

Sketchboard

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1 Basic concepts

1.1 With extra capital internal index on embedding fields

Promote embedding fields X^μ to have an internal group index with D values

$$\begin{aligned}
X^\mu &\rightarrow X^{\mu I} , \quad I = 0, \dots, d \\
g_{ab} &= 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{IJ} \sim f D_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu} \\
D_a X^{\mu I} &= \partial_a X^{\mu I} + \omega_{aJ}^I X^{\mu J} \\
e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ} , \quad i, j = 0, 1 \\
g = \det(g_{ab}) &\rightarrow g = \det(\text{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

1.2 With 1 extra small internal index on embedding fields

Promote embedding fields to have a internal group index with 2 values

$$\begin{aligned}
X^\mu &\rightarrow X^{\mu i} , \quad i = 0, 1 \\
g_{ab} &= 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{ij} \sim f D_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu} \\
D_a X^{\mu i} &= \partial_a X^{\mu i} + \omega_{aj}^i X^{\mu j} \\
e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij} \\
g = \det(g_{ab}) &\rightarrow g = \det(\text{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

1.3 With 2 extra small indices

Promote embedding fields X^μ to have two internal indices with 2 values

$$X^\mu \rightarrow X^{\mu ij} , \quad i, j = 0, 1$$

1.4 Without extra index on embedding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$\begin{aligned}
e_a^i e_b^j \eta_{ij} &= g_{ab} = 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\
g = \det(g_{ab}) &= \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int d\tau \wedge d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} ,$$

2.1 With capital internal index

... and promote partial derivative ∂_a to covariant derivative D_a , giving us our first attempt at modified Polyakov action

$$\begin{aligned}
S_{MP1} &= -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ} \\
&\quad \updownarrow \\
\mathcal{L}_{MP1} &= -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}
\end{aligned}$$

2.2 With small internal index

2.2.1 Without extra field, 1 internal index

... and promote derivatives to covariant D_a , giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

$$\updownarrow$$

$$\mathcal{L}_{MP2} = -\frac{T_0}{2} \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

2.2.2 With extra field

..., promote partial derivative to covariant and add extra internal WS field v^i leading us to

$$S_{MP3} = -\frac{T}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

$$\updownarrow$$

$$\mathcal{L}_{MP3} = -\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

2.2.3 Without extra field, 2 internal indices

... and promote derivatives to covariant D_a ,

$$S_{MP4} = -\frac{T}{2} \int d\tau \wedge d\sigma e e_i^a e^{bi} D_a X^{\mu jj'} \eta_{jj'} \eta_{kk'} D_b X^{\nu kk'} E_\mu^I E_{\nu I}$$

$$\mathcal{L}_{MP4} = -\frac{T}{2} e e_i^a e_j^b D_a X^{\mu ik} \eta_{kl} D_b X^{\nu lj} E_\mu^I E_{\nu I}$$

2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) .$$

$$\updownarrow$$

$$\mathcal{L}_{DP} = -\frac{T_0}{2} \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) .$$

2.4 Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of D -dimensional gamma matrices and bulk spinors

$$S_{LP} = -\frac{T}{2} \int d\tau \wedge d\sigma \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

$$\updownarrow$$

$$\mathcal{L}_{LP} = -\frac{T}{2} \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

3 EoMs

Start by writing $\det(e) = \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n$ and $g_{ab} = g_{ab}^{ij} \eta_{ij}$

3.1 With small internal index

3.1.1 Without extra field, w.r.t e

$$\begin{aligned}
\frac{\delta S_{MP2}}{\delta e_l^e} &= \frac{D\mathcal{L}_{MP2}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)} \right) \left(-\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \right) = \\
&= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (\eta^{ml} g_{ce} e_d^n e_i^a e_j^b + e_c^m \eta^{nl} g_{de} e_i^a e_j^b + \\
&+ e_c^m e_d^n \delta_e^a \delta_i^l e_j^b + e_c^m e_d^n e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} g_{ce} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} e_c^k e_{ek} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_i^l e_j^a e_k^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -T (-\det(e) e_e^l e_i^a e_j^b + \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \stackrel{!}{=} 0 \\
&\quad \downarrow \\
T_e^l &:= E_\mu^I E_{\nu I} (e_i^a D_a X^{\mu i} D_e X^{\nu l} - e_e^l e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j}) = 0 \\
&\quad \Downarrow \\
e_e^l &= f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I} , \\
\frac{1}{f} &= e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \\
&\quad \downarrow \\
e_e^l e_{fl} &= (f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I}) (f e_{i'}^{a'} D_{a'} X^{\mu' i'} D_f X_l^{\nu'} E_{\mu'}^{I'} E_{\nu'}^{I'}) = \\
&= f^2 D_e X^{\nu l} D_f X_l^{\nu'} E_\nu^{I'} E_{\nu'}^{I'} (e_i^a e_{i'}^{a'} D_a X^{\mu i} D_{a'} X^{\mu' i'} E_\mu^I E_{\mu'}^{I'}) = \\
&= f D_e X^{\mu i} D_f X_i^\nu E_\mu^I E_{\nu I} \\
&\quad \Downarrow \\
D_e X^{\mu i} D_f X_i^\nu E_\mu^I E_{\nu I} &= 2 \partial_e X^\mu \partial_f X^\nu G_{\mu\nu} \\
e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\
&\quad \Downarrow \\
D_a X^{\mu 0} &= -i \partial_a X^\mu \\
D_a X^{\mu 1} &= \partial_a X^\mu \\
&\quad \Downarrow \\
X^{\mu 0} &= -i X^{\mu 1}
\end{aligned}$$

3.1.2 With extra field, w.r.t e

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta e_o^e} &= \frac{D\mathcal{L}_{MP3}}{De_o^e} = \left(\frac{\partial}{\partial e_o^e} - \partial_f \frac{\partial}{\partial (\partial_f e_o^e)} \right) \left(-\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (g_{ce} \eta^{mo} e_d^n \eta^{ij} e_i^a e_j^b + e_c^m g_{de} \eta^{no} \eta^{ij} e_i^a e_j^b + \\
&+ e_c^m e_d^n \eta^{ij} \delta_e^a \delta_i^o e_j^b + e_c^m e_d^n \eta^{ij} e_i^a \delta_e^b \delta_j^o) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -\frac{T}{2} \varepsilon^{cd} \varepsilon_{mn} (e_c^p e_{ep} e_d^n \eta^{mo} \eta^{ij} e_i^a e_j^b + 2 \det(e) e^{ao} \delta_e^b) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -T(-\det(e) e_e^o \eta^{ij} e_i^a e_j^b + \det(e) e^{ao} \delta_e^b) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \stackrel{!}{=} 0
\end{aligned}$$

↓

$$\boxed{T_e^o := (e^{ao} D_a(X_k^\mu v^k) D_e(X_l^\nu v^l) - e_e^o \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l)) E_\mu^I E_\nu^J \eta_{IJ} = 0}$$

⇓

$$\begin{aligned}
e_e^o &= f e^{ao} D_a(X_k^\mu v^k) D_e(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} , \\
\frac{1}{f} &= \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}
\end{aligned}$$

3.1.3 With extra field, w.r.t ω

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta \omega_c^{mn}} &= \frac{D\mathcal{L}_{MP3}}{D\omega_c^{mn}} = \left(\frac{\partial}{\partial \omega_c^{mn}} - \partial_d \frac{\partial}{\partial (\partial_d \omega_c^{mn})} \right) \left(-\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T}{4} e g^{ab} ((-\delta_a^c \delta_{[m}^{k'} \eta_{n]k} X_k^\mu v^k) D_b X_l^\nu v^l + D_a X_k^\mu v^k (-\delta_b^c \delta_{[m}^{l'} \eta_{n]l} X_l^\nu v^l)) G_{\mu\nu} = \\
&= \frac{T}{4} e (g^{ca} X_{[m}^\mu v_{n]} D_a X_k^\nu v^k + g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_{n]}) G_{\mu\nu} = \\
&= \frac{T}{2} e g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_{n]} G_{\mu\nu} \stackrel{!}{=} 0
\end{aligned}$$

↓

$$\boxed{\mathcal{T}_{ab}^i := e^{ci} D_c X_k^\mu v^k X_{[m}^\nu v_{n]} e_a^m e_b^n G_{\mu\nu} = 0}$$

3.1.4 With extra field, w.r.t v

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta v^l} &= \frac{D\mathcal{L}_{MP3}}{Dv^l} = \left(\frac{\partial}{\partial v^l} - \partial_c \frac{\partial}{\partial (\partial_c v^l)} \right) \left(-\frac{T}{2} e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_\nu^I \right) = \\
&= -\frac{T}{2} g^{ab} (D_a X_j^\mu \delta_l^j D_b X_k^\nu v^k + D_a X_j^\mu v^j D_b X_k^\nu \delta_l^k) G_{\mu\nu} = \\
&= -T g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} \stackrel{!}{=} 0
\end{aligned}$$

↓

$$\boxed{g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} = 0}$$

3.1.5 With extra field, w.r.t X

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta X_l^\lambda} &= \frac{D\mathcal{L}_{MP3}}{DX_l^\lambda} = \left(\frac{\partial}{\partial X_l^\lambda} - \partial_c \frac{\partial}{\partial (\partial_c X_l^\lambda)} \right) \left(-\frac{T}{2} e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I} \right) = \\
&= -\frac{T}{2} \left(e g^{ab} \left((-\omega_{aj}^{j'} \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k - D_a X_j^\mu v^j \omega_{bk}^{k'} \delta_\lambda^\nu \delta_k^l v^k) E_\mu^I E_{\nu I} + \right. \right. \\
&\quad \left. \left. + e g^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \left(\frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} + E_\mu^I \frac{\partial E_{\nu I}}{\partial X_l^\lambda} \right) \right) - \right. \\
&\quad \left. - \partial_c (e g^{ab} (\delta_a^c \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k + D_a X_j^\mu v^j \delta_b^c \delta_\lambda^\nu \delta_k^l v^k) G_{\mu\nu}) \right) = \\
&= -T \left(-e g^{ab} D_a X_j^\mu v^j \omega_{bk}^l v^k G_{\mu\lambda} + e g^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - \partial_c (e g^{ac} v^l D_a X_j^\mu v^j G_{\mu\lambda}) \right) \\
&= -T \left(e g^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - D_b (e g^{ab} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) \right) \stackrel{!}{=} 0 \\
&\quad \downarrow \\
&\quad \boxed{D_b (e e_i^a e^{bi} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) = e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I}}
\end{aligned}$$

3.2 Without extra index

3.2.1 w.r.t e

$$\begin{aligned}
\frac{\delta S_{DP}}{\delta e_l^e} &= \frac{D\mathcal{L}_{DP}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial (\partial_f e_l^e)} \right) \left(-\frac{T_0}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T_0}{4}
\end{aligned}$$

3.3 Linear action

3.3.1 w.r.t e

$$\begin{aligned}
\frac{\delta S_{LP}}{\delta e_j^b} &= \frac{D\mathcal{L}_{LP}}{De_j^b} = \left(\frac{\partial}{\partial e_j^b} - \partial_e \frac{\partial}{\partial (\partial_e e_j^b)} \right) \left(-\frac{T}{2} \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi \right) = \\
&= -\frac{T}{4} \bar{\psi} \varepsilon^{cd} \varepsilon_{mn} ((g_{cb} \eta^{mj} e_d^n + e_c^m g_{db} \eta^{nj}) e_i^a + e_c^m e_d^n \delta_b^a \delta_i^j) D_a X^{\mu i} E_\mu^I \gamma_I \psi = \\
&= -\frac{T}{2} \bar{\psi} (-e e_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I + e D_b X^{\mu j} E_\mu^I \gamma_I) \psi \stackrel{!}{=} 0 \\
&\quad \downarrow \\
T_b^j &:= \bar{\psi} (D_b X^{\mu j} E_\mu^I \gamma_I - e_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I) \psi = 0 \\
&\quad \downarrow \\
e_b^j &= F \bar{\psi} D_b X^{\mu j} E_\mu^I \gamma_I \psi, \\
\frac{1}{F} &= \bar{\psi} e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi \\
&\quad \Downarrow \\
\bar{\psi} \psi &= 1, \quad \psi \bar{\psi} = \mathbb{1} \\
e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_\nu^J \eta_{IJ} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}
\end{aligned}$$

4 Inverse Area Action

4.1 Nambu-Goto

Start from Nambu-Goto action

$$S_{NG} = -T \int d^2x \sqrt{-h},$$

and make quantum geometry correction

$$\sqrt{-h} \rightarrow \sqrt{-(h + g\Delta)} \approx \sqrt{-h} \left(1 - \frac{g\Delta}{2(-h)} + \mathcal{O}\left(\frac{g^2}{h^2}\right) \right),$$

leading to modified NG action

$$\begin{aligned} S_{MNG} &= -T \int d^2x \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}} \right) = S_{NG} + S_{IA} \\ &\downarrow \\ \mathcal{L}_{MNG} &= -T \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}} \right) \end{aligned}$$

4.1.1 EoMs

Let $\mathcal{P}_\mu^\tau \equiv \partial \mathcal{L}_{MNG} / \partial \dot{X}^\mu$ and $\mathcal{P}_\mu^\sigma \equiv \partial \mathcal{L}_{MNG} / \partial X'^\mu$

$$\begin{aligned} \delta S_{MNG} &= \int d^2x \left(\frac{\partial \mathcal{L}_{MNG}}{\partial \dot{X}^\mu} \delta \dot{X}^\mu + \frac{\partial \mathcal{L}_{MNG}}{\partial X'^\mu} \delta X'^\mu \right) = \\ &= - \int d^2x \left(\partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma \right) \delta X^\mu + \int d\tau \mathcal{P}_\mu^\sigma \delta X^\mu \Big|_{\sigma=0}^{\sigma=\sigma_1} \stackrel{!}{=} 0 \\ &\downarrow \\ \text{EoM} : \partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma &= 0 \\ \text{B.C.} : \mathcal{P}_\mu^\sigma \delta X^\mu \Big|_{\sigma=0}^{\sigma=\sigma_1} &= 0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}_\mu^\tau &= \frac{\partial \mathcal{L}_{MNG}}{\partial \dot{X}^\mu} = -T \left(\frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} + \frac{g\Delta}{2} \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{(-h)^{3/2}} \right) = \\ &= -T \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} \left(1 + \frac{g\Delta}{2(-h)} \right) = \mathcal{P}_{\mu(NG)}^\tau \left(1 + \frac{g\Delta}{2(-h)} \right) \\ \mathcal{P}_\mu^\sigma &= \frac{\partial \mathcal{L}_{MNG}}{\partial X'^\mu} = -T \left(\frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} + \frac{g\Delta}{2} \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{(-h)^{3/2}} \right) = \\ &= -T \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} \left(1 + \frac{g\Delta}{2(-h)} \right) = \mathcal{P}_{\mu(NG)}^\sigma \left(1 + \frac{g\Delta}{2(-h)} \right) \end{aligned}$$

gauge fixing static gauge $\tau = t$ and transverse gauge $\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial s} \frac{ds}{d\sigma} = 0$ (s = length along string)

$$\begin{aligned} &\downarrow \\ \mathcal{P}_{(NG)}^{\tau\mu} &= T \frac{ds}{d\sigma} \gamma_{v\perp} \frac{\partial X^\mu}{\partial t} \\ \mathcal{P}_{(NG)}^{\sigma\mu} &= \frac{T}{\gamma_{v\perp}} \frac{\partial X^\mu}{\partial s} \\ -h &= \frac{1}{\gamma_{v\perp}^2} \left(\frac{ds}{d\sigma} \right)^2 \end{aligned}$$

string energy get's redefined to

$$\frac{\partial}{\partial t} \left(T \frac{ds}{d\sigma} \gamma_{v\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \right) = 0,$$

and for the spatial part we have

$$\partial_\tau \vec{\mathcal{P}}^\tau + \partial_\sigma \vec{\mathcal{P}}^\sigma = 0$$

↓

$$\frac{\partial}{\partial t} \left[T \frac{ds}{d\sigma} \gamma_{v\perp} \frac{\partial \vec{X}}{\partial t} \left(1 + \frac{g\Delta}{2(-h)} \right) \right] + \frac{ds}{d\sigma} \frac{\partial}{\partial s} \left[-\frac{T}{\gamma_{v\perp}} \frac{\partial \vec{X}}{\partial s} \left(1 + \frac{g\Delta}{2(-h)} \right) \right] = 0$$

↓

$$\mu \gamma_{v\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial}{\partial s} \left[\frac{T}{\gamma_{v\perp}} \left(1 + \frac{g\Delta}{2(-h)} \right) \frac{\partial \vec{X}}{\partial s} \right] = 0$$

\Rightarrow effective mass density becomes $\mu_{eff} = \mu \gamma_{v\perp} \left(1 - \frac{g\Delta}{2(-h)} \right)$ and effective tension becomes $T_{eff} = \frac{T}{\gamma_{v\perp}} \left(1 - \frac{g\Delta}{2(-h)} \right)$

↓

$$\mu_{eff} \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial}{\partial s} \left[T_{eff} \frac{\partial \vec{X}}{\partial s} \right] = 0$$

$$\begin{aligned} \mathcal{H} &= \dot{\vec{X}} \cdot \vec{\pi} - \mathcal{L} = \\ &= \vec{v}_\perp \cdot \left(T \frac{ds}{d\sigma} \gamma_{v\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \vec{v}_\perp \right) - \left(-T \frac{ds}{d\sigma} \frac{1}{\gamma_{v\perp}} \left(1 + \frac{g\Delta}{2(-h)} \right) \right) = \\ &= T \frac{ds}{d\sigma} \left(1 + \frac{g\Delta}{2(-h)} \right) \left(\gamma_{v\perp} v_\perp^2 + \frac{1}{\gamma_{v\perp}} \right) = \\ &= T \frac{ds}{d\sigma} \gamma_{v\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \end{aligned}$$

let $\left(1 + \frac{g\Delta}{2(-h)} \right) = F$

$$\frac{\partial^2 \vec{X}}{\partial t^2} - \frac{1}{F \gamma_{v\perp}} \frac{ds}{d\sigma} \frac{\partial}{\partial \sigma} \left[\frac{1}{\gamma_{v\perp}} F \frac{ds}{d\sigma} \frac{\partial \vec{X}}{\partial \sigma} \right] = 0$$

↓

$$A(\sigma) = \frac{\gamma_{v\perp}}{F} \frac{ds}{d\sigma} \stackrel{!}{=} 1$$

↓

$$d\sigma = \frac{\gamma_{v\perp}}{F} ds = \frac{1}{TF^2} dE \Rightarrow \sigma(q) = \frac{1}{T} \int_0^q \frac{1}{F^2} dE$$

↓

$$F^2 \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial^2 \vec{X}}{\partial \sigma^2} = 0$$

\Rightarrow speed of wave on the string gets modified by correction factor $v = c/F$.

