Chapter Two: Linear Transformation

Learning Objectives:
By the end of the chapter, students should be able to:
1. Describe the basic linear transformations such as reflection, scaling, shearing, rotation and translation.
2. Use matrices to represent objects and transformations in homogeneous coordinates.
3. Express a sequence of transformations as a composite transformation matrix.
4. Find the inverse of simple and composite transformation matrices.

Introduction

Today, literally everyone, from the very young to the elderly, is enthralled by all kinds of graphical displays on the computer screen. This is brought about not only by the advent of the digital computers but by the development of the field of computer graphics which is concerned with the creation and manipulation of graphical objects. These graphical objects include those that are *natural* as well as those that are *artificial*, that is, generated algorithmically. Natural objects are those that are digitized by a digital camera for example photographs, images from medical tomography, video or scanner. Artificial objects are those that are created by the users. Drawing, drafting, painting or images generated by some programs are just some examples. Computer graphics encompasses the whole field of image processing.

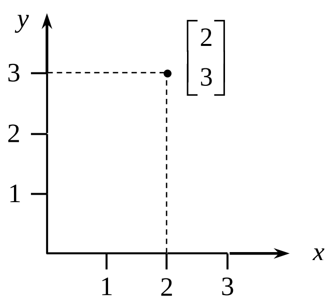
This chapter deals with some aspects of two-dimensional computer graphics. It introduces the mathematical representations of simple graphical objects like points and lines. The use of matrix transformation to manipulate these objects will be discussed.

2.1 Definition of a Point

A **column vector** can be used as a mathematical representation of a point in a two-dimensional (xy) plane. The entries in the column vector are the coordinates of the point, where the first row is the *x*-coordinate, and the second row is the *y*-coordinate.

For example, $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ may be viewed as a point as shown in the following figure.

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{matrix} \rightarrow \quad \boldsymbol{x}\text{-coordinate} \\ \rightarrow \quad \boldsymbol{y}\text{-coordinate} \end{matrix}$$



2.2 Definition of a Graphical Object

A graphical object consists of a few points, and can be represented by a matrix where each column contains the coordinates of each corners of the object. For example, a graphical object like a "house" shown below can be represented by a matrix $\mathbf{P}$ where each of the columns contains the coordinates of each of the 5 corners a to e. Thus,

$$\mathbf{P} = \begin{bmatrix} -1 & -3 & -3 & -2 & -1 \\ 1 & 1 & 3 & 4 & 3 \end{bmatrix}$$

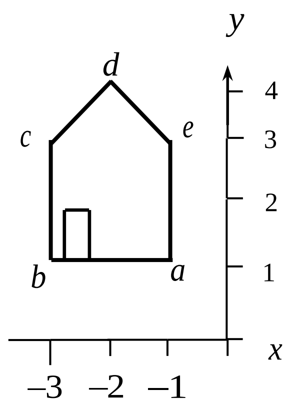


Figure 2.1 A house

2.3 Linear Transformation

In this section, we will discuss basic linear transformations. In general, each point $\mathbf{P} = \begin{bmatrix} x \\ y \end{bmatrix}$ on the plane is transformed to a new location $\mathbf{P}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ by multiplying with a matrix $\mathbf{T}$ such that

$$\mathbf{P'} = \mathbf{TP}$$

$\mathbf{T}$ is called a **transformation matrix** and since it transforms a point on the two-dimensional plane to another point on the plane, $\mathbf{T}$ must be a ___×___ matrix. Depending on the entries of the transformation matrix, different transformations are effected. We shall explore the effects of a number of transformation matrices.

2.3.1 Reflection

Let us first look at the geometrical effects of the following transformation matrices:

$$\mathbf{T}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{T}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathbf{T}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(i) Reflection about *y*-axis

In the first case, $\mathbf{T}_1$ changes the sign of the x-coordinate of every point and leaves the y-coordinate unchanged. So, for example, the point $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is transformed into $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, and vice versa.

In effect, the transformation matrix $\mathbf{T}_1$ gives a *reflection* that takes every point to its image on the opposite side of the y-axis, which is acting as a mirror.

$$\mathbf{P}' = \mathbf{T}_1\mathbf{P}$$

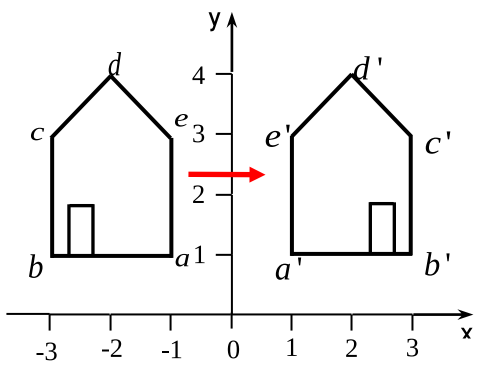
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}$$

To appreciate the effect of $\mathbf{T}_1$, let us apply it to the house in Figure 2.1 which is represented by the matrix $\mathbf{P} = \begin{bmatrix} -1 & -3 & -3 & -2 & -1 \\ 1 & 1 & 3 & 4 & 3 \end{bmatrix}$. Then the image is

$$\mathbf{P}' = \mathbf{T}_1\mathbf{P}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -3 & -3 & -2 & -1 \\ 1 & 1 & 3 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 1 & 1 & 3 & 4 & 3 \end{bmatrix}$$



(ii) Reflection about *x*-axis

In the second case, $\mathbf{T}_2$ changes the sign of the y-coordinate of every point and leaves the x-coordinate unchanged. So, for example, the point $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is transformed into $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, and vice versa.

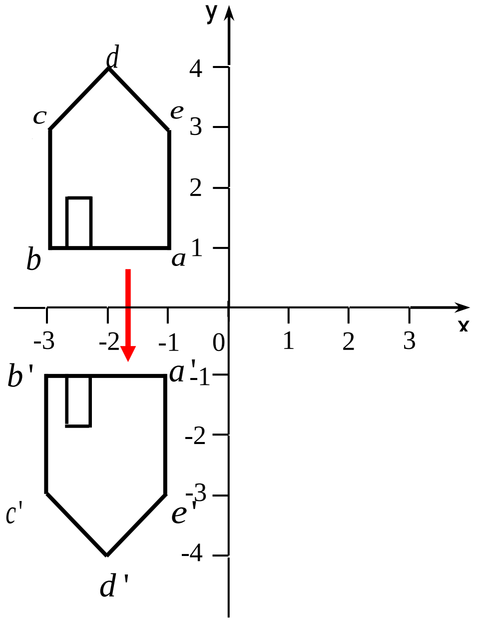
In effect, the transformation matrix $\mathbf{T}_2$ gives a *reflection* that takes every point to its image on the opposite side of the x-axis, which is acting as a mirror.

$$\mathbf{P}' = \mathbf{T}_2\mathbf{P}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}$$

