Sketchboard

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1   Basic concepts

1.1   With extra capital internal index on embbeding fields

Promote embedding fields X^μ to have an internal group index with D values

$$X^\mu \to X^{\mu I} \ , \quad I = 0, ..., d$$

$$g_{ab} = 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{IJ} \sim f D_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu}$$

$$D_a X^{\mu I} = \partial_a X^{\mu I} + \omega_{aJ}^I X^{\mu J}$$

$$e_a^i e_b^j \eta_{ij} = g_{ab} \to e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ} \ , \quad i, j = 0, 1$$

$$g = \det(g_{ab}) \to g = \det(\text{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.2   With 1 extra small internal index on embedding fields

Promote embbeding fields to have a internal group index with 2 values

$$X^\mu \to X^{\mu i} \ , \quad i = 0, 1$$

$$g_{ab} = 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{ij} \sim f D_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu}$$

$$D_a X^{\mu i} = \partial_a X^{\mu i} + \omega_{aj}^i X^{\mu j}$$

$$e_a^i e_b^j \eta_{ij} = g_{ab} \to e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij}$$

$$g = \det(g_{ab}) \to g = \det(\text{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.3   With 2 extra small indices

Promote embbeding fields X^μ to have two internal indices with 2 values

$$X^\mu \to X^{\mu ij} \ , \quad i, j = 0, 1$$

1.4   Without extra index on embbeding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$e_a^i e_b^j \eta_{ij} = g_{ab} = 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$g = \det(g_{ab}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

2   Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int \mathrm{d}\tau \wedge \mathrm{d}\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \ ,$$

2.1   With capital internal index

... and promote partial derivative ∂_a to covariant derivative D_a, giving us our first attempt at modified Polyakov action

$$S_{MP1} = -\frac{T_0}{2} \int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}$$

$$\updownarrow$$

$$\mathscr{L}_{MP1} = -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}$$

2.2   With small internal index

2.2.1   Without extra field, 1 internal index

... and promote derivatives to covariant D_a, giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e)e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

$$\updownarrow$$

$$\mathscr{L}_{MP2} = -\frac{T_0}{2}\det(e)e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

2.2.2   With extra field

..., promote partial derivative to covariant and add extra internal WS field v^i leading us to

$$S_{MP3} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e)\eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

$$\updownarrow$$

$$\mathscr{L}_{MP3} = -\frac{T}{2}\det(e)\eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

2.2.3   Without extra field, 2 internal indices

... and promote derivatives to covariant D_a,

$$S_{MP4} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma e e_i^a e^{bi} D_a X^{\mu j j'} \eta_{jj'} \eta_{kk'} D_b X^{\nu kk'} E_\mu^I E_{\nu I}$$

$$\mathscr{L}_{MP4} = -\frac{T}{2}e e_i^a e_j^b D_a X^{\mu ik} \eta_{kl} D_b X^{\nu lj} E_\mu^I E_{\nu I}$$

2.3   Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e)e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) \;.$$

$$\updownarrow$$

$$\mathscr{L}_{DP} = -\frac{T_0}{2}\det(e)e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) \;.$$

2.4   Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of D-dimensional gamma matrices and bulk spinors

$$S_{LP} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

$$\updownarrow$$

$$\mathscr{L}_{LP} = -\frac{T}{2}\overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

3   EoMs

Start by writing $\det(e) = \frac{1}{2}\varepsilon^{cd}\varepsilon_{mn} e_c^m e_d^n$ and $g_{ab} = g_{ab}^{ij}\eta_{ij}$

3.1 With small internal index

3.1.1 Without extra field, w.r.t e

$$\frac{\delta S_{MP2}}{\delta e_l^e} = \frac{D\mathscr{L}_{MP2}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e_c^m e_d^n e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I}\right) =$$

$$= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(\eta^{ml}g_{ce}e_d^n e_i^a e_j^b + e_c^m \eta^{nl}g_{de}e_i^a e_j^b +$$

$$+ e_c^m e_d^n \delta_e^a \delta_i^l e_j^b + e_c^m e_d^n e_i^a \delta_e^b \delta_j^l)D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} =$$

$$= -\frac{T}{2}\left(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}g_{ce}e_d^n e_i^a e_j^b + 2\det(e)e_i^a \delta_e^b \delta_j^l\right)D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} =$$

$$= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}e_c^k e_{ek}e_d^n e_i^a e_j^b + 2\det(e)e_i^a \delta_e^b \delta_j^l)D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} =$$

$$= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}e_c^m e_d^n e_e^l e_i^a e_j^b + 2\det(e)e_i^a \delta_e^b \delta_j^l)D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} =$$

$$= -T(-\det(e)e_e^l e_i^a e_j^b + \det(e)e_i^a \delta_e^b \delta_j^l)D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \overset{!}{=} 0$$

$$\downarrow$$

$$T_e^l := E_\mu^I E_{\nu I}(e_i^a D_a X^{\mu i} D_e X^{\nu l} - e_e^l e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j}) = 0$$

$$\Downarrow$$

$$e_e^l = f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I} ,$$

$$\frac{1}{f} = e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I}$$

$$\downarrow$$

$$e_e^l e_{fl} = (f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I})(f e_{i'}^{a'} D_{a'} X^{\mu' i'} D_f X_l^{\nu'} E_{\mu'}^{I'} E_{\nu' I'}) =$$

$$= f^2 D_e X^{\nu l} D_f X_l^{\nu'} E_\nu^{I'} E_{\nu' I'}(e_i^a e_{i'}^{a'} D_a X^{\mu i} D_{a'} X^{\mu' i'} E_\mu^I E_{\mu' I}) =$$

$$= f D_e X^{\mu i} D_f X_i^\nu E_\mu^I E_{\nu I}$$

$$\Downarrow$$

$$D_e X^{\mu i} D_f X_i^\nu E_\mu^I E_{\nu I} = 2\partial_e X^\mu \partial_f X^\nu G_{\mu\nu}$$

$$e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\Downarrow$$

$$D_a X^{\mu 0} = -i\partial_a X^\mu$$

$$D_a X^{\mu 1} = \partial_a X^\mu$$

$$\Downarrow$$

$$X^{\mu 0} = -iX^{\mu 1}$$

3.1.2 With extra field, w.r.t e

$$\frac{\delta S_{MP3}}{\delta e_o^e} = \frac{D\mathscr{L}_{MP3}}{De_o^e} = \left(\frac{\partial}{\partial e_o^e} - \partial_f \frac{\partial}{\partial(\partial_f e_o^e)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e_c^m e_d^n \eta^{ij}e_i^a e_j^b D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ}\right) =$$

$$= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(g_{ce}\eta^{mo}e_d^n \eta^{ij}e_i^a e_j^b + e_c^m g_{de}\eta^{no}\eta^{ij}e_i^a e_j^b +$$

$$+ e_c^m e_d^n \eta^{ij}\delta_e^a \delta_i^o e_j^b + e_c^m e_d^n \eta^{ij}e_i^a \delta_e^b \delta_j^o)D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} =$$

$$= -\frac{T}{2}\varepsilon^{cd}\varepsilon_{mn}(e_c^p e_{ep}e_d^n \eta^{mo}\eta^{ij}e_i^a e_j^b + 2\det(e)e^{ao}\delta_e^b)D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} =$$

$$= -T(-\det(e)e_e^o \eta^{ij}e_i^a e_j^b + \det(e)e^{ao}\delta_e^b)D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{T_e^o := (e^{ao}D_a(X_k^\mu v^k)D_e(X_l^\nu v^l) - e_e^o \eta^{ij}e_i^a e_j^b D_a(X_k^\mu v^k)D_b(X_l^\nu v^l))E_\mu^I E_\nu^J \eta_{IJ} = 0}$$

$$\Downarrow$$

$$e_e^o = f\, e^{ao}D_a(X_k^\mu v^k)D_e(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} \ ,$$

$$\frac{1}{f} = \eta^{ij}e_i^a e_j^b D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ}$$

3.1.3 With extra field, w.r.t ω

$$\frac{\delta S_{MP3}}{\delta \omega_c^{mn}} = \frac{D\mathscr{L}_{MP3}}{D\omega_c^{mn}} = \left(\frac{\partial}{\partial \omega_c^{mn}} - \partial_d \frac{\partial}{\partial(\partial_d \omega_c^{mn})}\right)\left(-\frac{T}{2}\det(e)\eta^{ij}e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}\right) =$$

$$= -\frac{T}{4}eg^{ab}((-\delta_a^c \delta_{[m}^{k'}\eta_{n]k}X_{k'}^\mu v^k)D_b X_l^\nu v^l + D_a X_k^\mu v^k(-\delta_b^c \delta_{[m}^{l'}\eta_{n]l}X_{l'}^\nu v^l))G_{\mu\nu} =$$

$$= \frac{T}{4}e(g^{ca}X_{[m}^\mu v_{n]}D_a X_k^\nu v^k + g^{ac}D_a X_k^\mu v^k X_{[m}^\nu v_{n]})G_{\mu\nu} =$$

$$= \frac{T}{2}eg^{ac}D_a X_k^\mu v^k X_{[m}^\nu v_{n]}G_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{\mathcal{T}_{ab}^i := e^{ci}D_c X_k^\mu v^k X_{[m}^\nu v_{n]}e_a^m e_b^n G_{\mu\nu} = 0}$$

3.1.4 With extra field, w.r.t v

$$\frac{\delta S_{MP3}}{\delta v^l} = \frac{D\mathscr{L}_{MP3}}{Dv^l} = \left(\frac{\partial}{\partial v^l} - \partial_c \frac{\partial}{\partial(\partial_c v^l)}\right)\left(-\frac{T}{2}ee_i^a e^{bi}D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I}\right) =$$

$$= -\frac{T}{2}g^{ab}(D_a X_j^\mu \delta_l^j D_b X_k^\nu v^k + D_a X_j^\mu v^j D_b X_k^\nu \delta_l^k)G_{\mu\nu} =$$

$$= -Tg^{ab}D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{g^{ab}D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} = 0}$$

3.1.5 With extra field, w.r.t X

$$\frac{\delta S_{MP3}}{\delta X_l^\lambda} = \frac{D\mathscr{L}_{MP3}}{DX_l^\lambda} = \left(\frac{\partial}{\partial X_l^\lambda} - \partial_c \frac{\partial}{\partial(\partial_c X_l^\lambda)}\right)\left(-\frac{T}{2}ee_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I}\right) =$$

$$= -\frac{T}{2}\left(eg^{ab}\left((-\omega_{aj}^{j'}\delta_\lambda^\mu \delta_{j'}^l, v^j D_b X_k^\nu v^k - D_a X_j^\mu v^j \omega_{bk}^{k'}\delta_\lambda^\nu \delta_{k'}^l, v^k)E_\mu^I E_{\nu I} + \right.\right.$$

$$\left.\left. + eg^{ab}D_a X_j^\mu v^j D_b X_k^\nu v^k \left(\frac{\partial E_\mu^I}{\partial X_l^\lambda}E_{\nu I} + E_\mu^I \frac{\partial E_{\nu I}}{\partial X_l^\lambda}\right)\right) - \right.$$

$$\left. - \partial_c(eg^{ab}(\delta_a^c \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k + D_a X_j^\mu v^j \delta_b^c \delta_\lambda^\nu \delta_k^l v^k)G_{\mu\nu})\right) =$$

$$= -T\left(-eg^{ab}D_a X_j^\mu v^j \omega_{bk}^l v^k G_{\mu\lambda} + eg^{ab}D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda}E_{\nu I} - \partial_c(eg^{ac}v^l D_a X_j^\mu v^j G_{\mu\lambda})\right)$$

$$= -T\left(eg^{ab}D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda}E_{\nu I} - D_b(eg^{ab}v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I})\right) \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{D_b(ee_i^a e^{bi} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) = ee_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda}E_{\nu I}}$$