4.2 Translation and Lorentz symmetry

Starting with global translations, define the transformation by

$$X^\mu \mapsto X^\mu + \epsilon^\mu.$$

This does not change the action since it only depends on derivatives of X^μ and so $\delta(\partial_a X^\mu) = \partial_a(\delta X^\mu) = \partial_a \epsilon^\mu = 0$.

As for Lorentz transformations, the infinitesimal for is

$$X^\mu \mapsto X^\mu + \epsilon^{\mu\nu} X_\nu,$$

with $\epsilon^{\mu\nu}$ anti-symmetric.

4.3 Verifying if there is longitudinal velocity

$$\begin{aligned}
-h &= \frac{1}{\gamma_{v_\perp}^2} \left(\frac{ds}{d\sigma} \right)^2 = \frac{F^2}{\gamma_{v_\perp}^4} = F^2 (1 - v_\perp^2)^2 \\
&\downarrow \\
-h &= \left(1 + \frac{g\Delta}{2(-h)} \right)^2 (1 - v_\perp^2)^2 \\
&\downarrow \\
\frac{-h}{\left(1 + \frac{g\Delta}{2(-h)} \right)^2} &= (1 - v_\perp^2)^2 \\
&\downarrow \text{(WolframAlpha)} \\
-h &= \frac{(1 - v_\perp^2)^2}{3} - f_1 - f_2,
\end{aligned}$$

where (let $a = \frac{g\Delta}{2}$ and $b = (1 - v_\perp^2)^2$)

$$\begin{aligned}
f_1 &= \frac{\sqrt[3]{-27a^2b + 3\sqrt{3}\sqrt{27a^4b^2 + 4a^3b^3} - 18ab^2 - 2b^3}}{3\sqrt[3]{2}} \\
f_2 &= \frac{(6ab + b^2)}{6f_1} \\
&\downarrow \\
F &= 1 + \frac{g\Delta}{2(-h)} = 1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)} \\
&\downarrow \\
v &= \frac{1}{F} = \frac{1}{1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}}
\end{aligned}$$

5 Bimetric Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

and promote to bimetric action

$$\downarrow$$

$$S_{BP} = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{T'}{2} \int d^2x \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

5.1 EoMs

$$\begin{aligned} \frac{\delta S_{BP}}{\delta g^{cd}} &= -\frac{T}{2} \left(\frac{\partial \sqrt{-g}}{\partial g^{cd}} g^{ab} + \sqrt{-g} \frac{\partial g^{ab}}{\partial g^{cd}} \right) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + 0 = \\ &= -\frac{T}{2} \left(-\frac{1}{2} \sqrt{-g} g_{cd} g^{ab} + \sqrt{-g} \delta_{(c}^a \delta_{d)}^b \right) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \stackrel{!}{=} 0 \end{aligned}$$

$$T_{cd}^{(G)} := \left(\partial_c X^\mu \partial_d X^\nu - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \right) G_{\mu\nu} = 0$$

\Downarrow

$$g_{cd} = 2f^{(G)} \partial_c X^\mu \partial_d X^\nu G_{\mu\nu},$$

$$\frac{1}{f^{(G)}} = g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\begin{aligned} \frac{\delta S_{BP}}{\delta h^{cd}} &= \dots = \\ &= -\frac{T'}{2} \left(-\frac{1}{2} \sqrt{-h} h_{cd} h^{ab} + \sqrt{-h} \delta_{(c}^a \delta_{d)}^b \right) \partial_a X^\mu \partial_b X^\nu H_{\mu\nu} \stackrel{!}{=} 0 \end{aligned}$$

\downarrow

$$T_{cd}^{(H)} := \left(\partial_c X^\mu \partial_d X^\nu - \frac{1}{2} h_{cd} h^{ab} \partial_a X^\mu \partial_b X^\nu \right) H_{\mu\nu} = 0$$

\Downarrow

$$h_{cd} = 2f^{(H)} \partial_c X^\mu \partial_d X^\nu H_{\mu\nu},$$

$$\frac{1}{f^{(H)}} = h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

If the auxiliary metrics g and h coincide with the induced metrics from G and H , then $f^{(G)} = f^{(H)} = 1/2$ and the action reduces to

$$S = -T \int d^2x \sqrt{-g} - T' \int d^2x \sqrt{-h},$$

where if $T' = -Tk\Delta/2$ and $h = g^{-1}$ we recover the NG area corrected action

$$S = -T \int d^2x \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right).$$

The condition $h = g^{-1}$ can be achieved by

$$h_{ab} = S_{ac} g^{cd} T_{db},$$

where $S, T \in \text{SL}(2, \mathbb{C})$, so we have a gauge freedom in choosing h_{ab} . Since we require $[h_{ab}] = [h_{ab}]^\dagger$ (h_{ab} must be symmetric and real), this means $[T_{db}] = [S_{db}]^\dagger$ and also that $[S_{ab}]^\dagger = [S_{ab}]$, thus

$$h_{ab} = S_{ac} g^{cd} S_{db}.$$

This restriction constrains our gauge group to just the boosts part of $SL(2, \mathbb{C})$,

$$S = e^{-n^i \sigma_i \beta}.$$

This also implies

$$\begin{aligned} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu} &= S_{ac} g^{cd} S_{db} = \\ &= S_{ac} \left(\frac{1}{2g} \varepsilon^{cc'} \varepsilon^{dd'} g_{c'd'} \right) S_{db} = \\ \delta_a^{c'} \delta_b^{d'} \partial_{c'} X^\mu \partial_{d'} X^\nu H_{\mu\nu} &= S_{ac} \left(\frac{1}{2g} \varepsilon^{cc'} \varepsilon^{dd'} \partial_{c'} X^\mu \partial_{d'} X^\nu G_{\mu\nu} \right) S_{db} \\ \delta_a^{c'} \delta_b^{d'} H_{\mu\nu} &= \frac{1}{2g} S_{ac} \varepsilon^{cc'} S_{db} \varepsilon^{dd'} G_{\mu\nu} \\ H_{\mu\nu} &= -\frac{1}{8g} S_{ac} \varepsilon^{ca} S_{db} \varepsilon^{bd} G_{\mu\nu} \\ H_{\mu\nu} &= \frac{1}{8(-g)} \text{Tr}(S\varepsilon) \text{Tr}(S\varepsilon) G_{\mu\nu} \\ H_{\mu\nu} &= \frac{1}{8(-g)} \text{Tr}(S\varepsilon)^2 G_{\mu\nu}. \end{aligned}$$

This restricts S to necessarily being not symmetric, otherwise the trace $\text{Tr}(S\varepsilon)$ vanishes and we would get $H_{\mu\nu} = 0$.

$$\begin{aligned} \frac{\delta S_{BP}}{\delta X^\lambda} &= \left(-\frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu - \\ &- \partial_c \left(-\frac{T}{2} \sqrt{-g} g^{ab} G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} H_{\mu\nu} \right) (\delta_a^c \delta_\lambda^\mu \partial_b X^\nu + \partial_a X^\mu \delta_b^c \delta_\lambda^\nu) = \\ &= \partial_a \left(T \sqrt{-g} g^{ab} G_{\mu\lambda} + T' \sqrt{-h} h^{ab} H_{\mu\lambda} \right) \partial_b X^\mu - \\ &- \left(\frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} + \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu \stackrel{!}{=} 0 \\ &\downarrow \\ \partial_a (F_{\mu\lambda}^{ab} \partial_b X^\mu) &= \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu, \\ F_{\mu\nu}^{ab} &= T \sqrt{-g} g^{ab} G_{\mu\nu} + T' \sqrt{-h} h^{ab} H_{\mu\nu}. \end{aligned}$$

Assuming the condition for $h = g^{-1}$ and flat background $G_{\mu\nu} = \eta_{\mu\nu}$, the EoM reduces to

$$\partial_a \left(\left(\sqrt{-g} g^{ab} - \frac{k\Delta}{2(-g)} \sqrt{-g}^{-1} (S^{-1})^{ac} g_{cd} (S^{-1})^{db} \right) \partial_b X^\mu \right) = 0,$$

which in conformal gauge $g_{ab} = \phi \eta_{ab}$ with $(S^{-1})^{ac} \eta_{cd} (S^{-1})^{db} = \eta^{ab}$ reduces further

$$\begin{aligned} \partial_a \left(\left(1 - \frac{k\Delta}{2\phi^2} \right) \eta^{ab} \partial_b X^\mu \right) &= 0 \\ \eta^{ab} \partial_a (F^- \partial_b X^\mu) &= 0 \\ F^- \eta^{ab} \partial_a \partial_b X^\mu + \eta^{ab} \partial_a F^- \partial_b X^\mu &= 0 \\ \eta^{ab} \partial_a \partial_b X^\mu + \eta^{ab} \partial_a \ln(F^-) \partial_b X^\mu &= 0. \end{aligned}$$

This is the exact same equation derived in the next section which leads to the Klein-Gordon eqn. The requirement that $(S^{-1})^{ac} \eta_{cd} (S^{-1})^{db} = \eta^{ab}$ implies

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix},$$

$$S = \begin{bmatrix} \cosh \beta - n^3 \sinh \beta & -(n^1 - in^2) \sinh \beta \\ -(n^1 + in^2) \sinh \beta & \cosh \beta + n^3 \sinh \beta \end{bmatrix},$$

$$S^{-1} = \begin{bmatrix} \cosh \beta + n^3 \sinh \beta & +(n^1 - in^2) \sinh \beta \\ +(n^1 + in^2) \sinh \beta & \cosh \beta - n^3 \sinh \beta \end{bmatrix}$$

so $n^3 = 0$. Since $(n^1)^2 + (n^2)^2 = 1$, we can parametrize as $n^1 = \cos \theta$ and $n^2 = \sin \theta$, thus S has the form

$$S = \begin{bmatrix} \cosh \beta & -e^{-i\theta} \sinh \beta \\ -e^{i\theta} \sinh \beta & \cosh \beta \end{bmatrix},$$

and $\text{Tr}(S\varepsilon) = -2i \sin \theta \sinh \beta$. Let's try to find more restrictions on S :

$$h^{ab} = (S^{-1})^{ac} g_{cd} (S^{-1})^{db}$$

$$h^{ab} = \frac{1}{2g^{-1}} \varepsilon^{aa'} \varepsilon^{bb'} \partial_{a'} X^\mu \partial_{b'} X^\nu \left(\frac{1}{8(-g)} \text{Tr}(S\varepsilon)^2 G_{\mu\nu} \right)$$

$$\Downarrow$$

$$(S^{-1})^{ac} g_{cd} (S^{-1})^{db} = \frac{1}{2g^{-1}} \varepsilon^{aa'} \varepsilon^{bb'} \frac{1}{8(-g)} \text{Tr}(S\varepsilon)^2 \partial_{a'} X^\mu \partial_{b'} X^\nu G_{\mu\nu}$$

$$(S^{-1})^{ac} g_{cd} (S^{-1})^{db} = -\frac{1}{16} \text{Tr}(S\varepsilon)^2 \varepsilon^{ac} \varepsilon^{bd} g_{cd}$$

$$(S^{-1})^{ac} g_{cd} (S^{-1})^{db} = \left(\frac{i}{4} \text{Tr}(S\varepsilon) \varepsilon^{ac} \right) g_{cd} \left(\frac{i}{4} \text{Tr}(S\varepsilon) \varepsilon^{db} \right)$$

$$g_{cd} = S_{ca} \left(\frac{i}{4} \text{Tr}(S\varepsilon)^* \varepsilon^{ac'} \right) g_{c'd'} \left(\frac{i}{4} \text{Tr}(S\varepsilon) \varepsilon^{d'b} \right) S_{bd}$$

$$\Downarrow$$

$$S_{ca} \left(\frac{i}{4} \text{Tr}(S\varepsilon)^* \varepsilon^{ac'} \right) = \delta_c^{c'}$$

$$\left(\frac{i}{4} \text{Tr}(S\varepsilon) \varepsilon^{d'b} \right) S_{bd} = \delta_d^{d'}$$

$$\Downarrow$$

$$(S^{-1})^{ab} = \left(\frac{i}{4} \text{Tr}(S\varepsilon) \varepsilon^{ab} \right),$$

thus we have that

$$\begin{bmatrix} \cosh \beta & e^{-i\theta} \sinh \beta \\ e^{i\theta} \sinh \beta & \cosh \beta \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \sin \theta \sinh \beta \\ -\frac{1}{2} \sin \theta \sinh \beta & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\theta = \frac{n\pi}{2}$$

$$\beta = i \frac{m\pi}{2}$$

6 Inverse Area Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

and make quantum geometry correction

$$\sqrt{-g} \rightarrow \sqrt{-(g+k\Delta)} \approx \sqrt{-g} \left(1 + \frac{k\Delta}{2g} + \mathcal{O}\left(\frac{k^2}{g^2}\right) \right)$$

leading to

$$S_{IAP} = -\frac{T}{2} \int d^2x \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

6.1 EoMs

$$\begin{aligned} \frac{\delta S_{IAP}}{\delta g^{cd}} &= \frac{\partial}{\partial g^{cd}} \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = \\ &= \left(-\frac{1}{2} \sqrt{-g} g_{cd} + \frac{k\Delta}{4\sqrt{-g}} g_{cd} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} = \\ &= -\frac{1}{2} \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} \stackrel{!}{=} 0 \end{aligned}$$

↓

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = 0$$

↓

$$\begin{aligned} g_{cd} &= 2f \partial_c X^\mu \partial_d X^\nu G_{\mu\nu}, \\ \frac{1}{f} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \end{aligned}$$

$$\begin{aligned} \frac{\delta S_{IAP}}{\delta X^\lambda} &= \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - \partial_c \left(\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) 2g^{ab} \delta_a^c \delta_b^\lambda \partial_b X^\nu G_{\mu\nu} \right) = \\ &= \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2 \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} - \\ &\quad - 2\partial_a \left(\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} = \\ &= \left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2 \left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} - \\ &\quad - 2\partial_a \left(\left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} \stackrel{!}{=} 0 \end{aligned}$$

↓

$$\begin{aligned} \left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} + \partial_a \left(\left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} = \\ \frac{1}{2} \left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu}, \end{aligned}$$

imposing conformal symmetry $g^{ab} = (\phi(x))^{-1} \eta^{ab}$ and let $F' = \left(1 - \frac{k\Delta}{2\phi^2} \right)$ leads to

$$\eta^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} + \frac{1}{F'} \eta^{ab} \partial_a F' \partial_b X^\nu G_{\lambda\nu} = \eta^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

assuming flat background $G_{\mu\nu} = \eta_{\mu\nu}$

$$\eta^{ab}\partial_a\partial_b X^\nu\eta_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F'\partial_b X^\nu\eta_{\lambda\nu} = 0$$

