Chapter Five: Logic

Learning Objectives:
By the end of the chapter, students should be able to:
1. List the logical values and operators.
2. Express propositions in logical notation form, and vice versa.
3. Construct a truth table to determine the truth values of logical expressions.
4. State the condition for two propositions to be logically equivalent.

Introduction

Logic is an ancient subject going back to the time of Aristotle relating to the study of reasoning. The rules of logic give precise meaning to mathematical statements. Besides its importance in mathematical reasoning, it has numerous applications in computer science. Logic is used in the design of computer circuits, the construction of computer programs and the verification of the correctness of the programs and in many other ways.

5.1 Simple Proposition

A **proposition** is a statement that can either be **true** or **false**, but not both.

Examples of propositions:	Examples of non-propositions:
(a) There are seven days in a week. (b) You are a student of Singapore Polytechnic. (c) $3+4=10$ (d) It rained in SP on 31 October 1999. (e) $x^2=9$ when $x=3$ or $x=-3$	(a) There are 365 days in a year. (b) What is your name? (c) What a great day! (d) $x^2=9$

There is a need to symbolize propositions and we will use lowercase letters such as *p*, *q*, and *r*, to represent them.

p: There are seven days in a week.

q: You are a student of Singapore Polytechnic.

r: $3+4=10$

The primary interest in each proposition is their truth or falsity and therefore each proposition is assigned a **truth value**. The truth value is T if the proposition is true and F if the proposition is false. Hence, the truth values of *p*, *q*, and *r* are _____, _____, and _____ respectively.

Example 5.1

Which of the following statements are propositions? If it is a proposition, identify the truth value of these propositions.

(a) There are 30 days in a month.

(b) $3+4=7$

(c) All prime numbers are odd.

(d) The sum of two prime numbers is even.

(e) $x+y>10$

(f) $x+y>10$ when $x>5$ and $y>2$

5.2 Compound Proposition

We can construct new propositions by the use of the following three fundamental logical operations: negation, conjunction, and disjunction. Any proposition formed by combining one or more propositions with the logical operators is called a **compound proposition**. The truth value of a compound proposition is determined by the truth values of its component propositions and by the logical operators. In other words, if you know the truth values of the pieces and how the pieces are connected then you know the truth value of the proposition.

5.2.1 Negation (NOT)

The negation of a proposition q, denoted as ______, is the proposition "**not** q".
It can also be read as "**it is false that** q".

Note that q and $\neg q$ always bear **opposite** truth values:
if $q = \mathrm{F}$, then $\neg q = \mathrm{T}$; and conversely if $q = \mathrm{T}$, then $\neg q = \mathrm{F}$.

The following **truth table** summarizes the truth value of the proposition $\neg q$:

q	$\neg q$
F	
T	

Example 5.2

Let the proposition q be

q: I am a student of Singapore Polytechnic,

then the truth value of q is __________.

The negation of q is the proposition

$\neg q$: ____ or

$\neg q$: ____

The truth value of $\neg q$ is __________.

Example 5.3

Write down the negation of the following statements:

(a) Today is Friday.

(b) At least two students from this class obtained 'A' grade in Maths in their O-level.

5.2.2 Conjunction (AND)

The conjunction of p with q, denoted by __________, is the proposition " p **and** q ".

The conjunction $p \wedge q$ is true only if both p and q are true.
The truth value of the proposition $p \wedge q$ is defined by the following **truth table**:

p	q	$p \wedge q$
F	F	
F	T	
T	F	
T	T	

Example 5.4
Let the following propositions

p: Today it is raining.

q: Today it is sunny.

The conjunction of p with q is therefore

$p \wedge q$: ____

- If the truth is that it is raining and it is sunny, then p is ____ and q is ____. Or we can conclude that $p \wedge q$ is _________.
- If the truth is that it is raining but it is not sunny, then p is ____ and q is ____. Or we can conclude that $p \wedge q$ is _________.
- If the truth is that it is not raining but it is sunny, then p is ____ and q is ____. Or we can conclude that $p \wedge q$ is _________.
- If the truth is that it is neither raining nor sunny, then p is ____ and q is ____. Or we can conclude that $p \wedge q$ is _________.

Example 5.5

Let the following propositions

r: Mathematics is my favorite subject.

s: Physics is my favorite subject.

