Capítulo 1
First

1.1. Control theory

1.1.1. Basics

Taylor expansion (Linearization) of two-variable nonlinear equation.

$$f(x,y) = f(\overline{x},\overline{y}) + \left[\frac{\partial f}{\partial x}(x-\overline{x}) + \frac{\partial f}{\partial y}(y-\overline{y})\right] + \ldots$$

Matlab command to convert state space to transfer function `[num,den]=ss2tf(A,B,C,D,iu)` where `iu` must be specified for systems with more than one input.

1.1.2. State space

$\mathbf{u}(t)$ is inputs vector and is of size $p \times 1$ for a given system, i.e: $\mathbf{u}(t) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ for two input system, $p = 2$.

$\mathbf{y}(t)$ is the output vector of size $q \times 1$.

$\mathbf{z}(t)$ is the *disturbance input*. Only applies to dynamical systems and is of size $r \times 1$

Thus we define the **state space variables** so that system output is purely in function of current system state variables and input variables.

$$\text{System Output} = f\,(\text{Current System State, System Input})$$

We will define X or $\mathbf{x}$ as our system state variables. There are some important aspects to note about state space variables such as

- System output $\mathbf{y}(t)$ will be a function of them
- They change over time
- They are internal to the system
- They may include system outputs (outputs will be a function of themselves in part)
- Their selection is inherent part of the system design process and there are different methods of selecting them.

- We will assume there is a minimal quantity of state variables that is sufficient to accurately describe the system
- If all system inputs u_j are defined beforehand for $t \geq t_0$ then $\mathbf{x}(t)$ defines all system states for time $t \geq t_0$

The mathematical representation of state space variables will be will be that of the **state vector** $\mathbf{x}(t)$ of size $n \times 1$.

To model our system we then define the equations that govern it in **state space**[1]. These are the **state-space equations** of the system. For a dynamic system these must include a variable that serves as memory of inputs for $t \geq t_1$. *Integrators* serve as memory devices for *continuous-time* models, however, our state-space representation is discrete! This is when state-space variables come in handy: The outputs of integrators can be considered as the variables that define the internal state of the dynamic system (Ogata).

For a system of size $p = q = n = 1$ one has the state-space representation defined as:

$$\dot{x}(t) = g\left[t_0, t, x(t), x(0), u(t)\right], \qquad y = h\left[t, x(t), u(t)\right]$$

For a *linear time-variant dynamical system* of arbitrary size it is convenient to represent it in it's linearized form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{E}(t)\mathbf{z}(t) \tag{1.1}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \tag{1.2}$$

where

$\mathbf{A}_{n\times n}$ System matrix. Relates future state change with current state. May be zero. Also referred to as the state matrix in some bibliographies.

$\mathbf{B}_{n\times p}$ Control/input matrix. How system input influences state change. May be zero.

$\mathbf{C}_{q\times n}$ Output matrix. How system state influences system output.

$\mathbf{D}_{q\times p}$ Feed forward or feedthrough matrix. How system input influences system output. Is usually zero for most physical systems.

$\mathbf{E}_{n\times r}$ Input matrix for disturbances. Applies only for dynamical systems.

the system is said to be **time-invariant** if the above matrices are not dependent of time. An example of a **time-variant** system is a spacecraft, whose mass changes due to fuel consumption.

One method of state space variable selection is **physical selection**. This method is based on energy accumulators. It can be said that *the minimum number of state-space variables needed to*

[1]State space can be thought of an n-dimensional space whose axes are the state variables $(x_1, x_2 \ldots)$

model the system accurately is equal to the number of independent energy accumulators. When state-space variable is not a energy variable it is said to be an augmented variable.

The general solution to the linear differential equation of state:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\ \mathbf{x}(t) + \mathbf{B}\ \mathbf{u}(t)$$

es

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x})0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\ \mathbf{u}(\tau)\mathrm{d}\tau$$

Definition 1.1 (Matrix Exponential)**.**

$$e^{\mathbf{A}t} = \mathbf{I} + \sum_{i=1}^{\infty} \frac{(\mathbf{A}t)^i}{i!} \tag{1.3}$$

A MATLAB function is provided to calculate the matrix exponential.

CÓDIGO 1.1: matrixexponential.m

```
A = rand(3);
t0 = 0.5;
fprintf('e^(At)=');disp(expt(t0,1e5,A));
function y = expt(t,n,A)
    y = eye(size(A));
    for i=1:n
        y = y + (A*t)^i/factorial(i);
    end
end
```