The image of the house after the transformation is

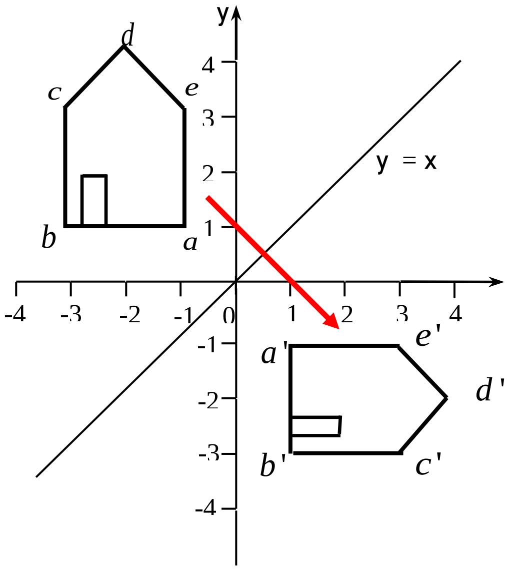
$$\mathbf{P}' = \mathbf{T}_2\mathbf{P}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -3 & -3 & -2 & -1 \\ 1 & 1 & 3 & 4 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -3 & -3 & -2 & -1 \\ -1 & -1 & -3 & -4 & -3 \end{bmatrix}$$



(iii) Reflection about the line $y = x$

In the third case, $\mathbf{T}_3$ exchanges the x and y coordinates of every point. So, for example, the point $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is transformed into $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and vice versa. In effect, the transformation matrix $\mathbf{T}_3$ gives a *reflection* that takes every point to its image on the opposite side of the line $y = x$, which is acting as a mirror.

$$\mathbf{P}' = \mathbf{T}_3\mathbf{P}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}$$



The image of the house after the transformation is

$$\mathbf{P}' = \mathbf{T}_3\mathbf{P}$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -3 & -3 & -2 & -1 \\ 1 & 1 & 3 & 4 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 3 & 4 & 3 \\ -1 & -3 & -3 & -2 & -1 \end{bmatrix}$$

So, the three transformation matrices $\mathbf{T}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{T}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{T}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ each takes every point to its image on the opposite side of the y-axis, x-axis and the line $y = x$ respectively.

2.3.2 Scaling (relative to the origin)

(i) Scaling in the *x*-direction

Let us next look at the geometrical effects of the following two transformation matrices which are diagonal matrices with positive diagonal elements:

$$\mathbf{T}_4 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{T}_5 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

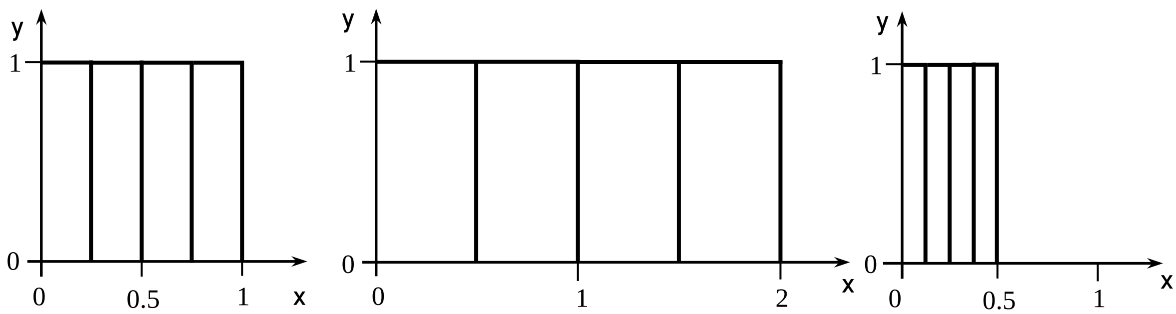
The transformation $\mathbf{T}_4$ doubles the x-coordinate of every point and leaves the y-coordinate unchanged. Every point along the y-axis such as $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is left unchanged, while every other point such as $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is moved in the x-direction:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

So, effectively, $\mathbf{T}_4$ produces an *expansion* by a factor of 2 in the x-direction as can be seen in Figure 2.2(b).

$$\mathbf{P}' = \mathbf{T}_4\mathbf{P}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}$$



(a) Initial (b) Transform by T_4 (c) Transform by T_5

Figure 2.2 Scaling in the x direction

Similarly, $\mathbf{T}_5 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ halves the x-coordinate of every point and leaves the y-coordinate unchanged. So effectively, $\mathbf{T}_5$ produces a *contraction* by a factor of $\frac{1}{2}$ in the x-direction as can be seen in Figure 2.2(c).

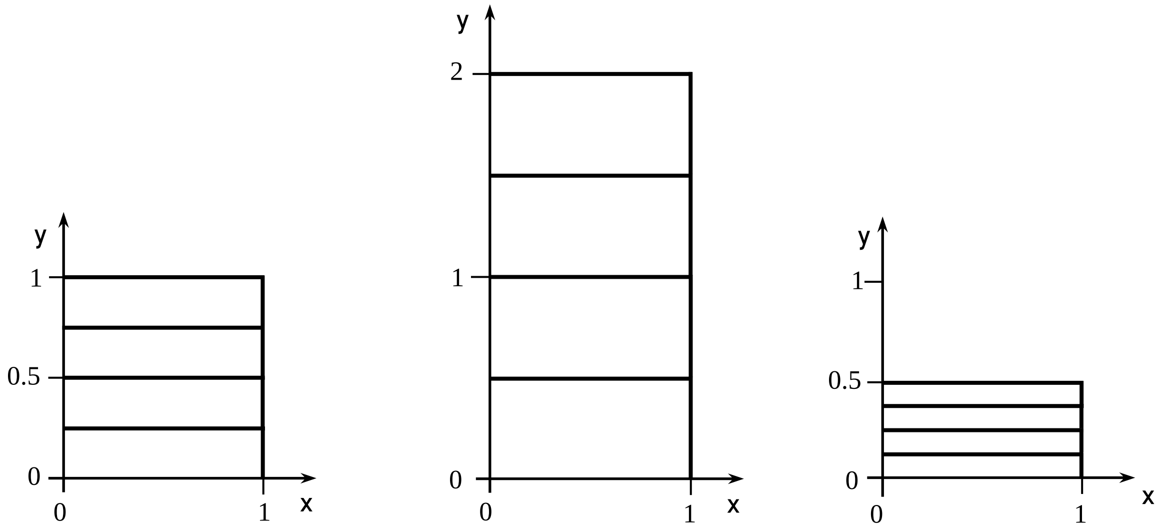
$$\mathbf{P}' = \mathbf{T}_5\mathbf{P}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}$$

(ii) Scaling in the *y*-direction
With a similar argument, the following transformation matrices

$$\mathbf{T_6} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{T_7} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

can be shown to produce an expansion by a factor of 2 in the y-direction and a contraction by a factor of $\frac{1}{2}$ in the y-direction respectively as shown in Figure 2.3.