\downarrow

$$\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\partial_a \ln(F')\partial_b X^\mu = 0,$$

use plane-wave ansatz $X^\mu = X_0^\mu e^{-i(E\tau - p\sigma)}$

$$(E^2 - p^2)X^\mu + \partial_\tau \ln(F')iEX^\mu - \partial_\sigma \ln(F')ipX^\mu = 0$$

\downarrow

$$(E^2 - p^2)X^\mu + i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F'))X^\mu = 0$$

\downarrow

$$i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F')) = -m^2,$$

$$-m^2 = -E^2 + p^2$$

\Downarrow

$$\partial_\tau \ln(F') = iE, \quad \partial_\sigma \ln(F') = ip$$

\Downarrow

$$\ln(F') = i(E\tau + p\sigma) + a, \quad a \in \mathbb{C}$$

\Downarrow

$$F' = Ae^{i(E\tau + p\sigma + \phi)}, \quad A > 0$$

\Downarrow

$$\phi(\tau, \sigma) = \sqrt{\frac{k\Delta}{2(1 - Ae^{i(E\tau + p\sigma + \phi)})}}$$

\downarrow

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2X^\mu = 0$$

7 Quantum geometry Polyakov

Since quantum geometry is expected to violate Lorentz invariance at the Planck scale by modifying the dispersion relation as in

$$E^2 = m^2 + p^2 + \sum_{n \geq 3} c_n \frac{p^n}{E_{pl}^{n-2}},$$

where E_{pl} is the Planck energy, we make a correction to Polyakov action in the form

$$S = -\frac{1}{4\pi\alpha'} \int d^2x \sqrt{-g} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 e^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right)$$

where T is the temperature of target-space, (in SI units) μ_0 has units of inverse area (perhaps $\mu_0^2 \sim (\text{quantum of area})^{-1}$? Maybe $\mu_0^2 = \mu^2/(M_{pl}^2 A_j)$, where A_j is the quantum of area with spin j) and $m_0 \gg m_1 \gg \dots \gg m_d$. This action has manifest WS reparameterisation invariance. As for background Poincaré symmetry and WS Weyl invariance, those hold approximately for the fields with $m_\mu \gg T$. The masses m_μ are constrained by the dispersion relations

$$\tilde{E}_\mu^2 = \tilde{p}_\mu^2 + \sum_{n=0}^d m_n^2 \frac{\tilde{p}_\mu^n}{E_{pl}^n}$$

where \tilde{E}_μ^2 does not mean $\tilde{E}^\mu \tilde{E}_\mu$, rather it is $(\tilde{E}_0)^2, (\tilde{E}_1)^2, \dots, (\tilde{E}_d)^2$, that is, the squared WS energy (and momentum) of each field X^μ .

7.1 EoMs

$$\begin{aligned} \frac{\delta S}{\delta g^{cd}} &\propto \frac{\partial \sqrt{-g}}{\partial g^{cd}} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 e^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right) + \sqrt{-g} \delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = \\ &= -\frac{1}{2} \sqrt{-g} g_{cd} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 e^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right) + \sqrt{-g} \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = 0 \\ T_{cd} &:= \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 e^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right) = 0 \\ g_{cd} &= 2f \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu}, \\ \frac{1}{f} &= \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 e^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right). \end{aligned}$$

From this, it is clear that $e^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} = 0$, thus the base mass μ_0^2 is a Lagrange multiplier, the auxiliary metric reduces to the induced metric and the action simplify back to regular NG action.

$$\begin{aligned} \frac{\delta S}{\delta X^\lambda} &\propto -2\sqrt{-g} \mu_0^2 e^{-m_\mu/T} \delta_\lambda^\mu X^\nu \eta_{\mu\nu} - 2\partial_c (\sqrt{-g} g^{ab} \delta_a^c \delta_\lambda^b \partial_b X^\nu \eta_{\mu\nu}) = \\ &= -2 \left(\partial_a (\sqrt{-g} g^{ab} \partial_b X^\nu \eta_{\lambda\nu}) + \sqrt{-g} \mu_0^2 e^{-m_\lambda/T} X^\nu \eta_{\lambda\nu} \right) = 0 \\ \partial_a (\sqrt{-g} g^{ab} \partial_b X^\mu) &+ \sqrt{-g} \mu_0^2 e^{-m_\mu/T} X^\mu = 0 \\ \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ab} \partial_b X^\mu) &+ \mu_0^2 e^{-m_\mu/T} X^\mu = 0 \\ g^{ab} \nabla_a \nabla_b X^\mu &+ \mu_0^2 e^{-m_\mu/T} X^\mu = 0. \end{aligned}$$

We see that 2d gravitational field of the WS appears naturally in metric-Christoffel symbols formulation. Perhaps THIS is the TRUE quantum gravitational field since in high enough energies only $m_0, m_1 \gg T$ and thus only X^0 and X^1 remain massless, while the others are massive and thus violate Poincaré invariance. To this end, we may postulate that $m_2 \sim E_{pl}$, such that at $T = T_{pl}$ we have massive X^2, X^3, \dots, X^d with (approximately) massless X^0 and X^1 . From that and the modified WS dispersion relations for each X^μ we can get the other masses.

From the X EoM before simplifying, we can use conformal gauge $g_{ab} = e^{\phi(x)} \eta_{ab}$ to get a variable-coefficient KG eqn

$$\begin{aligned} \eta^{ab} \partial_a \partial_b X^\mu &+ e^{\phi(x)} \mu_0^2 e^{-m_\mu/T} X^\mu = 0 \\ ((\partial_\sigma)^2 - (\partial_\tau)^2) X^\mu &+ \mu_0^2 e^{\phi_\mu(x)} X^\mu = 0, \end{aligned}$$

where $\phi_\mu(x) \equiv \phi(x) - m_\mu/T$.

8 Relating NG and Polyakov analysis

From NG analysis

$$(F^+)^2 \frac{\partial^2 X^\mu}{\partial t^2} - \frac{\partial^2 X^\mu}{\partial x^2} = 0,$$

make change of variables $\tau = \tau(t, x)$ and $\sigma = \sigma(t, x)$

$$\begin{aligned} (F^+)^2 \left(\frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial t} \frac{\partial}{\partial \sigma} \right) \left(\frac{\partial \tau}{\partial t} \frac{\partial X^\mu}{\partial \tau} + \frac{\partial \sigma}{\partial t} \frac{\partial X^\mu}{\partial \sigma} \right) - \left(\frac{\partial \tau}{\partial x} \frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial x} \frac{\partial}{\partial \sigma} \right) \left(\frac{\partial \tau}{\partial x} \frac{\partial X^\mu}{\partial \tau} + \frac{\partial \sigma}{\partial x} \frac{\partial X^\mu}{\partial \sigma} \right) &= 0 \\ (F^+)^2 \left(\left(\frac{\partial \tau}{\partial t} \right)^2 \frac{\partial^2 X^\mu}{\partial \tau^2} + 2 \frac{\partial \tau}{\partial t} \frac{\partial \sigma}{\partial t} \frac{\partial^2 X^\mu}{\partial \tau \partial \sigma} + \left(\frac{\partial \sigma}{\partial t} \right)^2 \frac{\partial^2 X^\mu}{\partial \sigma^2} \right) - \\ &- \left(\left(\frac{\partial \tau}{\partial x} \right)^2 \frac{\partial^2 X^\mu}{\partial \tau^2} + 2 \frac{\partial \tau}{\partial x} \frac{\partial \sigma}{\partial x} \frac{\partial^2 X^\mu}{\partial \tau \partial \sigma} + \left(\frac{\partial \sigma}{\partial x} \right)^2 \frac{\partial^2 X^\mu}{\partial \sigma^2} \right) + \\ &+ \left((F^+)^2 \frac{\partial^2 \tau}{\partial t^2} - \frac{\partial^2 \tau}{\partial x^2} \right) \frac{\partial X^\mu}{\partial \tau} + \left((F^+)^2 \frac{\partial^2 \sigma}{\partial t^2} - \frac{\partial^2 \sigma}{\partial x^2} \right) \frac{\partial X^\mu}{\partial \sigma} = 0 \\ ((F^+)^2 (\partial_t \tau)^2 - (\partial_x \tau)^2) \partial_\tau^2 X^\mu + ((F^+)^2 (\partial_t \sigma)^2 - (\partial_x \sigma)^2) \partial_\sigma^2 X^\mu + \\ &+ 2 ((F^+)^2 \partial_t \tau \partial_t \sigma - \partial_x \tau \partial_x \sigma) \partial_\tau \partial_\sigma X^\mu + ((F^+)^2 \partial_t^2 \tau - \partial_x^2 \tau) \partial_\tau X^\mu + ((F^+)^2 \partial_t^2 \sigma - \partial_x^2 \sigma) \partial_\sigma X^\mu = 0 \end{aligned}$$

Without loss of generality, we can set (just rescale/rotate the coordinates)

$$\begin{aligned} (F^+)^2 (\partial_t \sigma)^2 - (\partial_x \sigma)^2 &= (\partial_x \tau)^2 - (F^+)^2 (\partial_t \tau)^2 \\ (F^+)^2 \partial_t \tau \partial_t \sigma &= \partial_x \tau \partial_x \sigma, \end{aligned}$$

or more compactly

$$\begin{aligned} \partial_t \tau &= v \partial_x \sigma \\ \partial_t \sigma &= v \partial_x \tau, \end{aligned}$$

where $v = 1/F^+$, which is equivalent to

$$\begin{aligned} \partial_t^2 \tau &= \partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau + v^2 \partial_x^2 \tau \\ \partial_t^2 \sigma &= \partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma + v^2 \partial_x^2 \sigma, \end{aligned}$$

reducing the big equation to

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu + \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\tau X^\mu + \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\sigma X^\mu = 0.$$

Next introduce $X^\mu = \kappa(\tau, \sigma) Y^\mu(\tau, \sigma)$ s.t terms proportional to $\partial_\tau Y^\mu$ and $\partial_\sigma Y^\mu$ vanish:

$$\begin{aligned} \partial_\tau X^\mu &= \partial_\tau \kappa Y^\mu + \kappa \partial_\tau Y^\mu \\ \partial_\sigma X^\mu &= \partial_\sigma \kappa Y^\mu + \kappa \partial_\sigma Y^\mu \\ \partial_\tau^2 X^\mu &= \partial_\tau^2 \kappa Y^\mu + 2 \partial_\tau \kappa \partial_\tau Y^\mu + \kappa \partial_\tau^2 Y^\mu \\ \partial_\sigma^2 X^\mu &= \partial_\sigma^2 \kappa Y^\mu + 2 \partial_\sigma \kappa \partial_\sigma Y^\mu + \kappa \partial_\sigma^2 Y^\mu \end{aligned}$$

$$\begin{aligned} \kappa \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} + 2 \partial_\tau \kappa &\stackrel{!}{=} 0 \\ \kappa \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} - 2 \partial_\sigma \kappa &\stackrel{!}{=} 0 \end{aligned}$$

$$\begin{aligned}\partial_\tau \kappa &= -\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \\ \partial_\sigma \kappa &= \frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)},\end{aligned}$$

undoing chain rule for t by multiplying first eq by $\partial_t \tau$ and second by $\partial_t \sigma$ and summing gives

$$\begin{aligned}\partial_t \kappa &= \partial_t \tau \partial_\tau \kappa + \partial_t \sigma \partial_\sigma \kappa = \\ &= \partial_t \tau \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) + \partial_t \sigma \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) = \\ &= v \partial_x \sigma \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) + v \partial_x \tau \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) = \\ &= \frac{v \kappa \partial_t v ((\partial_x \tau)^2 - (\partial_x \sigma)^2)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} = \\ &= \frac{v \kappa \partial_t v ((\partial_x \tau)^2 - (\partial_t \tau / v)^2)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} = \\ &= -\frac{\kappa \partial_t v}{2v},\end{aligned}$$

or simply

$$\begin{aligned}\frac{\partial_t \kappa}{\kappa} &= -\frac{1}{2} \frac{\partial_t v}{v} \\ \partial_t \ln(\kappa) &= -\frac{1}{2} \partial_t \ln(v) \\ \ln(\kappa) &= -\frac{1}{2} \ln(v) + f(x).\end{aligned}$$

To determine $f(x)$, we undo the chain rule for x now:

$$\begin{aligned}\partial_x \kappa &= \partial_x \tau \partial_\tau \kappa + \partial_x \sigma \partial_\sigma \kappa = \\ &= \partial_x \tau \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) + \partial_x \sigma \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) = \\ &= \frac{\kappa v \partial_x v ((\partial_x \sigma)^2 - (\partial_x \tau)^2)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} = \\ &= \frac{\kappa v \partial_x v ((\partial_t \tau / v)^2 - (\partial_x \tau)^2)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} = \\ &= \frac{\kappa \partial_x v}{2v}\end{aligned}$$

$$\begin{aligned}\frac{\partial_x \kappa}{\kappa} &= \frac{1}{2} \frac{\partial_x v}{v} \\ \partial_x \ln(\kappa) &= \frac{1}{2} \partial_x \ln(v) \\ \ln(\kappa) &= \frac{1}{2} \ln(v), \\ \partial_x \left(-\frac{1}{2} \ln(v) + f(x) \right) &= \frac{1}{2} \partial_x \ln(v)\end{aligned}$$

$f'(x) = \partial_x \ln(v)$ (perhaps this means somewhere in the composition of functions time dependence is lost?)

$$f(x) = \ln(v)$$

giving us

$$\kappa = v^{\frac{1}{2}}, \quad v = v(x).$$

Substituting this into the earlier equation yields

$$\partial_\tau^2 Y^\mu - \partial_\sigma^2 Y^\mu + m^2 Y^\mu = 0,$$

with

$$\begin{aligned}
m^2 &= \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau) \partial_\tau \kappa + (\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma) \partial_\sigma \kappa}{\kappa ((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} = \\
&= \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{\partial_t v (\partial_x \sigma \partial_\tau \kappa + \partial_x \tau \partial_\sigma \kappa) + v \partial_x v \partial_x \kappa}{\kappa ((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} = \\
&= \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{(\partial_x v)^2}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)},
\end{aligned}$$

where by use of chain rule we have that

$$\partial_t^2 \kappa - v^2 \partial_x^2 \kappa - v \partial_x v \partial_x \kappa = ((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2) (\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa),$$

where since $\kappa = v^{\frac{1}{2}}$, m^2 reduces to

$$m^2 = \frac{(\partial_x v)^2 - 2v \partial_x^2 v}{4((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)},$$

with τ satisfying

$$\begin{aligned}
\partial_t \tau &= v \partial_x \sigma \\
v \partial_x \tau &= \partial_t \sigma
\end{aligned}$$

such that $-m^2$ is indeed constant.

