Sketchboard

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1 Basic concepts

1.1 With extra capital internal index on embbeding fields

Promote embedding fields X^μ to have an internal group index with D values

$$X^\mu \to X^{\mu I} \ , \quad I = 0, ..., d$$

$$g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{IJ} \sim f D_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu}$$
$$D_a X^{\mu I} = \partial_a X^{\mu I} + \omega_{aJ}^{I} X^{\mu J}$$
$$e_a^i e_b^j \eta_{ij} = g_{ab} \to e_a^i e_b^j \eta_{ij} = \mathrm{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ} \ , \quad i,j = 0,1$$
$$g = \det(g_{ab}) \to g = \det(\mathrm{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.2 With 1 extra small internal index on embedding fields

Promote embbeding fields to have a internal group index with 2 values

$$X^\mu \to X^{\mu i} \ , \quad i = 0, 1$$

$$g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{ij} \sim f D_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu}$$
$$D_a X^{\mu i} = \partial_a X^{\mu i} + \omega_{aj}^{i} X^{\mu j}$$
$$e_a^i e_b^j \eta_{ij} = g_{ab} \to e_a^i e_b^j \eta_{ij} = \mathrm{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij}$$
$$g = \det(g_{ab}) \to g = \det(\mathrm{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.3 With 2 extra small indices

Promote embbeding fields X^μ to have two internal indices with 2 values

$$X^\mu \to X^{\mu ij} \ , \quad i,j = 0,1$$

1.4 Without extra index on embbeding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$e_a^i e_b^j \eta_{ij} = g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$g = \det(g_{ab}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int \mathrm{d}\tau \wedge \mathrm{d}\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \ ,$$

2.1 With capital internal index

... and promote partial derivative ∂_a to covariant derivative D_a, giving us our first attempt at modified Polyakov action

$$S_{MP1} = -\frac{T_0}{2} \int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}$$
$$\updownarrow$$
$$\mathscr{L}_{MP1} = -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}$$

2.2 With small internal index

2.2.1 Without extra field, 1 internal index

... and promote derivatives to covariant D_a, giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

$$\updownarrow$$

$$\mathscr{L}_{MP2} = -\frac{T_0}{2} \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

2.2.2 With extra field

..., promote partial derivative to covariant and add extra internal WS field v^i leading us to

$$S_{MP3} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

$$\updownarrow$$

$$\mathscr{L}_{MP3} = -\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

2.2.3 Without extra field, 2 internal indices

... and promote derivatives to covariant D_a,

$$S_{MP4} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma e e_i^a e^{bi} D_a X^{\mu jj'} \eta_{jj'} \eta_{kk'} D_b X^{\nu kk'} E_\mu^I E_{\nu I}$$

$$\mathscr{L}_{MP4} = -\frac{T}{2} e e_i^a e_j^b D_a X^{\mu ik} \eta_{kl} D_b X^{\nu lj} E_\mu^I E_{\nu I}$$

2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) \ .$$

$$\updownarrow$$

$$\mathscr{L}_{DP} = -\frac{T_0}{2} \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) \ .$$

2.4 Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of D-dimensional gamma matrices and bulk spinors

$$S_{LP} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

$$\updownarrow$$

$$\mathscr{L}_{LP} = -\frac{T}{2} \overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

3 EoMs

Start by writing $\det(e) = \frac{1}{2}\varepsilon^{cd}\varepsilon_{mn} e_c^m e_d^n$ and $g_{ab} = g_{ab}^{ij}\eta_{ij}$

3.1 With small internal index

3.1.1 Without extra field, w.r.t e

$$
\begin{aligned}
\frac{\delta S_{MP2}}{\delta e^e_l} &= \frac{D\mathscr{L}_{MP2}}{De^e_l} = \left(\frac{\partial}{\partial e^e_l} - \partial_f \frac{\partial}{\partial(\partial_f e^e_l)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e^m_c e^n_d e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I}\right) = \\
&= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(\eta^{ml}g_{ce}e^n_d e^a_i e^b_j + e^m_c \eta^{nl} g_{de} e^a_i e^b_j + \\
&+ e^m_c e^n_d \delta^a_e \delta^l_i e^b_j + e^m_c e^n_d e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = \\
&= -\frac{T}{2}\left(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}g_{ce}e^n_d e^a_i e^b_j + 2\det(e) e^a_i \delta^b_e \delta^l_j\right) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = \\
&= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}e^k_c e_{ek} e^n_d e^a_i e^b_j + 2\det(e) e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = \\
&= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}e^m_c e^n_d e^l_e e^a_i e^b_j + 2\det(e) e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = \\
&= -T(-\det(e) e^l_e e^a_i e^b_j + \det(e) e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} \stackrel{!}{=} 0
\end{aligned}
$$