(a) Initial (b) Transform by T_6 (c) Transform by T_7

Figure 2.3 Scaling in the y direction

In general, the transformation matrix $\begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}$ produces a scaling relative to the origin in both the x and y directions. The scaling factors in the x and y directions are k_x and k_y respectively. A factor with value greater than 1 will result in expansion. On the other hand, a factor with a positive value less than 1 will result in contraction.

$$\mathbf{P}' = \mathbf{TP}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \underline{\qquad} \\ \underline{\qquad} \end{bmatrix}$$

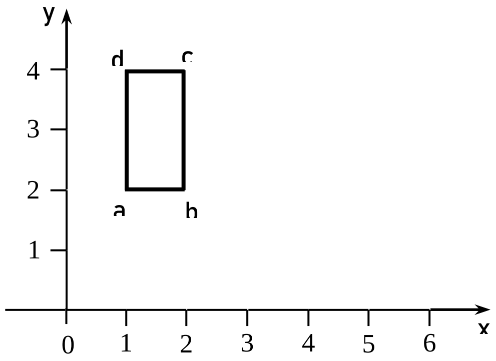
Example 2.1

(a) Find the transformation matrix $\mathbf{T}$ that will produce an expansion in the x-direction by a factor of 3 and a contraction in the y-direction by a factor of $\frac{1}{2}$.

(b) Find the image of the rectangle shown below after transformation $\mathbf{T}$, and superimpose it on the same diagram.

(c) Besides the scaling effects, what can you observe about the position of the image? Explain your observation.

$$\left(\begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}; \begin{bmatrix} 3 & 6 & 6 & 3 \\ 1 & 1 & 2 & 2 \end{bmatrix}\right)$$



2.3.3 Shearing

(i) Shearing in the *x*-direction
Now, let us look at the geometrical effects of the following two transformation matrices:

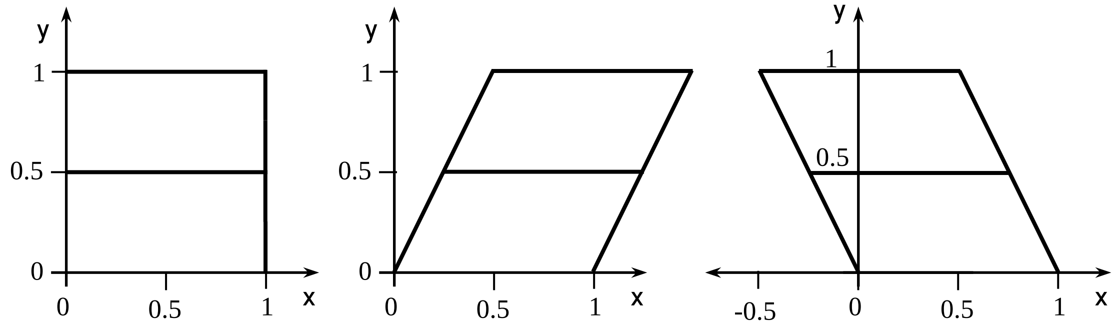
$$\mathbf{T_8} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \quad \mathbf{T_9} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

The transformation effected by $\mathbf{T_8} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ is a little harder to describe. It adds half of the y-coordinate of every point to the x-coordinate and leaves the y-coordinate unchanged.
Every point along the x-axis such as $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is left unchanged, while the point $\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$ is moved to $\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}$ and the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is moved further to $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

This transformation is known as **shearing** of the plane in the x-direction. It moves each point parallel to the x-axis by an amount proportional to its distance from the x-axis as shown in Figure 2.4(b). The amount of shear is defined by the constant of proportionality known as the **shear factor**.

For the transformation matrix $\mathbf{T_8}$, the shear factor is $\frac{1}{2}$. The transformation matrix $\mathbf{T_9}$ is similar to $\mathbf{T_8}$ except that the shear factor is $-\frac{1}{2}$ instead of $\frac{1}{2}$. The shearing is still in the x-direction but in the opposite direction as shown in Figure 2.4(c).



(a) Initial (b) Transform by T_8 (c) Transform by T_9

Figure 2.4 Shearing in the x direction

(ii) Shearing in the *y*-direction

Similarly, shearing in the y -direction can be effected by the following transformation matrices:

$$\mathbf{T}_{10} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}, \quad \mathbf{T}_{11} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Figure 2.5 shows the effect of shearing with these matrices.

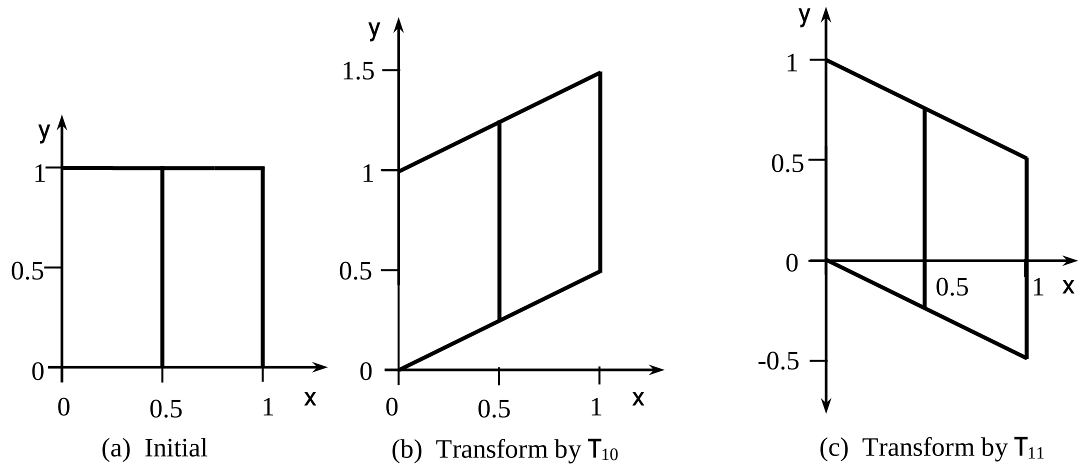


Figure 2.5 Shearing in the y direction

Therefore, in general, the transformation matrices $\begin{bmatrix} 1 & s_x \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ s_y & 1 \end{bmatrix}$ produce shearing in the x and y direction respectively.

$$\mathbf{P}' = \mathbf{TP}$$

- Shearing in the x direction: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & s_x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_ \end{bmatrix}$

$$\mathbf{P}' = \mathbf{TP}$$

- Shearing in the y direction: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_ \end{bmatrix}$

If the original image is in Quadrant 1, the effect of the shearing factors s_x and s_y will be as follows:

- If s_x is positive, the shearing is to the right.
  If s_x is negative, the shearing is to the left.
- If s_y is positive, the shearing is upward.
  If s_y is negative, the shearing is downward.

Example 2.2

A rectangle with coordinates $P(0,0)$, $Q(3,0)$, $R(3,2)$ and $S(0,2)$ is sheared to the right with a factor of 1. Find the image of the rectangle after the transformation.

$$\left(\begin{bmatrix} 0 & 3 & 5 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}\right)$$

2.3.4 Rotation (about the origin)

One very important transformation is the rotation about the origin. Let us find a 2×2 matrix, $\mathbf{R}=\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$ that will perform rotation by θ in the anti-clockwise direction.