3.2 Without extra index

3.2.1 w.r.t e

$$\frac{\delta S_{DP}}{\delta e_l^e} = \frac{D\mathscr{L}_{DP}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)}\right)\left(-\frac{T_0}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e_c^m e_d^n \eta^{ij}e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I E_\nu^J \eta_{IJ}\right) =$$

$$= -\frac{T_0}{4}$$

3.3 Linear action

3.3.1 w.r.t e

$$\frac{\delta S_{LP}}{\delta\delta e_j^b} = \frac{D\mathscr{L}_{LP}}{De_j^b} = \left(\frac{\partial}{\partial e_j^b} - \partial_e \frac{\partial}{\partial(\partial_e e_j^b)}\right)\left(-\frac{T}{2}\overline{\psi}ee_i^a D_a X^{\mu i}E_\mu^I \gamma_I \psi\right) =$$

$$= -\frac{T}{4}\overline{\psi}\varepsilon^{cd}\varepsilon_{mn}((g_{cb}\eta^{mj}e_d^n + e_c^m g_{db}\eta^{nj})e_i^a + e_c^m e_d^n \delta_b^a \delta_i^j)D_a X^{\mu i}E_\mu^I \gamma_I \psi =$$

$$= -\frac{T}{2}\overline{\psi}(-ee_b^j e_i^a D_a X^{\mu i}E_\mu^I \gamma_I + eD_b X^{\mu j}E_\mu^I \gamma_I)\psi \overset{!}{=} 0$$

$$\downarrow$$

$$T_b^j := \overline{\psi}(D_b X^{\mu j}E_\mu^I \gamma_I - e_b^j e_i^a D_a X^{\mu i}E_\mu^I \gamma_I)\psi = 0$$

$$\downarrow$$

$$e_b^j = F\overline{\psi}D_b X^{\mu j}E_\mu^I \gamma_I \psi,$$

$$\frac{1}{F} = \overline{\psi}e_i^a D_a X^{\mu i}E_\mu^I \gamma_I \psi$$

$$\Downarrow$$

$$\overline{\psi}\psi = 1, \ \psi\overline{\psi} = \mathbb{1}$$

$$e_i^a e_j^b D_a X^{\mu i}D_b X^{\nu j}E_\mu^I E_\nu^J \eta_{IJ} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

4 Inverse Area Action

4.1 Nambu-Goto

Start from Nambu-Goto action

$$S_{NG} = -T \int \mathrm{d}^2 x \sqrt{-h},$$

and make quantum geometry correction

$$\sqrt{-h} \to \sqrt{-(h + g\Delta)} \approx \sqrt{-h}\left(1 - \frac{g\Delta}{2(-h)} + \mathcal{O}\left(\frac{g^2}{h^2}\right)\right),$$

leading to modified NG action

$$S_{MNG} = -T \int \mathrm{d}^2 x \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}}\right) = S_{NG} + S_{IA}$$

$$\downarrow$$

$$\mathscr{L}_{MNG} = -T\left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}}\right)$$

4.1.1 EoMs

Let $\mathcal{P}_\mu^\tau \equiv \partial \mathscr{L}_{MNG}/\partial \dot{X}^\mu$ and $\mathcal{P}_\mu^\sigma \equiv \partial \mathscr{L}_{MNG}/\partial X'^\mu$

$$\delta S_{MNG} = \int \mathrm{d}^2 x \left(\frac{\partial \mathscr{L}_{MNG}}{\partial \dot{X}^\mu}\delta \dot{X}^\mu + \frac{\partial \mathscr{L}_{MNG}}{\partial X'^\mu}\delta X'^\mu\right) =$$

$$= -\int \mathrm{d}^2 x \left(\partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma\right)\delta X^\mu + \int \mathrm{d}\tau \mathcal{P}_\mu^\sigma \delta X^\mu\big|_{\sigma=0}^{\sigma=\sigma_1} \overset{!}{=} 0$$

$$\downarrow$$

$$\text{EoM} : \partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma = 0$$

$$\text{B.C.} : \mathcal{P}_\mu^\sigma \delta X^\mu\big|_{\sigma=0}^{\sigma=\sigma_1} = 0,$$

where

$$\mathcal{P}_\mu^\tau = \frac{\partial \mathscr{L}_{MNG}}{\partial \dot{X}^\mu} = -T\left(\frac{(\dot{X}\cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} + \frac{g\Delta}{2}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{(-h)^{3/2}}\right) =$$

$$= -T\frac{(\dot{X}\cdot X')X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}}\left(1 + \frac{g\Delta}{2(-h)}\right) = \mathcal{P}_{\mu(NG)}^\tau \left(1 + \frac{g\Delta}{2(-h)}\right)$$

$$\mathcal{P}_\mu^\sigma = \frac{\partial \mathscr{L}_{MNG}}{\partial X'^\mu} = -T\left(\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} + \frac{g\Delta}{2}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{(-h)^{3/2}}\right) =$$

$$= -T\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}}\left(1 + \frac{g\Delta}{2(-h)}\right) = \mathcal{P}_{\mu(NG)}^\sigma \left(1 + \frac{g\Delta}{2(-h)}\right)$$

gauge fixing static gauge $\tau = t$ and transverse gauge $\frac{\partial X}{\partial \tau}\cdot\frac{\partial X}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}\sigma} = 0$ ($s = $ length along string)

$$\downarrow$$

$$\mathcal{P}_{(NG)}^{\tau\mu} = T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\frac{\partial X^\mu}{\partial t}$$

$$\mathcal{P}_{(NG)}^{\sigma\mu} = \frac{T}{\gamma_{v_\perp}}\frac{\partial X^\mu}{\partial s}$$

$$-h = \frac{1}{\gamma_{v_\perp}^2}\left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2$$

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string energy get's redefined to

$$\frac{\partial}{\partial t}\left(T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\right)=0,$$

and for the spatial part we have

$$\partial_\tau \vec{\mathcal{P}}^\tau + \partial_\sigma \vec{\mathcal{P}}^\sigma = 0$$

$$\downarrow$$

$$\frac{\partial}{\partial t}\left[T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\frac{\partial \vec{X}}{\partial t}\left(1+\frac{g\Delta}{2(-h)}\right)\right]+\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{\partial}{\partial s}\left[-\frac{T}{\gamma_{v_\perp}}\frac{\partial \vec{X}}{\partial s}\left(1+\frac{g\Delta}{2(-h)}\right)\right]=0$$

$$\downarrow$$

$$\mu\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\frac{\partial^2 \vec{X}}{\partial t^2}-\frac{\partial}{\partial s}\left[\frac{T}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right)\frac{\partial \vec{X}}{\partial s}\right]=0$$

$\implies$ effective mass density becomes $\mu_{eff}=\mu\gamma_{v_\perp}\left(1-\frac{g\Delta}{2(-h)}\right)$ and effective tension becomes $T_{eff}=\frac{T}{\gamma_{v_\perp}}\left(1-\frac{g\Delta}{2(-h)}\right)$

$$\downarrow$$

$$\mu_{eff}\frac{\partial^2 \vec{X}}{\partial t^2}-\frac{\partial}{\partial s}\left[T_{eff}\frac{\partial \vec{X}}{\partial s}\right]=0$$

$$\mathscr{H}=\dot{\vec{X}}\cdot\vec{\pi}-\mathscr{L}=$$

$$=\vec{v}_\perp\cdot\left(T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\vec{v}_\perp\right)-\left(-T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{1}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right)\right)=$$

$$=T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\left(1+\frac{g\Delta}{2(-h)}\right)\left(\gamma_{v_\perp}v_\perp^2+\frac{1}{\gamma_{v_\perp}}\right)=$$

$$=T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)$$

let $\left(1+\frac{g\Delta}{2(-h)}\right)=F$

$$\frac{\partial^2 \vec{X}}{\partial t^2}-\frac{1}{F\gamma_{v_\perp}}\frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\partial}{\partial\sigma}\left[\frac{1}{\gamma_{v_\perp}}F\frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\partial \vec{X}}{\partial\sigma}\right]=0$$

$$\downarrow$$

$$A(\sigma)=\frac{\gamma_{v_\perp}}{F}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\overset{!}{=}1$$

$$\downarrow$$

$$\mathrm{d}\sigma=\frac{\gamma_{v_\perp}}{F}\mathrm{d}s=\frac{1}{TF^2}\mathrm{d}E\implies\sigma(q)=\frac{1}{T}\int_0^q\frac{1}{F^2}\mathrm{d}E$$

$$\downarrow$$

$$F^2\frac{\partial^2 \vec{X}}{\partial t^2}-\frac{\partial^2 \vec{X}}{\partial\sigma^2}=0$$

$\implies$ speed of wave on the string gets modified by correction factor $v=c/F$

$$-h=\frac{1}{\gamma_{v_\perp}^2}\left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2=\frac{F^2}{\gamma_{v_\perp}^4}=F^2\left(1-v_\perp^2\right)^2$$

$$\downarrow$$

$$-h=\left(1+\frac{g\Delta}{2(-h)}\right)^2\left(1-v_\perp^2\right)^2$$

$$\downarrow$$

$$\frac{-h}{\left(1 + \frac{g\Delta}{2(-h)}\right)^2} = \left(1 - v_\perp^2\right)^2$$

$$\downarrow \text{(WolframAlpha)}$$

$$-h = \frac{(1 - v_\perp^2)^2}{3} - f_1 - f_2,$$

where (let $a = \frac{g\Delta}{2}$ and $b = (1 - v_\perp^2)^2$)

$$f_1 = \frac{\sqrt[3]{-27a^2b + 3\sqrt{3}\sqrt{27a^4b^2 + 4a^3b^3} - 18ab^2 - 2b^3}}{3\sqrt[3]{2}}$$

$$f_2 = \frac{(6ab + b^2)}{6f_1}$$

$$\downarrow$$

$$F = 1 + \frac{g\Delta}{2(-h)} = 1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}$$

$$\downarrow$$

$$v = \frac{1}{F} = \frac{1}{1 + \dfrac{g\Delta}{2\left(\dfrac{b}{3} - f_1 - f_2\right)}}$$

5   Bimetric Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

and promote to bimetric action

$$\downarrow$$

$$S_{BP} = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{T'}{2}\int \mathrm{d}^2x\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

5.1   EoMs

$$\frac{\delta S_{BP}}{\delta g^{cd}} = -\frac{T}{2}\left(\frac{\partial\sqrt{-g}}{\partial g^{cd}}g^{ab} + \sqrt{-g}\frac{\partial g^{ab}}{\partial g^{cd}}\right)\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + 0 =$$

$$= -\frac{T}{2}\left(-\frac{1}{2}\sqrt{-g}g_{cd}g^{ab} + \sqrt{-g}\delta_{(c}^a\delta_{d)}^b\right)\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \overset{!}{=} 0$$

$$T_{cd}^{(G)} := \left(\partial_c X^\mu \partial_d X^\nu - \frac{1}{2}g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu\right)G_{\mu\nu} = 0$$

$$\Downarrow$$

$$g_{cd} = 2f^{(G)}\partial_c X^\mu \partial_d X^\nu G_{\mu\nu},$$

$$\frac{1}{f^{(G)}} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\frac{\delta S_{BP}}{\delta h^{cd}} = \ldots =$$

$$= -\frac{T'}{2}\left(-\frac{1}{2}\sqrt{-h}h_{cd}h^{ab} + \sqrt{-h}\delta_{(c}^a\delta_{d)}^b\right)\partial_a X^\mu \partial_b X^\nu H_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$T_{cd}^{(H)} := \left(\partial_c X^\mu \partial_d X^\mu - \frac{1}{2} h_{cd} h^{ab} \partial_a X^\mu \partial_b X^\nu \right) H_{\mu\nu} = 0$$

$$\Downarrow$$

$$h_{cd} = 2 f^{(H)} \partial_c X^\mu \partial_d X^\nu H_{\mu\nu},$$

$$\frac{1}{f^{(H)}} = h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

$$\frac{\delta S_{BP}}{\delta X^\lambda} = \left(-\frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu -$$

$$- \partial_c \left(-\frac{T}{2} \sqrt{-g} g^{ab} G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} H_{\mu\nu} \right) (\delta_a^c \delta_\lambda^\mu \partial_b X^\nu + \partial_a X^\mu \delta_b^c \delta_\lambda^\nu) =$$

$$= \partial_a \left(T \sqrt{-g} g^{ab} G_{\mu\lambda} + T' \sqrt{-h} h^{ab} H_{\mu\lambda} \right) \partial_b X^\mu -$$

$$- \left(\frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} + \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu \overset{!}{=} 0$$

$$\downarrow$$

$$\partial_a \left(F_{\mu\lambda}^{ab} \partial_b X^\mu \right) = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu,$$

$$F_{\mu\nu}^{ab} = T \sqrt{-g} g^{ab} G_{\mu\nu} + T' \sqrt{-h} h^{ab} H_{\mu\nu}$$

let $T' = Tk\Delta/2$ and $h = g^{-1}(H_{\mu\nu} = G_{\mu\nu})$

$$F_{\mu\nu}^{ab} = T \left(\sqrt{-g} g^{ab} G_{\mu\nu} + \frac{k\Delta}{2} \sqrt{-g^{-1}} U^a_{\ a'} (g^{-1})^{a'b'} U_b^{\ b} (G^{-1})_{\mu\nu} \right),$$

add new field coupled to derivative A_a transforming as

$$A_a g^{bc} \rightarrow \left(A_a g^{bc} \right)' = A_a U^b_{\ b'} g^{b'c'} U_{c'}^{\ c} - \frac{1}{\alpha} g^{b'c'} (\partial_a U^b_{\ b'} U_{c'}^{\ c} + U^b_{\ b'} \partial_a U_{c'}^{\ c}),$$

defining new WS covariant derivative

$$D_a g^{bc} = \partial_a g^{bc} + \alpha A_a g^{bc},$$

such that derivative ignores $SO(1,1)$ gauge freedom on inverse metric. Upgrade EoM to incorporate this derivative:

$$D_a (F_{\mu\lambda}^{ab} \partial_b X^\mu) = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu$$

$$\downarrow$$

$$F_{\mu\lambda}^{ab} \partial_a \partial_b X^\mu + D_a F_{\mu\lambda}^{ab} \partial_b X^\mu = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu,$$

imposing conformal symmetry $g^{ab} = (\phi(x))^{-1} \eta^{ab} \implies D_a F_{\mu\lambda}^{ab} = 0$

$$F_{\mu\lambda}^{ab} \partial_a \partial_b X^\mu = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu,$$

$$F_{\mu\lambda}^{ab} = T \left(G_{\mu\lambda} + \frac{k\Delta}{2} (G^{-1})_{\mu\lambda} \right) \eta^{ab}$$

$$(G^{-1})_{\mu\nu} \overset{?}{=} \frac{1}{-g} G_{\mu\nu}$$

6 Inverse Area Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2} \int \mathrm{d}^2 x \sqrt{-g}\, g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

and make quantum geometry correction

$$\sqrt{-g} \to \sqrt{-(g + k\Delta)} \approx \sqrt{-g}\left(1 + \frac{k\Delta}{2g} + \mathcal{O}\left(\frac{k^2}{g^2}\right)\right)$$

leading to

$$S_{IAP} = -\frac{T}{2} \int \mathrm{d}^2 x \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

