SOLUTIONS

SINGAPORE POLYTECHNIC
2022/2023 Semester 2 Examination

No.	SOLUTION
A1	Answer: **(d)** Notice that $\mathbf{A}^{2023}$ performs unit shearing in the y-direction for 2023 times, and this is equivalent to shearing in the y-direction by a factor of 2023. Similarly, $\mathbf{B}^{2023}$ performs unit shearing in the x-direction for 2023 times, and this is equivalent to shearing in the x-direction by a factor of 2023. $\mathbf{A}^{2023}+\mathbf{B}^{2023}=\begin{bmatrix}1 & 0\\ 2023 & 1\end{bmatrix}+\begin{bmatrix}1 & 2023\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 2023\\ 2023 & 2\end{bmatrix}$ Hence, $a+b+c+d=2+2023+2023+2=4050$.
A2	Answer: **(c)** There are three possible cases for any two sets A and B such that $\|A\|=\|B\|$. <u>Case 1:</u> $A=B$ <u>Case 2:</u> $A\neq B$, and they are disjoint <u>Case 3:</u> $A\neq B$, and they are not disjoint The only set relationship that applies for all three cases is the inclusion-exclusion principle, i.e. $\|A\cup B\|=\|A\|+\|B\|-\|A\cap B\|$. Since $\|A\|=\|B\|$, the terms in the inclusion-exclusion principle can be rearranged to become $\|A\cup B\|+\|A\cap B\|=2\|A\|$.
A3	Answer: **(a)** For the proposition "If x is an integer or x is positive, then x is an integer" to be false, we require the antecedent to be true and the consequent to be false. For the consequent to be false, this means that x must **not** be an integer. For the antecedent to be true, knowing that x must not be an integer, this means that x must be positive. Hence, $x=\frac{1}{2}$ is a possible value that makes the above proposition false.
A4	Answer: **(b)** There are $16C2=120$ ways to pair up every player with every other player. This corresponds to the total number of matches played if every player plays exactly **one** match against every other player. Hence, the total number of matches played if every player plays exactly **two** matches against every other player is $120\times 2=240$.

SOLUTIONS

SINGAPORE POLYTECHNIC
2022/2023 Semester 2 Examination

A5	Answer: **(c)** Since the card obtained has at least one green side, it has to be either Card 1, Card 2 or Card 3. Card 1 has two green sides, while Card 2 and Card 3 has one green side each, so there are four green sides in total. Hence, the probability that the card obtained is Card 1 (i.e. the other side is green) is 2/4 = 1/2. Alternative solution: $$\begin{aligned} P(\text{other side green} \mid \text{one side green}) &= \frac{P(\text{other side green} \cap \text{one side green})}{P(\text{one side green})} \\ &= \frac{P(\text{Card 1})}{P(\text{one side green})} \\ &= \frac{1/4}{4/8} \\ &= \frac{1}{2} \end{aligned}$$