$$\downarrow$$

$$T^l_e := E^I_\mu E_{\nu I}(e^a_i D_a X^{\mu i} D_e X^{\nu l} - e^l_e e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j}) = 0$$

$$\Downarrow$$

$$e^l_e = f e^a_i D_a X^{\mu i} D_e X^{\nu l} E^I_\mu E_{\nu I} \ ,$$

$$\frac{1}{f} = e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I}$$

$$\downarrow$$

$$
\begin{aligned}
e^l_e e_{fl} &= (f e^a_i D_a X^{\mu i} D_e X^{\nu l} E^I_\mu E_{\nu I})(f e^{a'}_{i'} D_{a'} X^{\mu' i'} D_f X^{\nu'}_l E^{I'}_{\mu'} E_{\nu' I'}) = \\
&= f^2 D_e X^{\nu l} D_f X^{\nu'}_l E^{I'}_\nu E_{\nu' I'}(e^a_i e^{a'}_{i'} D_a X^{\mu i} D_{a'} X^{\mu' i'} E^I_\mu E_{\mu' I}) = \\
&= f D_e X^{\mu i} D_f X^\nu_i E^I_\mu E_{\nu I}
\end{aligned}
$$

$$\Downarrow$$

$$D_e X^{\mu i} D_f X^\nu_i E^I_\mu E_{\nu I} = 2\partial_e X^\mu \partial_f X^\nu G_{\mu\nu}$$

$$e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\Downarrow$$

$$D_a X^{\mu 0} = -i\partial_a X^\mu$$

$$D_a X^{\mu 1} = \partial_a X^\mu$$

$$\Downarrow$$

$$X^{\mu 0} = -iX^{\mu 1}$$

3.1.2 With extra field, w.r.t e

$$\frac{\delta S_{MP3}}{\delta e^e_o} = \frac{D\mathscr{L}_{MP3}}{De^e_o} = \left(\frac{\partial}{\partial e^e_o} - \partial_f \frac{\partial}{\partial(\partial_f e^e_o)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e^m_c e^n_d \eta^{ij} e^a_i e^b_j D_a(X^\mu_k v^k) D_b(X^\nu_l v^l) E^I_\mu E^J_\nu \eta_{IJ}\right) =$$
$$= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(g_{ce}\eta^{mo}e^n_d\eta^{ij}e^a_i e^b_j + e^m_c g_{de}\eta^{no}\eta^{ij}e^a_i e^b_j +$$
$$+ e^m_c e^n_d \eta^{ij}\delta^a_e \delta^o_i e^b_j + e^m_c e^n_d \eta^{ij} e^a_i \delta^b_e \delta^o_j) D_a(X^\mu_k v^k) D_b(X^\nu_l v^l) E^I_\mu E^J_\nu \eta_{IJ} =$$
$$= -\frac{T}{2}\varepsilon^{cd}\varepsilon_{mn}(e^p_c e_{ep} e^n_d \eta^{mo}\eta^{ij} e^a_i e^b_j + 2\det(e) e^{ao}\delta^b_e) D_a(X^\mu_k v^k) D_b(X^\nu_l v^l) E^I_\mu E^J_\nu \eta_{IJ} =$$
$$= -T(-\det(e) e^o_e \eta^{ij} e^a_i e^b_j + \det(e) e^{ao}\delta^b_e) D_a(X^\mu_k v^k) D_b(X^\nu_l v^l) E^I_\mu E^J_\nu \eta_{IJ} \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{T^o_e := (e^{ao} D_a(X^\mu_k v^k) D_e(X^\nu_l v^l) - e^o_e \eta^{ij} e^a_i e^b_j D_a(X^\mu_k v^k) D_b(X^\nu_l v^l)) E^I_\mu E^J_\nu \eta_{IJ} = 0}$$