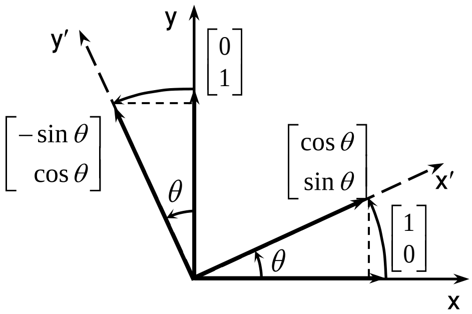


Figure 2.6 Rotation about Origin

Referring to Figure 2.6, we can observe that the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is rotated to $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ after the rotation, and the point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is rotated to $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$. Therefore,

$$\mathbf{P'} = \mathbf{RP}$$

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{21} \end{bmatrix}$$

$$\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_{12} \\ r_{22} \end{bmatrix}$$

Therefore, the matrix that will perform *rotation* by θ in the anti-clockwise direction is:

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Note that:

- θ is positive if the rotation is in the anti-clockwise direction, and
- θ is negative if the rotation is in the clockwise direction.

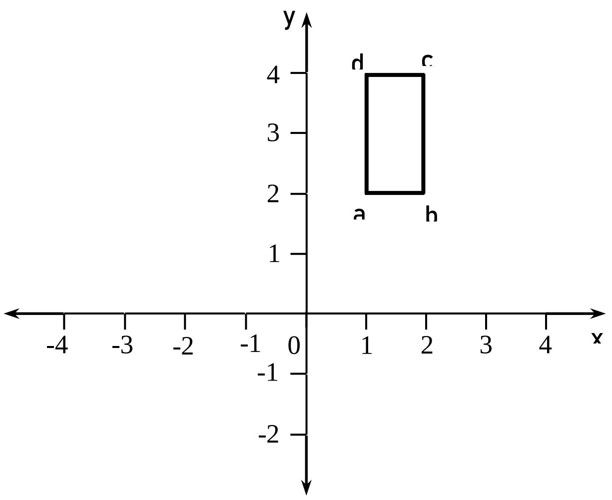
Values of sin, cos, tan for special angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta \quad \left(=\frac{\sin\theta}{\cos\theta}\right)$
$0°$ (0 rad)	0	1	0
$30°$ ($\frac{\pi}{6}$ rad)	$\frac{1}{2}=0.5$	$\frac{1}{2}\sqrt{3}\approx 0.866$	$\frac{1}{\sqrt{3}}\approx 0.577$
$45°$ ($\frac{\pi}{4}$ rad)	$\frac{1}{2}\sqrt{2}\approx 0.707$	$\frac{1}{2}\sqrt{2}\approx 0.707$	1
$60°$ ($\frac{\pi}{3}$ rad)	$\frac{1}{2}\sqrt{3}\approx 0.866$	$\frac{1}{2}=0.5$	$\sqrt{3}\approx 1.732$
$90°$ ($\frac{\pi}{2}$ rad)	1	0	∞

Example 2.3
Find the transformation matrix and the image of the rectangle shown below if it is rotated about the origin by:

(a) $90°$ in the anti-clockwise direction, and
(b) $90°$ in the clockwise direction.

In each case, superimpose the image on the diagram below.



$$\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix};\begin{bmatrix}-2 & -2 & -4 & -4\\ 1 & 2 & 2 & 1\end{bmatrix};\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix};\begin{bmatrix}2 & 2 & 4 & 4\\ -1 & -2 & -2 & -1\end{bmatrix}\right)$$

2.3.5 Translation

The last transformation we want to consider is translation of a point. Let us attempt to find a 2×2 matrix, $\mathbf{T}=\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ that will translate the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. In other words,

$$\mathbf{P'} = \mathbf{TP}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Immediately, we see that we are in trouble trying to find the matrix $\mathbf{T}=\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$. Putting it in another way, we just cannot find a 2×2 matrix that will translate every point in a plane.

To find a transformation matrix that works conveniently, we introduce the **homogeneous coordinates**, which essentially are point on the $z=1$ plane as shown in Figure 2.7. In other words, we represent a point by a 3×1 column vector $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ instead of a 2×1 column vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

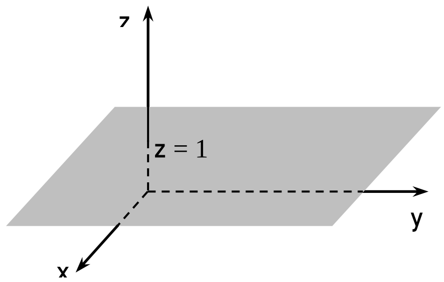


Figure 2.7 **z** = 1 plane

Let us now find a transformation matrix $\mathbf{T}$ that translates the point $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ by d_x in the x-direction and d_y in the y-direction. After the translation, the location of the point is at $\begin{bmatrix} x+d_x \\ y+d_y \\ 1 \end{bmatrix}$ as shown in Figure 2.8. Hence,

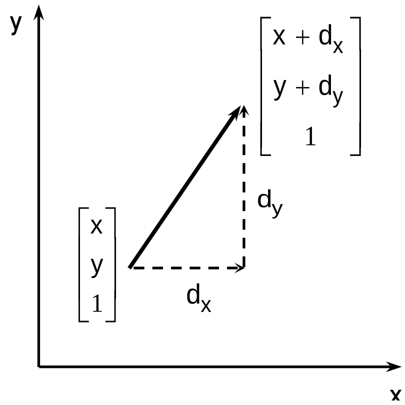


Figure 2.8 Translation of a point in homogeneous coordinates

$$\mathbf{P'} = \mathbf{TP}$$

$$\begin{bmatrix} x+d_x \\ y+d_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Since the vectors representing the points are 3×1 column vectors, by matrix conformability, the transformation matrix must be a 3×3 matrix. Hence, the translation matrix is

$$\mathbf{T}=\begin{bmatrix}1 & 0 & d_x\\ 0 & 1 & d_y\\ 0 & 0 & 1\end{bmatrix},$$

where d_x is the amount of shift in the x-direction and d_y is the amount of shift in the y-direction.

Note that:

- d_x is positive if the translation is to the right
  d_x is negative if the translation is to the left
- d_y is positive if the translation is upward
  d_y is negative if the translation is downward

Example 2.4

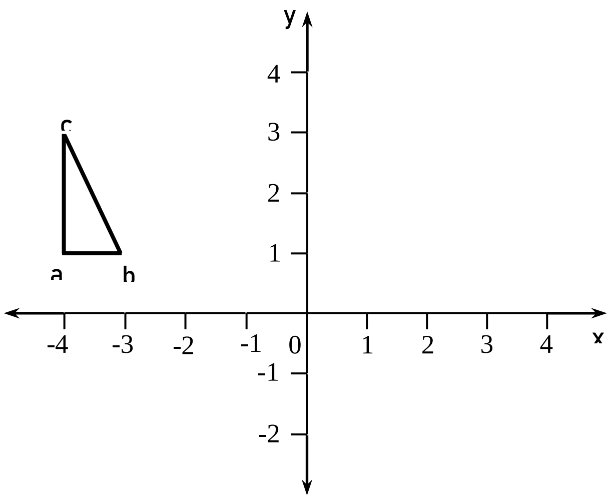
Find the transformation matrix and the image of the triangle shown below if:

(a) it is translated 6 units to the right and 2 units downwards, and

(b) the triangle is moved such that point a is moved to $(3,0)$.

