

# Sketchboard

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# 1 Basic concepts

## 1.1 With extra capital internal index on embedding fields

Promote embedding fields  $X^\mu$  to have an internal group index with  $D$  values

$$\begin{aligned}
X^\mu &\rightarrow X^{\mu I} , \quad I = 0, \dots, d \\
g_{ab} &= 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{IJ} \sim f D_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu} \\
D_a X^{\mu I} &= \partial_a X^{\mu I} + \omega_{aJ}^I X^{\mu J} \\
e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ} , \quad i, j = 0, 1 \\
g = \det(g_{ab}) &\rightarrow g = \det(\text{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

## 1.2 With 1 extra small internal index on embedding fields

Promote embedding fields to have a internal group index with 2 values

$$\begin{aligned}
X^\mu &\rightarrow X^{\mu i} , \quad i = 0, 1 \\
g_{ab} &= 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{ij} \sim f D_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu} \\
D_a X^{\mu i} &= \partial_a X^{\mu i} + \omega_{aj}^i X^{\mu j} \\
e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij} \\
g = \det(g_{ab}) &\rightarrow g = \det(\text{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

## 1.3 With 2 extra small indices

Promote embedding fields  $X^\mu$  to have two internal indices with 2 values

$$X^\mu \rightarrow X^{\mu ij} , \quad i, j = 0, 1$$

## 1.4 Without extra index on embedding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$\begin{aligned}
e_a^i e_b^j \eta_{ij} &= g_{ab} = 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\
g = \det(g_{ab}) &= \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

# 2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int d\tau \wedge d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} ,$$

## 2.1 With capital internal index

... and promote partial derivative  $\partial_a$  to covariant derivative  $D_a$ , giving us our first attempt at modified Polyakov action

$$\begin{aligned}
S_{MP1} &= -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ} \\
&\quad \updownarrow \\
\mathcal{L}_{MP1} &= -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}
\end{aligned}$$

## 2.2 With small internal index

### 2.2.1 Without extra field, 1 internal index

... and promote derivatives to covariant  $D_a$ , giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

$$\updownarrow$$

$$\mathcal{L}_{MP2} = -\frac{T_0}{2} \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

### 2.2.2 With extra field

..., promote partial derivative to covariant and add extra internal WS field  $v^i$  leading us to

$$S_{MP3} = -\frac{T}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

$$\updownarrow$$

$$\mathcal{L}_{MP3} = -\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

### 2.2.3 Without extra field, 2 internal indices

... and promote derivatives to covariant  $D_a$ ,

$$S_{MP4} = -\frac{T}{2} \int d\tau \wedge d\sigma e e_i^a e^{bi} D_a X^{\mu jj'} \eta_{jj'} \eta_{kk'} D_b X^{\nu kk'} E_\mu^I E_{\nu I}$$

$$\mathcal{L}_{MP4} = -\frac{T}{2} e e_i^a e_j^b D_a X^{\mu ik} \eta_{kl} D_b X^{\nu lj} E_\mu^I E_{\nu I}$$

## 2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) .$$

$$\updownarrow$$

$$\mathcal{L}_{DP} = -\frac{T_0}{2} \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) .$$

## 2.4 Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of  $D$ -dimensional gamma matrices and bulk spinors

$$S_{LP} = -\frac{T}{2} \int d\tau \wedge d\sigma \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

$$\updownarrow$$

$$\mathcal{L}_{LP} = -\frac{T}{2} \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

## 3 EoMs

Start by writing  $\det(e) = \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n$  and  $g_{ab} = g_{ab}^{ij} \eta_{ij}$