9 Solutions for the Polyakov KG eqn (WRONG ONE)

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi \eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta_{cd} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\begin{aligned} \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ \partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0. \end{aligned}$$

These simplify further by expanding the summation on a, b :

$$\begin{aligned} \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} (-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}) \\ \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}. \end{aligned}$$

The EoM is just a one-dimensional Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left(a^\mu(p) e^{-i(-E\tau + p\sigma)} + b^\mu(p) e^{i(-E\tau + p\sigma)} \right),$$

where $E(p) = \sqrt{p^2 + m^2}$. The derivatives are

$$\begin{aligned} \partial_\tau X^\mu &= \frac{i}{4\pi} \int dp \left(a^\mu(p) e^{-i(-E\tau + p\sigma)} - b^\mu(p) e^{i(-E\tau + p\sigma)} \right) \\ \partial_\sigma X^\mu &= \frac{i}{4\pi} \int dp \frac{p}{\sqrt{p^2 + m^2}} \left(-a^\mu(p) e^{-i(-E\tau + p\sigma)} + b^\mu(p) e^{i(-E\tau + p\sigma)} \right). \end{aligned}$$

Starting with the mixed constraint,

$$\begin{aligned} \partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp \left(a^\mu(p) e^{-i(-E(p)\tau + p\sigma)} - b^\mu(p) e^{i(-E(p)\tau + p\sigma)} \right) \times \\ &\quad \times \int dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\nu(p') e^{-i(-E(p')\tau + p'\sigma)} + b^\nu(p') e^{i(-E(p')\tau + p'\sigma)} \right) = \\ &= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left(a^\mu(p) e^{-i(-E(p)\tau + p\sigma)} - b^\mu(p) e^{i(-E(p)\tau + p\sigma)} \right) \times \\ &\quad \times \left(-a^\nu(p') e^{-i(-E(p')\tau + p'\sigma)} + b^\nu(p') e^{i(-E(p')\tau + p'\sigma)} \right) = \\ &= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\mu(p) a^\nu(p') e^{i(E(p) + E(p'))\tau} e^{-i(p + p')\sigma} + a^\mu(p) b^\nu(p') e^{i(E(p) - E(p'))\tau} e^{-i(p - p')\sigma} + \right. \\ &\quad \left. + b^\mu(p) a^\nu(p') e^{-i(E(p) - E(p'))\tau} e^{i(p - p')\sigma} - b^\mu(p) b^\nu(p') e^{-i(E(p) + E(p'))\tau} e^{i(p + p')\sigma} \right) = 0. \end{aligned}$$

If we integrate this expression over σ we obtain

$$\begin{aligned}
& -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} \int d\sigma e^{-i(p+p')\sigma} + \right. \\
& \quad + a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} \int d\sigma e^{-i(p-p')\sigma} + b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} \int d\sigma e^{i(p-p')\sigma} - \\
& \quad \left. - b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} \int d\sigma e^{i(p+p')\sigma} \right) = \\
& = -\frac{\eta_{\mu\nu}}{8\pi} \int dp dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} \delta(p+p') + a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} \delta(p-p') + \right. \\
& \quad \left. + b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} \delta(p-p') - b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} \delta(p+p') \right) = \\
& = -\frac{\eta_{\mu\nu}}{8\pi} \int dp \frac{p}{\sqrt{p^2 + m^2}} \left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} + a^\mu(p) b^\nu(p) + b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) = 0
\end{aligned}$$

which implies the whole expression inside the integral must vanish, thus we have (leaving $(a^\mu b^\nu + b^\mu a^\nu) \eta_{\mu\nu}$ as is since eventually these will not be numbers and may not commute, thus will not be equal to $2a^\mu b^\nu \eta_{\mu\nu}$)

$$\left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} + a^\mu(p) b^\nu(p) + b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu} = 0.$$

As for the other part of the constraint, let's start with the L.H.S

$$\begin{aligned}
\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{(4\pi)^2} \int dp dp' \left(a^\mu(p) e^{-i(-E(p)\tau+p\sigma)} - b^\mu(p) e^{i(-E(p)\tau+p\sigma)} \right) \times \\
& \quad \times \left(a^\nu(p') e^{-i(-E(p')\tau+p'\sigma)} - b^\nu(p') e^{i(-E(p')\tau+p'\sigma)} \right) = \\
& = -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \left(a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} e^{-i(p+p')\sigma} - a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} e^{-i(p-p')\sigma} - \right. \\
& \quad \left. - b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} e^{i(p-p')\sigma} + b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} e^{i(p+p')\sigma} \right),
\end{aligned}$$

which we integrate over σ to get

$$\begin{aligned}
& -\frac{\eta_{\mu\nu}}{8\pi} \int dp dp' \left(a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} \delta(p+p') - a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} \delta(p-p') - \right. \\
& \quad \left. - b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} \delta(p-p') + b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} \delta(p+p') \right) = \\
& = -\frac{\eta_{\mu\nu}}{8\pi} \int dp \left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} - a^\mu(p) b^\nu(p) - b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right),
\end{aligned}$$

and now for the R.H.S

$$\begin{aligned}
-\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{p}{\sqrt{p^2 + m^2}} \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\mu(p) e^{-i(-E\tau+p\sigma)} + b^\mu(p) e^{i(-E\tau+p\sigma)} \right) \times \\
& \quad \times \left(-a^\nu(p') e^{-i(-E(p')\tau+p'\sigma)} + b^\nu(p') e^{i(-E(p')\tau+p'\sigma)} \right) = \\
& = \frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{pp'}{E(p)E(p')} \left(a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} e^{-i(p+p')\sigma} - a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} e^{-i(p-p')\sigma} - \right. \\
& \quad \left. - b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} e^{i(p-p')\sigma} + b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} e^{i(p+p')\sigma} \right),
\end{aligned}$$

which once again we integrate over σ ,

$$\begin{aligned} \frac{\eta_{\mu\nu}}{8\pi} \int dp dp' \frac{pp'}{E(p)E(p')} & \left(a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} \delta(p+p') - a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} \delta(p-p') - \right. \\ & \left. - b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} \delta(p-p') + b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} \delta(p+p') \right) = \\ & = \frac{1}{8\pi} \int dp \frac{p^2}{p^2+m^2} \left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} - a^\mu(p) b^\nu(p) - b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu}, \end{aligned}$$

which when equated to the L.H.S yields

$$\int dp \left(\frac{p^2}{p^2+m^2} + 1 \right) \left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} - a^\mu(p) b^\nu(p) - b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu} = 0,$$

which implies

$$\left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} - a^\mu(p) b^\nu(p) - b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu} = 0.$$

We can sum and subtract this with the previous condition to simplify and obtain

$$\begin{aligned} \left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu} &= 0 \\ \left(a^\mu(p) b^\nu(p) + b^\mu(p) a^\nu(p) \right) \eta_{\mu\nu} &= 0, \end{aligned}$$

and for while a^μ and b^ν are number-valued, the second condition can be further simplified to

$$a^\mu(p) b^\nu(p) \eta_{\mu\nu} = 0,$$

meaning a^μ and b^ν are orthogonal. In general, this condition states that a^μ and b^ν anti-commute w.r.t Lorentz inner product. The first condition must hold for all values of τ , and since the exponentials are linearly independent, we thus have

$$\begin{aligned} a^\mu(p) a^\nu(-p) \eta_{\mu\nu} &= 0 \\ b^\mu(p) b^\nu(-p) \eta_{\mu\nu} &= 0, \end{aligned}$$

meaning that reflecting the argument of a^μ and b^ν creates orthogonal vectors. We can thus write

$$\begin{aligned} a^\mu(p) &= \Lambda^\mu{}_\nu \left(\frac{p}{4} \right) a^\nu(0) \\ b^\mu(p) &= \Upsilon^\mu{}_\nu \left(\frac{p}{4} \right) b^\nu(0), \end{aligned}$$

where $\Lambda, \Upsilon \in \text{SO}^+(1, d)$. Also. since the orthogonality under reflection must hold for all values of p , in particular we have

$$\begin{aligned} a^\mu(0) a^\nu(0) \eta_{\mu\nu} &= 0 \\ b^\mu(0) b^\nu(0) \eta_{\mu\nu} &= 0, \end{aligned}$$

meaning that the initial $a^\mu(0)$ and $b^\nu(0)$ are null vectors. This means that $a^\mu(p)$ and $b^\nu(p)$ are also null vectors since they are related to the initial values by Lorentz transformation. With this, the solution becomes

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left(\Lambda^\mu{}_\nu \left(\frac{p}{4} \right) a^\nu(0) e^{-i(-E\tau+p\sigma)} + \Upsilon^\mu{}_\nu \left(\frac{p}{4} \right) b^\nu(0) e^{i(-E\tau+p\sigma)} \right)$$

with $(a^\mu b^\nu + b^\nu a^\mu) \eta_{\mu\nu} = 0$. This condition implies

$$\begin{aligned} \left(\Lambda^\mu{}_\gamma \left(\frac{p}{4} \right) a^\gamma(0) \Upsilon^\nu{}_\rho \left(\frac{p}{4} \right) b^\rho(0) \right) \eta_{\mu\nu} &= 0 \\ \Upsilon^\mu{}_\nu \left(\frac{p}{4} \right) &= \Lambda^\mu{}_\nu \left(\frac{p}{4} \right) = \Lambda^\mu{}_\nu \left(-\frac{p}{4} \right), \end{aligned}$$

thus turning the solution into

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left(\Lambda^\mu{}_\nu \left(\frac{p}{4} \right) a^\nu(0) e^{-i(-E\tau+p\sigma)} + \Lambda^\mu{}_\nu \left(-\frac{p}{4} \right) b^\nu(0) e^{i(-E\tau+p\sigma)} \right).$$

Next, we choose background coordinates s.t $a^\mu(0) = (a_0, a_0, 0, \dots, 0) = a_0 x^+$ and $b^\nu(0) = (b_0, b_0, 0, \dots, 0) = b_0 x^+$, simplifying the solution to

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left(a_0 \Lambda^\mu{}_+ \left(\frac{p}{4} \right) e^{-i(-E\tau+p\sigma)} + b_0 \Lambda^\mu{}_+ \left(-\frac{p}{4} \right) e^{i(-E\tau+p\sigma)} \right) x^+$$

10 Solutions for the Polyakov KG eqn

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi \eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta_{cd} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\begin{aligned} \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ \partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0. \end{aligned}$$

These simplify further by expanding the summation on a, b :

$$\begin{aligned} \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} (-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}) \\ \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= 0. \end{aligned}$$

These can be combined by multiplying the previous one by 2 and summing, leaving us with

$$(\partial_\tau X^\mu + \partial_\sigma X^\mu)(\partial_\tau X^\nu + \partial_\sigma X^\nu) \eta_{\mu\nu} = 0.$$

The EoM is just a one-dimensional finite-space Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu e^{i(-E_n \tau + n\sigma)} + b_n^\mu e^{i(E_n \tau + n\sigma)} \right),$$

with $E_n = \sqrt{n^2 + m^2}$. The derivatives are

$$\begin{aligned} \partial_\tau X^\mu &= \frac{i}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \left(-a_n^\mu e^{i(-E_n \tau + n\sigma)} + b_n^\mu e^{i(E_n \tau + n\sigma)} \right) \\ \partial_\sigma X^\mu &= \frac{i}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{n}{E_n} \left(a_n^\mu e^{i(-E_n \tau + n\sigma)} + b_n^\mu e^{i(E_n \tau + n\sigma)} \right). \end{aligned}$$

Reality of X^μ implies

$$\begin{aligned} (X^\mu)^* &= \left(\frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu e^{i(-E_n \tau + n\sigma)} + b_n^\mu e^{i(E_n \tau + n\sigma)} \right) \right)^* = \\ &= \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left((a_n^\mu)^* e^{-i(-E_n \tau + n\sigma)} + (b_n^\mu)^* e^{-i(E_n \tau + n\sigma)} \right) \equiv X^\mu, \end{aligned}$$

giving us

$$\begin{aligned} (a_n^\mu)^* &= b_{-n}^\mu \\ (b_n^\mu)^* &= a_{-n}^\mu, \end{aligned}$$

so the solution becomes for closed strings

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu e^{i(-E_n \tau + n\sigma)} + (a_{-n}^\mu)^* e^{i(E_n \tau + n\sigma)} \right).$$

For open strings with free endpoints B.C we have general solution

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} (a_n^\mu e^{-iE_n\tau} + b_n^\mu e^{iE_n\tau}) \cos(n\sigma),$$

which reality condition implies $b_n^\mu = (a_n^\mu)^*$, thus open string with free endpoint solution becomes

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} (a_n^\mu e^{-iE_n\tau} + (a_n^\mu)^* e^{iE_n\tau}) \cos(n\sigma).$$

Analysing the constraints, for the closed string we have that

$$\partial_\tau X^\mu + \partial_\sigma X^\mu = \frac{i}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \left(\left(\frac{n}{E_n} - 1 \right) a_n^\mu e^{i(-E_n\tau+n\sigma)} + \left(\frac{n}{E_n} + 1 \right) b_n^\mu e^{i(E_n\tau+n\sigma)} \right),$$

so the constraint reads

$$\begin{aligned} (\partial_\tau X + \partial_\sigma X)^2 = & -\frac{\eta_{\mu\nu}}{(4\pi\alpha')^2} \sum_n \sum_p \left(\left(\frac{n}{E_n} - 1 \right) \left(\frac{p}{E_p} - 1 \right) a_n^\mu a_p^\nu e^{i(-(E_n+E_p)\tau+(n+p)\sigma)} + \right. \\ & + \left(\frac{n}{E_n} - 1 \right) \left(\frac{p}{E_p} + 1 \right) a_n^\mu (a_{-p}^\nu)^* e^{i(-(E_n-E_p)\tau+(n+p)\sigma)} + \left(\frac{n}{E_n} + 1 \right) \left(\frac{p}{E_p} - 1 \right) (a_{-n}^\mu)^* a_p^\nu e^{i((E_n-E_p)\tau+(n+p)\sigma)} + \\ & \left. + \left(\frac{n}{E_n} + 1 \right) \left(\frac{p}{E_p} + 1 \right) (a_{-n}^\mu)^* (a_{-p}^\nu)^* e^{i((E_n+E_p)\tau+(n+p)\sigma)} \right) \equiv 0. \end{aligned}$$

Since the exponentials are all linearly independent and we can't decouple the τ exponential from any of the sums, we have 3 cases to investigate: $n = p$, $n = -p$ and $n \neq \pm p$. For $n \neq \pm p$, we get

$$\begin{aligned} L_{n,p}^1 &:= \left(\frac{n}{E_n} - 1 \right) \left(\frac{p}{E_p} - 1 \right) a_n^\mu a_p^\nu \eta_{\mu\nu} = 0 \\ L_{n,p}^2 &:= \left(\frac{n}{E_n} - 1 \right) \left(\frac{p}{E_p} + 1 \right) a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0 \\ L_{n,p}^3 &:= \left(\frac{n}{E_n} + 1 \right) \left(\frac{p}{E_p} - 1 \right) (a_{-n}^\mu)^* a_p^\nu \eta_{\mu\nu} = 0 \\ L_{n,p}^4 &:= \left(\frac{n}{E_n} + 1 \right) \left(\frac{p}{E_p} + 1 \right) (a_{-n}^\mu)^* (a_{-p}^\nu)^* \eta_{\mu\nu} = 0, \end{aligned}$$

where $L_{n,p}^2$ and $L_{n,p}^3$ are just complex conjugates of each other with $n \mapsto -n$ and $p \mapsto -p$, so they are redundant. The same is true for $L_{n,p}^1$ and $L_{n,p}^4$, so in the end we have

$$\begin{aligned} L_{n,p} &:= \left(\frac{n}{E_n} - 1 \right) \left(\frac{p}{E_p} - 1 \right) a_n^\mu a_p^\nu \eta_{\mu\nu} = 0 \\ \tilde{L}_{n,p} &:= \left(\frac{n}{E_n} - 1 \right) \left(\frac{p}{E_p} + 1 \right) a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0. \end{aligned}$$

The coefficients of both expressions are actually not necessary, since for them to be 0 we would need $n \rightarrow \pm\infty$ or $p \rightarrow \pm\infty$, so we can redefine

$$\begin{aligned} L_{n,p} &:= a_n^\mu a_p^\nu \eta_{\mu\nu} = 0 \\ \tilde{L}_{n,p} &:= a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0, n \neq \pm p. \end{aligned}$$

For $n = -p$, the first and last terms are complex conjugates of each other with the relabeling $n \mapsto -n$ and $p \mapsto -p$, while the exponentials in the 2 middle terms are reduced to 1, so we get

$$\begin{aligned} & \left(\frac{n}{E_n} - 1 \right)^2 a_n^\mu a_{-n}^\nu \eta_{\mu\nu} = 0 \\ & \sum_n \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) a_n^\mu (a_n^\nu)^* \eta_{\mu\nu} \right) = 0, \end{aligned}$$

where in the first equation we can ignore the first factor once more. Finally for $n = p$, we get

$$\begin{aligned} \left(\frac{n}{E_n} - 1\right)^2 a_n^\mu a_n^\nu \eta_{\mu\nu} &= 0 \\ \left(\left(\frac{n}{E_n}\right)^2 - 1\right) a_n^\mu (a_{-n}^\nu)^* \eta_{\mu\nu} &= 0, \end{aligned}$$

where we again ignore the first factor because it's only 0 at infinity, thus collecting everything we have

$$\begin{aligned} L_{n,p} &:= a_n^\mu a_p^\nu \eta_{\mu\nu} = 0, \quad \forall n, p \in \mathbb{Z}, \\ \tilde{L}_{n,p} &:= a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0, \quad n \neq -p, \\ \sum_n \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) &= 0. \end{aligned}$$

One last thing before quantizing: the $n = 0$ term in the solution reads

$$\frac{1}{2\pi\alpha'} \frac{1}{2m} (a_0^\mu e^{-im\tau} + (a_0^\mu)^* e^{im\tau}),$$

which is divergent in the $m \rightarrow 0$ limit unless $\Re(a_0^\mu) \sim m$. Keeping the analogy to the $m \rightarrow 0$ regime, we conclude $a_0^\mu = 2\pi\alpha' m x^\mu + 4\pi i (\alpha')^2 p^\mu$. Since a_0^μ includes p^μ , we have that

$$M^2 = -p^\mu p_\mu = \frac{1}{16\pi^2 (\alpha')^4} ((2\pi\alpha')^2 x^\mu x^\nu - a_0^\mu (a_0^\nu)^*) \eta_{\mu\nu},$$

which after analysing closer the $n = -p$ condition

$$\begin{aligned} \sum_n \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) &= 0 \\ \sum_{n \neq 0} \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) - \tilde{L}_{0,0} &= 0 \\ \sum_{n \neq 0} \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) &= a_0^\mu (a_0^\nu)^* \eta_{\mu\nu} \equiv (4\pi(\alpha')^2)^2 p^\mu p_\mu + (2\pi\alpha' m)^2 x^\mu x_\mu, \end{aligned}$$

we have the string mass in terms of vibrational modes... (and center of mass position!?)

