Chapter One: Matrices

Learning Objectives:
By the end of the chapter, students should be able to:
1. Define a matrix, its order and elements.
2. Identify some types of special matrices.
3. Perform matrix operations such as transpose, addition, subtraction, scalar multiplication and matrix multiplication.
4. State the definition of inverse matrix.
5. Perform matrix algebra manipulation to rearrange matrix equations.

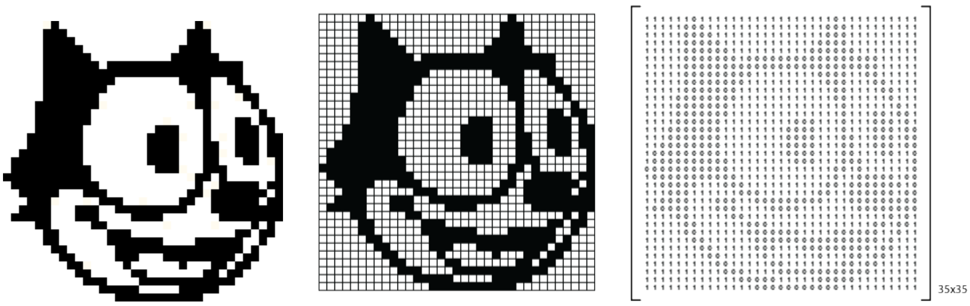
Introduction

In our daily life, information can often be conveniently presented as an array of rows and columns. Such an arrangement of information is called a **matrix** (plural: *matrices*). Applications of matrices are found in most scientific fields. For instance,

- in physics related applications, they are used in the study of the electrical properties of a circuit.
- in computer graphics, they are used to manipulate 3D models and project them onto a two-dimensional screen.
- in cryptography, they are used for encryption of message codes.
- in computer graphics, they can be used to represent transformations, as we shall see later.

Application of Matrices: Digital Images

The images you see on the web, and the photos you take with your mobile phone are examples of digital images. These digital images can be presented using matrices. For example, the small image of Felix the Cat (on the left) can be presented using a 35×35 matrix (grid or *pixels*) whose elements are numbers 0 and 1. A pixel is the smallest graphical element of a matrix image, which can take only one color at a time: the number 0 indicates black, and the number 1 indicates white. This is the simplest type of image because it only uses two colors: black and white. These images are called *binary images* or *Boolean images*.



Source: http://blog.kleinproject.org/?p=588

1.1 Definition of a Matrix

A **matrix** is a rectangular array of quantities, like numbers, enclosed by a pair of large brackets.
An example of a matrix is shown below:

$$\begin{array}{ll} \text{Column 1} & \\ \begin{bmatrix} 20 & 7 \\ 31 & 9 \\ 13 & 10 \end{bmatrix} & \begin{array}{l} \text{Row 1} \\ \\ \text{Element} \end{array} \end{array}$$

The horizontal lists of numbers are called the **rows**, while the vertical lists of numbers are called the **columns**. The above matrix is known as a 3×2 matrix, or a matrix of **order** 3×2.

In general, a matrix with m rows and n columns is called an m by n matrix, written as $m \times n$. This pair of numbers specifies the **order** (also called **size** or **dimension**) of the matrix.

A general matrix $\mathbf{A}$ can be presented in the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

An element in the matrix is identified by two subscripts, which specify the row and column of its location. The element a_{ij} is located in row i and column j of the matrix.

<u>Note:</u>

1. Matrices are denoted by capital letters.
2. The elements are enclosed in large square brackets or large round brackets.

Example 1.1

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{bmatrix} 1 & 3 & 7 \\ -9 & 0 & 4 \end{bmatrix}$.

(a) State the order of $\mathbf{A}$.
(b) State all the rows and columns in matrix $\mathbf{A}$.
(c) Find the elements a_{12}, a_{21} and a_{23}.

(2×3; 3, −9, 4)

1.2 Special Matrices

Matrices with special features or characteristics are given special names, making it convenient and easy for us to refer to them. We will list here some of the common ones.