### 3.1 With small internal index

#### 3.1.1 Without extra field, w.r.t $e$

$$\begin{aligned}
\frac{\delta S_{MP2}}{\delta e_l^e} &= \frac{D\mathcal{L}_{MP2}}{De_l^e} = \left( \frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)} \right) \left( -\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \right) = \\
&= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (\eta^{ml} g_{ce} e_d^n e_i^a e_j^b + e_c^m \eta^{nl} g_{de} e_i^a e_j^b + \\
&+ e_c^m e_d^n \delta_e^a \delta_i^l e_j^b + e_c^m e_d^n e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} g_{ce} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} e_c^k e_{ek} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_i^l e_j^a e_e^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -T (-\det(e) e_e^l e_i^a e_j^b + \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \stackrel{!}{=} 0 \\
&\quad \downarrow \\
T_e^l &:= E_\mu^I E_{\nu I} (e_i^a D_a X^{\mu i} D_e X^{\nu l} - e_e^l e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j}) = 0 \\
&\quad \Downarrow \\
e_e^l &= f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I} , \\
\frac{1}{f} &= e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \\
&\quad \downarrow \\
e_e^l e_{fl} &= (f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I}) (f e_{i'}^{a'} D_{a'} X^{\mu' i'} D_f X_{l'}^{\nu'} E_{\mu'}^{I'} E_{\nu'}^{I'}) = \\
&= f^2 D_e X^{\nu l} D_f X_{l'}^{\nu'} E_{\nu'}^{I'} E_{\nu'}^{I'} (e_i^a e_{i'}^{a'} D_a X^{\mu i} D_{a'} X^{\mu' i'} E_\mu^I E_{\mu'}^{I'}) = \\
&= f D_e X^{\mu i} D_f X_i^\nu E_\mu^I E_{\nu I} \\
&\quad \Downarrow \\
D_e X^{\mu i} D_f X_i^\nu E_\mu^I E_{\nu I} &= 2 \partial_e X^\mu \partial_f X^\nu G_{\mu\nu} \\
e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\
&\quad \Downarrow \\
D_a X^{\mu 0} &= -i \partial_a X^\mu \\
D_a X^{\mu 1} &= \partial_a X^\mu \\
&\quad \Downarrow \\
X^{\mu 0} &= -i X^{\mu 1}
\end{aligned}$$

### 3.1.2 With extra field, w.r.t $e$

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta e_o^e} &= \frac{D\mathcal{L}_{MP3}}{De_o^e} = \left( \frac{\partial}{\partial e_o^e} - \partial_f \frac{\partial}{\partial (\partial_f e_o^e)} \right) \left( -\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (g_{ce} \eta^{mo} e_d^n \eta^{ij} e_i^a e_j^b + e_c^m g_{de} \eta^{no} \eta^{ij} e_i^a e_j^b + \\
&+ e_c^m e_d^n \eta^{ij} \delta_e^a \delta_i^o e_j^b + e_c^m e_d^n \eta^{ij} e_i^a \delta_e^b \delta_j^o) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -\frac{T}{2} \varepsilon^{cd} \varepsilon_{mn} (e_c^p e_{ep} e_d^n \eta^{mo} \eta^{ij} e_i^a e_j^b + 2 \det(e) e^{ao} \delta_e^b) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -T(-\det(e) e_e^o \eta^{ij} e_i^a e_j^b + \det(e) e^{ao} \delta_e^b) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \stackrel{!}{=} 0
\end{aligned}$$

↓

$$\boxed{T_e^o := (e^{ao} D_a(X_k^\mu v^k) D_e(X_l^\nu v^l) - e_e^o \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l)) E_\mu^I E_\nu^J \eta_{IJ} = 0}$$

⇓

$$\begin{aligned}
e_e^o &= f e^{ao} D_a(X_k^\mu v^k) D_e(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} , \\
\frac{1}{f} &= \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}
\end{aligned}$$

### 3.1.3 With extra field, w.r.t $\omega$

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta \omega_c^{mn}} &= \frac{D\mathcal{L}_{MP3}}{D\omega_c^{mn}} = \left( \frac{\partial}{\partial \omega_c^{mn}} - \partial_d \frac{\partial}{\partial (\partial_d \omega_c^{mn})} \right) \left( -\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T}{4} e g^{ab} ((-\delta_a^c \delta_{[m}^{k'} \eta_{n]k} X_k^\mu v^k) D_b X_l^\nu v^l + D_a X_k^\mu v^k (-\delta_b^c \delta_{[m}^{l'} \eta_{n]l} X_l^\nu v^l)) G_{\mu\nu} = \\
&= \frac{T}{4} e (g^{ca} X_{[m}^\mu v_{n]} D_a X_k^\nu v^k + g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_{n]}) G_{\mu\nu} = \\
&= \frac{T}{2} e g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_{n]} G_{\mu\nu} \stackrel{!}{=} 0
\end{aligned}$$

