Quantizing the Bosonic String on a Loop Quantum Gravity Background

Deepak Vaid*
National Institute of Technology, Karnataka, India

Luigi Teixeira de Sousa†
Universidade Federal de São Carlos, Brasil

(Dated: June 8, 2024)

With the goal of understanding whether or not it is possible to construct a string theory which is consistent with loop quantum gravity (LQG), we study alternate versions of the Nambu-Goto action for a bosonic string. We consider two types of modifications. The first is a phenomenological action based on the observation that LQG tells us that areas of two-surfaces are operators in quantum geometry and are bounded from below. This leads us to a string action which is similar to that of bimetric gravity. We provide formulations of the bimetric string for both the Nambu-Goto (second order) and Polyakov (first order) formulations. We explore the classical solutions of this action and its quantization and relate it to the conventional string solutions.

The second is an action in which the background geometry is described in terms of the pullback of the connection which describes the bulk geometry to the worldsheet. The resulting action is in the form of a gauged sigma model, where the spacetime co-ordinates are now SO(D,1) vectors. We find that for the particular case of a constant background connection the action reduces to the bimetric action discussed above. We discuss classical solutions and quantization strategies for this action and its implications for the broader program of unifying string theory and loop quantum gravity.

CONTENTS

Todo list 1

I. Introduction 2

II. Strings in Background Fields and Emergent Gravity 3

III. Modified Nambu Goto Action 5
A. Modified Equations of Motion 6

IV. Inverse Metric Polyakov 6
A. Simplifying The Equations of Motion 7
B. Relation With The Nambu-Goto Analysis 8
C. Solving The String Klein-Gordon Equation 8
D. The Expansion Modes Algebra 9
E. The "Virasoro-Klein-Gordon" Algebra 9

V. The Quantum Klein-Gordon String 9

VI. Strings and Quantum Geometry 12
A. Gauging Lorentz Transformations 12
B. The Einstein-Cartan String 13
C. Two Expressions for Area 13
D. 14

VII. Physical Implications 14

VIII. Discussion 14
A. Emergent Dimensions and LQG 14

Acknowledgments 15

A. Worldsheet Action in Terms of Transverse Velocity 15

B. The Bimetric String 15
1. Bimetric Polyakov Action 16
2. The Bimetric Polyakov Equations 16

References 17

TODO LIST

insert refs 2
insert ref 3
Insert brief intro to spin nets 3
Does this make sense? Check! 4
insert refs 4
insert refs 4

* dvaid79@gmail.com
† luigi.tiraque@gmail.com

insert refs 4

I. INTRODUCTION

"Quantum gravity" refers to the broad enterprise dedicated to finding a complete, consistent theory in which quantum mechanics coexists peacefully with general relativity. There are two major approaches to quantum gravity which are widely recognized for their success in describing various aspects of physics at the Planck scale. The first of these is string theory, the quantum theory of a one dimensional object which is said to contain within it a complete description of all particles, forces and their interactions, including gravity. However, where precisely in the vast space of effective field theories which can possibly arise as in the low energy limit of a conjectured "M-Theory", our Universe with its collection of particles, forces and coupling constants, exists, is still a matter of vigorous debate.

One of the primary criticisms of string theory is its apparent lack of "background independence". This is more general than the principle of *general covariance* which states that physics should be independent of the choice of co-ordinates. Background independence is the statement that any *quantum* theory of gravity - which, presumably, should provide a description not only of classical spacetimes, but also of fluctuating, semiclassical geometries and those in the deep quantum regime which do not have any sensible description in terms of any Riemannian geometry - should provide a description of physics which is independent, not only of the choice of co-ordinates, but also of the choice of background manifold on which those co-ordinates live. On the face of it string theory, at least in its conventional form, does not satisfy this principle.

It is sometimes argued that string theory is, in fact, background independent since the equations of motion which arise when the (Bosnic) string is coupled to the metric of the "worldvolume" (the background manifold in which the string is propagating) turn out to include Einstein's field equations for gravity. In other words, Weyl invariance - one of the fundamental symmetries of the string worldsheet, *requires* that the background geometry must satisfy Einstein's equations. While technically correct, it begs the question, if general relativity "emerges" from string theory then why is it not possible to quantize the string on an arbitrary background to begin with? As anybody who has studied the basics of string theory knows, the standard quantization procedure *assumes* that the worldvolume has a flat metric. It is, thus far, not technically feasible to do away with this assumption.

insert refs

There is another sense in which string theory falls short of being a theory of quantum gravity. This, however, rests upon a philosophical perspective, about the nature of quantum gravity, which one may or may not choose to subscribe to. This is the viewpoint that in any quantum theory of gravity, geometry itself must be quantized. Again, one might say that gravitons - the quantum perturbations of the metric - are "quanta of geometry". However, gravitons *live* upon some background manifold. Moreover, as mentioned, they are *perturbative* by construction. Gravitons cannot provide a quantum description of geometry, anymore than quantizing water waves can provide a picture of the molecular structure of water[1]. These considerations bring us to the second major[2] approach towards a quantum theory of gravity, known as Loop Quantum Gravity or LQG for short.

LQG arose almost by accident, something it has in common with string theory. Traditionally, there are two ways in which a classical theory can be quantized. These are the Hamiltonian and Lagrangian approaches. The Lagrangian or path integral approach follows the prescription first suggested by Dirac and then made concrete by Feynman. There one views the classical action associated to a given evolution as corresponding to a *phase angle* which determines the complex weight of the associated evolution. This approach respects spacetime covariance since here the central object is the action which is invariant under spacetime transformations by construction. The Hamiltonian approach, on the other hand, involves making a choice of a spacelike surface Σ_t and a corresponding timelike vector t^μ, normal to Σ_t (here t is a continuous parameter which labels the family of surfaces). This approach, therefore, necessarily does not respect spacetime covariance since it involves making a specific choice of the foliation of the background into spacelike surfaces.

In the decades prior to the advent of string theory a great deal of effort was put into attempting to quantize gravity via both the Lagrangian and Hamiltonian approaches. The Hamiltonian approach involves, as stated earlier, a choice of the foliation of the background geometry into a collection of spacelike surfaces Σ_t. Central to this approach was the ADM (Arnowitt-Deser-Misner) formalism which allows one to construct a Hamiltonian H_{GR} for general relativity starting from the Einstein-Hilbert action S_{EH}. The resulting phase is co-ordinatized by a configuration variable which is the 3-metric h_{ab} of the "leaves" Σ_t of the foliation and a momentum variable which is a function of a the extrinsic curvature k_{ab} of the leaves[3]. The gravity Hamiltonian turns out to be a sum of two constraints known as the diffeomorphism

[1] This viewpoint has been most clearly articulated by Jacobson [1]

[2] the characterization of LQG as a "major" approach is the our choice and not necessarily reflective of the consensus in the broader quantum gravity community.

[3] Our notational convention is the following. Lowercase Greek letters from the middle of the alphabet $\mu, \nu, \ldots$ are spacetime indices which run from $(0 \ldots 3)$, whereas lowercase Roman letters $a, b, \ldots$ are spatial indices for quantities which live solely on the surfaces Σ_t and take values in $(1, 2, 3)$

(or "momentum") constraint H_{diff} and the Hamiltonian constraint H_{ham}:

$$H_{GR} = H_{diff} + H_{ham}, \tag{1}$$

and these are, in turn, functionals on the phase space of general relativity written in terms of h_{ab} and k_{ab}. The idea then is that physical states of the theory $|\Psi_{phys}\rangle$, which can be written as functionals $\Psi[h_{ab}]$ of the 3-metric h_{ab} , must be annihilated by these constraints:

$$H_{GR}|\Psi_{phys}\rangle \equiv 0 \tag{2}$$

While this procedure is straightforward in principle, in practice is was impossible to implement in the quantum theory due to highly complicated non-polynomial dependence of the diffeomorphism and Hamiltonian constraints on the configuration and momentum variables. This unfortunate state of affairs persisted until the 1980s when Abhay Ashtekar recast general relativity as a theory of a Minkowski tetrad E^I_μ and a self-dual $\mathfrak{sl}(2,\mathbb{C})$ connection A^{IJ}_μ. Ashtekar realised that the constraints of general relativity, when expressed in terms of these "new variables", simplified drastically and became polynomial functions of the tetrad and connection variables. It was quickly realised by various researchers that the resulting theory could be quantized using the same methods used to quantize Yang-Mills gauge theory and that the diffeomorphism constraint could be solved exactly in terms of so-called *spin-network* states.

insert ref

Following this, work by Rovelli and Smolin and by Ashtekar, Rovelli and Smolin demonstrated the most remarkable feature of this theory, which came to be known as "Loop Quantum Gravity", was that one could construct quantum operators for geometric quantities such as areas of two dimensional surfaces and volumes of three dimensional regions. Moreover, these operators could be diagonlized exactly in the spin-network basis and a lower bound on the smallest possible quantum of area and quantum of volume could be derived. This was the first time that physicists had discovered the "atoms of space" or the "quanta of geometry" in the true sense of the expression.

Insert brief intro to spin nets

The following is the outline of the paper. In section I we recall the Nambu-Goto action and its symmetries. We explain how the bulk (embedding manifold) geometry is coupled to the worldsheet geometry and the conventional argument for the statement that string theory describes not only the object but also its background (the geometry it is propagating) in and is therefore a complete theory of quantum gravity. We critique this claim and explain in what way LQG might provide a resolution of some of the puzzles of string theory. In section III we present a phenomenological modification of the Nambu-Goto action based on the observation from LQG that there exists a non-zero, positive, eigenvalue of the area operator. We show that this action can be expanded in terms of a small coupling constant $g = l_{pl}/l_s$ where the Planck length - the length at which quantum geometric effects become important - l_{pl} is much smaller than the string scale l_s and therefore $g \ll 1$. We point out that the lowest order correction to the NG action can be viewed as a bimetric action for the string. We derive the classical equations of motion for this area-corrected NG action. In section IV we construct the Polyakov version of the modified NG action, show that the equations of motion it yields are the same as those coming from the modified NG action. Further we solve the string equations of motion to find the fundamental modes and perform the quantization of the Bosonic string in the usual manner. In section VI we describe how to construct an action for the Bosonic string which encodes the geometry of the bulk spacetime in the language of the connection formulation of general relativity. We discuss possible avenues for quantizing this action and further studying its implications for the relationship between string theory and LQG.

In section B we discuss further the bimetric extension of the string action in both the Nambu-Goto and Polyakov version.

II. STRINGS IN BACKGROUND FIELDS AND EMERGENT GRAVITY

Let us begin by recalling the Polyakov action for the bosonic string.

$$S_P = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \tag{3}$$

Here $\mu, \nu \in \{0, 1, \dots, D-1\}$ are co-ordinates of the D dimensional worldvolume (the geometry in which the string propagates), $a, b \in \{0, 1\}$ are co-ordinates on the string worldsheet, X^μ are the embedding co-ordinates which specify the location of a point on the string worldsheet in the bulk worldvolume, $\eta_{\mu\nu}$ is the flat Minkowski metric on the worldvolume, g_{ab} is the metric on the string worldsheet, T is the string tension and τ, σ are the co-ordinates on the string worldsheet.