Translate the following compound propositions into English:

(a) $r \wedge s$

(b) $r \wedge \neg s$

(c) $\neg r \wedge s$

(d) $\neg r \wedge \neg s$

Look at the truth tables for “AND” and “NOT” that you have obtained previously.
Why do we have 4 rows for “AND” truth table but only 2 rows for “NOT” truth table? What decides the number of rows inside a truth table?

What happens if we have more than 2 variables? Draw a truth table for $p \wedge q \wedge r$.

5.2.3 Disjunction (OR)

"OR" has two meanings in English:

- **Inclusive OR**:

  "If Chelsea wins the League **or** the European Cup, then their manager will keep his job."

  In this case, if Chelsea wins only the League, their manager will keep his job. If Chelsea wins only the European Cup, their manager will also keep his job. And if Chelsea wins both, of course their manager will also keep his job. This is an "or" in an **inclusive** sense, which is denoted by $\vee$ in logic.

- **Exclusive OR:**

  "In this buffet, you may have ice cream **or** pudding as your dessert."

  This is actually an ambiguous statement. Clearly, you may have ice cream (only) for dessert, or pudding (only) for dessert. Can you have choose to have both for dessert? The answer is "don't know" --- the statement never clearly states so. However, in this case, judging by the context of "buffet", most likely you may **choose** to have *either* ice cream *or* pudding as your dessert, but you **cannot have both**. This is an "or" in an **exclusive** sense, which is usually denoted by $\underline{\vee}$ in logic.

  In normal English sentences, to emphasize the "exclusive OR" notion of a statement, the word "either" is often used. This is to remove the ambiguity with the inclusive OR. As such, the following sentence would be clearer to indicate exclusivity (and hence, less ambiguous):

  "In this buffet, you may have **either** ice cream **or** pudding as your dessert."

(i) Inclusive Disjunction (OR)

The (inclusive) disjunction of p with q, denoted by __________, is the proposition “ p **or** q ”.

The truth value of the proposition $p \vee q$ is defined by the following truth table:

p	q	$p \vee q$
F	F	
F	T	
T	F	
T	T	

Example 5.6

Let the following propositions

p: John takes “Java Programming” module this semester.

q: John takes “HTML Programming” module this semester.

The disjunction of p with q is therefore

$p \vee q$: ____

- If the truth is that John takes “Java Programming” and “HTML Programming” this semester, then p is ____ and q is ____. Or we can conclude that $p \vee q$ is _________.
- If the truth is that John takes “Java Programming” but not “HTML Programming” this semester, then p is ____ and q is ____. Or we can conclude that $p \vee q$ is _________.
- If the truth is that John takes “HTML Programming” but not “Java Programming” this semester, then p is ____ and q is ____. Or we can conclude that $p \vee q$ is _________.
- If the truth is that John takes neither “Java Programming” nor “HTML Programming” this semester, then p is ____ and q is ____. Or we can conclude that $p \vee q$ is _________.

(ii) Exclusive Disjunction (XOR)
The exclusive disjunction is similar to "either… or…" in English, which means we can have either one of the two, but not both.

The exclusive disjunction of p with q, denoted by __________, is the proposition " p **xor** q ".

The truth value of the proposition $p \veebar q$ is defined by the following truth table:

p	q	$p \veebar q$
F	F	
F	T	
T	F	
T	T	

Example 5.7
Let the following propositions

p: John takes "Java Programming" module this semester.

q: John takes "HTML Programming" module this semester.

The exclusive disjunction of p with q is therefore

$p \veebar q$: ____

- If the truth is that John takes "Java Programming" and "HTML Programming" this semester, then p is ____ and q is ____. Or we can conclude that $p \veebar q$ is _________.
- If the truth is that John takes "Java Programming" but not "HTML Programming" this semester, then p is ____ and q is ____. Or we can conclude that $p \veebar q$ is _________.
- If the truth is that John takes "HTML Programming" but not "Java Programming" this semester, then p is ____ and q is ____. Or we can conclude that $p \veebar q$ is _________.
- If the truth is that John takes neither "Java Programming" nor "HTML Programming" this semester, then p is ____ and q is ____. Or we can conclude that $p \veebar q$ is _________.

Restate the difference between OR and XOR in your own words!