↓

$$\boxed{\mathcal{T}_{ab}^i := e^{ci} D_c X_k^\mu v^k X_{[m}^\nu v_{n]} e_a^m e_b^n G_{\mu\nu} = 0}$$

### 3.1.4 With extra field, w.r.t $v$

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta v^l} &= \frac{D\mathcal{L}_{MP3}}{Dv^l} = \left( \frac{\partial}{\partial v^l} - \partial_c \frac{\partial}{\partial (\partial_c v^l)} \right) \left( -\frac{T}{2} e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_\nu^I \right) = \\
&= -\frac{T}{2} g^{ab} (D_a X_j^\mu \delta_l^j D_b X_k^\nu v^k + D_a X_j^\mu v^j D_b X_k^\nu \delta_l^k) G_{\mu\nu} = \\
&= -T g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} \stackrel{!}{=} 0
\end{aligned}$$

↓

$$\boxed{g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} = 0}$$

### 3.1.5 With extra field, w.r.t $X$

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta X_l^\lambda} &= \frac{D\mathcal{L}_{MP3}}{DX_l^\lambda} = \left( \frac{\partial}{\partial X_l^\lambda} - \partial_c \frac{\partial}{\partial (\partial_c X_l^\lambda)} \right) \left( -\frac{T}{2} e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I} \right) = \\
&= -\frac{T}{2} \left( e g^{ab} \left( (-\omega_{aj}^{j'} \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k - D_a X_j^\mu v^j \omega_{bk}^{k'} \delta_\lambda^\nu \delta_k^l v^k) E_\mu^I E_{\nu I} + \right. \right. \\
&\quad \left. \left. + e g^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \left( \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} + E_\mu^I \frac{\partial E_{\nu I}}{\partial X_l^\lambda} \right) \right) - \right. \\
&\quad \left. - \partial_c (e g^{ab} (\delta_a^c \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k + D_a X_j^\mu v^j \delta_b^c \delta_\lambda^\nu \delta_k^l v^k) G_{\mu\nu}) \right) = \\
&= -T \left( -e g^{ab} D_a X_j^\mu v^j \omega_{bk}^l v^k G_{\mu\lambda} + e g^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - \partial_c (e g^{ac} v^l D_a X_j^\mu v^j G_{\mu\lambda}) \right) \\
&= -T \left( e g^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - D_b (e g^{ab} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) \right) \stackrel{!}{=} 0 \\
&\quad \downarrow \\
&\quad \boxed{D_b (e e_i^a e^{bi} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) = e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I}}
\end{aligned}$$

## 3.2 Without extra index

### 3.2.1 w.r.t $e$

$$\begin{aligned}
\frac{\delta S_{DP}}{\delta e_l^e} &= \frac{D\mathcal{L}_{DP}}{De_l^e} = \left( \frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial (\partial_f e_l^e)} \right) \left( -\frac{T_0}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T_0}{4}
\end{aligned}$$

## 3.3 Linear action

### 3.3.1 w.r.t $e$

$$\begin{aligned}
\frac{\delta S_{LP}}{\delta e_j^b} &= \frac{D\mathcal{L}_{LP}}{De_j^b} = \left( \frac{\partial}{\partial e_j^b} - \partial_e \frac{\partial}{\partial (\partial_e e_j^b)} \right) \left( -\frac{T}{2} \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi \right) = \\
&= -\frac{T}{4} \bar{\psi} \varepsilon^{cd} \varepsilon_{mn} ((g_{cb} \eta^{mj} e_d^n + e_c^m g_{db} \eta^{nj}) e_i^a + e_c^m e_d^n \delta_b^a \delta_i^j) D_a X^{\mu i} E_\mu^I \gamma_I \psi = \\
&= -\frac{T}{2} \bar{\psi} (-e e_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I + e D_b X^{\mu j} E_\mu^I \gamma_I) \psi \stackrel{!}{=} 0 \\
&\quad \downarrow \\
T_b^j &:= \bar{\psi} (D_b X^{\mu j} E_\mu^I \gamma_I - e_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I) \psi = 0 \\
&\quad \downarrow \\
e_b^j &= F \bar{\psi} D_b X^{\mu j} E_\mu^I \gamma_I \psi, \\
\frac{1}{F} &= \bar{\psi} e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi \\
&\quad \Downarrow \\
\bar{\psi} \psi &= 1, \quad \psi \bar{\psi} = \mathbb{1} \\
e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_\nu^J \eta_{IJ} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}
\end{aligned}$$

