Chapter Eight: Probability

Learning Objectives:
By the end of the chapter, students should be able to:
1. Define basic terms used in probability theory such as experiment, sample space, event and probability.
2. State the elementary probability rules such as the addition rule, subtraction rule and multiplication rule.
3. Calculate the conditional probability of a given event.
4. State the properties of mutually exclusive events and independent events.
5. Apply the rules of probability to solve real-life problems.

Introduction

Many phenomena and happenings in our world are non-deterministic and random. We like to have answers to questions like, "Will it rain tomorrow?" or "Will there be an earthquake in a particular country we like to visit?" or "What are the chances of winning the lottery this week?" The one thing that is common in all these questions is that one cannot be certain of the outcomes in advance. To say that the outcome is uncertain, or non-deterministic, or random is not to say that the occurrence is completely haphazard. There is some sort of regularity within this randomness that can be observed after many tests or repetitions. Probability theory is a mathematical representation of randomness that helps us understand randomness and enables us to define measures of uncertainty or unpredictability.

8.1 Fundamentals

Let us consider a very simple random phenomenon like tossing a coin. What we are certain of is the outcome can be either a head or a tail, but what we are not certain of is that a head will appear.

We define some important terminologies here:

- **Experiment** is any process or action that generates data or observations;
- **Sample space** is the list of all possible outcomes of the experiment;
- **Event** is a subset of the sample space;
- **Probability** is the likelihood of the occurrence of the event with respect to its sample space

8.1.1 Experiment

Experiment is any process or action that generates data or observations. For example:

- Tossing a coin
- Tossing a die
- Tossing three dice
- Get 2 cards from a deck of poker cards, etc.

8.1.2 Sample Space

The **sample space** of an experiment is the set S of all possible outcomes of the experiment.

Example 8.1

Write down the sample space for each of the following experiment:

(a) Toss a coin once.

(b) Toss a coin three times.

(c) Toss two dice. (*Assume fair, six-sided dice are used)

8.1.3 Event

Very often, we are not interested in just a particular outcome but in a collection of outcomes that exhibits some general property. Such a collection of outcomes is called an **event** E, which in fact is a subset of the sample space S. Hence, $E \subset S$ and $|E| \leq |S|$.

Example 8.2

Consider an experiment of tossing a fair six-sided die once.

(a) List down the sample space S of the experiment.
(b) Write down the event A of getting an even number and event B of getting an odd number.
(c) Are events A and B subsets of the sample space S?
(d) Find the cardinality of A, B, S and verify that $|A| \leq |S|$ and $|B| \leq |S|$.

Mutually Exclusive Events

Two events A and B are said to be **mutually exclusive** if they cannot occur simultaneously. In this case, the two sets A and B are disjoint, having no outcomes in common, that is

$$A \cap B = \varnothing .$$

Example 8.3

Consider the experiment of tossing a coin three times. Let us define:

Event A is "getting 2 or more heads"
Event B is "getting 2 or more tails"

(a) List down the members of event A and B.
(b) Are they mutually exclusive? Why?

Note that the sample space S and the null set $\varnothing$ are themselves events, because $S \subset S$ and $\varnothing \subset S$.

- The sample space S is called the *certain* (or sure) event.
- The null set $\varnothing$ is called the *impossible* event.

8.1.4 Probability

Probability value measures the likelihood of occurrence of an event. We assign a notation $P(A)$ to be "the probability that event A occurs." There are three main approaches of assigning probabilities, namely the **classical approach**, the **relative frequency approach** and the **subjective approach**. You can Google them if you are interested. In this module, we will only deal with the classical approach.

The classical approach is the oldest way of measuring uncertainties that was developed originally in connection with the games of chance. This approach assumes that:

- The number of outcomes of an experiment is finite;
- All the outcomes are equally likely to occur;
- All events are pair-wise mutually exclusive, i.e. no two outcomes can occur at the same time.

