x + a_7y + a_8xy = N_1v_1 + \cdots + N_4v_4$$

Deformaciones

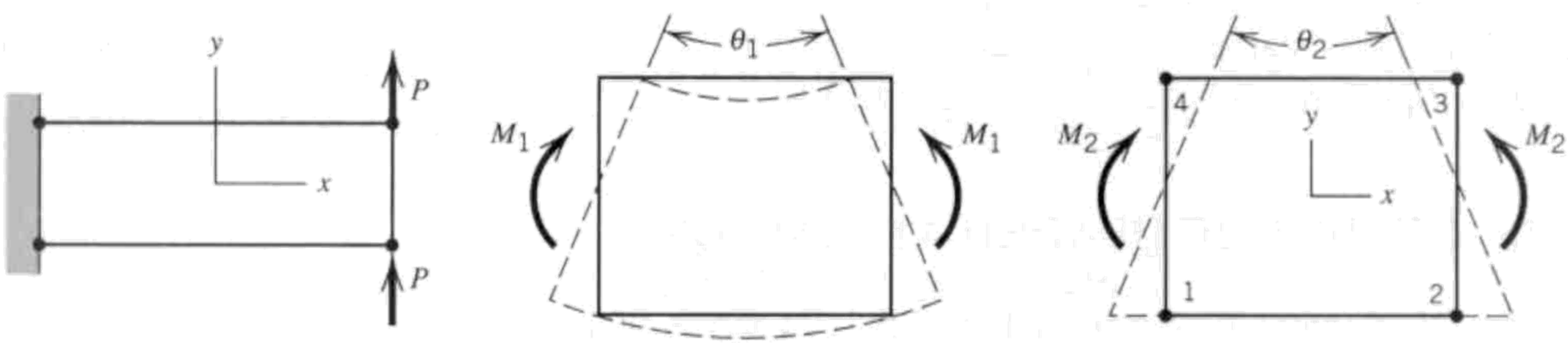
$$\varepsilon_x = \frac{\partial u}{\partial x} = a_2 + a_4 y$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = a_7 + a_8 x$$
$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = a_3 + a_6 + a_4 x + a_8 y$$





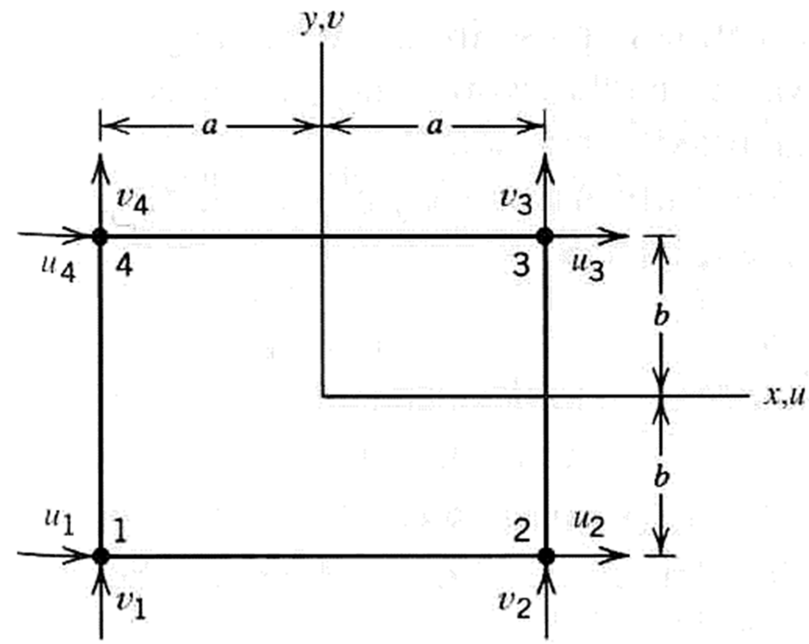
Corte Espúreo

No es bueno para simular vigas, mala deformación por corte.