## 4 Inverse Area Action

### 4.1 Nambu-Goto

Start from Nambu-Goto action

$$S_{NG} = -T \int d^2x \sqrt{-h},$$

and make quantum geometry correction

$$\sqrt{-h} \rightarrow \sqrt{-(h + g\Delta)} \approx \sqrt{-h} \left( 1 - \frac{g\Delta}{2(-h)} + \mathcal{O}\left(\frac{g^2}{h^2}\right) \right),$$

leading to modified NG action

$$\begin{aligned} S_{MNG} &= -T \int d^2x \left( \sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}} \right) = S_{NG} + S_{IA} \\ &\downarrow \\ \mathcal{L}_{MNG} &= -T \left( \sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}} \right) \end{aligned}$$

#### 4.1.1 EoMs

Let  $\mathcal{P}_\mu^\tau \equiv \partial \mathcal{L}_{MNG} / \partial \dot{X}^\mu$  and  $\mathcal{P}_\mu^\sigma \equiv \partial \mathcal{L}_{MNG} / \partial X'^\mu$

$$\begin{aligned} \delta S_{MNG} &= \int d^2x \left( \frac{\partial \mathcal{L}_{MNG}}{\partial \dot{X}^\mu} \delta \dot{X}^\mu + \frac{\partial \mathcal{L}_{MNG}}{\partial X'^\mu} \delta X'^\mu \right) = \\ &= - \int d^2x \left( \partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma \right) \delta X^\mu + \int d\tau \mathcal{P}_\mu^\sigma \delta X^\mu \Big|_{\sigma=0}^{\sigma=\sigma_1} \stackrel{!}{=} 0 \\ &\downarrow \\ \text{EoM} : \partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma &= 0 \\ \text{B.C.} : \mathcal{P}_\mu^\sigma \delta X^\mu \Big|_{\sigma=0}^{\sigma=\sigma_1} &= 0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}_\mu^\tau &= \frac{\partial \mathcal{L}_{MNG}}{\partial \dot{X}^\mu} = -T \left( \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} + \frac{g\Delta}{2} \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{(-h)^{3/2}} \right) = \\ &= -T \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} \left( 1 + \frac{g\Delta}{2(-h)} \right) = \mathcal{P}_{\mu(NG)}^\tau \left( 1 + \frac{g\Delta}{2(-h)} \right) \\ \mathcal{P}_\mu^\sigma &= \frac{\partial \mathcal{L}_{MNG}}{\partial X'^\mu} = -T \left( \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} + \frac{g\Delta}{2} \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{(-h)^{3/2}} \right) = \\ &= -T \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} \left( 1 + \frac{g\Delta}{2(-h)} \right) = \mathcal{P}_{\mu(NG)}^\sigma \left( 1 + \frac{g\Delta}{2(-h)} \right) \end{aligned}$$

gauge fixing static gauge  $\tau = t$  and transverse gauge  $\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial s} \frac{ds}{d\sigma} = 0$  ( $s$  = length along string)

$$\begin{aligned} &\downarrow \\ \mathcal{P}_{(NG)}^{\tau\mu} &= T \frac{ds}{d\sigma} \gamma_{v\perp} \frac{\partial X^\mu}{\partial t} \\ \mathcal{P}_{(NG)}^{\sigma\mu} &= \frac{T}{\gamma_{v\perp}} \frac{\partial X^\mu}{\partial s} \\ -h &= \frac{1}{\gamma_{v\perp}^2} \left( \frac{ds}{d\sigma} \right)^2 \end{aligned}$$

string energy get's redefined to

$$\frac{\partial}{\partial t} \left( T \frac{ds}{d\sigma} \gamma_{v_\perp} \left( 1 + \frac{g\Delta}{2(-h)} \right) \right) = 0,$$

and for the spatial part we have

$$\partial_\tau \vec{\mathcal{P}}^\tau + \partial_\sigma \vec{\mathcal{P}}^\sigma = 0$$