The classical probability of an event A is defined as

$$P(A) = \frac{\text{number of desired outcomes}}{\text{total number of possible outcomes}} = \frac{|A|}{|S|}$$

Example 8.4

In an experiment of rolling a pair of dice, find the probability of each of the following events:

(a) A is the event that the sum of the numbers on the dice is 12.

(b) B is the event that the sum of the numbers on the dice is less than 7.

(c) C is the event that the two dice show the same number.

($\frac{1}{36}$; $\frac{5}{12}$; $\frac{1}{6}$)

8.2 Basic Rules of Probability (I)

The following are some basic rules of probability. Some of these are axioms, and some others are important consequences of the axioms:

- The probability of an event E is a non-negative real number, i.e. $P(E) \geq 0$.
- There are no events outside the sample space, i.e. $P(S) = 1$.
- The probability of an impossible event is 0, i.e. $P(\varnothing) = 0$.
- The probability of any event E assumes a value between 0 and 1, i.e. $0 \leq P(E) \leq 1$.

8.2.1 Special Addition Rule

If A and B are **mutually exclusive** events, i.e. $A \cap B = \varnothing$, then

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

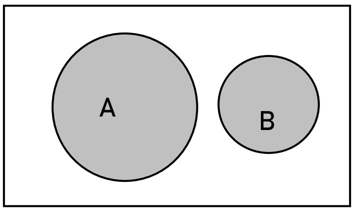

Figure 8.1 $P(A \cup B) = P(A) + P(B)$

8.2.2 General Addition Rule

If A and B are **not mutually exclusive** events, then A and B have some outcomes in common, or $A \cap B \neq \varnothing$.

When $P(A)$ and $P(B)$ are added together, the probability of the intersection, i.e. $P(A \cap B)$ is added twice. To compensate from this double addition, $P(A \cap B)$ needs to be subtracted from the sum:

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

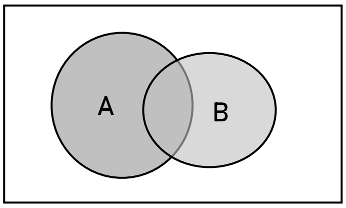

Figure 8.2
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example 8.5
In an experiment, a fair six-sided die is thrown. What is the probability that the number thrown is:
(a) ‘1’ or even?
(b) Even or divisible by 3?

($\frac{2}{3}$; $\frac{2}{3}$)

Example 8.6
The probability that a student passes Mathematics is $\frac{3}{4}$ and the probability that a student passes Java programming is $\frac{7}{8}$.
(a) If the probability of passing at least one of these two subjects is $\frac{15}{16}$, what is the probability of passing both subjects?
(b) Why is it impossible that these two events are mutually exclusive?

($\frac{11}{16}$)

Let M be the event that a student passes Mathematics. $P(M) =$

Let J be the even that a student passes Java programming. $P(J) =$

8.2.3 Subtraction Rule

If A is an event, then $\overline{A}$ is called the **complement** of A. It is the event that A does not occur, and consists of all the outcomes in S which are not in A.

Since A and $\overline{A}$ are mutually exclusive and $A \cup \overline{A} = S$, then

$$P\left(A \cup \overline{A}\right) = P(A) + P\left(\overline{A}\right) = P(S) = 1$$

Hence,

$$\boxed{P\left(\overline{A}\right) = 1 - P(A)}$$

Example 8.7

In an experiment, a fair coin is tossed six times in succession. What is the probability that:

(a) No heads occur?
(b) At least one head occurs?
(c) At least two heads occur?
(d) At least three heads occur?

($\frac{1}{64}$; $\frac{63}{64}$; $\frac{57}{64}$; $\frac{21}{32}$)

8.3 Conditional Probability

The probability of an event sometimes changes if additional information about another event is available.

The probability that event A occurs when it is known that some event B has occurred is called a **conditional probability** and is denoted by $P(A|B)$. The vertical line $|$ is read as "given" and so $P(A|B)$ is read as "the probability of A given B".

$$\boxed{P(A|B)=\frac{P(A\cap B)}{P(B)}}$$

Example 8.8

A six-sided die is rolled. You do not know what the outcome is, but somebody peeped and told you the outcome is an even number (assume she is telling the truth). What is the probability that the outcome is the number '4'?

