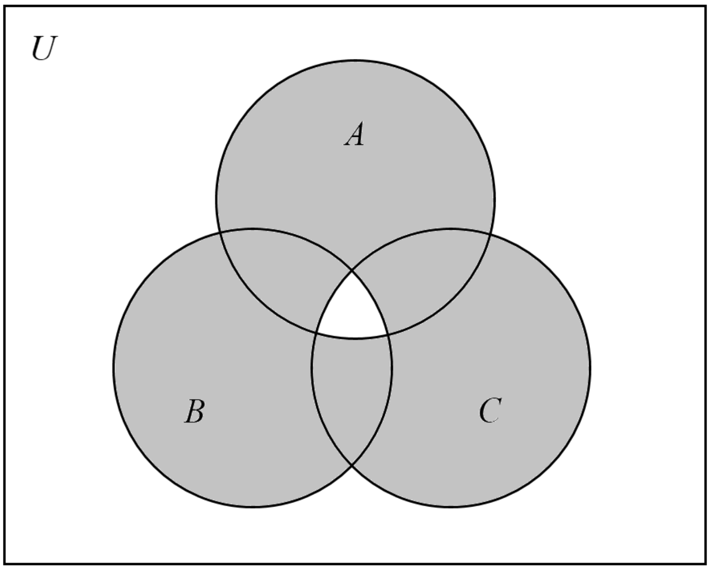
SOLUTIONS

SINGAPORE POLYTECHNIC

2022/2023 Semester 1 Examination

No.	SOLUTION
A1	Answer: **(a)** Expanding the expression $\left(\mathbf{ABA}^{-1}\right)^n$, we have $\mathbf{ABA}^{-1}\mathbf{ABA}^{-1}\mathbf{ABA}^{-1}\ldots$ (until n times). Since $\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$ and $\mathbf{BI}=\mathbf{B}$, the original expression simplifies to $\mathbf{AB}^n\mathbf{A}^{-1}$.
A2	Answer: **(d)**  As seen in the Venn diagram, $(A-B)\cup(B-C)\cup(C-A)=(A\cup B\cup C)-(A\cap B\cap C)$.
A3	Answer: **(c)** We will start by assuming that Gary is a knight. Gary, who must be telling the truth since he is a knight, says that both Gary and Gladys are knights. However, Gladys, who must also be telling the truth since she is a knight too, says that Gary is a knave, which contradicts our assumption that Gary is a knight. Hence, Gary is a knave. Similarly, we will first assume that Gladys is a knight. Gladys, who must be telling the truth since she is a knight, says that either of Gary or Gladys (or both) is a knave. This is true and does not contradict our assumption, since Gary is indeed a knave. Hence, Gladys is a knight.

SOLUTIONS

A4	Answer: **(c)** All positive integers smaller than 10^{16} can be completely represented using 16 digits. There are two cases where the sum of their digits is equal to 2: Case 1 (14 ‘0’s and 2 ‘1’s): $\frac{16!}{14!2!}=120$ Case 2 (15 ‘0’s and 1 ‘2’): $\frac{16!}{15!1!}=16$ Hence, the total number of integers = 120 + 16 = 136.
A5	Answer: **(d)** First, choose any one of the five seats for Amy to occupy. This leaves four remaining seats that Bernice can possibly occupy, of which two of them will always be beside Amy (one on her immediate left and one on her immediate right). Hence, the probablity that Amy and Bernice will be seated beside each other = 2/4 = 1/2.