$$\Downarrow$$

$$e^o_e = f\, e^{ao} D_a(X^\mu_k v^k) D_e(X^\nu_l v^l) E^I_\mu E^J_\nu \eta_{IJ}\ ,$$
$$\frac{1}{f} = \eta^{ij} e^a_i e^b_j D_a(X^\mu_k v^k) D_b(X^\nu_l v^l) E^I_\mu E^J_\nu \eta_{IJ}$$

3.1.3 With extra field, w.r.t ω

$$\frac{\delta S_{MP3}}{\delta \omega^{mn}_c} = \frac{D\mathscr{L}_{MP3}}{D\omega^{mn}_c} = \left(\frac{\partial}{\partial \omega^{mn}_c} - \partial_d \frac{\partial}{\partial(\partial_d \omega^{mn}_c)}\right)\left(-\frac{T}{2}\det(e)\eta^{ij} e^a_i e^b_j D_a X^\mu_k v^k D_b X^\nu_l v^l E^I_\mu E^J_\nu \eta_{IJ}\right) =$$
$$= -\frac{T}{4} e g^{ab}((-\delta^c_a \delta^{k'}_{[m}\eta_{n]k} X^\mu_{k'} v^k) D_b X^\nu_l v^l + D_a X^\mu_k v^k(-\delta^c_b \delta^{l'}_{[m}\eta_{n]l} X^\nu_{l'} v^l)) G_{\mu\nu} =$$
$$= \frac{T}{4} e(g^{ca} X^\mu_{[m} v_{n]} D_a X^\nu_k v^k + g^{ac} D_a X^\mu_k v^k X^\nu_{[m} v_{n]}) G_{\mu\nu} =$$
$$= \frac{T}{2} e g^{ac} D_a X^\mu_k v^k X^\nu_{[m} v_{n]} G_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{\mathcal{T}^i_{ab} := e^{ci} D_c X^\mu_k v^k X^\nu_{[m} v_{n]} e^m_a e^n_b G_{\mu\nu} = 0}$$

3.1.4 With extra field, w.r.t v

$$\frac{\delta S_{MP3}}{\delta v^l} = \frac{D\mathscr{L}_{MP3}}{Dv^l} = \left(\frac{\partial}{\partial v^l} - \partial_c \frac{\partial}{\partial(\partial_c v^l)}\right)\left(-\frac{T}{2} e e^a_i e^{bi} D_a X^\mu_j v^j D_b X^\nu_k v^k E^I_\mu E_{\nu I}\right) =$$
$$= -\frac{T}{2} g^{ab}(D_a X^\mu_j \delta^j_l D_b X^\nu_k v^k + D_a X^\mu_j v^j D_b X^\nu_k \delta^k_l) G_{\mu\nu} =$$
$$= -T g^{ab} D_a X^\mu_l D_b X^\nu_j v^j G_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{g^{ab} D_a X^\mu_l D_b X^\nu_j v^j G_{\mu\nu} = 0}$$

3.1.5 With extra field, w.r.t X

$$\frac{\delta S_{MP3}}{\delta X^\lambda_l} = \frac{D\mathscr{L}_{MP3}}{DX^\lambda_l} = \left(\frac{\partial}{\partial X^\lambda_l} - \partial_c \frac{\partial}{\partial(\partial_c X^\lambda_l)}\right)\left(-\frac{T}{2} e e^a_i e^{bi} D_a X^\mu_j v^j D_b X^\nu_k v^k E^I_\mu E_{\nu I}\right) =$$

$$= -\frac{T}{2}\Bigg(e g^{ab}\Bigg((-\omega^{j'}_{aj}\delta^\mu_\lambda \delta^l_{j'} v^j D_b X^\nu_k v^k - D_a X^\mu_j v^j \omega^{k'}_{bk}\delta^\nu_\lambda \delta^l_{k'} v^k) E^I_\mu E_{\nu I} +$$

