Chapter Seven: Techniques of Counting

Learning Objectives:
By the end of the chapter, students should be able to:
1. Count a set of objects using the addition principle, inclusion-exclusion principle and/or multiplication principle.
2. Calculate the number of permutations, *r*-permutations and *r*-combinations of a set of objects.
3. Distinguish between cases where the order of the objects is important and cases where the order is not important.
4. Distinguish between cases where repetition of objects is allowed and cases where repetition is not allowed.
5. Apply techniques of counting to solve real-life problems.

Introduction

When we mention the word "counting", many of us would probably say "I have learnt counting in my kindergarten and first grade of my primary school!" In fact, many of us associate counting with a simple counting 1, 2, 3, and so on. However, the "counting" we are concerned in this chapter is more involved than that. In real life, there is often a need to count, at least approximately, the number of ways that a process can happen or the number of steps required in an algorithm. These are important information because they help in determining the potential cost and the feasibility of the computation.

In this chapter we will consider some basic techniques for counting such numbers without direct enumeration. Such techniques are sometimes called **combinatorics** or combinatorial analysis. This knowledge is very useful in real life, especially when combined with the concept of probability introduced in the next chapter.

Combinatorics is a branch of pure mathematics concerning the study of discrete (and usually finite) objects. Counting the number of ways that certain patterns can be formed is the central problem of combinatorics.

7.1 The Cardinality of a Union of Sets

In Chapter 4 (Set Theory), we have said that the number of elements in a finite set A is known as the **cardinality** of A. Given a certain number of finite sets $A_1, A_2, \cdots, A_n$ and suppose we know the number of elements in each of these sets. What is then the cardinality of the union of these sets? In other words, how many elements are there in the union of these sets? The answer to this question depends on whether the sets are disjoint or not. If they are disjoint sets, then the answer can be obtained by the *Addition Principle*. If they are not disjoint sets, then the more complicated *Inclusion-Exclusion Principle* need to be sought.

7.1.1 The Addition Principle

If A_1 and A_2 are two disjoint finite sets, i.e. $A_1 \cap A_2 = \varnothing$, then

$$|A_1 \cup A_2| = |A_1| + |A_2|$$

This is the **Addition Principle** (for two sets).

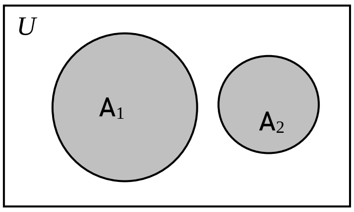


Figure 7.1 $|A_1 \cup A_2| = |A_1| + |A_2|$

Example 7.1

Let $A_1 = \{a, b, c, d, e\}$ and $A_2 = \{g, h, k\}$.

How many elements are there altogether in sets A_1 and A_2? (8)

$|A_1| =$ ______

$|A_2| =$ ______

$|A_1 \cup A_2| = |A_1| + |A_2| =$

Example 7.2

How many ways are there to draw an ace or a queen from an ordinary deck of playing cards? (8)

Let A be the set of aces and Q be the set of queens.

$|A| =$ ______

$|Q| =$ ______

$|A \cup Q| = |A| + |Q| =$

Example 7.3

When two dice are rolled, how many ways are there to get a sum of 3 or of 5? (6)

This principle can be extended to any number of disjoint finite sets. If $A_1, A_2, \cdots, A_n$ are disjoint finite sets, i.e. $A_i \cap A_j = \varnothing$ for all $i \neq j$, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

In words, the number of elements in the union of disjoint sets equals the sum of the sizes of all the individual sets.

7.1.2 The Inclusion-Exclusion Principle

The Inclusion-Exclusion Principle gives the number of elements in the union of sets that are not disjoint. Let us consider this principle for 2 sets A_1 and A_2 whereby $A_1 \cap A_2 \neq \varnothing$.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

This is called the Inclusion-Exclusion Principle for two sets.

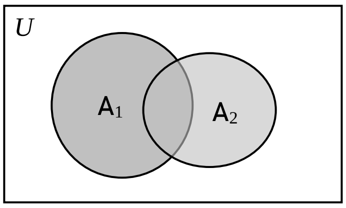


Figure 7.2 $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Notice that when $|A_1|$ is added to $|A_2|$, the elements in $|A_1 \cap A_2|$ are counted twice. In order to get the correct count, each element in $|A_1 \cap A_2|$ should be counted once only. Therefore, $|A_1 \cap A_2|$ needs to be subtracted from the sum $|A_1| + |A_2|$ to get the correct count for $|A_1 \cup A_2|$.