SOLUTIONS

SINGAPORE POLYTECHNIC
2022/2023 Semester 1 Examination

B1a	$\mathbf{X}=\mathbf{A}-2\mathbf{B}^T=\begin{bmatrix}5&8&-2\\-1&3&4\end{bmatrix}-\begin{bmatrix}4&10&-4\\-6&2&12\end{bmatrix}=\begin{bmatrix}1&-2&2\\5&1&-8\end{bmatrix}$
B1b	$\mathbf{CD}=\begin{bmatrix}2&1\\7&4\end{bmatrix}\begin{bmatrix}4&-1\\-7&2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ Since $\mathbf{CD}=\mathbf{I}$, therefore $\mathbf{C}^{-1}=\mathbf{D}$ and $\mathbf{D}^{-1}=\mathbf{C}$. Hence, $\mathbf{C}^{-1}-\mathbf{D}^{-1}=\mathbf{D}-\mathbf{C}=\begin{bmatrix}4&-1\\-7&2\end{bmatrix}-\begin{bmatrix}2&1\\7&4\end{bmatrix}=\begin{bmatrix}2&-2\\-14&-2\end{bmatrix}$
B2a	$\mathbf{T}_1=\begin{bmatrix}1&0&2\\0&1&-3\\0&0&1\end{bmatrix}$; $\mathbf{T}_2=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}$ $\mathbf{C}=\mathbf{T}_2\mathbf{T}_1=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&2\\0&1&-3\\0&0&1\end{bmatrix}=\begin{bmatrix}0&-1&3\\1&0&2\\0&0&1\end{bmatrix}$
B2b	$\mathbf{P}'=\mathbf{CP}=\begin{bmatrix}0&-1&3\\1&0&2\\0&0&1\end{bmatrix}\begin{bmatrix}0&3&3&0\\0&0&2&2\\1&1&1&1\end{bmatrix}=\begin{bmatrix}3&3&1&1\\2&5&5&2\\1&1&1&1\end{bmatrix}$
B2c	$\mathbf{T}_1^{-1}=\begin{bmatrix}1&0&-2\\0&1&3\\0&0&1\end{bmatrix}$; $\mathbf{T}_2^{-1}=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}$ $\mathbf{C}^{-1}=\mathbf{T}_1^{-1}\mathbf{T}_2^{-1}=\begin{bmatrix}1&0&-2\\0&1&3\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&1&-2\\-1&0&3\\0&0&1\end{bmatrix}$
B3a	<u>Convert to BIN:</u> $3E8.4_{16}=(0011)(1110)(1000).(0100)_2=1111101000.01_2$ <u>Convert to DEC:</u> $3E8.4_{16}=3\times16^2+14\times16^1+8\times16^0+4\times16^{-1}=1000.25_{10}$

SOLUTIONS

SINGAPORE POLYTECHNIC

2022/2023 Semester 1 Examination

B3b

Integral part:		
16	8071	
16	504	7
16	31	8
16	1	15
	0	1

Fractional part:		
16	0.55	
2	0.8	8
2	0.8	12
2	0.8	12
2	0.8 (rep)	12

$\therefore 8071.55_{10} = 1F87.8\overline{C}_{16}$

B4a

$U = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{2, 4, 6, 8, 10\}$

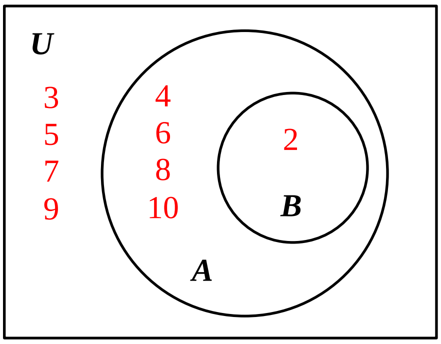
$B = \{2\}$

B4b

$\overline{A \cup B} = \{3, 5, 7, 9\}$

$|A \cap B| = 1$

B4c



B5a

p	q	$\neg q$	$p \Rightarrow \neg q$	$p \wedge \neg q$	$p \Leftrightarrow (p \wedge \neg q)$
T	T	F	F	F	F
T	F	T	T	T	T
F	T	F	T	F	T
F	F	T	T	F	T