5.2.4 Implication (IF)

An "if… then…" statement in English is denoted as *implication* in propositional logic.

If p and q are propositions, the compound proposition

$$\text{if } p \text{ then } q$$

is called an **implication** or a **conditional proposition** and is denoted by ________________.

The proposition p is called the **hypothesis** (or **antecedent**), and the proposition q is called the **conclusion** (or **consequent**).

The implication $p \Rightarrow q$ can be read in one of the following ways:

- if p then q
- p implies q
- q if p
- q follows from p

Example 5.8

In each of the following propositions containing implication, identify the antecedent and consequent.

(a) ***If*** it rains, then I am not going to the beach.

Antecedent: ____

Consequent: ____

(b) John will go to school ***if*** his mother asks him to do so.

Antecedent: ____

Consequent: ____

The truth value of the proposition $p \Rightarrow q$ is defined by the following truth table:

p	q	$p \Rightarrow q$
F	F	
F	T	
T	F	
T	T	

Example 5.9
Let the following propositions:

p: I finish work early.

q: I will fetch you home.

Translate $p \Rightarrow q$ into English sentence:

____

- If the truth is that I finish work early and I fetch you home, then p is ____ and q is ____. Or we can conclude that $p \Rightarrow q$ is _________.
- If the truth is that I finish work early but I do not fetch you home, then p is ____ and q is ____. Or we can conclude that $p \Rightarrow q$ is _________.
- If the truth is that I do not finish work early but I fetch you home, then p is ____ and q is ____. Or we can conclude that $p \Rightarrow q$ is _________.
- If the truth is that I do not finish work early and I do not fetch you home, then p is ____ and q is ____. Or we can conclude that $p \Rightarrow q$ is _________.

Take note that the promise is only valid with the condition "If I finish work early". The promise will be broken if **in fact I finish work early and yet I do not fetch you home**. Nothing has been promised if in fact I do not finish work early. I may or may not fetch you home.

Note:

1. Implication only gives a false outcome if the antecedent is T (i.e. what is promised has occurred) and yet the consequent is F (i.e. the consequence of the promise is not fulfilled).

   If the antecedent is F (i.e. what is promised has not occurred), then by default it does not matter if the consequent is T or F (i.e. whether the consequent of the promise is fulfilled or not). The outcome of this implication is **by default** T, as it is out of scope of the original promise. In other words, we cannot say that the original promise has been broken.

2. In everyday usage, the hypothesis and conclusion in an implication are normally related in subject matter. However, logic is concerned primarily with the form of propositions and the relation of propositions to each other and not with the subject matter itself. Hence, it allows a compound proposition such as:

   If $3+4=10$, then there are seven days in a week.

   In this example, since the hypothesis is false, the implication is true. Note that a true implication is not the same as an implication with a true conclusion.

Given an implication $p \Rightarrow q$, other variations of the proposition can be formed by swapping the position of p and q, and also by inserting negations. Three of these with their respective names are tabulated below.

Name	Logical Form
Converse of $p \Rightarrow q$	$q \Rightarrow p$
Inverse of $p \Rightarrow q$	$\neg p \Rightarrow \neg q$
Contrapositive of $p \Rightarrow q$	$\neg q \Rightarrow \neg p$

Example 5.10
Let the following propositions:

p: Brian gets a scholarship.

q: Brian is clever.

Translate the following logical symbols into English:

(a) $p \Rightarrow q$

(b) The converse of $p \Rightarrow q$

(c) The inverse of $p \Rightarrow q$

(d) The contrapositive of $p \Rightarrow q$

Example 5.11

Let p: $3>7$, and q: $2<6$.

Find the truth value of

- the implication $p \Rightarrow q$,

- the converse of $p \Rightarrow q$,

- the inverse of $p \Rightarrow q$, and

- the contrapositive of $p \Rightarrow q$.

It is important to note that the implication and its converse, contrapositive and inverse are four different propositions. Although there are logical relationships between some of them which we shall explore later, the propositions themselves are different propositions.

Supplementary Material: *only if*

The word "only if" is a bit confusing. The implication $p \Rightarrow q$ can be read as either:

- if p then q
- q if p
- p only if q

The following example will illustrate this idea better.

Example 5.12

In each of the following propositions containing implication, identify the antecedent and consequent.