↓

$$\frac{\partial}{\partial t} \left[ T \frac{ds}{d\sigma} \gamma_{v_\perp} \frac{\partial \vec{X}}{\partial t} \left( 1 + \frac{g\Delta}{2(-h)} \right) \right] + \frac{ds}{d\sigma} \frac{\partial}{\partial s} \left[ -\frac{T}{\gamma_{v_\perp}} \frac{\partial \vec{X}}{\partial s} \left( 1 + \frac{g\Delta}{2(-h)} \right) \right] = 0$$

↓

$$\mu \gamma_{v_\perp} \left( 1 + \frac{g\Delta}{2(-h)} \right) \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial}{\partial s} \left[ \frac{T}{\gamma_{v_\perp}} \left( 1 + \frac{g\Delta}{2(-h)} \right) \frac{\partial \vec{X}}{\partial s} \right] = 0$$

$\Rightarrow$  effective mass density becomes  $\mu_{eff} = \mu \gamma_{v_\perp} \left( 1 - \frac{g\Delta}{2(-h)} \right)$  and effective tension becomes  $T_{eff} = \frac{T}{\gamma_{v_\perp}} \left( 1 - \frac{g\Delta}{2(-h)} \right)$

↓

$$\mu_{eff} \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial}{\partial s} \left[ T_{eff} \frac{\partial \vec{X}}{\partial s} \right] = 0$$

$$\begin{aligned} \mathcal{H} &= \dot{\vec{X}} \cdot \vec{\pi} - \mathcal{L} = \\ &= \vec{v}_\perp \cdot \left( T \frac{ds}{d\sigma} \gamma_{v_\perp} \left( 1 + \frac{g\Delta}{2(-h)} \right) \vec{v}_\perp \right) - \left( -T \frac{ds}{d\sigma} \frac{1}{\gamma_{v_\perp}} \left( 1 + \frac{g\Delta}{2(-h)} \right) \right) = \\ &= T \frac{ds}{d\sigma} \left( 1 + \frac{g\Delta}{2(-h)} \right) \left( \gamma_{v_\perp} v_\perp^2 + \frac{1}{\gamma_{v_\perp}} \right) = \\ &= T \frac{ds}{d\sigma} \gamma_{v_\perp} \left( 1 + \frac{g\Delta}{2(-h)} \right) \end{aligned}$$

let  $\left( 1 + \frac{g\Delta}{2(-h)} \right) = F$

$$\frac{\partial^2 \vec{X}}{\partial t^2} - \frac{1}{F \gamma_{v_\perp}} \frac{d\sigma}{ds} \frac{\partial}{\partial \sigma} \left[ \frac{1}{\gamma_{v_\perp}} F \frac{d\sigma}{ds} \frac{\partial \vec{X}}{\partial \sigma} \right] = 0$$

↓

$$A(\sigma) = \frac{\gamma_{v_\perp}}{F} \frac{ds}{d\sigma} \stackrel{!}{=} 1$$

↓

$$d\sigma = \frac{\gamma_{v_\perp}}{F} ds = \frac{1}{TF^2} dE \Rightarrow \sigma(q) = \frac{1}{T} \int_0^q \frac{1}{F^2} dE$$

↓

$$F^2 \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial^2 \vec{X}}{\partial \sigma^2} = 0$$

$\Rightarrow$  speed of wave on the string gets modified by correction factor  $v = c/F$

$$-h = \frac{1}{\gamma_{v_\perp}^2} \left( \frac{ds}{d\sigma} \right)^2 = \frac{F^2}{\gamma_{v_\perp}^4} = F^2 (1 - v_\perp^2)^2$$

↓

$$-h = \left( 1 + \frac{g\Delta}{2(-h)} \right)^2 (1 - v_\perp^2)^2$$

↓



$$\frac{-h}{\left(1 + \frac{g\Delta}{2(-h)}\right)^2} = (1 - v_\perp^2)^2$$

↓ (WolframAlpha)

$$-h = \frac{(1 - v_\perp^2)^2}{3} - f_1 - f_2,$$

where (let  $a = \frac{g\Delta}{2}$  and  $b = (1 - v_\perp^2)^2$ )

$$f_1 = \frac{\sqrt[3]{-27a^2b + 3\sqrt{3}\sqrt{27a^4b^2 + 4a^3b^3 - 18ab^2 - 2b^3}}}{3\sqrt[3]{2}}$$

$$f_2 = \frac{(6ab + b^2)}{6f_1}$$

↓

$$F = 1 + \frac{g\Delta}{2(-h)} = 1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}$$