SOLUTIONS

SINGAPORE POLYTECHNIC
2022/2023 Semester 2 Examination

B1a (i)	$\mathbf{A}-2\mathbf{B}=\begin{bmatrix}1 & 5\\-6 & 2\end{bmatrix}-\begin{bmatrix}-4 & 2\\-8 & 6\end{bmatrix}=\begin{bmatrix}5 & 3\\2 & -4\end{bmatrix}$
B1a (ii)	$\mathbf{AB}^T=\begin{bmatrix}1 & 5\\-6 & 2\end{bmatrix}\begin{bmatrix}-2 & -4\\1 & 3\end{bmatrix}=\begin{bmatrix}3 & 11\\14 & 30\end{bmatrix}$
B1b	$\mathbf{CD}=\begin{bmatrix}-2 & 0 & -3\\-1 & 1 & -2\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}4 & -3 & 3\\2 & -4 & -1\\-1 & 2 & -2\end{bmatrix}=\begin{bmatrix}-5 & 0 & 0\\0 & -5 & 0\\0 & 0 & -5\end{bmatrix}$ Since $\mathbf{CD}=-5\mathbf{I}$: $\mathbf{CDD}^{-1}=-5\mathbf{ID}^{-1}$ $\Rightarrow \mathbf{C}=-5\mathbf{D}^{-1}$ $\therefore \mathbf{D}^{-1}=-\frac{1}{5}\mathbf{C}=-\frac{1}{5}\begin{bmatrix}-2 & 0 & -3\\-1 & 1 & -2\\0 & 1 & 2\end{bmatrix}$
B2a	$\mathbf{T}_1=\begin{bmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$; $\mathbf{T}_2=\begin{bmatrix}1 & 0 & -3\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}$ $\mathbf{C}=\mathbf{T}_2\mathbf{T}_1=\begin{bmatrix}1 & 0 & -3\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2 & -3\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}$
B2b	$\mathbf{P}'=\mathbf{CP}=\begin{bmatrix}1 & 2 & -3\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-2 & 3\\2 & 1\\1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 2\\4 & 3\\1 & 1\end{bmatrix}$
B2c	$\mathbf{T}_1^{-1}=\begin{bmatrix}1 & -2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$; $\mathbf{T}_2^{-1}=\begin{bmatrix}1 & 0 & 3\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}$ $\mathbf{C}^{-1}=\mathbf{T}_1^{-1}\mathbf{T}_2^{-1}=\begin{bmatrix}1 & -2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 3\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & -2 & 7\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}$
B3a	Convert to HEX: $1011110.01_2=(0101)(1110).(0100)_2=5E.4_{16}$ Convert to DEC: $1011110.01_2=2^6+2^4+2^3+2^2+2^1+2^{-2}=94.25_{10}$

SOLUTIONS

SINGAPORE POLYTECHNIC
2022/2023 Semester 2 Examination

B3b

Integral part:		
2	632	
2	316	0
2	158	0
2	79	0
2	39	1
2	19	1
2	9	1
2	4	1
2	2	0
2	1	0
	0	1

Fractional part:		
2	0.95	
2	0.9	1
2	0.8	1
2	0.6	1
2	0.2	1
2	0.4	0
2	0.8	0
2	0.6	1
2	0.2	1
2	0.4	0
2	0.8	0
2	0.6 (rep)	1

$\therefore 632.95_{10} = 1001111000.11\overline{1100}_2$

B4a

$U = \{1,2,3,4,5,6,7,8,9,10\}$

$A = \{2,3,4,5,6\}$

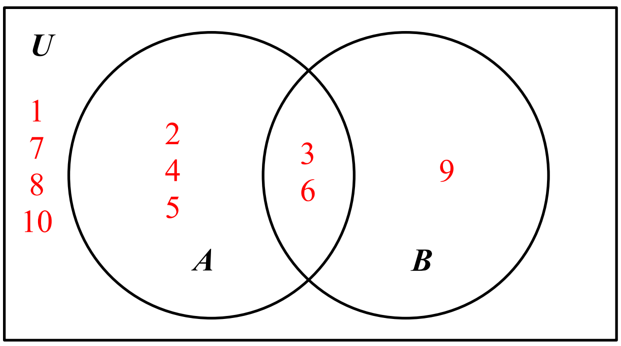
$B = \{3,6,9\}$

B4b

$A \cap \overline{B} = \{2,4,5\}$

$|A \cup B| = 6$

B4c



B5a

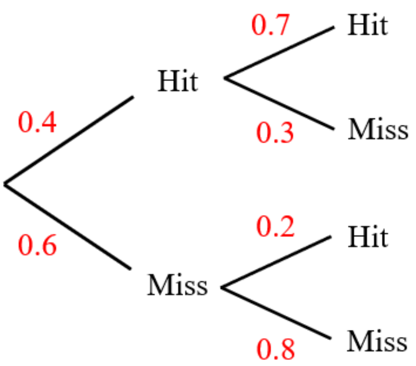
John goes to university if and only if his GPA is above 3.8 and he does not secure a full-time job.