6.1 EoMs

$$\frac{\delta S_{IAP}}{\delta g^{cd}} = \frac{\partial}{\partial g^{cd}}\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right)\delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} =$$

$$= \left(-\frac{1}{2}\sqrt{-g}\,g_{cd} + \frac{k\Delta}{4\sqrt{-g}}g_{cd}\right) g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right)\partial_c X^\mu \partial_d X^\nu G_{\mu\nu} =$$

$$= -\frac{1}{2}\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g_{cd} g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right)\partial_c X^\mu \partial_d X^\nu G_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} - \frac{1}{2}g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = 0$$

$$\downarrow$$

$$g_{cd} = 2f\partial_c X^\mu \partial_d X^\nu G_{\mu\nu},$$

$$\frac{1}{f} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\frac{\delta S_{IAP}}{\delta X^\lambda} = \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - \partial_c \left(\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) 2g^{ab}\delta_a^c \delta_\lambda^\mu \partial_b X^\nu G_{\mu\nu}\right) =$$

$$= \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\partial_a \partial_b X^\nu G_{\lambda\nu} -$$

$$- 2\partial_a \left(\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\right)\partial_b X^\nu G_{\lambda\nu} =$$

$$= \left(1 - \frac{k\Delta}{2(-g)}\right)\sqrt{-g}\,g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2\left(1 - \frac{k\Delta}{2(-g)}\right)\sqrt{-g}\,g^{ab}\partial_a \partial_b X^\nu G_{\lambda\nu} -$$

$$- 2\partial_a \left(\left(1 - \frac{k\Delta}{2(-g)}\right)\sqrt{-g}\,g^{ab}\right)\partial_b X^\nu G_{\lambda\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$\left(1 - \frac{k\Delta}{2(-g)}\right)\sqrt{-g}\,g^{ab}\partial_a \partial_b X^\nu G_{\lambda\nu} + \partial_a \left(\left(1 - \frac{k\Delta}{2(-g)}\right)\sqrt{-g}\,g^{ab}\right)\partial_b X^\nu G_{\lambda\nu} =$$

$$\frac{1}{2}\left(1 - \frac{k\Delta}{2(-g)}\right)\sqrt{-g}\,g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

imposing conformal symmetry $g^{ab} = (\phi(x))^{-1}\eta^{ab}$ and let $F' = \left(1 - \frac{k\Delta}{2\phi^2}\right)$ leads to

$$\eta^{ab}\partial_a \partial_b X^\nu G_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F' \partial_b X^\nu G_{\lambda\nu} = \eta^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

assuming flat background $G_{\mu\nu} = \eta_{\mu\nu}$

$$\eta^{ab}\partial_a\partial_b X^\nu \eta_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F'\partial_b X^\nu \eta_{\lambda\nu} = 0$$

$$\downarrow$$

$$\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\partial_a \ln(F')\partial_b X^\mu = 0,$$

use plane-wave ansatz $X^\mu = X_0^\mu e^{-i(E\tau - p\sigma)}$

$$(E^2 - p^2)X^\mu + \partial_\tau \ln(F')iEX^\mu - \partial_\sigma \ln(F')ipX^\mu = 0$$

$$\downarrow$$

$$(E^2 - p^2)X^\mu + i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F'))X^\mu = 0$$

$$\downarrow$$

$$i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F')) = -m^2,$$
$$-m^2 = -E^2 + p^2$$

$$\Downarrow$$

$$\partial_\tau \ln(F') = iE, \ \partial_\sigma \ln(F') = ip$$

$$\Downarrow$$

$$\ln(F') = i(E\tau + p\sigma) + a, \ a \in \mathbb{C}$$

$$\Downarrow$$

$$F' = Ae^{i(E\tau + p\sigma + \phi)}, \ A > 0$$

$$\Downarrow$$

$$\phi(\tau,\sigma) = \sqrt{\frac{k\Delta}{2\left(1 - Ae^{i(E\tau + p\sigma + \phi)}\right)}}$$

$$\downarrow$$

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

7 Klein-Gordon String Action

Regular string EoM is a wave eqn, so promote to KG eqn from quantum geometry effects

$$(\partial_\sigma)^2 X^\mu - (\partial_\tau)^2 X^\mu + \mu^2 X^\mu = 0,$$

which can be achieved by the action

$$S = -\frac{T}{2}\int \mathrm{d}^2 x \sqrt{-g}\left(g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu^2 X^\mu X^\nu \eta_{\mu\nu}\right).$$

This action has manifest Lorentz invariance, but not translation invariance. This is fine since the string is not propagating in a vacuum. It also has WS Weyl symmetry.

7.1 EoMs

$$\frac{\delta S}{\delta g^{cd}} \propto \frac{\partial \sqrt{-g}}{\partial g^{cd}}\left(g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu^2 X^\mu X^\nu \eta_{\mu\nu}\right) + \sqrt{-g}\delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} =$$

$$= -\frac{1}{2}\sqrt{-g}g_{cd}\left(g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu^2 X^\mu X^\nu \eta_{\mu\nu}\right) + \sqrt{-g}\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = 0$$

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2}g_{cd}\left(g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu^2 X^\mu X^\nu \eta_{\mu\nu}\right) = 0$$

$$g_{cd} = 2f\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu},$$

$$\frac{1}{f} = \left(g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu^2 X^\mu X^\nu \eta_{\mu\nu}\right).$$

This implies $X^\mu X^\nu \eta_{\mu\nu} = 0$, so the string coordinates are on the light-cone and the auxiliary metric coincides with the induced metric, thus the action reduces back to regular NG action after imposing this EoM.

$$\frac{\delta S}{\delta X^\lambda} \propto -2\sqrt{-g}\mu^2 X^\mu \delta_\lambda^\nu \eta_{\mu\nu} - 2\partial_c\left(\sqrt{-g}g^{ab}\delta_a^c \delta_\lambda^\mu \partial_b X^\nu \eta_{\mu\nu}\right) =$$

$$= -2\sqrt{-g}\mu^2 X^\mu \eta_\lambda - 2\partial_a\left(\sqrt{-g}g^{ab}\partial_b X^\mu \eta_{\mu\lambda}\right) = 0$$

$$\partial_a\left(\sqrt{-g}g^{ab}\partial_b X^\mu\right) + \sqrt{-g}\mu^2 X^\mu = 0$$

$$\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}g^{ab}\partial_b X^\mu\right) + \mu^2 X^\mu = 0$$

$$g^{ab}\nabla_a \nabla_b X^\mu + \mu^2 X^\mu = 0$$

8 Relating NG and Polyakov analysis

From NG analysis

$$(F^+)^2 \frac{\partial^2 X^\mu}{\partial t^2} - \frac{\partial^2 X^\mu}{\partial x^2} = 0,$$

make change of variables $\tau = \tau(t,x)$ and $\sigma = \sigma(t,x)$

$$(F^+)^2 \left(\frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial t} \frac{\partial}{\partial \sigma} \right) \left(\frac{\partial \tau}{\partial t} \frac{\partial X^\mu}{\partial \tau} + \frac{\partial \sigma}{\partial t} \frac{\partial X^\mu}{\partial \sigma} \right) - \left(\frac{\partial \tau}{\partial x} \frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial x} \frac{\partial}{\partial \sigma} \right) \left(\frac{\partial \tau}{\partial x} \frac{\partial X^\mu}{\partial \tau} + \frac{\partial \sigma}{\partial x} \frac{\partial X^\mu}{\partial \sigma} \right) = 0$$

$$(F^+)^2 \left(\left(\frac{\partial \tau}{\partial t}\right)^2 \frac{\partial^2 X^\mu}{\partial \tau^2} + 2 \frac{\partial \tau}{\partial t} \frac{\partial \sigma}{\partial t} \frac{\partial^2 X^\mu}{\partial \tau \partial \sigma} + \left(\frac{\partial \sigma}{\partial t}\right)^2 \frac{\partial^2 X^\mu}{\partial \sigma^2} \right) - $$
$$- \left(\left(\frac{\partial \tau}{\partial x}\right)^2 \frac{\partial^2 X^\mu}{\partial \tau^2} + 2 \frac{\partial \tau}{\partial x} \frac{\partial \sigma}{\partial x} \frac{\partial^2 X^\mu}{\partial \tau \partial \sigma} + \left(\frac{\partial \sigma}{\partial x}\right)^2 \frac{\partial^2 X^\mu}{\partial \sigma^2} \right) + $$
$$+ \left((F^+)^2 \frac{\partial^2 \tau}{\partial t^2} - \frac{\partial^2 \tau}{\partial x^2} \right) \frac{\partial X^\mu}{\partial \tau} + \left((F^+)^2 \frac{\partial^2 \sigma}{\partial t^2} - \frac{\partial^2 \sigma}{\partial x^2} \right) \frac{\partial X^\mu}{\partial \sigma} = 0$$

$$\left((F^+)^2 (\partial_t \tau)^2 - (\partial_x \tau)^2 \right) \partial_\tau^2 X^\mu + \left((F^+)^2 (\partial_t \sigma)^2 - (\partial_x \sigma)^2 \right) \partial_\sigma^2 X^\mu + $$
$$+ 2 \left((F^+)^2 \partial_t \tau \partial_t \sigma - \partial_x \tau \partial_x \sigma \right) \partial_\tau \partial_\sigma X^\mu + \left((F^+)^2 \partial_t^2 \tau - \partial_x^2 \tau \right) \partial_\tau X^\mu + \left((F^+)^2 \partial_t^2 \sigma - \partial_x^2 \sigma \right) \partial_\sigma X^\mu = 0$$

Without loss of generality, we can set (just rescale/rotate the coordinates)

$$(F^+)^2 (\partial_t \sigma)^2 - (\partial_x \sigma)^2 = (\partial_x \tau)^2 - (F^+)^2 (\partial_t \tau)^2$$
$$(F^+)^2 \partial_t \tau \partial_t \sigma = \partial_x \tau \partial_x \sigma,$$

or more compactly

$$\partial_t \tau = v \partial_x \sigma$$
$$\partial_t \sigma = v \partial_x \tau,$$

where $v = 1/F^+$, which is equivalent to

$$\partial_t^2 \tau = \partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau + v^2 \partial_x^2 \tau$$
$$\partial_t^2 \sigma = \partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma + v^2 \partial_x^2 \sigma,$$

reducing the big equation to

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu + \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\tau X^\mu + \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\sigma X^\mu = 0.$$

Next introduce $X^\mu = \kappa(\tau, \sigma) Y^\mu(\tau, \sigma)$ s.t terms proportional to $\partial_\tau Y^\mu$ and $\partial_\sigma Y^\mu$ vanish:

$$\partial_\tau X^\mu = \partial_\tau \kappa Y^\mu + \kappa \partial_\tau Y^\mu$$
$$\partial_\sigma X^\mu = \partial_\sigma \kappa Y^\mu + \kappa \partial_\sigma Y^\mu$$
$$\partial_\tau^2 X^\mu = \partial_\tau^2 \kappa Y^\mu + 2 \partial_\tau \kappa \partial_\tau Y^\mu + \kappa \partial_\tau^2 Y^\mu$$
$$\partial_\sigma^2 X^\mu = \partial_\sigma^2 \kappa Y^\mu + 2 \partial_\sigma \kappa \partial_\sigma Y^\mu + \kappa \partial_\sigma^2 Y^\mu$$

$$\kappa \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} + 2 \partial_\tau \kappa \overset{!}{=} 0$$
$$\kappa \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} - 2 \partial_\sigma \kappa \overset{!}{=} 0$$

