SINGAPORE POLYTECHNIC
2022/2023 SEMESTER ONE EXAMINATION

Common Infocomm Technology Programme (CITP)
Diploma in Applied AI & Analytics (DAAA)
Diploma in Infocomm Security Management (DISM)
Diploma in Information Technology (DIT)

MS0105 – Mathematics

Time allowed: 2 hours 10 minutes

Instructions to Candidates

1. The SP examination rules are to be complied with.
Any candidate who cheats or attempts to cheat will face disciplinary action.

2. This paper consists of **8** printed pages (including the cover page and formula sheet).

3. This paper consists of three sections (100 marks in total):

   Section A: 5 multiple-choice questions (10 marks)
   Answer all questions behind the cover page of the answer booklet.

   Section B: 7 structured questions (50 marks)
   The total mark of the questions in this section is 70 marks. You may answer as many questions as you wish. The marks from all questions you answered will be added, but the maximum mark you may obtain from this section is 50 marks.

   Section C: 3 structured questions (40 marks)
   Answer all questions.

4. Unless otherwise stated, all **non-exact** decimal answers should be rounded to **at least two** decimal places.

5. Except for sketches, graphs and diagrams, no solutions are to be written in pencil. Failure to comply may result in loss of marks.

SECTION A (10 marks)

Answer ALL **FIVE** questions. Each question carries 2 marks.
Tick the choice of answer for each question in the box of the MCQ answer sheet provided in the answer booklet.

A1. Let $\mathbf{A}$ and $\mathbf{B}$ be invertible matrices.
Which of the following expressions correctly represents $\left(\mathbf{ABA}^{-1}\right)^n$ for all $n \in \mathbb{N}$?

(a) $\mathbf{AB}^n\mathbf{A}^{-1}$

(b) $\mathbf{A}^{-1}\mathbf{B}^n\mathbf{A}$

(c) $\mathbf{A}^n\mathbf{B}^n\left(\mathbf{A}^{-1}\right)^n$

(d) $\left(\mathbf{A}^{-1}\right)^n\mathbf{B}^n\mathbf{A}^n$

A2. If A, B and C are three finite sets, then $(A-B)\cup(B-C)\cup(C-A)$ is equal to:

(a) $A\cap B\cap C$

(b) $A\cup B\cup C$

(c) $(A\cap B\cap C)-(A\cup B\cup C)$

(d) $(A\cup B\cup C)-(A\cap B\cap C)$

A3. On a certain island, there are only two types of inhabitants – knights and knaves. The knights always tell the truth, while the knaves always lie.

Two of the island's inhabitants, Gary and Gladys, say the following:

- Gary says "*Gladys is a knight and I am a knight.*"
- Gladys says "*Gary is a knave or I am a knave.*"

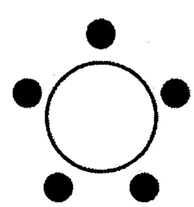
What types of inhabitants are Gary and Gladys?

(a) Gary is a knight and Gladys is a knight.
(b) Gary is a knight and Gladys is a knave.
(c) Gary is a knave and Gladys is a knight.
(d) Gary is a knave and Gladys is a knave.

A4. How many positive integers smaller than 10^{16} have the sum of their digits equal to 2?

(a) 105 (b) 120 (c) 136 (d) 153

A5. Five people, Amy, Bernice, Charles, Daniel and Evelyn, randomly occupy five seats around a circular table, as shown in the figure on the right.



What is the probability that Amy and Bernice are seated beside each other?

(a) $\frac{1}{5}$ (b) $\frac{1}{4}$ (c) $\frac{2}{5}$ (d) $\frac{1}{2}$

SECTION B (50 marks)

Each question carries 10 marks. The total mark of all the questions in this section is 70 marks. You may answer as many questions as you wish. The marks from all questions you answered will be added, but the maximum mark you may obtain from this section is 50 marks.