($\frac{1}{3}$)

Let E be the event that the outcome is even.
Let F be the event that the outcome is the number '4'.

Example 8.9

A coin is tossed twice.

(a) Write down the sample space of this experiment.
(b) Find the probability of obtaining:
  (i) Two heads.
  (ii) Two heads, given that at least one of the outcome is a head.
  (iii) Two heads, given that the first outcome is a head.

($\frac{1}{4}$; $\frac{1}{3}$; $\frac{1}{2}$)

Note that when we talk about the conditional probability $P(A|B)$, it is always assumed that B is not an impossible event, i.e. $P(B)>0$. If B is an impossible event, it is senseless to talk about the probability that event A occurs **given** that event B has occurred.

Similarly, the conditional probability of B given A (provided $P(A)>0$) is given by

$$\boxed{P(B|A)=\frac{P(A\cap B)}{P(A)}}$$

The main idea behind conditional probability is that **the additional information given caused the sample space to be reduced**. In turns, the probability of each outcome is proportionately modified so that the sum of the probabilities of all the outcomes in the reduced space equals to one.

Example 8.10

A couple has two children. Assume that the probability of having a boy $P(B)=0.4$ and the probability of having a girl $P(G)=0.6$. Find the probability that:

(a) The couple has two boys.
(b) The couple has two boys, given that at least one of the children is a boy.
(c) The couple has two boys, given that the first child is a boy.

(0.16; 0.25; 0.4)

The sample space is $S=\{\quad,\quad,\quad,\quad\}$

8.4 Basic Rules of Probability (II)

8.4.1 General Multiplication Rule

An immediate consequence of the definition of conditional probability is the general multiplication rule of probability. In the previous section, we have seen that the following two formulas are true for conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Rearranging the two formulas, we have:

$$\boxed{\begin{aligned} &P(A \cap B) = \\ &P(A \cap B) = \end{aligned}}$$

These are called the **general multiplication rule**.

8.4.2 Special Multiplication Rule

Independent Events

Two events A and B are said to be **independent**, if and only if the probability of event B occurring is not affected by whether A has or has not occurred. In other words, event B is independent of A if the probability of B equals the conditional probability of B given A, and vice versa:

$$P(B) = P(B|A)$$

$$P(A) = P(A|B)$$

Substituting this into the general multiplication rule, the probability that both events A and B occur is:

$$\boxed{P(A \cap B) = }$$

This is called the **special multiplication rule**.

Important:

Note that this formula can only be used if events A and B are known to be independent.
If you are not sure whether the events are independent, the general multiplication rule should be used.

Example 8.11
Two cards are drawn at random from a well-shuffled deck of 52 standard playing cards. Find the probability that they are both kings, if the first card is:
(a) replaced.
(b) not replaced.

($\frac{1}{169}$; $\frac{1}{221}$)

Let K_1 be the event that the first draw is a king, and
let K_2 be the event that the second draw is a king.

(a) If the first card is replaced:

$P(K_1) =$ ________

$P(K_2|K_1) =$ ________ ; $P(K_2|\overline{K_1}) =$ ________ ; $P(K_2) =$ ________

Since we know that $P(K_2|K_1) = P(K_2)$, events K_1 and K_2 are **independent** events.
Hence, the special multiplication rule can be applied:

$$\begin{aligned} P(K_1 \cap K_2) &= P(K_1) \times P(K_2|K_1) \\ &= P(K_1) \times P(K_2) \\ &= \end{aligned}$$

(b) If the first card is not replaced:

$P(K_1) =$ ________

$P(K_2|K_1) =$ ________ ; $P(K_2|\overline{K_1}) =$ ________

We can easily see that events K_1 and K_2 are not independent events in this case, as the conditional probability of K_2 will change depending on whether K_1 has or has not happened.
Hence, the general multiplication rule must be used:

$$P(K_1 \cap K_2) = P(K_1) \times P(K_2|K_1) =$$

Conditions for Independence

Two events A and B are independent if and only if:

- $P(B) = P(B|A)$
- $P(A) = P(A|B)$
- $P(A \cap B) = P(A) \times P(B)$

Example 8.12

Out of the 300 students who took the English and Mathematics tests,

30 failed English but not Mathematics,
50 failed Mathematics but not English, and
10 failed both English and Mathematics.