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$$\partial_\tau \kappa = -\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)}$$

$$\partial_\sigma \kappa = \frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)},$$

undoing chain rule for t by multiplying first eq by $\partial_t \tau$ and second by $\partial_t \sigma$ and summing gives

$$\partial_t \kappa = \partial_t \tau \partial_\tau \kappa + \partial_t \sigma \partial_\sigma \kappa =$$

$$= \partial_t \tau \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)}\right) + \partial_t \sigma \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)}\right) =$$

$$= v \partial_x \sigma \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)}\right) + v \partial_x \tau \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)}\right) =$$

$$= \frac{v \kappa \partial_t v \left((\partial_x \tau)^2 - (\partial_x \sigma)^2\right)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} =$$

$$= \frac{v \kappa \partial_t v \left((\partial_x \tau)^2 - (\partial_t \tau / v)^2\right)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} =$$

$$= -\frac{\kappa \partial_t v}{2v},$$

or simply

$$\frac{\partial_t \kappa}{\kappa} = -\frac{1}{2}\frac{\partial_t v}{v}$$

$$\partial_t \ln(\kappa) = -\frac{1}{2}\partial_t \ln(v)$$

$$\ln(\kappa) = -\frac{1}{2}\ln(v) + f(x).$$

To determine $f(x)$, we undo the chain rule for x now:

$$\partial_x \kappa = \partial_x \tau \partial_\tau \kappa + \partial_x \sigma \partial_\sigma \kappa =$$

$$= \partial_x \tau \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)}\right) + \partial_x \sigma \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)}\right) =$$

$$= \frac{\kappa v \partial_x v \left((\partial_x \sigma)^2 - (\partial_x \tau)^2\right)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} =$$

$$= \frac{\kappa v \partial_x v \left((\partial_t \tau / v)^2 - (\partial_x \tau)^2\right)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} =$$

$$= \frac{\kappa \partial_x v}{2v}$$

$$\frac{\partial_x \kappa}{\kappa} = \frac{1}{2}\frac{\partial_x v}{v}$$

$$\partial_x \ln(\kappa) = \frac{1}{2}\partial_x \ln(v)$$

$$\ln(\kappa) = \frac{1}{2}\ln(v),$$

$$\partial_x \left(-\frac{1}{2}\ln(v) + f(x)\right) = \frac{1}{2}\partial_x \ln(v)$$

$f'(x) = \partial_x \ln(v)$ (perhaps this means somewhere in the composition of functions time dependence is lost?)

$$f(x) = \ln(v)$$

giving us

$$\kappa = v^{\frac{1}{2}}, \quad v = v(x).$$

Substituting this into the earlier equation yields

$$\partial_\tau^2 Y^\mu - \partial_\sigma^2 Y^\mu + m^2 Y^\mu = 0,$$

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with

$$m^2 = \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)\partial_\tau \kappa + (\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)\partial_\sigma \kappa}{\kappa \left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} =$$

$$= \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{\partial_t v (\partial_x \sigma \partial_\tau \kappa + \partial_x \tau \partial_\sigma \kappa) + v \partial_x v \partial_x \kappa}{\kappa \left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} =$$

$$= \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{(\partial_x v)^2}{\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)},$$

where by use of chain rule we have that

$$\partial_t^2 \kappa - v^2 \partial_x^2 \kappa - v \partial_x v \partial_x \kappa = ((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)(\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa),$$

where since $\kappa = v^{\frac{1}{2}}$, m^2 reduces to

$$m^2 = \frac{(\partial_x v)^2 - 2v \partial_x^2 v}{4\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)},$$

with τ satisfying

$$\partial_t \tau = v \partial_x \sigma$$
$$v \partial_x \tau = \partial_t \sigma$$

such that $-m^2$ is indeed constant.

9 Solutions for the Polyakov KG eqn (WRONG ONE)

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2}g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi\eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2}\eta_{cd}\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2}\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \frac{1}{2}\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0.$$

These simplify further by expanding the summation on a, b:

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2}\left(-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}\right)$$

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}.$$

The EoM is just a one-dimensional Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi}\int dp \frac{1}{2E(p)}\left(a^\mu(p)e^{-i(-E\tau+p\sigma)} + b^\mu(p)e^{i(-E\tau+p\sigma)}\right),$$

where $E(p) = \sqrt{p^2 + m^2}$. The derivatives are

$$\partial_\tau X^\mu = \frac{i}{4\pi}\int dp \left(a^\mu(p)e^{-i(-E\tau+p\sigma)} - b^\mu(p)e^{i(-E\tau+p\sigma)}\right)$$

$$\partial_\sigma X^\mu = \frac{i}{4\pi}\int dp \frac{p}{\sqrt{p^2+m^2}}\left(-a^\mu(p)e^{-i(-E\tau+p\sigma)} + b^\mu(p)e^{i(-E\tau+p\sigma)}\right).$$

Starting with the mixed constraint,

$$\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = -\frac{\eta_{\mu\nu}}{(4\pi)^2}\int dp \left(a^\mu(p)e^{-i(-E(p)\tau+p\sigma)} - b^\mu(p)e^{i(-E(p)\tau+p\sigma)}\right) \times$$

$$\times \int dp' \frac{p'}{\sqrt{p'^2+m^2}}\left(-a^\nu(p')e^{-i(-E(p')\tau+p'\sigma)} + b^\nu(p')e^{i(-E(p')\tau+p'\sigma)}\right) =$$

$$= -\frac{\eta_{\mu\nu}}{(4\pi)^2}\int dpdp' \frac{p'}{\sqrt{p'^2+m^2}}\left(a^\mu(p)e^{-i(-E(p)\tau+p\sigma)} - b^\mu(p)e^{i(-E(p)\tau+p\sigma)}\right) \times$$

$$\times \left(-a^\nu(p')e^{-i(-E(p')\tau+p'\sigma)} + b^\nu(p')e^{i(-E(p')\tau+p'\sigma)}\right) =$$

$$= -\frac{\eta_{\mu\nu}}{(4\pi)^2}\int dpdp' \frac{p'}{\sqrt{p'^2+m^2}}\left(-a^\mu(p)a^\nu(p')e^{i(E(p)+E(p'))\tau}e^{-i(p+p')\sigma} + a^\mu(p)b^\nu(p')e^{i(E(p)-E(p'))\tau}e^{-i(p-p')\sigma}+\right.$$

$$\left.+ b^\mu(p)a^\nu(p')e^{-i(E(p)-E(p'))\tau}e^{i(p-p')\sigma} - b^\mu(p)b^\nu(p')e^{-i(E(p)+E(p'))\tau}e^{i(p+p')\sigma}\right) = 0.$$

If we integrate this expression over σ we obtain

$$-\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\int \mathrm{d}\sigma\mathrm{e}^{-i(p+p')\sigma}+$$

$$+a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\int \mathrm{d}\sigma\mathrm{e}^{-i(p-p')\sigma}+b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\int \mathrm{d}\sigma\mathrm{e}^{i(p-p')\sigma}-$$

$$-b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\int \mathrm{d}\sigma\mathrm{e}^{i(p+p')\sigma}\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p')+a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p')+$$

$$+b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p')-b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p')\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\frac{p}{\sqrt{p^2+m^2}}\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}+a^\mu(p)b^\nu(p)+b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)=0$$

which implies the whole expression inside the integral must vanish, thus we have (leaving $(a^\mu b^\nu+b^\mu a^\nu)\eta_{\mu\nu}$ as is since eventually these will not be numbers and may not commute, thus will not be equal to $2a^\mu b^\nu \eta_{\mu\nu}$)

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}+a^\mu(p)b^\nu(p)+b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu}=0.$$

As for the other part of the constraint, let's start with the L.H.S

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu}=-\frac{1}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)\mathrm{e}^{-i(-E(p)\tau+p\sigma)}-b^\mu(p)\mathrm{e}^{i(-E(p)\tau+p\sigma)}\Big)\times$$

$$\times \Big(a^\nu(p')\mathrm{e}^{-i(-E(p')\tau+p'\sigma)}-b^\nu(p')\mathrm{e}^{i(-E(p')\tau+p'\sigma)}\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\mathrm{e}^{-i(p+p')\sigma}-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\mathrm{e}^{-i(p-p')\sigma}-$$

$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\mathrm{e}^{i(p-p')\sigma}+b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\mathrm{e}^{i(p+p')\sigma}\Big),$$

which we integrate over σ to get

$$-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p')-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p')-$$

$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p')+b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p')\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}-a^\mu(p)b^\nu(p)-b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big),$$

and now for the R.H.S

$$-\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}=\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{p}{\sqrt{p^2+m^2}}\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)\mathrm{e}^{-i(-E\tau+p\sigma)}+b^\mu(p)\mathrm{e}^{i(-E\tau+p\sigma)}\Big)\times$$

$$\times \Big(-a^\nu(p')\mathrm{e}^{-i(-E(p')\tau+p'\sigma)}+b^\nu(p')\mathrm{e}^{i(-E(p')\tau+p'\sigma)}\Big)=$$

$$=\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{pp'}{E(p)E(p')}\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\mathrm{e}^{-i(p+p')\sigma}-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\mathrm{e}^{-i(p-p')\sigma}-$$

$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\mathrm{e}^{i(p-p')\sigma}+b^\mu(p)b^\nu(p)\mathrm{e}^{-i(E(p)+E(p'))\tau}\mathrm{e}^{i(p+p')\sigma}\Big),$$

which once again we integrate over σ,

$$\frac{\eta_{\mu\nu}}{8\pi} \int \mathrm{d}p\mathrm{d}p' \frac{pp'}{E(p)E(p')} \Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p') - a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p')-$$

$$- b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p') + b^\mu(p)b^\nu(p)\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p') \Big) =$$

$$= \frac{1}{8\pi} \int \mathrm{d}p \frac{p^2}{p^2+m^2} \Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau} \Big) \eta_{\mu\nu},$$

which when equated to the L.H.S yields

$$\int \mathrm{d}p \left(\frac{p^2}{p^2+m^2} + 1 \right) \Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau} \Big) \eta_{\mu\nu} = 0,$$

which implies

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau} \Big) \eta_{\mu\nu} = 0.$$

We can sum and subtract this with the previous condition to simplify and obtain

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau} \Big) \eta_{\mu\nu} = 0$$

$$\Big(a^\mu(p)b^\nu(p) + b^\mu(p)a^\nu(p) \Big) \eta_{\mu\nu} = 0,$$

and for while a^μ and b^ν are number-valued, the second condition can be further simplified to

$$a^\mu(p)b^\nu(p)\eta_{\mu\nu} = 0,$$

meaning a^μ and b^ν are orthogonal. In general, this condition states that a^μ and b^ν anti-commute w.r.t Lorentz inner product. The first condition must hold for all values of τ, and since the exponentials are linearly independent, we thus have

$$a^\mu(p)a^\nu(-p)\eta_{\mu\nu} = 0$$

$$b^\mu(p)b^\nu(-p)\eta_{\mu\nu} = 0,$$

meaning that reflecting the argument of a^μ and b^ν creates orthogonal vectors. We can thus write

$$a^\mu(p) = \Lambda^\mu{}_\nu \left(\frac{p}{4} \right) a^\nu(0)$$

$$b^\mu(p) = \Upsilon^\mu{}_\nu \left(\frac{p}{4} \right) b^\nu(0),$$

where $\Lambda, \Upsilon \in \mathrm{SO}^+(1,d)$. Also. since the orthogonality under reflection must hold for all values of p, in particular we have

$$a^\mu(0)a^\nu(0)\eta_{\mu\nu} = 0$$

$$b^\mu(0)b^\nu(0)\eta_{\mu\nu} = 0,$$

meaning that the initial $a^\mu(0)$ and $b^\nu(0)$ are null vectors. This means that $a^\mu(p)$ and $b^\nu(p)$ are also null vectors since they are related to the initial values by Lorentz transformation. With this, the solution becomes

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi} \int \mathrm{d}p \frac{1}{2E(p)} \left(\Lambda^\mu{}_\nu \left(\frac{p}{4} \right) a^\nu(0)\mathrm{e}^{-i(-E\tau+p\sigma)} + \Upsilon^\mu{}_\nu \left(\frac{p}{4} \right) b^\nu(0)\mathrm{e}^{i(-E\tau+p\sigma)} \right)$$

with $(a^\mu b^\nu + b^\nu a^\mu)\eta_{\mu\nu} = 0$. This condition implies

$$\left(\Lambda^\mu{}_\gamma \left(\frac{p}{4} \right) a^\gamma(0)\Upsilon^\nu{}_\rho \left(\frac{p}{4} \right) b^\rho(0) \right) \eta_{\mu\nu} = 0$$

$$\Upsilon^\mu{}_\nu \left(\frac{p}{4} \right) = \Lambda_\nu{}^\mu \left(\frac{p}{4} \right) = \Lambda^\mu{}_\nu \left(-\frac{p}{4} \right),$$

