Chapter Six: Boolean Algebra

Learning Objectives:
By the end of the chapter, students should be able to:
1. List the Boolean values and operators.
2. Determine the dual of a Boolean expression using the duality principle.
3. Simplify Boolean expressions using the laws of Boolean algebra.
4. Derive Boolean expressions from truth tables using the sum-of-products method and/or the product-of-sums method.
5. Apply knowledge of Boolean algebra in simple digital systems contexts.

Introduction

The idea on “Boolean algebra” started with the investigation of the laws of thought. The inventor, George Boole constructed a “logical algebra”, which is then named “Boolean algebra”. His work received little attention for many years, until an American mathematician Claude E. Shannon made a major application of Boolean algebra to switching circuits in the 1930s. Since then, Boolean algebra became an indispensable tool for the analysis and design of modern computers. It has been fundamental in the development of digital electronics, and is provided for in all programming languages. It is also used in set theory and statistics.

In this chapter, we shall discuss the propositional logic, Boolean operators, truth tables, the laws of Boolean algebra and the methods of representing and simplifying logic expressions. We shall also show briefly how Boolean algebra applies to the operation of computer devices.

6.1 Boolean Algebra

6.1.1 Definition

In Boolean algebra, a **Boolean variable** is any variable that assumes values only from the set $\mathsf{B} = \{0,1\}$ (in contrast with the elementary algebra where the values of the variables are numbers from the set of real numbers $\mathbb{R}$). The 0 and 1 correspond to the truth values ________ and _________ respectively. Immediately, we see that Boolean variables are equivalent to propositions in logic.

Instead of elementary algebra where the main operations are addition, subtraction, multiplication and division, the main operations of Boolean algebra are as follows:

- **Disjunction (OR)**, denoted by $\vee$ in propositional logic, is now denoted by $+$ (**logical sum**)
- **Conjunction (AND)**, denoted by $\wedge$ in propositional logic, is now denoted by $\bullet$ (**logical product**)
- **Negation (NOT)**, denoted by $\neg$ in propositional logic, is now denoted by a unary operator $\overline{}$ (**complement**)

$$\vee \leftrightarrow +$$
$$\wedge \leftrightarrow \bullet$$
$$\neg \leftrightarrow \overline{}$$

Example 6.1
Rewrite the following logical expressions using Boolean algebra:

(a) $p \vee q \equiv q \vee p \quad \leftrightarrow$

(b) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r \quad \leftrightarrow$

(c) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \quad \leftrightarrow$

(d) $\neg(p \vee q) \equiv \neg p \wedge \neg q \quad \leftrightarrow$

The rules of precedence for Boolean operators, in order of precedence, are:

- Complement ($\overline{}$)
- Logical products ($\bullet$)
- Logical sums ($+$)

Example 6.2
Find the truth values of the following expressions:

(a) $1 \bullet 0 + \overline{1 + 0} =$

(b) $\overline{\overline{1+0} + \overline{0} \bullet 1} =$

6.1.2 Basic Laws of Boolean Algebra

Now, having defined the Boolean variables and having introduced the Boolean operators, it is time to look at the laws for Boolean algebra.

In the following laws, x, y and z are Boolean variables.
(Recall: what does it mean by Boolean variables?)

Name	**Identity**
Commutative Laws	$x \bullet y = y \bullet x$ $x + y = y + x$
Associative Laws	$x \bullet (y \bullet z) = (x \bullet y) \bullet z = x \bullet y \bullet z$ $x + (y + z) = (x + y) + z = x + y + z$
Distributive Laws	$x \bullet (y + z) = (x \bullet y) + (x \bullet z)$ $x + (y \bullet z) = (x + y) \bullet (x + z)$
Identity Laws	$x \bullet 1 = x$ $x + 0 = x$
Complement Laws	$x \bullet \overline{x} = 0$ $x + \overline{x} = 1$
Involution Law	$\overline{\overline{x}} = x$
Idempotent Laws	$x \bullet x = x$ $x + x = x$
Bound Laws	$x \bullet 0 = 0$ $x + 1 = 1$
De Morgan's Laws	$\overline{x \bullet y} = \overline{x} + \overline{y}$ $\overline{x + y} = \overline{x} \bullet \overline{y}$
Absorption Laws	$x \bullet (x + y) = x$ $x + (x \bullet y) = x$ $x \bullet (\overline{x} + y) = x \bullet y$ $x + (\overline{x} \bullet y) = x + y$

Each of the laws can be verified using a truth table or by the use of other laws.