Suppose a student is selected randomly, and E is the event that a student failed English and M is the event that a student failed Mathematics.

(a) What is the probability that a student passes both English and Mathematics?
(b) Are events E and M mutually exclusive? Why?
(c) Are events E and M independent? Why?

(0.7; No; No)

Example 8.13

A class consists of 10 girls and 30 boys altogether. It is known that 50% of the girls and 80% of the boys in the class play table tennis.

(a) If a student is randomly selected from the class, what is the probability that the student selected plays table tennis?

(b) What is the probability that the student selected is a boy, given that

  (i) the student selected plays table tennis?

  (ii) the student selected does not play table tennis?

($\frac{29}{40}$; $\frac{24}{29}$; $\frac{6}{11}$)

Tutorial 8 – Probability

Section A (Basic)

1. In an experiment, two fair six-sided dice are rolled.
   (a) Write down the sample space S of this experiment.
   (b) Find the probabilities of the following events.
      (i) A, the event that the first die shows a 2
      (ii) B, the event of rolling a double, i.e. both dice show the same number
      (iii) C, the event that the sum is even
      (iv) D, the event that the sum is odd
      (v) E, the event that the sum is 11
   (c) Referring to your answers in part (b), find the probabilities of the following events.
      (i) The sum is not 11
      (ii) Rolling doubles or the sum is odd
      (iii) The sum is even or the sum is odd
      (iv) The sum is odd and the first die is 2
      (v) The sum is even and the first die is 2
      (vi) Rolling doubles and the sum is even and the first die is 2
      (vii) The sum is 11, given that the sum is odd
      (viii) The first die is 2, given that the sum is even
   (d) Referring to your answers in part (b), answer the following questions:
      (i) Are events B and D mutually exclusive?
      (ii) Are events A and D independent?
      (iii) What is the relationship between events A and C?

2. A box contains 100 items, out of which 27 are oversized and 16 are undersized. An item is taken from the box, tested and **replaced**. A second item is then treated in a similar manner. What is the probability that
   (a) both items are acceptable?
   (b) the first is oversized and the second undersized?
   (c) one is oversized and the other is undersized?

3. Jack and Jill each have a standard deck of 52 playing cards. Each of them flip over a randomly selected card.
   (a) If they are only interested in the suits of the cards, what is the sample space of this experiment?
   (b) What is the probability that
      (i) the cards are of the same suit?
      (ii) at least one card is a heart?
      (iii) neither card is a spade?
      (iv) neither card is a spade nor a club?

4. Suppose $P(A)=0.7$, $P(\overline{B})=0.4$ and $P(A\cup B)=0.8$. Find the following:
   (a) $P(B)$
   (b) $P(A\cap B)$

5. A fair six-sided die is rolled three times in succession. What is the probability that
   (a) the outcome ‘6’ occurs zero times?
   (b) the outcome ‘6’ occurs at least once?
   (c) the outcome ‘6’ occurs three times?
   (d) the outcome ‘6’ occurs at most twice?

6. Two cards are drawn one after another without replacement from a standard deck of 52 playing cards. Suppose A is the event that the first card drawn is an ace and B is the event that the second card drawn is an ace. Find $P(A)$, $P(B|A)$, $P(A\cap B)$ and $P(B)$.

7. Suppose A and B are two events from a sample space S such that $P(A)=\frac{1}{2}$, $P(B)=\frac{1}{3}$ and $P(A\cap B)=\frac{1}{4}$. Find the following probabilities.