$$\begin{aligned} M^2 &= \frac{1}{(4\pi(\alpha')^2)^2} \left((2\pi\alpha' m)^2 x^\mu x_\mu - \sum_{n \neq 0} \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) \right) = \\ &= \frac{1}{(4\pi(\alpha')^2)^2} \left((2\pi\alpha' m)^2 x^\mu x_\mu - \sum_{n \neq 0} \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) a_n^\mu (a_n^\nu)^* \eta_{\mu\nu} \right) \right). \end{aligned}$$

As for the open string, its derivatives are

$$\begin{aligned} \partial_\tau X^\mu &= \frac{i}{4\pi\alpha'} \sum_n (-a_n^\mu e^{-iE_n\tau} + (a_n^\mu)^* e^{iE_n\tau}) \cos(n\sigma) \\ \partial_\sigma X^\mu &= -\frac{1}{4\pi\alpha'} \sum_n \frac{n}{E_n} (a_n^\mu e^{-iE_n\tau} + (a_n^\mu)^* e^{iE_n\tau}) \sin(n\sigma) \end{aligned}$$

and the constraint reads

$$\partial_\tau X^\mu + \partial_\sigma X^\mu = \frac{1}{4\pi\alpha'} \sum_n \left(-a_n^\mu e^{-iE_n\tau} \left(i \cos(n\sigma) + \frac{n}{E_n} \sin(n\sigma) \right) + (a_n^\mu)^* e^{iE_n\tau} \left(i \cos(n\sigma) - \frac{n}{E_n} \sin(n\sigma) \right) \right)$$

$$\begin{aligned}
(\partial_\tau X + \partial_\sigma X)^2 &= \frac{\eta_{\mu\nu}}{(4\pi\alpha')^2} \sum_n \sum_p \left(a_n^\mu a_p^\nu e^{-i(E_n+E_p)\tau} \left(i \cos(n\sigma) + \frac{n}{E_n} \sin(n\sigma) \right) \left(i \cos(p\sigma) + \frac{p}{E_p} \sin(p\sigma) \right) - \right. \\
&\quad - a_n^\mu (a_p^\nu)^* e^{-i(E_n-E_p)\tau} \left(i \cos(n\sigma) + \frac{n}{E_n} \sin(n\sigma) \right) \left(i \cos(p\sigma) - \frac{p}{E_p} \sin(p\sigma) \right) - \\
&\quad - (a_n^\mu)^* a_p^\nu e^{i(E_n-E_p)\tau} \left(i \cos(n\sigma) - \frac{n}{E_n} \sin(n\sigma) \right) \left(i \cos(p\sigma) + \frac{p}{E_p} \sin(p\sigma) \right) + \\
&\quad \left. + (a_n^\mu)^* (a_p^\nu)^* e^{i(E_n+E_p)\tau} \left(i \cos(n\sigma) - \frac{n}{E_n} \sin(n\sigma) \right) \left(i \cos(p\sigma) - \frac{p}{E_p} \sin(p\sigma) \right) \right) = 0.
\end{aligned}$$

Once more, we see that the middle 2 terms and outer 2 terms are complex conjugate pairs. Unlike the closed string, here all cases lead to

$$\begin{aligned}
L_{n,p} &:= a_n^\mu a_p^\nu \eta_{\mu\nu} = 0 \\
\tilde{L}_{n,p} &:= a_n^\mu (a_p^\nu)^* \eta_{\mu\nu} = 0,
\end{aligned}$$

where again to be more in line with ST literature we remap $p \mapsto p - n$

$$\begin{aligned}
L_{n,p} &= a_n^\mu a_{p-n}^\nu \eta_{\mu\nu} = 0 \\
\tilde{L}_{n,p} &= a_n^\mu (a_{p-n}^\nu)^* \eta_{\mu\nu} = 0.
\end{aligned}$$

Since we don't have a summed constraint, the mass of the open string can't be written in terms of vibrational modes, only in terms of the 0-th mode

$$\begin{aligned}
M^2 &= -p^\mu p_\mu = \frac{1}{(4\pi(\alpha')^2)} a_0^\mu a_0^\nu \eta_{\mu\nu} = \\
&= \frac{1}{(4\pi(\alpha')^2)} L_{0,0} \equiv 0
\end{aligned}$$

10.1 The Expansion Modes Algebra

10.1.1 Closed String

Before starting this analysis, let's rename the vibrational modes to α_n^μ to be more in line with the literature. Start by finding the canonical momentum

$$\begin{aligned}
\Pi_\lambda &= \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\lambda)} = -\frac{T}{2} \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \delta_a^\tau \delta_\lambda^\mu \partial_b X^\nu \eta_{\mu\nu} = \\
&= -\frac{1}{4\pi\alpha'} \left(1 - \frac{k\Delta}{2\phi^2} \right) \eta^{\tau b} \partial_b X^\nu \eta_{\lambda\nu} = \\
&= \frac{1}{4\pi\alpha'} F^- \partial_\tau X^\nu \eta_{\lambda\nu} = F^- \frac{i}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \left(-\alpha_n^\nu e^{i(-E_n\tau+n\sigma)} + (\alpha_{-n}^\nu)^* e^{i(E_n\tau+n\sigma)} \right) \eta_{\nu\lambda}.
\end{aligned}$$

We then proceed to the canonical Poisson bracket relations

$$\begin{aligned}
\{X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} &= \delta(\sigma - \sigma') \eta^{\mu\nu} \\
\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} &= \{\Pi^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} = 0,
\end{aligned}$$

where by expanding the first relation we get

$$\begin{aligned}
\{X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} &= \left\{ \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left(\alpha_n^\mu e^{i(-E_n\tau+n\sigma)} + (\alpha_{-n}^\mu)^* e^{i(E_n\tau+n\sigma)} \right), \right. \\
&\quad \left. F^- \frac{i}{4\pi\alpha'} \sum_{p \in \mathbb{Z}} \left(-\alpha_p^\nu e^{i(-E_p\tau+p\sigma')} + (\alpha_{-p}^\nu)^* e^{i(E_p\tau+p\sigma')} \right) \right\} =
\end{aligned}$$

$$= \frac{i}{8\pi^2(\alpha')^2} \sum_n \sum_p \frac{1}{2E_n} \left(-F^- \{ \alpha_n^\mu, \alpha_p^\nu \} e^{-i(E_n+E_p)\tau} e^{i(n\sigma+p\sigma')} + F^- \{ \alpha_n^\mu, (\alpha_{-p}^\nu)^* \} e^{-i(E_n-E_p)\tau} e^{i(n\sigma+p\sigma')} - \right. \\ \left. - F^- \{ (\alpha_{-n}^\mu)^*, \alpha_p^\nu \} e^{i(E_n-E_p)\tau} e^{i(n\sigma+p\sigma')} + F^- \{ (\alpha_{-n}^\mu)^*, (\alpha_{-p}^\nu)^* \} e^{i(E_n+E_p)\tau} e^{i(n\sigma+p\sigma')} \right) = \delta(\sigma - \sigma') \eta^{\mu\nu}$$

$$\left| \frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{-ik\sigma} \right.$$

$$\frac{i}{8\pi^2(\alpha')^2} \sum_n \sum_p \frac{1}{2E_n} \left(-F^- \{ \alpha_n^\mu, \alpha_p^\nu \} e^{-i(E_n+E_p)\tau} \delta_{n,k} e^{ip\sigma'} + F^- \{ \alpha_n^\mu, (\alpha_{-p}^\nu)^* \} e^{-i(E_n-E_p)\tau} \delta_{n,k} e^{ip\sigma'} - \right. \\ \left. - F^- \{ (\alpha_{-n}^\mu)^*, \alpha_p^\nu \} e^{i(E_n-E_p)\tau} \delta_{n,k} e^{ip\sigma'} + F^- \{ (\alpha_{-n}^\mu)^*, (\alpha_{-p}^\nu)^* \} e^{i(E_n+E_p)\tau} \delta_{n,k} e^{ip\sigma'} \right) = \frac{\eta^{\mu\nu}}{2\pi} e^{-ik\sigma'}$$

$$\frac{i}{8\pi^2(\alpha')^2} \sum_p \frac{1}{2E_k} \left(-F^- \{ \alpha_k^\mu, \alpha_p^\nu \} e^{-i(E_k+E_p)\tau} e^{ip\sigma'} + F^- \{ \alpha_k^\mu, (\alpha_{-p}^\nu)^* \} e^{-i(E_k-E_p)\tau} e^{ip\sigma'} - \right. \\ \left. - F^- \{ (\alpha_{-k}^\mu)^*, \alpha_p^\nu \} e^{i(E_k-E_p)\tau} e^{ip\sigma'} + F^- \{ (\alpha_{-k}^\mu)^*, (\alpha_{-p}^\nu)^* \} e^{i(E_k+E_p)\tau} e^{ip\sigma'} \right) = \frac{\eta^{\mu\nu}}{2\pi} e^{-ik\sigma'}$$

$$\left| \frac{1}{2\pi} \int_0^{2\pi} d\sigma' e^{im\sigma'} \right.$$

$$\frac{1}{2\pi} \int_0^{2\pi} d\sigma' F^- e^{i(p+m)\sigma'}, \quad F^- = 1 - \frac{k\Delta}{2\phi^2}, \quad \text{assume } \phi = \text{constant}$$

$$\frac{i}{8\pi^2(\alpha')^2} \frac{F^-}{2E_k} \left(- \{ \alpha_k^\mu, \alpha_{-m}^\nu \} e^{-i(E_k+E_m)\tau} + \{ \alpha_k^\mu, (\alpha_m^\nu)^* \} e^{-i(E_k-E_m)\tau} - \right. \\ \left. - \{ (\alpha_{-k}^\mu)^*, \alpha_{-m}^\nu \} e^{i(E_k-E_m)\tau} + \{ (\alpha_{-k}^\mu)^*, (\alpha_m^\nu)^* \} e^{i(E_k+E_m)\tau} \right) = \frac{\eta^{\mu\nu}}{2\pi} \delta_{m,k}.$$

The only way for this to hold is

$$\{ \alpha_k^\mu, \alpha_{-m}^\nu \} = \{ (\alpha_{-k}^\mu)^*, (\alpha_m^\nu)^* \} = 0 \\ \{ \alpha_k^\mu, (\alpha_m^\nu)^* \} = - \{ (\alpha_{-k}^\mu)^*, \alpha_{-m}^\nu \} = -i \frac{4\pi(\alpha')^2}{F^-} E_k \eta^{\mu\nu} \delta_{k,m},$$

thus rescaling we get harmonic oscillator bracket relations

$$a_k^\mu := \frac{1}{2\alpha'} \sqrt{\frac{F^-}{\pi}} \frac{1}{\sqrt{E_k}} \alpha_k^\mu \\ (a_k^\mu)^* := \frac{1}{2\alpha'} \sqrt{\frac{F^-}{\pi}} \frac{1}{\sqrt{E_k}} (\alpha_k^\mu)^* \\ \Downarrow$$

$$\{ a_k^\mu, (a_p^\nu)^* \} = -i \eta^{\mu\nu} \delta_{k,p} \\ \{ a_k^\mu, a_p^\nu \} = \{ (a_k^\mu)^*, (a_p^\nu)^* \} = 0.$$

For the $k = p = 0$ case, we also get the expected relations for x^μ and p^ν :

$$\{ \alpha_0^\mu, (\alpha_0^\nu)^* \} = -i \frac{4\pi(\alpha')^2}{F^-} E_0 \eta^{\mu\nu}$$

$$\{ 2\pi\alpha' \mu x^\mu + 4\pi i(\alpha')^2 p^\mu, 2\pi\alpha' \mu x^\nu - 4\pi i(\alpha')^2 p^\nu \} = -i \frac{4\pi(\alpha')^2}{F^-} \mu \eta^{\mu\nu}$$

$$(2\pi\alpha')^2 \left(\mu^2 \{x^\mu, x^\nu\} - 2i\alpha' \mu \{x^\mu, p^\nu\} + 2i\alpha' \mu \{p^\mu, x^\nu\} + (2\alpha') \{p^\mu, p^\nu\} \right) = -i \frac{4\pi(\alpha')^2}{F^-} \mu \eta^{\mu\nu}$$

\Downarrow

$$\begin{aligned} \{x^\mu, x^\nu\} &= \{p^\mu, p^\nu\} = 0 \\ \{x^\mu, p^\nu\} &= \frac{1}{4\pi\alpha' F^-} \eta^{\mu\nu}. \end{aligned}$$

Since $F^- = 1 - k\Delta/(2\phi^2)$, if we choose

$$\phi = \sqrt{\frac{k\Delta}{2\left(1 - \frac{1}{4\pi\alpha'}\right)}}$$

then the bracket relation simplifies to the expected

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu}.$$

10.2 The Constraint Algebra

With the bracket relations between the vibrational modes in hand, we can now calculate the algebra generated by the constraints. For simplicity, since $E_n = \sqrt{n^2 + \mu^2} \neq 0$, we can switch the constraints to be between the normalized harmonic oscillator modes a_n^μ instead of the α_n^μ .

$$\{L_{n,m}, L_{k,p}\} = \left\{ a_n^\mu a_m^\nu \eta_{\mu\nu}, a_k^{\mu'} a_p^{\nu'} \eta_{\mu'\nu'} \right\} = 0$$

$$\begin{aligned} \{L_{n,m}, \tilde{L}_{k,p}\} &= \left\{ a_n^\mu a_m^\nu \eta_{\mu\nu}, a_k^{\mu'} (a_{-p}^{\nu'})^* \eta_{\mu'\nu'} \right\} = \\ &= \eta_{\mu\nu} \left(a_m^\nu \left\{ a_n^\mu, (a_{-p}^{\nu'})^* \right\} a_k^{\mu'} + a_n^\mu \left\{ a_m^\nu, (a_{-p}^{\nu'})^* \right\} a_k^{\mu'} \right) \eta_{\mu'\nu'} = \\ &= \eta_{\mu\nu} \left(a_m^\nu (-i\eta^{\mu\nu'} \delta_{n,-p}) a_k^{\mu'} + a_n^\mu (-i\eta^{\nu\nu'} \delta_{m,-p}) a_k^{\mu'} \right) \eta_{\mu'\nu'} = \\ &= -i (a_m^\mu a_k^\nu \eta_{\mu\nu} \delta_{n,-p} + a_n^\mu a_k^\nu \eta_{\mu\nu} \delta_{m,-p}) = \\ &= -i (L_{m,k} \delta_{n,-p} + L_{n,k} \delta_{m,-p}) \end{aligned}$$

$$\begin{aligned} \{\tilde{L}_{n,m}, \tilde{L}_{k,p}\} &= \left\{ a_n^\mu (a_{-m}^\nu)^* \eta_{\mu\nu}, a_k^{\mu'} (a_{-p}^{\nu'})^* \eta_{\mu'\nu'} \right\} = \\ &= \eta_{\mu\nu} \left((a_{-m}^\nu)^* \left\{ a_n^\mu, (a_{-p}^{\nu'})^* \right\} a_k^{\mu'} + a_n^\mu \left\{ (a_{-m}^\nu)^*, a_k^{\mu'} \right\} (a_{-p}^{\nu'})^* \right) \eta_{\mu'\nu'} = \\ &= \eta_{\mu\nu} \left((a_{-m}^\nu)^* (-i\eta^{\mu\nu'} \delta_{n,-p}) a_k^{\mu'} - a_n^\mu (-i\eta^{\nu\mu'} \delta_{-m,k}) (a_{-p}^{\nu'})^* \right) \eta_{\mu'\nu'} = \\ &= -i (a_k^\mu (a_{-m}^\nu)^* \eta_{\mu\nu} \delta_{n,-p} - a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} \delta_{-m,k}) = \\ &= -i (\tilde{L}_{k,m} \delta_{n,-p} - \tilde{L}_{n,p} \delta_{-m,k}). \end{aligned}$$

11 Into the Quantum Realm

We now proceed to quantize the KG string through canonical quantization. The string X^μ and it's momentum Π^ν are promoted to operators \hat{X}^μ and $\hat{\Pi}^\nu$, and we turn the Poisson bracket $\{.,.\}$ into commutator between operators $i\hbar[.,.]$. The equal time bracket relations turn into equal time commutation relations

$$\begin{aligned} [\hat{X}^\mu(\tau, \sigma), \hat{\Pi}^\nu(\tau, \sigma')] &= i\delta(\sigma - \sigma')\eta^{\mu\nu} \\ [\hat{X}^\mu, \hat{X}^\nu] &= [\hat{\Pi}^\mu, \hat{\Pi}^\nu] = 0 \\ [\hat{a}_n^\mu, \hat{a}_m^\nu] &= [(\hat{a}_n^\mu)^\dagger, (\hat{a}_m^\nu)^\dagger] = 0 \\ [\hat{a}_n^\mu, (\hat{a}_m^\nu)^\dagger] &= \eta^{\mu\nu}\delta_{n,m} \\ [\hat{x}^\mu, \hat{x}^\nu] &= [\hat{p}^\mu, \hat{p}^\nu] = 0 \\ [\hat{x}^\mu, \hat{p}^\nu] &= i\eta^{\mu\nu}. \end{aligned}$$

Thus, the rescaled vibrational mode operators \hat{a}_n^μ and $(\hat{a}_m^\nu)^\dagger$ are annihilation and creations operators, respectively.