Example 7.4

Let $A_1 = \{a, b, c, d, e\}$ and $A_2 = \{c, d, e, g, h, k\}$.

How many elements are there altogether in sets A_1 and A_2?

(8)

$|A_1| =$ ______

$|A_2| =$ ______

$|A_1 \cap A_2| =$ ______

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| =$

Alternatively, try to list down the members inside $A_1 \cup A_2$.

$A_1 \cup A_2 =$

$|A_1 \cup A_2| =$

Example 7.5
In a small town, 32 people save paper or bottles for recycling. It is known that 30 save paper and 14 save bottles. Find the number of people who save

(a) both paper and bottles
(b) paper only
(c) bottles only

(12; 18; 2)

***Common Error:** When you read the question that 30 people save paper, you may have taken this to mean "30 people save paper only and not bottles". However, this is not what it meant. The category of people who save paper include all who save paper, regardless of whether they save bottles or not.

Important Operation in Counting (1)

Modulo Operation

The module operation finds the remainder after division of one number by another. Given two positive integers x and y, the mathematical notation $x \bmod y$ means the remainder when x is divided by y. The integer x is called the dividend and the integer y is the divisor.

For example, $5 \bmod 2 = 1$, because when 5 is divided by 2, the remainder is 1.

Example

Find the results of the following operations:

(a) $17 \bmod 3$

(b) $25 \bmod 4$

(c) $35 \bmod 7$

(d) $39 \bmod 8$

(2; 1; 0; 7)

Note that:

- If $r = x \bmod y$, the range of r is between 0 and $y-1$ (inclusive).
- $x \bmod 1$ is always zero.
- x is perfectly divisible by y if and only if $x \bmod y = 0$.

Important Operation in Counting (2)

Floor and Ceiling Functions

There are two functions which are particularly useful in discrete mathematics and computer science:

1. **Floor function**:
   $floor(x) = \lfloor x \rfloor =$ the greatest integer less than or equal to x.
   For example, $\lfloor 2 \rfloor = 2$, $\lfloor 2.4 \rfloor = 2$, $\lfloor 2.9 \rfloor = 2$, $\lfloor -2.1 \rfloor = -3$

2. **Ceiling function**:
   $ceiling(x) = \lceil x \rceil =$ the smallest integer greater than or equal to x.
   For example, $\lceil 2 \rceil = 2$, $\lceil 2.4 \rceil = 3$, $\lceil 2.9 \rceil = 3$, $\lceil -2.1 \rceil = -2$

Example

(a) $\lfloor 2.499 \rfloor =$

(b) $\lfloor 2.5 \rfloor =$

(c) $\lfloor 2.7 \rfloor =$

(d) $\lfloor 2.999 \rfloor =$

(e) $\lfloor 2 \rfloor =$

(f) $\lfloor -2.1 \rfloor =$

(g) $\lceil 2.1 \rceil =$

(h) $\lceil 2 \rceil =$

(i) $\lceil -2.9 \rceil =$

(2; 2; 2; 2; 2; -3; 3; 2; -2)

Note that:

- For any real number x, if $\lfloor x \rfloor = \lceil x \rceil$, then x is an integer.
- If x and y are positive numbers, $x \bmod y = x - y\left\lfloor \dfrac{x}{y} \right\rfloor$.

Example

Let $x = 27$ and $y = 11$. Find the following values:

(a) $x \bmod y$

(b) $x - y\left\lfloor \dfrac{x}{y} \right\rfloor$

(c) Are the results in part (a) and (b) the same?

(5; 5; yes)

Example 7.6

Find the number of integers between 1 and 50 inclusive that are

(a) divisible by 3 or 7

(b) not divisible by 3 or 7

(21; 29)

Example 7.7

Find the number of integers between 70 and 150 inclusive that are

(a) divisible by 2
(b) divisible by 3
(c) divisible by 6
(d) divisible by 2 and 3
(e) divisible by 2 or 3
(f) divisible by 3 and 6
(g) divisible by 3 or 6
(h) not divisible by 2, 3 or 6

(41; 27; 14; 14; 54; 14; 27; 27)

7.2 The Multiplication Principle

Example 7.8
Four students are given the choice of five projects. How many different ways can they choose a project if

(a) No one may choose the same project?
(b) Everyone can choose any project that he likes?