Now, one proceeds in the usual way by determining the symmetries of the action (B6), varying the action to find the equations of motion, fixing the gauge using the Weyl freedom of the worldsheet and then solving the classical equations of motion. Imposition of (bosonic) commutation relations on the operator versions of the embedding fields $\hat{X}^\mu$ then leads us to description of the quantum state of the bosonic string in terms of an infinite ladder of harmonic oscillators which obey the Virasoro algebra.

The obvious drawback of this approach is that the background metric is non-dynamical and is fixed to be the flat Minkowski metric $\eta_{\mu\nu}$. Clearly, one would like to be able to understand the physics of a string propagating on an arbitrary curved background. It wouldn't make much sense to refer to string theory as a theory of "quantum gravity" if strings can only be described on flat backgrounds. The way this is accomplished is by treating the metric of the worldvolume as a "background field" $G_{\mu\nu}$, in terms of which the Polyakov action becomes:

$$S'_P = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X), \quad (4)$$

where the bulk metric is now a function of the bulk coordinates $G_{\mu\nu}(X)$. Now this metric is still non-dynamical because the action (4) does not contain any terms with time derivatives of $G_{\mu\nu}$. However, if we view the action purely as a two-dimensional theory of D scalar fields, then the bulk metric can be viewed as a collection of *coupling constants*, rather than a dynamical entity which exists independent of the string. One can now proceed in the usual manner for any field theory and calculate the beta function of the coupling constants of the theory as a function of the energy scale.

Then, as shown long ago by Friedan [2] (see also [3], [4] or [5, Sec 3.7], [6, Sec 7.2] for a more pedagogical explanation), that the garviton beta function is proportional to $R_{\mu\nu}$, the Ricci curvature of the bulk geometry. The requirement of Weyl invariance of the string worldsheet implies that this beta function should vanish:

$$\beta(G_{\mu\nu}) \propto R_{\mu\nu} = 0. \quad (5)$$

Therefore we find that conformal invariance of the string worldsheet *implies* that the background geometry in which the string propagates satisfies the *vacuum* Einstein equations. It is this result which is often cited as evidence for the claim that string theory is a *background independent* theory of quantum gravity.

In our opinion, however, this result is insufficient evidence for asserting that string theory is background independent. Our reasons are the following:

1. While the conformal invariance of the string worldsheet implies that the background is Ricci flat, this, in itself, is not very surprising. After all conformal invariance of worldsheet is equivalent to the statement of diffeomorphism invariance of the worldsheet, and in order for this to be consistent with the bulk geometry, the bulk geometry should also have a description in terms of diffeomorphism invariant physics. The simplest form of this requirement is that the background should satisfy the vacuum Einstein equations .

Does this make sense? Check

2. One can ask what is the quantum mechanical description of the bulk geometry? It is often suggested that the bulk geometry corresponds to a coherent state or a graviton condensate, where the graviton are themselves excitations of the fundamental string. However, there does not appear to be an actual mathematical realization of this statement in terms of non-perturbative physics.

3. Finally, as mentioned in the introduction, in any *quantum* theory of gravity one would expect to be able to describe the "atoms of geometry". There is no sense in which this is realised in this picture of gravity emerging from stringy physics. Even though one can calculate higher order corrections to the graviton beta function and this provides us with the quantum corrected effective action for the bulk geometry, all the physical quantities are still defined in terms of continuous fields such as the metric. For *e.g.* , to the best of our knowledge, there is no clear sense in which one construct a *non-perturbative* state which corresponds to a superposition of geometries using the quantum theory of a bosonic string alone.

Despite all the concerns string theory has emerged as the primary candidate for a theory of quantum gravity. There are good reasons for this. AdS/CFT - the most concrete version of a *non-perturbative* theory of quantum gravity - arose out of string theory. AdS/CFT is also far from being a complete quantum theory of quantum gravity. Our universe, for instance, is described by a positive cosmological constant $\Lambda > 0$ rather than $\Lambda < 0$ as is implied by the "Anti-deSitter" part of the correspondence. However, the correspondence has passed numerous checks and is now considered part of the established canon. Moreoever, it has led to numerous theoretical insights. We have gained a deep understanding of the properties of real world systems referred to as "strange metals" by studying their dual gravitational theories. The significance of quantum error correction and quantum information for a theory of quantum gravity was also first most clearly illuminated via AdS/CFT . For these, are other reasons too numerous to mention here, most theorists still regard string theory as the primary (or even the sole) candidate for a theory of quantum gravity.

insert refs

insert refs

On the other hand LQG is also built upon a rigorous mathematical framework which follows the well-established rules of quantization[4]. Moreoever it allows us to construct a non-perturbative framework for quantum geometry which makes it possible to probe regimes which do not appear to be accessible via string theory.

[4] Though there are those who disagree with this assertion

insert refs

Thus, we are now at an impasse. We have two theories, both built upon rigorous mathematical foundations, but which do not appear to be concordant with each other. There are two possibilities. The first is that one of the two approaches is fundamentally flawed in some way and must be rejected. The second is that we are missing some crucial ingredient which would allow us to find a missing link between the two theories. We feel that it worthwhile to explore the second option in greater detail, if for no other reason then to rule out the possibility of such a link between these two theories.

Before proceeding let us state our "central dogma". We take as physically well-motivated the central result of LQG that at the Planck scale well-defined geometric operators - such as those which measure the area and volume of a given region of space - exist, have discrete spectra and are gapped, *i.e.* , the smallest eigenvalue is greater than zero. We now wish to understand how to construct a worldsheet action which will reflect this central dogma.

III. MODIFIED NAMBU GOTO ACTION

In a very real sense string theory is a quantum theory of geometry. After all, the Nambu-Goto action:

$$S_{NG} = -T \int \mathrm{d}^2 x \sqrt{-\det h} \tag{6}$$

is nothing more than the *area* of the string worldsheet. So when we are constructing the quantum theory of the string we are constructing a theory in which the fundamental excitations are geometric in nature. What we would like to do is to modify the string action in a way which incorporates the insight from LQG that at the Planck scale there is a minimum quantum of area. One simple way in which to do this is to modify the integrand in (6) as follows:

$$\sqrt{-h} \to \sqrt{-(h + g\Delta)}.$$

Here, $g = l_{pl}/l_s$, where l_{pl} is the Planck scale, the scale at which quantum geometric effects become important; l_s is the string scale where quantum geometric effects due to area quantization become small; Δ is a scalar density of weight 2 which can be taken to be the determinant of some worldsheet tensor $\Delta_{ab}(x)$ and corresponds to the square of the quantum of area at the point x where the spin foam pierces the WS. In the limit that $l_s \gg l_{pl}$, $g \ll 1$ and the string action reduces to the conventional Nambu-Goto action with small corrections coming from quantum geometry. In this limit we can expand the integrand as follows:

$$\sqrt{-(h + g\Delta)} \sim \sqrt{-h}\left(1 - \frac{1}{2}\frac{g\Delta}{(-h)} + \mathcal{O}(g^2/h^2)\right). \tag{7}$$

Presumably the limit $g \ll 1$ corresponds to when the area of the string worldsheet is large compared to Δ, $h \gg \Delta$. Thus in this limit $g^2 \ll 1$ and $h^2 \gg 1$ because of which we can drop the $\mathcal{O}(g^2/h^2)$ corrections. The final area corrected Nambu-Goto action becomes:

$$S'_{NG} = -T \int \mathrm{d}^2 x \left(\sqrt{-h} - \frac{1}{2}\frac{g\Delta}{\sqrt{-h}}\right). \tag{8}$$

This action has, to the best of our knowledge, not been explored in the string theory literature. However, such an inverse determinant term arises very naturally in the LQG approach and goes by the name of "inverse triad" corrections, in the context of the 3D volume of spatial hypersurfaces.

The action eq (8) has manifest background Poincaré invariance since just like the regular Nambu-Goto action since all its terms are derivatives of X^μ with all Lorentz indices contracted, thus are Lorentz scalars. At first glance it might seem to not be WS reparameterization invariant because of the second term, however, this is corrected by the fact that Δ is a scalar *density* of weight 2, thus counteracting the inverse metric determinant. Moreover, if we look at that expression we see that it can be written as:

$$S'_{NG}[h_{ab}] = -T \int \mathrm{d}^2 x \left(\sqrt{-h} - \frac{1}{2} g\Delta\sqrt{-(h^{-1})}\right), \tag{9}$$

where h^{-1} is now the determinant of the *inverse* metric h^{ab}. As such, in the expression (8) we can see a duality between the first and second terms. If we replace the metric h_{ab} by its inverse $h'_{ab} \equiv h^{ab}$, then the action in terms of h'_{ab} is the same as (8), but with the difference that the factor of $g\Delta/2$ is mapped to $2/(g\Delta)$:

$$\begin{aligned} S'_{NG}[h'_{ab}] &= -T' \int \mathrm{d}^2 x' \left(\sqrt{-h'} - \frac{2}{g\Delta}\sqrt{-(h')^{-1}}\right) \\ &= -T' \int \mathrm{d}^2 x' \left(\sqrt{-h'} - \frac{1}{2} g'\Delta'\sqrt{-(h')^{-1}}\right), \end{aligned} \tag{10}$$

which has the same form as the original action (8), as long as make the replacements $T \to T' = \frac{1}{2}gT$, $\frac{1}{2}g\Delta \to \frac{1}{2}g'\Delta' = \frac{2}{g\Delta}$ and $\mathrm{d}^2x \to \mathrm{d}^2x' = \Delta \mathrm{d}^2 x$ (or equivalently $\mathrm{d}x^a \to \mathrm{d}x'_a = \Delta_{ab}\mathrm{d}x^b$). This is a duality between the physics at the Planck scale and the physics at the string scale, *i.e.* between strong coupling ($g' \gg 1$) and weak coupling ($g \ll 1$), small quantum of area (Δ) and large quantum of area (Δ') and between large string tension T and small string tension T'.