We define a vacuum state of the string to obey

$$\hat{a}_n^\mu|0\rangle = 0, \text{ for } n \neq 0.$$

For $n = 0$, we have the center of mass position and momentum operators, so the vacuum also obeys

$$\begin{aligned} \hat{x}^\mu|0; x\rangle &= x^\mu|0; x\rangle \\ \hat{p}_\mu|0; x\rangle &= -i\frac{\partial}{\partial x^\mu}|0; x\rangle \end{aligned}$$

in position representation or alternatively

$$\begin{aligned} \hat{x}^\mu|0; p\rangle &= -i\frac{\partial}{\partial p_\mu}|0; p\rangle \\ \hat{p}_\mu|0; p\rangle &= p_\mu|0; p\rangle \end{aligned}$$

in momentum representation.

A generic state arises from a sequence of creation operators on the vacuum

$$((a_1^{\mu_1})^\dagger)^{n_{\mu_1}} ((a_2^{\mu_2})^\dagger)^{n_{\mu_2}} \dots ((a_{-1}^{\nu_1})^\dagger)^{n_{\nu_1}} ((a_{-2}^{\mu_2})^\dagger)^{n_{\nu_2}} \dots |0\rangle.$$

This (should) give rise to particles.

As in regular ST, we have ghosts arising from the Minkowski metric

$$[\hat{a}_n^\mu, (\hat{a}_m^\nu)^\dagger] = \eta^{\mu\nu}\delta_{nm}.$$

For constraints, classically we have

$$\begin{aligned} L_{n,p} &= a_n^\mu a_p^\nu \eta_{\mu\nu} = 0 \\ \tilde{L}_{n,p} &= a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0 \end{aligned}$$

which because of the existence of ghosts we require to have vanishing matrix elements when sandwiched between physical states

$$\langle \text{phys}' | \hat{L}_{n,p} | \text{phys} \rangle = 0 = \langle \text{phys}' | \hat{\tilde{L}}_{n,p} | \text{phys} \rangle.$$

When translating the constraints into quantum operators, in $L_{n,p}$ we have no ambiguity of ordering since \hat{a} commutes with itself. As for $\tilde{L}_{n,p}$, there is an ambiguity for $p = -n \neq 0$, so we pick normal ordering with the annihilation operators moved to the right

$$\hat{\tilde{L}}_{n,-n} = (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu}.$$

The ambiguity manifests in the imposition of this constraint as

$$\langle \text{phys}' | \left(\sum_n \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) - c \right) | \text{phys} \rangle = 0,$$

for some constant c . Since classically

$$\begin{aligned} M^2 &= \frac{1}{(4\pi(\alpha')^2)^2} \left((2\pi\alpha'\mu)^2 x^\mu x_\mu - \sum_{n \neq 0} \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) \right) = \\ &= \frac{1}{(4\pi(\alpha')^2)^2} \left((2\pi\alpha'\mu)^2 x^\mu x_\mu - \sum_{n \neq 0} \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) a_n^\mu (a_n^\nu)^* \eta_{\mu\nu} \right) \right). \end{aligned}$$

we see that the string mass spectrum will be affected by this constant

$$\hat{M}^2 = \frac{1}{(4\pi(\alpha')^2)^2} \left((2\pi\alpha'\mu)^2 \hat{x}^\mu \hat{x}_\mu - \left(\sum_{n \neq 0} \left(\left(\frac{n}{E_n} \right)^2 - 1 \right) (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu} - c \right) \right)$$

The commutation relations between the L 's are inherited from the bracket relations

$$\begin{aligned} [\hat{L}_{n,m}, \hat{L}_{k,p}] &= 0 \\ [\hat{L}_{n,m}, \hat{\tilde{L}}_{k,p}] &= \hat{L}_{m,k} \delta_{n,-p} + \hat{L}_{n,k} \delta_{m,-p} \\ [\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}] &= \hat{\tilde{L}}_{k,m} \delta_{n,-p} - \hat{\tilde{L}}_{n,p} \delta_{-m,k}. \end{aligned}$$

Since $[\hat{L}_{n,m}, \hat{\tilde{L}}_{p,k}]$ only has annihilation operators, it has no ordering ambiguities. $[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{p,k}]$ have ordering ambiguities for $m = -k$ and/or $p = -n$, so we add the anomalous terms

$$[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}] = \hat{\tilde{L}}_{k,m} \delta_{n,-p} - \hat{\tilde{L}}_{n,p} \delta_{-m,k} + C_n \delta_{n,-p} + D_k \delta_{-m,k}$$

Clearly, $C_0 = 0 = D_0$, since in these cases the equivalent terms are composed by $\hat{\tilde{L}}_{0,0} \sim \hat{x}^\mu \hat{x}_\mu + \hat{p}^\mu \hat{p}_\mu$. Also, because of the δ , we also have that $C_{-n} = -C_n$ and $D_{-k} = -D_k$. We get from the Jacobi identity

$$[\hat{\tilde{L}}_{n,m}, [\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{r,s}]] + [\hat{\tilde{L}}_{r,s}, [\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}]] + [\hat{\tilde{L}}_{k,p}, [\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{n,m}]] = 0$$

$$\begin{aligned} &[\hat{\tilde{L}}_{n,m}, (\hat{\tilde{L}}_{r,p} \delta_{k,-s} - \hat{\tilde{L}}_{k,s} \delta_{-p,r} + C_k \delta_{k,-s} + D_r \delta_{-p,r})] + \\ &+ [\hat{\tilde{L}}_{r,s}, (\hat{\tilde{L}}_{k,m} \delta_{n,-p} - \hat{\tilde{L}}_{n,p} \delta_{-m,k} + C_n \delta_{n,-p} + D_k \delta_{-m,k})] + \\ &+ [\hat{\tilde{L}}_{k,p}, (\hat{\tilde{L}}_{n,s} \delta_{r,-m} - \hat{\tilde{L}}_{r,m} \delta_{-s,n} + C_r \delta_{r,-m} + D_n \delta_{-s,n})] = 0 \end{aligned}$$

$$\begin{aligned} &[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{r,p}] \delta_{k,-s} - [\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,s}] \delta_{-p,r} + [\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{k,m}] \delta_{n,-p} - \\ &- [\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{n,p}] \delta_{-m,k} + [\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{n,s}] \delta_{r,-m} - [\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{r,m}] \delta_{-s,n} = 0 \end{aligned}$$

$$\begin{aligned} &(\hat{\tilde{L}}_{r,m} \delta_{n,-p} - \hat{\tilde{L}}_{n,p} \delta_{-m,r} + C_n \delta_{n,-p} + D_r \delta_{-m,r}) \delta_{k,-s} - (\hat{\tilde{L}}_{k,m} \delta_{n,-s} - \hat{\tilde{L}}_{n,s} \delta_{-m,k} + C_n \delta_{n,-s} + D_k \delta_{-m,k}) \delta_{-p,r} + \\ &+ (\hat{\tilde{L}}_{k,s} \delta_{r,-m} - \hat{\tilde{L}}_{r,m} \delta_{-s,k} + C_r \delta_{r,-m} + D_k \delta_{-s,k}) \delta_{n,-p} - (\hat{\tilde{L}}_{n,s} \delta_{r,-p} - \hat{\tilde{L}}_{r,p} \delta_{-s,n} + C_r \delta_{r,-p} + D_n \delta_{-s,n}) \delta_{-m,k} + \\ &+ (\hat{\tilde{L}}_{n,p} \delta_{k,-s} - \hat{\tilde{L}}_{k,s} \delta_{-p,n} + C_k \delta_{k,-s} + D_n \delta_{-p,n}) \delta_{r,-m} - (\hat{\tilde{L}}_{r,p} \delta_{k,-m} - \hat{\tilde{L}}_{k,m} \delta_{-p,r} + C_k \delta_{k,-m} + D_r \delta_{-p,r}) \delta_{-s,n} = 0. \end{aligned}$$

For $k = -s$, $n = -p$ and $r = -m$

$$\begin{aligned} &(\hat{\tilde{L}}_{r,-r} - \hat{\tilde{L}}_{n,-n} + C_n + D_r) - (\hat{\tilde{L}}_{k,-r} \delta_{p,s} - \hat{\tilde{L}}_{n,-k} \delta_{r,k} + C_n \delta_{n,k} + D_k \delta_{r,k}) \delta_{n,m} + \\ &+ (\hat{\tilde{L}}_{k,-k} - \hat{\tilde{L}}_{r,-r} + C_r + D_k) - (\hat{\tilde{L}}_{n,-k} \delta_{r,n} - \hat{\tilde{L}}_{r,-n} \delta_{k,n} + C_r \delta_{r,n} + D_n \delta_{k,n}) \delta_{r,k} + \\ &+ (\hat{\tilde{L}}_{n,-n} - \hat{\tilde{L}}_{k,-k} + C_k + D_n) - (\hat{\tilde{L}}_{r,-n} \delta_{k,r} - \hat{\tilde{L}}_{k,-r} \delta_{n,r} + C_k \delta_{k,r} + D_r \delta_{n,r}) \delta_{k,n} = 0. \end{aligned}$$

Assuming $n \neq m$, $r \neq k$ and $k \neq n$ leaves us with

$$C_n + C_r + C_k + D_r + D_k + D_n = 0,$$

which for $n + k + r = 0$ becomes

$$C_n + C_k + C_{-n-k} + D_n + D_k + D_{-n-k} = 0$$

$$C_n + C_k - C_{n+k} + D_n + D_k - D_{n+k} = 0,$$

and now setting $k = 1$ gives

$$C_n - C_{n+1} + C_1 + D_n - D_{n+1} + D_1 = 0$$

$$C_{n+1} - C_n - C_1 = -(D_{n+1} - D_n - D_1),$$

meaning C_n and D_n are negatives of each other differing by some constant β , so it suffices to find C_n and later determine the constant β . It is not difficult to see that the solution to this difference eqn is $C_n = An$, so $D_n = -An + \beta$. Since we know $D_{-n} = -D_n$, then $\beta = 0$, and $A = C_1$. We now need to determine the value of C_1 . By now, the commutator of \hat{L} is

$$\left[\hat{L}_{n,m}, \hat{L}_{k,p} \right] = \hat{L}_{k,m} \delta_{n,-p} - \hat{L}_{n,p} \delta_{-m,k} + C_1 n \delta_{n,-p} - C_1 k \delta_{-m,k}.$$

Let's calculate the VEV of this commutator for $m = k \neq 0$ with $n = -p = 1$

$$\langle 0 | \left[\hat{L}_{1,k}, \hat{L}_{k,-1} \right] | 0 \rangle = \langle 0 | \hat{L}_{1,k} \hat{L}_{k,-1} | 0 \rangle$$

$$\langle 0 | \hat{L}_{k,k} + C_1 | 0 \rangle = \langle 0 | (a_{-k}^\mu)^\dagger a_1^\nu \eta_{\mu\nu} (a_1^{\mu'})^\dagger a_k^{\nu'} \eta_{\mu'\nu'} | 0 \rangle$$

$$\langle 0 | C_1 | 0 \rangle = 0 \iff C_1 = 0,$$

so in actuality the constraint algebra has no anomalies.

Let us now denote the ground state of momentum p^μ as $|0; p\rangle$. The mass-shell condition

$$\hat{M}^2 = \frac{1}{(4\pi(\alpha')^2)^2} \left((2\pi\alpha'\mu)^2 \hat{x}^\mu \hat{x}_\mu - \left(\sum_{n \neq 0} \left(\left(\frac{n}{E_n} \right)^2 - 1 \right) (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu} - c \right) \right)$$

implies that

$$\langle 0; p | \hat{M}^2 | 0; p \rangle = \frac{1}{(4\pi(\alpha')^2)^2} \left((2\pi\alpha'\mu)^2 \langle 0; p | \hat{x}^\mu \hat{x}_\mu | 0; p \rangle - \left(\sum_{n \neq 0} \left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \langle 0; p | (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu} | 0; p \rangle - \langle 0; p | c | 0; p \rangle \right) \right)$$

$$M^2 = \frac{1}{(4\pi(\alpha')^2)^2} \left((2\pi\alpha'\mu)^2 \frac{\partial \psi^*(p)}{\partial p_\mu} \frac{\partial \psi(p)}{\partial p^\mu} + c \right)$$

$$(4\pi(\alpha')^2)^2 p^2 = - \left((2\pi\alpha'\mu)^2 \left| \frac{\partial \psi(p)}{\partial p} \right|^2 + c \right)$$

Now looking at the first excited state $\zeta_\mu (\hat{a}_1^\mu)^\dagger |0; p\rangle$ with $\zeta_\mu = \zeta_\mu(p)$ being the polarization covector, the mass-shell now reads

$$\langle 0; p | \hat{a}_1^\mu \zeta_\mu \hat{M}^2 \zeta_\nu (\hat{a}_1^\nu)^\dagger | 0; p \rangle = \frac{1}{(4\pi(\alpha')^2)^2} \left((2\pi\alpha'\mu)^2 \langle 0; p | \hat{a}_1^\mu \zeta_\mu \hat{x}^{\mu'} \hat{x}_{\mu'} \zeta_\nu (\hat{a}_1^\nu)^\dagger | 0; p \rangle - \left(\sum_{n \neq 0} \left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \langle 0; p | \hat{a}_1^\mu \zeta_\mu (\hat{a}_n^{\mu'})^\dagger \hat{a}_n^{\nu'} \eta_{\mu'\nu'} \zeta_\nu (\hat{a}_1^\nu)^\dagger | 0; p \rangle - \langle 0; p | \hat{a}_1^\mu \zeta_\mu c \zeta_\nu (\hat{a}_1^\nu)^\dagger | 0; p \rangle \right) \right)$$

$$M^2 \zeta^\mu \zeta_\mu = \frac{1}{(4\pi(\alpha')^2)^2} \left((2\pi\alpha'\mu)^2 \left| \frac{\partial \psi(p)}{\partial p} \right|^2 \zeta^\mu \zeta_\mu - \left(\frac{1}{1 + \mu^2} - 1 \right) \zeta^\mu \zeta_\mu + c \zeta^\mu \zeta_\mu \right)$$

$$(4\pi(\alpha')^2)^2 p^2 = - \left((2\pi\alpha'\mu)^2 \left| \frac{\partial\psi(p)}{\partial p} \right|^2 - \left(\frac{1}{1+\mu^2} - 1 \right) + c \right).$$

The auxiliary $\hat{\tilde{L}}_{1,0}\zeta_{\mu'}(\hat{a}_1^{\mu'})^\dagger|0;p\rangle = 0$ condition implies

$$\begin{aligned} \hat{\tilde{L}}_{1,0}\zeta_{\mu'}(\hat{a}_1^{\mu'})^\dagger|0;p\rangle &= 0 \\ (\hat{a}_0^\mu)^\dagger\hat{a}_1^\nu\eta_{\mu\nu}\zeta_{\mu'}(\hat{a}_1^{\mu'})^\dagger|0;p\rangle &= 0 \\ \zeta_\mu(\hat{a}_0^\mu)^\dagger|0;p\rangle &= 0 \\ \zeta_\mu(2\pi\alpha'\mu\hat{x}^\mu - 4\pi(\alpha')^2i\hat{p}^\mu)|0;p\rangle &= 0 \\ -2\pi\alpha'\mu i\zeta_\mu\frac{\partial\psi(p)}{\partial p_\mu} &= 4\pi(\alpha')^2i\zeta_\mu p^\mu \\ \zeta_\mu p^\mu &= -\frac{\mu}{2\alpha'}\zeta_\mu\frac{\partial\psi(p)}{\partial p_\mu}. \end{aligned}$$