(a) John will go to school ***if*** his mother asks him to do so.

Antecedent: ______________________________

Consequent: ______________________________

(b) John will go to school ***only if*** his mother asks him to do so.

Antecedent: ______________________________

Consequent: ______________________________

Example 5.13

Let the following propositions:

p: I will buy this bag.

q: There is a sale.

Use "if" to translate the implication $q \Rightarrow p$ into English:

________________________________ **if** _____________________.

When will this statement be true? When will it be false?

What is the difference between your answer above and the following statement?

*I will buy this bag **only if** there is a sale.*

Note that this statement inherently means that "If there is **NO** sale, I will **NOT** buy this bag."

Now, when will this statement be true? When will it be false?

In other words, the following statements are all the same:

*I will buy this bag **only if** there is a sale.*
If there is no sale, I will not buy this bag.
I will not buy this bag if there is no sale.
If I buy this bag, then there is a sale.

Hence, "*p* only if *q*" is equivalent to "*q* if *p*" (or "if *p* then *q*").

5.2.5 Equivalence (IFF)

Another useful compound proposition is

p if and only if q.

This can also be abbreviated as " p **iff** q ".

This compound proposition is considered true only when p and q have the same truth values (i.e., p and q are both true or p and q are both false). It is called an **equivalence** or a **bi-conditional proposition** and is denoted by

______________.

It can be read in one of the following ways:

- p is equivalent to q
- p if and only if q

Example 5.14

Translate the following into logical expressions:

(a) p if q

(b) p only if q

(c) p if and only if q

The truth value for equivalence is defined by the following truth table:

p	q	$p \Rightarrow q$	$p \Leftarrow q$	$p \Leftrightarrow q$
F	F			
F	T			
T	F			
T	T			

Note that $p \Leftrightarrow q$ is true when both p and q are **simultaneously** true or false.

Example 5.15
Let p and q be propositions and P and Q be compound propositions made up of propositions p and q. Construct the truth tables for $P = \neg(p \vee q)$ and $Q = \neg p \wedge \neg q$.

p	q	$\neg p$	$\neg q$	$p \vee q$	$P = \neg(p \vee q)$	$Q = \neg p \wedge \neg q$

Note that either P and Q are both true or P and Q are both false. Hence, we say that P and Q are **logically equivalent** and write

$$P \equiv Q$$

or

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

This is the first of the **De Morgan's laws** for Logic, the second is $\neg(p \wedge q) \equiv \neg p \vee \neg q$.
We will revisit these laws in Chapter 6 (Boolean Algebra).

Example 5.16
Construct the truth table for the following:

- the implication $p \Rightarrow q$,
- the converse of $p \Rightarrow q$,
- the inverse of $p \Rightarrow q$, and
- the contrapositive of $p \Rightarrow q$.

Which of these four are logically equivalent?

p	q	$\neg p$	$\neg q$				

The above truth table shows that the implication $p \Rightarrow q$ is logically equivalent to its ____________,
and the converse of $p \Rightarrow q$ is logically equivalent to ____________________.

Example 5.17
A **tautology** is a propositional formula that is **T (true)** for *all* possible truth values of its propositional variables. A **contradiction** is a propositional formula that is **F (false)** for *all* possible truth values of its propositional variables.

(a) Show that the logical form $(p \vee q) \vee (\neg p \vee \neg q)$ is a tautology.

(b) Show that the logical form $(p \wedge q) \wedge (\neg p \wedge \neg q)$ is a contradiction.

<u>Summary: Truth Table for all logical operators</u>

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \Rightarrow q$	$p \Leftrightarrow q$
F	F						
F	T						
T	F						
T	T						

Tutorial 5 – Logic

Section A (Basic)

1. Classify the following statements as propositions or non-propositions. If the statement is a proposition, state its truth value.
   (a) There are 30 days in October.
   (b) What a wonderful day!
   (c) All cows are brown.
   (d) The earth is further from the sun than Venus.
   (e) There are male and female students in Singapore Polytechnic.
   (f) $x+2=2x$ when $x=1$
   (g) If a and b are real numbers, then $(a+b)(a-b)=a^2-b^2$.