In each case, superimpose the image on the diagram below.

$$\left(\begin{bmatrix}1 & 0 & 6\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}; \begin{bmatrix}2 & 3 & 2\\ -1 & -1 & 1\\ 1 & 1 & 1\end{bmatrix}; \begin{bmatrix}1 & 0 & 7\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}; \begin{bmatrix}3 & 4 & 3\\ 0 & 0 & 2\\ 1 & 1 & 1\end{bmatrix}\right)$$

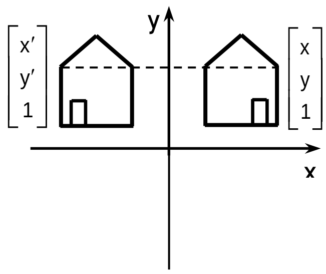


2.4 Transformation Matrices in Homogeneous Coordinates

For consistency we will perform all the different transformations that we have discussed so far (reflection, scaling, shearing, rotation and translation) in terms of homogeneous coordinates. In this section, we will list down all the transformation matrices as 3×3 matrices.

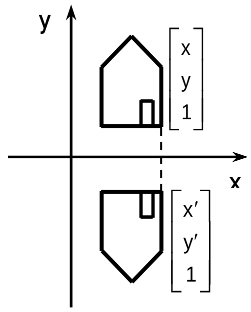
Reflection

- **in the *y*-axis**



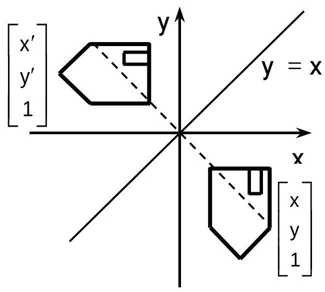
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **in the *x*-axis**



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **in the line $y = x$**

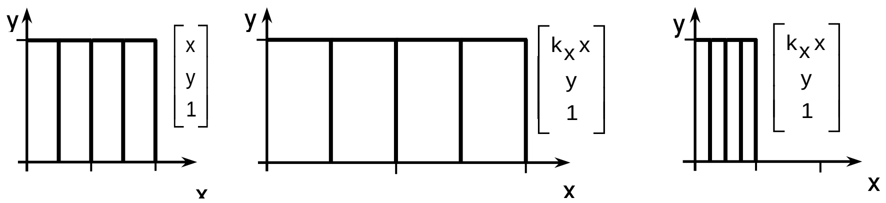


$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling (relative to the origin)

- **in the *x* direction**

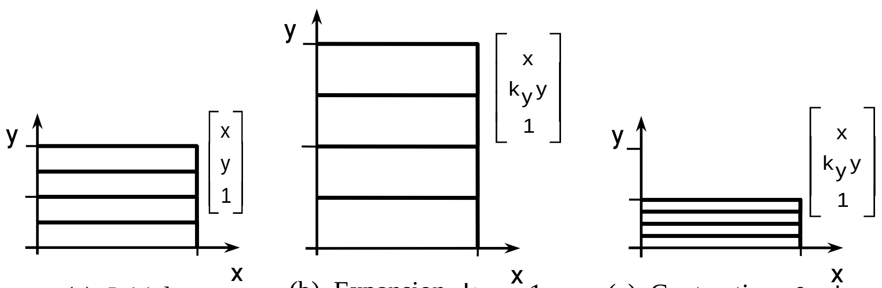
$$\begin{bmatrix} k_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



(a) Initial (b) Expansion, $k_x > 1$ (c) Contraction, $0 < k_x < 1$

- **in the *y* direction**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

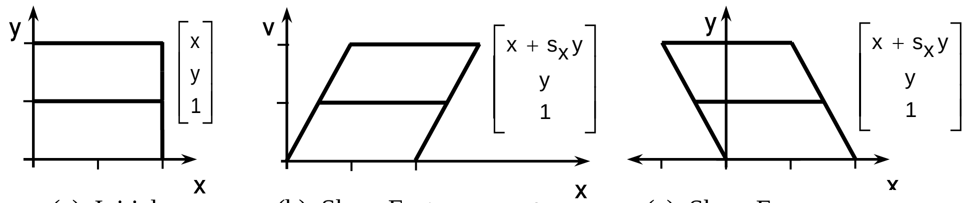


(a) Initial (b) Expansion, $k_y > 1$ (c) Contraction, $0 < k_y < 1$

Shearing

- **in the *x* direction**

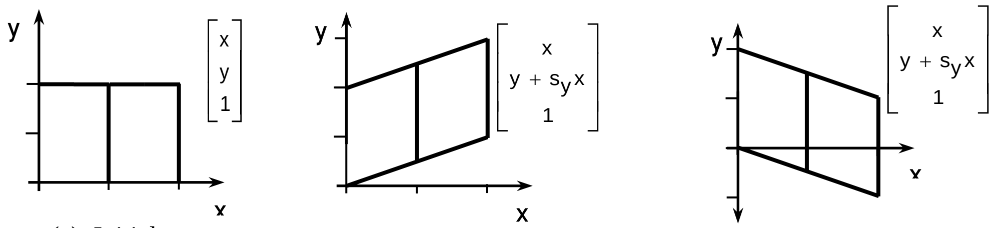
$$\begin{bmatrix} 1 & s_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



(a) Initial (b) Shear Factor, $s_x > 0$ (c) Shear Factor, $s_x < 0$

- **in the *y* direction**

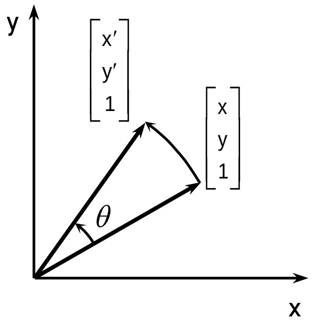
$$\begin{bmatrix} 1 & 0 & 0 \\ s_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



(a) Initial (b) Shear Factor, $s_y > 0$ (c) Shear Factor, $s_y < 0$

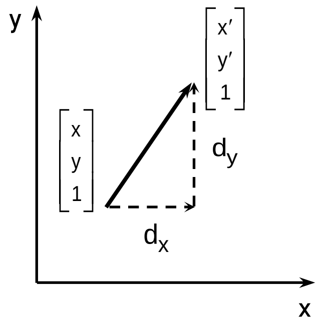
Rotation (about the origin)

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Translation

$$\begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix}$$



Example 2.5

Describe the transformation effected by the following transformation matrices:

(a) $\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\mathbf{T}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\mathbf{T}_3 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\mathbf{T}_4 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\mathbf{T}_5 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(f) $\mathbf{T}_6 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2.5 Composition of Transformations

We have considered five simple transformation matrices, namely reflection, scaling, shearing, rotation and translation. These matrices are used all the time in computer graphics. Other more complicated transformations can be obtained by performing several of these simple transformations in sequence. This is called "composition" of transformations.

To facilitate our discussion, let us consider the following example.