El corte espúreo se lleva la energía de deformación a corte, que para estas cuentas no existe. Quiere decir que se va a curvar menos de lo que realmente debería.

Problema Plano – Tensión Plana



Matriz Constitutiva (E=1, ν=0.33):

$$C := \begin{pmatrix} 1.125 & 0.375 & 0 \\ 0.375 & 1.125 & 0 \\ 0 & 0 & 0.375 \end{pmatrix}$$

Funciones de Forma:

$$N1(x,y) = \frac{(a-x)\cdot(b-y)}{4a\cdot b} \qquad N2 = \frac{(a+x)\cdot(b-y)}{4a\cdot b}$$

$$N3 = \frac{(a+x)\cdot(y+b)}{4a\cdot b} \qquad N4 = \frac{(a-x)\cdot(y+b)}{4a\cdot b}$$

$$N(x,y) = \begin{pmatrix} N1(x,y) & 0 & N2(x,y) & 0 & N3(x,y) & 0 & N4(x,y) & 0 \\ 0 & N1(x,y) & 0 & N2(x,y) & 0 & N3(x,y) & 0 & N4(x,y) \end{pmatrix}$$

Matriz Desplazamiento- Deformación: [B]=[∂][N]

$$B(x,y) := \begin{pmatrix} \frac{d}{dx}N(x,y)_1 & 0 & \frac{d}{dx}N(x,y)_2 & 0 & \frac{d}{dx}N(x,y)_3 & 0 & \frac{d}{dx}N(x,y)_4 & 0 \\ 0 & \frac{d}{dy}N(x,y)_1 & 0 & \frac{d}{dy}N(x,y)_2 & 0 & \frac{d}{dy}N(x,y)_3 & 0 & \frac{d}{dy}N(x,y)_4 \\ \frac{d}{dy}N(x,y)_1 & \frac{d}{dx}N(x,y)_1 & \frac{d}{dy}N(x,y)_2 & \frac{d}{dx}N(x,y)_2 & \frac{d}{dy}N(x,y)_3 & \frac{d}{dx}N(x,y)_3 & \frac{d}{dy}N(x,y)_4 & \frac{d}{dx}N(x,y)_4 \end{pmatrix}$$

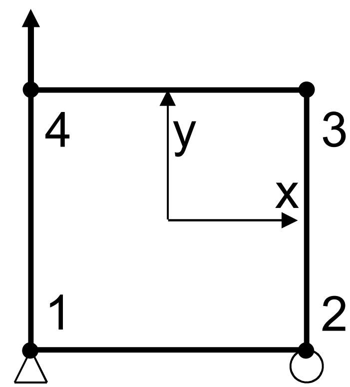
Problema Plano – Tensión Plana

Adoptando a=1 y b=1

$$B(x,y) \rightarrow \begin{pmatrix} \frac{-1}{4}+\frac{1}{4}\cdot y & 0 & \frac{1}{4}-\frac{1}{4}\cdot y & 0 & \frac{1}{4}\cdot y+\frac{1}{4} & 0 & \frac{-1}{4}\cdot y-\frac{1}{4} & 0 \\ 0 & \frac{-1}{4}+\frac{1}{4}\cdot x & 0 & \frac{-1}{4}-\frac{1}{4}\cdot x & 0 & \frac{1}{4}+\frac{1}{4}\cdot x & 0 & \frac{1}{4}-\frac{1}{4}\cdot x \\ \frac{-1}{4}+\frac{1}{4}\cdot x & \frac{-1}{4}+\frac{1}{4}\cdot y & \frac{-1}{4}-\frac{1}{4}\cdot x & \frac{1}{4}-\frac{1}{4}\cdot y & \frac{1}{4}+\frac{1}{4}\cdot x & \frac{1}{4}\cdot y+\frac{1}{4} & \frac{1}{4}-\frac{1}{4}\cdot x & \frac{-1}{4}\cdot y-\frac{1}{4} \end{pmatrix}$$