1. Row Matrix

A matrix with only one row is called a **row matrix** or **row vector**.

For example, $\begin{bmatrix} 2 & -3 & 0 \end{bmatrix}$ is a 1×3 row matrix.

2. Column Matrix

A matrix with only one column is called a **column matrix** or **column vector**.

For example, $\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$ is a 3×1 column matrix.

3. Zero Matrix

A matrix in which all its entries are zero is called a **zero matrix** or a **null matrix**.

It is usually denoted by $\mathbf{0_{mn}}$ or simply by $\mathbf{0}$ if the order is obvious from the context.

For example, $\mathbf{0_{23}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a 2×3 zero matrix.

4. Square Matrix

A matrix with equal number of rows and columns is called a **square matrix**.

For example, $\begin{bmatrix} 3 & 0 & 5 \\ 1 & -2 & 4 \\ 6 & -7 & 2 \end{bmatrix}$ is a 3×3 square matrix.

5. Symmetric Matrix

A square matrix, such that $a_{ij} = a_{ji}$ for all values of i and j, is called a **symmetric matrix**.

For example, the following matrix is a symmetric matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

This is called the **principal diagonal** of the matrix.
For symmetric matrix, it acts as a "mirror"

$a_{12} = a_{21} =$ ____

$a_{13} = a_{\_} =$ ____

$a_{23} = a_{\_} =$ ____

Example 1.2

Find the values of a, b and c such that the matrix $\begin{bmatrix} 1 & 0 & 1 \\ c & b & 2b \\ a & a+c & 2 \end{bmatrix}$ is a symmetric matrix.

(1; 0.5; 0)

6. Diagonal Matrix

A square matrix, whose elements along the principal diagonal that goes from the top left corner to the bottom right corner are not all zero while all the other off-diagonal elements are all zero is called a **diagonal matrix**.

For example, $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a diagonal matrix.

Example 1.3

Find the values of a, b and c such that the matrix $\begin{bmatrix} 1 & 0 & a \\ 0 & 0 & 0 \\ c^2 & 2a+b-1 & 2 \end{bmatrix}$ is a diagonal matrix.

(0; 1; 0)

7. Identity Matrix

A diagonal matrix whose entries along the principal diagonal are all one is called an **identity matrix** or **unit matrix**. It is usually denoted by $\mathbf{I_n}$ or simply by $\mathbf{I}$ if the order is obvious from the context.

For example, $\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3×3 identity matrix.

Example 1.4

Write down $\mathbf{I_2}$ and $\mathbf{I_4}$.

$\mathbf{I_2} =$

$\mathbf{I_4} =$

Note: **I** is a reserved name for the identity matrix (i.e. it cannot be used as a representation for other matrices).

1.3 Matrix Operations

1.3.1 Transpose of a Matrix

The transpose $\mathbf{A}^{\mathbf{T}}$ of a matrix $\mathbf{A}$ is the matrix whose rows are the corresponding columns of $\mathbf{A}$.

$$\text{If } \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{array}{l} \Leftarrow \text{Row 1} \\ \\ \\ \\ \\ \end{array} \text{ is an } m \times n \text{ matrix, then}$$

$$\mathbf{A}^{\mathbf{T}} = \begin{array}{l} \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix} \\ \;\Uparrow \\ \text{Column 1} \end{array} \text{ is an } n \times m \text{ matrix.}$$

Example 1.5

Given that $\mathbf{A} = \begin{bmatrix} 2 & 6 & -3 \\ 7 & -4 & 9 \end{bmatrix}$, find $\mathbf{A}^{\mathbf{T}}$ and $\left(\mathbf{A}^{\mathbf{T}}\right)^{\mathbf{T}}$.

$$\left(\begin{bmatrix} 2 & 7 \\ 6 & -4 \\ -3 & 9 \end{bmatrix}; \begin{bmatrix} 2 & 6 & -3 \\ 7 & -4 & 9 \end{bmatrix} \right)$$

<u>Note:</u>

1. $\left(\mathbf{A}^{\mathbf{T}}\right)^{\mathbf{T}} = \mathbf{A}$
2. If $\mathbf{A}$ is symmetric, then $\mathbf{A}^{\mathbf{T}} = \mathbf{A}$. Give yourself an example!

