

Quantizing the Bosonic String on a Loop Quantum Gravity Background

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We write down an action for a bosonic string propagating in a bulk background whose geometry is specified in terms of connection and tetrad (or “vierbein”) variables

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I. INTRODUCTION

“Quantum gravity” refers to the broad enterprise dedicated to finding a complete, consistent theory in which quantum mechanics coexists peacefully with general relativity. There are two major approaches to quantum gravity which are widely recognized for their success in describing various aspects of physics at the Planck scale. The first of these is string theory, the quantum theory of a one dimensional object which is said to contain within it a complete description of all particles, forces and their interactions, including gravity. However, where precisely in the vast space of effective field theories which can possibly arise as in the low energy limit of a conjectured “M-Theory”, our Universe with its collection of particles, forces and coupling constants, exists, is still a matter of vigorous debate.

One of the primary criticisms of string theory is its apparent lack of “background independence”. This is more general than the principle of *general covariance* which states that physics should be independent of the choice of co-ordinates. Background independence is the statement that any *quantum* theory of gravity - which, presumably, should provide a description not only of classical spacetimes, but also of fluctuating, semiclassical geometries and those in the deep quantum regime which do not have any sensible description in terms of any Riemannian geometry - should provide a description of physics which is independent, not only of the choice of co-ordinates, but also of the choice of background manifold on which those co-ordinates live. On the face of it string theory, at least in its conventional form, does not satisfy this principle.

It is sometimes argued that string theory is, in fact, background independent since the equations of motion which arise when the (Bosnic) string is coupled to the metric of the “worldvolume” (the background manifold in which the string is propagating) turn out to include Einstein’s field equations for gravity. In other words, Weyl invariance - one of the fundamental symmetries of the string worldsheet, *requires* that the background geometry must satisfy Einstein’s equations. While technically correct, it begs the question, if general relativity “emerges” from string theory then why is it not possi-

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ble to quantize the string on an arbitrary background to begin with? As anybody who has studied the basics of string theory knows, the standard quantization procedure *assumes* that the worldvolume has a flat metric. It is, thus far, not technically feasible to do away with this assumption.

There is another sense in which string theory falls short of being a theory of quantum gravity. This, however, rests upon a philosophical perspective, about the nature of quantum gravity, which one may or may not choose to subscribe to. This is the viewpoint that in any quantum theory of gravity, geometry itself must be quantized. Again, one might say that gravitons - the quantum perturbations of the metric - are “quanta of geometry”. However, gravitons *live* upon some background manifold. Moreover, as mentioned, they are *perturbative* by construction. Gravitons cannot provide a quantum description of geometry, anymore than quantizing water waves can provide a picture of the molecular structure of water¹. These considerations bring us to the second major² approach towards a quantum theory of gravity, known as Loop Quantum Gravity or LQG for short.

LQG arose almost by accident, something it has in common with string theory. Traditionally, there are two ways in which a classical theory can be quantized. These are the Hamiltonian and Lagrangian approaches. The Lagrangian or path integral approach follows the prescription first suggested by Dirac and then made concrete by Feynman. There one views the classical action associated to a given evolution as corresponding to a *phase angle* which determines the complex weight of the associated evolution. This approach respects spacetime covariance since here the central object is the action which is invariant under spacetime transformations by construction. The Hamiltonian approach, on the other hand, involves making a choice of a spacelike surface Σ_t and a corresponding timelike vector t^μ , normal to Σ_t (here t is a continuous parameter which labels the family of surfaces). This approach, therefore, necessarily does not respect spacetime covariance since it involves making a specific choice of the foliation of the background into spacelike surfaces.

In the decades prior to the advent of string theory a great deal of effort was put into attempting to quantize gravity via both the Lagrangian and Hamiltonian approaches. The Hamiltonian approach involves, as stated earlier, a choice of the foliation of the background geometry into a collection of spacelike surfaces Σ_t . Central to this approach was the ADM (Arnowitt-Deser-Misner) formalism which allows one to construct a Hamiltonian H_{GR} for general relativity starting from the Einstein-Hilbert action S_{EH} . The resulting phase is co-ordinatized

by a configuration variable which is the 3-metric h_{ab} of the “leaves” Σ_t of the foliation and a momentum variable which is a function of the extrinsic curvature k_{ab} of the leaves³. The gravity Hamiltonian turns out to be a sum of two constraints known as the diffeomorphism (or “momentum”) constraint H_{diff} and the Hamiltonian constraint H_{ham} :

$$H_{GR} = H_{diff} + H_{ham}, \quad (1)$$

and these are, in turn, functionals on the phase space of general relativity written in terms of h_{ab} and k_{ab} . The idea then is that physical states of the theory $|\Psi_{phys}\rangle$, which can be written as functionals $\Psi[h_{ab}]$ of the 3-metric h_{ab} , must be annihilated by these constraints:

$$H_{GR}|\Psi_{phys}\rangle \equiv 0 \quad (2)$$

While this procedure is straightforward in principle, in practice it was impossible to implement in the quantum theory due to highly complicated non-polynomial dependence of the diffeomorphism and Hamiltonian constraints on the configuration and momentum variables. This unfortunate state of affairs persisted until the 1980s when Abhay Ashtekar recast general relativity as a theory of a Minkowski tetrad E_μ^I and a self-dual $\mathfrak{sl}(2, \mathbb{C})$ connection A_μ^{IJ} . Ashtekar realised that the constraints of general relativity, when expressed in terms of these “new variables”, simplified drastically and became polynomial functions of the tetrad and connection variables. It was quickly realised by various researchers that the resulting theory could be quantized using the same methods used to quantize Yang-Mills gauge theory and that the diffeomorphism constraint could be solved exactly in terms of so-called *spin-network* states.