Now, one might object and say that having an action which is the sum of Nambu-Goto actions for two different metrics seems strange. However, such actions have been extensively studied under the heading of "bimetric gravity", with the earliest work dating as far back as 1940 in two papers [7, 8] by Einstein's future collaborator Nathan Rosen (the 'R' in 'EPR' and 'ER'). In 2010, bimetric gravity gained great popularity due to the seminal paper [9] which showed that by introducing a second *reference* metric into a model for massive gravity one can get rid of the Boulware-Deser ghost [10] which otherwise plagues theories with massive gravitons. The *reference* metric introduced in the work [9] was taken to be a flat metric. However later work [11, 12] showed that the second metric need not be flat and that an action of the following form:

$$S = M_g^2 \int \mathrm{d}^4x\sqrt{-g}R^{(g)} + M_f^2 \int \mathrm{d}^4x\sqrt{-f}R^{(f)} + 2m^2 M_{eff}^2 \int \mathrm{d}^4x\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n(\sqrt{g^{-1}f}), \tag{11}$$

where $g_{\mu\nu}$ and $f_{\mu\nu}$ are *arbitrary* metrics, M_g^2, M_f^2 are the Planck masses for the two sectors respectively. $R^{(g)}, R^{(f)}$ are the respective Ricci scalars for each sector. The third term is an interaction term which is responsible for making one of the gravitons massive while the other remains massless. M_{eff}^2 is an *effective* Planck mass given by:

$$M_{eff}^2 = \left(\frac{1}{M_g^2} + \frac{1}{M_f^2}\right)^{-1} \tag{12}$$

A. Modified Equations of Motion

Following the procedure done in [13], the correction(8) does not change the form of the equations of motion and neither of the boundary conditions, i.e they are still given by

$$\text{EoM} : \partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0 \tag{13}$$

and

$$\text{B.C.} : \mathcal{P}^\sigma_\mu \delta X^\mu\Big|_{\sigma=0}^{\sigma=\sigma_1} = 0, \tag{14}$$

where

$$\begin{aligned}\mathcal{P}^\tau_\mu &= \frac{\partial \mathscr{L}'_{NG}}{\partial \dot{X}^\mu}\\ &= -T\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{-h}}\left(1+\frac{g\Delta}{2(-h)}\right)\\ &= \mathcal{P}^\tau_{\mu(NG)}\left(1+\frac{g\Delta}{2(-h)}\right)\end{aligned} \tag{15}$$

$$\begin{aligned}\mathcal{P}^\sigma_\mu &= \frac{\partial \mathscr{L}'_{NG}}{\partial X'^\mu}\\ &= -T\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}}\left(1+\frac{g\Delta}{2(-h)}\right)\\ &= \mathcal{P}^\sigma_{\mu(NG)}\left(1+\frac{g\Delta}{2(-h)}\right)\end{aligned} \tag{16}$$

and $\mathcal{P}^\tau_{\mu(NG)}$, $\mathcal{P}^\sigma_{\mu(NG)}$ are the regular Nambu-Goto worldsheet currents.

In the following, we parametrize the string WS using the static gauge $\tau = t$ and transverse gauge

$$\frac{\partial X}{\partial s}\cdot\frac{\partial X}{\partial t} = 0, \tag{17}$$

where s is the length along the string parameter since this gives the motion of the string in terms of transverse velocity, which is a natural dynamical variable of the string. We find that the new form for the conserved energy per infinitesimal piece of the string is the regular energy of the Nambu-Goto string times the correction factor

$$E\mathrm{d}\sigma = T\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\mathrm{d}s, \tag{18}$$

and the form of the wave equation maintains its shape as in

$$\mu_{eff}\frac{\partial^2\vec{X}}{\partial t^2} - \frac{\partial}{\partial s}\left[T_{eff}\frac{\partial\vec{X}}{\partial s}\right] = 0, \tag{19}$$

where the effective mass density and effective tension are given by

$$\mu_{eff} = \mu\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right) \tag{20}$$

$$T_{eff} = \frac{T}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right). \tag{21}$$

Rewriting eq(19) in terms of σ derivatives and parameterizing σ as (let $F^+ = 1 + g\Delta/2(-h)$)

$$\sigma(q) = \frac{1}{T}\int_0^q \mathrm{d}E\frac{1}{(F^+)^2}, \tag{22}$$

we find the variable-speed wave equation

$$(F^+)^2\frac{\partial^2\vec{X}}{\partial t^2} - \frac{\partial^2\vec{X}}{\partial\sigma^2} = 0, \tag{23}$$

from which we can see that the correction factor modifies the speed of propagation of the wave in the string as

$$v = \frac{c}{F^+}. \tag{24}$$

IV. INVERSE METRIC POLYAKOV

In the same way we considered the quantum geometry inverse metric correction to the Nambu-Goto action, it is also natural to consider such correction to the Polyakov action. We start with the action(4) and proceed with the inverse area correction(7), which yields our corrected Polyakov action

$$S_{IAP} = -\frac{T}{2}\int \mathrm{d}^2x\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right)\times \times g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}. \tag{25}$$

One might think this action is incomplete since we only applied the correction to the determinant and didn't change the inverse metric g^{ab}, however the correction 7 is a correction to the *area* of the WS, not its metric. Moreover, there are other good reasons for this form of the Polyakov action: the first is that the action in this form, as we shall see, reduces neatly to the area-corrected Nambu-Goto action (8), and the second comes from a more rigorous bimetric Polyakov string analysis which will be done in appendix B, where we show that starting from a bimetric Polyakov string one can derive the action (25).

This action is even more manifestly background Poincaré invariant, since now the X^μ dependence is explicit with this being a first-order action with g_{ab} and X^μ treated as independent variables. This time, however, we lose WS Weyl symmetry, since the second term has inverse determinant multiplied by inverse metric.

The equations of motion for the world-sheet metric are completely unchanged by this correction,

$$\frac{\delta S_{IAP}}{\delta g^{cd}} = 0 \tag{26}$$

which leads to the stress-energy tensor for the worldsheet:

$$\begin{aligned} T_{cd} := \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} - \\ - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = 0, \end{aligned} \tag{27}$$

from which we can extract the metric g_{cd} as

$$g_{cd} = 2f \partial_c X^\mu \partial_d X^\nu G_{\mu\nu}, \tag{28}$$

where the function f is given by

$$\frac{1}{f} = g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}. \tag{29}$$

As for variation w.r.t the embbeding fields X^λ, we have

$$\frac{\delta S_{IAP}}{\delta X^\lambda} = 0 \tag{30}$$

$$\begin{aligned} F^- \sqrt{-g} g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} + \partial_a \left(F^- \sqrt{-g} g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} = \\ = \frac{1}{2} F^- \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu}, \end{aligned} \tag{31}$$

where

$$F^- \equiv 1 - \frac{k\Delta}{2(-g)}. \tag{32}$$

Since all 2-dimensional metrics are conformally flat [14], we use conformal gauge $g_{ab} = e^{\phi(x)} \eta_{ab}$ and assume flat background $G_{\mu\nu} = \eta_{\mu\nu}$, to simplify eq (31) into

$$\eta^{ab} \partial_a \partial_b X^\mu + \frac{1}{F^-} \eta^{ab} \partial_a F^- \partial_b X^\mu = 0. \tag{33}$$

By expanding the definition of F^- we get

$$\eta^{ab} \partial_a \partial_b X^\mu + \eta^{ab} \frac{(2\partial_a \phi - \partial_a \ln(\Delta))}{\frac{2e^{2\phi}}{k\Delta} - 1} \partial_b X^\mu = 0, \tag{34}$$

revealing a wave equation sourced by a coupling between the WS conformal factor, quantum of area field Δ and the embedding fields much akin to the dilaton field [15]. We'll come back to this in section VIII.

A. Simplifying The Equations of Motion

Equation(33) seems a bit odd, but we can polish it by choosing a plane-wave ansatz for the embedding fields $X^\mu = X_0^\mu e^{-i(E\tau - p\sigma)}$ and using it to fix a form for the conformal factor $e^{\phi(x)}$ appearing in F^-. By using the ansatz in eq(33), it becomes

$$(E^2 - p^2) X^\mu + i(E\partial_\tau \ln(F^-) - p\partial_\sigma \ln(F^-)) X^\mu = 0, \tag{35}$$

from which we can use the freedom in the conformal factor to fix the second term as $-m^2 X^\mu$, with $-m^2 = p^2 - E^2$ on-shell and this, in turn, implies that the derivatives of $\ln(F^-)$ are given by

$$\partial_\tau \ln(F^-) = iE, \;\; \partial_\sigma \ln(F^-) = ip, \tag{36}$$

thus we have

$$\ln(F^-) = i(E\tau + p\sigma) + a, \;\; a \in \mathbb{C} \tag{37}$$

$$F^- = A e^{i(E\tau + p\sigma)}, \;\; A \in \mathbb{C} \tag{38}$$

$$e^{\phi(x)} = \sqrt{\frac{k\Delta}{2\left(1 - A e^{i(E\tau + p\sigma)}\right)}}. \tag{39}$$

At first glance, it seems problematic that we have a complex conformal factor, but the complex structure comes from our ansatz for the embedding fields, and when we choose either the real or imaginary part of the latter it should also pick the respective component of e^ϕ. With this choice, we thus turned eq(33) into a Klein-Gordon equation

$$\left(\partial_\sigma^2 - \partial_\tau^2\right) X^\mu - \mu^2 X^\mu = 0. \tag{40}$$

The imposition that

$$i(E\partial_\tau \ln(F^-) - p\partial_\sigma \ln(F^-)) = -\mu^2 \tag{41}$$

can be used to find the explicit dependence of μ^2 with the quantum geometry parameters $k\Delta$ as in

$$\begin{aligned} \mu^2 = \frac{i}{\frac{2e^{2\phi}}{k\Delta} - 1} \left(p\partial_\sigma \ln\left(e^{2\phi}/\Delta\right) - \right. \\ \left. - E\partial_\tau \ln\left(e^{2\phi}/\Delta\right) \right). \end{aligned} \tag{42}$$

B. Relation With The Nambu-Goto Analysis

At first glance, our recent analysis of the Polyakov string with inverse metric correction seems to not be related to what we found in the analysis of the Nambu-Goto string. However, as pointed out in [16], wave equations with a variable speed (the result of the Nambu-Goto analysis) can be turned into Klein-Gordon equations (what we just found). While their work deals only with space-varying wave speed, it is straightforward to generalise the procedure to also include time dependence.