Proof of the Laws:
(Use this page to prove all the Boolean algebra laws stated in the previous page.)

6.1.3 The Duality Principle

It should be obvious from the list of laws that the laws appear in pairs. Either one can be obtained from the other by interchanging the operations of $+$ and $\bullet$, as well as the elements 0 and 1. This is known as the **duality principle**.

$$+ \leftrightarrow \bullet$$
$$1 \leftrightarrow 0$$

Example 6.3

(a) Look at the distributive law. State the dual of $x \bullet (y+z) = (x \bullet y) + (x \bullet z)$.

(b) Look at the identity law. State the dual of $x \bullet 1 = x$.

The duality principle states that **if a Boolean expression is true, then its dual is also true**.

Example 6.4

Give the dual for the following identities and verify them:

(a) $x \bullet (\overline{x} + y) = x \bullet y$

(b) $\overline{x} + x \bullet y = \overline{x} + y$

6.2 Simplification of Boolean Expressions

A Boolean expression may be a mathematical expression of a logic circuit. The laws of Boolean algebra can be used to change a Boolean expression into its simpler form. The reason for performing such simplification is **to find another equivalent one, in which the logic circuit can be constructed more easily and economically**. In fact, this is the reason why modern calculators (and computers) can be priced so economically, despite the fact that it can perform a wide range of functions.

Let us now look at some examples that demonstrate the simplification of Boolean expressions.

Example 6.5
Simplify the following Boolean expressions:

(a) $x\overline{y}z + x\overline{y}\,\overline{z}$

(b) $\left(\overline{x} + y\right)\left(x + y\right)$

(c) $x + y\left(x + y\right) + x\left(\overline{x} + y\right)$

Example 6.6

Simplify the following Boolean expressions:

(a) $rsp + rs\bar{p}q$

(b) $\overline{\bar{x} + \overline{yz} + \bar{z}y}$

(c) $x + \overline{\bar{y}\bar{z} + \bar{y}\bar{u}} + \bar{x}u$

6.3 Boolean Functions

Let us recall how we construct truth table, given a Boolean expression $f(x, y)$.

Example 6.7

Construct the truth table for the Boolean function $f(x, y) = x + \overline{y}$, where x and y are Boolean variables.

x	y	$\overline{y}$	$x+\overline{y}$	$f(x,y)$
0	0			
0	1			
1	0			
1	1			

In the previous example, the truth table of the Boolean function was constructed knowing the Boolean expression. But, in practical application, most of the time the designer of a digital circuit often knows the truth table describing what the circuit should do, and the task is largely to determine what type of circuit will perform the function described in the truth table. So the goal of this section is the reverse process: **given the truth table of a Boolean function, determine the Boolean expression**.

Example 6.8

To save energy, you decide to design a logic circuit to control a smart lighting system inside your room. The smart lighting should dim the light at all times, except when it is night time and there is a person (or a few people) inside your room.

- Let Boolean variable x represent the input from a clock. When it is day time, $x = 1$ and when it is night time, $x = 0$.
- Let Boolean variable y represent the input from a presence sensor. When presence is detected, $y = 1$ and when presence is not detected, $y = 0$.
- Let Boolean output $f(x, y)$ represent the lighting control. When $f(x, y) = 1$, the lighting is dimmed and when $f(x, y) = 0$ the lighting is not dimmed.

Construct the truth table for the logic circuit, and determine its Boolean expression.

x	y	$f(x,y)$

$f(x, y) =$

There are two methods to obtain a Boolean expression from the truth table, namely the sum-of-products method (SOP) and the product-of-sums method (POS).