B1. Let $\mathbf{A}=\begin{bmatrix} 5 & 8 & -2 \\ -1 & 3 & 4 \end{bmatrix}$, $\mathbf{B}=\begin{bmatrix} 2 & -3 \\ 5 & 1 \\ -2 & 6 \end{bmatrix}$, $\mathbf{C}=\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ and $\mathbf{D}=\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$.

(a) Find matrix $\mathbf{X}$ such that $\mathbf{A}-\mathbf{X}=2\mathbf{B}^T$.

(4 marks)

(b) Evaluate $\mathbf{CD}$. Hence, find $\mathbf{C}^{-1}-\mathbf{D}^{-1}$.

(6 marks)

B2. **Solve this question using homogeneous coordinates.**

Rectangle $\mathbf{P}$ with coordinates $(0,0)$, $(3,0)$, $(3,2)$ and $(0,2)$ undergoes the following sequence of transformations:

$\mathbf{T}_1$: translation 2 units to the right and 3 units downwards, followed by

$\mathbf{T}_2$: rotation 90° anticlockwise about the origin.

(a) Write down the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$. Hence, compute the composite matrix $\mathbf{C}$ for the above sequence of transformations.

(4 marks)

(b) Find $\mathbf{P}'$, the image matrix of rectangle $\mathbf{P}$ after undergoing the above sequence of transformations.

(2 marks)

(c) Write down the inverse transformation matrices $\mathbf{T}_1^{-1}$ and $\mathbf{T}_2^{-1}$. Hence, compute the composite matrix $\mathbf{C}^{-1}$ that transforms $\mathbf{P}'$ back to $\mathbf{P}$.

(4 marks)

B3. **Show your working clearly for this question.**

(a) Convert $3E8.4_{16}$ to its binary and decimal representations.

(4 marks)

(b) Convert 8071.55_{10} to its hexadecimal representation.
Express your answer in **exact** form, showing the recursion clearly for the fractional part, if any.

(6 marks)

B4. Let the universal set $U = \{x \in \mathbb{Z} \mid 2 \le x \le 10\}$ and define the following sets within U :

$$A = \{x \mid x \text{ is even}\}$$

$$B = \{5 - 3x \mid x \in \mathbb{N}\}$$

(a) Rewrite sets U, A and B by listing.

(3 marks)

(b) Find $\overline{A \cup B}$ and $|A \cap B|$.

(4 marks)

(c) Draw a Venn diagram showing sets U, A and B, indicating all the elements clearly.

(3 marks)

B5. (a) Two compound propositions A and B are defined as follows:

$A: p \Rightarrow \neg q$

$B: p \Leftrightarrow (p \wedge \neg q)$

By constructing a truth table, determine whether or not $A \equiv B$.

(5 marks)

(b) Simplify the Boolean expression $\overline{\overline{x} + y} + x\left(\overline{x} + \overline{y}\right)$.

(5 marks)

B6. How many ways can the letters of the word `ALGORITHM` be arranged, if

(a) there are no restrictions?

(2 marks)

(b) the word must begin with the string '`ALGO`'?

(2 marks)

(c) the word must **not** contain the string '`ALGO`'?

(3 marks)

(d) the letters `A`, `L`, `G` and `O` can only appear within the first five letters of the word?

(3 marks)

B7. A bag contains 5 white balls and 7 black balls.

Two balls are drawn from the bag at random, one after another **without replacement**.

(a) Find the probabilities of the following events:

(i) X, the event that both balls drawn are white

(ii) Y, the event that at least one ball drawn is black

(iii) Z, the event that both balls drawn are of the same colour

(6 marks)

(b) Determine if Y and Z are **independent**. Show your working clearly.

(4 marks)

<u>SECTION C</u> (40 marks)

Answer ALL **THREE** questions.

1. (a) A digital circuit comprises of an output display controlled by three input switches, x, y and z. The circuit is designed in such a way that the output display only lights up when **exactly two** switches are turned on.