Following the procedure found in [16], we start with eq(23) obtained in sec(III A), renaming σ as x and dividing by $(F^+)^2$ such that we have v^2 on the second term like the mentioned work, and perform a change of variables $(t,x) \to (\tau(t,x), \sigma(t,x))$, where without loss of generality (one needs just to rescale and/or rotate the coordinates) we set

$$\begin{aligned}(\partial_t\sigma)^2 - v^2(\partial_x\sigma)^2 = v^2(\partial_x\tau)^2 - (\partial_t\tau)^2 \\ \partial_t\tau\partial_t\sigma = v^2\partial_x\tau\partial_x\sigma,\end{aligned} \tag{43}$$

or more compactly,

$$\begin{aligned}\partial_t\tau = v\partial_x\sigma \\ \partial_t\sigma = v\partial_x\tau\end{aligned} \quad , \tag{44}$$

which is equivalent to

$$\begin{aligned}\partial_t^2\tau - v^2\partial_x^2\tau = \partial_t v\partial_x\sigma + v\partial_x v\partial_x\tau \\ \partial_t^2\sigma - v^2\partial_x^2\sigma = \partial_t v\partial_x\tau + v\partial_x v\partial_x\sigma.\end{aligned} \tag{45}$$

This turns eq(23) into

$$\begin{aligned}\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu + \\ + \frac{\partial_t v\partial_x\sigma + v\partial_x v\partial_x\tau}{((\partial_t\tau)^2 - v^2(\partial_x\tau)^2)}\partial_\tau X^\mu + \\ + \frac{\partial_t v\partial_x\tau + v\partial_x v\partial_x\sigma}{((\partial_t\tau)^2 - v^2(\partial_x\tau)^2)}\partial_\sigma X^\mu = 0.\end{aligned} \tag{46}$$

Next, we introduce $X^\mu = \kappa(\tau,\sigma)W^\mu(\tau,\sigma)$ such that terms proportional to first derivatives of W^μ cancel out. This is obtained by choosing $\kappa = v^{1/2}$ and concluding that v only depends on x. With this, eq(46) turns into a Klein-Gordon equation

$$\partial_\tau^2 W^\mu - \partial_\sigma^2 W^\mu + \mu^2 W^\mu = 0, \tag{47}$$

with "mass" squared given by

$$\mu^2 = \frac{(\partial_x v)^2 - 2v\partial_x^2 v}{4\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)} \tag{48}$$

and $\tau(t,x)$ satisfying constraints(44). This shows that our action(25) is indeed in agreement with the inverse area corrected Nambu-Goto action(8).

C. Solving The String Klein-Gordon Equation

In the following, we focus on the analysis of the closed string unless otherwise stated.

From our analysis of the inverse-area corrected Polyakov action, we got D finite-space one-dimensional Klein-Gordon equations (40), with constraints given by eq(27). The constraints are nothing new, they're precisely the same as the ones of regular String Theory. What's new in our analysis is the μ^2 term, which we will assume to be constant for a first analysis, making it so the solution is given by Fourier transform

$$\begin{aligned}X^\mu(\tau,\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_{p\in\mathbb{Z}}\frac{1}{2E_p}\Big(\alpha_p^\mu e^{i(E_p\tau - p\sigma)} + \\ + \beta_p^\mu e^{-i(E_p\tau - p\sigma)}\Big),\end{aligned} \tag{49}$$

with $E_p = \sqrt{p^2 + \mu^2}$. From here on, following an analogous procedure to the one found in [17], reality of X^μ implies $(X^\mu)^* = X^\mu$, from which we get

$$\beta_n^\mu = (\alpha_n^\mu)^*. \tag{50}$$

When looking at the constraints, we see that we can't decouple the exponentials in τ from any of the sums, thus we get two-index "Virasoro modes"

$$L_{n,p} := \alpha_n^\mu\alpha_p^\nu\eta_{\mu\nu} = 0,\ \forall\, n,p\in\mathbb{Z}, \tag{51}$$

$$\widetilde{L}_{n,p} := \alpha_n^\mu(\alpha_p^\nu)^*\eta_{\mu\nu} = 0,\ n\neq p, \tag{52}$$

$$\sum_n\left(\left(1+\frac{n}{E_n}\right)^2\alpha_n^\mu(\alpha_n^\nu)^*\eta_{\mu\nu}\right) = 0. \tag{53}$$

Constraint (52) implies that we only have D independent expansion modes for each field X^μ, which for consistency as we will see later means $\alpha_{-p}^\mu = \alpha_p^\mu,\ p\leq d = D-1$ and $\alpha_n^\mu = 0,\ \forall\ |n| > d$, so we have a truncated solution. This is not surprising, since because space-time is discrete we expect wavelengths smaller than the granular structure to be ruled out.

Before proceeding, we have one remark: the 0-th term of the solution reads

$$\sqrt{\frac{\alpha'}{2}}\frac{1}{2\mu}\left(\alpha_0^\mu e^{-i\mu\tau} + (\alpha_0^\mu)^* e^{i\mu\tau}\right), \tag{54}$$

which is clearly divergent in the $\mu \to 0$ limit unless $\mathfrak{Re}(\alpha_0^\mu) \sim \mu$. Moreover, if we want to recover the classic String Theory expression in this regime, it is not difficult to see that we require

$$\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}}\mu x^\mu + i\frac{1}{2}\sqrt{\frac{\alpha'}{2}}p^\mu. \tag{55}$$

This in actuality will differ from the normal ST literature by a factor of 1/2 in the momentum term, but this is needed for the Poisson brackets to give the expected results.

Looking closer at our Virasoro-like "modes", in particular the $n = p$ condition, we get the mass of the string in terms of its vibrational modes and center of mass position

$$M^2 = -p^\mu p_\mu = \frac{4}{\alpha'}\left(\sum_{n\neq 0}\left(\left(1+\frac{n}{E_n}\right)^2 \widetilde{L}_{n,n}\right) - \right.$$
$$\left. - \frac{2\mu^2}{\alpha'}x^\mu x_\mu\right) = \quad (56)$$

$$= \frac{4}{\alpha'}\left(\sum_{n\neq 0}\left(\left(1+\frac{n}{E_n}\right)^2 \alpha_n^\mu(\alpha_n^\nu)^*\eta_{\mu\nu}\right) - \right.$$
$$\left. - \frac{2\mu^2}{\alpha'}x^\mu x_\mu\right). \quad (57)$$

D. The Expansion Modes Algebra

We start by calculating the canonical momentum density conjugate to the fields $X^\mu(x)$, $\Pi^\nu(x)$, given by

$$\Pi^\nu(x) = F^- \frac{i}{8\pi\alpha'}\sum_n \left(\alpha_n^\nu \mathrm{e}^{i(E_n\tau - n\sigma)} - \right.$$
$$\left. - (\alpha_n^\nu)^* \mathrm{e}^{-i(E_n\tau - n\sigma)}\right) \quad (58)$$

where

$$F^- = 1 - \frac{k\Delta}{2\mathrm{e}^{2\phi(x)}} =$$
$$= \sum_l \left(A_l \mathrm{e}^{i(E_l\tau + l\sigma)} + (A_l)^* \mathrm{e}^{-i(E_l\tau + l\sigma)}\right), \quad (59)$$

and then proceed to use the equal-time Poisson bracket relation at $\tau = 0$

$$\{X^\mu(0,\sigma), \Pi^\nu(0,\sigma')\} = \delta(\sigma - \sigma')\eta^{\mu\nu} \quad (60)$$

to calculate the bracket relations between the expansion modes α_n^μ and $(\alpha_p^\nu)^*$. After using the freedom we still have in choosing the expansion modes of F^- to gauge fix it to satisfy $\mathfrak{Re}(A_0) = -2\alpha'$ and realizing that $\alpha_{-n}^\mu = \alpha_n^\mu$, we get the desired

$$\left\{\alpha_n^\mu, \alpha_p^\nu\right\} = 0 = \left\{(\alpha_n^\mu)^*, (\alpha_p^\nu)^*\right\} \quad (61)$$

$$\left\{\alpha_n^\mu, (\alpha_p^\nu)^*\right\} = -iE_n\delta_{|n|,|p|}\eta^{\mu\nu}, \quad (62)$$

from which we can normalize the expansion modes to get harmonic oscillator modes and bracket relations

$$a_n^\mu := \frac{1}{\sqrt{E_n}}\alpha_n^\mu \quad (63)$$

$$(a_n^\mu)^* := \frac{1}{\sqrt{E_n}}(\alpha_n^\mu)^* \quad (64)$$

$$\left\{a_n^\mu, (a_p^\nu)^*\right\} = -i\delta_{|m|,|l|}\eta^{\mu\nu}. \quad (65)$$

The bracket relations for α_0^μ can be used to get the expected bracket relations between x^μ and p^ν

$$\{x^\mu, x^\nu\} = 0 = \{p^\mu, p^\nu\} \quad (66)$$
$$\{x^\mu, p^\nu\} = \eta^{\mu\nu}. \quad (67)$$

E. The "Virasoro-Klein-Gordon" Algebra

Having the bracket relations between the vibrational modes of the string, we can now calculate the constraint algebra, which we shall refer to as the Virasoro-Klein-Gordon Algebra. For simplicity, since $E_n = \sqrt{n^2+\mu^2} > 0$, we shall work with the normalized harmonic oscillator modes a_n^μ instead of the expansion modes α_n^μ. The algebra is given by

$$\{L_{n,m}, L_{k,p}\} = 0 \quad (68)$$

$$\left\{L_{n,m}, \widetilde{L}_{k,p}\right\} = -i\left(L_{m,k}\delta_{|n|,|p|} + L_{n,k}\delta_{|m|,|p|}\right) \quad (69)$$

$$\left\{\widetilde{L}_{n,m}, \widetilde{L}_{k,p}\right\} = -i\left(\widetilde{L}_{k,m}\delta_{|n|,|p|} - \widetilde{L}_{n,p}\delta_{|m|,|k|}\right). \quad (70)$$

V. THE QUANTUM KLEIN-GORDON STRING

We now seek to quantize the Klein-Gordon string by means of covariant canonical quantization, in a manner much similar to those found in ([17],[18]). The string coordinates X^μ and it's conjugate momentum Π^ν are turned into operators $\hat{X}^\mu$ and $\hat{\Pi}^\nu$, with the Poisson brackets $\{.,.\}$ being replaced by commutators $[.,.] = i\hbar\{.,.\}$:

$$\left[\hat{X}^\mu(\tau,\sigma), \hat{\Pi}^\nu(\tau,\sigma')\right] = i\delta(\sigma-\sigma')\eta^{\mu\nu} \quad (71)$$

$$\left[\hat{X}^\mu, \hat{X}^\nu\right] = 0 = \left[\hat{\Pi}^\mu, \hat{\Pi}^\nu\right] \quad (72)$$

$$\left[\hat{a}_n^\mu, \hat{a}_p^\nu\right] = 0 = \left[(\hat{a}_n^\mu)^\dagger, (\hat{a}_p^\nu)^\dagger\right] \quad (73)$$

$$\left[\hat{a}_n^\mu, (\hat{a}_p^\nu)^\dagger\right] = \delta_{|n|,|p|}\eta^{\mu\nu} \quad (74)$$

$$[\hat{x}^\mu, \hat{x}^\nu] = 0 = [\hat{p}^\mu, \hat{p}^\nu] \quad (75)$$

$$[\hat{x}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu} \quad (76)$$

As usual, we define the vacuum state of the string to obey

$$\hat{a}_p^\mu|0\rangle = 0, \text{ for } p \neq 0. \quad (77)$$

For $p = 0$ we have center of mass position and momentum, so the vacuum has extra structure obeying

$$\hat{x}^\mu|0;x\rangle = x^\mu|0;x\rangle \quad (78)$$

$$\hat{p}_\mu|0;0\rangle = -i\frac{\partial}{\partial x^\mu}|0;x\rangle \quad (79)$$

in position representation and

$$\hat{x}^\mu|0;p\rangle = -i\frac{\partial}{\partial p_\mu}|0;p\rangle \tag{80}$$

$$\hat{p}_\mu|0;p\rangle = p_\mu|0;p\rangle \tag{81}$$

in momentum representation.

