SOLUTIONS

SINGAPORE POLYTECHNIC
2020 / 2021 Semester One Examination

No.	SOLUTION
A1	B
A2	C
A3	D
A4	B
A5	B
B1a (i)	$\mathbf{A}-2\mathbf{B}=\begin{bmatrix}1 & 4\\ -2 & -3\end{bmatrix}-2\begin{bmatrix}-3 & -4\\ 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 4\\ -2 & -3\end{bmatrix}-\begin{bmatrix}-6 & -8\\ 4 & 2\end{bmatrix}=\begin{bmatrix}7 & 12\\ -6 & -5\end{bmatrix}$
B1a (ii)	<u>Answer:</u> Not possible to evaluate / Not conformable. <u>Reason:</u> The number of columns in $\mathbf{B}$ is not equal to the number of rows in $\mathbf{C}^T$.
B1b	$\mathbf{AB}=\begin{bmatrix}1 & 4\\ -2 & -3\end{bmatrix}\begin{bmatrix}-3 & -4\\ 2 & 1\end{bmatrix}=\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}$ Since $\mathbf{AB}=5\mathbf{I}$, $\mathbf{A}^{-1}\mathbf{AB}=5\mathbf{A}^{-1}\mathbf{I}$ $\mathbf{B}=5\mathbf{A}^{-1}$ $\mathbf{A}^{-1}=\frac{1}{5}\mathbf{B}=\frac{1}{5}\begin{bmatrix}-3 & -4\\ 2 & 1\end{bmatrix}$
B2a	$\mathbf{T}_1=\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$; $\mathbf{T}_2=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$ $\mathbf{C}=\mathbf{T}_2\mathbf{T}_1=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$
B2b	$\mathbf{P}'=\mathbf{CP}=\begin{bmatrix}0 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & 3\\ 1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 3\\ 0 & 7\\ 1 & 1\end{bmatrix}$
B2c	$\mathbf{T}_1^{-1}=\begin{bmatrix}1 & -2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$; $\mathbf{T}_2^{-1}=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$ $\mathbf{C}^{-1}=\mathbf{T}_1^{-1}\mathbf{T}_2^{-1}=\begin{bmatrix}1 & -2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-2 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$

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B3a

Convert to decimal:

$1011.101_2 = 2^3 + 2^1 + 2^0 + 2^{-1} + 2^{-3} = 11.625_{10}$

Convert to hexadecimal:

$1011.101_2 = 1011.1010_2 = \mathrm{B.A}_{16}$

B3b

Integral part:		
2	43	
2	21	1
2	10	1
2	5	0
2	2	1
2	1	0
	0	1

Fractional part:		
2	0.7	
2	0.4	1
2	0.8	0
2	0.6	1
2	0.2	1
2	0.4	0
2	0.8	0
2	0.6	1
2	0.2	1
2	0.4	0
	0.8 (rep)	0

