SOLUTIONS

SINGAPORE POLYTECHNIC

2020/2021 Semester 2 Mid-Semester Test

<table>
<tr><th>No.</th><th>SOLUTION</th></tr>
<tr><td>1(a)</td><td>For symmetric matrix, $\mathbf{A}_{ij} = \mathbf{A}_{ji}$.
$c = 2$
$b = -c = -2$
$a = \frac{b}{2} = -1$</td></tr>
<tr><td>1(b)</td><td>$2\mathbf{B} - 3\mathbf{F} = 4\mathbf{I}_2$
$\mathbf{F} = \frac{1}{3}(2\mathbf{B} - 4\mathbf{I}_2) = \frac{1}{3}\left(2\begin{bmatrix}2 & 1\\4 & 2\end{bmatrix} - 4\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix}4 & 2\\8 & 4\end{bmatrix} - \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\right) = \frac{2}{3}\begin{bmatrix}0 & 1\\4 & 0\end{bmatrix}$</td></tr>
<tr><td>1(c)(i)</td><td>$\mathbf{BD} = \begin{bmatrix}2 & 1\\4 & 2\end{bmatrix}\begin{bmatrix}1 & -1 & 4\\2 & -3 & 1\end{bmatrix} = \begin{bmatrix}4 & -5 & 9\\8 & -10 & 18\end{bmatrix}$</td></tr>
<tr><td>1(c)(ii)</td><td>$(\mathbf{B}+\mathbf{E})^T = \left(\begin{bmatrix}2 & 1\\4 & 2\end{bmatrix} + \begin{bmatrix}-1 & 3\\-2 & 2\end{bmatrix}\right)^T = \begin{bmatrix}1 & 4\\2 & 4\end{bmatrix}^T = \begin{bmatrix}1 & 2\\4 & 4\end{bmatrix}$</td></tr>
<tr><td>1(c)(iii)</td><td>Not possible to evaluate, because the number of columns in $\mathbf{D}^T$ (or $\mathbf{AD}^T$) is not equal to the number of rows in $\mathbf{C}$.</td></tr>
<tr><td>1(d)(i)</td><td>$\mathbf{C}^{-1}\mathbf{C} = \mathbf{I}$
$\begin{bmatrix}x & 12 & 5\\2 & y & -1\\3 & -3 & z\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\1 & 4 & 1\\0 & -3 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$
$(\mathbf{C}^{-1}\mathbf{C})_{11} = x + 12 + 0 = 1 \Rightarrow x = -11$
$(\mathbf{C}^{-1}\mathbf{C})_{22} = 6 + 4y + 3 = 1 \Rightarrow y = -2$
$(\mathbf{C}^{-1}\mathbf{C})_{33} = 6 - 3 + 2z = 1 \Rightarrow z = -1$</td></tr>
<tr><td>1(d)(ii)</td><td>Given $\mathbf{G}^T\mathbf{C} = \mathbf{C}^T$,
$(\mathbf{G}^T\mathbf{C})^T = (\mathbf{C}^T)^T$
$\mathbf{C}^T\mathbf{G} = \mathbf{C}$
$\mathbf{C}^T\mathbf{G}\mathbf{G}^{-1} = \mathbf{C}\mathbf{G}^{-1}$
$\mathbf{C}^{-1}\mathbf{C}^T = \mathbf{C}^{-1}\mathbf{C}\mathbf{G}^{-1}$
$\Rightarrow \mathbf{G}^{-1} = \mathbf{C}^{-1}\mathbf{C}^T$
Hence, $\mathbf{G}^{-1} = \begin{bmatrix}-11 & 12 & 5\\2 & -2 & -1\\3 & -3 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\3 & 4 & -3\\2 & 1 & 2\end{bmatrix} = \begin{bmatrix}35 & 42 & -26\\-6 & -7 & 4\\-8 & -10 & 7\end{bmatrix}$</td></tr>
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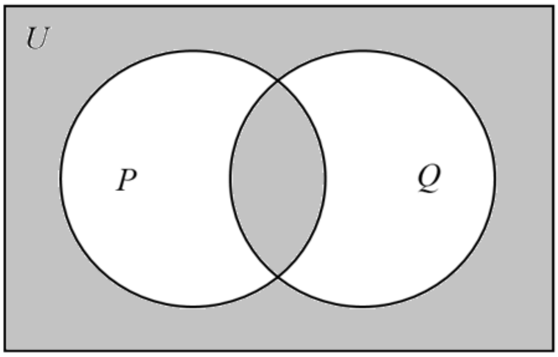
2(a) (i)	$U = \{1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$ $B = \{7,8,9,10\}$ $C = \{6,9\}$
2(a) (ii)	$\lvert B \cap C \rvert = 1$ $B - C = \{7,8,10\}$ $A \cup \overline{B} = \{1,2,3,4,5,6\}$ $\overline{A} \cap B = \{7,8,9,10\}$
2(b) (i)	
2(b) (ii)	$U - (P - Q) - (Q - P)$

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3(a)

Integral part:		
2	2570	
2	1285	0
2	642	1
2	321	0
2	160	1
2	80	0
2	40	0
2	20	0
2	10	0
2	5	0
2	2	1
2	1	0
	0	1

Fractional part:		
2	0.35	
2	0.7	0
2	0.4	1
2	0.8	0
2	0.6	1
2	0.2	1
2	0.4	0
2	0.8	0
2	0.6	1
2	0.2	1
2	0.4 (rep)	0

$$\begin{aligned}\therefore 2570.35_{10} &= 1010\ 0000\ 1010.01\overline{0110}_2 \\ &= \text{A0A.5}\overline{9}_{16}\end{aligned}$$

