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No.	SOLUTION
A1	Answer: **(d)** The order of $A_{m\times n}A^T{}_{n\times m}$ is $m\times m$
A2	Answer: **(c)** $A\cap B=B$ and $\lvert A-B\rvert=1 \rightarrow B=\{3,6\},\{3,9\},\{6,9\}$ $B\subset C$ and $C\cup A=A \rightarrow C=\{3,6\},\{3,9\},\{6,9\},\{3,6,9\}$
A3	Answer: **(c)**
A4	Answer: **(b)** Assume moving rightwards =1 & downwards = 0, we will need 3 ‘1’s and 3 ‘0’s. With 2 missing lines, all possible ways are: 111000, 110100, 110010, 110001, 011100, 011010, 011001, 010110, 010101, 001110, 001101, 000111 Therefore, there are 12 paths.
A5	Answer: **(c)** $P(biased\mid heads)=\dfrac{P(biased)\times P(heads\mid biased)}{P(biased)\times P(heads\mid biased)+P(fair)\times P(heads\mid fair)}$ $=\dfrac{\frac{1}{3}\times 0.7}{\frac{1}{3}\times 0.7+\frac{2}{3}\times 0.5}=\dfrac{7}{17}\approx 0.41$
B1a (i)	$\mathbf{3}\begin{bmatrix}3 & -2\\ 4 & 1\end{bmatrix}+\begin{bmatrix}-2 & 3\\ 5 & -6\end{bmatrix}=\begin{bmatrix}7 & -3\\ 17 & -3\end{bmatrix}$
B1a (ii)	$\mathbf{BA}^{\mathrm{T}}=\begin{bmatrix}-2 & 3\\ 5 & -6\end{bmatrix}\begin{bmatrix}3 & 4\\ -2 & 1\end{bmatrix}=\begin{bmatrix}-12 & -5\\ 27 & 14\end{bmatrix}$

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B1b	$\mathbf{CD}=\begin{bmatrix}3&0&2\\2&0&-2\\0&1&1\end{bmatrix}\begin{bmatrix}1&1&0\\-1&\frac{3}{2}&5\\1&-\frac{3}{2}&0\end{bmatrix}=\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$ Since $\mathbf{CD}=5\mathbf{I}$: $\mathbf{C}^{-1}\mathbf{CD}=5\mathbf{C}^{-1}\mathbf{I}$ $\Rightarrow \mathbf{D}=5\mathbf{C}^{-1}$ $\therefore \mathbf{C}^{-1}=\frac{1}{5}\mathbf{D}=\frac{1}{5}\begin{bmatrix}1&1&0\\-1&\frac{3}{2}&5\\1&-\frac{3}{2}&0\end{bmatrix}=\begin{bmatrix}\frac{1}{5}&\frac{1}{5}&0\\-\frac{1}{5}&\frac{3}{10}&1\\\frac{1}{5}&-\frac{3}{10}&0\end{bmatrix}$
B2a	$\mathbf{T}_1=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_2=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$ $\mathbf{C}=\mathbf{T}_2\mathbf{T}_1=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-1&0&0\\2&-1&0\\0&0&1\end{bmatrix}$
B2b	$\mathbf{P}'=\mathbf{CP}=\begin{bmatrix}-1&0&0\\2&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&-1&0\\0&1&2\\1&1&1\end{bmatrix}=\begin{bmatrix}-1&1&0\\2&-3&-2\\1&1&1\end{bmatrix}$
B2c	$\mathbf{T}_1^{-1}=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_2^{-1}=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$ $\mathbf{C}^{-1}=\mathbf{T}_1^{-1}\mathbf{T}_2^{-1}=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-1&0&0\\-2&-1&0\\0&0&1\end{bmatrix}$
B3a	Convert to HEX $1100101001.101_2=329.\text{A}_{16}$ Convert to DEC $1100101001.101_2=329.\text{A}_{16}=3\times16^2+2\times16^1+9\times16^0+10\times16^{-1}=809.625_{10}$

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B3b

Integral part:		
2	264	
2	132	0
2	66	0
2	33	0
2	16	1
2	8	0
2	4	0
2	2	0
2	1	0
	0	1