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thus turning the solution into

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int \mathrm{d}p \frac{1}{2E(p)} \left(\Lambda^\mu{}_\nu \left(\frac{p}{4}\right) a^\nu(0)\mathrm{e}^{-i(-E\tau+p\sigma)} + \Lambda^\mu{}_\nu \left(-\frac{p}{4}\right) b^\nu(0)\mathrm{e}^{i(-E\tau+p\sigma)} \right).$$

Next, we choose background coordinates s.t $a^\mu(0) = (a_0, a_0, 0, ..., 0) = a_0 x^+$ and $b^\nu(0) = (b_0, b_0, 0, ..., 0) = b_0 x^+$, simplifying the solution to

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int \mathrm{d}p \frac{1}{2E(p)} \left(a_0 \Lambda^\mu{}_+ \left(\frac{p}{4}\right) \mathrm{e}^{-i(-E\tau+p\sigma)} + b_0 \Lambda^\mu{}_+ \left(-\frac{p}{4}\right) \mathrm{e}^{i(-E\tau+p\sigma)} \right) x^+$$

10 Solutions for the Polyakov KG eqn

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi \eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta_{cd} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0.$$

These simplify further by expanding the summation on a, b:

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2} \left(-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} \right)$$

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = 0.$$

These can be combined by multiplying the previous one by 2 and summing/subtracting, leaving us with

$$(\partial_\tau X \pm \partial_\sigma X)^2 = 0.$$

The EoM is just a one-dimensional finite-space Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu e^{i(E_n \tau - n\sigma)} + b_n^\mu e^{i(E_n \tau + n\sigma)} \right),$$

with $E_n = \sqrt{n^2 + m^2}$.

Reality of X^μ implies

$$(X^\mu)^* = \left(\frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu e^{i(E_n \tau - n\sigma)} + b_n^\mu e^{i(E_n \tau + n\sigma)} \right) \right)^* =$$

$$= \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left((a_n^\mu)^* e^{-i(E_n \tau - n\sigma)} + (b_n^\mu)^* e^{-i(E_n \tau + n\sigma)} \right) \equiv X^\mu,$$

giving us

$$(a_n^\mu)^* = b_n^\mu$$
$$(b_n^\mu)^* = a_n^\mu,$$

and also that actually only the $\cos(E_n \tau)$ is used, so the solution becomes for closed strings

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu e^{-in\sigma} + (a_n^\mu)^* e^{in\sigma} \right) \cos(E_n \tau).$$

For open strings with free endpoints B.C we have general solution

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} a_n^\mu \cos(E_n \tau) \cos(n\sigma).$$

Analysing the constraints, for the closed string we have the derivatives

$$\partial_\tau X^\mu = -\frac{1}{4\pi\alpha'}\sum_n \left(a_n^\mu e^{-in\sigma} + (a_n^\mu)^* e^{in\sigma}\right)\sin(E_n\tau)$$

$$\partial_\sigma X^\mu = \frac{i}{4\pi\alpha'}\sum_n \frac{n}{E_n}\left(-a_n^\mu e^{-in\sigma} + (a_n^\mu)^* e^{in\sigma}\right)\cos(E_n\tau),$$

$$\partial_\tau X^\mu \pm \partial_\sigma X^\mu = \frac{1}{4\pi\alpha'}\sum_n \left(-a_n^\mu\left(\sin(E_n\tau) \pm i\frac{n}{E_n}\cos(E_n\tau)\right)e^{-in\sigma} + (a_n^\mu)^*\left(-\sin(E_n\tau) \pm i\frac{n}{E_n}\cos(E_n\tau)\right)e^{in\sigma}\right)$$

so the constraint reads

$$(\partial_\tau X \pm \partial_\sigma X)^2 = \frac{\eta_{\mu\nu}}{(4\pi\alpha')^2}\sum_n\sum_p \left(a_n^\mu a_p^\nu\left(\sin(E_n\tau) \pm i\frac{n}{E_n}\cos(E_n\tau)\right)\left(\sin(E_p\tau) \pm i\frac{p}{E_p}\cos(E_p\tau)\right)e^{-i(n+p)\sigma}-\right.$$

$$- a_n^\mu(a_p^\nu)^*\left(\sin(E_n\tau) \pm i\frac{n}{E_n}\cos(E_n\tau)\right)\left(-\sin(E_p\tau) \pm i\frac{p}{E_p}\cos(E_p\tau)\right)e^{-i(n-p)\sigma}-$$

$$- (a_n^\mu)^* a_p^\nu\left(-\sin(E_n\tau) \pm i\frac{n}{E_n}\cos(E_n\tau)\right)\left(\sin(E_p\tau) \pm i\frac{p}{E_p}\cos(E_p\tau)\right)e^{i(n-p)\sigma}+$$

$$\left.+ (a_n^\mu)^*(a_p^\nu)^*\left(-\sin(E_n\tau) \pm i\frac{n}{E_n}\cos(E_n\tau)\right)\left(-\sin(E_p\tau) \pm i\frac{p}{E_p}\cos(E_p\tau)\right)e^{i(n+p)\sigma}\right) \equiv 0$$

Since the exponentials are all linearly independent and we can't decouple the τ exponential from any of the sums, we have 3 cases to investigate: $n = p$, $n = -p$ and $n \neq \pm p$. For $n \neq \pm p$, we get

$$L_{n,p}^1 := \left(\frac{n}{E_n} - 1\right)\left(\frac{p}{E_p} - 1\right)a_n^\mu a_p^\nu \eta_{\mu\nu} = 0$$

$$L_{n,p}^2 := \left(\frac{n}{E_n} - 1\right)\left(\frac{p}{E_p} + 1\right)a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0$$

$$L_{n,p}^3 := \left(\frac{n}{E_n} + 1\right)\left(\frac{p}{E_p} - 1\right)(a_{-n}^\mu)^* a_p^\nu \eta_{\mu\nu} = 0$$

$$L_{n,p}^4 := \left(\frac{n}{E_n} + 1\right)\left(\frac{p}{E_p} + 1\right)(a_{-n}^\mu)^*(a_{-p}^\nu)^* \eta_{\mu\nu} = 0,$$

where $L_{n,p}^2$ and $L_{n,p}^3$ are just complex conjugates of each other with $n \mapsto -n$ and $p \mapsto -p$, so they are redundant. The same is true for $L_{n,p}^1$ and $L_{n,p}^4$, so in the end we have

$$L_{n,p} := \left(\frac{n}{E_n} - 1\right)\left(\frac{p}{E_p} - 1\right)a_n^\mu a_p^\nu \eta_{\mu\nu} = 0$$

$$\widetilde{L}_{n,p} := \left(\frac{n}{E_n} - 1\right)\left(\frac{p}{E_p} + 1\right)a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0.$$

The coefficients of both expressions are actually not necessary, since for them to be 0 we would need $n \to \pm\infty$ or $p \to \pm\infty$, so we can redefine

$$L_{n,p} := a_n^\mu a_p^\nu \eta_{\mu\nu} = 0$$

$$\widetilde{L}_{n,p} := a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0, n \neq \pm p.$$

For $n = -p$, the first and last terms are complex conjugates of each other with the relabeling $n \mapsto -n$ and $p \mapsto -p$, while the exponentials in the 2 middle terms are reduced to 1, so we get

$$\left(\frac{n}{E_n} - 1\right)^2 a_n^\mu a_{-n}^\nu \eta_{\mu\nu} = 0$$

$$\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right)a_n^\mu (a_n^\nu)^* \eta_{\mu\nu}\right) = 0,$$

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where in the first equation we can ignore the first factor once more. Finally for $n = p$, we get

$$\left(\frac{n}{E_n} - 1\right)^2 a_n^\mu a_n^\nu \eta_{\mu\nu} = 0$$

$$\left(\left(\frac{n}{E_n}\right)^2 - 1\right) a_n^\mu (a_{-n}^\nu)^* \eta_{\mu\nu} = 0,$$

where we again ignore the first factor because it's only 0 at infinity, thus collecting everything we have

$$L_{n,p} := a_n^\mu a_p^\nu \eta_{\mu\nu} = 0, \ \forall \ n, p \in \mathbb{Z},$$

$$\widetilde{L}_{n,p} := a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0, \ n \neq -p,$$

$$\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) \widetilde{L}_{n,-n}\right) = 0.$$

To make it more in line with regular ST literature, let's also map $p \mapsto p - n$:

$$L_{n,p} = a_n^\mu a_{p-n}^\nu \eta_{\mu\nu} = 0$$

$$\widetilde{L}_{n,p} = a_n^\mu (a_{n-p}^\nu)^* \eta_{\mu\nu} = 0$$

$$\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) \widetilde{L}_{n,0}\right) = 0.$$

One last thing before quantizing: the $n = 0$ term in the solution reads

$$\frac{1}{2\pi\alpha'} \frac{1}{2m} \left(a_0^\mu e^{-im\tau} + (a_0^\mu)^* e^{im\tau}\right),$$

which is divergent in the $m \to 0$ limit unless $\mathfrak{Re}(a_0^\mu) = 0$, thus a_0^μ must be purely imaginary. Keeping the analogy to the $m \to 0$ regime, we conclude $a_0^\mu = 4\pi i (\alpha')^2 p^\mu$. Since a_0^μ is proportional to p^μ, we have that

$$M^2 = -p^\mu p_\mu = \frac{1}{16\pi^2 (\alpha')^4} a_0^\mu a_0^\nu \eta_{\mu\nu},$$

which after analysing closer the $n = -p$ condition

$$\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) \widetilde{L}_{n,0}\right) = 0$$

$$\sum_{n \neq 0} \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) \widetilde{L}_{n,0}\right) - \widetilde{L}_{0,0} = 0$$

$$\sum_{n \neq 0} \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) \widetilde{L}_{n,0}\right) = a_0^\mu (a_0^\nu)^* \eta_{\mu\nu} \equiv (4\pi(\alpha')^2)^2 p^\mu p_\mu,$$

we have the string mass in terms of vibrational modes

$$M^2 = -\frac{1}{(4\pi(\alpha')^2)^2} \sum_{n \neq 0} \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) \widetilde{L}_{n,0}\right) =$$

$$= -\frac{1}{(4\pi(\alpha')^2)^2} \sum_{n \neq 0} \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) a_n^\mu (a_n^\nu)^* \eta_{\mu\nu}\right).$$

However, since

$$L_{n,p} = 0, \ \forall \ n, p \in \mathbb{Z},$$

in particular we have $L_{0,0} = a_0^\mu a_0^\nu \eta_{\mu\nu} = 0$, thus the center of mass momentum p^μ is a null vector.