$\therefore 43.7_{10} = 101011.1\overline{0110}_2$

B4a

$U = \{1,2,3,4,5,6,7,8,9\}$

$A = \{1,2,3,4,5,6\}$

$B = \{3,6,9\}$

B4b

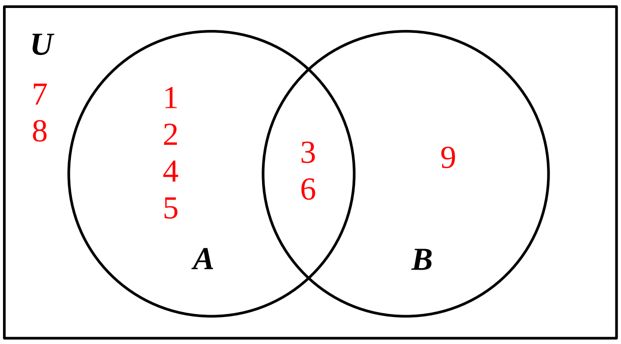
$\therefore \bar{A} = \{7,8,9\}$

$\therefore A \cap B = \{3,6\}$

$A \cup B = \{1,2,3,4,5,6,9\}$

$\therefore |A \cup B| = 7$

B4c



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<table>
<tr><td>B5a (i)</td><td>For $p \wedge q$: $\mathrm{T} \wedge \mathrm{F}$ is FALSE
For $\neg(p \wedge q)$: $\neg\mathrm{F}$ is TRUE</td></tr>
<tr><td>B5a (ii)</td><td>For $p \Rightarrow q$: $\mathrm{T} \Rightarrow \mathrm{F}$ is FALSE
For $(p \Rightarrow q) \vee p$: $\mathrm{F} \vee \mathrm{T}$ is TRUE</td></tr>
<tr><td>B5b</td><td>
<table>
<tr><th>p</th><th>q</th><th>$\neg p$</th><th>$\neg p \vee q$</th><th>$(\neg p \vee q) \wedge p$</th><th>$\neg p \wedge q$</th><th>$(\neg p \wedge q) \vee p$</th></tr>
<tr><td>T</td><td>T</td><td>F</td><td>T</td><td>T</td><td>F</td><td>T</td></tr>
<tr><td>T</td><td>F</td><td>F</td><td>F</td><td>F</td><td>F</td><td>T</td></tr>
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</table>
From the truth table above, we conclude that $M \not\equiv N$.</td></tr>
<tr><td>B6a</td><td>$6^4 = 1296$</td></tr>
<tr><td>B6b</td><td>${}^6P_4 = 360$</td></tr>
<tr><td>B6c</td><td>${}^4C_1 \times {}^5P_3 = 240$</td></tr>
<tr><td>B6d</td><td>${}^5C_2 \times \dfrac{4!}{2!} = 120$</td></tr>
<tr><td>B7a</td><td>There are 6 possible scenarios: $(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)$
Probability = $\dfrac{6}{36} = \dfrac{1}{6}$</td></tr>
<tr><td>B7b</td><td>Probability = $1 - \dfrac{1}{6} = \dfrac{5}{6}$</td></tr>
<tr><td>B7c</td><td>There are 5 possible scenarios: $(4,6),(5,5),(6,4),(5,6),(6,5)$
Probability = $\dfrac{5}{36}$</td></tr>
<tr><td>B7d</td><td>There are 4 possible scenarios: $(1,5),(2,6),(5,1),(6,2)$
Probability = $\dfrac{4}{36} = \dfrac{1}{9}$</td></tr>
<tr><td>C1a</td><td>
<table>
<tr><th>x</th><th>y</th><th>z</th><th>$f(x,y,z)$</th></tr>
<tr><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>1</td><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>0</td><td>1</td></tr>
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SINGAPORE POLYTECHNIC
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C1b	$f(x,y,z) = (\bar{x}+\bar{y}+z)\bullet(x+\bar{y}+z)\bullet(x+y+z)$ $= (\bar{y}+z+(\bar{x}\bullet x))\bullet(x+y+z)$ $= (\bar{y}+z)\bullet(x+y+z)$ $= z+\bar{y}\bullet(x+y)$ $= z+x\bar{y}$
C1c	From the truth table, when $z=0$ and $f(x,y,z)=1$, we see that $x=1$ (i.e. the game is running continuously for > 2 hours). Hence, $t>2$.
C2a	$\mathbf{T}_1$: rotation $135°$ anticlockwise about the origin $\mathbf{T}_2$: scaling relative to the origin by a factor of $\sqrt{2}$ in the *x*-direction and a factor of $\frac{1}{\sqrt{2}}$ in the *y*-direction $\mathbf{T}_3$: shearing by a factor of 1 in the *x*-direction $\mathbf{T}_1 = \begin{bmatrix} \cos 135° & -\sin 135° & 0 \\ \sin 135° & \cos 135° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\mathbf{T}_2 = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\mathbf{T}_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
C2b	$\mathbf{T} = \mathbf{T}_3\mathbf{T}_2\mathbf{T}_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\mathbf{P}' = \mathbf{TP} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -3 & -2 \\ 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ (verified)
C2c	For parallelogram $A'B'C'D'$ to transform to a rhombus by vertical scaling only, the lengths of both $A'B'$ and $C'D'$ must be equal to 2 units. Pythagoras Theorem: $2^2 = 1^2 + k^2 \Rightarrow k = \pm\sqrt{3}$
C3a	For each person to hold on to the baton exactly once, each person will have one less person to pass the baton to for every subsequent pass. Answer $= 5! = 120$
C3b	Notice that Bob can only hold on to the baton for a maximum of three times in total, and this happens when: $A\ B\ \_\ B\ \_\ B$. No. of ways where Bob held on to the baton three times $= 5^2$ No. of ways where there are no restrictions $= 5^5$ Answer $= 5^5 - 5^2 = 3100$

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C3c	No. of ways to form a group of three people $= {}^5C_2$ No. of ways where there are no restrictions (within the group of three people) $= 2^5$ However, the 2^5 ways calculated above will always include 2 ways where only two people pass the baton back and forth among themselves (e.g. $A\,B\,A\,B\,A\,B$, $A\,C\,A\,C\,A\,C$, etc.). Hence, we need to subtract 2 ways from the 2^5 ways. Answer $= {}^5C_2 \times \left(2^5 - 2\right) = 300$
C3d	For the baton to start and end with Alice, we require: $A\,O\ \_\ \_\ O\,A$, where O represents anyone else (other than Alice) holding on to the baton. There are 3 distinct cases: Case 1 ($A\,O\,O\,O\,O\,A$): $5\times4\times4\times4\times1=320$ ways Case 2 ($A\,O\,O\,A\,O\,A$): $5\times4\times1\times5\times1=100$ ways Case 3 ($A\,O\,A\,O\,O\,A$): $5\times1\times5\times4\times1=100$ ways Answer $=320+100+100=520$