6.3.1 The Sum-of-Products Method (SOP)

The following are the steps for the sum-of-products method:

1. Examine the truth table for any rows where the output is ‘1’.
2. Write a Boolean product term that would equal a value of ‘1’ given those input conditions.
3. Join these Boolean product expressions together by addition, to create a single Boolean expression describing the truth table as a whole.
4. Simplify the obtained expression using the laws of Boolean algebra, if necessary.

Example 6.9

Given the following truth table, determine the Boolean expression $f(x,y)$ by the sum-of-products method.

($f(x,y)=x+\overline{y}$)

x	y	$f(x,y)$
0	0	1
0	1	0
1	0	1
1	1	1

Example 6.10

Given the following truth table, determine the Boolean expression $f(x,y,z)$ by the sum-of-products method.

($f(x,y,z)=\bar{x}z+yz+x\bar{y}\bar{z}$)

x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

6.3.2 The Product-of-Sums Method (POS)

The following are the steps for the product-of-sums method:

1. Examine the truth table for any rows where the output is ‘0’.
2. Write a Boolean sum term that would equal a value of ‘0’ given those input conditions.
3. Join these Boolean sum expressions together by multiplication, to create a single Boolean expression describing the truth table as a whole.
4. Simplify the obtained expression using the laws of Boolean algebra, if necessary.

Example 6.11

Given the following truth table, determine the Boolean expression $f(x, y)$ by the product-of-sums method.

($f(x, y) = x + \overline{y}$)

x	y	$f(x, y)$
0	0	1
0	1	0
1	0	1
1	1	1

Example 6.12

Given the following truth table, determine the Boolean expression $f(x, y, z)$ by the product-of-sums method.

($f(x, y, z) = \overline{x}z + yz + x\overline{y}\overline{z}$)

x	y	z	$f(x, y, z)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Tutorial 6 – Boolean Algebra

Section A (Basic)

1. Suppose $p=q=r=1$ and $x=y=z=0$, find the value for the following Boolean expressions:
   (a) $\overline{p}+q$
   (b) $\overline{q}+x$
   (c) $px+qy$
   (d) $\overline{pq}+xy$
   (e) $\overline{x\overline{y}}+q\overline{r}$
   (f) $(p+q+r)\ \overline{p+(q+r)}$

2. Construct the truth tables for the following Boolean expressions:
   (a) $xyz+\overline{x}\,\overline{y}\,\overline{z}$
   (b) $pqr+p\overline{q}\,\overline{r}+\overline{p}\,\overline{q}r$

3. By laws of Boolean algebra, prove the following identities:
   (a) $\overline{x}y+x\overline{y}=\overline{xy}(x+y)$
   (b) $\overline{p}\,\overline{q}+p\overline{q}+pq=p+\overline{q}$
   (c) $(\overline{a}\,\overline{b}+c)(a+b)(\overline{\overline{b}+ac})=\overline{a}bc$

4. Write down the **dual** for the following Boolean equalities:
   (a) $x\bullet 1+(0+\overline{x})=1$
   (b) $x+\overline{(\overline{y}+x)y}=1$
   (c) $\overline{y}(\overline{x}+\overline{y})=\overline{y}$
   (d) $(\overline{x}+y)(x+\overline{y})=\overline{x}\,\overline{y}+xy$

5. Simplify the following Boolean expressions:
   (a) $ac+ab\overline{c}+a$
   (b) $a\overline{b}\,\overline{c}+ab\overline{c}+\overline{a}\,\overline{c}$
   (c) $\overline{\overline{x(x+y)}+\overline{y+yz}}$
   (d) $xy(\overline{x}y\overline{z}+x\overline{y}\,\overline{z}+\overline{x}\,\overline{y}\,\overline{z})$
   (e) $qmnp+qm\overline{p}+q\overline{m}n+\overline{q}m\overline{p}+qn+q\overline{m}np$

6. Simplify the following Boolean expressions:
   (a) $p\overline{q}+pq+\overline{p}$
   (b) $\overline{p\overline{q}+pq}+\overline{p}$
   (c) $p\overline{q}+\overline{pq+\overline{p}}$
   (d) $\overline{p\overline{q}+pq+\overline{p}}$
   (e) $pq+p+p\overline{q}r+\overline{pqr+p(q+r)}$

7. By using the **sum-of-products** method, derive the Boolean expression for a logic network that will have an output of 1 when

$$x=1,\ y=0,\ z=0;$$
$$x=1,\ y=1,\ z=0;$$
$$x=1,\ y=1,\ z=1.$$

The circuit will have an output of 0 for all other sets of input values. Simplify the expression derived.