$\therefore A \equiv B$

SOLUTIONS

SINGAPORE POLYTECHNIC

2022/2023 Semester 1 Examination

B5b	$\overline{\overline{x}+y}+x\left(\overline{x}+\overline{y}\right)=\overline{\overline{x}}\,\overline{y}+x\left(\overline{x}+\overline{y}\right)$ $=x\,\overline{y}+x\left(\overline{x}+\overline{y}\right)$ $=x\,\overline{y}+x\,\overline{x}+x\,\overline{y}$ $=x\,\overline{y}+x\,\overline{y}$ $=x\,\overline{y}$
B6a	$9!=362,880$
B6b	$5!=120$
B6c	$9!-6!=362,160$
B6d	$5\times 5!\times 4!=14,400$
B7a (i)	$P\left(X\right)=\frac{5}{12}\times\frac{4}{11}=\frac{5}{33}$
B7a (ii)	$P\left(Y\right)=1-\frac{5}{33}=\frac{28}{33}$
B7a (iii)	$P\left(Z\right)=\frac{5}{33}+\frac{7}{12}\times\frac{6}{11}=\frac{31}{66}$
B7b	$P\left(Y\right)\times P\left(Z\right)=\frac{28}{33}\times\frac{31}{66}=\frac{434}{1089}$ $P\left(Y\cap Z\right)=\frac{7}{12}\times\frac{6}{11}=\frac{7}{22}$ Since $P\left(Y\cap Z\right)\neq P\left(Y\right)\times P\left(Z\right)$, therefore Y and Z are **not** independent.

SOLUTIONS

SINGAPORE POLYTECHNIC

2022/2023 Semester 1 Examination

C1a

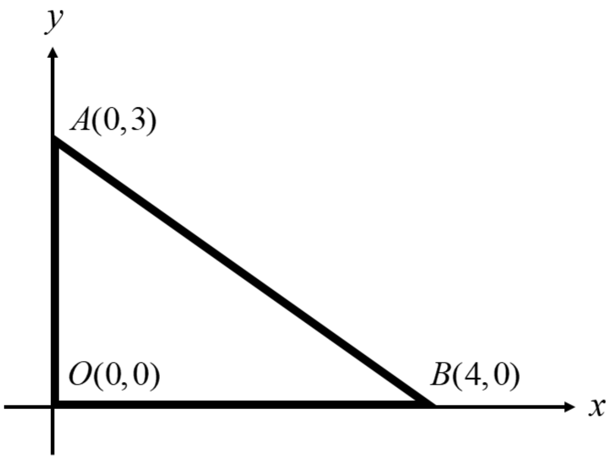
x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

SOP expression: $f(x,y,z) = \bar{x}\,y\,z + x\,\bar{y}\,z + x\,y\,\bar{z}$

C1b

$$\begin{aligned} a+b &= \overline{\overline{a+b}} \\ &= \overline{\bar{a}\bullet\bar{b}} \\ &= \overline{\left(\overline{a\bullet a}\right)\bullet\left(\overline{b\bullet b}\right)} \\ &= \overline{(a\,|\,a)\bullet(b\,|\,b)} \\ &= (a\,|\,a)\,\big|\,(b\,|\,b) \end{aligned}$$

C2a



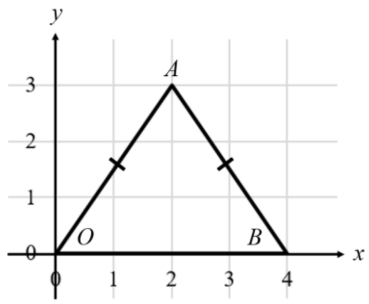
SOLUTIONS

SINGAPORE POLYTECHNIC

2022/2023 Semester 1 Examination

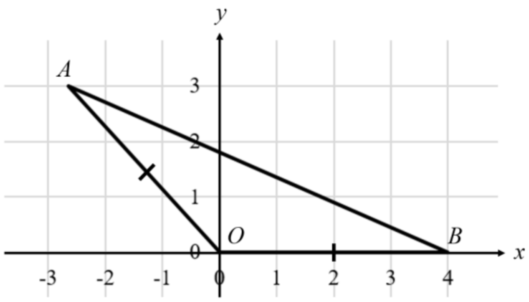
C2b

Case 1: $OA = AB$ (the x-coordinate of point A becomes 2)

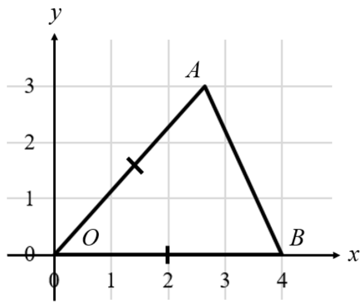


For Case 1, the shear factor $s = \dfrac{2}{3}$.

