SOLUTIONS

SINGAPORE POLYTECHNIC

2021/2022 Semester 2 Mid-Semester Test

No.	SOLUTION
1(a)	For symmetric matrix, $\mathbf{A}_{ij} = \mathbf{A}_{ji}$. $\mathbf{A}_{13} = \mathbf{A}_{31}:\ b = -2$ $\mathbf{A}_{12} = \mathbf{A}_{21}:\ 2a + 1 = 5$ $\Rightarrow a = 2$ $\mathbf{A}_{23} = \mathbf{A}_{32}:\ 2c = 3a + 4b$ $2c = 6 - 8 = -2$ $\Rightarrow c = -1$
1(b)	$\mathbf{E}^T + 3\mathbf{X} = 2\mathbf{F}$ $3\mathbf{X} = 2\mathbf{F} - \mathbf{E}^T$ $\mathbf{X} = \frac{1}{3}\left(2\mathbf{F} - \mathbf{E}^T\right)$ $= \frac{1}{3}\left(2\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 7 & 4 \end{bmatrix}^T\right) = \frac{1}{3}\left(\begin{bmatrix} 6 & 4 \\ 8 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 7 \\ -1 & 4 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 3 & -3 \\ 9 & -6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$
1(c) (i)	$\mathbf{ED} = \begin{bmatrix} 3 & -1 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} -5 & 1 & 3 & -2 \\ 2 & -6 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -17 & 9 & 10 & -10 \\ -27 & -17 & 17 & 2 \end{bmatrix}$
1(c) (ii)	$\mathbf{F}^T - 4\mathbf{I}_2 = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}^T - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & -5 \end{bmatrix}$
1(d) (i)	$\mathbf{BC} = \begin{bmatrix} 2 & -1 & 1 \\ -5 & 3 & 2 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 9 & -2 & -5 \\ 13 & -2 & -9 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$
1(d) (ii)	Given $(\mathbf{C} + \mathbf{Y})(\mathbf{CY} - \mathbf{I}_3) = \mathbf{YCY}$, $\mathbf{C}^2\mathbf{Y} - \mathbf{C} + \mathbf{YCY} - \mathbf{Y} = \mathbf{YCY}$ $\mathbf{C}^2\mathbf{Y} - \mathbf{C} - \mathbf{Y} = \mathbf{0}$ $\mathbf{C}^2\mathbf{Y}\mathbf{Y}^{-1} - \mathbf{C}\mathbf{Y}^{-1} - \mathbf{Y}\mathbf{Y}^{-1} = \mathbf{0}$ $\mathbf{C}^2 - \mathbf{CY}^{-1} - \mathbf{I}_3 = \mathbf{0}$ $\mathbf{C}^{-1}\mathbf{C}^2 - \mathbf{C}^{-1}\mathbf{CY}^{-1} - \mathbf{C}^{-1}\mathbf{I}_3 = \mathbf{0}$ $\mathbf{C} - \mathbf{Y}^{-1} - \mathbf{C}^{-1} = \mathbf{0}$ $\mathbf{Y}^{-1} = \mathbf{C} - \mathbf{C}^{-1}$ From part (i), $\mathbf{BC} = 8\mathbf{I} \Rightarrow \mathbf{C}^{-1} = \frac{1}{8}\mathbf{B}$ Hence, $\mathbf{Y}^{-1} = \mathbf{C} - \frac{1}{8}\mathbf{B} = \begin{bmatrix} 9 & -2 & -5 \\ 13 & -2 & -9 \\ 3 & 2 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} 2 & -1 & 1 \\ -5 & 3 & 2 \\ 4 & -3 & 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 70 & -15 & -41 \\ 109 & -19 & -74 \\ 20 & 19 & 7 \end{bmatrix}$

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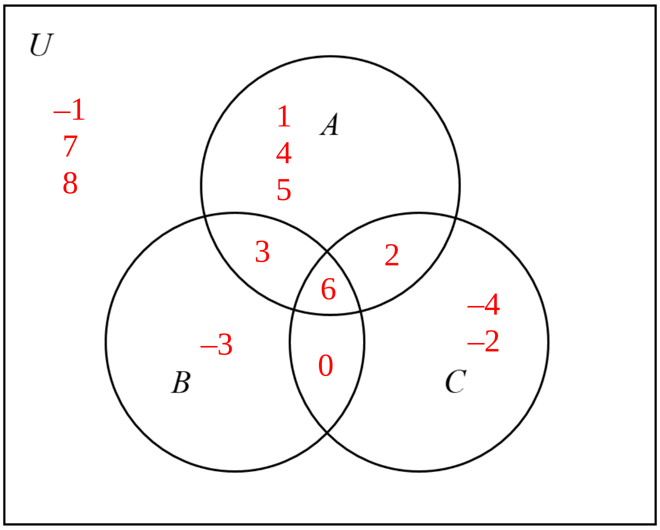
2(a) (i)	$U=\{-4,-3,-2,-1,0,1,2,3,4,5,6,7,8\}$ $A=\{1,2,3,4,5,6\}$ $B=\{-3,0,3,6\}$ $C=\{-4,-2,0,2,6\}$
2(a) (ii)	$A\cap\bar{B}=\{1,2,4,5\}$ $\bar{A}\cup C=\{-4,-3,-2,-1,0,2,6,7,8\}$
2(a) (iii)	
2(b)	$X\cap\bar{P}=\varnothing\Rightarrow X\subset P\Rightarrow X\subset\{1,2,3,4,5\}$ $X\cup(P-Q)=X\Rightarrow(P-Q)\subset X\Rightarrow\{1,2,3\}\subset X$ This means that set X must contain elements 1, 2 and 3, but must not contain any other elements in addition to 1, 2, 3, 4 and 5. Therefore, $X=\{1,2,3\}$ or $\{1,2,3,4\}$ or $\{1,2,3,5\}$ or $\{1,2,3,4,5\}$.