(120; 625)

Example 7.9
Suppose you are asked to create a system password consisting of three alphabets (can be upper or lower case).

(a) How many such unique passwords can you possibly create?
(b) If a hacking software uses brute force method to attempt accessing the system, in the worst case how long does the software need to be able to access the system if it can generate one random password in 20μs?

(140,608; 2.8s)

Example 7.10
Suppose a car license plate contains two distinct capital letters followed by 4 digits and the letters ‘O’ and ‘I’ are not used. Find the number of possible different license plates if the first digit is not zero and the digits cannot be repeated.

(2,503,872)

7.3 The Tree Diagram

As we have seen, both the addition and multiplication principles are very useful in counting the number of ways something can be done. As a matter of fact, these two principles are the basic "building blocks" for most counting problems.

Under certain scenarios where events occur sequentially (one after another), a tree diagram can be used to list all possibilities arising from a sequence of events in a systematic way.

Example 7.11

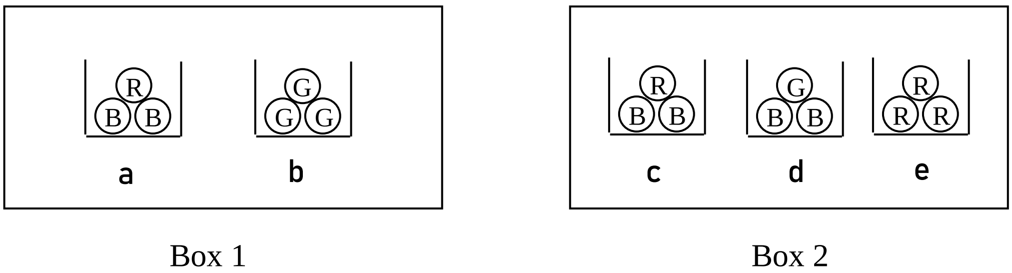
A fair coin is tossed three times. With the aid of the tree diagram, count how many times heads occur more than tails.

(4)

Example 7.12

There are two boxes. Box 1 contains 2 urns and Box 2 contains 3 urns. The urns contain balls of different colors as shown in the figure. An experiment involves first selecting a box, then an urn in the box and then a ball from the urn. With the help of a tree diagram, find the number of possible outcomes of the experiment.

(8)



7.4 Permutation & Combination

Let us consider selecting a fixed number of r elements from a given set A. Two factors need to be considered when the selections are made:

(a) whether repetition of the elements are allowed, and
(b) whether order matters.

In the first instant, let us make the restriction that **no repetition of the elements is allowed** in the selection. We will name the two ways of selecting elements from a set A and construct the list of the elements depending on whether order matters or not. The list of all possible selection of r elements from a given set A is called r **-permutations** of A if order matters, and r **-combinations** of A if order does not matter.

Example 7.13

Suppose $A=\{a,b,c,d\}$. We want to form 3-letter words from the elements in set A. How many ways can we do so, if repetition is not allowed?

If order does not matter: **combination**				
If order matters: **permutation**				

*Note:

1. For permutations, the sequence abc and acb are considered two different instances, even though they consist of the same 3 elements. They are distinguished by the order in which the elements are selected. The 3- **permutations** of A is an **ordered selection** or **arrangement** of 3 elements of A.
2. For combinations, the sequence abc and acb are considered the same instance, because they consist of the same 3 elements. In other words, the 3- **combinations** of A is an **unordered selection** of 3 elements of A.

You would have realized by now that writing down the lists of r -permutations or r -combinations can be a very tedious and tiresome task even for small sets – not to mention sets with large number of elements. Fortunately, in most cases we are interested in the number of r -permutations or r -combinations than the actual lists. So in the subsequent section we will next develop formulas for these numbers. Before that, let us consider a special product known as the factorials.

7.4.1 Factorials

Factorial is a mathematical notation to denote a certain pattern of product. The product $4\times3\times2\times1$ is represented by the notation $4!$ and is read as "4 factorial". In general, for any positive integer n,

$$n! = n\times(n-1)\times(n-2)\times\cdots\times1$$

So, $n!$ (*n* factorial) is the product of all integers from 1 to *n*.