Following this, work by Rovelli and Smolin and by Ashtekar, Rovelli and Smolin demonstrated the most remarkable feature of this theory, which came to be known as “Loop Quantum Gravity”, was that one could construct quantum operators for geometric quantities such as areas of two dimensional surfaces and volumes of three dimensional regions. Moreover, these operators could be diagonalized exactly in the spin-network basis and a lower bound on the smallest possible quantum of area and quantum of volume could be derived. This was the first time that physicists had discovered the “atoms of space” or the “quanta of geometry” in the true sense of the expression.

Insert brief intro to spin nets

¹ This viewpoint has been most clearly articulated by Jacobson [1]

² the characterization of LQG as a “major” approach is the our choice and not necessarily reflective of the consensus in the broader quantum gravity community.

³ Our notational convention is the following. Lowercase Greek letters from the middle of the alphabet μ, ν, \dots are spacetime indices which run from $(0 \dots 3)$, whereas lowercase Roman letters a, b, \dots are spatial indices for quantities which live solely on the surfaces Σ_t and take values in $(1, 2, 3)$

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II. STRINGS IN BACKGROUND FIELDS AND EMERGENT GRAVITY

Let us begin by recalling the Polyakov action for the bosonic string.

$$S_P = -\frac{T}{2} \int d\tau \wedge d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \quad (3)$$

Here $\mu, \nu \in \{0, 1, \dots, D-1\}$ are co-ordinates of the D dimensional worldvolume (the geometry in which the string propagates), $a, b \in \{0, 1\}$ are co-ordinates on the string worldsheet, X^μ are the embedding co-ordinates which specify the location of a point on the string worldsheet in the bulk worldvolume, $\eta_{\mu\nu}$ is the flat Minkowski metric on the worldvolume, g_{ab} is the metric on the string worldsheet, T is the string tension and τ, σ are the co-ordinates on the string worldsheet.

Now, one proceeds in the usual way by determining the symmetries of the action (81), varying the action to find the equations of motion, fixing the gauge using the Weyl freedom of the worldsheet and then solving the classical equations of motion. Imposition of (bosonic) commutation relations on the operator versions of the embedding fields \hat{X}^μ then leads us to description of the quantum state of the bosonic string in terms of an infinite ladder of harmonic oscillators which obey the Virasoro algebra.

The obvious drawback of this approach is that the background metric is non-dynamical and is fixed to be the flat Minkowski metric $\eta_{\mu\nu}$. Clearly, one would like to be able to understand the physics of a string propagating on an arbitrary curved background. It wouldn't make much sense to refer to string theory as a theory of "quantum gravity" if strings can only be described on flat backgrounds. The way this is accomplished is by treating the metric of the worldvolume as a "background field" $G_{\mu\nu}$, in terms of which the Polyakov action becomes:

$$S'_P = -\frac{T}{2} \int d\tau \wedge d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X), \quad (4)$$

where the bulk metric is now a function of the bulk co-ordinates $G_{\mu\nu}(X)$. Now this metric is still non-dynamical because the action (4) does not contain any terms with time derivatives of $G_{\mu\nu}$. However, if we view the action purely as a two-dimensional theory of D scalar fields, then the bulk metric can be viewed as a collection of *coupling constants*, rather than a dynamical entity which exists independent of the string. One can now proceed in the usual manner for any field theory and calculate the beta function of the coupling constants of the theory as a function of the energy scale.

Then, as shown long ago by Friedan [2] (see also [3], [4] or [5, Sec 3.7], [6, Sec 7.2] for a more pedagogical explanation), that the graviton beta function is proportional to $R_{\mu\nu}$, the Ricci curvature of the bulk geometry. The requirement of Weyl invariance of the string worldsheet

implies that this beta function should vanish:

$$\beta(G_{\mu\nu}) \propto R_{\mu\nu} = 0. \quad (5)$$

Therefore we find that conformal invariance of the string worldsheet *implies* that the background geometry in which the string propagates satisfies the *vacuum* Einstein equations. It is this result which is often cited as evidence for the claim that string theory is a *background independent* theory of quantum gravity.

In our opinion, however, this result is insufficient evidence for asserting that string theory is background independent. Our reasons are the following:

1. While the conformal invariance of the string worldsheet implies that the background is Ricci flat, this, in itself, is not very surprising. After all conformal invariance of worldsheet is equivalent to the statement of diffeomorphism invariance of the worldsheet, and in order for this to be consistent with the bulk geometry, the bulk geometry should also have a description in terms of diffeomorphism invariant physics. The simplest form of this requirement is that the background should satisfy the vacuum Einstein equations .
2. One can ask what is the quantum mechanical description of the bulk geometry? It is often suggested that the bulk geometry corresponds to a coherent state or a graviton condensate, where the graviton are themselves excitations of the fundamental string. However, there does not appear to be an actual mathematical realization of this statement in terms of non-perturbative physics.
3. Finally, as mentioned in the introduction, in any *quantum* theory of gravity one would expect to be able to describe the "atoms of geometry". There is no sense in which this is realised in this picture of gravity emerging from stringy physics. Even though one can calculate higher order corrections to the graviton beta function and this provides us with the quantum corrected effective action for the bulk geometry, all the physical quantities are still defined in terms of continuous fields such as the metric. For *e.g.* , to the best of our knowledge, there is no clear sense in which one construct a *non-perturbative* state which corresponds to a superposition of geometries using the quantum theory of a bosonic string alone.