B5b

$(\neg q \vee r) \Rightarrow \neg p$

SOLUTIONS

SINGAPORE POLYTECHNIC
2022/2023 Semester 2 Examination

<table>
<tr><td>B5c</td><td>
<table>
<tr><th>p</th><th>q</th><th>$p \vee q$</th><th>$\neg(p \vee q)$</th><th>$\neg q$</th><th>$p \wedge \neg q$</th><th>$\neg(p \vee q) \wedge (p \wedge \neg q)$</th></tr>
<tr><td>T</td><td>T</td><td>T</td><td>F</td><td>F</td><td>F</td><td>F</td></tr>
<tr><td>T</td><td>F</td><td>T</td><td>F</td><td>T</td><td>T</td><td>F</td></tr>
<tr><td>F</td><td>T</td><td>T</td><td>F</td><td>F</td><td>F</td><td>F</td></tr>
<tr><td>F</td><td>F</td><td>F</td><td>T</td><td>T</td><td>F</td><td>F</td></tr>
</table>
$\therefore \neg(p \vee q) \wedge (p \wedge \neg q)$ is a contradiction.
</td></tr>
<tr><td>B6a</td><td>$^{12}C_5 = 792$</td></tr>
<tr><td>B6b</td><td>$^{11}C_4 = 330$</td></tr>
<tr><td>B6c</td><td>$^{11}C_5 = 462 \quad \left(\text{or } ^{12}C_5 - {}^{11}C_4 = 462\right)$</td></tr>
<tr><td>B6d</td><td>$^{10}C_3 = 120$</td></tr>
<tr><td>B6e</td><td>$^{10}C_4 = 210$</td></tr>
<tr><td>B7a</td><td>


</td></tr>
<tr><td>B7b (i)</td><td>$P(\text{hit exactly once}) = 0.4 \times 0.3 + 0.6 \times 0.2 = 0.24 \quad (\text{or } 24\%)$</td></tr>
<tr><td>B7b (ii)</td><td>$P\left(\text{hit } 1^{\text{st}} \mid \text{hit } 2^{\text{nd}}\right) = \dfrac{P\left(\text{hit } 1^{\text{st}} \cap \text{hit } 2^{\text{nd}}\right)}{P\left(\text{hit } 2^{\text{nd}}\right)} = \dfrac{0.4 \times 0.7}{0.4 \times 0.7 + 0.6 \times 0.2} = 0.7 \quad (\text{or } 70\%)$</td></tr>
</table>

SOLUTIONS

SINGAPORE POLYTECHNIC
2022/2023 Semester 2 Examination

C1a	Since $p \vee \neg p$ is a tautology, it is always true. Since $(p \vee \neg p) \Leftrightarrow (p \wedge q \wedge r)$ is true and $p \vee \neg p$ is always true, $p \wedge q \wedge r$ must be true. Therefore, $p = q = r = \mathrm{T}$.
C1b (i)	see table below
C1b (ii)	$f(x,y,z) = (x+y+z)(x+y+\overline{z})(x+\overline{y}+z)$ $= (x+y+z\overline{z})(x+\overline{y}+z)$ $= (x+y)(x+\overline{y}+z)$ $= x + y(\overline{y}+z)$ $= x + yz$
C2a	The word QUEENSTOWN has 1 'Q', 1 'U', 2 'E', 2 'N', 1 'S', 1 'T', 1 'O', 1 'W'. Case 1 (all 4 letters distinct): ${}^{8}P_{4} = 1{,}680$ Case 2 (2 'E' + 2 distinct letters): $\dfrac{{}^{7}C_{2} \times 4!}{2!} = 252$ Case 3 (2 'N' + 2 distinct letters): $\dfrac{{}^{7}C_{2} \times 4!}{2!} = 252$ Case 4 (2 'E' + 2 'N'): $\dfrac{4!}{2!2!} = 6$ $\therefore$ Total number of ways $= 1{,}680 + 252 + 252 + 6 = 2{,}190$

C1b (i):

x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

SOLUTIONS

SINGAPORE POLYTECHNIC
2022/2023 Semester 2 Examination

<table>
<tr><td>C2b
(i)</td><td>Let b and w represent the number of black marbles and white marbles respectively in the urn.