As for the open string, its derivatives are

$$\partial_\tau X^\mu = \frac{i}{4\pi\alpha'} \sum_n \left(-a_n^\mu e^{-iE_n\tau} + (a_n^\mu)^* e^{iE_n\tau}\right) \cos(n\sigma)$$

$$\partial_\sigma X^\mu = -\frac{1}{4\pi\alpha'} \sum_n \frac{n}{E_n} \left(a_n^\mu e^{-iE_n\tau} + (a_n^\mu)^* e^{iE_n\tau}\right) \sin(n\sigma)$$

and the constraint reads

$$\partial_\tau X^\mu + \partial_\sigma X^\mu = \frac{1}{4\pi\alpha'} \sum_n \left(-a_n^\mu e^{-iE_n\tau}\left(i\cos(n\sigma) + \frac{n}{E_n}\sin(n\sigma)\right) + (a_n^\mu)^* e^{iE_n\tau}\left(i\cos(n\sigma) - \frac{n}{E_n}\sin(n\sigma)\right)\right)$$

$$(\partial_\tau X + \partial_\sigma X)^2 = \frac{\eta_{\mu\nu}}{(4\pi\alpha')^2} \sum_n \sum_p \left(a_n^\mu a_p^\nu e^{-i(E_n+E_p)\tau}\left(i\cos(n\sigma) + \frac{n}{E_n}\sin(n\sigma)\right)\left(i\cos(p\sigma) + \frac{p}{E_p}\sin(p\sigma)\right) - \right.$$

$$- a_n^\mu (a_p^\nu)^* e^{-i(E_n-E_p)\tau}\left(i\cos(n\sigma) + \frac{n}{E_n}\sin(n\sigma)\right)\left(i\cos(p\sigma) - \frac{p}{E_p}\sin(p\sigma)\right) -$$

$$- (a_n^\mu)^* a_p^\nu e^{i(E_n-E_p)\tau}\left(i\cos(n\sigma) - \frac{n}{E_n}\sin(n\sigma)\right)\left(i\cos(p\sigma) + \frac{p}{E_p}\sin(p\sigma)\right) +$$

$$\left. + (a_n^\mu)^* (a_p^\nu)^* e^{i(E_n+E_p)\tau}\left(i\cos(n\sigma) - \frac{n}{E_n}\sin(n\sigma)\right)\left(i\cos(p\sigma) - \frac{p}{E_p}\sin(p\sigma)\right)\right) = 0.$$

Once more, we see that the middle 2 terms and outer 2 terms are complex conjugate pairs. Unlike the closed string, here all cases lead to

$$L_{n,p} := a_n^\mu a_p^\nu \eta_{\mu\nu} = 0$$

$$\widetilde{L}_{n,p} := a_n^\mu (a_p^\nu)^* \eta_{\mu\nu} = 0,$$

where again to be more in line with ST literature we remap $p \mapsto p - n$

$$L_{n,p} = a_n^\mu a_{p-n}^\nu \eta_{\mu\nu} = 0$$

$$\widetilde{L}_{n,p} = a_n^\mu (a_{p-n}^\nu)^* \eta_{\mu\nu} = 0.$$

Since we don't have a summed constraint, the mass of the open string can't be written in terms of vibrational modes, only in terms of the 0-th mode

$$M^2 = -p^\mu p_\mu = \frac{1}{(4\pi(\alpha')^2)^2} a_0^\mu a_0^\nu \eta_{\mu\nu} =$$

$$= \frac{1}{(4\pi(\alpha')^2)^2} L_{0,0} \equiv 0.$$

10.1 The Expansion Modes Algebra

10.1.1 Closed String

Before starting this analysis, let's rename the vibrational modes to α_n^μ to be more in line with the literature
Start by finding the canonical momentum

$$\Pi_\lambda = \frac{\partial \mathscr{L}}{\partial(\partial_\tau X^\lambda)} = -\frac{T}{2}\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab} \delta_a^\tau \delta_\lambda^\mu \partial_b X^\nu \eta_{\mu\nu} =$$

$$= -\frac{1}{4\pi\alpha'}\left(1 - \frac{k\Delta}{2\phi^2}\right) \eta^{\tau b} \partial_b X^\nu \eta_{\lambda\nu} =$$

$$= \frac{1}{4\pi\alpha'} F^- \partial_\tau X^\nu \eta_{\lambda\nu} = F^- \frac{i}{4\pi\alpha'} \sum_{n\in\mathbb{Z}} \left(-\alpha_n^\nu e^{i(-E_n\tau+n\sigma)} + (\alpha_{-n}^\nu)^* e^{i(E_n\tau+n\sigma)}\right) \eta_{\nu\lambda}.$$

We then proceed to the cannonical Poisson bracket relations

$$\{X^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')\} = \delta(\sigma-\sigma')\eta^{\mu\nu}$$

$$\{X^\mu(\tau,\sigma),X^\nu(\tau,\sigma')\} = \{\Pi^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')\} = 0,$$

where by expanding the first relation we get

$$\{X^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')\} = \left\{ \frac{1}{2\pi\alpha'}\sum_{n\in\mathbb{Z}}\frac{1}{2E_n}\left(\alpha_n^\mu e^{i(-E_n\tau+n\sigma)} + (\alpha_{-n}^\mu)^* e^{i(E_n\tau+n\sigma)}\right), \right.$$

$$\left. F^-\frac{i}{4\pi\alpha'}\sum_{p\in\mathbb{Z}}\left(-\alpha_p^\nu e^{i(-E_p\tau+p\sigma')} + (\alpha_{-p}^\nu)^* e^{i(E_p\tau+p\sigma')}\right)\right\} =$$

$$= \frac{i}{8\pi^2(\alpha')^2}\sum_n\sum_p\frac{1}{2E_n}\left(-F^-\{\alpha_n^\mu,\alpha_p^\nu\}e^{-i(E_n+E_p)\tau}e^{i(n\sigma+p\sigma')} + F^-\{\alpha_n^\mu,(\alpha_{-p}^\nu)^*\}e^{-i(E_n-E_p)\tau}e^{i(n\sigma+p\sigma')} - \right.$$

$$\left. - F^-\{(\alpha_{-n}^\mu)^*,\alpha_p^\nu\}e^{i(E_n-E_p)\tau}e^{i(n\sigma+p\sigma')} + F^-\{(\alpha_{-n}^\mu)^*,(\alpha_{-p}^\nu)^*\}e^{i(E_n+E_p)\tau}e^{i(n\sigma+p\sigma')}\right) = \delta(\sigma-\sigma')\eta^{\mu\nu}$$

$$\left\lfloor \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma e^{-ik\sigma}\right.$$

$$\frac{i}{8\pi^2(\alpha')^2}\sum_n\sum_p\frac{1}{2E_n}\left(-F^-\{\alpha_n^\mu,\alpha_p^\nu\}e^{-i(E_n+E_p)\tau}\delta_{n,k}e^{ip\sigma'} + F^-\{\alpha_n^\mu,(\alpha_{-p}^\nu)^*\}e^{-i(E_n-E_p)\tau}\delta_{n,k}e^{ip\sigma'} - \right.$$

$$\left. - F^-\{(\alpha_{-n}^\mu)^*,\alpha_p^\nu\}e^{i(E_n-E_p)\tau}\delta_{n,k}e^{ip\sigma'} + F^-\{(\alpha_{-n}^\mu)^*,(\alpha_{-p}^\nu)^*\}e^{i(E_n+E_p)\tau}\delta_{n,k}e^{ip\sigma'}\right) = \frac{\eta^{\mu\nu}}{2\pi}e^{-ik\sigma'}$$

$$\frac{i}{8\pi^2(\alpha')^2}\sum_p\frac{1}{2E_k}\left(-F^-\{\alpha_k^\mu,\alpha_p^\nu\}e^{-i(E_k+E_p)\tau}e^{ip\sigma'} + F^-\{\alpha_k^\mu,(\alpha_{-p}^\nu)^*\}e^{-i(E_k-E_p)\tau}e^{ip\sigma'} - \right.$$

$$\left. - F^-\{(\alpha_{-k}^\mu)^*,\alpha_p^\nu\}e^{i(E_k-E_p)\tau}e^{ip\sigma'} + F^-\{(\alpha_{-k}^\mu)^*,(\alpha_{-p}^\nu)^*\}e^{i(E_k+E_p)\tau}e^{ip\sigma'}\right) = \frac{\eta^{\mu\nu}}{2\pi}e^{-ik\sigma'}$$

$$\left\lfloor \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma' e^{im\sigma'}\right.$$

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma' F^- e^{i(p+m)\sigma'}, \ \ F^- = \sum_l\left(A_l e^{i(E_l\tau+l\sigma'+\theta_l)} + (A_{-l})^* e^{i(-E_l\tau+l\sigma'-\theta_l)}\right)$$

$$\frac{i}{8\pi^2(\alpha')^2}\sum_p\frac{\left(A_{-(p+m)}e^{i(E_{p+m}\tau+\theta_{-(p+m)})} + (A_{p+m})^* e^{i(-E_{p+m}\tau-\theta_{-(p+m)})}\right)}{2E_k}\left(-\{\alpha_k^\mu,\alpha_p^\nu\}e^{-i(E_k+E_p)\tau} + \right.$$

$$+ \{\alpha_k^\mu,(\alpha_{-p}^\nu)^*\}e^{-i(E_k-E_p)\tau} - \{(\alpha_{-k}^\mu)^*,\alpha_p^\nu\}e^{i(E_k-E_p)\tau} + $$

$$\left. + \{(\alpha_{-k}^\mu)^*,(\alpha_{-p}^\nu)^*\}e^{i(E_k+E_p)\tau}\right) = \frac{1}{2\pi}\eta^{\mu\nu}\delta_{k,m}$$

$$\frac{i}{8\pi^2(\alpha')^2}\sum_p\frac{1}{2E_k}\left(-A_{-(p+m)}e^{i\theta_{-(p+m)}}\{\alpha_k^\mu,\alpha_p^\nu\}e^{i(E_{p+m}-E_k-E_p)\tau} + \right.$$

$$+ A_{-(p+m)}e^{i\theta_{-(p+m)}}\{\alpha_k^\mu,(\alpha_{-p}^\nu)^*\}e^{i(E_{p+m}-E_k+E_p)\tau} - A_{-(p+m)}e^{i\theta_{-(p+m)}}\{(\alpha_{-k}^\mu)^*,\alpha_p^\nu\}e^{i(E_{p+m}+E_k-E_p)\tau} +$$

$$+ A_{-(p+m)}e^{i\theta_{-(p+m)}}\{(\alpha_{-k}^\mu)^*,(\alpha_{-p}^\nu)^*\}e^{i(E_{p+m}+E_k+E_p)\tau} - (A_{p+m})^* e^{-i\theta_{-(p+m)}}\{\alpha_k^\mu,\alpha_p^\nu\}e^{-i(E_{p+m}+E_k+E_p)\tau} +$$

$$+ (A_{p+m})^* e^{-i\theta_{-(p+m)}}\{\alpha_k^\mu,(\alpha_{-p}^\nu)^*\}e^{-i(E_{p+m}+E_k-E_p)\tau} - (A_{p+m})^* e^{-i\theta_{-(p+m)}}\{(\alpha_{-k}^\mu)^*,\alpha_p^\nu\}e^{-i(E_{p+m}-E_k+E_p)\tau} +$$

$$\left. + (A_{p+m})^* e^{-i\theta_{-(p+m)}}\{(\alpha_{-k}^\mu)^*,(\alpha_{-p}^\nu)^*\}e^{-i(E_{p+m}-E_k-E_p)\tau}\right) = \frac{1}{2\pi}\eta^{\mu\nu}\delta_{k,m}$$

11   Into the Quantum Realm

Start by finding the canonical momentum

$$\Pi_\lambda = \frac{\partial \mathscr{L}}{\partial(\partial_\tau X^\lambda)} = -\frac{T}{2}\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\delta_a^\tau \delta_\lambda^\mu \partial_b X^\nu \eta_{\mu\nu} =$$

$$= -\frac{1}{4\pi\alpha'}\left(1 - \frac{k\Delta}{2\phi^2}\right)\eta^{\tau b}\partial_b X^\nu \eta_{\lambda\nu} =$$

$$= \frac{1}{4\pi\alpha'}F^-\partial_\tau X^\nu \eta_{\lambda\nu} = F^-\frac{i}{(4\pi\alpha')^2}\sum_p\left(-\alpha_p^\nu e^{i(-E_p\tau+p\sigma)} + (\alpha_{-p}^\nu)^* e^{i(E_p\tau+p\sigma)}\right)\eta_{\nu\lambda}.$$

Translating the equal-time Poisson bracket relations

$$\{X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\} = \delta(\sigma-\sigma')\eta^{\mu\nu}$$

$$\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\} = \{\Pi^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\} = 0$$

into equal time commutation relations for the now operators $\hat{X}^\mu$ and $\hat{\Pi}^\nu$

$$\left[\hat{X}^\mu(\tau,\sigma), \hat{\Pi}^\nu(\tau,\sigma')\right] = i\delta(\sigma-\sigma')\eta^{\mu\nu}$$

$$\left[\hat{X}^\mu(\tau,\sigma), \hat{X}^\nu(\tau,\sigma')\right] = \left[\hat{\Pi}^\mu(\tau,\sigma), \hat{\Pi}^\nu(\tau,\sigma')\right] = 0,$$

we start by the non-zero one

$$\left[\hat{X}^\mu(\tau,\sigma), \hat{\Pi}^\nu(\tau,\sigma')\right] = \left[\frac{1}{2\pi\alpha'}\sum_{p\in\mathbb{Z}}\frac{1}{2E_p}\left(\hat{\alpha}_p^\mu e^{i(-E_p\tau+p\sigma)} + (\hat{\alpha}_{-p}^\mu)^\dagger e^{i(E_p\tau+p\sigma)}\right),\right.$$

$$\left. F^-\frac{i}{(4\pi\alpha')^2}\sum_{n\in\mathbb{Z}}\left(-\hat{\alpha}_n^\nu e^{i(-E_n\tau+n\sigma')} + (\hat{\alpha}_{-n}^\nu)^\dagger e^{i(E_n\tau+n\sigma')}\right)\right] =$$

$$= F^-\frac{i}{(4\pi\alpha')^3}\sum_p\sum_n\frac{1}{E_p}\left(-\left[\hat{\alpha}_p^\mu, \hat{\alpha}_n^\nu\right]e^{-i(E_p+E_n)\tau}e^{i(p\sigma+n\sigma')} + \left[\hat{\alpha}_p^\mu, (\hat{\alpha}_{-n}^\nu)^\dagger\right]e^{-i(E_p-E_n)\tau}e^{i(p\sigma+n\sigma')}-\right.$$