Note: $0! = 1$.

Example 7.14

(a) Find the value of $n!$ for $2 \le n \le 10$.

(b) What do you observe about the growth of the value of $n!$ as n increases?

(2; 6; 24; 120; 720; 5040; 40,320; 362,880; 3,628,800)

$2! = 2\times1 =$

$3! = 3\times2\times1 =$

$4! = 4\times3\times2\times1 =$

$5! = 5\times4\times3\times2\times1 =$

(a) $6! = 6\times5\times4\times3\times2\times1 =$

$7! = 7\times6\times5\times4\times3\times2\times1 =$

$8! = 8\times7\times6\times5\times4\times3\times2\times1 =$

$9! = 9\times8\times7\times6\times5\times4\times3\times2\times1 =$

$10! = 10\times9\times8\times7\times6\times5\times4\times3\times2\times1 =$

(b)

Example 7.15

Find the values of the following:

(a) $\frac{8!}{5!} =$

(b) $\frac{9!}{5!4!} =$

(c) $\frac{9!}{3!} =$

(d) $\frac{9}{3}! =$

(e) $8! - 7! =$

(336; 126; 60,480; 6; 35,280)

7.4.2 Number of *r*-Permutations (without repetition)

The number of r-permutations from a set with n distinct elements is denoted by ${}_nP_r$. It is read as "the permutations of n elements taken r at a time", "r-permutation of n" or simply "n **-permute-** r".

Note: Instead of ${}_nP_r$, some textbooks use the following symbols: ${}^nP_r$, P_r^n, $P_{n,r}$ or $P(n,r)$.

Recall that **for permutation, the order of selection matters**.

Suppose $A=\{a,b,c,d\}$. What is the number of 4-permutations of A, or simply, the permutations of A? In other words, what is ${}_4P_4$ or $P(4,4)$? We can find this number without first writing out the list of permutations. Instead, we can use the multiplication principle and consider the counting problem of filling 4 "slots" with elements without repetition from set A.

Slot 1	Slot 2	Slot 3	Slot 4

The first slot can be filled by any of the 4 elements of A. Since repetitions are not allowed, there are 3 choices for the second slot, 2 choices for the third slot and 1 choice for the fourth slot:

4	**3**	**2**	**1**
Slot 1	Slot 2	Slot 3	Slot 4

So, using the multiplication principle, ${}_4P_4 = 4\times3\times2\times1=4!=24$

In general, we can deduce that the number of permutations of A if it contains n distinct elements is

$$\boxed{{}_nP_n = n!}$$

In a similar manner, we can find ${}_nP_r$ by using the multiplication principle to the counting problem of filling r slots with elements without repetition from a set containing n distinct elements. The first slot can be filled by any of the n elements of the set. Since repetitions are not allowed, there are $n-1$ choices for the second slot, $n-2$ choices for the third slot and this continues until the last slot.

n	$n-1$	$n-2$	$n-3$	$\cdots$	$n-(r-1)$
Slot 1	Slot 2	Slot 3	Slot 4	…	Slot r

Therefore,

$$\begin{aligned}{}_nP_r &= n(n-1)(n-2)\cdots(n-r+1)\\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots1}{(n-r)(n-r-1)\cdots1} = \frac{n!}{(n-r)!}\end{aligned}$$

$$\boxed{{}_nP_r = \frac{n!}{(n-r)!}}$$

Example 7.16

There are 5 students named Anne, Ben, Carol, David, Eric. They have to sit in a row of 5 chairs. (Assume that Anne and Carol are girls and the rest are boys). In how many ways can they sit, if

(a) They can sit anywhere?
(b) The boys and the girls are each to sit together?
(c) The girls are to sit together?
(d) The girls cannot sit together?
(e) All the boys are to sit separately?

(120; 24; 48; 72; 12)

7.4.3 Number of *r*-Combinations (without repetition)

The number of r-combinations from a set with n distinct elements is denoted by ${}_nC_r$. It is read as "the combinations of n elements taken r at a time", "r-combination of n" or simply "n**-choose-**r".

<u>*Note:</u> Instead of ${}_nC_r$, some textbooks use the following symbols: ${}^nC_r$, C_r^n, $C_{n,r}$, $C(n,r)$ or $\begin{pmatrix} n \\ r \end{pmatrix}$.

Recall that **for combination, the order of selection does not matter**.