8. By using the **product-of-sums** method, derive the Boolean expression for a logic network that will have an output of 0 when

$$x=1,\ y=0,\ z=0;$$
$$x=1,\ y=1,\ z=0;$$
$$x=1,\ y=1,\ z=1.$$

The circuit will have an output of 1 for all other sets of input values. Simplify the expression derived.

Section B (Intermediate/Challenging)

9. Find the **complement** of the following expressions and simplify your results:
   (a) $x\,\overline{y}\,\overline{z}+\overline{x}\,\overline{y}\,z$
   (b) $*\left(y\,\overline{z}+\overline{x}\,w\right)\left(x\,\overline{y}+\overline{w}\,z\right)$

10. **(1314S1/C2)** Let Boolean function f be defined as

$$f\left(x,y,z\right)=x\,y+\overline{y}\,z\,.$$

   (a) Find the dual of f.
   (b) With the help of a truth table, determine the **sum-of-products** expression of f.
   (c) *Rewrite f using the OR $(+)$ and NOT $\left(\overline{}\right)$ operators only.

11. *It is given that $a\oplus b$ means "a XOR b".
   (a) Express $a\oplus b$ using the NOT $\left(\overline{}\right)$, AND $(\bullet)$ and OR $(+)$ operators only.
   (Hint: You can use either the SOP or POS method.)
   (b) Hence, or otherwise, prove the identity $\overline{a\oplus b}=\overline{a}\oplus b$.

12. *By using laws of Boolean algebra, prove the identity $x\,\overline{y}+\overline{y}\,z+\overline{x}\,z=x\,\overline{y}+\overline{x}\,z$.

13. You are tasked to design a smart system to prevent unauthorized access to a high-security laboratory. The design is such that the door of the laboratory will only open under the following circumstances:
    - The scanned thumbprint matches any of those authorized to enter, **or**
    - The scanned access card matches any of those authorized to enter **and** the password entered matches that of the scanned access card.

    In all other circumstances, the door will not open.

    The Boolean variables are defined as follows:

Inputs	x = scanned thumbprint (1 = matches, 0 = does not match) y = scanned access card (1 = matches, 0 = does not match) z = password entered (1 = matches, 0 = does not match)
Output	$f(x,y,z)$ = status of door (1 = open, 0 = closed)

    a) Construct a truth table for the design of the smart system.
    b) Use an appropriate method to find the simplified Boolean expression for the smart system.

14. **(1213S1/C2)** A logic circuit for three switches is to be designed to control a programmable light.
    - Let Boolean variables x, y and z represent the three switches. When the Boolean variable is 1, the switch is closed. When the Boolean variable is 0, the switch is open.
    - Let Boolean function $f(x,y,z)$ represent the light. When $f(x,y,z)=1$, the light is turned on. When $f(x,y,z)=0$, the light is turned off.

    The circuit is to be designed such that the light is turned on if switches x and y are both closed, or if switch x is open and switch z is closed. Otherwise, the light is turned off.

    (a) Construct a truth table for the logic circuit.
    (b) *By using the **product-of-sums** method, show that the simplified expression of the Boolean function $f(x,y,z)$ for the logic circuit is given by

$$f(x,y,z) = xy + \overline{x}z .$$

Section C (MCQ)

15. Let p, q and r be Boolean variables.
    Which of the following Boolean expressions is a **complement** of $p \bullet q + r$?
    (a) $p + q \bullet r$ (c) $\overline{p} + \overline{q} \bullet \overline{r}$
    (b) $(p+q) \bullet r$ (d) $(\overline{p} + \overline{q}) \bullet \overline{r}$

