5.4 Comparison Tests

Ex1 $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$ $\qquad a_n = \dfrac{\cos^2 n}{n^2}$. What should b_n be?

(usually $b_n = \dfrac{1}{n^p}$ works as a p-series.)

$0 \le \cos^2 n \le 1$ $\qquad$ Try $b_n = \dfrac{1}{n^2}$

(1) Make comparison $\quad 0 \le \underbrace{\dfrac{\cos^2 n}{n^2}}_{= a_n} \le \underbrace{\dfrac{1}{n^2}}_{= b_n}$

(2) Note $\quad \sum b_n = \sum \dfrac{1}{n^2}$ converges by p-series since $p = 2 > 1$.

(3) Thus $\quad \sum a_n = \sum \dfrac{\cos^2 n}{n^2}$ converges by Comparison Test.

Ex2 $\quad a_n = \dfrac{\sin\left(\frac{1}{n}\right)}{\sqrt{n}}$. Determine whether $\sum a_n$ converges or diverges.

Try Divergence Test: as $n \to \infty$ $\quad \sin\left(\frac{1}{n}\right) \to \sin(0) = 0$

$\dfrac{1}{\sqrt{n}} \to 0$

$a_n \to 0 \cdot 0 = 0$

Divergence Test does not apply.

Series may converge, or not.

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1 = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} \quad *$$

$x = \frac{1}{n}$

$n \to \infty$

$$\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} < 1$$

get $\frac{\sin\left(\frac{1}{n}\right)}{\sqrt{n}}$

multiply by $\frac{1}{n^{3/2}}$

on both sides

$$\Longleftrightarrow \quad n \cdot \sin\left(\frac{1}{n}\right) < 1$$

$$\Longleftrightarrow \quad \frac{1}{n^{3/2}} \cdot n \cdot \sin\left(\frac{1}{n}\right) < \frac{1}{n^{3/2}}$$

$$\Longleftrightarrow \quad a_n = \frac{\sin\left(\frac{1}{n}\right)}{\sqrt{n}} < \frac{1}{n^{3/2}} = b_n \qquad (1)$$

By p-series, $p = \frac{3}{2} > 1$, $\sum b_n$ converges. (2)

By Comparison Test, since $a_n < b_n$ as $n \to \infty$

(for all $n \geq N$) we have that $\sum a_n$ converges (3)

<u>Exercise for you</u> <u>Ex 3</u> $\sum \frac{1}{n+1}$. $a_n = \frac{1}{n+1}$.

Should $b_n = \frac{1}{n}$ or $b_n = \frac{1}{2n}$.

And do these series converge or diverge??

Limit Comparison Test

Ex 4 $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ $\qquad a_n = \frac{\ln n}{n^2}$

$\ln n << n^2$ so by growth rates $\frac{\ln n}{n^2} \to 0$

So Divergence Test gives no info on this series.

choose b_n ① option $b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = \left(\frac{\ln n}{n^2}\right) \times \frac{n^2}{1} = \ln n \longrightarrow \infty \quad \times$$

does not show convergence

$$\frac{a_n}{b_n} > M \implies a_n > M b_n$$

So a_n terms are bigger than terms of a convergent series

$\implies$ NO INFO.

② option $b_n = \frac{1}{n^{1.1}}$

$$\frac{a_n}{b_n} = \left(\frac{\ln n}{n^2}\right)\left(\frac{n^{1.1}}{1}\right) = \frac{\ln n}{n^{.9}} \longrightarrow 0 \quad ①$$

by $\ln n << n^{.9}$

$$\frac{a_n}{b_n} < \varepsilon$$

$a_n < b_n \cdot \varepsilon$ (proof ..)

so a_n terms are <u>smaller</u> than terms of a convergent series.

By Limit comparison Test, since $\frac{a_n}{b_n} \to 0$

and $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n^{1.1}}$ converges by p-series ②

$p = 1.1 > 1$

we conclude $\sum_1^\infty a_n = \sum_1^\infty \frac{\ln n}{n^2}$ converges. ③

Ex 5

① $\sum_1^\infty \frac{n^0}{n^2+1}$ $\frac{n^1}{n^2} = \frac{1}{n} = b_n$ What b_n?

$a_n =$

② $\sum_1^\infty \frac{n^0+1}{n^3-1}$ $\frac{n^1}{n^3} = \frac{1}{n^2} = b_n$

$a_n =$

Do ①

①

$\frac{a_n}{b_n} = \frac{n}{n^2+1} \times \frac{n}{1} = \frac{0\,n^2}{0\,n^2+1} \longrightarrow \frac{1}{1} = 1 = L \neq 0$

bc 1 is the ratio of the leading coefficients.

and deg numer = deg denom.

By Limit Comp Test $\sum_1^\infty a_n$ $\sum_1^\infty b_n$ either

③ both converge, or

both diverge

(2) $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n}$ diverges by p-series $p=1 \not> 1$.

(3) Therefore $\sum_1^\infty a_n = \sum_1^\infty \frac{n}{n^2+1}$ diverges as well.

Pls try (2).

Benefits of LC in (1) To establish divergence using $\boxed{b_n = \frac{1}{n}}$ would need

$$b_n < a_n \qquad \text{i.e.}$$

$$\frac{1}{n} < \frac{n}{n^2+1} \quad \leftrightarrow$$

$$n^2+1 < n^2 \quad \text{false.}$$

So a more nuanced b_n is required.

like $b_n = \frac{1}{2n}$.

$\sum_1^\infty \frac{1}{2n}$ diverges bc it is a constant multiple of $\sum_1^\infty \frac{1}{n}$, which diverges.

$$\frac{1}{2n} = b_n < a_n = \frac{n}{n^2+1} \quad \leftrightarrow$$

$$n^2+1 < 2n^2 \quad \leftrightarrow$$

$$-n^2 \qquad\qquad -n^2$$

$1 < n^2$ true.

using $\boxed{b_n = \frac{1}{2n}}$, CT can be used to show

$\sum a_n$ diverges.