If the momentum wave function is chosen to be (need to justify this)

$$\psi(p) = \frac{\alpha'}{\mu}p^2,$$

we get that $\zeta_\mu p^\mu = 0$. Using this in the mass-shell condition for the first excited state yields

$$\begin{aligned} (4\pi(\alpha')^2)^2 p^2 &= - \left((2\pi\alpha'\mu)^2 4 \frac{(\alpha')^2}{\mu^2} p^2 - \left(\frac{1}{1+\mu^2} - 1 \right) + c \right) \\ 2(4\pi(\alpha')^2)^2 p^2 &= \frac{1}{1+\mu^2} - 1 - c, \end{aligned}$$

and for the ground state

$$2(4\pi(\alpha')^2)^2 p^2 = -c.$$

In general, the n -th excited state will have momentum given by

$$2(4\pi(\alpha')^2)^2 p^2 = \frac{n^2}{n^2 + \mu^2} - 1 - c.$$

The norm of the states is given by $\zeta_\mu\zeta^\mu$, thus if we choose p to lie on the $(0,1)$ -plane, then the $D-2$ spacelike polarizations normal to that plane clearly have positive norm. If we choose c such that the $n=d$, $d=(D-1)$ (we choose this because the solution only has $n=0,1,2,\dots,d$ terms by the \tilde{L} constraints) excited state is a tachyon with $p^2 > 0$, then p can be taken to have no time component and thus ζ is time-like with negative norm. If $p^2 < 0$, p can be chosen to ONLY have a time component, leaving ζ spacelike with positive norm again. At last, if $p^2 = 0$, then ζ is proportional to p with zero norm. Thus, for the absence of ghosts we require

$$c \geq \frac{d^2}{d^2 + \mu^2} - 1.$$

In the boundary case, the 3rd excited vector state is massless and the rest (including the scalar ground state) are tachyons... This analysis, however, is to get spin 1 particles from the closed string. The actual massless particle we want from closed strings is a symmetric spin 2 particle, so let's consider now the state $\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle$ where $\zeta_{\mu\nu} = \zeta_{\nu\mu}$ is the polarization tensor. The mass-shell is

$$\begin{aligned} \langle 0;p|\hat{a}_d^\alpha\hat{a}_d^\beta\zeta_{\alpha\beta}\hat{M}^2\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle &= \frac{1}{(4\pi(\alpha')^2)^2} \left((2\pi\alpha'\mu)^2 \langle 0;p|\hat{a}_d^\alpha\hat{a}_d^\beta\zeta_{\alpha\beta}\hat{x}^{\mu'}\hat{x}_{\mu'}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle - \right. \\ &\quad \left. - \left(\sum_{n \neq 0} \left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \langle 0;p|\hat{a}_d^\alpha\hat{a}_d^\beta\zeta_{\alpha\beta}(\hat{a}_n^{\mu'})^\dagger\hat{a}_n^{\nu'}\eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle - \langle 0;p|\hat{a}_d^\alpha\hat{a}_d^\beta\zeta_{\alpha\beta}c\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle \right) \right) \end{aligned}$$

$$\begin{aligned}
M^2 \zeta_{\mu\nu} \zeta^{\mu\nu} &= \frac{1}{(4\pi(\alpha')^2)^2} \left((2\pi\alpha'\mu)^2 \left| \frac{\partial\psi(p)}{\partial p} \right|^2 \zeta_{\mu\nu} \zeta^{\mu\nu} - \left(2 \left(\frac{d^2}{d^2 + \mu^2} - 1 \right) \zeta_{\mu\nu} \zeta^{\mu\nu} - c \zeta_{\mu\nu} \zeta^{\mu\nu} \right) \right) \\
(4\pi(\alpha')^2)^2 p^2 &= - \left((2\pi\alpha'\mu)^2 \left| \frac{\partial\psi(p)}{\partial p} \right|^2 - \left(2 \left(\frac{d^2}{d^2 + \mu^2} - 1 \right) - c \right) \right) \\
2(4\pi(\alpha')^2)^2 p^2 &= 2 \left(\frac{d^2}{d^2 + \mu^2} - 1 \right) - c.
\end{aligned}$$

Using the auxiliary $\hat{\tilde{L}}_{d,0} \zeta_{\mu\nu} (\hat{a}_d^\mu)^\dagger (\hat{a}_d^\nu)^\dagger |0; p\rangle = 0$ condition implies

$$\begin{aligned}
\hat{\tilde{L}}_{d,0} \zeta_{\mu\nu} (\hat{a}_d^\mu)^\dagger (\hat{a}_d^\nu)^\dagger |0; p\rangle &= 0 \\
(\hat{a}_0^{\mu'})^\dagger \hat{a}_d^{\nu'} \eta_{\mu'\nu'} \zeta_{\mu\nu} (\hat{a}_d^\mu)^\dagger (\hat{a}_d^\nu)^\dagger |0; p\rangle &= 0 \\
\left((\hat{a}_0^{\mu'})^\dagger \eta_{\mu'\nu'} \zeta_{\mu\nu} \eta^{\nu'\mu} (\hat{a}_d^\nu)^\dagger + (\hat{a}_0^{\mu'})^\dagger \eta_{\mu'\nu'} \zeta_{\mu\nu} (\hat{a}_d^\mu)^\dagger \hat{a}_d^{\nu'} (\hat{a}_d^\nu)^\dagger \right) |0; p\rangle &= 0 \\
\left((\hat{a}_0^{\mu'})^\dagger \eta_{\mu'\nu'} \zeta_{\mu\nu} \eta^{\nu'\mu} (\hat{a}_d^\nu)^\dagger + (\hat{a}_0^{\mu'})^\dagger \eta_{\mu'\nu'} \zeta_{\mu\nu} (\hat{a}_d^\mu)^\dagger \eta^{\nu'\nu} \right) |0; p\rangle &= 0 \\
2\zeta_{\mu\nu} (\hat{a}_0^\mu)^\dagger (\hat{a}_d^\nu)^\dagger |0; p\rangle &= 0 \\
\zeta_{\mu\nu} (\hat{a}_d^\nu)^\dagger (2\pi\alpha'\mu\hat{x}^\mu - 4\pi(\alpha')^2 i\hat{p}^\mu) |0; p\rangle &= 0 \\
\zeta_{\mu\nu} \left(-2\pi\alpha'\mu i \frac{\partial\psi(p)}{\partial p_\mu} - 4\pi(\alpha')^2 i p^\mu \right) (\hat{a}_d^\nu)^\dagger |0\rangle &= 0 \\
\zeta_{\mu\nu} p^\mu (\hat{a}_d^\nu)^\dagger |0\rangle &= 0,
\end{aligned}$$

implying that

$$\zeta_{\mu\nu} p^\mu = \zeta_{\mu\nu} p^\nu = 0.$$

Since the norm of the spin-2 states is given by $\zeta^{\mu\nu} \zeta_{\mu\nu}$, we need those to not be negative. Since ζ is symmetric and we work in $(-, +, +, \dots)$ signature, $\zeta^{\mu\nu} \zeta_{\mu\nu} \geq 0$ always hold, so we just need to choose c such that we get at least 1 massless spin-2 state. For that, we see that

$$c = 2 \left(\frac{n^2}{n^2 + \mu^2} - 1 \right)$$

does the trick for the n -th state, and since for the non-existence of ghosts in the spin-1 states we required

$$c \geq \frac{d^2}{d^2 + \mu^2} - 1,$$

which is negative, we can thus choose

$$c \equiv \frac{2d^2}{d^2 + \mu^2} - 2,$$

leaving the d -th spin-2 state massless, the rest being tachyons... This, however, is if we impose that one spin-2 state have to be massless. Since granular space-time affects wave propagation speed, in the effective limit gravitons might be effectively massive, thus we can get rid of the tachyons by letting $c > 0$ if need be.

The fact that $\zeta_{\mu\nu} = \zeta_{\nu\mu}$ leaves $\zeta_{\mu\nu}$ with $D(D+1)/2$ independent components. The traceless condition $\eta^{\mu\nu} \zeta_{\mu\nu} = 0$ takes out 1 from that, leaving $(D(D+1)/2) - 1$ allowed polarizations. Finally, the condition $\zeta_{\mu\nu} p^\mu = 0$ takes D out of those, leaving $(D(D-1)/2) - 1$ allowed polarizations.

$$\frac{D(D-1)}{2} - 1 = \frac{D^2 - D - 2}{2} = \frac{(D-2)(D+1)}{2}.$$

This leaves 5 polarizations in $D = 4$.

Let's now analyse spurious states to see the conditions to maximize 0-norm states. A physical state $|\phi\rangle$ is a state satisfying the constraints

$$\begin{aligned}\hat{L}_{n,m}|\phi\rangle &= 0, \quad \forall n, m \in \mathbb{Z} \\ \hat{\tilde{L}}_{n,m}|\phi\rangle &= 0, \quad n \neq -m \\ \left(\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) \hat{\tilde{L}}_{n,-n}\right) - c\right) |\phi\rangle &= 0.\end{aligned}$$

A spurious state $|\psi\rangle$ is a state satisfying the last constraint and orthogonal to all physical states, i.e,

$$\begin{aligned}\left(\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) \hat{\tilde{L}}_{n,-n}\right) - c\right) |\psi\rangle &= 0, \\ \langle\phi|\psi\rangle &= 0.\end{aligned}$$

Spurious states can be written in the form

$$|\psi\rangle = \sum_{n \neq -m} \hat{\tilde{L}}_{n,m} |\chi_{n,m}\rangle$$

with $|\chi_{n,m}\rangle$ satisfying

$$\left(\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) \hat{\tilde{L}}_{n,-n}\right) - c - k - p\right) |\chi_{k,p}\rangle = 0.$$

To see that this state is indeed orthogonal to $|\phi\rangle$, we perform the inner product

$$\langle\phi|\psi\rangle = \sum_{n \neq -m} \langle\phi|\hat{\tilde{L}}_{n,m}|\chi_{n,m}\rangle = \sum_{n \neq -m} \langle\chi_{n,m}|\hat{\tilde{L}}_{-m,-n}|\phi\rangle^* \equiv 0.$$

The interesting case is when $|\psi\rangle$ is both physical AND spurious, since by construction those states have 0-norm, being orthogonal to every physical state, including themselves. Such states can be constructed by considering spurious states of the form

$$|\psi_{1,n}\rangle = \sum_m \hat{\tilde{L}}_{n,m} |\tilde{\chi}_{1,m}\rangle$$

where $|\tilde{\chi}_{1,m}\rangle$ satisfies

$$\begin{aligned}\hat{\tilde{L}}_{k,p}|\tilde{\chi}_{1,m}\rangle &= 0, \quad k \neq -p \\ \left(\sum_{n'} \left(\left(\left(\frac{n'}{E_{n'}}\right)^2 - 1\right) \hat{\tilde{L}}_{n',-n'}\right) - c - 1 - m\right) |\tilde{\chi}_{1,m}\rangle &= 0.\end{aligned}$$

These states are annihilated by $\hat{\tilde{L}}_{k,p}$, for $p \neq -n$, and for this case we have that

$$\begin{aligned}&\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) \hat{\tilde{L}}_{2,-n} |\psi_{1,n}\rangle\right) = \\ &= \sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) \hat{\tilde{L}}_{2,-n} \sum_m \hat{\tilde{L}}_{n,m} |\tilde{\chi}_{1,m}\rangle\right) = \\ &= \sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right) \hat{\tilde{L}}_{n,-n}\right) |\tilde{\chi}_{1,-2}\rangle,\end{aligned}$$

which vanish if $c = 1$.

12 Polyakov Action in terms of LQG Variables

Turn embedding fields X^μ into vector in the spin-1 representation of $\mathbb{R}^{d+1} \rtimes \text{Spin}(d, 1)$, X^I , and promote partial derivative to covariant derivative $\partial_a \mapsto \mathcal{D}_a$ acting as

$$\mathcal{D}_a X^I = \partial_a X^I + k \mathcal{A}_{aJ}^I X^J,$$

where the WS $\mathbb{R}^{d+1} \rtimes \text{Spin}(d, 1)$ connection \mathcal{A}_a with $I, J = 0, \dots, d, d+1$ is given by

$$\mathcal{A}_a X = \begin{bmatrix} 0 & \beta_a^{01} & \beta_a^{02} & \dots & \beta_a^{0d} & p_a^0/l \\ \beta_a^{01} & 0 & -\theta_a^{12} & \dots & -\theta_a^{1d} & p_a^1/l \\ \beta_a^{02} & \theta_a^{12} & 0 & \dots & -\theta_a^{2d} & p_a^2/l \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_a^{0d} & \theta_a^{1d} & \theta_a^{2d} & \dots & 0 & p_a^d/l \\ -\epsilon p_a^0/l & \epsilon p_a^1/l & \epsilon p_a^2/l & \dots & \epsilon p_a^d/l & 0 \end{bmatrix} \begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ \vdots \\ X^d \\ 1 \end{bmatrix} = \left(\beta_a^{0\alpha} B_{0\alpha} + \frac{1}{2} \theta_a^{\alpha\beta} J_{\alpha\beta} + \frac{p_a^\mu}{l} P_\mu \right) X,$$

where $\epsilon = \text{sgn}(\Lambda)$, i.e the sign of the background cosmological constant with $\alpha, \beta = 1, \dots, d$ and $\mu = 0, \dots, d$, so the action becomes

$$S = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ},$$

where the metrics can be recast in terms of auxiliary and bulk vielbein fields as in

$$S = -\frac{T}{2} \int d^2x e e_I^a e_J^b \eta^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ},$$

where

$$e := \sqrt{-\det(e_a^I e_b^J \eta_{IJ})} = \sqrt{-\frac{1}{2} \varepsilon^{ff'} \varepsilon^{gg'} (e_f^M e_g^N \eta_{MN}) (e_{f'}^{M'} e_{g'}^{N'} \eta_{M'N'})}.$$

12.1 Checking the derivative

Since the connection is non-abelian, it should transform as

$$\mathcal{A}'_a = \mathcal{P} \mathcal{A}_a \mathcal{P}^{-1} - \frac{1}{k} (\partial_a \mathcal{P}) \mathcal{P}^{-1},$$

where $\mathcal{P} = e^{\beta^{0\alpha} B_{0\alpha} + \theta^{\alpha\beta} J_{\alpha\beta}/2 + p^\mu P_\mu/l}$. Putting this into a gauge transformed derivative we have

$$\begin{aligned} \mathcal{D}'_a X' &= \partial_a X' + k \mathcal{A}'_a X' = \\ &= \partial_a (\mathcal{P} X) + k (\mathcal{P} \mathcal{A}_a \mathcal{P}^{-1} - \frac{1}{k} (\partial_a \mathcal{P}) \mathcal{P}^{-1}) (\mathcal{P} X) = \\ &= \partial_a \mathcal{P} X + \mathcal{P} \partial_a X + k \mathcal{P} \mathcal{A}_a X - \partial_a \mathcal{P} X \equiv \mathcal{P} \mathcal{D}_a X, \end{aligned}$$

so this is a proper covariant derivative for the Poincaré group.

Since under gauge transformations $\mathcal{D}_a X$ transforms covariantly, the action is not invariant under diffeomorphisms. In fact, it transforms as

$$\begin{aligned} S' &= -\frac{T}{2} \int d^2x e' e'^a_I e'^b_J \eta^{I'J'} \mathcal{D}'_a X'^I \mathcal{D}'_b X'^J \eta_{IJ} = \\ &= -\frac{T}{2} \int d^2x e e'^a_{K'} (\mathcal{P}^{-1})^{K'I'} e'^b_{L'} (\mathcal{P}^{-1})^{L'J'} \eta^{I'J'} \mathcal{P}^I_{K'} \mathcal{D}_a X^K \mathcal{P}^J_{L'} \mathcal{D}_b X^L \eta_{IJ}, \end{aligned}$$

so we promote the killing forms η_{IJ} and $\eta^{I'J'}$ to doubly co(ntra)-variant forms K_{IJ} and $K^{I'J'}$ such that $K'_{IJ} = (\mathcal{P}^{-1})^M_I (\mathcal{P}^{-1})^N_J K_{MN}$ and $K'^{I'J'} = \mathcal{P}^{I'}_{M'} \mathcal{P}^{J'}_{N'} K^{M'N'}$, so the new action is

$$S = -\frac{T}{2} \int d^2x e e'^a_I e'^b_J K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ}.$$

For consistency, let's assume K is a background field $K = K(X)$ like the metric G .