   (a) $P(A\cup B)$
   (b) $P(A|B)$
   (c) $P(B|A)$
   (d) $P(\overline{A}\cap\overline{B})$
   (e) $P(\overline{A}|\overline{B})$
   (f) $P(\overline{B}|\overline{A})$

8. In a finite sample space, the random events A, B and C are related as follows:
   - Events A and B are mutually exclusive
   - Events A and C are independent
   - $P(A)=\frac{1}{5}$, $P(B)=\frac{1}{10}$, $P(A\cup C)=\frac{7}{15}$ and $P(B\cup C)=\frac{23}{60}$

   Find the following:
   (a) $P(A\cup B)$
   (b) $P(A\cap C)$
   (c) $P(B\cap C)$
   (d) Are events B and C independent?

9. A man is dealt four spade cards from a standard deck of 52 playing cards. If he is given another three additional cards, what is the probability that
   (a) at least one of the additional cards is a spade?
   (b) at most two of the additional cards are spades?

10. In a certain class, 75% of the students own a tablet, 85% own a handphone, and 10% own neither. A student is selected at random from this class.
    (a) What is the probability that this student owns a tablet or a handphone?
    (b) What is the probability that this student owns a tablet and a handphone?
    (c) If this student owns a tablet, what is the probability that he also owns a handphone?
    (d) If this student owns a tablet, what is the probability that he does not own a handphone?
    (e) If this student does not own a tablet, what is the probability that he owns a handphone?

Section B (Intermediate/Challenging)

11. At each of the five defense stations, the probability of striking an attacking airplane is 0.1. If an airplane has to pass all five defense stations before arriving at its target, what is the probability that
    (a) the airplane gets shot down at the third defense station?
    (b) the airplane gets shot down before it reaches its target?

12. There are two boxes. Box I contains 3 white balls and 2 black balls, while Box II contains 2 white balls and 4 black balls. A fair six-sided die is rolled. If the outcome is a '6', then a ball is randomly selected from Box I. If the outcome is not a '6', then a ball is randomly selected from Box II.
    (a) What is the probability that a white ball is selected?
    (b) If a white ball is selected, what is the probability that it came from Box I?

13. A man is dealt 5 cards at random from a standard deck of 52 playing cards.
    (a) Find the probability that the 5 cards dealt are all hearts.
    (b) Find the probability of being dealt 4 aces.
    (c) Find the probability of being dealt 'four of a kind' (example: 3-3-3-3-A, J-J-J-J-2, etc.).

14. *Bayes' Theorem states that for any two events A and B,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}, \text{ provided } P(B) \neq 0.$$

    Bayes' Theorem has applications in Bayesian spam filtering – a statistical technique used to filter out spam emails based on certain suspicious words detected.

    Given the following information:
    - 95% of all emails that contain a suspicious word are spam
    - 90% of all emails that do **not** contain a suspicious word are **not** spam
    - 40% of all emails contain a suspicious word

    (a) What is the probability that an email is spam?
    (b) What is the probability that an email contains a suspicious word, given that
        (i) the email is spam?
        (ii) the email is **not** spam?

15. **(1516S2/C3a)** A machine is used to generate codes consisting of 4 random characters. A character can be either one of the 26 letters of the alphabet 'a' to 'z' (lower-case), 'A' to 'Z' (upper-case), or one of the 10 digits '0' to '9'. Each of the 62 characters is equally likely to be generated, and each character can be generated more than once.

    What is the probability that a randomly generated code has
    (a) no digits?
    (b) no repeated characters?
    (c) the first two characters consisting of two digits?
    (d) *the first two characters consisting of two digits and the first digit is larger than the second digit?
    (e) *two different letters^ and two different digits?
    ^Note: Upper-case and lower-case are considered as different letters.