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Despite all the concerns string theory has emerged as the primary candidate for a theory of quantum gravity. There are good reasons for this. AdS/CFT - the most concrete version of a *non-perturbative* theory of quantum gravity - arose out of string theory. AdS/CFT is also far from being a complete quantum theory of quantum gravity. Our universe, for instance, is described by a positive cosmological constant $\Lambda > 0$ rather than $\Lambda < 0$ as

is implied by the “Anti-deSitter” part of the correspondence. However, the correspondence has passed numerous checks and is now considered part of the established canon. Moreover, it has led to numerous theoretical insights. We have gained a deep understanding of the properties of real world systems referred to as “strange metals” by studying their dual gravitational theories. The significance of quantum error correction and quantum information for a theory of quantum gravity was also first most clearly illuminated via AdS/CFT⁴. For these, are other reasons too numerous to mention here, most theorists still regard string theory as the primary (or even the sole) candidate for a theory of quantum gravity.

On the other hand LQG is also built upon a rigorous mathematical framework which follows the well-established rules of quantization⁴. Moreover it allows us to construct a non-perturbative framework for quantum geometry which makes it possible to probe regimes which do not appear to be accessible via string theory.

Thus, we are now at an impasse. We have two theories, both built upon rigorous mathematical foundations, but which do not appear to be concordant with each other. There are two possibilities. The first is that one of the two approaches is fundamentally flawed in some way and must be rejected. The second is that we are missing some crucial ingredient which would allow us to find a missing link between the two theories. We feel that it worthwhile to explore the second option in greater detail, if for no other reason then to rule out the possibility of such a link between these two theories.

Before proceeding let us state our “central dogma”. We take as physically well-motivated the central result of LQG that at the Planck scale well-defined geometric operators - such as those which measure the area and volume of a given region of space - exist, have discrete spectra and are gapped, *i.e.*, the smallest eigenvalue is greater than zero. We now wish to understand how to construct a worldsheet action which will reflect this central dogma.

III. MODIFIED NAMBU GOTO ACTION

In a very real sense string theory is a quantum theory of geometry. After all, the Nambu-Goto action:

$$S_{NG} = -T \int d^2x \sqrt{-\det h} \quad (6)$$

is nothing more than the *area* of the string worldsheet. So when we are constructing the quantum theory of the string we are constructing a theory in which the fundamental excitations are geometric in nature. What we

would like to do is to modify the string action in a way which incorporates the insight from LQG that at the Planck scale there is a minimum quantum of area. One simple way in which to do this is to modify the integrand in (6) as follows:

$$\sqrt{-h} \rightarrow \sqrt{-(h + g\Delta)}.$$

Here, $g = l_{pl}/l_s$, where l_{pl} is the Planck scale, the scale at which quantum geometric effects become important; l_s is the string scale where quantum geometric effects due to area quantization become small; Δ is a constant which corresponds to the square of the minimal quantum of area. In the limit that $l_s \gg l_{pl}$, $g \ll 1$ and the string action reduces to the conventional Nambu-Goto action with small corrections coming from quantum geometry. In this limit we can expand the integrand as follows:

$$\sqrt{-(h + g\Delta)} \sim \sqrt{-h} \left(1 - \frac{1}{2} \frac{g\Delta}{(-h)} + \mathcal{O}(g^2/h^2) \right). \quad (7)$$

Presumably the limit $g \ll 1$ corresponds to when the area of the string worldsheet is large compared to $\sqrt{h} \gg \Delta$. Thus in this limit $g^2 \ll 1$ and $h^2 \gg 1$ because of which we can drop the $\mathcal{O}(g^2/h^2)$ corrections. The final area corrected Nambu-Goto action becomes:

$$S'_{NG} = -T \int d^2x \left(\sqrt{-h} - \frac{1}{2} \frac{g\Delta}{\sqrt{-h}} \right). \quad (8)$$

This action has, to the best of our knowledge, not been explored in the string theory literature. However, such an inverse area term arises very naturally in the LQG approach and goes by the name of “inverse triad” corrections, in the context of the 3D volume of spatial hypersurfaces.

The action (8) has some interesting properties. First, on the face of it it does not appear to satisfy diffeomorphism invariance of the string worldsheet because of the second term. Given that we are assuming the existence of a minimum quantum of area, it is obvious that diffeomorphism invariance, at least in the classical sense, will no longer be an exact symmetry of the theory. However, if we look at that expression we see that it can be written as:

$$S'_{NG}[h_{ab}] = -T \int d^2x \left(\sqrt{-h} - \frac{1}{2} g\Delta \sqrt{-(h^{-1})} \right), \quad (9)$$

where h^{-1} is now the determinant of the *inverse* metric h^{ab} . So, while the second term is not diffeomorphism invariant *w.r.t.* the original metric h_{ab} it is clearly diffeomorphism invariant *w.r.t.* the inverse metric h^{ab} .