↓

$$v = \frac{1}{F} = \frac{1}{1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}}$$

## 5 Bimetric Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

and promote to bimetric action

↓

$$S_{BP} = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{T'}{2} \int d^2x \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

### 5.1 EoMs

$$\begin{aligned} \frac{\delta S_{BP}}{\delta g^{cd}} &= -\frac{T}{2} \left( \frac{\partial \sqrt{-g}}{\partial g^{cd}} g^{ab} + \sqrt{-g} \frac{\partial g^{ab}}{\partial g^{cd}} \right) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + 0 = \\ &= -\frac{T}{2} \left( -\frac{1}{2} \sqrt{-g} g_{cd} g^{ab} + \sqrt{-g} \delta_{(c}^a \delta_{d)}^b \right) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \stackrel{!}{=} 0 \end{aligned}$$

$$T_{cd}^{(G)} := \left( \partial_c X^\mu \partial_d X^\nu - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \right) G_{\mu\nu} = 0$$

⇓

$$g_{cd} = 2f^{(G)} \partial_c X^\mu \partial_d X^\nu G_{\mu\nu},$$

$$\frac{1}{f^{(G)}} = g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\begin{aligned} \frac{\delta S_{BP}}{\delta h^{cd}} &= \dots = \\ &= -\frac{T'}{2} \left( -\frac{1}{2} \sqrt{-h} h_{cd} h^{ab} + \sqrt{-h} \delta_{(c}^a \delta_{d)}^b \right) \partial_a X^\mu \partial_b X^\nu H_{\mu\nu} \stackrel{!}{=} 0 \end{aligned}$$

$$\downarrow$$

$$T_{cd}^{(H)} := \left( \partial_c X^\mu \partial_d X^\mu - \frac{1}{2} h_{cd} h^{ab} \partial_a X^\mu \partial_b X^\nu \right) H_{\mu\nu} = 0$$

$$\downarrow$$

$$h_{cd} = 2f^{(H)} \partial_c X^\mu \partial_d X^\nu H_{\mu\nu},$$

$$\frac{1}{f^{(H)}} = h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

$$\begin{aligned} \frac{\delta S_{BP}}{\delta X^\lambda} &= \left( -\frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu - \\ &\quad - \partial_c \left( -\frac{T}{2} \sqrt{-g} g^{ab} G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} H_{\mu\nu} \right) (\delta_a^c \delta_\lambda^\mu \partial_b X^\nu + \partial_a X^\mu \delta_b^c \delta_\lambda^\nu) = \\ &= \partial_a \left( T \sqrt{-g} g^{ab} G_{\mu\lambda} + T' \sqrt{-h} h^{ab} H_{\mu\lambda} \right) \partial_b X^\mu - \\ &\quad - \left( \frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} + \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu \stackrel{!}{=} 0 \end{aligned}$$

$$\downarrow$$

$$\partial_a (F_{\mu\lambda}^{ab} \partial_b X^\mu) = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu,$$

$$F_{\mu\nu}^{ab} = T \sqrt{-g} g^{ab} G_{\mu\nu} + T' \sqrt{-h} h^{ab} H_{\mu\nu}$$

let  $T' = Tk\Delta/2$  and  $h = g^{-1}(H_{\mu\nu} = G_{\mu\nu})$

$$F_{\mu\nu}^{ab} = T \left( \sqrt{-g} g^{ab} G_{\mu\nu} + \frac{k\Delta}{2} \sqrt{-g^{-1}} U_{a'}^a (g^{-1})^{a'b'} U_{b'}^b (G^{-1})_{\mu\nu} \right),$$