$$+ e g^{ab} D_a X^\mu_j v^j D_b X^\nu_k v^k \left(\frac{\partial E^I_\mu}{\partial X^\lambda_l} E_{\nu I} + E^I_\mu \frac{\partial E_{\nu I}}{\partial X^\lambda_l}\right)\Bigg) -$$

$$- \partial_c (e g^{ab}(\delta^c_a \delta^\mu_\lambda \delta^l_j v^j D_b X^\nu_k v^k + D_a X^\mu_j v^j \delta^c_b \delta^\nu_\lambda \delta^l_k v^k) G_{\mu\nu})\Bigg) =$$

$$= -T\left(- e g^{ab} D_a X^\mu_j v^j \omega^l_{bk} v^k G_{\mu\lambda} + e g^{ab} D_a X^\mu_j v^j D_b X^\nu_k v^k \frac{\partial E^I_\mu}{\partial X^\lambda_l} E_{\nu I} - \partial_c (e g^{ac} v^l D_a X^\mu_j v^j G_{\mu\lambda})\right)$$

$$= -T\left(e g^{ab} D_a X^\mu_j v^j D_b X^\nu_k v^k \frac{\partial E^I_\mu}{\partial X^\lambda_l} E_{\nu I} - D_b(e g^{ab} v^l D_a X^\mu_j v^j E^I_\mu E_{\lambda I})\right) \stackrel{!}{=} 0$$

$$\downarrow$$

$$\boxed{D_b(e e^a_i e^{bi} v^l D_a X^\mu_j v^j E^I_\mu E_{\lambda I}) = e e^a_i e^{bi} D_a X^\mu_j v^j D_b X^\nu_k v^k \frac{\partial E^I_\mu}{\partial X^\lambda_l} E_{\nu I}}$$

3.2 Without extra index

3.2.1 w.r.t e

$$\frac{\delta S_{DP}}{\delta e^e_l} = \frac{D\mathscr{L}_{DP}}{De^e_l} = \left(\frac{\partial}{\partial e^e_l} - \partial_f \frac{\partial}{\partial(\partial_f e^e_l)}\right)\left(-\frac{T_0}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn} e^m_c e^n_d \eta^{ij} e^a_i e^b_j \partial_a X^\mu \partial_b X^\nu E^I_\mu E^J_\nu \eta_{IJ}\right) =$$

$$= -\frac{T_0}{4}$$

3.3 Linear action

3.3.1 w.r.t e

$$\frac{\delta S_{LP}}{\delta \delta e^b_j} = \frac{D\mathscr{L}_{LP}}{De^b_j} = \left(\frac{\partial}{\partial e^b_j} - \partial_e \frac{\partial}{\partial(\partial_e e^b_j)}\right)\left(-\frac{T}{2}\overline{\psi} e e^a_i D_a X^{\mu i} E^I_\mu \gamma_I \psi\right) =$$

$$= -\frac{T}{4}\overline{\psi}\varepsilon^{cd}\varepsilon_{mn}((g_{cb}\eta^{mj} e^n_d + e^m_c g_{db}\eta^{nj}) e^a_i + e^m_c e^n_d \delta^a_b \delta^j_i) D_a X^{\mu i} E^I_\mu \gamma_I \psi =$$

$$= -\frac{T}{2}\overline{\psi}(-e e^j_b e^a_i D_a X^{\mu i} E^I_\mu \gamma_I + e D_b X^{\mu j} E^I_\mu \gamma_I)\psi \stackrel{!}{=} 0$$

$$\downarrow$$

$$T^j_b := \overline{\psi}(D_b X^{\mu j} E^I_\mu \gamma_I - e^j_b e^a_i D_a X^{\mu i} E^I_\mu \gamma_I)\psi = 0$$

$$\downarrow$$

$$e^j_b = F\overline{\psi} D_b X^{\mu j} E^I_\mu \gamma_I \psi,$$

$$\frac{1}{F} = \overline{\psi} e^a_i D_a X^{\mu i} E^I_\mu \gamma_I \psi$$

$$\Downarrow$$

$$\overline{\psi}\psi = 1, \ \psi\overline{\psi} = \mathbb{1}$$

$$e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E^J_\nu \eta_{IJ} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

