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No.	SOLUTION
A1	Answer: **(c)** Students may assume a Matrix **A** and multiply with its transpose. For example: let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\mathbf{AA}^{\mathrm{T}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$ (symmetric matrix) Note: this is valid for any matrix with more than 1 row.
A2	Answer: **(d)**
A3	Answer: **(a)** Since Alice is the engineer, we can have Alice sitting next to Bob, and that means Claire can only be the artist and must either sit on the left or right. On the contrary, if Claire is the computer scientist, that means Alice and Bob cannot be sitting next to each other, and thus must sit on the left and right, with Claire being in the middle. Combining these two ideas, if one can see that Alice sits in the middle, she must sit next to Bob and Claire. So either: • Claire is the artist and Bob is the computer scientist, contradicting the first clue, or • Claire is the computer scientist and Bob is the artist, contradicting the second clue. Therefore, Alice cannot be sitting in the middle.
A4	Answer: **(c)** There are ${}^9C_5$ ways not to include the couple and ${}^9C_3$ ways to include them. ${}^9C_5 + {}^9C_3 = 210$
A5	Answer: **(b)** Assume n items to pack into a box. $P(no\,defect) = 0.999^n$ $P(\geq 1\,defects) = 1 - 0.999^n \leq 0.1$ $0.999^n \geq 0.9$ $n \leq \dfrac{\log 0.9}{\log 0.999} = 105.31$ $\therefore n_{\max} = 105$

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B1a	$2\mathbf{A}+\begin{bmatrix}4&3&2\\1&5&8\end{bmatrix}=3\begin{bmatrix}2&5&8\\7&3&6\end{bmatrix}$ $2\mathbf{A}=\begin{bmatrix}6&15&24\\21&9&18\end{bmatrix}-\begin{bmatrix}4&3&2\\1&5&8\end{bmatrix}$ $\mathbf{A}=\begin{bmatrix}1&6&11\\10&2&5\end{bmatrix}$
B1b	$\Rightarrow 2k-1=k+2$ $k=3$ $\Rightarrow n=3k$ $=3(3)$ $=9$
B1c	$\begin{bmatrix}2&x\\3&1\end{bmatrix}\begin{bmatrix}2&1\\1&4\end{bmatrix}=\begin{bmatrix}3x+2&7-x\\7&7\end{bmatrix}$ $\begin{bmatrix}4+x&2+4x\\7&7\end{bmatrix}=\begin{bmatrix}3x+2&7-x\\7&7\end{bmatrix}$ $\Rightarrow 4+x=3x+2$ $x=1$
B2a	$\mathbf{T}_1=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_2=\begin{bmatrix}2&0&0\\0&3&0\\0&0&1\end{bmatrix}$ $\mathbf{C}=\mathbf{T}_2\mathbf{T}_1=\begin{bmatrix}2&0&0\\0&3&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&-2&0\\3&0&0\\0&0&1\end{bmatrix}$
B2b	$\mathbf{P}'=\mathbf{CP}=\begin{bmatrix}0&-2&0\\3&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&2\\1&1&3\\1&1&1\end{bmatrix}=\begin{bmatrix}-2&-2&-6\\3&6&6\\1&1&1\end{bmatrix}$
B2c	$\mathbf{T}_1^{-1}=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_2^{-1}=\begin{bmatrix}1/2&0&0\\0&1/3&0\\0&0&1\end{bmatrix}$ $\mathbf{C}^{-1}=\mathbf{T}_1^{-1}\mathbf{T}_2^{-1}=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1/2&0&0\\0&1/3&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&1/3&0\\-1/2&0&0\\0&0&1\end{bmatrix}$