Second, in the expression (8) we can see a duality between the first and second terms. If we replace the metric h_{ab} by its inverse $h'_{ab} \equiv h^{ab}$, then the action in terms of h'_{ab} is the same as (8), but with the difference that the

⁴ Though there are those who disagree with this assertion

factor of $g\Delta/2$ is mapped to $2/(g\Delta)$:

$$\begin{aligned} S'_{NG}[h'_{ab}] &= -T' \int d^2x \left(\sqrt{-h'} - \frac{2}{g\Delta} \sqrt{-(h')^{-1}} \right) \\ &= -T' \int d^2x \left(\sqrt{-h'} - \frac{1}{2} g' \Delta' \sqrt{-(h')^{-1}} \right), \end{aligned} \quad (10)$$

which has the same form as the original action (8), as long as make the replacements $T \rightarrow T' = \frac{1}{2}g\Delta T$ and $\frac{1}{2}g\Delta \rightarrow \frac{1}{2}g'\Delta' = \frac{2}{g\Delta}$. This is a duality between the physics at the Planck scale and the physics at the string scale, *i.e.* between strong coupling ($g \gg 1$) and weak coupling ($g \ll 1$), small quantum of area (Δ) and large quantum of area (Δ') and between large string tension T and small string tension T' .

Now, one might object and say that having an action which is the sum of Nambu-Goto actions for two different metrics seems strange. However, such actions have been extensively studied under the heading of “bimetric gravity”, with the earliest work dating as far back as 1940 in two papers [7, 8] by Einstein’s future collaborator Nathan Rosen (the ‘R’ in ‘EPR’ and ‘ER’). In 2010, bimetric gravity gained great popularity due to the seminal paper [9] which showed that by introducing a second *reference* metric into a model for massive gravity one can get rid of the Boulware-Deser ghost [10] which otherwise plagues theories with massive gravitons. The *reference* metric introduced in the work [9] was taken to be a flat metric. However later work [11, 12] showed that the second metric need not be flat and that an action of the following form:

$$\begin{aligned} S &= M_g^2 \int d^4x \sqrt{-g} R^{(g)} + M_f^2 \int d^4x \sqrt{-f} R^{(f)} \\ &\quad + 2m^2 M_{eff}^2 \int d^4x \sqrt{-g} \sum_{n=0}^4 \beta_n e_n(\sqrt{g^{-1}f}), \end{aligned} \quad (11)$$

where $g_{\mu\nu}$ and $f_{\mu\nu}$ are *arbitrary* metrics, M_g^2, M_f^2 are the Planck masses for the two sectors respectively. $R^{(g)}, R^{(f)}$ are the respective Ricci scalars for each sector. The third term is an interaction term which is responsible for making one of the gravitons massive while the other remains massless. M_{eff}^2 is an *effective* Planck mass given by:

$$M_{eff}^2 = \left(\frac{1}{M_g^2} + \frac{1}{M_f^2} \right)^{-1} \quad (12)$$

A. Modified Equations of Motion

Following the procedure done in [13], the correction(8) does not change the form of the equations of motion and neither of the boundary conditions, *i.e.* they are still given by

$$\text{EoM} : \partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma = 0 \quad (13)$$

and

$$\text{B.C.} : \mathcal{P}_\mu^\sigma \delta X^\mu \big|_{\sigma=0}^{\sigma=\sigma_1} = 0, \quad (14)$$

where

$$\begin{aligned} \mathcal{P}_\mu^\tau &= \frac{\partial \mathcal{L}'_{NG}}{\partial \dot{X}^\mu} \\ &= -T \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} \left(1 + \frac{g\Delta}{2(-h)} \right) \\ &= \mathcal{P}_{\mu(NG)}^\tau \left(1 + \frac{g\Delta}{2(-h)} \right) \end{aligned} \quad (15)$$

$$\begin{aligned} \mathcal{P}_\mu^\sigma &= \frac{\partial \mathcal{L}'_{NG}}{\partial X'^\mu} \\ &= -T \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} \left(1 + \frac{g\Delta}{2(-h)} \right) \\ &= \mathcal{P}_{\mu(NG)}^\sigma \left(1 + \frac{g\Delta}{2(-h)} \right) \end{aligned} \quad (16)$$

and $\mathcal{P}_{\mu(NG)}^\tau, \mathcal{P}_{\mu(NG)}^\sigma$ are the regular Nambu-Goto world-sheet currents.

Using the static gauge $\tau = t$ and the transverse gauge with $\frac{\partial \vec{X}}{\partial s} \cdot \frac{\partial \vec{X}}{\partial t} = 0$, where s is the length along the string parameter, we find that the new form for the conserved energy per infinitesimal piece of the string is the regular energy of the Nambu-Goto string times the correction factor

$$Ed\sigma = T\gamma_{v\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) ds, \quad (17)$$

and the form of the wave equation maintains its shape as in

$$\mu_{eff} \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial}{\partial s} \left[T_{eff} \frac{\partial \vec{X}}{\partial s} \right] = 0, \quad (18)$$

where the effective mass density and effective tension are given by

$$\mu_{eff} = \mu\gamma_{v\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \quad (19)$$

$$T_{eff} = \frac{T}{\gamma_{v\perp}} \left(1 + \frac{g\Delta}{2(-h)} \right). \quad (20)$$

Rewriting eq(18) in terms of σ derivatives and parameterizing σ as (let $F^+ = 1 + g\Delta/2(-h)$)

$$\sigma(q) = \frac{1}{T} \int_0^q dE \frac{1}{(F^+)^2} = \frac{E(q)}{T(F^+)^2}, \quad (21)$$

we find the wave equation

$$(F^+)^2 \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial^2 \vec{X}}{\partial \sigma^2} = 0, \quad (22)$$

from which we can see that the correction factor modifies the speed of propagation of the wave in the string as

$$v = \frac{c}{F^+}. \quad (23)$$

IV. INVERSE METRIC POLYAKOV

In the same way we considered the quantum geometry inverse metric correction to the Nambu-Goto action, it is also natural to consider such correction to the Polyakov string. we start with the action(4) and proceed with the inverse area correction(7), which yields our corrected Polyakov action

$$S_{IAP} = -\frac{T}{2} \int d^2x \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \times \\ \times g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}. \quad (24)$$