1.3.2 Equality of Matrices

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be equal, i.e. $\mathbf{A}=\mathbf{B}$, if:

1. Order of $\mathbf{A}$ = order of $\mathbf{B}$, and
2. $a_{ij}=b_{ij}$ for all values of i and j

Example 1.6

Find the values of x, y and z, if:

(a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -x \\ x^2 \end{bmatrix}$ (b) $\begin{bmatrix} x+1 & y^2 \\ \frac{y}{3} & 2z \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ -1 & 4 \end{bmatrix}$

(-3; 3; 9; 0; -3; 2)

1.3.3 Matrix Addition (and Subtraction)

Two matrices $\mathbf{A}$ and $\mathbf{B}$ can be added (or subtracted) if $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same size. The sum $\mathbf{A}+\mathbf{B}$ is the matrix obtained by adding the corresponding elements of matrices $\mathbf{A}$ and $\mathbf{B}$, and the difference $\mathbf{A}-\mathbf{B}$ is the matrix obtained by subtracting the corresponding elements of matrices $\mathbf{A}$ and $\mathbf{B}$.

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

<u>Note:</u>

1. $\mathbf{A}+\mathbf{0}=\mathbf{0}+\mathbf{A}=\mathbf{A}$
2. $\mathbf{A}-\mathbf{A}=\mathbf{0}$

Example 1.7
Perform the following matrix addition:

(a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} =$

(b) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix} =$

(c) $\begin{bmatrix} 5 & 8 & -4 \\ 6 & 9 & -5 \\ 4 & 7 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 5 \\ 4 & -1 & 3 \\ 9 & 6 & 5 \end{bmatrix} =$

(d) $\begin{bmatrix} 5 & 8 & -4 \\ 6 & 9 & -5 \\ 4 & 7 & -3 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 5 \\ 4 & -1 & 3 \\ 9 & 6 & 5 \end{bmatrix} =$

($\begin{bmatrix} 4 \\ 6 \end{bmatrix}$; $\begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix}$; $\begin{bmatrix} 8 & 10 & 1 \\ 10 & 8 & -2 \\ 13 & 13 & 2 \end{bmatrix}$; $\begin{bmatrix} 2 & 6 & -9 \\ 2 & 10 & -8 \\ -5 & 1 & -8 \end{bmatrix}$)

1.3.4 Scalar Multiplication

When a matrix $\mathbf{A}$ is multiplied by a scalar or a number k, each element in the matrix a_{ij} is multiplied by the scalar k.

$$k\mathbf{A} = k\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}$$

Note:

1. If $\mathbf{A}$ is an $m \times n$ matrix, $k\mathbf{A}$ is also an $m \times n$ matrix.
2. $k\mathbf{A} = \mathbf{A}k$
3. $(-1)\mathbf{A} = -\mathbf{A}$
4. $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}$

Example 1.8

Evaluate $2\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}+3\begin{bmatrix}-1 & 2\\ 0 & 1\end{bmatrix}$.

($\begin{bmatrix}-1 & 10\\ 4 & 9\end{bmatrix}$)

Example 1.9

Find the values of x and y such that

$$3\begin{bmatrix}x & 0\\ 2 & 2y\end{bmatrix}-4\begin{bmatrix}3 & -1\\ 1 & y\end{bmatrix}=2\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}$$

(6; 4)

Example 1.10

Find matrix $\mathbf{A}$ such that $\begin{bmatrix}2 & 1 & -3\\ 1 & 5 & 0\end{bmatrix}+\mathbf{A}^{\mathbf{T}}=2\begin{bmatrix}-1 & 4 & 3\\ -2 & 0 & 4\end{bmatrix}$.

($\begin{bmatrix}-4 & -5\\ 7 & -5\\ 9 & 8\end{bmatrix}$)

1.3.5 Matrix Multiplication

(i) Matrix Conformability
Not any two matrices can be used for matrix multiplication. Two matrices can be multiplied only when they are conformable for matrix multiplication.