Section C (MCQ)

16. If A, B and C are three events such that $P(A)=0.5$, $P(A|B)=1$ and $P(A|C)=0.5$, which of the following statements is **false**?
    (a) Events A and B are mutually exclusive
    (b) Event A is not independent of event B
    (c) Event A is independent of event C
    (d) $P(A\cap B)=P(B)$

17. **(1213S1/A4)** From an experiment of tossing a fair coin 10 times. i.e. $P(\text{head})=P(\text{tail})=0.5$, which of the following statements is true?
    (a) $P(10\text{ heads})=P(5\text{ heads and }5\text{ tails})$
    (b) $P(10\text{ heads})>P(5\text{ heads and }5\text{ tails})$
    (c) $P(10\text{ heads})<P(5\text{ heads and }5\text{ tails})$
    (d) The relationship between $P(10\text{ heads})$ and $P(5\text{ heads and }5\text{ tails})$ cannot be determined, as it varies with time.

18. **(1314S2/A5)** Two fair dice, each with 6 sides, are tossed simultaneously. Out of the following events, which one is the **most likely** to happen?
    (a) Exactly one of the two dice is even
    (b) At least one of the two dice is even
    (c) None of the two dice is even
    (d) The sum of the two dice is even

<u>Tutorial 8 – Answers</u>

1. (a) $S=\left\{\begin{matrix}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\end{matrix}\right\}$

   (b) (i) $\frac{1}{6}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{2}$ (iv) $\frac{1}{2}$ (v) $\frac{1}{18}$

   (c) (i) $\frac{17}{18}$ (ii) $\frac{2}{3}$ (iii) 1 (iv) $\frac{1}{12}$ (v) $\frac{1}{12}$ (vi) $\frac{1}{36}$
   (vii) $\frac{1}{9}$ (viii) $\frac{1}{6}$

   (d) (i) Yes (ii) Yes (iii) Events A and C are independent

2. (a) 0.3249 (b) 0.0432 (c) 0.0864

3. (a) $S = \left\{ \begin{array}{l} (\clubsuit,\clubsuit),(\clubsuit,\heartsuit),(\clubsuit,\diamondsuit),(\clubsuit,\spadesuit), \\ (\heartsuit,\clubsuit),(\heartsuit,\heartsuit),(\heartsuit,\diamondsuit),(\heartsuit,\spadesuit), \\ (\diamondsuit,\clubsuit),(\diamondsuit,\heartsuit),(\diamondsuit,\diamondsuit),(\diamondsuit,\spadesuit), \\ (\spadesuit,\clubsuit),(\spadesuit,\heartsuit),(\spadesuit,\diamondsuit),(\spadesuit,\spadesuit) \end{array} \right\}$

(b) (i) $\frac{1}{4}$ (ii) $\frac{7}{16}$ (iii) $\frac{9}{16}$ (iv) $\frac{1}{4}$

4. (a) 0.6 (b) 0.5
5. (a) $\frac{125}{216}$ (b) $\frac{91}{216}$ (c) $\frac{1}{216}$ (d) $\frac{215}{216}$
6. $\frac{1}{13}$; $\frac{1}{17}$; $\frac{1}{221}$; $\frac{1}{13}$
7. (a) $\frac{7}{12}$ (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) $\frac{5}{12}$ (e) $\frac{5}{8}$ (f) $\frac{5}{6}$
8. (a) $\frac{3}{10}$ (b) $\frac{1}{15}$ (c) $\frac{1}{20}$ (d) Not independent
9. (a) $\frac{8157}{17296}$ (b) $\frac{4303}{4324}$
10. (a) $\frac{9}{10}$ (b) $\frac{7}{10}$ (c) $\frac{14}{15}$ (d) $\frac{1}{15}$ (e) $\frac{3}{5}$
11. (a) 0.081 (b) 0.4095
12. (a) $\frac{17}{45}$ (b) $\frac{9}{34}$
13. (a) $\frac{33}{66640}$ (b) $\frac{1}{54145}$ (c) $\frac{1}{4165}$
14. (a) 0.44 (b) (i) 0.8636 (ii) 0.0357
15. (a) 0.4948 (b) 0.9061 (c) 0.0260 (d) 0.0117 (e) 0.0969
16. a 17. c 18. b