$$\left. -\left[(\hat{\alpha}_{-p}^\mu)^\dagger, \hat{\alpha}_n^\nu\right]e^{i(E_p-E_n)\tau}e^{i(p\sigma+n\sigma')} + \left[(\hat{\alpha}_{-p}^\mu)^\dagger, (\hat{\alpha}_{-n}^\nu)^\dagger\right]e^{i(E_p+E_n)\tau}e^{i(p\sigma+n\sigma')}\right) \equiv i\delta(\sigma-\sigma')\eta^{\mu\nu},$$

from which we conclude that

$$\left[\hat{\alpha}_p^\mu, (\hat{\alpha}_{-n}^\nu)^\dagger\right] = \eta^{\mu\nu}\delta_{p,-n}$$

$$\left[\hat{\alpha}_p^\mu, \hat{\alpha}_n^\nu\right] = 0 = \left[(\hat{\alpha}_{-p}^\mu)^\dagger, (\hat{\alpha}_{-n}^\nu)^\dagger\right],$$

giving us the harmonic oscillator operators

$$\hat{a}_n^\mu := \hat{\alpha}_n^\mu$$

$$(\hat{a}_n^\mu)^\dagger := (\hat{\alpha}_n^\mu)^\dagger, \ n \neq 0.$$

We define a vacuum state of the string to obey

$$\hat{a}_n^\mu|0\rangle = 0, \ \text{for } n \neq 0.$$

(What does $(\hat{a}_{-n}^\mu)^\dagger|0\rangle$ mean? Also, note how in our case, each field X^μ only gives rise to one tower of creation and annihilation operators...) For $n = 0$, we have the center of mass momentum operator, so the vacuum also obeys

$$\hat{p}^\mu|0\rangle = p^\mu|0\rangle.$$

A generic state arises from a sequence of creation operators on the vacuum

$$((a_1^{\mu_1})^\dagger)^{n_{\mu_1}}((a_2^{\mu_2})^\dagger)^{n_{\mu_2}}...((a_1^{\nu_1})^\dagger)^{n_{\nu_1}}((a_2^{\mu_2})^\dagger)^{n_{\nu_2}}...|0\rangle.$$

This (should) give rise to particles.

As in regular ST, we have ghosts arising from the Minkowski metric

$$\left[\hat{a}_n^\mu, (\hat{a}_m^\nu)^\dagger\right] = \eta^{\mu\nu}\delta_{nm}.$$

For constraints, classically we have

$$L_{n,p} = \alpha_n^\mu \alpha_{p-n}^\nu \eta_{\mu\nu} = 0$$
$$\widetilde{L}_{n,p} = \alpha_n^\mu (\alpha_{n-p}^\nu)^* \eta_{\mu\nu} = 0$$

which because of the existence of ghosts we require to have vanishing matrix elements when sandwiched between physical states

$$\langle \text{phys}'|\hat{L}_{n,p}|\text{phys}\rangle = 0 = \langle \text{phys}'|\hat{\widetilde{L}}_{n,p}|\text{phys}\rangle.$$

When translating the constraints into quantum operators, in $L_{n,p}$ we have no ambiguity of ordering since $\hat{\alpha}$ commutes with itself. As for $\widetilde{L}_{n,p}$, there is an ambiguity for $p = 0$ and $n \neq 0$, so we pick normal ordering with the annihilation operators moved to the right

$$\hat{\widetilde{L}}_{n,0} = (\hat{\alpha}_n^\mu)^\dagger \hat{\alpha}_n^\nu \eta_{\mu\nu}.$$

The ambiguity manifests in the imposition of this constraint as

$$\langle \text{phys}'|(\hat{\widetilde{L}}_{n,0} - c_n)|\text{phys}\rangle = 0,$$

for some sequence c_n with $c_0 = 0$ since $\hat{\widetilde{L}}_{0,0}$ has no ordering ambiguities. Since classically

$$M^2 = -p^\mu p_\mu = -\frac{1}{(4\pi(\alpha')^2)^2}\alpha_0^\mu (\alpha_0^\nu)^* \eta_{\mu\nu} = \frac{1}{(4\pi(\alpha')^2)^2}\widetilde{L}_{0,0},$$

we see that the string mass spectrum will be on-shell

$$\hat{M}^2 = -\frac{1}{(4\pi(\alpha')^2)^2}(\hat{\alpha}_0^\mu)^\dagger \hat{\alpha}_0^\nu \eta_{\mu\nu} = -\hat{p}^\mu \hat{p}_\mu.$$

The commutation relations between the L's are

$$\left[\hat{L}_{n,m}, \hat{L}_{p,k}\right] = 0 \text{ (this has only annihilation operators)}$$

$$\left[\hat{L}_{n,m}, \hat{\widetilde{L}}_{p,k}\right] = \left[\hat{\alpha}_n^\mu \hat{\alpha}_{m-n}^\nu \eta_{\mu\nu}, (\hat{\alpha}_{p-k}^{\mu'})^\dagger \hat{\alpha}_p^{\nu'} \eta_{\mu'\nu'}\right] =$$
$$= \eta_{\mu\nu}\left(\hat{\alpha}_n^\mu \left[\hat{\alpha}_{m-n}^\nu, (\hat{\alpha}_{p-k}^{\mu'})^\dagger \hat{\alpha}_p^{\nu'}\right] + \left[\hat{\alpha}_n^\mu, (\hat{\alpha}_{p-k}^{\mu'})^\dagger \hat{\alpha}_p^{\nu'}\right]\hat{\alpha}_{m-n}^\nu\right)\eta_{\mu'\nu'} =$$
$$= \eta_{\mu\nu}\left(\hat{\alpha}_n^\mu \eta^{\nu\mu'}\delta_{m-n-p+k,0}\hat{\alpha}_p^{\nu'} + \eta^{\mu\mu'}\delta_{n-p+k,0}\hat{\alpha}_p^{\nu'}\hat{\alpha}_{m-n}^\nu\right)\eta_{\mu'\nu'} =$$
$$= \hat{\alpha}_n^\mu \hat{\alpha}_p^\nu \eta_{\mu\nu}\delta_{m-n-p+k} + \hat{\alpha}_p^\mu \hat{\alpha}_{m-n}^\nu \eta_{\mu\nu}\delta_{n-p+k} =$$
$$= \hat{L}_{n,p+n}\delta_{m-n-p+k,0} + \hat{L}_{p,m-n+p}\delta_{n-p+k,0}$$

$$\left[\hat{\widetilde{L}}_{n,m}, \hat{\widetilde{L}}_{p,k}\right] = \left[(\hat{\alpha}_{n-m}^\mu)^\dagger \hat{\alpha}_n^\nu \eta_{\mu\nu}, (\hat{\alpha}_{p-k}^{\mu'})^\dagger \hat{\alpha}_p^{\nu'} \eta_{\mu'\nu'}\right] =$$
$$= \eta_{\mu\nu}\left((\hat{\alpha}_{p-k}^{\mu'})^\dagger \left[(\hat{\alpha}_{n-m}^\mu)^\dagger \hat{\alpha}_n^\nu, \hat{\alpha}_p^{\nu'}\right] + \left[(\hat{\alpha}_{n-m}^\mu)^\dagger \hat{\alpha}_n^\nu, (\hat{\alpha}_{p-k}^{\mu'})^\dagger\right]\hat{\alpha}_p^{\nu'}\right)\eta_{\mu'\nu'} =$$
$$= \eta_{\mu\nu}\left((\hat{\alpha}_{p-k}^{\mu'})^\dagger(-\eta^{\mu\nu'}\delta_{p-n+m,0})\hat{\alpha}_n^\nu + (\hat{\alpha}_{n-m}^\mu)^\dagger \eta^{\nu\mu'}\delta_{n-p+k,0}\hat{\alpha}_p^{\nu'}\right)\eta_{\mu'\nu'} =$$
$$= -(\hat{\alpha}_{p-k}^\mu)^\dagger \hat{\alpha}_n^\nu \eta_{\mu\nu}\delta_{p-n+m,0} + (\hat{\alpha}_{n-m}^\mu)^\dagger \hat{\alpha}_p^\nu \eta_{\mu\nu}\delta_{n-p+k,0} =$$
$$= -\hat{\widetilde{L}}_{n,n-p+k}\delta_{p-n+m,0} + \hat{\widetilde{L}}_{p,p-n+m}\delta_{n-p+k,0}$$

Since $[\hat{L}_{n,m}, \hat{\widetilde{L}}_{p,k}]$ only has annihilation operators, it has no ordering ambiguities. $[\hat{\widetilde{L}}_{n,m}, \hat{\widetilde{L}}_{p,k}]$ have ordering ambiguities for $n - p + k = 0$ with $k + m = 0$ and/or $p - n + m = 0$ with $m + k = 0$, so we add the anomalous terms

$$\left[\hat{\widetilde{L}}_{n,m}, \hat{\widetilde{L}}_{p,k}\right] = -\hat{\widetilde{L}}_{n,n-p+k}\delta_{p-n+m,0} + \hat{\widetilde{L}}_{p,p-n+m}\delta_{n-p+k,0} + C_n\delta_{n-p+k,0} + D_p\delta_{p-n+m,0}.$$

Clearly, $C_0 = 0 = D_0$, since in these cases the equivalent terms are composed by $\hat{\widetilde{L}}_{0,0} = \hat{p}^\mu \hat{p}_\mu$.

12 Polyakov Action in terms of LQG Variables

Turn embbeding fields X^μ into vector in the spin-1 representation of $\text{Spin}(d,1)$, X^I, and promote partial derivative to covariant derivative $\partial_a \mapsto \mathcal{D}_a$ acting as

$$\mathcal{D}_a X^I = \partial_a X^I + k\mathcal{A}^I_{aJ} X^J,$$

where $\mathcal{A}^I_{aJ}(x) = (A^{\alpha\beta}_a(x)T_{\alpha\beta})^I{}_J$ is the background $\text{Spin}(d,1)$ connection pulledback to the WS, so the action becomes

$$S = -\frac{T}{2} \int \mathrm{d}^2 x \sqrt{g} g^{ab} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ},$$

where the WS metric can be recast in terms of auxiliary zweibein fields as

$$S = -\frac{T}{2} \int \mathrm{d}^2 x \, e e^a_i e^{bi} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ}.$$

12.1 EoMs

12.1.1 W.r.t e

$$\frac{\delta S}{\delta e^c_k} = \frac{T}{2} \left(\frac{\partial}{\partial e^c_k} e e^a_i e^{bi} + e \frac{\partial}{\partial e^c_k} (e^a_i e^{bi}) \right) \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ} =$$

$$= T \left(-e e^k_c e^a_i e^{bi} + e \delta^a_c \delta^k_i e^{bi} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ} =$$

$$= T \left(-e e^k_c e^a_i e^{bi} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ} + e e^{bk} \mathcal{D}_c X^I \mathcal{D}_b X^J \eta_{IJ} \right) \overset{!}{=} 0$$

$$T^k_c := e^{bk} \mathcal{D}_c X^I \mathcal{D}_b X^J \eta_{IJ} - e^k_c e^a_i e^{bi} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ} = 0$$

$$e^k_c = f e^{bk} \mathcal{D}_c X^I \mathcal{D}_b X^J \eta_{IJ},$$

$$\frac{1}{f} = e^a_i e^{bi} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ}$$