Our Fock space is built from the vacuum state $|0\rangle$ by operating with a sequence of creation operators

$$|\psi\rangle = \prod_{i\neq 0}((\hat{a}_i^{\mu_i})^\dagger)^{n_i}|0\rangle. \tag{82}$$

As usual, the appearance of the Minkowski metric in the non-zero commutator

$$\left[\hat{a}_p^\mu, (\hat{a}_k^\nu)^\dagger\right] = \eta^{\mu\nu}\delta_{|p|,|k|}, \tag{83}$$

brings ghosts into the theory which must be dealt with.

When translating the constraints as quantum operators, we require that they have vanishing matrix elements when sandwiched by physical states ψ and ϕ:

$$\langle\phi|\hat{L}_{p,k}|\psi\rangle = 0 \tag{84}$$

$$\langle\phi|\hat{\tilde{L}}_{p,k}|\psi\rangle = 0. \tag{85}$$

$\hat{L}_{p,k}$ has no ambiguities since it's composed of only annihilation operators. As for $\tilde{L}_{p,k}$, we have an ambiguity for $p = k \neq 0$, so we pick normal ordering with the annihilation operator to the right

$$\hat{\tilde{L}}_{p,p} = (\hat{\alpha}_p^\mu)^\dagger\hat{\alpha}_p^\nu\eta_{\mu\nu}. \tag{86}$$

This ambiguity manifests in the imposition of the constraint as

$$\langle\phi|\left(\sum_{n\neq 0}\left(\left(1+\frac{n}{E_n}\right)^2\hat{\tilde{L}}_{n,n}\right) - a\right)|\psi\rangle = 0, \tag{87}$$

for some constant a. Classically, we had eq (56) for the string rest mass, from which we can now see that the string mass spectrum will get shifted by

$$\hat{M}^2 = \frac{8}{\alpha'}\left(\frac{2\mu^2}{\alpha'}\hat{x}^\mu\hat{x}_\mu - \right.$$
$$\left. - \left(\sum_{n\neq 0}\left(\left(1+\frac{n}{E_n}\right)^2 E_n(\hat{a}_n^\mu)^\dagger\hat{a}_n^\nu\eta_{\mu\nu}\right) - a\right)\right). \tag{88}$$

This constant can be calculated by imposing normal-ordering on the summed constraint since $n \in (-d, d)$ is finite, and the result is

$$a = -D\sum_{n\neq 0}\left(1+\frac{n}{E_n}\right)^2 E_n =$$
$$= -D\sum_{n>0}\left(1+\frac{n^2}{E_n^2}\right)E_n. \tag{89}$$

The commutation relations between the constraints are inherited by the Poisson brackets

$$\left[\hat{L}_{n,m}, \hat{L}_{k,p}\right] = 0 \tag{90}$$

$$\left[\hat{L}_{n,m}, \hat{\tilde{L}}_{k,p}\right] = \hat{L}_{m,k}\delta_{|n|,|p|} + \hat{L}_{n,k}\delta_{|m|,|p|} \tag{91}$$

$$\left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}\right] = \hat{\tilde{L}}_{k,m}\delta_{|n|,|p|} - \hat{\tilde{L}}_{n,p}\delta_{|m|,|k|}. \tag{92}$$

The commutator (91) ends up with only annihilation operators in the R.H.S, so it has no ordering ambiguities, meanwhile the commutator (92) have ordering ambiguities for $|m| = |k|$ and/or $|n| = |p|$, so we add the anomalous terms

$$\left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}\right] = \hat{\tilde{L}}_{k,m}\delta_{|n|,|p|} - \hat{\tilde{L}}_{n,p}\delta_{|m|,|k|} +$$
$$+ C_n\delta_{|n|,|p|} + D_k\delta_{|m|,|k|}. \tag{93}$$

Clearly, $C_0 = D_0 = 0$, since when all indices are 0 we have only the position and momentum operators, and thus the commutator should vanish identically. Also, because of the symmetry in the indices we have that $C_{-n} = C_n$ and $D_{-k} = D_k$. Using the Jacobi identity

$$\left[\hat{\tilde{L}}_{n,m}, \left[\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{r,s}\right]\right] +$$
$$+ \left[\hat{\tilde{L}}_{r,s}, \left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}\right]\right] +$$
$$+ \left[\hat{\tilde{L}}_{k,p}, \left[\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{n,m}\right]\right] = 0 \tag{94}$$

we get the difference equation

$$F_{n+1} + F_n + F_1 = 0 \tag{95}$$

for the sum of both sequences $F_n = C_n + D_n$. This difference equation has solution

$$F_n = \frac{F_1}{2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right), \tag{96}$$

and since both C_n and D_n need to satisfy the initial value of 0, each of then is half of F_n,

$$C_n = D_n = \frac{F_n}{2} = \frac{F_1}{4}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right). \tag{97}$$

Next, by calculating the vacuum expectation value (VEV) of the commutator (92) with $|m| \neq |k|$ and $n = p = 1$, we find that $F_1 = 0$, thus both sequences $C_n = D_n = 0$ vanish and we conclude that the quantum version of our algebra *has no anomalies*. This is a relevant result, since it could mean our model doesn't need a specific number of dimensions to work, which will be explored again later in section VIII.

A quick calculation reveals that the WS Hamiltonian and momentum operators are given by

$$\hat{H} = -\frac{1}{2}\sum_n \hat{\tilde{L}}_{n,n} \tag{98}$$

$$\hat{P} = -\frac{1}{2}\sum_n \frac{n}{E_n}\hat{\tilde{L}}_{n,n} \tag{99}$$

since those generate τ and σ translations respectively

$$\left[\hat{H}, \hat{X}^{\mu}\right] = -i\partial_{\tau}\hat{X}^{\mu} \tag{100}$$

$$\left[\hat{P}, \hat{X}^{\mu}\right] = i\partial_{\sigma}\hat{X}^{\mu}. \tag{101}$$

Clearly, $\hat{P}$ does not necessarily annihilate physical states (see (99)), thus those are not required to be invariant under rigid rotations of the string. *Physical states for the KG string have non-conserved WS momentum.* Also, we see that there is no level-matching between left and right-moving modes since those are not distinct in our model.

Let us denote the ground state with momentum p^{μ} by $|0;p\rangle$. Then, the mass-shell condition (88) implies that

$$\frac{\alpha'}{8}p^2 = \frac{2\mu^2}{\alpha'}\left|\frac{\partial\psi(p)}{\partial p}\right|^2 + a. \tag{102}$$

Next we look at the n-th excited vector state $\zeta_{\mu}(\hat{a}^{\mu}_{n})^{\dagger}$ with $\zeta_{\mu} = \zeta_{\mu}(p)$ being the polarization covector. The reason we consider closed string vector states unlike normal String Theory is that in our model, the closed string solution doesn't have distinct left and right sectors, thus permitting spin-1 states. We'll discuss these closed string single excitation states in section VIII. The mass-shell now reads

$$\frac{\alpha'}{8}M^2 = \left(\frac{2\mu^2}{\alpha'}\left|\frac{\partial\psi(p)}{\partial p}\right|^2 - \right.$$
$$\left. -\left(1+\frac{n^2}{n^2+\mu^2}\right)\sqrt{n^2+\mu^2} + a\right), \tag{103}$$

while the auxiliary $\langle 0;p|\hat{\tilde{L}}_{n,0}\zeta_{\mu'}(\hat{a}^{\mu'}_{d})^{\dagger}|0;p\rangle = 0$ implies that

$$\zeta_{\mu}p^{\mu} = -\frac{4\mu}{\alpha'}\zeta_{\mu}\frac{\partial\psi(p)}{\partial p_{\mu}}. \tag{104}$$

If the momentum wave-function is taken to be

$$\psi(p) = -\frac{\alpha'}{8\sqrt{2}\mu}p^{\mu}p_{\mu}, \tag{105}$$

eq (104) reduces to $\zeta_{\mu}p^{\mu} = 0$. Putting this back in the mass-shell yields

$$\frac{\alpha'}{16}p^2 = a = -D\sum_{k>0}\left(1+\frac{k^2}{E_k^2}\right)E_k \tag{106}$$

for the ground state and

$$\frac{\alpha'}{16}p^2 = a - \left(1+\frac{n^2}{n^2+\mu^2}\right)\sqrt{n^2+\mu^2} =$$
$$= -D\sum_{k>0}\left(1+\frac{k^2}{E_k^2}\right)E_k - \left(1+\frac{n^2}{E_n^2}\right)E_n \tag{107}$$

for the n-th state. The norm of these states is given by $\zeta_{\mu}\zeta^{\mu}$, and since by eq (107) we see that $p^2 < 0$, thus $\zeta^2 > 0$ always holds and we have no ghosts in the closed string spin-1 spectrum. This analysis, however, is for vector states from the closed string. The actual particle we want from the closed string is a symmetric, traceless spin-2 particle, so let's now consider the state $\zeta_{\mu\nu}(\hat{a}^{\mu}_{n})^{\dagger}(\hat{a}^{\nu}_{n})^{\dagger}|0;p\rangle$ where $\zeta_{\mu\nu}$ is the polarization tensor. The mass-shell reads

$$\frac{\alpha'}{16}p^2 = -D\sum_{k>0}\left(1+\frac{k^2}{E_k^2}\right)E_k - 4\left(1+\frac{n^2}{E_n^2}\right)E_n. \tag{108}$$

The auxiliary $\langle 0;p|\xi_{\alpha}\hat{a}^{\alpha}_{n}\hat{\tilde{L}}_{n,0}(\hat{a}^{\mu}_{n})^{\dagger}(\hat{a}^{\nu}_{n})^{\dagger}\zeta_{\mu\nu}|0;p\rangle = 0$ condition where ξ_{α} is a arbitrary polarization for the auxiliary vector excitation implies that

$$(\zeta_{\mu\nu} + \zeta_{\nu\mu})p^{\mu} = 0. \tag{109}$$

We can decompose the polarization into a symmetric traceless part $G_{\mu\nu}$, an anti-symmetric part $B_{\mu\nu}$ and a trace part $\Phi\eta_{\mu\nu}$ given by

$$G_{\mu\nu} = \frac{1}{2}(\zeta_{\mu\nu} + \zeta_{\nu\mu}) - \frac{1}{D}\eta^{\alpha\beta}\zeta_{\alpha\beta}\eta_{\mu\nu} \tag{110}$$

$$B_{\mu\nu} = \frac{1}{2}(\zeta_{\mu\nu} - \zeta_{\nu\mu}) \tag{111}$$

$$\Phi = \frac{1}{D}\eta^{\alpha\beta}\zeta_{\alpha\beta}, \tag{112}$$

$\zeta_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu} + \Phi\eta_{\mu\nu}$, giving rise to 3 well-known types of background fields: the spacetime metric $G_{\mu\nu}$, Kalb-Ramond 2-form $B_{\mu\nu}$ and dilaton Φ, all of those having squared mass given by eq (108). We thus see that the symmetric part $S_{\mu\nu} = G_{\mu\nu} + \Phi\eta_{\mu\nu}$ of $\zeta_{\mu\nu}$ is orthogonal to the momentum of the string, $S_{\mu\nu}p^{\mu} = 0$, and since $p^2 < 0$, we can choose a reference frame where $p^{\mu} = (p, 0, ..., 0)$ only has a time component, leaving us with $S_{0\nu} = 0$, implying that $\zeta_{0\nu} = -\zeta_{\nu 0}$. This condition guarantees that the timelike components of $\zeta_{\mu\nu}$ have a positive contribution to its square $\zeta^{\mu\nu}\zeta_{\mu\nu} = \zeta^{ij}\zeta_{ij} + b$, $b = \zeta^{0i}\zeta_{0i} > 0$, and the remaining part is spacelike thus must also have positive square, so we get that $\zeta^{\mu\nu}\zeta_{\mu\nu} > 0$, so we are free of ghosts in the spin-2 spectrum.