The equations of motion for the world-sheet metric are completely unchanged by this correction, so they are

$$\frac{\delta S_{IAP}}{\delta g^{cd}} \stackrel{!}{=} 0 \quad (25)$$

from which we obtain the stress-energy tensor for the worldsheet:

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} - \\ - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = 0, \quad (26)$$

from which we can extract the metric g_{cd} as

$$g_{cd} = 2f \partial_c X^\mu \partial_d X^\nu G_{\mu\nu}, \quad (27)$$

where the function f is given by

$$\frac{1}{f} = g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}. \quad (28)$$

As for variation w.r.t the embedding fields X^λ , we have

$$\frac{\delta S_{IAP}}{\delta X^\lambda} \stackrel{!}{=} 0 \quad (29)$$

$$F^- \sqrt{-g} g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} + \partial_a (F^- \sqrt{-g} g^{ab}) \partial_b X^\nu G_{\lambda\nu} = \\ = \frac{1}{2} F^- \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu}, \quad (30)$$

where $F^- = 1 - k\Delta/2(-g)$. Using conformal gauge $g^{ab} = (\phi(x))^{-1} \eta^{ab}$ and assuming flat background $G_{\mu\nu} = \eta_{\mu\nu}$, eq(30) simplify to

$$\eta^{ab} \partial_a \partial_b X^\mu + \eta^{ab} \partial_a \ln(F^-) \partial_b X^\mu = 0. \quad (31)$$

A. Simplifying The Equations of Motion

Equation(31) seems a bit odd, but we can polish it by choosing a plane-wave ansatz for the embedding fields $X^\mu = X_0^\mu e^{-i(E\tau - p\sigma)}$ and using it to fix a form for the conformal factor $\phi(x)$ appearing in F^- . By using the ansatz in eq(31), it becomes

$$(E^2 - p^2) X^\mu + i(E\partial_\tau \ln(F^-) - p\partial_\sigma \ln(F^-)) X^\mu = 0, \quad (32)$$

from which we can use the freedom in the conformal function to fix the second term as $-\mu^2 X^\mu$, with $-\mu^2 = p^2 - E^2$ on-shell, and this in turn implies that the derivatives of $\ln(F^-)$ are given by

$$\partial_\tau \ln(F^-) = iE, \quad \partial_\sigma \ln(F^-) = ip, \quad (33)$$

thus we have

$$\ln(F^-) = i(E\tau + p\sigma) + a, \quad a \in \mathbb{C} \quad (34)$$

$$F^- = A e^{i(E\tau + p\sigma + \varphi)}, \quad A > 0 \quad (35)$$

$$\phi(\tau, \sigma) = \sqrt{\frac{k\Delta}{2(1 - A e^{i(E\tau + p\sigma + \varphi)})}}. \quad (36)$$

At first glance it seems problematic that we have a complex conformal function $\phi(x)$, but the complex structure comes from our ansatz for the embedding fields, and when we choose either the real or imaginary part of the latter it should also pick the respective component of ϕ . With this choice, we thus turned eq(31) into a Klein-Gordon equation

$$(\partial_\sigma^2 - \partial_\tau^2) X^\mu - \mu^2 X^\mu = 0. \quad (37)$$

B. Relation With The Nambu-Goto Analysis

At first glance, our recent analysis of the Polyakov string with inverse metric correction seems to not be related to what we found in the analysis of the Nambu-Goto string. However, as pointed out in [14], wave equations with a variable speed (the result of the Nambu-Goto analysis) can be turned into Klein-Gordon equations (what we just found). While their work deals only with space-varying wave speed, it is straightforward to generalise the procedure to also include time dependence.

Following the procedure found in [14], we start with eq(22) obtained in sec(III A), renaming σ as x and dividing by $(F^+)^2$ such that we have v^2 on the second term like the mentioned work, and perform a change of variables $(t, x) \rightarrow (\tau(t, x), \sigma(t, x))$, where without loss of generality (one needs just to rescale and/or rotate the coordinates) we set

$$(\partial_t \sigma)^2 - v^2 (\partial_x \sigma)^2 = v^2 (\partial_x \tau)^2 - (\partial_t \tau)^2 \\ \partial_t \tau \partial_t \sigma = v^2 \partial_x \tau \partial_x \sigma, \quad (38)$$

or more compactly,

$$\begin{aligned} \partial_t \tau &= v \partial_x \sigma \\ \partial_t \sigma &= v \partial_x \tau \end{aligned}, \quad (39)$$

which is equivalent to

$$\begin{aligned} \partial_t^2 \tau - v^2 \partial_x^2 \tau &= \partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau \\ \partial_t^2 \sigma - v^2 \partial_x^2 \sigma &= \partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma. \end{aligned} \quad (40)$$

This turns eq(22) into

$$\begin{aligned} \partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu + \\ + \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\tau X^\mu + \\ + \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\sigma X^\mu = 0. \end{aligned} \quad (41)$$

Next, we introduce $X^\mu = \kappa(\tau, \sigma) W^\mu(\tau, \sigma)$ such that terms proportional to first derivatives of W^μ cancel out. This is obtained by choosing $\kappa = v^{1/2}$ and concluding that v only depends on x . With this, eq(41) turns into a Klein-Gordon equation

$$\partial_\tau^2 W^\mu - \partial_\sigma^2 W^\mu + \mu^2 W^\mu = 0, \quad (42)$$

with “mass” squared given by

$$\mu^2 = \frac{(\partial_x v)^2 - 2v \partial_x^2 v}{4((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \quad (43)$$

and $\tau(t, x)$ satisfying constraints(39). This shows that our action(24) is indeed in agreement with the inverse area corrected Nambu-Goto action(8).