Matriz de Rigidez

$$K = \int_{-a}^{a}\int_{-b}^{b} B(x,y)^{T}\cdot C\cdot B(x,y)\,dx\,dy$$



$$K = \begin{pmatrix} 0.5 & 0.188 & -0.313 & 0 & -0.25 & -0.188 & 0.063 & 0 \\ 0.188 & 0.5 & 0 & 0.063 & -0.188 & -0.25 & 0 & -0.313 \\ -0.313 & 0 & 0.5 & -0.188 & 0.063 & 0 & -0.25 & 0.188 \\ 0 & 0.063 & -0.188 & 0.5 & 0 & -0.313 & 0.188 & -0.25 \\ -0.25 & -0.188 & 0.063 & 0 & 0.5 & 0.188 & -0.313 & 0 \\ -0.188 & -0.25 & 0 & -0.313 & 0.188 & 0.5 & 0 & 0.063 \\ 0.063 & 0 & -0.25 & 0.188 & -0.313 & 0 & 0.5 & -0.188 \\ 0 & -0.313 & 0.188 & -0.25 & 0 & 0.063 & -0.188 & 0.5 \end{pmatrix}$$

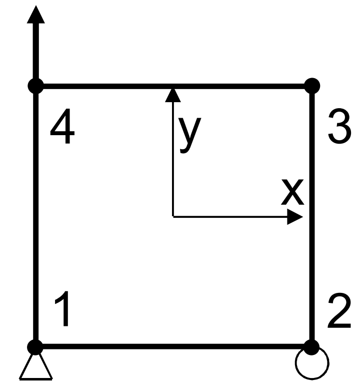
Problema Plano – Tensión Plana

Matriz Reducida

$$K = \begin{bmatrix} 0.5 & 0.0625 & 0 & -0.25 & 0.1875 \\ 0.0625 & 0.5 & 0.1875 & -0.3125 & 0 \\ 0 & 0.1875 & 0.5 & 0 & 0.0625 \\ -0.25 & -0.3125 & 0 & 0.5 & -0.1875 \\ 0.1875 & 0 & 0.0625 & -0.1875 & 0.5 \end{bmatrix}$$

Vector de Cargas:

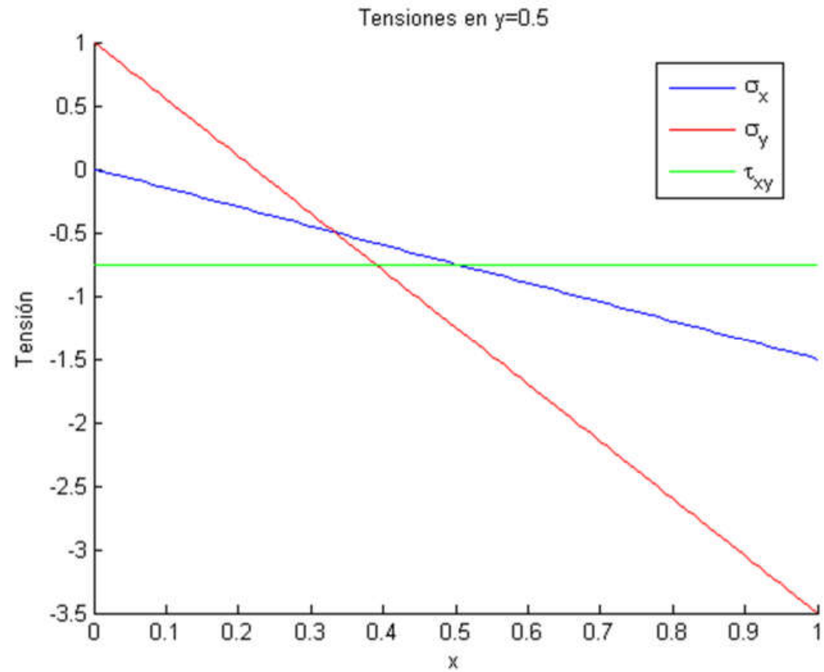
$$R = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$