Use the following convention to answer this part:

	'0'	'1'
Input switches	Turned off	Turned on
Output display	Does not light up	Lights up

Construct a truth table for the digital circuit and obtain its **sum-of-products** (SOP) expression. Do not simplify the expression obtained.

(6 marks)

(b) The *Sheffer stroke* is a Boolean operator denoted by '$|$' and is defined as: $a|b = \overline{a \bullet b}$.

Rewrite the Boolean expression $a+b$ using **only** Sheffer strokes.

(5 marks)

2. Triangle OAB has vertices $O(0,0)$, $A(0,3)$ and $B(4,0)$.

(a) Sketch triangle OAB, labelling clearly the x-axis, y-axis and coordinates of OAB's vertices.

(2 marks)

(b) Triangle OAB undergoes the transformation effected by the following matrix, and is subsequently transformed to $O'A'B'$.

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

If $O'A'B'$ is an isosceles triangle, find the possible value(s) of s.

(<u>Note:</u> An isosceles triangle is a triangle that has two equal sides.)

(6 marks)

(c) The same triangle OAB now undergoes the transformation effected by the following matrix instead, and is subsequently transformed to $O''A''B''$.

$$\frac{1}{6}\begin{bmatrix} 3k & 0 \\ 0 & 4k \end{bmatrix}$$

The value of k is determined by rolling two fair six-sided dice and taking the sum of the outcomes of the two die rolls.

What is the probability that the area of $O''A''B''$ is smaller than its perimeter?

(6 marks)

Q7. (a) Ten **distinct** balls are to be distributed across five **identical** boxes.

Assuming that each box must contain the same number of balls, how many ways are there to distribute the balls?

(4 marks)

(b) Ten **identical** balls are to be distributed across five **distinct** boxes.

Assuming that each box must contain at least one ball, how many ways are there to distribute the balls?

(5 marks)

(c) How many solutions does the following equation have?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 50 \text{, such that } x_n \in \{ x \in \mathbb{Z} \mid x \geq n \}$$

(<u>Hint:</u> Use your answer in part (b).)

(6 marks)

***** **END OF PAPER** *****

Formula Sheet

Transformation Matrices

Reflection	about the y-axis	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
	about the x-axis	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
	about the line $y = x$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Scaling	relative to the origin	$\begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Shearing	in the x-direction	$\begin{bmatrix} 1 & s_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
	in the y-direction	$\begin{bmatrix} 1 & 0 & 0 \\ s_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Rotation	about the origin	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Translation		$\begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}$

Boolean Algebra

Name	Identity
Commutative Laws	$x \bullet y = y \bullet x$ $x + y = y + x$
Associative Laws	$x \bullet (y \bullet z) = (x \bullet y) \bullet z = x \bullet y \bullet z$ $x + (y+z) = (x+y) + z = x + y + z$
Distributive Laws	$x \bullet (y+z) = (x \bullet y) + (x \bullet z)$ $x + (y \bullet z) = (x+y) \bullet (x+z)$
Identity Laws	$x \bullet 1 = x$ $x + 0 = x$
Complement Laws	$x \bullet \overline{x} = 0$ $x + \overline{x} = 1$
Involution Law	$\overline{\overline{x}} = x$
Idempotent Laws	$x \bullet x = x$ $x + x = x$
Bound Laws	$x \bullet 0 = 0$ $x + 1 = 1$
De Morgan's Laws	$\overline{x \bullet y} = \overline{x} + \overline{y}$ $\overline{x + y} = \overline{x} \bullet \overline{y}$
Absorption Laws	$x \bullet (x + y) = x$ $x + (x \bullet y) = x$ $x \bullet (\overline{x} + y) = x \bullet y$ $x + (\overline{x} \bullet y) = x + y$

Probability Rules

Addition	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Subtraction	$P(\overline{A}) = 1 - P(A)$
Multiplication	$P(A \cap B) = P(A)P(B)$ if A and B are independent events
Conditional Probability	$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$