Case 2: $OA = OB$ (the x-coordinate of point A becomes $\pm\sqrt{4^2 - 3^2} = \pm\sqrt{7}$)

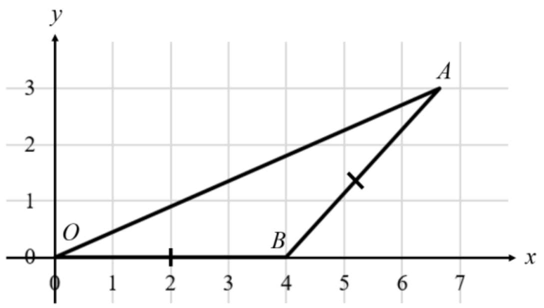


or

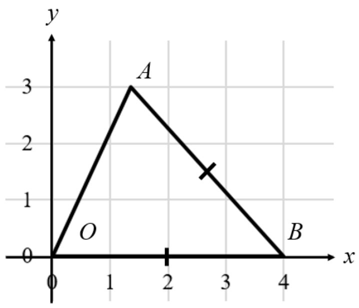


For Case 2, the shear factor $s = \pm\dfrac{\sqrt{7}}{3}$.

Case 3: $OB = AB$ (the x-coordinate of point A becomes $4 \pm \sqrt{4^2 - 3^2} = 4 \pm \sqrt{7}$)



or



For Case 3, the shear factor $s = \dfrac{4 \pm \sqrt{7}}{3}$.

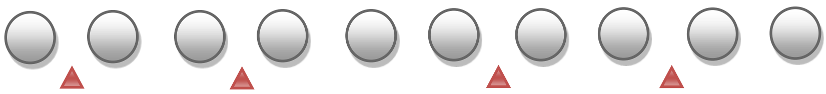
In total, there are five possible values of s: $-\dfrac{\sqrt{7}}{3}$, $\dfrac{4-\sqrt{7}}{3}$, $\dfrac{2}{3}$, $\dfrac{\sqrt{7}}{3}$ and $\dfrac{4+\sqrt{7}}{3}$.

Since the question only asked for three possible values of s, therefore finding any three of the above five values of s will suffice.

SOLUTIONS

SINGAPORE POLYTECHNIC

2022/2023 Semester 1 Examination

C2c	The transformed triangle $O''A''B''$ has vertices $O''(0,0)$, $A''(0,2k)$, $B''(2k,0)$. Area of $O''A''B'' = \frac{1}{2} \times 2k \times 2k = 2k^2$ Perimeter of $O''A''B'' = 2k + 2k + \sqrt{(2k)^2 + (2k)^2} = 4k + \sqrt{8k^2} = \left(4+\sqrt{8}\right)k$ Area < Perimeter $\Rightarrow 2k^2 < \left(4+\sqrt{8}\right)k \Rightarrow k < \frac{4+\sqrt{8}}{2} \Rightarrow k < 2+\sqrt{2} \Rightarrow k = 2$ or 3 Die roll outcomes for $k = 2$ or 3: $(1,1),(1,2),(2,1)$ $\therefore$ Probability$= \frac{3}{36} = \frac{1}{12}$
C3a	$\frac{10C2 \times 8C2 \times 6C2 \times 4C2 \times 2C2}{5!} = 945$
C3b	Since 10 identical balls are to be distributed across 5 distinct boxes, where each box must contain at least one ball, we can insert 4 dividers between the gaps of 10 balls (there are 9 gaps in total):  $\therefore 9C4 = 126$
C3c	Define $y_n = x_n - n + 1$, the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 50$, such that $x_n \in \{ x \in \mathbb{Z} \mid x \ge n \}$ can be modified to: $y_1 + y_2 + y_3 + y_4 + y_5 = 40$, such that $y_n \in \{ y \in \mathbb{Z} \mid y \ge 1 \}$ The number of solutions that satisfies this modified equation is equivalent to the number of ways to distribute 40 identical balls across 5 distinct boxes. Hence, the same concept used in part (b) can also be applied here. $\therefore 39C4 = 82,251$