It seems we're in trouble since eq (108) tells us we have no massless 2 excitation states, but in fact, this is not a problem and is actually expected since quantum geometric effects are expected to violate Lorentz invariance near the Planck scale [19], making it so massless particles effectively travel slightly slower than c, with corrections on the order of $\sim E_{pl}^{-4}$.

The fact that $G_{\mu\nu}$ is symmetric, traceless and satisfies $G_{\mu\nu}p^{\mu} = 0$ leaves it with

$$\frac{1}{2}D(D-1) - 1 \tag{113}$$

possible polarizations, which shows that indeed those polarizations fit nicely in a representation of $\mathrm{SO}(D-1,1)$, *i.e.* the traceless symmetric tensor representation, thus all our excited spin-2 states are consistent with Lorentz symmetry at the macroscopic scale even tho near the Planck scale they deviate a bit by having an effective mass.

Although our method is quite different from the one in [20], we got similar results: a quantum Virasoro-like algebra with no anomalies, unconstrained background dimensionality D and no negative-norm states in physical processes.

VI. STRINGS AND QUANTUM GEOMETRY

Uptil now we have studied a phenomenological modification of the string action. However, ultimately all phenomenological ideas must have a grounding in some exact, microscopic description of the system. Further, we have neglected one very important aspect of this whole endeavor, which is the description of the background geometry in the connection formalism and how to couple that to the Bosonic string. In this section we present a possible action in which the bulk geometry is encoded in the string action in terms of a connection living on the worldsheet.

First let us recall the form of the Polyakov action for the bosonic string (3):

$$S_P = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}.$$

If one was to ask a beginning graduate student as to how one can incorporate a gauge field in the above expression, the natural answer would be that we can promote the X^μ to vectors which live in the representation of some gauge group $\mathcal{G}$ and promote the ordinary derivative to a gauge covariant derivative *w.r.t.* a connection valued in $\mathcal{G}$. The resulting action would now have the form:

$$S_P = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \sqrt{-g} g^{ab} \mathcal{D}_a X^\mu \mathcal{D}_b X^\nu \eta_{\mu\nu}, \tag{114}$$

where the covariant derivative $\mathcal{D}_a$ is of the form:

$$\mathcal{D}_a X^\mu(\tau,\sigma) = \partial_a X^\mu + g \mathcal{A}^{\mu}_{a\,\nu} X^\nu, \tag{115}$$

where $\mathcal{A}^{\mu}_{a\,\nu}$ is the gauge field and g is the gauge coupling. A more experienced researcher would object saying that we know that the X^μ are spacetime co-ordinates and not vectors transforming

Let us begin by writing down the action of a string propagating in a background geometry. The background geometry is specified not in terms of a metric, but in terms of a connection A^{IJ}_μ and vielbein fields E^I_μ. Further, we will require that the bulk connection and the bulk vielbein can be pulled back onto a connection $\mathcal{A}^{ij}_a$ and a dyad e^a_i, respectively, living on the string worldsheet Σ. $X^\mu(\tau,\sigma)$ are the space-time coordinates of a point (τ,σ) on the worldsheet. Now if one takes the sigma model viewpoint then the X's can be viewed as $d+1$ scalar fields living on a $1+1$ worldsheet. But these fields are not really scalars. They are components of a $d+1$ dim vector since they transform with the fundamental representation of $\mathrm{SO}(d,1)$ under global Lorentz transformations acting on the ambient space-time, which we take to be flat for the time being.

The key point here is that the Lorentz transformations acting on the X's are **global**. If the background geometry is taken to be Minkowski then the X's become points on this manifold and $\mathrm{SO}(d,1)$ just takes one point in Minkowski to another point in Minkowski. The overall geometry is invariant under $\mathrm{SO}(d,1)$. Now, consider the case when the background geometry is not flat. Immediately you can see that the X's can no longer be treated as elements of a $\mathrm{SO}(d,1)$ vector under **global** Lorentz transformations.

A. Gauging Lorentz Transformations

Instead of viewing the X's as elements of a Minkowski space, we think of them as elements of the tangent space $T_p(M)$ at a given point p of the background manifold M. When we travel along some path $\mathcal{C} \subset M$ from point p to point p', the local Lorentz frames at the two points are, in general, related by an element of $g(p,p') \in \mathrm{SO}(d,1)$. This group element arises from taking the holonomy of a $\mathfrak{so}(d,1)$ Lie algebra valued connection living on M, along a the path $\mathcal{C}$.

Let us remind ourselves what we're trying to do here. We want a gauge field, which lives on the background geometry and which encodes the background geometry, to somehow make an appearance in the action of a Bosonic string. There is an obvious connection $\mathcal{A}^{IJ}_a$ living on the worldsheet which is the pullback from M to Σ of the connection A^{IJ}_μ.

The end result is that we now have with a gauge field $\mathcal{A}^{IJ}_a(\tau,\sigma)$ living on the worldsheet Σ. Here the lower index a is the worldsheet index and I,J labels the generators of the Lie algebra $\mathfrak{so}(d,1)$. Remember this is a gauge field which is now living on the worldsheet. The consequence for the string action is that in order for the action to be invariant under local gauge transformations of $\mathcal{A}$, we have to replace the ordinary derivatives of X^μ with the gauge covariant derivative $\partial_a X^\mu \to \mathcal{D}_a X^\mu$, which is defined as:

$$\mathcal{D}_a X^I(\tau,\sigma) = \partial_a X^I + k \mathcal{A}^{I}_{aJ} X^J, \tag{116}$$

where $\mathcal{A}_a^{IJ}$ is a field living on Σ which takes values in the Lie algebra $\mathfrak{so}(d,1)$. We should therefore think of the embedding fields as $d+1$ dimensional vectors living in the fundamental representation of $\mathrm{SO}(d,1)$.

This viewpoint on the embedding fields, as elements of the tangent space $T_p(M)$, might appear to be at odds with their prescribed usage as co-ordinate functions on M. However, some reflection shows that this is not the case. For instance, one can take the embedding manifold to be a two-dimensional sphere S^2. Then the co-ordinates of a point on the string worldsheet Σ can be expressed in terms of the polar and azimuthal angles θ and ϕ. The action for this system would be:

$$S_{S^2} = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a\theta^\mu\partial_b\theta^\nu G_{\mu\nu}, \tag{117}$$

where $\theta^\mu(\sigma,\tau) \in \{\theta,\phi\}$ are the embedding fields, and $G_{\mu\nu}$ is *a* metric living on S^2.

B. The Einstein-Cartan String

The careful reader will have noticed that there is a serious difficulty in the above argument. The bulk geometry is $d+1$ dimensional, therefore there are $d+1$ embedding fields. We have taken these fields X^μ and cast them as components of a $SO(d,1)$ vector. Let us pick an arbitrary point P_0 on the bulk manifold and call it the "origin". At P_0 the co-ordinates are all zero $\vec{X}_0 = (0,0,\ldots,0)$. If we start acting with infinitesimal $SO(d,1)$ transformations on $\vec{X}_0$, the resulting surface will correspond to a "constant energy" hyperboloid in the bulk. This is because $SO(d,1)$ contains only boosts and rotations. We also need translations to generate the entirety of the bulk starting from a single base point.

This is where we run into a problem. There is no way to express translations in $d+1$ dimensions as the action of a $d+1$ dimensional matrix valued group on a point living in $\mathbb{R}^{d+1}$. In order to incorporate translations we have to resort to a trick and introduce an extra "dummy" dimension for the background manifold. Now we have $d+2$ embedding fields which we can write as:

$$X^A := \{X^0, \vec{X}, X^r\} = \{X^\mu, X^r\}, \tag{118}$$

where $X^\mu = \{X^0, \vec{X}\}$ are the fields we had originally and X^r is our new "dummy" co-ordinate.

treating the X's as co-ordinates for the bulk background and also as

C. Two Expressions for Area

There is another explanation for why it makes sense to promote the X's to local fields. This can be found in terms of the expression for the area of a 2-surface. We have a manifold $\mathcal{M}$, within which is embedded a two dimensional submanifold $\mathcal{N}$. Let (τ,σ) be the location of a point on the worldsheet and $X^\mu(\tau,\sigma)$ be the location of that same point in terms of co-ordinates on $\mathcal{M}$. Then we can define the induced metric on $\mathcal{N}$, h_{ab} in terms of the embedding fields and their derivatives:

$$h_{ab} = \frac{\partial X^\mu}{\partial x^a}\frac{\partial X^\nu}{\partial x^b}g_{\mu\nu}, \tag{119}$$

where $g_{\mu\nu}$ is the metric in the ambient space $\mathcal{M}$. Now, we can calculate the area of a small patch of $\mathcal{P}\in\mathcal{N}$ by taking the square root of the determinant of the induced metric $\sqrt{\det(h)}$ and integrating over that patch:

$$A^{(1)}_{\mathcal{P}} = \int_{\mathcal{P}} d^2x\,\sqrt{\det(h)}. \tag{120}$$

This is, of course, the expression for the Nambu-Goto action - modulo some constants - which we all know and love. In LQG we don't have a notion of embedding fields. Since we are dealing with gauge invariant observables constructed from holonomies of a connection along a 1d-curve and the flux of a tetrad across a 2-surface, the embedding co-ordinates of the curves and surface turn to not be needed. When promoting classical observables into quantum operators acting on lines and surfaces the embedding co-ordinates of the lines and surfaces are not relevant as long as we work with gauge invariant quantities.