To determine ${}_nC_r$, we observe that for each r-combination, there are $r!$ of r-permutations (revisit Example 7.14). For instance, the 3-combination abc has six $(3!)$ of 3-permutations: abc, acb, bac, bca, cab, cba. Hence,

$$\boxed{{}_nP_r = {}_nC_r \times r!}$$

or

$$\boxed{{}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}}$$

Example 7.17

There is a need to form a committee of 7 persons from 6 men and 5 women. How many ways are there to form the committee, if

(a) There are no restrictions?
(b) The committee must consist of 4 men and 3 women?
(c) The committee must consist of at least 3 women?

(330; 150; 265)

7.4.4 Number of *r*-Permutations (with repetition)

Let us now remove the restriction we imposed earlier of not allowing any repetition of the elements in the set.

Example 7.18

Suppose $A = \{a, b, c, d\}$.

(a) List the 2-permutations of set A, if repetition is allowed.
(b) How many ways are there to select 2 elements from set A?

(16)

To count the number of r-permutations from a set of n elements with repetition allowed, we can use the multiplication principle to the counting problem of filling r slots with elements from a set containing n distinct elements. The first slot can be filled by any of the n elements of the set. Since repetitions are allowed, there are n choices for the second slot, n choices for the third slot and this continues until the last slot.

n	n	n	n	…	n
Slot 1	Slot 2	Slot 3	Slot 4	…	Slot r

In general, the number of r-permutations from a set of n elements with repetition allowed is n^r.

Example 7.19

An urn contains 9 balls numbered 1 to 9. If we sample 5 times from the urn, find the number of unique number sequences obtained, if the balls are taken

(a) without replacement.
(b) with replacement.

(15,120; 59,049)

7.4.5 Number of *r*-Permutations (with partial repetition)

Frequently, we want to know the number of r-permutations when some elements of set A are repeated a specified number of times.

Example 7.20

Find the number of ways to arrange the letters in the word

(a) COCOON

(b) PREPOSSESSED

(60; 1,663,200)

7.4.6 *Number of *r*-Combinations (with repetition)

This topic is outside the scope of this module, as it is rather complicated. You may want to Google about it if you want to find out more.

In general, if we want to find r-combinations from a set containing n elements, it can be done in $\binom{n+r-1}{r}$ ways.

Summary of Techniques

From n items, take r	Repetition not allowed	Repetition allowed	Partial Repetition
Order matters			
Order does not matter		*Outside the scope of this module	

Online Supplementary Resources

(1) Easy-to-understand introduction to permutation and combination:
https://www.mathsisfun.com/combinatorics/combinations-permutations.html

Tutorial 7A – Techniques of Counting (Addition, Inclusion-Exclusion, Multiplication Principle)

Section A (Basic)

1. Assuming one card is drawn, how many ways are there to draw each of the following cards from an ordinary deck of playing cards?
   (a) A heart or a club.
   (b) A heart or a king.
   (c) A card with a red suit or an ace.

2. When two dice are rolled, how many ways are there to get a sum that is
   (a) less than 6?
   (b) less than 6 or more than 10?

3. In a class of 45 students, 28 own smartphones, 23 own laptops, and 2 own neither. Find the number of students who own
   (a) both smartphones and laptops.
   (b) only smartphones.
   (c) only laptops.

4. Find the number of integers between 1 and 700 inclusive that are
   (a) divisible by 3 or 9.
   (b) divisible by neither 3 nor 9.
   (c) divisible by 3 but not 9.

5. A psychologist preparing three-letter words for use in a memory test chooses the first letter from among the consonants ***q***, ***w***, ***x*** and ***z***; the second letter from among the vowels ***e***, ***i*** and ***u***; and the third letter from among the consonants ***c***, ***f***, ***p*** and ***v***. How many different three-letter words can he construct?

6. In an English class, the students are given the choice of eight different essay topics. How many different ways can four students each choose a topic if
   (a) no student may choose the same topic?
   (b) there is no restriction on the choice of topic?

7. How many three-digit numbers can be formed using the digits 2, 4, 5, 6, 7 and 9 if
   (a) repetition is allowed?
   (b) repetition is not allowed?

8. How many different four-digit numbers can be formed with 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 if the last digit must be zero and repetitions are not allowed?