add new field coupled to derivative  $A_a$  transforming as

$$A_a g^{bc} \rightarrow (A_a g^{bc})' = A_a U_{b'}^b g^{b'c'} U_{c'}^c - \frac{1}{\alpha} g^{b'c'} (\partial_a U_{b'}^b U_{c'}^c + U_{b'}^b \partial_a U_{c'}^c),$$

defining new WS covariant derivative

$$D_a g^{bc} = \partial_a g^{bc} + \alpha A_a g^{bc},$$

such that derivative ignores  $\text{SO}(1,1)$  gauge freedom on inverse metric. Upgrade EoM to incorporate this derivative:

$$D_a (F_{\mu\lambda}^{ab} \partial_b X^\mu) = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu$$

$$\downarrow$$

$$F_{\mu\lambda}^{ab} \partial_a \partial_b X^\mu + D_a F_{\mu\lambda}^{ab} \partial_b X^\mu = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu,$$

imposing conformal symmetry  $g^{ab} = (\phi(x))^{-1} \eta^{ab} \implies D_a F_{\mu\lambda}^{ab} = 0$

$$F_{\mu\lambda}^{ab} \partial_a \partial_b X^\mu = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu,$$

$$F_{\mu\lambda}^{ab} = T \left( G_{\mu\lambda} + \frac{k\Delta}{2} (G^{-1})_{\mu\lambda} \right) \eta^{ab}$$

$$(G^{-1})_{\mu\nu} \stackrel{?}{=} \frac{1}{-g} G_{\mu\nu}$$

## 6 Inverse Area Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

and make quantum geometry correction

$$\sqrt{-g} \rightarrow \sqrt{-(g+k\Delta)} \approx \sqrt{-g} \left( 1 + \frac{k\Delta}{2g} + \mathcal{O}\left(\frac{k^2}{g^2}\right) \right)$$

leading to

$$S_{IAP} = -\frac{T}{2} \int d^2x \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

### 6.1 EoMs

$$\begin{aligned} \frac{\delta S_{IAP}}{\delta g^{cd}} &= \frac{\partial}{\partial g^{cd}} \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = \\ &= \left( -\frac{1}{2} \sqrt{-g} g_{cd} + \frac{k\Delta}{4\sqrt{-g}} g_{cd} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} = \\ &= -\frac{1}{2} \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} \stackrel{!}{=} 0 \end{aligned}$$

↓

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = 0$$

↓

$$\begin{aligned} g_{cd} &= 2f \partial_c X^\mu \partial_d X^\nu G_{\mu\nu}, \\ \frac{1}{f} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \end{aligned}$$

$$\begin{aligned} \frac{\delta S_{IAP}}{\delta X^\lambda} &= \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - \partial_c \left( \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) 2g^{ab} \delta_a^c \delta_b^\lambda \partial_b X^\nu G_{\mu\nu} \right) = \\ &= \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2 \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} - \\ &\quad - 2\partial_a \left( \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} = \\ &= \left( 1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2 \left( 1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} - \\ &\quad - 2\partial_a \left( \left( 1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} \stackrel{!}{=} 0 \end{aligned}$$

↓

$$\begin{aligned} \left( 1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} + \partial_a \left( \left( 1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} = \\ \frac{1}{2} \left( 1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu}, \end{aligned}$$

imposing conformal symmetry  $g^{ab} = (\phi(x))^{-1} \eta^{ab}$  and let  $F' = \left( 1 - \frac{k\Delta}{2\phi^2} \right)$  leads to

$$\eta^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} + \frac{1}{F'} \eta^{ab} \partial_a F' \partial_b X^\nu G_{\lambda\nu} = \eta^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

assuming flat background  $G_{\mu\nu} = \eta_{\mu\nu}$

$$\eta^{ab}\partial_a\partial_b X^\nu\eta_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F'\partial_b X^\nu\eta_{\lambda\nu} = 0$$

$\downarrow$

$$\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\partial_a \ln(F')\partial_b X^\mu = 0,$$

use plane-wave ansatz  $X^\mu = X_0^\mu e^{-i(E\tau - p\sigma)}$

$$(E^2 - p^2)X^\mu + \partial_\tau \ln(F')iEX^\mu - \partial_\sigma \ln(F')ipX^\mu = 0$$

$\downarrow$

$$(E^2 - p^2)X^\mu + i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F'))X^\mu = 0$$

