SOLUTIONS

SINGAPORE POLYTECHNIC
2022/2023 Semester 1 Mid-Semester Test

No.	SOLUTION
1(a)	For symmetric matrix, $\mathbf{A}_{ij} = \mathbf{A}_{ji}$. $\mathbf{A}_{12} = \mathbf{A}_{21}: \ 5a = 10 \Rightarrow a = 2$ $\mathbf{A}_{13} = \mathbf{A}_{31}: \ 2a - 3b = 7$, $4 - 3b = 7 \Rightarrow b = -1$ $\mathbf{A}_{23} = \mathbf{A}_{32} \quad a + b + c = 5$, $2 - 1 + c = 5 \Rightarrow c = 4$
1(b)	$\mathbf{X}^T = \frac{1}{4}(3\mathbf{E} - 5\mathbf{D}) = \frac{1}{4}\left(\begin{bmatrix} -9 & 3 \\ 6 & 15 \end{bmatrix} - \begin{bmatrix} 15 & -25 \\ -10 & 35 \end{bmatrix}\right) = \frac{1}{4}\begin{bmatrix} -24 & 28 \\ 16 & -20 \end{bmatrix} = \begin{bmatrix} -6 & 7 \\ 4 & -5 \end{bmatrix}$ $\therefore \mathbf{X} = \begin{bmatrix} -6 & 4 \\ 7 & -5 \end{bmatrix}$
1(c)(i)	$\mathbf{BD} = \begin{bmatrix} 6 & 1 \\ -1 & 4 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 16 & -23 \\ -11 & 33 \\ 13 & -29 \end{bmatrix}$
1(c)(ii)	$(\mathbf{EB}^T\mathbf{C})^2$ cannot be evaluated because $\mathbf{EB}^T\mathbf{C}$ does not have an equal number of rows and columns (or because $\mathbf{EB}^T\mathbf{C}$ is not a square matrix).
1(d)(i)	$\mathbf{E}^2 - 2\mathbf{E} - 9\mathbf{I} = \begin{bmatrix} -3 & 1 \\ 2 & 5 \end{bmatrix}^2 - 2\begin{bmatrix} -3 & 1 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 11 & 2 \\ 4 & 27 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ -4 & -10 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$
1(d)(ii)	$(\mathbf{E} + 3\mathbf{I})(\mathbf{E} + p\mathbf{I}) + q\mathbf{I} = \mathbf{E}^2 + (p+3)\mathbf{E} + (3p+q)\mathbf{I} = \mathbf{E}^2 - 2\mathbf{E} - 9\mathbf{I}$ Comparing coefficients: Equation (1): $p + 3 = -2$ Equation (2): $3p + q = -9$ Solving equations (1) and (2): $\therefore p = -5$ and $q = 6$
1(d)(iii)	From part (i) and part (ii), we have: $(\mathbf{E} + 3\mathbf{I})(\mathbf{E} - 5\mathbf{I}) + 6\mathbf{I} = 8\mathbf{I}$ $\Rightarrow (\mathbf{E} + 3\mathbf{I})(\mathbf{E} - 5\mathbf{I}) = 2\mathbf{I}$ $\therefore (\mathbf{E} + 3\mathbf{I})^{-1} = \frac{1}{2}(\mathbf{E} - 5\mathbf{I}) = \frac{1}{2}\left(\begin{bmatrix} -3 & 1 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} -8 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

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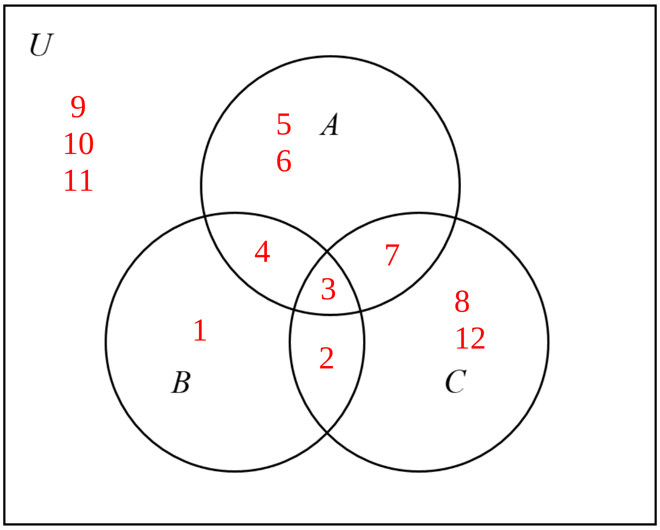
2(a)	$U=\{1,2,3,4,5,6,7,8,9,10,11,12\}$ $A=\{3,4,5,6,7\}$ $B=\{1,2,3,4\}$ $C=\{2,3,7,8,12\}$
2(b)	$\lvert A\rvert=5$ $\overline{A}\cap B=\{1,2\}$ $C-A=\{2,8,12\}$
2(c)	
2(d)	$A\cap(B\cup C)=\{3,4,7\}\Rightarrow$ possible values of $y=3,4,7$ $D\subset U\Rightarrow$ possible values of $x=1,2,3,4,5,6,7,8,9,10,11,12$ For each value of x, check if $\frac{x}{y}\notin\mathbb{Z}$ is satisfied for all values of y. If satisfied, then that value of x belongs to D. Otherwise, it does not belong to D. $\therefore D=\{1,2,5,10,11\}$