No. of ways to get two marbles of same colour $= {}^{b}C_2 + {}^{w}C_2 = \dfrac{b(b-1)}{2} + \dfrac{w(w-1)}{2}$

No. of ways to get two marbles of different colours $= {}^{b}C_1 \times {}^{w}C_1 = bw$

$\dfrac{b(b-1)}{2} + \dfrac{w(w-1)}{2} = bw$
$b(b-1) + w(w-1) = 2bw$
$b^2 - b + w^2 - w = 2bw$
$b^2 - 2bw + w^2 = b + w$
$(b-w)^2 = b + w = N$
$b - w = \sqrt{N}$ (reject $-\sqrt{N}$ since $b > w$)
Solving the two simultaneous equations for b and w:
$b + w = N$ ----- (1)
$b - w = \sqrt{N}$ ----- (2)
Therefore, $b = \dfrac{N+\sqrt{N}}{2}$ and $w = \dfrac{N-\sqrt{N}}{2}$.</td></tr>
<tr><td>C2b
(ii)</td><td>$\left\{x^2 \,\middle|\, x \in \mathbb{Z}, x \geq 2\right\}$ or $\{4, 9, 16, 25, 36, \cdots\}$</td></tr>
</table>

SOLUTIONS

SINGAPORE POLYTECHNIC
2022/2023 Semester 2 Examination

C3a	Transformation matrix $\mathbf{T}_a$ can be decomposed into a sequence of three transformations $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$: $\mathbf{T}_1$: rotation by an angle of α clockwise about the origin, followed by $\mathbf{T}_2$: reflection about the x-axis, followed by $\mathbf{T}_3$: rotation by an angle of α anticlockwise about the origin. Using the two properties $\sin(-\alpha) = -\sin\alpha$ and $\cos(-\alpha) = \cos\alpha$: $\mathbf{T}_1 = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\therefore \mathbf{T}_a = \mathbf{T}_3\mathbf{T}_2\mathbf{T}_1$ $= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ \sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 2\sin\alpha\cos\alpha & 0 \\ 2\sin\alpha\cos\alpha & \sin^2\alpha - \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $= \begin{bmatrix} \cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & -\cos 2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (shown)

SOLUTIONS

SINGAPORE POLYTECHNIC
2022/2023 Semester 2 Examination

C3b	Transformation matrix $\mathbf{T}_b$ can be decomposed into a sequence of three transformations $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$: $\mathbf{T}_1$: translation by c units downwards $\mathbf{T}_2$: reflection about the line $y = mx$ $\mathbf{T}_3$: translation by c units upwards Using the answer in part (a), a line inclined at an angle of α with respect to the positive x-axis can be expressed in terms of its gradient m. If we let $x = 1$, then $y = m$, and this forms a right-angled triangle with the following properties: $\cos\alpha = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1+m^2}}$ and $\sin\alpha = \frac{\text{opp}}{\text{hyp}} = \frac{m}{\sqrt{1+m^2}}$ With the above properties, we can express $\mathbf{T}_2$ in terms of m only: $\mathbf{T}_2 = \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 2\sin\alpha\cos\alpha & 0 \\ 2\sin\alpha\cos\alpha & \sin^2\alpha - \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} & 0 \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\therefore \mathbf{T}_b = \mathbf{T}_3\mathbf{T}_2\mathbf{T}_1$ $= \frac{1}{1+m^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1-m^2 & 2m & 0 \\ 2m & m^2-1 & 0 \\ 0 & 0 & 1+m^2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$ $= \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m & 0 \\ 2m & m^2-1 & c+m^2c \\ 0 & 0 & 1+m^2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$ $= \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m & -2mc \\ 2m & m^2-1 & 2c \\ 0 & 0 & 1+m^2 \end{bmatrix}$ (shown)