Suppose we want to transform an image in the following sequence:

- First, we want to perform a rotation of $90°$ in the anti-clockwise direction about the origin
- Second, we want to perform a translation of 1 unit to the right and 2 units upwards
- Third, we want to perform a reflection in the line $y = x$

Assume that P is a general point on the plane before the transformation, and that P', P'' and P''' are the points after the first, second and third transformation respectively. Also assume that $\mathbf{T_1}$, $\mathbf{T_2}$ and $\mathbf{T_3}$ are the three transformation matrices corresponding to the first, second and third transformation respectively. Then we have

$$\mathbf{T_1} = \begin{bmatrix} \cos 90° & -\sin 90° & 0 \\ \sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T_3} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $P = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, then $P' = \mathbf{T_1}P = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$,

$$P'' = \mathbf{T_2}P' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix},$$

$$P''' = \mathbf{T_3}P'' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}.$$

Let us look at the previous calculation once again:

$$P' = \mathbf{T}_1 P$$
$$P'' = \mathbf{T}_2 P' = \mathbf{T}_2\mathbf{T}_1 P$$
$$P''' = \mathbf{T}_3 P'' = \mathbf{T}_3\mathbf{T}_2\mathbf{T}_1 P$$

The third equation provides a “shortcut” to compute the final image P''' given the original point P before transformation. If we let a matrix $\mathbf{C}$ as follows:

$$\mathbf{C} = \mathbf{T}_3\mathbf{T}_2\mathbf{T}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

then $\mathbf{C}$ is a **composite matrix** of the three transformations that takes the point P to the final image P''' :

$$P''' = \mathbf{C}P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

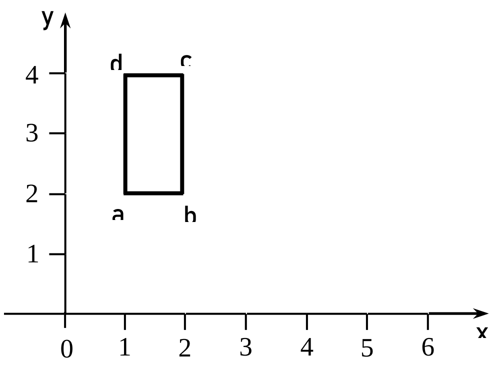
This idea of composite matrix can be extended to any number of transformations. However, it is important to take note of the order in which the individual transformation matrices are multiplied together. In the composite matrix $\mathbf{C} = \mathbf{T}_3\mathbf{T}_2\mathbf{T}_1$, the first transformation matrix $\mathbf{T}_1$ is on the far right, while the second transformation matrix $\mathbf{T}_2$ is to the left of $\mathbf{T}_1$ and finally the third transformation matrix $\mathbf{T}_3$ is to the left of $\mathbf{T}_2$.

Example 2.6

The rectangle shown below is going through the following sequence of three simple transformations:

1. a translation such that point a is moved to the origin,
2. an expansion in the x-direction by a factor of 3 and a contraction in the y-direction by a factor of $\frac{1}{2}$,
3. a translation that takes point a back to $(1,2)$.

(a) Write down each of the three transformation matrices.
(b) Find the composite matrix of the above sequence of three transformations.
(c) Find the image after the three transformations, and superimpose it on the same diagram.
(d) Describe the transformation that the composite matrix effectively produces.
(e) How does the image differ from that of Example 2.1?



$$\left(\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix};\begin{bmatrix}3 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix};\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix};\begin{bmatrix}3 & 0 & -2\\ 0 & \frac{1}{2} & 1\\ 0 & 0 & 1\end{bmatrix};\begin{bmatrix}1 & 4 & 4 & 1\\ 2 & 2 & 3 & 3\\ 1 & 1 & 1 & 1\end{bmatrix}\right)$$

2.6 Inverse of Transformation Matrices

2.6.1 Inverse of Simple Transformation Matrices

Two important facts need to be recalled in this section:

- Firstly, the identity matrix $\mathbf{I}$ takes every point on the plane to the same location. In other words, the transformation matrix $\mathbf{I}$ leaves all the points on the plane unmoved.
- Secondly, in chapter 1, we have seen that $\mathbf{A}^{-1}$ is the inverse of matrix $\mathbf{A}$, and that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

Example 2.7

Find the inverses of the following matrices: (<u>Hint:</u> describe each transformation matrix first, then undo)

(a) $\mathbf{T}_1 = \begin{bmatrix} \cos 90° & -\sin 90° & 0 \\ \sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathbf{T}_1$ is the transformation matrix that performs ____.

To undo $\mathbf{T}_1$, we need to perform ____.

Hence, $\mathbf{T}_1^{-1} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathbf{T}_2$ is the transformation matrix that performs ____.

To undo $\mathbf{T}_2$, we need to perform ____.

Hence, $\mathbf{T}_2^{-1} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\mathbf{T}_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathbf{T}_3$ is the transformation matrix that performs ____.

To undo $\mathbf{T}_3$, we need to perform ____.

Hence, $\mathbf{T}_3^{-1} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\mathbf{T}_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathbf{T}_4$ is the transformation matrix that performs ____.

To undo $\mathbf{T}_4$, we need to perform ____.

Hence, $\mathbf{T}_4^{-1} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\mathbf{T}_5 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathbf{T}_5$ is the transformation matrix that performs ____.

To undo $\mathbf{T}_5$, we need to perform ____.

Hence, $\mathbf{T}_5^{-1} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ 0 & 0 & 1 \end{bmatrix}$

(f) $\mathbf{T}_6 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\mathbf{T}_6$ is the transformation matrix that performs ____.

To undo $\mathbf{T}_6$, we need to perform ____.

Hence, $\mathbf{T}_6^{-1} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ 0 & 0 & 1 \end{bmatrix}$

2.6.2 Inverse of Composite Transformation Matrices

The inverse of a composite matrix $\mathbf{C} = \mathbf{T_3 T_2 T_1}$ is $\mathbf{C}^{-1} = \left(\mathbf{T_3 T_2 T_1}\right)^{-1} = \mathbf{T_1^{-1} T_2^{-1} T_3^{-1}}$.

Example 2.8

Find the inverse of the composite matrix found in Example 2.6(b).

$$\left(\begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{T}_1^{-1} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

$$\mathbf{T}_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{T}_2^{-1} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

$$\mathbf{T}_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{T}_3^{-1} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

$$\mathbf{C}^{-1} = \left(\mathbf{T_3 T_2 T_1}\right)^{-1} = \mathbf{T_1^{-1} T_2^{-1} T_3^{-1}}$$

$$= \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

Tutorial 2 – Linear Transformation

(Solve all the questions in this tutorial using **homogenous coordinates**, i.e. $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.)