How then does one define the area of a surface in LQG? It is defined as the magnitude of the area two form. Recall that given a tetradic basis e^I_μ for a local Lorentz frame whose metric is given by $g_{\mu\nu}$, we can write the metric in terms of the tetrad in the usual manner:

$$g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}, \tag{121}$$

where η_{IJ} is the flat Minkowski metric on the "internal" flat space. If we consider any two dimensional surface in the local Lorentz frame, then its area can be written in terms of the vielbein as:

$$A^{(2)}_{\mathcal{P}} = \int_{\mathcal{P}} \mathrm{Tr}(e_\mu \wedge e_\nu)dx^\mu \wedge dx^\nu, \tag{122}$$

the trace is taken over the "internal" Lorentz indices.

Now, if we compare the two expressions (119) and (121) we notice an obvious parallel between the derivatives $\partial X^\mu/\partial x_a$ of the embedding co-ordinates and the vielbein e^I_μ. This similarity becomes more apparent once we write down the expression for the area of a 2-surface by taking the determinant in (120) and comparing that to the expression obtained using frame fields.

From Embedding Fields:

We take the bulk metric $g_{\mu\nu}$ to be flat. Let $\{\tau,\sigma\} := \{x^0,x^1\}$ be the spacetime co-ordinates on the worldsheet.

Derivatives of the X's in terms of these co-ordinates can be written as:

$$\dot{X} := \left\{ \frac{\partial X^\mu}{\partial \tau} \right\}; \quad X' := \left\{ \frac{\partial X^\mu}{\partial \sigma} \right\} \tag{123}$$

we obtain the following form for the determinant of the worldsheet metric ((120)) in terms of the embedding fields:

$$\begin{aligned} \det h = \begin{vmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{vmatrix} &= \begin{vmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{vmatrix} \\ &= \dot{X}^2 X'^2 - (\dot{X}\cdot X')^2 \end{aligned} \tag{124}$$

From Vielbein:

We can proceed to obtain the area of a 2-surface from the bulk vielbein as follows. Starting with the bulk vielbein e^I_μ, we perform a local Lorentz transformation such that two "legs" of the vielbein are tangent to the worldsheet surface. We then project down the vielbein e^I_μ to a dyad e^i_a living in the tangent space of the worldsheet, in terms of which the worldsheet metric can be written as:

$$h_{ab} = e^i_a e^j_b \eta_{ij}, \tag{125}$$

where η_{ij} is the flat $(1+1)$ Minkowski metric. The determinant then becomes:

$$\begin{aligned} \det h = \begin{vmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{vmatrix} &= \begin{vmatrix} e_x{}^i e_x{}^j & e_x{}^i e_y{}^j \\ e_y{}^i e_x{}^j & e_y{}^i e_y{}^j \end{vmatrix} \eta_{ij} \\ &= \vec{e}_x{}^2 \vec{e}_y{}^2 - (\vec{e}_x \cdot \vec{e}_y)^2, \end{aligned} \tag{126}$$

where $\vec{e}_x \cdot \vec{e}_y = e_x{}^i e_y{}^j \eta_{ij}$. It is clear by comparing the two expressions (124) and (126) that we have the following correspondence between the vielbein the X derivatives:

$$\vec{e}_\tau \equiv \frac{\partial \vec{X}}{\partial \tau}; \quad \vec{e}_\sigma \equiv \frac{\partial \vec{X}}{\partial \sigma}. \tag{127}$$

In other words we can identify the vielbein with the derivatives of the embedding fields. Of course, this is not altogether surprising. If we consider any manifold on which we have some set of fields $\{X^I(x^\mu)\}$, where I is valued in the Lie algebra $\}$ of some group $\mathcal{G}$, as a function of some "fiducial" co-ordinates $\{x^\mu\}$, then we can always define a set of $\}$ valued vector fields:

$$e^I_\mu := \frac{\partial X^I}{\partial x^\mu}.$$

In the usual formulation of the string action the embedding fields are *not* Lie-algebra valued but are co-ordinate fields on what is generally taken to be a flat background. In order for this correspondence between vielbein and X derivatives to work we have to view the X fields as taking values in a Lie algebra. But these same X fields *also* serve as local co-ordinates for the bulk. In order for the co-ordinates to change from point to point in the bulk geometry, we must therefore require that the connection $\mathcal{A}$ in (116) can never vanish completely! If $\mathcal{A}$ vanishes then that will imply that the fields X don't change from one point to the next and cannot therefore be viewed as co-ordinate functions. Thus the amplitude $|\mathcal{A}|$ can be arbitrarily small but it can never be zero.

D.

VII. PHYSICAL IMPLICATIONS

VIII. DISCUSSION

We have presented in this work a possible way to formulate string theory in background independent manner.

In section V we saw that our model allows closed string spin-1 excitations at the first level, which is due to our solution not having independent left and right sectors. Closed string excitations are usually related to the gravitational field and background spacetime itself, so what could the first level vector states of the closed string relate to? Well, in section VI A we discussed about a possible formulation of ST in terms of a Lorentz connection $\mathcal{A}_\mu$, which is an ingredient in describing the gravitational field in tetrad-connection [19] formulation, and so this seems to be the natural candidate as to what the first level vector excitation of the closed string could be,

$$\mathcal{A}_\mu \sim \eta_{\mu\nu}(\hat{a}^\nu_1)^\dagger|0\rangle. \tag{128}$$

Another candidate for what the vector excitations could relate to is the Ashtekar electric fields $\widetilde{E}^\mu_I$,

$$\widetilde{E}^\mu_I \sim (\hat{a}^\mu_I)^\dagger|0\rangle. \tag{129}$$

A. Emergent Dimensions and LQG

The actions (8) and (25) used in sections III and IV have a clear disadvantage: they lost WS Weyl symmetry, which makes the equations extracted from them have a dependence on the scale factor $\phi(x)$. If we want to formulate an effective stringy action that takes LQG quantum geometric effects into consideration and retains WS Weyl invariance while giving similar results to the analysis done in section IV, we arrive at a similar model as [21], with the action being

$$\begin{aligned} S_{ED} = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2x\sqrt{-g}\big(&g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \\ &+ \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu}\big) \end{aligned} \tag{130}$$

where T is the *temperature* of target space and the parameters m_μ satisfy an ordering relation $m_0 \gg m_1 \gg \ldots \gg m_d$. This is Afshordi's model [21]. Where our approach diverges is that we realize the parameters m_μ are constrained by the modified dispersion relations of each field X^μ, since as [19] points out, LQG quantum geometry breaks Lorentz invariance near the Planck scale through modifying the dispersion relation by constraining the free parameters of the model as

$$E^2 = m^2 + p^2 + \sum_{n\geq 3} c_n \frac{p^n}{E_{pl}^{n-2}}, \tag{131}$$

so we note that the mass parameters of the fields X^μ will be constrained as

$$E_\alpha^2 = p_\alpha^2 + \sum_{n=3}^{D-1} \frac{m_n}{T} \frac{p_\alpha^n}{E_{pl}^{n-2}}, \tag{132}$$

where E_α^2 does not mean $E_\alpha E^\alpha$, rather it means $(E_3)^2, (E_4)^2, ..., (E_{D-1})^2$, *i.e.* , the squared WS energy (and momentum) of each field X^μ.

Afshordi's work notes that action (130) only has background Poincaré invariance and WS Weyl symmetry for the fields with $m_\mu \gg T$, thus the "emergent spacetime and evolving dimensions" by interpreting the fields with effective mass as being inaccessible due to not obeying Poincaré symmetry. We go one step further and note that since eq (132) leaves m_0, m_1 and m_2 unconstrained, the related dimensions might be regarded as being *fundamental* and thus the true quantum gravitational field should be $(2+1)$-dimensional, thus we postulate that $m_3 \sim T_{pl}$, which together with the constraints (132) should give us a way to calculate the other mass parameters given the WS energy and momentum of the fields, making it possible to predict in which scales the other dimensions should start to open up.

By analysing the equations of motion for the metric g_{cd} from the action (130) we get

$$g_{cd} = 2f \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu}, \tag{133}$$

with f given by

$$\frac{1}{f} = g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu}, \tag{134}$$

from which we see that $\mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} = 0$ is actually a constraint and thus the base mass μ_0^2 is a Lagrange multiplier. By dimensional analysis, we see that in SI units μ_0^2 has units of inverse area, so it could be that $\mu_0^2 \sim A_j^{-1}$, where A_j is the eigenvalue of the area operator with spin j, which depends on where the spin network punctures the WS, thus once more connecting Afshordi's work to LQG quantum geometry.

One could also consider a more general form of action (130) by swapping the flat Minkowski metric $\eta_{\mu\nu}$ by a more general curved metric $G_{\mu\nu}$ and considering off-diagonal mass parameters $m_{\mu\nu}$:

$$S'_{ED} = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2x \sqrt{-g} \big(g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \\ + \mu_0^2 \mathrm{e}^{-m_{\mu\nu}/T} X^\mu X^\nu G_{\mu\nu} \big), \tag{135}$$

where the mass parameters will be constrained as in

$$E_\alpha^2 = p_\alpha^2 + \sum_{n=3}^{D-1} \frac{m_{n\alpha}}{T} \frac{p_\alpha^n}{E_{pl}^n}, \tag{136}$$

since those are the ones that show up in the X^μ equations of motion

$$g^{ab} \nabla_a \nabla_b X^\mu G_{\mu\lambda} - \mu_0^2 \mathrm{e}^{-m_{\mu\lambda}/T} X^\mu G_{\mu\lambda} = \\ = \frac{1}{2} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu}. \tag{137}$$

In section IV we arrived at eq (34) and claimed the coupling between the WS conformal factor $\phi(x)$ and the embedding fields X^μ to be akin to the dilaton field $\Phi(X)$. We now explore this claim: since we're now dealing with emergent dimensions with an action (130) that only has Weyl symmetry as an approximate symmetry, we believe that the conformal factor $\phi(x)$ can be viewed as a WS dilaton, which sets the scale of the string dictating which dimensions it can interact with, thus the coupling "constant" of the masses may be related to the conformal factor

$$\mu_0^2 \sim \mathrm{e}^{-\phi(x)}, \tag{138}$$

where the minus sign is due to the fact that the dimensions open up at larger and larger scales, so we require a larger and larger string to interact with each new dimension.

ACKNOWLEDGMENTS

Appendix A: Worldsheet Action in Terms of Transverse Velocity

Appendix B: The Bimetric String

It would be natural to suspect that the action (9) might be related to bimetric gravity in the target space of the string. We can see that this is indeed the case as follows.