9. How many four-digit numbers, which are not divisible by 5, can be formed using the digits 2, 4, 5 and 7 if the digits are not repeated?

10. Paul's goal is to study a total of 6 hours in 4 consecutive days. On any given day, he can spend 0, 1 or 2 hours studying. With the aid of a tree diagram, find the number of ways he can achieve his goal.

11. A security system prompts you to create a password with the following restrictions: The password must contain 4-6 characters, in which only (small) letters, digits 0…9 and special characters # and $ can be used. How many different passwords can be created?

12. How many different ways can four people A, B, C and D form a queue if
    (a) there are no restrictions?
    (b) B must stand directly in front of A?
    (c) B must stand directly in front of A, and C must stand directly in front of D?
    (d) B must stand in front of A, and C must stand in front of D?
    (e) A must be in front of the queue?

Section B (Intermediate/Challenging)

13. Find the number of integers between 249 and 1000 inclusive that are
    (a) divisible by 2.
    (b) divisible by 3.
    (c) divisible by 4.
    (d) divisible by 6.
    (e) divisible by 2 and 3.
    (f) divisible by 2 or 3.
    (g) divisible by 2 and 4.
    (h) divisible by 2 or 4.
    (i) *not divisible by 2, 3 or 6.
    (j) *divisible by 4 or 6.

14. By drawing a Venn diagram of 3 sets or otherwise, where all 3 sets are not disjoint of one another, derive the formula for the Inclusion-Exclusion Principle for 3 sets.

15. *Find the number of integers between 1 and 1500 inclusive that are divisible by 2, 3 or 5.
    (Hint: Use the Inclusion-Exclusion Principle for 3 sets.)

16. Find the number of different 4-digit numbers that can be formed from the digits 3, 4, 6 and 8 with no repeated digit, and if the numbers must be
    (a) divisible by 2.
    (b) *divisible by 4.
    (c) *divisible by 3. (Hint: A number is divisible by 3 if the sum of all the digits is divisible by 3.)

17. The letters A, B, C, D, E and F are to be arranged in a straight line. How many ways are there of arranging these letters, if
    (a) there are no restrictions?
    (b) A and B are next to each other?
    (c) *A is before B?

Tutorial 7A – Answers

1. (a) 26 (b) 16 (c) 28
2. (a) 10 (b) 13
3. (a) 8 (b) 20 (c) 15
4. (a) 233 (b) 467 (c) 156
5. 48
6. (a) 1680 (b) 4096
7. (a) 216 (b) 120
8. 504
9. 18
10. 10
11. 3,092,256,688
12. (a) 24 (b) 6 (c) 2 (d) 6 (e) 6
13. (a) 376 (b) 251 (c) 188 (d) 125 (e) 125 (f) 502 (g) 188 (h) 376
    (i) 250 (j) 250
14. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
15. 1100
16. (a) 18 (b) 10 (c) 24
17. (a) 720 (b) 240 (c) 360

Tutorial 7B – Techniques of Counting (Permutations and Combinations)

Section A (Basic)

1. A jar contains 6 red, 3 blue and 1 green marbles. How many ways can a boy choose 2 red, 1 blue and no green marbles?

2. To fill a number of vacancies, the Human Resource manager of a company has to choose three secretaries from among ten applicants and two book-keepers from among five applicants. How many different ways can the manager fill the vacancies?

3. Two good batteries and one defective batteries are to be taken from a carton of 12 which contains 2 that are defective. How many different ways can these batteries be chosen?

4. Find the number of ways the letters in the following words can be arranged:
   (a) VICISSITUDES
   (b) SOCIOLOGICAL

5. How many signals, each is composed by 9 flags hung in a vertical line, can be formed from 3 identical yellow flags, 4 identical blue flags and 2 identical red flags?

6. In a particular examination, a student has to answer 6 out of 8 questions. How many ways can she answer the 6 questions, if
   (a) there are no restrictions?
   (b) the first 3 questions are compulsory?
   (c) she must answer at least 3 of the first 4 questions?

Section B (Intermediate/Challenging)

7. Suppose there are 40 students in a particular class. How many different ways are there to
   (a) choose four students to be class rep, assistant class rep, secretary and treasurer respectively?
   (b) choose a team of 4 students to represent the class in a coding competition?
   (c) *choose two teams of 2 students each to represent the class in a coding competition?