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be **conformable for multiplication**, i.e. $\mathbf{AB}$ exists, if the number of columns in $\mathbf{A}$ is equal to the number of rows in $\mathbf{B}$, i.e.

$$\mathbf{A}_{m\times n} \times \mathbf{B}_{n\times p} = (\mathbf{AB})_{m\times p}$$

Example 1.11
Are the two matrices $\mathbf{A}$ and $\mathbf{B}$ of the following sizes conformable for matrix multiplication $\mathbf{AB}$?
If yes, what is the order of $\mathbf{AB}$?

(a) $\mathbf{A}_{3\times 5}$, $\mathbf{B}_{5\times 4}$

(b) $\mathbf{A}_{3\times 5}$, $\mathbf{B}_{3\times 4}$

(Yes, 3×4; No)

(ii) Matrix Multiplication
If two matrices are conformable for multiplication, they can be multiplied together.

If $\mathbf{A}$ is a $1\times n$ row matrix, and $\mathbf{B}$ is a $n\times 1$ column matrix, then $\mathbf{AB}$ is a 1×1 matrix (or simply, a scalar). To find the product, multiply each element in $\mathbf{A}$ (from left to right) by the corresponding element in $\mathbf{B}$ (from top to bottom) and add the results.

$$\mathbf{AB} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Example 1.12

Given that $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$, find $\mathbf{AB}$.

(4)

In general, let $\mathbf{A}$ be a $m\times n$ matrix and $\mathbf{B}$ be a $n\times p$ matrix. Then $\mathbf{C = AB}$ is an $m\times p$ matrix**,** such that the element c_{ij} in row i and column j is the sum of products of the corresponding elements of row i (from left to right) of $\mathbf{A}$ and column j (from top to bottom) of $\mathbf{B}$, i.e.

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

where $a_{i1}, a_{i2}, \cdots, a_{in}$ are the elements in row i of $\mathbf{A}$ and $b_{1j}, b_{2j}, \cdots, b_{nj}$ are the elements in column j of $\mathbf{B}$. This is illustrated below:

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots b_{2p} \\ \vdots & \ddots & \vdots & \ddots \ \vdots \\ \vdots & \ddots & \vdots & \ddots \ \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1j} & \cdots c_{1p} \\ c_{21} & \cdots & \vdots & \cdots c_{2p} \\ \vdots & \ddots & \boxed{c_{ij}} & \ddots \ \vdots \\ \vdots & \ddots & \vdots & \ddots \ \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots c_{mp} \end{bmatrix}$$

Row i

Element (i, j)

Column j

Example 1.13

Given $\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 4 \\ 3 & 1 \\ 5 & 0 \end{bmatrix}$,

(a) Find $\mathbf{AB}$ and $\mathbf{BA}$.
(b) In general, is matrix multiplication commutative?

($\begin{bmatrix} 29 & 7 \\ 5 & 8 \end{bmatrix}$; $\begin{bmatrix} 8 & 0 & 4 \\ 5 & 9 & 13 \\ 5 & 15 & 20 \end{bmatrix}$; No)

Example 1.14

Given $\mathbf{A}=\begin{bmatrix}1 & 1\\ 2 & 2\\ 3 & 3\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$,

(a) Find $\mathbf{AB}$.
(b) What conclusion can you make from your result?

($\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$)

Example 1.15

Given $\mathbf{A}=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}$ and $\mathbf{C}=\begin{bmatrix}-2 & 7\\ 5 & -1\end{bmatrix}$,

(a) Find $\mathbf{AB}$ and $\mathbf{AC}$.
(b) What conclusion can you make from your result?

($\begin{bmatrix}8 & 5\\ 16 & 10\end{bmatrix}$)

Example 1.16

Given $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 3 & 2 \end{bmatrix}$,

(a) Find $\mathbf{AI_2}$ and $\mathbf{I_3A}$.

(b) What are your observations about multiplying a matrix with the identity matrix?