C. Solving The String Klein-Gordon Equation

From our analysis of the inverse-area corrected Polyakov action, we got D finite-space one-dimensional Klein-Gordon equations (37), with constraints given by eq(26). The constraints are nothing new, they’re precisely the same as the ones of regular String Theory. What’s new in our analysis is the μ^2 term, which makes it so the solution is given by Fourier transform

$$\begin{aligned} X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{p \in \mathbb{Z}} \frac{1}{2E_p} \left(\alpha_p^\mu e^{i(-E_p \tau + p\sigma)} + \right. \\ \left. + \beta_p^\mu e^{i(E_p \tau + p\sigma)} \right), \end{aligned} \quad (44)$$

with $E_p = \sqrt{p^2 + \mu^2}$. From here on, following an analogous procedure to the one found in [15], reality of X^μ implies $(X^\mu)^* = X^\mu$, from which we get

$$\beta_n^\mu = (\alpha_{-n}^\mu)^*. \quad (45)$$

When looking at the constraints, we see that we can’t decouple the exponentials in τ from any of the sums, thus we get two-index “Virasoro modes”

$$L_{n,p} := \frac{n-p}{E_{n-p}} \alpha_p^\mu \alpha_{n-p}^\nu \eta_{\mu\nu} = 0 \quad (46)$$

$$\tilde{L}_{n,p} := \frac{n-p}{E_{n-p}} \alpha_p^\mu (\alpha_{p-n}^\nu)^* \eta_{\mu\nu} = 0. \quad (47)$$

Before proceeding to quantization, we have one last remark: the 0-th term of the solution reads

$$\frac{1}{2\pi\alpha'} \frac{1}{2\mu} (\alpha_0^\mu e^{-i\mu\tau} + (\alpha_0^\mu)^* e^{i\mu\tau}), \quad (48)$$

which is clearly divergent in the $\mu \rightarrow 0$ limit unless $\Re(\alpha_0^\mu) = 0$. Moreover, if we want to recover the classic String Theory expression in this regime, it is not difficult to see that we require

$$\alpha_0^\mu = 4\pi i (\alpha')^2 p^\mu. \quad (49)$$

Since our Virasoro-like “modes” are double-indexed without summation, the string rest mass M^2 is given only in terms of the 0-th oscillator mode

$$M^2 = -p^\mu p_\mu = \frac{1}{16\pi^2 (\alpha')^4} \alpha_0^\mu \alpha_0^\nu \eta_{\mu\nu}. \quad (50)$$

Collecting our results, we have that the solution for our string is given by

$$\begin{aligned} X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{p \in \mathbb{Z}} \frac{1}{2E_p} \left(\alpha_p^\mu e^{i(-E_p \tau + p\sigma)} + \right. \\ \left. + (\alpha_{-p}^\mu)^* e^{i(E_p \tau + p\sigma)} \right) \end{aligned} \quad (51)$$

subject to the infinite family of constraints

$$L_{n,p} = \tilde{L}_{n,p} = 0 \quad (52)$$

and 0-th oscillation mode related to center of mass momentum

$$\alpha_0^\mu = 4\pi i (\alpha')^2 p^\mu.$$

V. THE QUANTUM KLEIN-GORDON STRING

In the following, we proceed to quantize the Klein-Gordon string via canonical covariant quantization, in a manner much similar to those found in [15] and [16]. To start, we calculate the momentum canonical to the embedding fields X^μ , which reads

$$\begin{aligned} \Pi_\mu(\tau, \sigma) = F^- \frac{i}{(4\pi\alpha')^2} \sum_p \left(-\alpha_p^\nu e^{i(-E_p \tau + p\sigma)} + \right. \\ \left. + (\alpha_{-p}^\nu)^* e^{i(E_p \tau + p\sigma)} \right) \eta_{\nu\mu}, \end{aligned} \quad (53)$$

and then we proceed to turn the equal-time Poisson bracket relations

$$\{X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} = \delta(\sigma - \sigma') \eta^{\mu\nu} \quad (54)$$

$$\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = 0 = \{\Pi^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} \quad (55)$$

into equal-time commutation relations between the now operators \hat{X}^μ and $\hat{\Pi}^\nu$:

$$[\hat{X}^\mu(\tau, \sigma), \hat{\Pi}^\nu(\tau, \sigma')] \equiv i\delta(\sigma - \sigma') \eta^{\mu\nu} \quad (56)$$

$$[\hat{X}^\mu(\tau, \sigma), \hat{X}^\nu(\tau, \sigma')] = 0 = [\hat{\Pi}^\mu(\tau, \sigma), \hat{\Pi}^\nu(\tau, \sigma')]. \quad (57)$$

From these relations, we get commutation relations for the vibrational modes $\hat{\alpha}_p^\mu$ and $(\hat{\alpha}_{-k}^\nu)^\dagger$:

$$[\hat{\alpha}_p^\mu, (\hat{\alpha}_{-k}^\nu)^\dagger] = \eta^{\mu\nu} \delta_{p,-k} \quad (58)$$

$$[\hat{\alpha}_p^\mu, \hat{\alpha}_k^\nu] = 0 = [(\hat{\alpha}_{-p}^\mu)^\dagger, (\hat{\alpha}_{-k}^\nu)^\dagger], \quad (59)$$

which are just harmonic oscillator relations with

$$\hat{a}_p^\mu = \hat{\alpha}_p^\mu \quad (60)$$

$$(\hat{a}_p^\mu)^\dagger = (\hat{\alpha}_p^\mu)^\dagger, \text{ for } p \neq 0. \quad (61)$$