Section A (Basic)

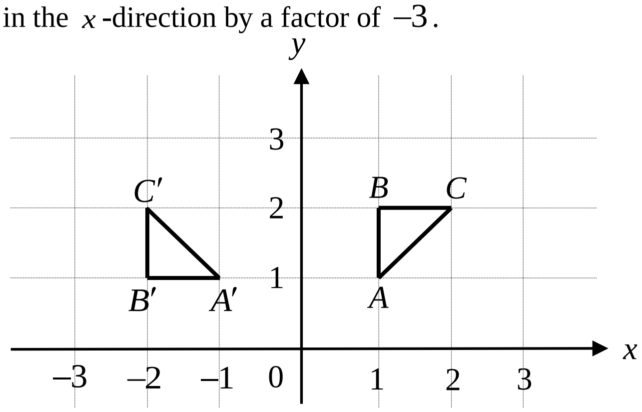
1. Write down the matrix $\mathbf{P}$ that represents each of the following graphical objects.
   (a) A point A with coordinate $(3, 5)$
   (b) A line AB with coordinates $(1, 3)$ and $(-3, -2)$
   (c) A triangle XYZ with coordinates $(3, 2)$, $(4, 1)$ and $(2, 5)$
   (d) A rectangle $ABCD$ with coordinates $(2, 0)$, $(4, 0)$, $(4, 1)$ and $(2, 1)$

2. The rectangle $ABCD$ in Question 1d undergoes the following transformations (they are all separate transformations). For each of the following transformations, find the transformation matrix $\mathbf{T}$, the images after each transformation $A'B'C'D'$ and sketch them.
   (a) Reflection in the y-axis
   (b) Reflection in the line $y = x$
   (c) Compression in the x-direction by a factor of $\frac{1}{2}$ and expansion in the y-direction by a factor of 2
   (d) Shearing in the x-direction by a factor of 1
   (e) Rotation of $90°$ anticlockwise about the origin

3. An object ABC is going through the following sequence of two simple transformations:

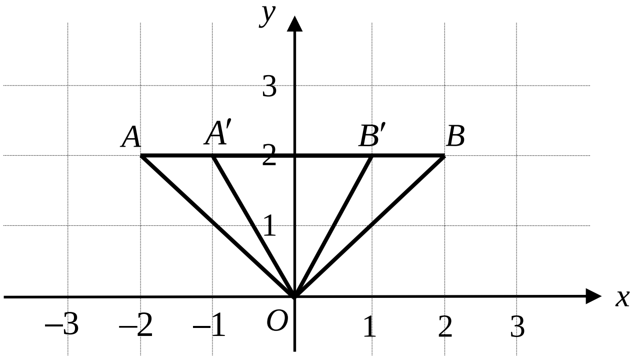
   $\mathbf{T}_1$: transforms ABC to $A'B'C'$ as shown in the diagram below, followed by

   $\mathbf{T}_2$: a shear in the x-direction by a factor of -3.



   (a) Write down the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$.
   (b) Find the composite matrix for the above sequence of transformations.
   (c) Find the composite matrix if the above **sequence** of transformations is **reversed**.

4. The triangle OAB is transformed by $\mathbf{T}_1$ that maps it to $OA'B'$. Then $OA'B'$ is subsequently transformed by $\mathbf{T}_2$ that translates the triangle such that the origin is moved to point $(2,-1)$.



(a) Describe the transformation matrix $\mathbf{T}_1$ that maps triangle OAB to $OA'B'$.

(b) Write down the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$.

(c) Find the composite transformation matrix $\mathbf{C}$ for the above sequence of transformations and the final image of the triangle OAB.

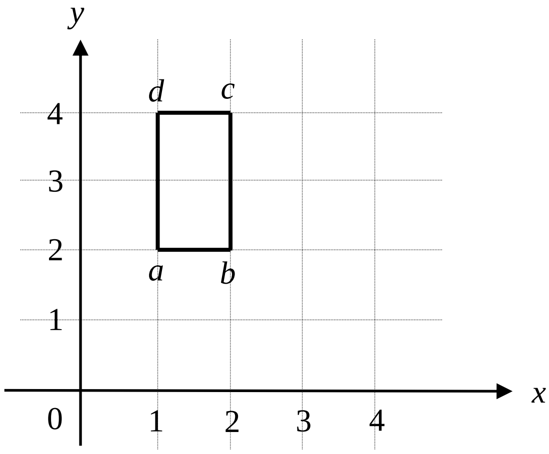
(d) Find the inverse of the composite matrix $\mathbf{C}^{-1}$.

5. The rectangle below is going through the following sequence of three transformations:

$\mathbf{T}_1$: translation such that point a is moved to the origin,

$\mathbf{T}_2$: rotation of $90°$ clockwise about the origin,

$\mathbf{T}_3$: translation that takes point a back to $(1,2)$



(a) Write down each of the three transformation matrices.

(b) Find the composite matrix of the above sequence of three transformations. What does it effectively produce?

(c) Find the image matrix of the rectangle after undergoing the three transformations.

(d) Find the inverse of the composite matrix.

6. For each of the transformations in Question 2, find the transformation matrix that will transform the image $A'B'C'D'$ back to the original object $ABCD$.

7. An object translates from position $(1,2)$ to $(5,2)$, and then rotates $90°$ in the anticlockwise direction about the origin. Find the composite matrix of the above sequence of transformations and the final position of the object.

8. Write down the inverses of the following transformation matrices:

(a) $\mathbf{T_1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\mathbf{T_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\mathbf{T_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

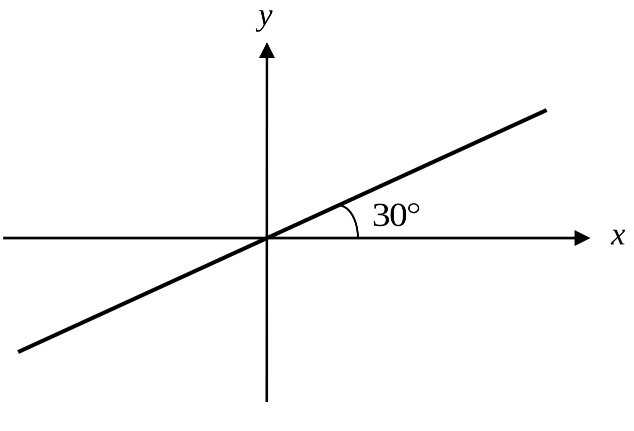
(d) $\mathbf{T_4} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (e) $\mathbf{T_5} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (f) $\mathbf{T_6} = \begin{bmatrix} \cos 30° & -\sin 30° & 0 \\ \sin 30° & \cos 30° & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Section B (Intermediate/Challenging)

9. Write down the inverses of the following transformation matrices:

(a) $\mathbf{T_1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\mathbf{T_2} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. What is the transformation matrix that can effect a reflection about a line that is inclined at an angle of $30°$ about the x-axis, as shown in the diagram below?
(Hint: Use a composite transformation to achieve that.)



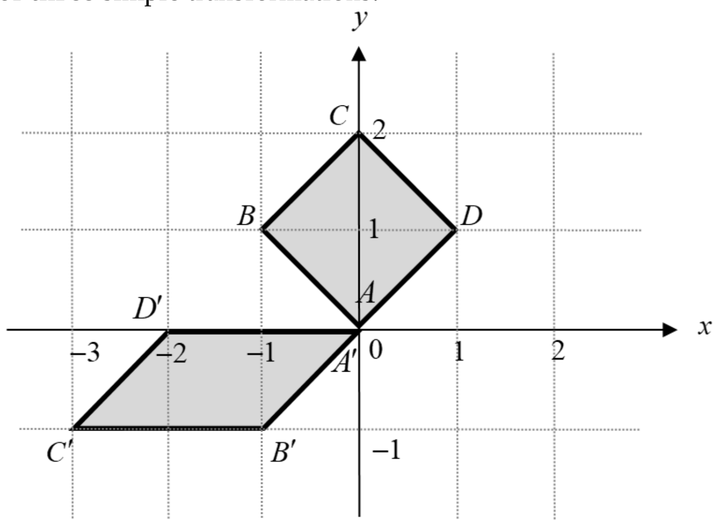
11. *What is the transformation matrix that can effect a rotation of angle θ anticlockwise about a point (a,b)?