(Same)

<u>Note:</u>

Assuming matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{0}$ (zero matrix) and $\mathbf{I}$ (identity matrix) are all conformable matrices and k is any scalar (number), then:

1. $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
2. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
3. $k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = \mathbf{A}(k\mathbf{B})$
4. $\mathbf{A0} = \mathbf{0A} = \mathbf{0}$
5. $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$
6. $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$

1.4 Inverse Matrix

Suppose $\mathbf{A}$ is a square matrix of size $n \times n$ (that means it has *n* rows and *n* columns), and there is another square matrix $\mathbf{B}$ of the same size. If you multiply $\mathbf{AB}$ (or $\mathbf{BA}$) and you realize the result is an identity matrix, $\mathbf{I}$, then in this case, we say that $\mathbf{A}$ is the **inverse matrix** of $\mathbf{B}$, or simply, $\mathbf{A} = \mathbf{B}^{-1}$. Conversely, we can also say that $\mathbf{B}$ is the **inverse matrix** of $\mathbf{A}$, or simply, $\mathbf{B} = \mathbf{A}^{-1}$.

In this case, we can say that both matrices $\mathbf{A}$ and $\mathbf{B}$ are **invertible**. The word "invertible" means "has inverse". Note that not all square matrices are invertible.

> Let $\mathbf{A}$ be a square matrix of order $n \times n$. If there exists a matrix $\mathbf{B}$ such that $\mathbf{AB = BA = I}$ where $\mathbf{I}$ is the $n \times n$ identity matrix, then $\mathbf{A} = \mathbf{B}^{-1}$ and $\mathbf{B} = \mathbf{A}^{-1}$.

Example 1.17

Given matrices $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix}$, find $\mathbf{AB}$. Hence, determine the inverse of $\mathbf{A}$.

Example 1.18

Suppose $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) Given that $\mathbf{AB} = \mathbf{I}_2$, find $\mathbf{B}$.

(b) Hence, find $\mathbf{B}^{-1}$.

$\left(\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}; \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \right)$

Note: Assuming matrices $\mathbf{A}$ and $\mathbf{B}$ are invertible, then:

1. $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$
2. $\left(\mathbf{AB}\right)^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Example 1.19

Given that $\mathbf{ACA}^{-1} = \mathbf{B}$, express $\mathbf{C}$ in terms of $\mathbf{A}$ and $\mathbf{B}$.

($\mathbf{C} = \mathbf{A}^{-1}\mathbf{BA}$)

Example 1.20

In this example, assume that all matrices are invertible.

(a) Show that $\left(\mathbf{AB}\right)^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

(b) Hence, deduce $\left(\mathbf{ABC}\right)^{-1}$.

(c) Given that $\mathbf{AB} = 3\mathbf{I}$, find $\mathbf{A}^{-1}$ in terms of $\mathbf{B}$.

($\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$; $\frac{1}{3}\mathbf{B}$)

Tutorial 1 – Matrices

Section A (Basic)

1. State the order of the following matrices.

(a) $\begin{bmatrix} 2 & -3 & 1 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 2 \\ 5 \\ 6 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 6 \\ -1 & -3 \\ 2 & 0 \end{bmatrix}$

2. Find the elements a_{22}, a_{23} and a_{32} of the matrix $\begin{bmatrix} 1 & 6 & 2 \\ -1 & -3 & 5 \\ 2 & 0 & 7 \end{bmatrix}$.

3. Find the transpose of each of the matrices in Question 1.

4. Find the values of a and b such that the matrix $\begin{bmatrix} 1 & a & b \\ b & 2 & a+b \\ a & 4 & 3 \end{bmatrix}$ is a symmetric matrix.

5. Find the possible value(s) of k such that the matrix $\begin{bmatrix} 1 & 0 & 2k+3 \\ 0 & k & 2 \\ k^2 & 2 & 3 \end{bmatrix}$ is a symmetric matrix.