For $p = 0$ we have the center of mass momentum operator \hat{p}^μ . As usual, we define a vacuum state for the string to obey

$$\hat{a}_p^\mu |0\rangle = 0, \text{ for } p \neq 0, \quad (62)$$

and for $p = 0$

$$\hat{p}^\mu |0\rangle = p^\mu |0\rangle. \quad (63)$$

Our Fock space is built from the vacuum state $|0\rangle$ by operating with a sequence of creation operators

$$|\psi\rangle = ((a_1^{\mu_1})^\dagger)^{n_{\mu_1}} ((a_2^{\mu_2})^\dagger)^{n_{\mu_2}} \dots \times ((a_1^{\nu_1})^\dagger)^{n_{\nu_1}} ((a_2^{\nu_2})^\dagger)^{n_{\nu_2}} \dots |0\rangle. \quad (64)$$

As usual, by the appearance of the Minkowski metric in the non-zero commutator

$$[\hat{a}_p^\mu, (\hat{a}_k^\nu)^\dagger] = \eta^{\mu\nu} \delta_{pk}, \quad (65)$$

we have ghosts in the theory which must be dealt with. When translating the constraints as quantum operators, we require that they have vanishing matrix elements when sandwiched by physical states ψ and ϕ :

$$\langle \phi | \hat{L}_{p,k} | \psi \rangle = 0 \quad (66)$$

$$\langle \phi | \hat{\tilde{L}}_{p,k} | \psi \rangle = 0. \quad (67)$$

$\hat{L}_{p,k}$ has no ambiguities since it's composed of only annihilation operators, however, we have an ambiguity for $\hat{\tilde{L}}_{p,0}$. To resolve this issue, we pick

$$\hat{\tilde{L}}_{p,0} = \frac{-p}{E_{-p}} (\hat{\alpha}_p^\mu)^\dagger \hat{\alpha}_p^\nu \eta_{\mu\nu} \quad (68)$$

with the annihilation operator to the right, which manifests in the imposition of the constraint as

$$\langle \phi | \hat{\tilde{L}}_{p,0} - c_p | \psi \rangle = 0, \quad (69)$$

for some sequence c_p . The reason for it to be a sequence of p and not a constant like in the regular String Theory literature is that in the usual theory, we would have a

sum over p and thus it would end up being a constant, but in our analysis we have this extra freedom, which we'll now see ends up being important. Classically, we had eq(50) for the string rest mass, which we can rewrite as

$$M^2 = -\frac{1}{(4\pi(\alpha')^2)^2} \lim_{n \rightarrow 0} \left(-\frac{E_p}{p} \tilde{L}_{p,0} \right), \quad (70)$$

from where we can now see that the string mass spectrum will get shifted by

$$\hat{M}^2 = -\frac{1}{(4\pi(\alpha')^2)^2} \left((\hat{\alpha}_0^\mu)^\dagger \hat{\alpha}_0^\nu \eta_{\mu\nu} + \lim_{p \rightarrow 0} \frac{E_p}{p} c_p \right). \quad (71)$$

Since $E_0 = \mu$, it seems like the natural candidate for c_p would be some function $f(\mu, p)$ such that

$$\lim_{p \rightarrow 0} \frac{c_p}{p} = \mu \quad (72)$$

like $c_p = \mu p$, $c_p = \sin(\mu p)$, $c_p = \tan(\mu p)$, etc, making it so the mass spectrum of the string gets shifted by the “mass” term of the Klein-Gordon equation (37), μ^2 .

The “Virasoro-Klein-Gordon” algebra our constraint modes satisfy is given by

$$[\hat{L}_{n,m}, \hat{L}_{p,k}] = 0 \quad (73)$$

$$[\hat{L}_{n,m}, \hat{\tilde{L}}_{p,k}] = \frac{m-n}{E_{m-n}} \left(\hat{L}_{n+k-p,n} \delta_{n-m,k} + \hat{L}_{n-m+k-p,n-m} \delta_{n,k} \right) \quad (74)$$

$$\begin{aligned} [\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{p,k}] &= \frac{(m-n)}{E_{m-n}} \frac{(k-p)}{E_{k-p}} \times \\ &\times \left(\frac{E_{-n}}{(-n)} \hat{\tilde{L}}_{k-p-n,k-p} \delta_{n-m,k} + \frac{E_{-k}}{(-k)} \hat{\tilde{L}}_{n-m-k,n-m} \delta_{n,k-p} \right) \end{aligned} \quad (75)$$

VI. THE BIMETRIC STRING

It would be natural to suspect that the action (9) might be related to bimetric gravity in the target space of the string. We can see that this is indeed the case as follows.

Consider a string propagating in a target space whose geometry is described by two bulk metrics. There are two ways that a string with two metrics might arise. One possibility is that there are two sets of embedding fields X^μ and Y^μ . As a result, the induced metrics on the string worldsheet would also be different for both sets of embedding fields. However, in this case, we would not be talking about *one* string but two *different* strings

described by the respective embedding functions. One could then proceed along the lines of (11) and introduce a third term for the worldsheet action which describes a coupling between the two sets of embedding fields, of the form:

$$S_{int} = -T_{eff} \int d^2x \Phi(g, h) \Psi(\mathbf{X}, \mathbf{Y}). \quad (76)$$

Here $\Phi(g, h)$ is some function of the *induced* metrics g_{ab} and h_{ab} on the two worldsheets and $\Psi(\mathbf{X}, \mathbf{Y})$ is some function of the embedding fields. The resulting action would be of the form:

$$S_{BNG} = -T \int d^2x \sqrt{-h} - T' \int d^2x \sqrt{-g} + S_{int}. \quad (77)$$

Here the subscript *BNG* stands for “Bimetric Nambu-Goto”.