12. **(1415S1/MST/2b)** Let $\mathbf{P} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \frac{2}{\sqrt{2}} \end{bmatrix}$.

    (a) Give a complete geometrical description of the transformation represented by matrix $\mathbf{P}$.

    (b) *Hence, find the smallest positive integer n for which $\mathbf{P}^n = \mathbf{I}$, and briefly explain how you obtain your answer.

13. ***(2021S1/C2)** Square $ABCD$ in the figure below is transformed to parallelogram $A'B'C'D'$ through a sequence of **three** simple transformations.



    (a) Describe, in words, the three transformations needed to transform square $ABCD$ to parallelogram $A'B'C'D'$, and write down the corresponding transformation matrices.

    (b) Hence, derive the composite matrix $\mathbf{T}$ for the above sequence of transformations and verify that $\mathbf{T}$ successfully transforms square $ABCD$ to parallelogram $A'B'C'D'$.

Section C (MCQ)

14. **(1314S2/A2)** A triangle XYZ is transformed to $X'Y'Z'$ by the transformation matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

    If the area of XYZ is A, the area of $X'Y'Z'$ is ___________.

    (a) $\frac{1}{6}A$ (b) $3A$ (c) $5A$ (d) $6A$

15. **(1415S2/A1)** A triangle ABC is transformed to $A'B'C'$ by the transformation matrix $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

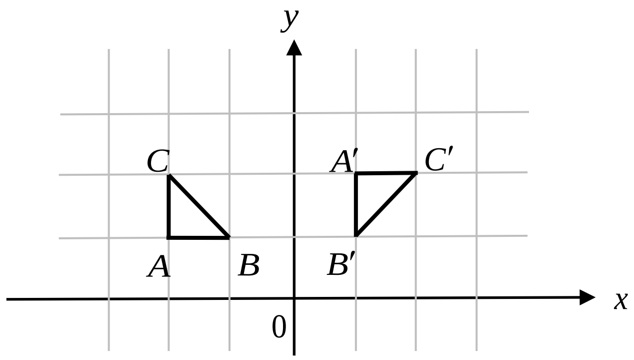
    If the area of ABC is y, the area of $A'B'C'$ is ___________.

    (a) $\frac{1}{3}y$ (b) y (c) $3y$ (d) $9y$

16. **(1314S1/A2)** A rectangular image of size $3'' \times 4''$ undergoes the sequence of transformations below:
    (i) Rotation 90° anticlockwise,
    (ii) Scaling twice in the x-direction and half in the y-direction,
    (iii) Reflection with respect to the line $y = x$.

    The final image will have a size of ___________.
    (a) $6'' \times 2''$ (b) $2'' \times 6''$ (c) $8'' \times 1.5''$ (d) $1.5'' \times 8''$

17. **(1617S1/A1)** Which of the following transformation matrices can transform the triangle ABC to $A'B'C'$, as shown below?



(a) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

18. **(1617S2/A1)** Which of the following transformation matrices $\mathbf{T}$ has the property of $\mathbf{T}^2 = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix?

(a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

<u>Tutorial 2 – Answers</u>

1. (a) $\mathbf{P}=\begin{bmatrix}3\\5\\1\end{bmatrix}$ (b) $\mathbf{P}=\begin{bmatrix}1 & -3\\3 & -2\\1 & 1\end{bmatrix}$ (c) $\mathbf{P}=\begin{bmatrix}3 & 4 & 2\\2 & 1 & 5\\1 & 1 & 1\end{bmatrix}$ (d) $\mathbf{P}=\begin{bmatrix}2 & 4 & 4 & 2\\0 & 0 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix}$

2. (a) $\mathbf{T}=\begin{bmatrix}-1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$; $A'B'C'D'=\begin{bmatrix}-2 & -4 & -4 & -2\\0 & 0 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix}$

   (b) $\mathbf{T}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$; $A'B'C'D'=\begin{bmatrix}0 & 0 & 1 & 1\\2 & 4 & 4 & 2\\1 & 1 & 1 & 1\end{bmatrix}$

   (c) $\mathbf{T}=\begin{bmatrix}\frac{1}{2} & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}$; $A'B'C'D'=\begin{bmatrix}1 & 2 & 2 & 1\\0 & 0 & 2 & 2\\1 & 1 & 1 & 1\end{bmatrix}$

   (d) $\mathbf{T}=\begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$; $A'B'C'D'=\begin{bmatrix}2 & 4 & 5 & 3\\0 & 0 & 1 & 1\\1 & 1 & 1 & 1\end{bmatrix}$

   (e) $\mathbf{T}=\begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$; $A'B'C'D'=\begin{bmatrix}0 & 0 & -1 & -1\\2 & 4 & 4 & 2\\1 & 1 & 1 & 1\end{bmatrix}$

3. (a) $\mathbf{T_1}=\begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$; $\mathbf{T}_2=\begin{bmatrix}1 & -3 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ (b) $\begin{bmatrix}-3 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$ (c) $\begin{bmatrix}0 & -1 & 0\\1 & -3 & 0\\0 & 0 & 1\end{bmatrix}$

4. (a) Scaling relative to origin in the horizontal direction by a factor of $\frac{1}{2}$.

   (b) $\mathbf{T_1}=\begin{bmatrix}\frac{1}{2} & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$; $\mathbf{T}_2=\begin{bmatrix}1 & 0 & 2\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}$ (c) $\mathbf{C}=\begin{bmatrix}\frac{1}{2} & 0 & 2\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}$; $OA'B'=\begin{bmatrix}2 & 1 & 3\\-1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$

   (d) $\mathbf{C}^{-1}=\begin{bmatrix}2 & 0 & -4\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$

5. (a) $\mathbf{T}_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$; $\mathbf{T}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\mathbf{T}_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$; rotation of 90° clockwise about point a

(c) $\begin{bmatrix} 1 & 1 & 3 & 3 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

6. (a) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$

8. (a) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. (a) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

11. $\begin{bmatrix} \cos\theta & -\sin\theta & -a\cos\theta + b\sin\theta + a \\ \sin\theta & \cos\theta & -a\sin\theta - b\cos\theta + b \\ 0 & 0 & 1 \end{bmatrix}$

12. (a) $\mathbf{P}$ represents a 45° anti-clockwise rotation about the origin.
(b) $n = 8$

13. (a) $\mathbf{T}_1$: rotation $135°$ anticlockwise about the origin

$\mathbf{T}_2$: scaling relative to the origin by a factor of $\sqrt{2}$ in the *x*-direction and $\frac{1}{\sqrt{2}}$ in the *y*-direction

$\mathbf{T}_3$: shearing by a factor of 1 in the *x*-direction

$$\mathbf{T}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix};\ \mathbf{T}_2 = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix};\ \mathbf{T}_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $$\mathbf{T} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

14. d   15. b   16. d   17. b   18. c