6. Given that $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$, find the following (whenever possible).

(a) $\mathbf{A} + \mathbf{B}$ (c) $2\mathbf{A} + 3\mathbf{B}$

(b) $\mathbf{A} - \mathbf{B}$ (d) $\mathbf{A} + \mathbf{B}^{\mathrm{T}}$

7. Find the matrix $\mathbf{X}$ such that

(a) $2\begin{bmatrix} 2 & 1 \\ -4 & 5 \end{bmatrix} - 4\mathbf{X} = 3\begin{bmatrix} 8 & -2 \\ 4 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} - 2\mathbf{X}^{\mathrm{T}} = -3\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

8. Find the values of a and b such that

$$3\begin{bmatrix} a & 2 \\ -1 & 2b \end{bmatrix} + 2\begin{bmatrix} a+b & 2 \\ 3 & a-b \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 6 \end{bmatrix}$$

9. If $\mathbf{A}$ is a 2×3 matrix and $\mathbf{B}$ is a 3×2 matrix, which of the following matrix multiplications can be performed? $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{AB^T}$, $\mathbf{BA^T}$, $\mathbf{A^TB^T}$, $\mathbf{B^TA^T}$, $\mathbf{A^2}$, $\mathbf{B^2}$

10. If $\mathbf{A}$ is a 2×3 matrix and the matrix product $\mathbf{AB}$ is a 2×4 matrix, find the order of the matrix $\mathbf{B}$.

11. Evaluate the following matrix products (whenever possible).

(a) $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 6 & 2 \\ -1 & -3 & 5 \\ 2 & 0 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 2 \\ 5 \\ 6 \end{bmatrix}\begin{bmatrix} 2 & -3 & 1 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -3 & 1 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 5 \\ 6 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 3 \\ -2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 4 & -1 & 2 \\ 3 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 5 & 7 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 6 & 3 \\ 2 & 1 \end{bmatrix}$

12. If $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$, find $\mathbf{A}^2$ and $\mathbf{A}^3$.

13. Suppose $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \\ -1 & 7 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$. Determine the following:

(a) $(\mathbf{AB})\mathbf{C}$

(b) $\mathbf{A}(\mathbf{B}+\mathbf{C})$

(c) $\mathbf{B^TA^T}$

(d) $\mathbf{B}^2 + 3\mathbf{I}_2$

14. Find the values of p, q and r such that

$$\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} q & -7 \\ p & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 6 & -3r \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

15. If $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}$, find the following:

(a) $(\mathbf{AB})^\mathbf{T}$ (b) $(\mathbf{BA})^\mathbf{T}$ (c) $\mathbf{A^TB^T}$ (d) $\mathbf{B^TA^T}$

Which of these results are the same?

Section B (Intermediate/Challenging)

16. Find k such that the product $\begin{bmatrix} 1 & 2 & 3 \\ 4 & k & 1 \\ 2k & -4 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1+k & 6 \\ 2 & k & -4 \\ k & 1 & -2 \end{bmatrix}$ is a symmetric matrix.

17. Given the matrix $\mathbf{A} = \begin{bmatrix} k & k \\ -2 & k \\ 1 & 1 \end{bmatrix}$, find the value(s) of k such that $\mathbf{A}^{\mathbf{T}}\mathbf{A}$ is a diagonal matrix.

18. Given the matrices $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -2 \\ 1 & 4 \end{bmatrix}$, show that $\mathbf{AB} \neq \mathbf{BA}$.

    *Hence, explain why $\mathbf{A}^2 - \mathbf{B}^2 \neq (\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})$.

19. Suppose $x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$, where x and y are constants.

    (a) ***Without** evaluating x and y, find matrix $\mathbf{A}$ such that $\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$.

    (b) Find the values of x and y.

20. Suppose the following system of linear equations can be expressed in the matrix form

$$\mathbf{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{b},$$

where $\mathbf{A}$ is a square matrix and $\mathbf{b}$ is a column vector of the right-hand side of the equation.

Find $\mathbf{A}$ and $\mathbf{b}$.