12.2 EoMs

12.2.1 W.r.t e

$$\begin{aligned}
\frac{\delta S}{\delta e_K^c} &\propto \left(\frac{\partial e}{\partial e_K^c} e_{I'}^a e_{J'}^b K^{I'J'} + e \frac{\partial}{\partial e_K^c} (e_{I'}^a e_{J'}^b K^{I'J'}) \right) \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} = \\
&= \left(-e e_c^K e_{I'}^a e_{J'}^b K^{I'J'} + 2e \delta_c^a \delta_{I'}^K e_{J'}^b K^{I'J'} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} = \\
&= -e e_c^K e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} + 2e e_{J'}^b K^{KJ'} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ} = 0 \\
T_c^K &:= 2e_{J'}^b K^{KJ'} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ} - e_c^K e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} = 0 \\
e_c^K &= 2f e^{bK} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ}, \\
\frac{1}{f} &= e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ}
\end{aligned}$$

12.2.2 W.r.t A

$$\begin{aligned}
\frac{\delta S}{\delta \mathcal{A}_c^{KL}} &= \frac{T}{2} \left(e e_{I'}^a e_{J'}^b K^{I'J'} \frac{\partial}{\partial \mathcal{A}_c^{KL}} (\mathcal{D}_a X^I \mathcal{D}_b X^J) K_{IJ} \right) = \\
&= T \left(e e_{I'}^a e_{J'}^b K^{I'J'} \delta_a^c \delta_{[K}^I \eta_{L]I'} X^{I'} \mathcal{D}_b X^J K_{IJ} \right) = \\
&= T e e_{I'}^c e_{J'}^b K^{I'J'} \mathcal{D}_b X_{[K} X_{L]} \stackrel{!}{=} 0 \\
\mathcal{T}_a^{IJ} &= \mathcal{D}_a X^{[I} X^{J]} = 0
\end{aligned}$$

12.2.3 W.r.t X

$$\begin{aligned}
\frac{\delta S}{\delta X^K} &= -\frac{T}{2} \left(\partial_c \left(e e_{I'}^a e_{J'}^b K^{I'J'} \frac{\partial}{\partial (\partial_c X^K)} (\mathcal{D}_a X^I \mathcal{D}_b X^J) K_{IJ} \right) - e e_{I'}^a e_{J'}^b K^{I'J'} \frac{\partial}{\partial X^K} (\mathcal{D}_a X^I \mathcal{D}_b X^J) K_{IJ} - \right. \\
&\quad \left. - e e_{I'}^a e_{J'}^b \frac{\partial K^{I'J'}}{\partial X^K} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} - e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \frac{\partial K_{IJ}}{\partial X^K} \right) = \\
&= -T \left(\left(\partial_c \left(e e_{I'}^a e_{J'}^b K^{I'J'} \delta_a^c \delta_K^I \mathcal{D}_b X^J K_{IJ} \right) - e e_{I'}^a e_{J'}^b K^{I'J'} k_{aL}^I \delta_L^{I'} \mathcal{D}_b X^J K_{IJ} \right) - \right. \\
&\quad \left. - e e_{I'}^a e_{J'}^b \partial_K K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} - e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \partial_K K_{IJ} \right) = \\
&= -T \left(\mathcal{D}_a \left(e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_b X^I K_{IK} \right) - e e_{I'}^a e_{J'}^b \left(\partial_K K^{I'J'} K_{IJ} + K^{I'J'} \partial_K K_{IJ} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J \right) \stackrel{!}{=} 0 \\
\mathcal{D}_a \left(e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_b X^I K_{IK} \right) &= e e_{I'}^a e_{J'}^b \left(\partial_K K^{I'J'} K_{IJ} + K^{I'J'} \partial_K K_{IJ} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J
\end{aligned}$$

In “constant gauge” $K_{IJ}(X) = \eta_{IJ}$ and in conformal gauge $e e_{I'}^a e_{J'}^b \eta^{I'J'} = \eta^{ab}$, this reduces to

$$\eta^{ab} \mathcal{D}_a \mathcal{D}_b X^I = 0.$$

More explicitly,

$$\begin{aligned}
\eta^{ab} \mathcal{D}_a \left(\partial_b X^I + k \mathcal{A}_{bJ}^I X^J \right) &= 0 \\
\eta^{ab} \left((\partial_a \partial_b X^I + k \mathcal{A}_{aJ}^I \partial_b X^J) + \left(k \partial_a (\mathcal{A}_{bJ}^I X^J) + k^2 \mathcal{A}_{aI'}^I \mathcal{A}_{bJ}^{I'} X^J \right) \right) &= 0 \\
\eta^{ab} \left(\partial_a \partial_b X^I + 2k \mathcal{A}_{aJ}^I \partial_b X^J + k \partial_a \mathcal{A}_{bJ}^I X^J + k^2 \mathcal{A}_{aI'}^I \mathcal{A}_{bJ}^{I'} X^J \right) &= 0
\end{aligned}$$

$$\eta^{ab} (\delta_J^I \partial_a \partial_b + 2k \mathcal{A}_{aJ}^I \partial_b) X^J = - \left(k \eta^{ab} \partial_a \mathcal{A}_{bJ}^I X^J + k^2 \eta^{ab} \mathcal{A}_{aI'}^I \mathcal{A}_{bJ}^{I'} X^J \right),$$

which has implicit solution given by

$$X^K(x) = - \int d^2 x' G_I^K(x, x') \left(k \eta^{a'b'} \partial_{a'} \mathcal{A}_{b'J}^I(x') X^J(x') + k^2 \eta^{a'b'} \mathcal{A}_{a'I'}^I(x') \mathcal{A}_{b'J}^{I'}(x') X^J(x') \right),$$

where the Green's function matrix $G_J^K(x, x')$ satisfies

$$\eta^{ab} (\delta_K^I \partial_a \partial_b + 2k \mathcal{A}_{aK}^I(x) \partial_b) G_J^K(x, x') = \delta_J^I \delta(x, x').$$

Boundary conditions come from

$$\begin{aligned} \delta S &= \int d^2 x \left(\frac{\partial \mathcal{L}}{\partial X^K} \delta X^K + \frac{\partial \mathcal{L}}{\partial (\partial_c X^K)} \delta (\partial_c X^K) + \dots \right) = \\ &= \int d^2 x \left((\text{EoM}) \delta X^K + \partial_c \left(\frac{\partial \mathcal{L}}{\partial (\partial_c X^K)} \delta X^K \right) + \dots \right) = \\ &= \int d^2 x (\text{EoM}) \delta X^K + \int d\tau \left(\frac{\partial \mathcal{L}}{\partial (\partial_\sigma X^K)} \delta X^K \right) \Big|_0^{\sigma_1} + \dots = 0, \end{aligned}$$

thus

$$\begin{aligned} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\sigma X^K)} \delta X^K \right) \Big|_0^{\sigma_1} &= 0 \\ ((-T e e_i^\sigma e^{bi} \mathcal{D}_b X_K) \delta X^K) \Big|_0^{\sigma_1} &= 0 \\ ((\mathcal{D}_\sigma X_K) \delta X^K) \Big|_0^{\sigma_1} &= 0, \end{aligned}$$

so either $\delta X^K(\tau, \sigma_*) = 0$, $\sigma_* = 0, \sigma_1$ (Dirichlet B.C) or $\mathcal{D}_\sigma X^K(\tau, \sigma_*) = 0$, $\sigma_* = 0, \sigma_1$ (free end-point B.C). For closed strings, $X^K(\tau, \sigma) = X^K(\tau, \sigma + 2\pi)$.

When connection is trivial EoM reduces to regular wave eqn, which have the regular string solution

$$X_0^I(\tau, \sigma) = X_{0L}^I(\sigma^+) + X_{0R}^I(\sigma^-),$$

with

$$\begin{aligned} X_{0L}^I(\sigma^+) &= \frac{1}{2} x^I + \frac{1}{2} \alpha' p^I \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^I e^{-in\sigma^+}, \quad \sigma^+ = \tau + \sigma \\ X_{0R}^I(\sigma^-) &= \frac{1}{2} x^I + \frac{1}{2} \alpha' p^I \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^I e^{-in\sigma^-}, \quad \sigma^- = \tau - \sigma. \end{aligned}$$

This, however, does not amount to any motion in actual space-time, since with trivial connection the vector does not change after parallel transport. For a more interesting case, let's consider $A_\tau^{\alpha\beta} = A_\sigma^{\alpha\beta} = a \varepsilon^{\alpha\beta}$, which amounts to flat space-time. The EoM turn into

$$\begin{aligned} (\partial_\sigma^2 - \partial_\tau^2) X^I + 2ka(\varepsilon^{\alpha\beta} T_{\alpha\beta})^I_J (\partial_\sigma - \partial_\tau) X^J &= 0 \\ (\partial_\sigma^2 - \partial_\tau^2) \begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ \vdots \\ X^d \end{bmatrix} + 4ka(\partial_\sigma - \partial_\tau) \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & -1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -1 & \dots & 0 \end{bmatrix} \begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ \vdots \\ X^d \end{bmatrix} &= 0 \\ \begin{bmatrix} (\partial_\sigma^2 - \partial_\tau^2) X^0 + 4ka(\partial_\sigma - \partial_\tau) \sum_{i>0} X^i \\ (\partial_\sigma^2 - \partial_\tau^2) X^1 + 4ka(\partial_\sigma - \partial_\tau) \sum_{i \neq 1} X^i \\ (\partial_\sigma^2 - \partial_\tau^2) X^2 + 4ka(\partial_\sigma - \partial_\tau) (X^0 - X^1 + \sum_{i>2} X^i) \\ \vdots \\ (\partial_\sigma^2 - \partial_\tau^2) X^d + 4ka(\partial_\sigma - \partial_\tau) (X^0 - \sum_{i=1}^{d-1} X^i) \end{bmatrix} &= 0 \end{aligned}$$

$$\begin{aligned}
& (\partial_\sigma^2 - \partial_\tau^2) X^I + 4ka (\partial_\sigma - \partial_\tau) \left((\text{sgn}(I))^2 X^0 + \sum_{i>0} \text{sgn}(i-I) X^i \right) = 0 \\
& (\partial_\sigma - \partial_\tau) \left((\partial_\sigma + \partial_\tau) X^I + 4ka \left((\text{sgn}(I))^2 X^0 + \sum_{i>0} \text{sgn}(i-I) X^i \right) \right) = 0 \\
& (\partial_\sigma + \partial_\tau) X^I + 4ka \left((\text{sgn}(I))^2 X^0 + \sum_{i>0} \text{sgn}(i-I) X^i \right) = C_1^I \cdot (\tau + \sigma), \quad C^I \in \mathbb{C} \\
& (\partial_\sigma + \partial_\tau) X^I = C_1^I \cdot (\tau + \sigma) - 4ka \left((\text{sgn}(I))^2 X^0 + \sum_{i>0} \text{sgn}(i-I) X^i \right) \\
& X^I = \frac{1}{2} C_1^I \cdot (\tau^2 + \sigma^2) + C_2^I \cdot (\tau - \sigma) - 4ka \int d^2 x' \left((\text{sgn}(I))^2 X^0 + \sum_{i>0} \text{sgn}(i-I) X^i \right).
\end{aligned}$$

If $D = 2$, we get a symmetric system of eqns

$$\begin{aligned}
X^0 &= \frac{1}{2} C_1^0 \cdot (\tau^2 + \sigma^2) + C_2^0 \cdot (\tau - \sigma) - 4ka \int d^2 x' X^1 \\
X^1 &= \frac{1}{2} C_1^1 \cdot (\tau^2 + \sigma^2) + C_2^1 \cdot (\tau - \sigma) - 4ka \int d^2 x' X^0 \\
X^0 &= \frac{1}{2} C_1^0 \cdot (\tau^2 + \sigma^2) + C_2^0 \cdot (\tau - \sigma) - 4ka \int d^2 x' \left(\frac{1}{2} C_1^1 \cdot (\tau'^2 + \sigma'^2) + C_2^1 \cdot (\tau' - \sigma') - 4ka \int d^2 x'' X^0 \right) \\
X^0 - 16k^2 a^2 \int d^2 x' \int d^2 x'' X^0 &= \frac{1}{2} C_1^0 \cdot (\tau^2 + \sigma^2) + C_2^0 \cdot (\tau - \sigma) - \frac{1}{2} C_1^1 \left(\frac{\tau^3}{3} \sigma + \tau \frac{\sigma^3}{3} \right) - C_2^1 \left(\frac{\tau^2}{2} \sigma - \tau \frac{\sigma^2}{2} \right).
\end{aligned}$$

This integral equation is tricky to deal with, but we can turn it into a differential equation by applying $\partial_\sigma \partial_\tau$ twice:

$$(\partial_\sigma \partial_\tau)^2 X^0 - 16k^2 a^2 X^0 = 0.$$

This has general solution given by sum of plane waves and exponentials

$$X^0 = \sum_{n \neq 0} \frac{1}{n^4} \left(a_n^0 e^{i2\sqrt{ka}n(\tau+\sigma)} + b_n^0 e^{i2\sqrt{ka}n(\tau-\sigma)} + c_n^0 e^{2\sqrt{ka}n(\tau+\sigma)} + d_n^0 e^{2\sqrt{ka}n(\tau-\sigma)} \right).$$

In actuality, only the $n = 1$ and $n = -1$ terms satisfy the differential equation, so we have just

$$\begin{aligned}
X^0 &= a_1^0 e^{i2\sqrt{ka}(\tau+\sigma)} + b_1^0 e^{i2\sqrt{ka}(\tau-\sigma)} + c_1^0 e^{2\sqrt{ka}(\tau+\sigma)} + d_1^0 e^{2\sqrt{ka}(\tau-\sigma)} + \\
&\quad + a_{-1}^0 e^{-i2\sqrt{ka}(\tau+\sigma)} + b_{-1}^0 e^{-i2\sqrt{ka}(\tau-\sigma)} + c_{-1}^0 e^{-2\sqrt{ka}(\tau+\sigma)} + d_{-1}^0 e^{-2\sqrt{ka}(\tau-\sigma)}
\end{aligned}$$

13 Polyakov-LQG V2

Same idea as before, but now contract the dyads directly with the $\mathcal{D}_a X^I$ factors

$$S = -\frac{T}{4} \int d^2 x e e_I^a e_J^b \mathcal{D}_a X^I \mathcal{D}_b X^J.$$

13.1 EoMs

13.1.1 W.r.t e

$$\begin{aligned} \frac{\delta S}{\delta e_K^c} &\propto \left(\frac{\partial e}{\partial e_K^c} e_I^a e_J^b + e \frac{\partial}{\partial e_K^c} (e_I^a e_J^b) \right) \mathcal{D}_a X^I \mathcal{D}_b X^J = \\ &= (-e e_c^K e_I^a e_J^b + 2e \delta_I^K \delta_c^a e_J^b) \mathcal{D}_a X^I \mathcal{D}_b X^J = \\ &= -e e_c^K e_I^a e_J^b \mathcal{D}_a X^I \mathcal{D}_b X^J + 2e e_J^b \mathcal{D}_c X^K \mathcal{D}_b X^J = 0 \\ T_c^K &:= 2e_J^b \mathcal{D}_c X^K \mathcal{D}_b X^J - e_c^K e_I^a e_J^b \mathcal{D}_a X^I \mathcal{D}_b X^J = 0, \\ e_c^K &= 2f \mathcal{D}_c X^K, \\ \frac{1}{f} &= e_J^b \mathcal{D}_b X^J. \end{aligned}$$

13.1.2 W.r.t A

$$\begin{aligned} \frac{\delta S}{\delta \mathcal{A}_c^{KL}} &\propto e e_I^a e_J^b \frac{\partial}{\partial \mathcal{A}_c^{KL}} (\mathcal{D}_a X^I \mathcal{D}_b X^J) = \\ &= 2e e_I^a e_J^b \delta_a^c \delta_{[K}^I \eta_{L]}^{J'} X^{I'} \mathcal{D}_b X^J = \\ &= 2e e_{[K}^c X_{L]} e_J^b \mathcal{D}_b X^J = 0 \\ \mathcal{T}_{KL}^c &:= e_{[K}^c X_{L]} = 0. \end{aligned}$$

13.1.3 W.r.t X

$$\begin{aligned} \frac{\delta S}{\delta X^K} &\propto 2e e_I^a e_J^b k_{aI'} \delta_K^{I'} \mathcal{D}_b X^J - 2\partial_c (e e_I^a e_J^b \delta_a^c \delta_K^I \mathcal{D}_b X^J) = \\ &= 2e e_I^a e_J^b k_{aK} \mathcal{D}_b X^J - 2\partial_a (e e_K^a e_J^b \mathcal{D}_b X^J) = 0 \\ \mathcal{D}_a (e e_K^a e_J^b \mathcal{D}_b X^J) &= 0. \end{aligned}$$

If conformal gauge makes it so that $e e_K^a e_J^b = \eta^{ab} \eta_{KJ}$, this reduces to

$$\eta^{ab} \mathcal{D}_a \mathcal{D}_b X^I = 0$$