Desplazamientos Globales:

$$D = K^{-1}R \qquad D = \begin{Bmatrix} 0 \\ 0 \\ -1/3 \\ 0 \\ 5/3 \\ -1 \\ 2 \\ 3 \end{Bmatrix}$$

Variación Tensiones:



Deformaciones Elementales:

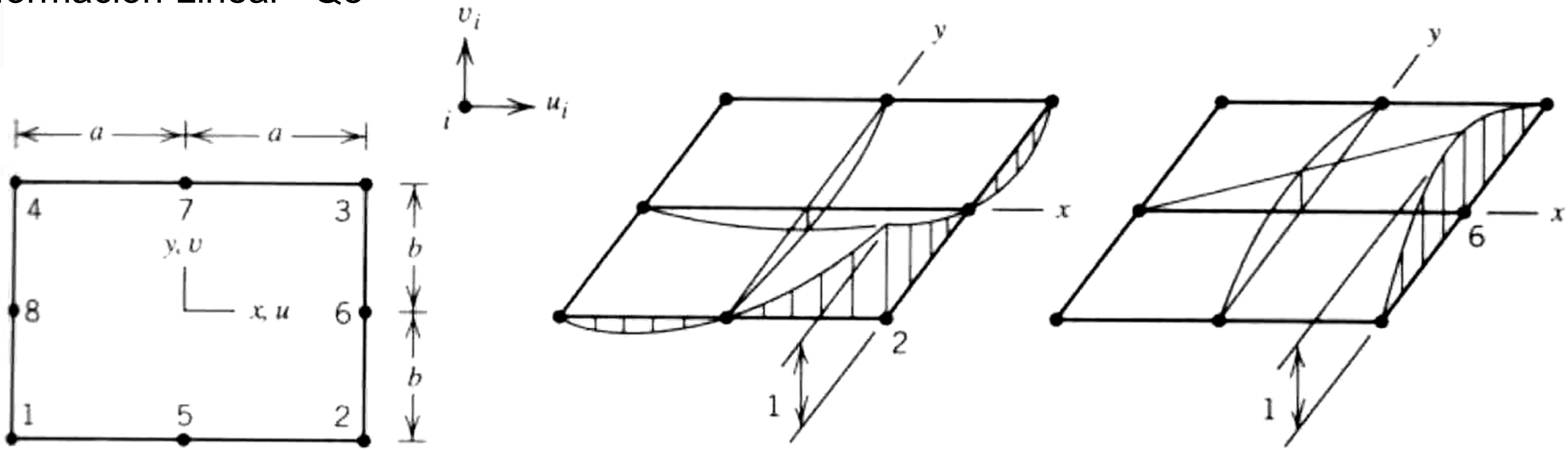
$$\varepsilon(x,y) = B(x,y)D \qquad \varepsilon(x,y) = \begin{Bmatrix} -1/3 \\ 1-4x \\ -4y \end{Bmatrix}$$

Tensiones Elementales:

$$\sigma(x,y) = CB(x,y)D \qquad \sigma(x,y) = \begin{Bmatrix} -\frac{3}{2}x \\ 1-\frac{9}{2}x \\ -\frac{3}{2}y \end{Bmatrix}$$

Elementos Rectangulares

Deformación Lineal - Q8



$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^2 y + a_8 xy^2$$

$$v = a_9 + a_{10} x + a_{11} y + a_{12} x^2 + a_{13} xy + a_{14} y^2 + a_{15} x^2 y + a_{16} xy^2$$

Deformaciones

$$\varepsilon_x = \frac{\partial u}{\partial x} = a_2 + 2a_4 x + a_5 y + 2a_7 xy + a_8 y^2$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = a_{11} + a_{13} x + 2a_{14} y + a_{15} x^2 + 2a_{16} xy$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = a_3 + a_{10} + (a_5 + 2a_{12})x + (2a_6 + a_{13})y + a_7 x^2 + 2(a_8 + a_{15})xy + a_{16} y^2$$