3(a)

Integral part:		
2	718	
2	359	0
2	179	1
2	89	1
2	44	1
2	22	0
2	11	0
2	5	1
2	2	1
2	1	0
	0	1

Fractional part:		
2	0.65	
2	0.3	1
2	0.6	0
2	0.2	1
2	0.4	0
2	0.8	0
2	0.6	1
2	0.2	1
2	0.4	0
2	0.8	0
2	0.6 (rep)	1

$$\begin{aligned} \therefore 718.65_{10} &= 10\,1100\,1110.10\,\overline{1001}_2 \\ &= 2CE.A\overline{6}_{16} \end{aligned}$$

3(b)

$$\begin{aligned} \text{Truncation error} &= 0.65_{10} - 0.1010_2 \\ &= 0.65_{10} - 0.625_{10} \\ &= 0.025_{10} \end{aligned}$$

3(c)

Convert to base-4: $0.625_{10} = 0.22_4$ and $0.875_{10} = 0.32_4$

Count in base-4 with 2 fractional digits: $0.23_4, 0.30_4, 0.31_4$

Convert back to decimal: $0.23_4 = 0.6875_{10}$, $0.30_4 = 0.75_{10}$, $0.31_4 = 0.8125_{10}$

$\therefore 0.6875_{10}, 0.75_{10}, 0.8125_{10}$

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4(a) (i)	$\mathbf{T}_1=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_2=\begin{bmatrix}1&0&2\\0&1&-3\\0&0&1\end{bmatrix}$; $\mathbf{T}_3=\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}$
4(a) (ii)	$\mathbf{C}=\mathbf{T}_3\mathbf{T}_2\mathbf{T}_1=\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&2\\0&1&-3\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}2&0&4\\-1&1&-3\\0&0&1\end{bmatrix}$
4(a) (iii)	$\mathbf{P}'=\mathbf{CP}=\begin{bmatrix}2&0&4\\-1&1&-3\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&4\\-1&3&1\\1&1&1\end{bmatrix}=\begin{bmatrix}6&8&12\\-5&-2&-6\\1&1&1\end{bmatrix}$
4(a) (iv)	$\mathbf{T}_1^{-1}=\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_2^{-1}=\begin{bmatrix}1&0&-2\\0&1&3\\0&0&1\end{bmatrix}$; $\mathbf{T}_3^{-1}=\begin{bmatrix}\frac{1}{2}&0&0\\0&1&0\\0&0&1\end{bmatrix}$
4(a) (v)	$\mathbf{C}^{-1}=\mathbf{T}_1^{-1}\mathbf{T}_2^{-1}\mathbf{T}_3^{-1}=\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-2\\0&1&3\\0&0&1\end{bmatrix}\begin{bmatrix}\frac{1}{2}&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&0&-2\\\frac{1}{2}&1&1\\0&0&1\end{bmatrix}$
4(b) (i)	$\mathbf{T}_a$: Rotation 45° clockwise about the origin $\mathbf{T}_b$: Scaling in the y-direction by a factor of $\frac{1}{2}$ relative to the origin $\mathbf{T}_c$: Rotation 45° anticlockwise about the origin $\mathbf{T}_a=\begin{bmatrix}\cos(-45°)&-\sin(-45°)&0\\\sin(-45°)&\cos(-45°)&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}&0\\-\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_b=\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}$ $\mathbf{T}_c=\begin{bmatrix}\cos(45°)&-\sin(45°)&0\\\sin(45°)&\cos(45°)&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{2}&0\\\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}&0\\0&0&1\end{bmatrix}$
4(b) (ii)	$\mathbf{T}=\mathbf{T}_c\mathbf{T}_b\mathbf{T}_a=\begin{bmatrix}\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{2}&0\\\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}&0\\-\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{3}{4}&\frac{1}{4}&0\\\frac{1}{4}&\frac{3}{4}&0\\0&0&1\end{bmatrix}$ $\mathbf{U}'=\mathbf{TU}=\begin{bmatrix}\frac{3}{4}&\frac{1}{4}&0\\\frac{1}{4}&\frac{3}{4}&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&4&4&0\\0&0&4&4\\1&1&1&1\end{bmatrix}=\begin{bmatrix}0&3&4&1\\0&1&4&3\\1&1&1&1\end{bmatrix}$ (verified)