$\downarrow$

$$i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F')) = -m^2,$$

$$-m^2 = -E^2 + p^2$$

$\Downarrow$

$$\partial_\tau \ln(F') = iE, \quad \partial_\sigma \ln(F') = ip$$

$\Downarrow$

$$\ln(F') = i(E\tau + p\sigma) + a, \quad a \in \mathbb{C}$$

$\Downarrow$

$$F' = Ae^{i(E\tau + p\sigma + \phi)}, \quad A > 0$$

$\Downarrow$

$$\phi(\tau, \sigma) = \sqrt{\frac{k\Delta}{2(1 - Ae^{i(E\tau + p\sigma + \phi)})}}$$

$\downarrow$

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2X^\mu = 0$$

## 7 Inverse Area Polyakov v2

Quantum geometry correction

$$\begin{aligned}
\sqrt{-g} &\rightarrow \sqrt{-(g+k\Delta)} \approx \sqrt{-g} \left( 1 + \frac{k\Delta}{2g} + \mathcal{O}\left(\frac{k^2}{g^2}\right) \right) \\
&\Downarrow \\
g_{ab} &\rightarrow \tilde{g}_{ab} = g_{ab} + k\Delta\delta_{ab} \\
&\Downarrow \\
g^{ab} &\rightarrow \tilde{g}^{ab} \approx g^{ab} - k\Delta\delta^{ab} \\
&\Downarrow \\
S &= -\frac{T}{2} \int d^2x \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) (g^{ab} - k\Delta\delta^{ab}) \partial_a X^\mu \partial_b X^\nu \tilde{G}_{\mu\nu}, \\
&\quad \tilde{G}_{\mu\nu} = G_{\mu\nu} + k\Theta\delta_{\mu\nu},
\end{aligned}$$

where  $\Theta$  is related to minimal hyper-volume of bulk quantum geometry

### 7.1 EoMs

$$\begin{aligned}
\frac{\delta S}{\delta g^{cd}} &= \frac{\partial}{\partial g^{cd}} \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) (g^{ab} - k\Delta\delta^{ab}) \partial_a X^\mu \partial_b X^\nu \tilde{G}_{\mu\nu} + \\
&+ \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \frac{\partial}{\partial g^{cd}} (g^{ab} - k\Delta\delta^{ab}) \partial_a X^\mu \partial_b X^\nu \tilde{G}_{\mu\nu} = \\
&= \left( -\frac{1}{2}\sqrt{-g}g_{cd} + \frac{k\Delta}{4\sqrt{-g}}g_{cd} \right) (g^{ab} - k\Delta\delta^{ab}) \partial_a X^\mu \partial_b X^\nu \tilde{G}_{\mu\nu} + \\
&+ \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu \tilde{G}_{\mu\nu} = \\
&= -\frac{1}{2} \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g_{cd} (g^{ab} - k\Delta\delta^{ab}) \partial_a X^\mu \partial_b X^\nu \tilde{G}_{\mu\nu} + \\
&+ \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \partial_c X^\mu \partial_d X^\nu \tilde{G}_{\mu\nu} \stackrel{!}{=} 0 \\
&\Downarrow \\
T_{cd} &:= \partial_c X^\mu \partial_d X^\nu \tilde{G}_{\mu\nu} - \frac{1}{2} g_{cd} (g^{ab} - k\Delta\delta^{ab}) \partial_a X^\mu \partial_b X^\nu \tilde{G}_{\mu\nu} = 0 \\
&\quad g_{cd} = 2f \partial_c X^\mu \partial_d X^\nu \tilde{G}_{\mu\nu},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{f} &= (g^{ab} - k\Delta\delta^{ab}) \partial_a X^\mu \partial_b X^\nu \tilde{G}_{\mu\nu} = \\
&= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - k\Theta g^{ab} \partial_a X^\mu \partial_b X^\nu \delta_{\mu\nu} - \\
&- k\Delta\delta^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - k^2\Delta\Theta\delta^{ab} \partial_a X^\mu \partial_b X^\nu \delta_{\mu\nu} \approx \\
&\approx g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - k\Theta g^{ab} \partial_a X^\mu \partial_b X^\nu \delta_{\mu\nu} - k\Delta\delta^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}
\end{aligned}$$