(a) $\begin{aligned} 6x+8y-2z&=1 \\ 4x-2y+3z&=5 \\ 3x+4y+5z&=4 \end{aligned}$

(b) $\begin{aligned} 5y+2x-4z&=2 \\ -2z+3x+4y&=7 \\ 4z+3y+6x&=3 \end{aligned}$

21. Given that $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$, find the matrix product $\mathbf{AB}$.

    Hence, find $\mathbf{A}^{-1}$.

22. Given that $\mathbf{A}=\begin{bmatrix}1&2&-1\\3&5&-1\\-2&-1&-2\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}11&-5&-3\\-8&4&2\\-7&3&1\end{bmatrix}$, find the matrix product $\mathbf{AB}$.

Hence, find $\mathbf{B}^{-1}$.

23. Let $\mathbf{A}$ and $\mathbf{B}$ be invertible matrices and let m and n be non-zero constants. For each of the following matrix equations, find an expression for $\mathbf{A}^{-1}$ in terms of $\mathbf{A}$, $\mathbf{B}$, m and n.
(a) $\mathbf{AB}=m\mathbf{I}$
(b) $\mathbf{A}^2+m\mathbf{A}+n\mathbf{I}=\mathbf{0}$
(c) * $\mathbf{A}+\mathbf{B}=n\mathbf{AB}$

24. (a) *By using the definition of inverse matrix $\mathbf{AA}^{-1}=\mathbf{I}$, show that the inverse of a general 2×2 matrix $\mathbf{A}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ is given by $\mathbf{A}^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$.
(b) Hence, find the inverse of $\begin{bmatrix}2&1\\3&4\end{bmatrix}$.

Section C (MCQ)

25. **(1314S2/A1)** If $\mathbf{A}$ is a 3×3 invertible matrix and $\mathbf{B}$ is a 4×3 matrix, which of the following is **NOT** defined?
(a) $\mathbf{BA}^{\mathbf{T}}\mathbf{A}$ (b) $\mathbf{BAB}^{\mathbf{T}}$ (c) $\mathbf{B}^{\mathbf{T}}\mathbf{BA}^{-1}$ (d) $\mathbf{A}^{-1}\mathbf{AB}$

26. **(1415S1/A1)** Given $\mathbf{A}$ and $\mathbf{B}$ are invertible square matrices and k is a non-zero constant, which of the following matrix properties is always true for all matrices $\mathbf{A}$ and $\mathbf{B}$?
(a) $\mathbf{AB}=\mathbf{BA}$
(b) $(\mathbf{A}+\mathbf{B})^2=\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2$
(c) $(k\,\mathbf{AB})^{-1}=k\,\mathbf{B}^{-1}\mathbf{A}^{-1}$
(d) $(k\,\mathbf{AB})^T=k\,\mathbf{B}^T\mathbf{A}^T$

27. **(1516S2/A1)** If $\mathbf{A}$ and $\mathbf{B}$ are invertible $n\times n$ matrices, then the inverse of $\mathbf{ABA}^{-1}$ is
(a) $\mathbf{ABA}^{-1}$ (b) $\mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{A}$ (c) $\mathbf{AB}^{-1}\mathbf{A}^{-1}$ (d) $\mathbf{B}^{-1}$

Tutorial 1 – Answers

1. (a) 1×4 (b) 4×1 (c) 2×3 (d) 3×2
2. -3; 5; 0
3. (a) $\begin{bmatrix}2\\-3\\1\\7\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2 & 5 & 6\end{bmatrix}$ (c) $\begin{bmatrix}2 & 0\\-1 & 4\\3 & 1\end{bmatrix}$ (d) $\begin{bmatrix}1 & -1 & 2\\6 & -3 & 0\end{bmatrix}$
4. 2; 2
5. -1 or 3
6. (a) $\begin{bmatrix}0 & 3 & 5\\3 & 3 & 2\end{bmatrix}$ (b) $\begin{bmatrix}2 & 1 & 1\\1 & -1 & 4\end{bmatrix}$ (c) $\begin{bmatrix}-1 & 7 & 12\\7 & 8 & 3\end{bmatrix}$