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<table>
<tr><td>B3a</td><td>
Convert to DEC
$8\times16^{2}+14\times16^{1}+15\times16^{0}+10\times16^{-1}+3\times16^{-2}$
$= 2287.636719_{10}$
Convert to BIN
$8EF.A3_{16} = 100011101111.10100011_{2}$
</td></tr>
<tr><td>B3b</td><td>
<table>
<tr><th colspan="3">Integral part:</th></tr>
<tr><td>2</td><td>125</td><td></td></tr>
<tr><td>2</td><td>62</td><td>1</td></tr>
<tr><td>2</td><td>31</td><td>0</td></tr>
<tr><td>2</td><td>15</td><td>1</td></tr>
<tr><td>2</td><td>7</td><td>1</td></tr>
<tr><td>2</td><td>3</td><td>1</td></tr>
<tr><td>2</td><td>1</td><td>1</td></tr>
<tr><td></td><td>0</td><td>1</td></tr>
</table>
<table>
<tr><th colspan="3">Fractional part:</th></tr>
<tr><td>2</td><td>0.3</td><td></td></tr>
<tr><td>2</td><td>0.6</td><td>0</td></tr>
<tr><td>2</td><td>0.2 (rep)</td><td>1</td></tr>
<tr><td>2</td><td>0.4</td><td>0</td></tr>
<tr><td>2</td><td>0.8</td><td>0</td></tr>
<tr><td>2</td><td>0.6</td><td>1</td></tr>
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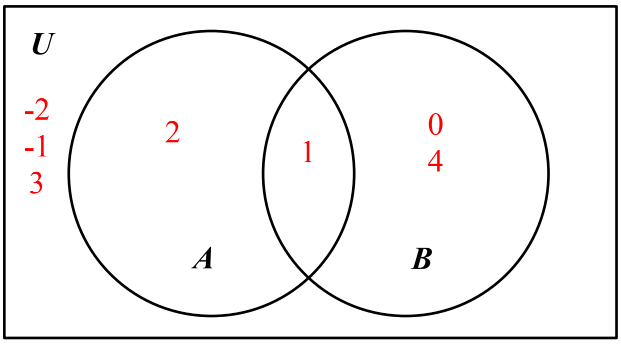
<tr><td>2</td><td>0.8</td><td>0</td></tr>
<tr><td>2</td><td>0.6</td><td>1</td></tr>
</table>
$\therefore 125.3_{10} = 1111101.0\overline{1001}_{2}$
</td></tr>
<tr><td>B4a</td><td>
$U=\{-2,-1,0,1,2,3,4\}$
$A=\{1,2\}$
$B=\{0,1,4\}$
</td></tr>
<tr><td>B4b</td><td>
$A\cup\overline{B}=\{-2,-1,1,2,3\}$
$\overline{A-B}=\{-2,-1,0,1,3,4\}$
</td></tr>
<tr><td>B4c</td><td>


</td></tr>
</table>

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B5a

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

$(p \wedge q) \wedge \neg(p \vee q)$ is a **contradiction**.

B5b

$\overline{(x+\overline{yz})} + (\overline{xy}+z) = \overline{x} \cdot yz + \overline{x} + \overline{y} + z$

$= \overline{x}yz + \overline{x} + \overline{y} + z$

$= \overline{x} + \overline{y} + z$

B6a

$26^6 (\approx 3.09 \times 10^8)$

B6b

${}^{26}P_6 = 26 \times 25 \times 24 \times 23 \times 22 \times 21 = 165765600$

B6c

Once 3 letters and 3 digits are chosen, 2 ways to arrange them - in LDLDLD and DLDLDL manner.

$2 \times (26 \times 25 \times 24) \times (10 \times 9 \times 8)$

$= 22464000$

B7a

Venti wins on the following events:

$$E = \left\{ \begin{array}{l} (1,1),(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2), \\ (3,4),(4,1),(4,3),(5,2),(5,6),(6,1),(6,5) \end{array} \right\}$$

B7b

$P(\text{Venti wins}) = \dfrac{|E|}{|S|} = \dfrac{15}{36}$ (or 41.7%)

B7c (i)

$P(\text{Venti wins all three matches}) = \left(P(\text{Venti wins})\right)^3$

$= \left(\dfrac{15}{36}\right)^3 = \dfrac{3375}{46656} = 0.0723$ (or 7.23%)

B7c (ii)

$P(\text{Venti wins at least once}) = 1 - P(\text{Venti loses all three matches})$

$= 1 - \left(P(\text{Venti loses one match})\right)^3$

$= 1 - \left(1 - \dfrac{15}{36}\right)^3 = 0.802$ (or 80.2%)

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<table>
<tr><td>C1a</td><td>

x	y	$x \downarrow y$
0	0	1
0	1	0
1	0	0
1	1	0

</td></tr>
<tr><td>C1b</td><td>
Sum of products: $\overline{x}\,\overline{y}$
Product-of-sums: $(x+\overline{y})(\overline{x}+y)(\overline{x}+\overline{y})$
</td></tr>
<tr><td>C1c</td><td>
$(x+\overline{y})(\overline{x}+y)(\overline{x}+\overline{y})$
$=(x+\overline{y})(\overline{x}+y\overline{y})$ Distributive Law 2
$=(x+\overline{y})(\overline{x}+0)$ Complement Law 1
$=\overline{x}(x+\overline{y})$ Identity Law 2 + Commutative law 1
$=\overline{x}\,\overline{y}$ Absorption Law 3
</td></tr>
<tr><td>C1d</td><td>
$p \downarrow q = \overline{p} \cdot \overline{q}$ From part (b)
$= \overline{p+q}$ De Morgan's Law 2