16. Let p, q and r be Boolean variables.
    Which of the following Boolean expressions is the **dual** of $p \bullet q + r$?
    (a) $p + q \bullet r$ (c) $\overline{p} + \overline{q} \bullet \overline{r}$
    (b) $(p+q) \bullet r$ (d) $(\overline{p} + \overline{q}) \bullet \overline{r}$

17. **(0809S2/A6)** The dual of the Boolean identity $(x+y)(y+z) = xz + y$ is
    (a) $(x \cdot y)(y \cdot z) = x + z \cdot y$ (c) $(x \cdot y) + (y \cdot z) = (x+z)y$
    (b) $(\overline{x} \cdot \overline{y})(\overline{y} \cdot \overline{z}) = \overline{x} + \overline{z} \cdot \overline{y}$ (d) $(x \cdot y) + (y \cdot z) = x + z \cdot y$

18. **(0910S1/A6)** The Boolean expression $x \cdot x + \bar{x} \cdot y$ can be simplified to

(a) $\bar{x} + \bar{y}$
(b) $x + y$
(c) $\bar{x} + y$
(d) $x + \bar{y}$

19. **(1011S1/A3)** Which one of the following Boolean expression is **not** equivalent to $\bar{x}y + \bar{x}yz$?

(a) $\bar{x}y$
(b) $1 + z$
(c) $\bar{x}y(1+z)$
(d) $\bar{x}yz + \bar{x}y\bar{z}$

20. **(1415S2/A4)** Given that A, B, C, D, E, F are all Boolean variables, the Boolean expression $A + AB + ABC + ABCD + ABCDE + ABCDEF$ can be simplified as:

(a) A
(b) $ABCDEF$
(c) 1
(d) $A + B + C + D + E + F$

<u>Tutorial 6 – Answers</u>

1. (a) 1 (b) 0 (c) 0 (d) 0 (e) 1 (f) 0
2. (a)

x	y	z	$xyz + \bar{x}\bar{y}\bar{z}$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

(b)

p	q	r	$pqr + p\bar{q}\bar{r} + \bar{p}\bar{q}r$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

3. (Proving)
4. (a) $(x+0)\bullet(1\bullet\bar{x}) = 0$

(b) $x \bullet \overline{(\bar{y}\bullet x) + y} = 0$

(c) $\bar{y} + (\bar{x}\bullet\bar{y}) = \bar{y}$

(d) $(\bar{x}\bullet y) + (x\bullet\bar{y}) = (\bar{x}+\bar{y})\bullet(x+y)$

5. (a) a (b) $\bar{c}$ (c) xy (d) 0 (e) $qn + m\bar{p}$
6. (a) 1 (b) $\bar{p}$ (c) $p\bar{q}$ (d) 0 (e) 1
7. $x\bar{z} + xy$
8. $\bar{x} + \bar{y}z$
9. (a) $y + \bar{x}\bar{z} + xz$ (b) 1

10. (a) $(x+y)(\overline{y}+z)$

(b)

x	y	z	$\overline{y}$	xy	$\overline{y}z$	$xy+\overline{y}z$	Products
0	0	0	1	0	0	0	
0	0	1	1	0	1	1	$\overline{x}\,\overline{y}\,z$
0	1	0	0	0	0	0	
0	1	1	0	0	0	0	
1	0	0	1	0	0	0	
1	0	1	1	0	1	1	$x\overline{y}z$
1	1	0	0	1	0	1	$xy\overline{z}$
1	1	1	0	1	0	1	xyz

$f(x,y,z)=\overline{x}\,\overline{y}\,z+x\,\overline{y}\,z+x\,y\,\overline{z}+x\,y\,z$

(c) $xy+\overline{y}z=\overline{\overline{x}+\overline{y}}+\overline{y+\overline{z}}$

11. (a) $a\oplus b=\overline{a}\bullet b+a\bullet\overline{b}$

(b) (Proving)

12. (Proving)

13. (a)

x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

(b) $f(x,y,z)=x+yz$

14. (a)

x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

(b) (Proving)

15. d 16. b 17. c 18. b 19. b 20. a