   (d) Not possible $\because$ $\mathbf{A}$ and $\mathbf{B}^{\mathbf{T}}$ are of different sizes
7. (a) $\begin{bmatrix}-5 & 2\\-5 & -2\end{bmatrix}$ (b) $\begin{bmatrix}\frac{5}{2} & \frac{7}{2}\\6 & \frac{7}{2}\end{bmatrix}$
8. 1; 1
9. $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}}$, $\mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$
10. 3×4
11. (a) $\begin{bmatrix}9 & 15 & 20\\-2 & -12 & 27\end{bmatrix}$ (b) $\begin{bmatrix}2 & -3 & 1 & 7\\4 & -6 & 2 & 14\\10 & -15 & 5 & 35\\12 & -18 & 6 & 42\end{bmatrix}$ (c) 43

    (d) $\begin{bmatrix}1 & 2\\0 & 5\\-2 & 5\end{bmatrix}$ (e) Not conformable for multiplication (f) $\begin{bmatrix}6 & 3\\2 & 1\\-8 & -5\end{bmatrix}$
12. $\begin{bmatrix}7 & -10\\-15 & 22\end{bmatrix}$; $\begin{bmatrix}-37 & 54\\81 & -118\end{bmatrix}$
13. (a) $\begin{bmatrix}3 & 10\\1 & 10\\-1 & 12\end{bmatrix}$ (b) $\begin{bmatrix}14 & 4\\8 & 8\\3 & 14\end{bmatrix}$ (c) $\begin{bmatrix}16 & 12 & 10\\-3 & -1 & 1\end{bmatrix}$ (d) $\begin{bmatrix}17 & -4\\8 & 1\end{bmatrix}$
14. 1; -3; -5
15. (a) $\begin{bmatrix}12 & 7\\5 & -3\end{bmatrix}$ (b) $\begin{bmatrix}2 & 6 & 1\\-1 & 4 & 10\\3 & 10 & 3\end{bmatrix}$ (c) $\begin{bmatrix}2 & 6 & 1\\-1 & 4 & 10\\3 & 10 & 3\end{bmatrix}$ (d) $\begin{bmatrix}12 & 7\\5 & -3\end{bmatrix}$

    $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$; $(\mathbf{BA})^{\mathbf{T}} = \mathbf{A}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}}$

16. -4 or 3
17. 1
18. $\mathbf{AB}=\begin{bmatrix}-1 & -10\\ 1 & 0\end{bmatrix}$; $\mathbf{BA}=\begin{bmatrix}-4 & -2\\ 11 & 3\end{bmatrix}$
19. (a) $\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}$ (b) 5; -3
20. (a) $\begin{bmatrix}6 & 8 & -2\\ 4 & -2 & 3\\ 3 & 4 & 5\end{bmatrix}$; $\begin{bmatrix}1\\ 5\\ 4\end{bmatrix}$ (b) $\begin{bmatrix}2 & 5 & -4\\ 3 & 4 & -2\\ 6 & 3 & 4\end{bmatrix}$; $\begin{bmatrix}2\\ 7\\ 3\end{bmatrix}$
21. $\mathbf{AB}=\begin{bmatrix}-2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2\end{bmatrix}$; $\mathbf{A}^{-1}=-\frac{1}{2}\begin{bmatrix}-1 & 1 & -1\\ 1 & -1 & -1\\ -1 & -1 & 1\end{bmatrix}$
22. $\mathbf{AB}=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$; $\mathbf{B}^{-1}=\frac{1}{2}\begin{bmatrix}1 & 2 & -1\\ 3 & 5 & -1\\ -2 & -1 & -2\end{bmatrix}$
23. (a) $\mathbf{A}^{-1}=\frac{1}{m}\mathbf{B}$ (b) $\mathbf{A}^{-1}=-\frac{1}{n}\left(\mathbf{A}+m\mathbf{I}\right)$ (c) $\mathbf{A}^{-1}=n\mathbf{I}-\mathbf{B}^{-1}$
24. (a) (Proving) (b) $\begin{bmatrix}0.8 & -0.2\\ -0.6 & 0.4\end{bmatrix}$
25. d 26. d 27. c