3(b)

Largest 6-digit hexadecimal number $= \text{FFFF.FF}_{16}$

$$\begin{aligned}\text{FFFF.FF}_{16} &= 15\times\left(16^3+16^2+16^1+16^0+16^{-1}+16^{-2}\right) \\ &= 65535.99609375_{10} \\ &\approx 65535.9961_{10}\end{aligned}$$

3(c)

$$357_k = 3k^2+5k+7 = 1855$$

$$\Rightarrow 3k^2+5k-1848=0$$

$$(k-24)(3k+77)=0$$

$$k=24 \text{ or } k=-\frac{77}{3} \text{ (rejected)}$$

Therefore, $k=24$.

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4(a) (i)	$\mathbf{P}=\begin{bmatrix}2&6&4\\4&4&8\\1&1&1\end{bmatrix}$
4(a) (ii)	$\mathbf{T}_1=\begin{bmatrix}1&0&-2\\0&1&3\\0&0&1\end{bmatrix}$; $\mathbf{T}_2=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_3=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}$
4(a) (iii)	$\mathbf{C}=\mathbf{T}_3\mathbf{T}_2\mathbf{T}_1=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-2\\0&1&3\\0&0&1\end{bmatrix}=\begin{bmatrix}0&1&3\\1&2&4\\0&0&1\end{bmatrix}$
4(a) (iv)	$\mathbf{P}'=\mathbf{CP}=\begin{bmatrix}0&1&3\\1&2&4\\0&0&1\end{bmatrix}\begin{bmatrix}2&6&4\\4&4&8\\1&1&1\end{bmatrix}=\begin{bmatrix}7&7&11\\14&18&24\\1&1&1\end{bmatrix}$
4(a) (v)	$\mathbf{T}_1^{-1}=\begin{bmatrix}1&0&2\\0&1&-3\\0&0&1\end{bmatrix}$; $\mathbf{T}_2^{-1}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_3^{-1}=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}$ $\mathbf{C}^{-1}=\mathbf{T}_1^{-1}\mathbf{T}_2^{-1}\mathbf{T}_3^{-1}=\begin{bmatrix}1&0&2\\0&1&-3\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-2&1&2\\1&0&-3\\0&0&1\end{bmatrix}$

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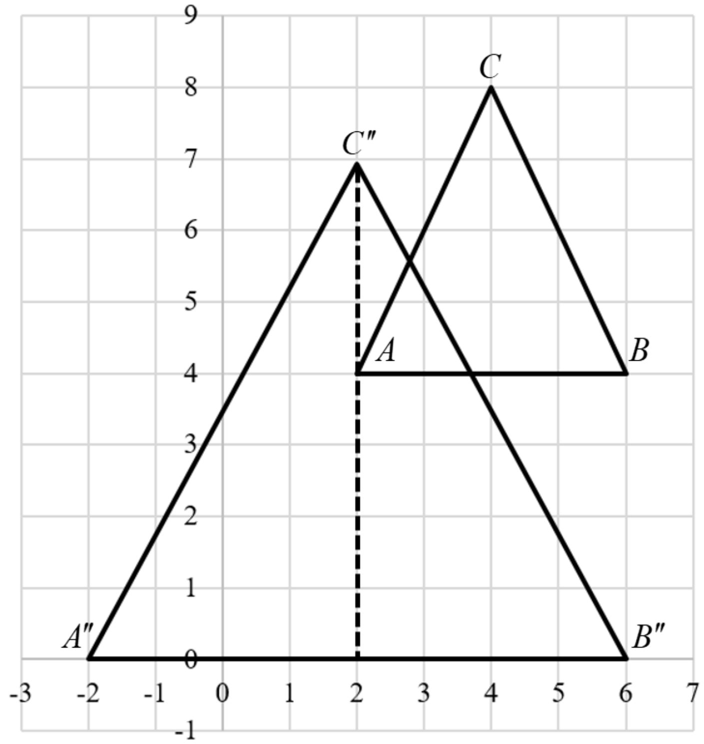
<tr><td>4(b)
(i)</td><td>


$p = \dfrac{6-2}{2} = 2$
Pythagoras' Theorem: $8^2 = 4^2 + q^2$
$q = \sqrt{8^2 - 4^2} = 4\sqrt{3}$ (reject $q = -4\sqrt{3}$ since $q > 0$)
</td></tr>
<tr><td>4(b)
(ii)</td><td>
$\mathbf{T}_a$: Translation 3 units to the left and 4 units downwards
$\mathbf{T}_b$: Scaling relative to the origin by a factor of 2 in the x-direction and a factor of $\sqrt{3}$ in the y-direction
$\mathbf{T}_a = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$; $\mathbf{T}_b = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
</td></tr>
<tr><td>4(b)
(iii)</td><td>
$\mathbf{T} = \mathbf{T}_b\mathbf{T}_a = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -6 \\ 0 & \sqrt{3} & -4\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}$
$\mathbf{P}'' = \mathbf{TP} = \begin{bmatrix} 2 & 0 & -6 \\ 0 & \sqrt{3} & -4\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 6 & 4 \\ 4 & 4 & 8 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 2 \\ 0 & 0 & 4\sqrt{3} \\ 1 & 1 & 1 \end{bmatrix}$ (verified)
</td></tr>
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