2. Propositions p, q and r are defined as

   p : John is good in mathematics

   q : John is good in programming

   r : John is happy

   Write the following compound propositions in words:
   (a) $\neg p$
   (b) $p \vee q$
   (c) $\neg p \wedge q$
   (d) $\neg p \wedge \neg q$
   (e) $p \wedge q \wedge \neg r$
   (f) $(p \wedge q) \Rightarrow r$
   (g) $r \Leftrightarrow (p \wedge q)$

3. Propositions p, q and r are defined as

   p : Sally studies hard for her mathematics exam

   q : Sally obtains 'A' for her mathematics exam

   r : Sally is happy

   Express the following statements in logical notations:
   (a) Despite the fact that Sally studied hard for her mathematics exam, she failed to obtain an 'A'.
   (b) Sally is happy if she obtains 'A' for her mathematics exam.
   (c) Sally is happy if and only if she obtains 'A' for her mathematics exam.
   (d) If Sally studies hard for her mathematics exam, she will obtain 'A'. And if she obtains an 'A', then she will be happy.

4. Find the truth value of each of the following compound propositions, if the truth values for propositions p, q and r are F, T and F respectively.
   (a) $p \vee q \vee r$
   (b) $\neg p \wedge r$
   (c) $\neg p \vee \neg(q \vee r)$
   (d) $\neg(p \vee q) \wedge (\neg p \vee r)$
   (e) $\neg p \Rightarrow (q \Rightarrow r)$
   (f) $\neg(p \vee q) \Leftrightarrow (\neg p \vee r)$

5. Assuming that the truth values of propositions p, q, r and s are F, T, F and T respectively, find the truth value of each of the following compound propositions.
   (a) $p \Rightarrow (p \Rightarrow q)$
   (b) $(p \Rightarrow q) \wedge (q \Rightarrow r)$
   (c) $(s \Rightarrow (p \wedge \neg r)) \wedge ((p \Rightarrow (r \vee q)) \wedge s)$

6. Construct the truth table of each of the following propositions.
   (a) $p \wedge \neg q$
   (b) $\neg p \vee (p \wedge q)$
   (c) $\neg (p \wedge q) \vee (r \wedge p)$

7. Write the converse, inverse and contrapositive of the following statements.
   (a) If it rains, then I am not going to the beach.
   (b) I treat the whole class if I score Distinction for this module.
   (c) If n is even, then n^2 is even.

8. For each pair of propositions P and Q, state whether or not $P \equiv Q$.
   (a) $P = p$, $Q = p \vee q$
   (b) $P = p \Rightarrow q$, $Q = \neg p \vee q$
   (c) $P = p \wedge (\neg q \vee r)$, $Q = p \vee (q \wedge \neg r)$

9. **(1213S1/B4b)** For the pair of propositions $P = (p \Rightarrow q) \Rightarrow r$ and $Q = p \Rightarrow (q \Rightarrow r)$, show by constructing a truth table whether or not $P \equiv Q$.

10. Let p and q be propositions. By constructing a truth table, show that the compound proposition $(p \Rightarrow q) \vee (p \wedge \neg q)$ is a tautology.

11. **(1415S1/B5c)** Suppose p and q are propositions. By constructing a truth table, determine whether the compound proposition $(p \wedge q) \vee (\neg p \vee (p \wedge \neg q))$ is a tautology or a contradiction.

Section B (Intermediate/Challenging)

12. **(1314S1/B5a)** Suppose s and t are propositions such that $s \vee t$ is false. Find the truth value of
   (a) $s \wedge t$
   (b) $\neg s \Rightarrow t$
   (c) the converse of $s \Rightarrow t$

13. Suppose you know that $\neg q \Rightarrow (p \wedge \neg p)$ is true. What is the truth value of q, and why?

14. *Which of the following statement(s) is/are equivalent to the following statement?

If I finish my work early, then I will fetch you.

(Hint: There is no need to prove the equivalence of all of the statements using a truth table. Use common sense first, and only if needed, use a truth table to aid understanding.)

(a) I will fetch you only if I finish my work early.
(b) I will fetch you if and only if I finish my work early.
(c) If I finish my work early, then I will not fetch you.
(d) If I do not finish my work early, then I will fetch you.
(e) If I do not finish my work early, then I will not fetch you.
(f) If I fetch you, then I finish my work early.
(g) If I do not fetch you, then I finish my work early.
(h) If I do not fetch you, then I do not finish my work early.
(i) I finish my work early but I will not fetch you.
(j) I finish my work early only if I fetch you.
(k) I do not finish my work early but I will fetch you.
(l) I do not finish my work early or I will fetch you.
(m) Either I do not finish my work early or I will fetch you.