Consider a string propagating in a target space whose geometry is described by two bulk metrics. There are two ways that a string with two metrics might arise. One possibility is that there are two sets of embedding fields X^μ and Y^μ. As a result, the induced metrics on the string worldsheet would also be different for both sets of embedding fields. However, in this case, we would not be talking about *one* string but two *different* strings described by the respective embedding functions. One could then proceed along the lines of (11) and introduce a third term for the worldsheet action which describes a coupling between the two sets of embedding fields, of the form:

$$S_{int} = -T_{eff} \int \mathrm{d}^2x\, \Phi(g,h) \Psi(\mathbf{X}, \mathbf{Y}). \tag{B1}$$

Here $\Phi(g,h)$ is some function of the *induced* metrics g_{ab} and h_{ab} on the two worldsheets and $\Psi(\mathbf{X},\mathbf{Y})$ is some function of the embedding fields. The resulting action would be of the form:

$$S_{BNG} = -T\int \mathrm{d}^2x\,\sqrt{-h} - T'\int \mathrm{d}^2x\,\sqrt{-g} + S_{int}. \quad \text{(B2)}$$

Here the subscript BNG stands for "Bimetric Nambu-Goto".

The second possibility is the following. Rather than considering two different sets of embedding fields, and consequently two different worldsheets, we can work with one set of embeddings $X^\mu(\tau,\sigma)$ and *two different* bulk metrics $G_{\mu\nu}$ and $H_{\mu\nu}$. As a result, we would now get two different induced metrics on the *same* string worldsheet, given by:

$$g_{ab} = \frac{\partial X^\mu}{\partial x^a}\frac{\partial X^\nu}{\partial x^b}G_{\mu\nu} \quad \text{(B3a)}$$

$$h_{ab} = \frac{\partial X^\mu}{\partial x^a}\frac{\partial X^\nu}{\partial x^b}H_{\mu\nu}. \quad \text{(B3b)}$$

We can then write down the combined action for the string worldsheet as:

$$S'_{BNG} = -T\int \mathrm{d}^2x\,\sqrt{-h} - T'\int \mathrm{d}^2x\,\sqrt{-g} + S'_{int}, \quad \text{(B4)}$$

where now $S'_{int}[g,h,\mathbf{X}]$ is a term describing the interaction between the two metrics living on the *same* worldsheet.

1. Bimetric Polyakov Action

The expression (B4) is not ideal because it does not retain any memory of the form of the bulk metrics unless these are included in the interaction term S'_{int}. One way to manifestly exhibit the dependence of the respective worldsheet actions on the two bulk metrics is to work with the Polaykov action.

Following the general procedure given in [4], we can couple the worldsheet to a bulk metric as follows:

$$S_P = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a X^\mu\partial_b X^\nu G_{\mu\nu}, \quad \text{(B5)}$$

where $G_{\mu\nu}$ is now the bulk or target space metric. Using this form we can now write down the worldsheet Polyakov bimetric action for two bulk metrics:

$$\begin{aligned} S_{PB} = &-\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a X^\mu\partial_b X^\nu G_{\mu\nu}\\ &-\frac{T'}{2}\int \mathrm{d}^2x\sqrt{-h}h^{ab}\partial_a X^\mu\partial_b X^\nu H_{\mu\nu}+\\ &+S_{int}[g,h,X]. \end{aligned} \quad \text{(B6)}$$

2. The Bimetric Polyakov Equations

Following the usual procedure such as the one found in [17], we find two copies of the string energy-momentum tensor:

$$\frac{\delta S_{PB}}{\delta g^{cd}} = 0 \quad \text{(B7)}$$

$$\begin{aligned} T^{(G)}_{cd} := G_{\mu\nu}\Big(&\partial_c X^\mu\partial_d X^\nu -\\ &-\frac{1}{2}g_{cd}g^{ab}\partial_a X^\mu\partial_b X^\nu\Big) = \frac{2}{T}\frac{\delta S_{int}}{\delta g^{cd}}, \end{aligned} \quad \text{(B8)}$$

and completely analogous for the $H_{\mu\nu}$ metric,

$$\frac{\delta S_{PB}}{\delta h^{cd}} = 0 \quad \text{(B9)}$$

$$\begin{aligned} T^{(H)}_{cd} := H_{\mu\nu}\Big(&\partial_c X^\mu\partial_d X^\mu -\\ &-\frac{1}{2}h_{cd}h^{ab}\partial_a X^\mu\partial_b X^\nu\Big) = \frac{2}{T'}\frac{\delta S_{int}}{\delta h^{cd}}. \end{aligned} \quad \text{(B10)}$$

If we assume no interaction term $S_{int} = 0$, we get the expected expressions for the auxiliary metrics g_{cd} and h_{cd} as in

$$g_{cd} = 2f^{(G)}\partial_c X^\mu\partial_d X^\nu G_{\mu\nu}, \quad \text{(B11)}$$

where the function $f^{(G)}$ is given by

$$\frac{1}{f^{(G)}} = g^{ab}\partial_a X^\mu\partial_b X^\nu G_{\mu\nu} \quad \text{(B12)}$$

and for h_{cd}

$$h_{cd} = 2f^{(H)}\partial_c X^\mu\partial_d X^\nu H_{\mu\nu}, \quad \text{(B13)}$$

with $f^{(H)}$ given by

$$\frac{1}{f^{(H)}} = h^{ab}\partial_a X^\mu\partial_b X^\nu H_{\mu\nu}. \quad \text{(B14)}$$

If the auxiliary metrics g_{cd} and h_{cd} coincide with the pullback metrics from $G_{\mu\nu}$ and $H_{\mu\nu}$, the action with no interaction reduces to

$$S = -T\int \mathrm{d}^2x\sqrt{-g} - T'\int \mathrm{d}^2x\sqrt{-h}, \quad \text{(B15)}$$

and from here we note that if $T' = -Tk/2$ and $h = \Delta^2/g$, we recover the Nambu-Goto area-corrected action (8)

$$S = -T\int \mathrm{d}^2x\left(\sqrt{-g} - \frac{1}{2}\frac{k\Delta}{\sqrt{-g}}\right). \quad \text{(B16)}$$

The condition $h = \Delta^2/g$ can be achieved by

$$h_{ab} = \Delta_{ac}\Delta_{bd}g^{cd}. \quad \text{(B17)}$$

This implies that

$$\left(\delta_a^{c'}\delta_b^{d'}H_{\mu\nu}-\frac{1}{2g}\Delta_{ac}\widetilde{\varepsilon}^{cc'}\Delta_{bd}\widetilde{\varepsilon}^{dd'}G_{\mu\nu}\right)\times \\ \times\,\partial_{c'}X^\mu\partial_{d'}X^\nu=0, \quad \text{(B18)}$$

which can be achieved if

$$\Delta_{ac}=\sqrt{\Delta}\widetilde{\varepsilon}_{ac} \quad \text{(B19)}$$

and if the background metrics are related by

$$H_{\mu\nu}=\frac{\Delta}{2g}G_{\mu\nu}, \quad \text{(B20)}$$

which makes sense since Δ_{ab} is related to area, so it should be a 2-form.

As for the embedding fields equation, continuing to assume no interaction term we get a non-homogeneous wave equation for the coupled metric $F^{ab}_{\mu\nu}=T\sqrt{-g}g^{ab}G_{\mu\nu}+T'\sqrt{-h}h^{ab}H_{\mu\nu}$

$$\frac{\delta S_{PB}}{\delta X^\lambda}=0 \quad \text{(B21)}$$

$$\partial_a\left(F^{ab}_{\mu\lambda}\partial_bX^\mu\right)=\frac{1}{2}\partial_\lambda F^{ab}_{\mu\nu}\partial_aX^\mu\partial_bX^\nu, \quad \text{(B22)}$$

whereby assuming the conditions for $h=\Delta^2/g$ (B17), (B19) and (B20) and flat background $G_{\mu\nu}=\eta_{\mu\nu}$ gets simplified to

$$\partial_a\left(\left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_bX^\mu\right)=0, \quad \text{(B23)}$$

which in conformal gauge $g_{ab}=\mathrm{e}^{\phi(x)}\eta_{ab}$ simplifies further to eq (33) derived in section IV. In fact, if we plug in the conditions (B17), (B19) and (B20) plus $T'=-Tk/2$ into the bimetric Polyakov action (B6) we recover precisely the action (25) of section IV, thus showing that the form of area corrected Polyakov action discussed there can be cast as a special case of the interactionless bimetric string.

[1] T. Jacobson, *Thermodynamics of spacetime:The Einstein equation of state*, *Physical Review Letters* **75** (June, 1995).

[2] D. H. Friedan, *Nonlinear Models in Two + Epsilon Dimensions*, *Annals Phys.* **163** (1985) 318.

[3] C. G. Callan, Jr. and L. Thorlacius, *Sigma Models and String Theory*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Particles, Strings and Supernovae (TASI 88)*, Mar., 1989.

[4] C. G. Callan, D. Friedan, E. J. Martinec, and M. J. Perry, *Strings in background fields*, *Nuclear Physics, Section B* **262** (1985), no. 4 593–609.

[5] J. Polchinski, *String Theory (Cambridge Monographs on Mathematical Physics) (Volume 1).* Cambridge University Press, 1 ed., Oct., 1998.

[6] D. Tong, *Lectures on String Theory*, arXiv:0908.0333.

[7] N. Rosen, *General Relativity and Flat Space. I*, *Physical Review* **57** (Jan., 1940) 147–150.

[8] N. Rosen, *General Relativity and Flat Space. II*, *Physical Review* **57** (Jan., 1940) 150–153.

[9] C. de Rham, G. Gabadadze, and A. J. Tolley, *Resummation of Massive Gravity*, *Physical Review Letters* **106** (June, 2011) 231101, [arXiv:1011.1232].

[10] D. G. Boulware and S. Deser, *Can Gravitation Have a Finite Range?*, *Physical Review D* **6** (Dec., 1972) 3368–3382.

[11] S. F. Hassan and R. A. Rosen, *Bimetric Gravity from Ghost-free Massive Gravity*, *Journal of High Energy Physics* **2012** (Feb., 2012) 126, [arXiv:1109.3515].

[12] S. F. Hassan, R. A. Rosen, and A. Schmidt-May, *Ghost-free Massive Gravity with a General Reference Metric*, *Journal of High Energy Physics* **2012** (Feb., 2012) 26, [arXiv:1109.3230].

[13] B. Zwiebach, *A First Course in String Theory.* Cambridge University Press, 2nd ed., 2009.

[14] M. Schottenloher, *A Mathematical Introduction to Conformal Field Theory.* Springer Berlin, Heidelberg, Berlin, 2nd edition ed., 2008.

[15] D. K. Bruno Balthazar, Jinwei Chu, *On time-dependent backgrounds in 1+1 dimensional string theory*, .

[16] R. Grimshaw, D. Pelinovsky, and E. Pelinovsky, *Homogenization of the variable-speed wave equation*, *Wave Motion* **47** (Dec., 2010) 496–507.

[17] D. Tong, *Lectures on String Theory*, arXiv:0908.0333.

[18] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory: 25th Anniversary Edition.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2012.

[19] N. Bodendorfer, *An elementary introduction to loop quantum gravity*, .

[20] T. Thiemann, *The LQG-string: Loop quantum gravity quantization of string theory i. flat target space*, Jan., 2004.

[21] D. S. Niayesh Afshordi, *Emergent spacetime in stochastically evolving dimensions*, .