12.1.2 W.r.t A

$$\frac{\delta S}{\delta \mathcal{A}^{KL}_c} = \frac{T}{2} \left(e e^a_i e^{bi} \frac{\partial}{\partial \mathcal{A}^{KL}_c} \left(\mathcal{D}_a X^I \mathcal{D}_b X^J \right) \eta_{IJ} \right) =$$

$$= T \left(e e^a_i e^{bi} \delta^c_a \delta^I_{[K} \eta_{L]I'} X^{I'} \mathcal{D}_b X^J \eta_{IJ} \right) =$$

$$= T e e^c_i e^{bi} \mathcal{D}_b X_{[K} X_{L]} \overset{!}{=} 0$$

$$\mathcal{T}^{IJ}_a = \mathcal{D}_a X^{[I} X^{J]} = 0$$

12.1.3 W.r.t X

$$\frac{\delta S}{\delta X^K} = -\frac{T}{2} \left(\partial_c \left(e e^a_i e^{bi} \frac{\partial}{\partial(\partial_c X^K)} \left(\mathcal{D}_a X^I \mathcal{D}_b X^J \right) \eta_{IJ} \right) - e e^a_i e^{bi} \frac{\partial}{\partial X^K} \left(\mathcal{D}_a X^I \mathcal{D}_b X^J \right) \eta_{IJ} \right) =$$

$$= -T \left(\partial_c \left(e e^a_i e^{bi} \delta^c_a \delta^I_K \mathcal{D}_b X^J \eta_{IJ} \right) - e e^a_i e^{bi} k \mathcal{A}^I_{aI'} \delta^{I'}_K \mathcal{D}_b X^J \eta_{IJ} \right) =$$

$$= -T \left(\partial_a \left(e e^a_i e^{bi} \mathcal{D}_b X^J \eta_{KJ} \right) - k \mathcal{A}^I_{aK} \left(e e^a_i e^{bi} \mathcal{D}_b X^J \eta_{IJ} \right) \right) =$$

$$= -T \mathcal{D}_a \left(e e^a_i e^{bi} \mathcal{D}_b X_K \right) \overset{!}{=} 0$$

$$\mathcal{D}_a \left(e e^a_i e^{bi} \mathcal{D}_b X^I \right) = 0$$

In conformal gauge $ee_i^a e^{bi} = \eta^{ab}$, thus

$$\eta^{ab}\mathcal{D}_a\mathcal{D}_b X^I = 0.$$

More explicitly,

$$\eta^{ab}\mathcal{D}_a\left(\partial_b X^I + k\mathcal{A}_{bJ}^I X^J\right) = 0$$

$$\eta^{ab}\left(\left(\partial_a\partial_b X^I + k\mathcal{A}_{aJ}^I\partial_b X^J\right) + \left(k\partial_a\left(\mathcal{A}_{bJ}^I X^J\right) + k^2\mathcal{A}_{aI'}^I\mathcal{A}_{bJ}^{I'}X^J\right)\right) = 0$$

$$\eta^{ab}\left(\partial_a\partial_b X^I + 2k\mathcal{A}_{aJ}^I\partial_b X^J + k\partial_a\mathcal{A}_{bJ}^I X^J + k^2\mathcal{A}_{aI'}^I\mathcal{A}_{bJ}^{I'}X^J\right) = 0$$

$$\eta^{ab}\left(\delta_J^I\partial_a\partial_b + 2k\mathcal{A}_{aJ}^I\partial_b\right)X^J = -\left(k\eta^{ab}\partial_a\mathcal{A}_{bJ}^I X^J + k^2\eta^{ab}\mathcal{A}_{aI'}^I\mathcal{A}_{bJ}^{I'}X^J\right),$$

which has implicit solution given by

$$X^K(x) = -\int \mathrm{d}^2 x' G_I^K(x,x')\left(k\eta^{a'b'}\partial_{a'}\mathcal{A}_{b'J}^I(x')X^J(x') + k^2\eta^{a'b'}\mathcal{A}_{a'I'}^I(x')\mathcal{A}_{b'J}^{I'}(x')X^J(x')\right),$$

where the Green's function matrix $G_J^K(x,x')$ satisfies

$$\eta^{ab}\left(\delta_K^I\partial_a\partial_b + 2k\mathcal{A}_{aK}^I(x)\partial_b\right)G_J^K(x,x') = \delta_J^I\delta(x,x').$$

Boundary conditions come from

$$\delta S = \int \mathrm{d}^2 x\left(\frac{\partial\mathscr{L}}{\partial X^K}\delta X^K + \frac{\partial\mathscr{L}}{\partial(\partial_c X^K)}\delta(\partial_c X^K) + ...\right) =$$

$$= \int \mathrm{d}^2 x\left((\text{EoM})\delta X^K + \partial_c\left(\frac{\partial\mathscr{L}}{\partial(\partial_c X^K)}\delta X^K\right) + ...\right) =$$

$$= \int \mathrm{d}^2 x(\text{EoM})\delta X^K + \int \mathrm{d}\tau\left(\frac{\partial\mathscr{L}}{\partial(\partial_\sigma X^K)}\delta X^K\right)\Bigg|_0^{\sigma_1} + ... = 0,$$

thus

$$\left(\frac{\partial\mathscr{L}}{\partial(\partial_\sigma X^K)}\delta X^K\right)\Bigg|_0^{\sigma_1} = 0$$

$$\left(\left(-Tee_i^\sigma e^{bi}\mathcal{D}_b X_K\right)\delta X^K\right)\Big|_0^{\sigma_1} = 0$$

$$\left((\mathcal{D}_\sigma X_K)\delta X^K\right)\Big|_0^{\sigma_1} = 0,$$

so either $\delta X^K(\tau,\sigma_*) = 0$, $\sigma_* = 0,\sigma_1$ (Dirichlet B.C) or $\mathcal{D}_\sigma X^K(\tau,\sigma_*) = 0$, $\sigma_* = 0,\sigma_1$ (free end-point B.C). For closed strings, $X^K(\tau,\sigma) = X^K(\tau,\sigma+2\pi)$.

When connection is trivial EoM reduces to regular wave eqn, which have the regular string solution

$$X_0^I(\tau,\sigma) = X_{0L}^I(\sigma^+) + X_{0R}^I(\sigma^-),$$

with

$$X_{0L}^I(\sigma^+) = \frac{1}{2}x^I + \frac{1}{2}\alpha'p^I\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^I e^{-in\sigma^+}, \ \sigma^+ = \tau+\sigma$$

$$X_{0R}^I(\sigma^-) = \frac{1}{2}x^I + \frac{1}{2}\alpha'p^I\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I e^{-in\sigma^-}, \ \sigma^- = \tau-\sigma.$$

For Schwarzchild space-time, we start by calculating the bulk vielbein field from the metric

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s^{d-2}}{r^{d-2}}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{r_s^{d-2}}{r^{d-2}}}\mathrm{d}r^2 + r^2\left((\mathrm{d}\phi^1)^2 + \sum_{i=2}^{d-1}\prod_{j=1}^{i-1}\sin^2(\phi^j)(\mathrm{d}\phi^i)^2\right)$$

$$E^0_\mu \mathrm{d}X^\mu = \left(1 - \left(\frac{r_s}{r}\right)^{d-2}\right)\mathrm{d}t$$

$$E^1_\mu \mathrm{d}X^\mu = \left(\frac{1}{1 - \left(\frac{r_s}{r}\right)^{d-2}}\right)\mathrm{d}r$$

$$E^2_\mu \mathrm{d}X^\mu = r\mathrm{d}\phi^1$$

$$E^3_\mu \mathrm{d}X^\mu = r\sin(\phi^1)\mathrm{d}\phi^2$$

$$\vdots$$

$$E^d_\mu \mathrm{d}X^\mu = r\prod_{j=1}^{d-2}\sin(\phi^j)\mathrm{d}\phi^{d-1},$$

from which we can get the spin connection as

$$A^{AB}_\mu = \frac{1}{2}E^{\nu A}(\partial_\mu E^B_\nu - \partial_\nu E^B_\mu) - \frac{1}{2}E^{\nu B}(\partial_\mu E^A_\nu - \partial_\nu E^A_\mu) - \frac{1}{2}E^{\lambda A}E^{\rho B}(\partial_\lambda E_{\rho C} - \partial_\rho E_{\lambda C})E^C_\mu$$

For FLRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (a(t))^2\left(\sum_{i=1}^d (\mathrm{d}X^i)^2\right)$$

the vielbein is

$$E^0_\mu \mathrm{d}X^\mu = \mathrm{d}t$$

$$E^{(i)}_\mu \mathrm{d}X^\mu = a(t)\mathrm{d}X^i, \;\; i = 1, ..., d,$$

so the connection is

$$A^{0(i)}_t = -A^{(i)0}_t = 0$$

$$A^{0(i)}_i = -A^{(i)0}_i = -\partial_t a(t)$$

$$A^{0(i)}_j = -A^{(i)0}_j = 0$$

$$A^{(i)(j)}_t = -A^{(j)(i)}_t = 0$$

$$A^{(i)(j)}_i = -A^{(j)(i)}_i = -A^{(i)(j)}_j = A^{(j)(i)}_j = 0,$$

all other components being 0. From this, the full connection is

$$\mathcal{A}_t = 0$$

$$\mathcal{A}_i = A^{AB}_i T_{AB} = A^{0(i)}_i T_{0(i)} = -\partial_t a(t)\sum_{i=1}^d T_{0i} = -\begin{bmatrix} 0 & \partial_t a(t) & \partial_t a(t) & ... & \partial_t a(t) \\ \partial_t a(t) & 0 & 0 & ... & 0 \\ \partial_t a(t) & 0 & 0 & ... & 0 \\ \vdots & & & & \vdots \\ \partial_t a(t) & 0 & 0 & ... & 0 \end{bmatrix}, \;\; i = 1, ..., d.$$

Assuming pulledback connection retains its form,

$$\mathcal{A}_\tau = 0$$

$$\mathcal{A}_\sigma = -\partial_\tau a(\tau)\begin{bmatrix} 0 & 1 & 1 & ... & 1 \\ 1 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & ... & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & ... & 0 \end{bmatrix}$$

$$\mathcal{A}_\sigma \mathcal{A}_\sigma = (\partial_\tau a(\tau))^2 \begin{bmatrix} d & 0 & \ldots & 0 \\ 0 & d & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & d \end{bmatrix}$$

$$\partial_\tau \mathcal{A}_\sigma = -\partial_a \partial_\tau a(\tau) \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

$$\partial_\sigma \mathcal{A}_\sigma = 0$$

$$\mathcal{A}_\sigma X = -\partial_\tau a(\tau) \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ \vdots \\ X^d \end{bmatrix} = -\partial_\tau a(\tau) \begin{bmatrix} \sum_{i=1}^d X^i \\ X^0 \\ X^0 \\ \vdots \\ X^0 \end{bmatrix}.$$

Solution thus becomes

$$X(x) = -k^2 d \int \mathrm{d}^2 x' (\partial_\tau a(\tau'))^2 \underline{G}(x;x') X(x')$$

where the Green's matrix satisfies

$$\left(\mathbb{1}\eta^{ab}\partial_a\partial_b - 2k\partial_\tau a(\tau) \sum_i T_{0(i)}\partial_\sigma \right) \underline{G}(x;x') = \mathbb{1}\delta(x;x').$$

$$\sum_i T_{0(i)}\underline{G} = \begin{bmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} G^0{}_0 & G^0{}_1 & \ldots & G^0{}_d \\ G^1{}_0 & G^1{}_1 & \ldots & G^1{}_d \\ \vdots & \vdots & & \vdots \\ G^d{}_0 & G^d{}_1 & \ldots & G^d{}_d \end{bmatrix} = \begin{bmatrix} \sum_i G^{(i)}{}_0 & \sum_i G^{(i)}{}_1 & \ldots & \sum_i G^{(i)}{}_d \\ G^0{}_0 & G^0{}_1 & \ldots & G^0{}_d \\ \vdots & \vdots & & \vdots \\ G^0{}_0 & G^0{}_1 & \ldots & G^0{}_d \end{bmatrix},$$

thus the Green's matrix condition becomes

$$\eta^{ab}\partial_a\partial_b \begin{bmatrix} G^0{}_0 & G^0{}_1 & \ldots & G^0{}_d \\ G^1{}_0 & G^1{}_1 & \ldots & G^1{}_d \\ \vdots & \vdots & & \vdots \\ G^d{}_0 & G^d{}_1 & \ldots & G^d{}_d \end{bmatrix} - 2k\partial_\tau a(\tau)\partial_\sigma \begin{bmatrix} \sum_i G^{(i)}{}_0 & \sum_i G^{(i)}{}_1 & \ldots & \sum_i G^{(i)}{}_d \\ G^0{}_0 & G^0{}_1 & \ldots & G^0{}_d \\ \vdots & \vdots & & \vdots \\ G^0{}_0 & G^0{}_1 & \ldots & G^0{}_d \end{bmatrix} = \delta(x;x') \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}.$$

For $x \neq x'$, we thus get

$$\eta^{ab}\partial_a\partial_b G^0{}_J - 2k\dot{a}\partial_\sigma \sum_i G^{(i)}{}_J = 0,$$

$$\eta^{ab}\partial_a\partial_b G^{(i)}{}_J - 2k\dot{a}\partial_\sigma G^0{}_J = 0.$$

The second eqn tells us that $G^1{}_J = G^2{}_J = \ldots = G^d{}_J$, since the functions all satisfy the same eqn with same B.C, reducing the first eqn to

$$\eta^{ab}\partial_a\partial_b G^0{}_J - 2kd\dot{a}\partial_\sigma G^1{}_J = 0,$$

leaving us with the almost symmetric system of eqns

$$\eta^{ab}\partial_a\partial_b G^0{}_J - 2kd\dot{a}\partial_\sigma G^1{}_J = 0$$
$$\eta^{ab}\partial_a\partial_b G^1{}_J - 2k\dot{a}\partial_\sigma G^0{}_J = 0.$$

These can be solved with the use of the 1D wave Green's function $\Theta(|\sigma| - \tau)$ as in

$$G^0{}_J = 2kd \int_0^{\sigma_1} \mathrm{d}^2 x'$$