Fractional part:		
2	0.425	
2	0.85	0
2	0.7	1
2	0.4	1
2	0.8 (rep)	0
2	0.6	1
2	0.2	1
2	0.4	0
2	0.8 (rep)	0
2	0.6	1
2	0.2	1
2	0.4	0

$\therefore 264.425_{10} = 100001000.011\overline{0110}_2$

B4a

$U = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

$A = \{-2, -1, 1, 2, 3\}$

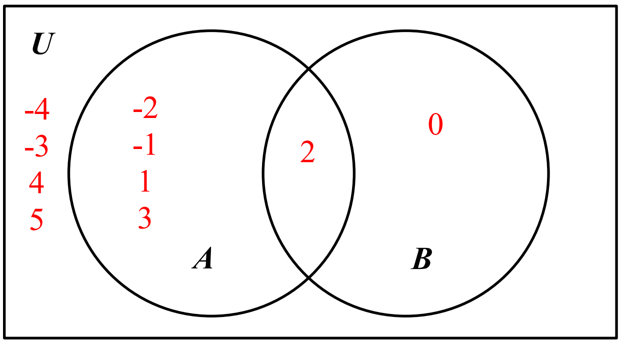
$B = \{0, 2\}$

B4b

$A \cap \overline{B} = \{-2, -1, 1, 3\}$

$\overline{A \cup B} = \{-4, -3, 4, 5\}$

B4c



B5a (i)

$(p \wedge \neg q) \Rightarrow r \rightarrow (\mathbf{F} \wedge \neg \mathbf{T}) \Rightarrow \mathbf{F} = (\mathbf{F} \wedge \mathbf{F}) \Rightarrow \mathbf{F} = \mathbf{F} \Rightarrow \mathbf{F} = \mathbf{T}$

B5a (ii)

$(p \vee r) \Leftrightarrow q \rightarrow (\mathbf{F} \vee \mathbf{F}) \Leftrightarrow \mathbf{T} = \mathbf{F} \Leftrightarrow \mathbf{T} = \mathbf{F}$

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B5b	p | q | $p \wedge q$ | $\neg(p \wedge q)$ | $\neg p$ | $\neg q$ | $\neg p \vee \neg q$ | $(\neg(p \wedge q)) \Leftrightarrow (\neg p \vee \neg q)$ T | T | T | F | F | F | F | T T | F | F | T | F | T | T | T F | T | F | T | T | F | T | T F | F | F | T | T | T | T | T $(\neg(p \wedge q)) \Leftrightarrow (\neg p \vee \neg q)$ is a tautology.
B6a	(i) $16! = 2.09 \times 10^{13}$ (ii) $8! \times 2! = 80640$
B6b	(i) $\left\lfloor \frac{800}{3} \right\rfloor - \left\lfloor \frac{150}{3} \right\rfloor + 1 = 266 - 50 + 1 = 217$ (ii) $\left\lfloor \frac{800}{6} \right\rfloor - \left\lfloor \frac{150}{6} \right\rfloor + 1 = 133 - 25 + 1 = 109$ (iii) $\lvert A_3 \cup A_6 \rvert = \lvert A_3 \rvert = 217$
B7a	$\frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} = \frac{1}{55} \approx 0.0182$
B7b	$1 - \frac{1}{55} = \frac{54}{55} \approx 0.982$
B7c	Only 1 combination to give sum of 10: 3, 3, 4. $P(3,3,4) = \frac{8}{12} \times \frac{7}{11} \times \frac{4}{10} = \frac{28}{165} \approx 0.170$
B7d	$P(sum = 10 \cap \geq 1\ black) = P(sum = 10) - P(sum = 10 \cap \text{all white})$ $= \frac{28}{165} - \frac{2}{12} \times \frac{1}{11} \times \frac{2}{10}$ $= \frac{1}{6} \approx 0.167$

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C1a

x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

$f(x,y,z) = (x+y+z)(x+y+\bar{z})(x+\bar{y}+z)$

C1b

p	q	pq	$\bar{p}$	$\bar{q}$	$\overline{pq}$	$p \odot q$
0	0	0	1	1	1	1
0	1	0	1	0	0	0
1	0	0	0	1	0	0
1	1	1	0	0	0	1