$p+q = \overline{\overline{p+q}}$
$= \overline{p \downarrow q}$
$= \overline{(p \downarrow q)+(p \downarrow q)}$
$= (p \downarrow q) \downarrow (p \downarrow q)$
</td></tr>
<tr><td>C2a</td><td>
$LHS = \left\lfloor \dfrac{n^2}{4} \right\rfloor = \left\lfloor \dfrac{(2k+1)^2}{4} \right\rfloor = \left\lfloor k^2+k+\dfrac{1}{4} \right\rfloor = k^2+k$
$RHS = \left(\dfrac{n+1}{2}\right)\left(\dfrac{n-1}{2}\right) = \left(\dfrac{2k+2}{2}\right)\left(\dfrac{2k}{2}\right) = k(k+1) = k^2+k$
$\therefore LHS = RHS$ (shown)
</td></tr>
<tr><td>C2b (i)</td><td>
Label the 20 points as follows:

The horizontal points as $X=\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10},x_{11}\}$
The vertical points as $Y=\{y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8\}$
The intersection point between the x-axis and y-axis as $O=\{o_1\}$

A triangle is formed by either using any two points from $Y \cup O$ and any one point from X, or using any two points from X and any one point from Y.
$\therefore$ Number of triangles $= {}^{9}C_2 \times {}^{11}C_1 + {}^{11}C_2 \times {}^{8}C_1 = 836$
</td></tr>
</table>

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C2b (ii)

A quadrilateral is formed by using any two points from X and any two points from Y.

$\therefore$ Number of quadrilaterals = ${}^{11}C_2 \times {}^{8}C_2 = 1540$

C3a

Based on the information given in the question, and knowing that the two die rolls are independent of each other, we form the following two equations:

$$\frac{2}{m}\left(1-\frac{2}{n}\right)=\frac{1}{8} \text{ ----- (1)}$$

$$\frac{2}{n}\left(1-\frac{2}{m}\right)=\frac{5}{24} \text{ ----- (2)}$$

Rearranging equation (1), we have: $m=16\left(1-\frac{2}{n}\right)$ ----- (3)

Substitute equation (3) into equation (2):

$$\frac{2}{n}\left(1-\frac{2}{16\left(1-\frac{2}{n}\right)}\right)=\frac{5}{24}$$

$$\frac{2}{n}\left(\frac{16n-32}{16n-32}-\frac{2n}{16n-32}\right)=\frac{5}{24}$$

$$\frac{2}{n}\left(\frac{14n-32}{16n-32}\right)=\frac{5}{24}$$

$$5n(16n-32)=48(14n-32)$$

$$80n^2-160n=672n-1536$$

$$5n^2-52n+96=0$$

$$(n-8)(5n-12)=0$$

$n=8$ or $n=\frac{12}{5}$ (rejected since n must be an integer)

Substituting $n=8$ back into equation (3), we obtain $m=16\left(1-\frac{2}{8}\right)=12$.

Hence, $m=12$ and $n=8$.

<u>Additional fun facts:</u>

	
A fair 8-sided die - *octahedron*.	A fair 12-sided die - *dodecahedron*.

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<table>
<tr><td>C3b</td><td>
Let R, Y and G represent the red light, yellow light and green light respectively.
Consider an arbitrary 3-minute window:
$P(R \text{ blinks}) = 1$, $P(Y \text{ blinks}) = \frac{3}{4}$ and $P(G \text{ blinks}) = \frac{3}{5}$
In this 3-minute window, there are four possible cases of Y and G blinking (or not), along with R blinking for sure.
<u>Case 1: R, Y and G blink</u>
$P(\text{Case } 1) = 1 \times \frac{3}{4} \times \frac{3}{5} = \frac{9}{20}$
<u>Case 2: R and Y blink, but G does not blink</u>
$P(\text{Case } 2) = 1 \times \frac{3}{4} \times \left(1 - \frac{3}{5}\right) = \frac{3}{10}$
<u>Case 3: R and G blink, but Y does not blink</u>
$P(\text{Case } 3) = 1 \times \frac{3}{5} \times \left(1 - \frac{3}{4}\right) = \frac{3}{20}$
<u>Case 4: R blinks, but Y and G do not blink</u>
$P(\text{Case } 4) = 1 \times \left(1 - \frac{3}{4}\right) \times \left(1 - \frac{3}{5}\right) = \frac{1}{10}$
Now, we can find the probability that R blinks first by taking the weighted sum of the four cases above.
$P(R \text{ blinks first}) = \sum_{X=1}^{4} \left[P(R \text{ blinks first} \mid \text{Case } X) \times P(\text{Case } X) \right]$
$= \frac{1}{3} \times \frac{9}{20} + \frac{1}{2} \times \frac{3}{10} + \frac{1}{2} \times \frac{3}{20} + 1 \times \frac{1}{10}$
$= \frac{19}{40}$ (or 47.5%)
</td></tr>
</table>