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3(a)

Integral part:		
2	1685	
2	842	1
2	421	0
2	210	1
2	105	0
2	52	1
2	26	0
2	13	0
2	6	1
2	3	0
2	1	1
	0	1

Fractional part:		
2	0.9	
2	0.8	1
2	0.6	1
2	0.2	1
2	0.4	0
2	0.8	0
2	0.6	1
2	0.2	1
2	0.4	0
2	0.8	0
2	0.6 (rep)	1

$$\therefore 1685.9_{10} = 110\,1001\,0101.1\overline{1100}_2$$
$$= 695.\text{E}\overline{6}_{16}$$

3(b)

Let $x = 0.1\overline{10010}_2$.

With the hint provided, we note that multiplying x by 2^n will shift the binary point of x to the right by n places.

Form two equations in x that have the same recurring fractional part:

Equation (1): $2x = 1.\overline{10010}_2$

Equation (2): $2^6 x = 110010.\overline{10010}_2$

Subtract equation (1) from equation (2):

$$64x - 2x = 110010.\overline{10010}_2 - 1.\overline{10010}_2$$
$$62x = 110010_2 - 1_2$$
$$62x = 50 - 1 = 49$$
$$\therefore x = \frac{49}{62}$$

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4(a) (i)	$\mathbf{T}_1=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_2=\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_3=\begin{bmatrix}1&0&-1\\0&1&2\\0&0&1\end{bmatrix}$
4(a) (ii)	$\mathbf{C}=\mathbf{T}_3\mathbf{T}_2\mathbf{T}_1=\begin{bmatrix}1&0&-1\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-1&-1&-1\\0&-1&2\\0&0&1\end{bmatrix}$
4(a) (iii)	$\mathbf{P}'=\mathbf{CP}=\begin{bmatrix}-1&-1&-1\\0&-1&2\\0&0&1\end{bmatrix}\begin{bmatrix}1&-3&4\\0&1&2\\1&1&1\end{bmatrix}=\begin{bmatrix}-2&1&-7\\2&1&0\\1&1&1\end{bmatrix}$
4(a) (iv)	$\mathbf{T}_1^{-1}=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_2^{-1}=\begin{bmatrix}1&-1&0\\0&1&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_3^{-1}=\begin{bmatrix}1&0&1\\0&1&-2\\0&0&1\end{bmatrix}$
4(a) (v)	$\mathbf{C}^{-1}=\mathbf{T}_1^{-1}\mathbf{T}_2^{-1}\mathbf{T}_3^{-1}=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&-1&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&-2\\0&0&1\end{bmatrix}=\begin{bmatrix}-1&1&-3\\0&-1&2\\0&0&1\end{bmatrix}$
4(b) (i)	$\mathbf{T}_a$: Scaling relative to origin by a factor of 2 in the *x*-direction and $\frac{1}{2}$ in the *y*-direction $\mathbf{T}_b$: Reflection about the $y=x$ line $\mathbf{T}_c$: Translation 3 units to the left and 2 units downwards $\mathbf{T}_a=\begin{bmatrix}2&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_b=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$; $\mathbf{T}_c=\begin{bmatrix}1&0&-3\\0&1&-2\\0&0&1\end{bmatrix}$
4(b) (ii)	$\mathbf{T}=\mathbf{T}_c\mathbf{T}_b\mathbf{T}_a=\begin{bmatrix}1&0&-3\\0&1&-2\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&\frac{1}{2}&-3\\2&0&-2\\0&0&1\end{bmatrix}$ $\mathbf{U}'=\mathbf{TU}=\begin{bmatrix}0&\frac{1}{2}&-3\\2&0&-2\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&3&2\\2&0&2&4\\1&1&1&1\end{bmatrix}=\begin{bmatrix}-2&-3&-2&-1\\0&2&4&2\\1&1&1&1\end{bmatrix}$ (verified)