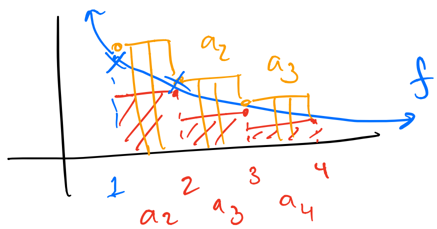
213 5.3 Applications

$f(n)=a_n$

f cont, pos, decr



add the red rectangles $= a_2+a_3+a_4+\cdots = \sum_{n=2}^{\infty} a_n$

add orange rectangles $= a_1+a_2+a_3+\cdots = \sum_{n=1}^{\infty} a_n$

$$\int_2^{\infty} f(x)\,dx < a_2+a_3+a_4\cdots$$

$$a_2+a_3+a_4+\cdots < \int_1^{\infty} f(x)\,dx$$

$$\int_2^{\infty} f(x)\,dx < \sum_{n=2}^{\infty} a_n < \int_1^{\infty} f(x)\,dx$$

More generally, replace 2 by N+1:

$$\int_{N+1}^{\infty} f(x)\,dx < \underbrace{\sum_{n=N+1}^{\infty} a_n}_{R_N := \text{tail (error)}} < \int_{N}^{\infty} f(x)\,dx$$

$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{N} a_n}_{S_N} + \underbrace{\sum_{n=N+1}^{\infty} a_n}_{R_N} = S \text{ (if series converges)}$$

S_N: sum of first N terms

R_N: error, when using S_N instead of S

<u>Ex</u> (like #173)

Estimate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to within $< .001$ using N terms. What is the least such N guaranteed by the theory above?

know: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (this is special...!)

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want: error $R_N < .001 = 10^{-3}$

suffices to show

$$R_N < \int_N^{\infty} \frac{1}{x^2}\,dx < 10^{-3}$$

<u>calculate</u>

$$\int_N^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left.\frac{x^{-1}}{-1}\right|_N^b$$

$$= \lim_{b\to\infty} \cancelto{0}{\frac{-1}{b}} + \frac{1}{N} = \frac{1}{N}$$

want : $\frac{1}{N} < 10^{-3}$

i.e. $10^3 < N$

so $N = 1001$ terms, suffices.

$$\sum_{n=1}^{1001} \frac{1}{n^2} = 1.6439356 = S_N = \text{Sum}(\text{sequence}(a_n, n, 1, 1001))$$

(Geogebra)

$S = \frac{\pi^2}{6}$: $S - S_N = .0009985 < .001 = 10^{-3}$ ✓

(Geogebra)