Section C (MCQ)

15. **(1112S1/A3)** Let p and q be the following propositions: $p: 4 < 2$ and $q: 7 > 5$.
Which of the following statement is true?
(a) The implication $p \Rightarrow q$ is false.
(b) The conjunction of p with q is false.
(c) The disjunction of p with q is false.
(d) The negation of p is false.

16. **(1213S2/A3)** Given two propositions p and q, where $p: 3 \in \mathbb{Z}$ and $q: 3 \in \{x \mid x = 2y,\ y \in \mathbb{Z}\}$,
which of the following compound propositions is TRUE?
(a) $p \Rightarrow q$ (b) $q \Rightarrow p$ (c) $p \Leftrightarrow q$ (d) $\neg p \vee q$

17. **(1314S2/A4)** The compound proposition $p \Rightarrow (q \Rightarrow r)$ is known to be false.
The truth values of p and q are _______ and ________ respectively.
(a) T, T (b) T, F (c) F, T (d) F, F

18. ***(1213S1/A2)** Which of the following statements is logically equivalent to the following proposition?

If I ace this exam, then I will be happy.

(a) If I do not ace this exam, then I will not be happy.
(b) If I am happy, then I aced this exam.
(c) If I am not happy, then I did not ace this exam.
(d) I will not be happy unless I ace this exam.

19. ***(1415S1/A4)** Which of the following propositions is logically equivalent to $p \Leftrightarrow q$?
    (a) $(p \wedge q) \vee \neg(p \wedge q)$
    (b) $(\neg p \wedge q) \vee (p \wedge \neg q)$
    (c) $(p \wedge q) \vee (\neg p \wedge \neg q)$
    (d) $(p \wedge q) \wedge (\neg p \wedge \neg q)$

20. ***(1516S1/A3)** Let p, q and r be propositions and suppose $(p \Rightarrow q) \wedge (q \Rightarrow p)$ is false.
    What is the truth value of $p \wedge q \wedge r$?
    (a) True, if and only if r is true
    (b) True, regardless of the truth value of r
    (c) False, regardless of the truth value of r
    (d) Cannot be determined, because the truth value of r is unknown

Tutorial 5 – Answers

1. (a) proposition, F
   (b) non-proposition
   (c) proposition, F
   (d) proposition, T
   (e) proposition, T
   (f) proposition, F
   (g) proposition, T
2. (a) John is not good in mathematics.
   (b) John is good in mathematics or programming.
   (c) John is not good in mathematics, but good in programming.
   (d) John is not good in mathematics and programming.
   (e) John is good in mathematics and programming, but he is not happy.
   (f) If John is good in mathematics and programming, then he is happy.
   (g) John is happy if and only if he is good in mathematics and programming.
3. (a) $p \wedge \neg q$
   (b) $q \Rightarrow r$
   (c) $q \Leftrightarrow r$
   (d) $(p \Rightarrow q) \wedge (q \Rightarrow r)$
4. (a) T (b) F (c) T (d) F (e) F (f) F
5. (a) T (b) F (c) F
6. (a) FFTF (b) TTFT (c) TTTT TTFT
7. (a) Converse: If I am not going to the beach, then it rains.
   Inverse: If it does not rain, then I am going to the beach.
   Contrapositive: If I am going to the beach, then it does not rain.
   (b) Converse: If I treat the whole class, then I score Distinction for this module.
   Inverse: If I do not score Distinction for this module, then I do not treat the whole class.
   Contrapositive: If I do not treat the whole class, then I do not score Distinction for this module.
   (c) Converse: If n^2 is even, then n is even.
   Inverse: If n is odd, then n^2 is odd.
   Contrapositive: If n^2 is odd, then n is odd.
8. (a) N (b) Y (c) N
9. N
10. (Proving)

11. Tautology
12. (a) F (b) F (c) T
13. Since $p \wedge \neg p$ is a contradiction and $F \Rightarrow F$ is T, therefore q is T.
14. Statements (h), (j) and (l) only.
15. b 16. b 17. a 18. c 19. c 20. c