8. A group of 6 people consisting of 3 married couples are to be seated together in a straight row. How many different ways are there of seating the 6 people, if
   (a) there are no restrictions?
   (b) each couple has to sit together?
   (c) all the husbands are to sit together and all the wives are to sit together?
   (d) all the wives are to sit together?
   (e) *no husband is to sit next to another husband?

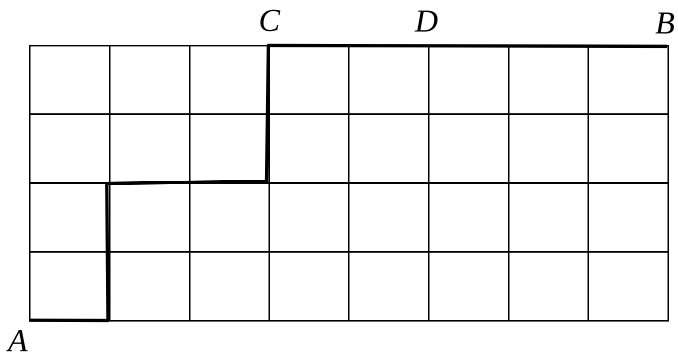
9. I have a group of 10 people and I need to select 6 of them to pose for a photograph. 2 of the 10 people are named Ken and Karen. How many ways can I arrange 6 people out of the group of 10 in a straight row for the photograph, if
   (a) Ken and Karen must be in the photograph?
   (b) Ken and Karen must be in the photograph, and they must appear next to each other in the photo?

10. A 4-digit number^ is to be formed with digits from the set $\{0,1,2,3,4,5,6,7,8,9\}$. How many ways are there to form this 4-digit number, if
    (a) there are no restrictions?
    (b) the number must be even?
    (c) *the number does not contain sequence "23"?
    (d) *the number contains at least two odd digits?
    (e) *the digits of the number strictly increase from left to right? (example: 1234, 0459, etc)
    ^Note: In this question, assume repetition is not allowed, and a number can start with 0. For example, 0123 is considered a 4-digit number, and not a 3-digit number.

11. **(1415S1/C3)** *How many ways can the letters of the word MINIMUM be arranged if
    (a) there are no restrictions?
    (b) all the 'M's are together?
    (c) the first letter is 'M' and the last letter is 'I'?
    (d) the consonants are separated by the vowels?

12. *Find the number of arrangements of the 7-letter word "*abcdefg*" that do **not** contain the sequence "*abc*", "*de*" or "*fg*". (Hint: Use the Inclusion-Exclusion Principle for 3 sets derived in Tut 7A Q14.)

13. **(1112S2/15)** *The figure below shows a 4×8 rectangular grid. We are required to move from the lower-left corner A to the upper-right corner B. However, we are restricted to move either rightwards or upwards only.



    (a) How many possible paths are there?
    (b) How many possible paths are there such that the **first** point reached **on the top edge** is at C?
    (c) How many possible paths are there such that the **first** point reached **on the top edge** is at either C or D?

Section C (MCQ)

14. **(1213S2/A4)** How many 5-digit numbers can be formed from the digits 1, 2, 3, 4, 5 and 6 if no digit can be used more than once?

(a) ${}^{6}P_{5}$ (b) ${}^{6}C_{5}$ (c) 6^{5} (d) $6!\times 5!$

15. **(1415S2/A5)** A sports club wants to send a team of 3 people to participate in a competition. The eligible members consist of x males and y females. How many ways can a team be formed that comprises of exactly 1 female?

(a) $\dfrac{x(x-1)(x-2)}{6}$ (b) $\dfrac{x(x-1)y}{2}$

(c) $\dfrac{y(y-1)(y-2)}{6}$ (d) $\dfrac{y(y-1)x}{2}$

<u>Tutorial 7B – Answers</u>

1. 45
2. 1200
3. 90
4. (a) 13,305,600 (b) 9,979,200
5. 1260
6. (a) 28 (b) 10 (c) 22
7. (a) 2,193,360 (b) 91,390 (c) 274,170
8. (a) 720 (b) 48 (c) 72 (d) 144 (e) 144
9. (a) 50,400 (b) 16,800
10. (a) 5040 (b) 2520 (c) 4872 (d) 3720 (e) 210
11. (a) 420 (b) 60 (c) 60 (d) 12
12. 3642
13. (a) 495 (b) 20 (c) 76
14. a 15. b