C1c

$p \odot (p \odot q) \odot q$	
$= p \odot (pq + \bar{p}\bar{q}) \odot q$	Substitution
$= (p \bullet (pq + \bar{p}\bar{q}) + \bar{p} \bullet \overline{pq + \bar{p}\bar{q}}) \odot q$	Substitution
$= (pq + \bar{p} \bullet (\overline{pq} \bullet \overline{\bar{p}\bar{q}})) \odot q$	De Morgan's Laws
$= (pq + \bar{p} \bullet (\bar{p} + \bar{q}) \bullet (p + q)) \odot q$	De Morgan's Laws
$= (pq + \bar{p}q) \odot q$	Associative+Idempotent Laws
$= q \odot q$	Distributive Laws
$= q \bullet q + \bar{q} \bullet \bar{q}$	Substitution
$= 1$	Idempotent Laws

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C2a

Let $\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}$ (since $\mathbf{Q}$ is a symmetric matrix, we have $q_{ij} = q_{ji}$)

$$\mathbf{PQP}^T = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} q_{11}x + q_{12}y + q_{13}z & q_{12}x + q_{22}y + q_{23}z & q_{13}x + q_{23}y + q_{33}z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= q_{11}x^2 + q_{12}xy + q_{13}xz + q_{12}xy + q_{22}y^2 + q_{23}yz + q_{13}xz + q_{23}yz + q_{33}z^2$$
$$= q_{11}x^2 + q_{22}y^2 + q_{33}z^2 + 2q_{12}xy + 2q_{13}xz + 2q_{23}yz$$

Comparing coefficients with the expression $2x^2 + 5y^2 + 3z^2 + 2xy - 2xz + 8yz$, we have $q_{11} = 2,\ q_{22} = 5,\ q_{33} = 3,\ q_{12} = 1,\ q_{13} = -1$ and $q_{23} = 4$.

$$\therefore \mathbf{Q} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 5 & 4 \\ -1 & 4 & 3 \end{bmatrix}$$

C2b

From Clue No. 1 & 2, Jack must be in the 4th place with 2B1G before him and 1B1G after him. From Clue No. 3, Zack must be in the 6th place (3B2G before him). From Clue No. 4 & 5, Amanda must be in the 1st place and Emma in the 5th place. From Clue No. 7, Victor must be in the 3rd place. As such, from Clue No. 6, Nick must be in the 2nd place. Therefore, the sequence of escape is

Sequence	Name
1	Amanda
2	Nick
3	Victor
4	Jack
5	Emma
6	Zack

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C3a	Let A_i be the event that a green ball appears for the first time at ith picking. Yasir can only win if i is an even number. $i=2 \to P(A_2)=\frac{7}{12}\times\frac{5}{11}=\frac{35}{132}$ $i=4 \to P(A_4)=\frac{7}{12}\times\frac{6}{11}\times\frac{5}{10}\times\frac{5}{9}=\frac{35}{396}$ $i=6 \to P(A_6)=\frac{7}{12}\times\frac{6}{11}\times\frac{5}{10}\times\frac{4}{9}\times\frac{3}{8}\times\frac{5}{7}=\frac{5}{264}$ $i=8 \to P(A_8)=\frac{7}{12}\times\frac{6}{11}\times\frac{5}{10}\times\frac{4}{9}\times\frac{3}{8}\times\frac{2}{7}\times\frac{1}{6}\times\frac{5}{5}=\frac{1}{792}$ Hence, $P(Yasir\ wins)=\frac{35}{132}+\frac{35}{396}+\frac{5}{264}+\frac{1}{792}=\frac{37}{99}$.
C3b (i)	Define the following possible paths from n_1 to n_4: $A=(l_1,l_4)\quad B=(l_2,l_5)\quad C=(l_1,l_3,l_5)$ Inclusion-exclusion formula for 3-event scenario: $P(A\cup B\cup C)$ $=P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$ $=p_1p_4+p_2p_5+p_1p_3p_5-p_1p_2p_4p_5-p_1p_3p_4p_5-p_1p_2p_3p_5+p_1p_2p_3p_4p_5$
C3b (ii)	$P(A\cup B\cup C)$ $=P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$ $=p_1p_4+p_2p_5+p_1p_3p_5-p_1p_2p_4p_5-p_1p_3p_4p_5-p_1p_2p_3p_5+p_1p_2p_3p_4p_5$ $=0.65^2\times2+0.65^3-0.65^4\times3+0.65^5$ $=0.700$