The second possibility is the following. Rather than considering two different sets of embedding fields, and consequently two different worldsheets, we can work with one set of embeddings $X^\mu(\tau, \sigma)$ and *two different* bulk metrics $G_{\mu\nu}$ and $H_{\mu\nu}$. As a result, we would now get two different induced metrics on the *same* string worldsheet, given by:

$$g_{ab} = \frac{\partial X^\mu}{\partial x^a} \frac{\partial X^\nu}{\partial x^b} G_{\mu\nu} \quad (78a)$$

$$h_{ab} = \frac{\partial X^\mu}{\partial x^a} \frac{\partial X^\nu}{\partial x^b} H_{\mu\nu}. \quad (78b)$$

We can then write down the combined action for the string worldsheet as:

$$S'_{BNG} = -T \int d^2x \sqrt{-h} - T' \int d^2x \sqrt{-g} + S'_{int}, \quad (79)$$

where now $S'_{int}[g, h, \mathbf{X}]$ is a term describing the interaction between the two metrics living on the *same* worldsheet.

A. Bimetric Polyakov Action

The expression (79) is not ideal because it does not retain any memory of the form of the bulk metrics unless these are included in the interaction term S'_{int} . One way to manifestly exhibit the dependence of the respective worldsheet actions on the two bulk metrics is to work with the Polyakov action.

Following the general procedure, given in [4], we can couple the worldsheet to a bulk metric as follows:

$$S_P = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}, \quad (80)$$

where $G_{\mu\nu}$ is now the bulk or target space metric. Using this form we can now write down the worldsheet Polyakov bimetric action for two bulk metrics:

$$S_{PB} = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{T'}{2} \int d^2x \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}. \quad (81)$$

B. The Bimetric Polyakov Equations

Following the usual procedure such as the one found in [15] with the bulk metric fixed, we find two copies of the string energy-momentum tensor being set to zero, which can be used to find an expression for the induced metrics:

$$\frac{\delta S_{PB}}{\delta g^{cd}} \stackrel{!}{=} 0 \quad (82)$$

$$T_{cd}^{(G)} := G_{\mu\nu} \left(\partial_c X^\mu \partial_d X^\nu - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \right) = 0 \quad (83)$$

$$g_{cd} = 2f^{(G)} \partial_c X^\mu \partial_d X^\nu G_{\mu\nu}, \quad (84)$$

where the function $f^{(G)}$ is given by

$$\frac{1}{f^{(G)}} = g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}, \quad (85)$$

and completely analogously for the $H_{\mu\nu}$ metric,

$$\frac{\delta S_{PB}}{\delta h^{cd}} \stackrel{!}{=} 0 \quad (86)$$

$$T_{cd}^{(H)} := H_{\mu\nu} \left(\partial_c X^\mu \partial_d X^\nu - \frac{1}{2} h_{cd} h^{ab} \partial_a X^\mu \partial_b X^\nu \right) = 0 \quad (87)$$

$$h_{cd} = 2f^{(H)} \partial_c X^\mu \partial_d X^\nu H_{\mu\nu}, \quad (88)$$

with

$$\frac{1}{f^{(H)}} = h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}. \quad (89)$$

As for the embedding fields equation, we get a non-homogeneous wave equation for the coupled metric

$$\frac{\delta S_{PB}}{\delta X^\lambda} \stackrel{!}{=} 0 \quad (90)$$

VII. STRINGS AND QUANTUM GEOMETRY

Let us begin by writing down the action of a string propagating in a background geometry which is specified not in terms of a metric, but in terms of a connection A_μ^{IJ} and vielbein fields E_μ^I . Further, we will require that the bulk connection and the bulk vielbein can be pulled back onto a connection ω_a^{ij} and a dyad e_i^a , respectively, living on the string worldsheet Σ . In order for the bulk connection A_μ^{IJ} , and its pullback ω_a^{ij} on Σ , to have an effect

on the string dynamics we will also promote the embedding fields X^μ to Lie algebra valued vector fields X_I^μ . However, due to the fact that parallel transport of the embedding fields on the worldsheet should only involve the worldsheet connection ω_a^{ij} , we can replace X_I^μ with X_i^μ , and to not lose the physical meaning of embedding fields as string coordinates, we include a new internal world-sheet field v^i , and define the string coordinates to be $\tilde{X}^\mu \equiv X_i^\mu v^i$.

where $e = \det(e_a^i)$ and D_a is the covariant derivative with respect to the worldsheet connection, and is given by:

$$D_a X_i^\mu = \partial_a X_i^\mu - \omega_a^j{}_i X_j^\mu, \quad (92)$$

where $\omega_a^i{}_j$ is the pullback of the bulk connection to the worldsheet.

VIII. PHYSICAL IMPLICATIONS

IX. DISCUSSION

ACKNOWLEDGMENTS

Appendix A

$$S_{PA} = -\frac{T}{2} \int d\tau \wedge d\sigma e e_a^i e^{bi} D_a X_j^\mu v^j \times \\ \times D_b X_k^\nu v^k E_\mu^I E_{\nu I}, \quad (91)$$

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