4 Inverse Area Action

4.1 Nambu-Goto

Start from Nambu-Goto action

$$S_{NG} = -T \int \mathrm{d}^2 x \sqrt{-h},$$

and make quantum geometry correction

$$\sqrt{-h} \to \sqrt{-(h + g\Delta)} \approx \sqrt{-h}\left(1 - \frac{g\Delta}{2(-h)} + \mathcal{O}\left(\frac{g^2}{h^2}\right)\right),$$

leading to modified NG action

$$S_{MNG} = -T \int \mathrm{d}^2 x \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}}\right) = S_{NG} + S_{IA}$$

$$\downarrow$$

$$\mathscr{L}_{MNG} = -T\left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}}\right)$$

4.1.1 EoMs

Let $\mathcal{P}^\tau_\mu \equiv \partial \mathscr{L}_{MNG}/\partial \dot{X}^\mu$ and $\mathcal{P}^\sigma_\mu \equiv \partial \mathscr{L}_{MNG}/\partial X'^\mu$

$$\begin{aligned} \delta S_{MNG} &= \int \mathrm{d}^2 x \left(\frac{\partial \mathscr{L}_{MNG}}{\partial \dot{X}^\mu}\delta \dot{X}^\mu + \frac{\partial \mathscr{L}_{MNG}}{\partial X'^\mu}\delta X'^\mu\right) = \\ &= -\int \mathrm{d}^2 x \left(\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu\right)\delta X^\mu + \int \mathrm{d}\tau \mathcal{P}^\sigma_\mu \delta X^\mu\Big|_{\sigma=0}^{\sigma=\sigma_1} \overset{!}{=} 0 \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} \text{EoM :} &\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0 \\ \text{B.C. :} &\mathcal{P}^\sigma_\mu \delta X^\mu\Big|_{\sigma=0}^{\sigma=\sigma_1} = 0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}^\tau_\mu = \frac{\partial \mathscr{L}_{MNG}}{\partial \dot{X}^\mu} &= -T\left(\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{-h}} + \frac{g\Delta}{2}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{(-h)^{3/2}}\right) = \\ &= -T\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{-h}}\left(1 + \frac{g\Delta}{2(-h)}\right) = \mathcal{P}^\tau_{\mu(NG)}\left(1 + \frac{g\Delta}{2(-h)}\right) \\ \mathcal{P}^\sigma_\mu = \frac{\partial \mathscr{L}_{MNG}}{\partial X'^\mu} &= -T\left(\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} + \frac{g\Delta}{2}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{(-h)^{3/2}}\right) = \\ &= -T\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}}\left(1 + \frac{g\Delta}{2(-h)}\right) = \mathcal{P}^\sigma_{\mu(NG)}\left(1 + \frac{g\Delta}{2(-h)}\right) \end{aligned}$$

gauge fixing static gauge $\tau = t$ and transverse gauge $\frac{\partial X}{\partial \tau}\cdot\frac{\partial X}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}\sigma} = 0$ (s = length along string)

$$\downarrow$$

$$\begin{aligned} \mathcal{P}^{\tau\mu}_{(NG)} &= T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\frac{\partial X^\mu}{\partial t} \\ \mathcal{P}^{\sigma\mu}_{(NG)} &= \frac{T}{\gamma_{v_\perp}}\frac{\partial X^\mu}{\partial s} \\ -h &= \frac{1}{\gamma^2_{v_\perp}}\left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2 \end{aligned}$$

string energy get's redefined to

$$\frac{\partial}{\partial t}\left(T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\right)=0,$$

and for the spatial part we have

$$\partial_\tau\vec{\mathcal{P}}^\tau+\partial_\sigma\vec{\mathcal{P}}^\sigma=0$$

$$\downarrow$$

$$\frac{\partial}{\partial t}\left[T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\frac{\partial\vec{X}}{\partial t}\left(1+\frac{g\Delta}{2(-h)}\right)\right]+\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{\partial}{\partial s}\left[-\frac{T}{\gamma_{v_\perp}}\frac{\partial\vec{X}}{\partial s}\left(1+\frac{g\Delta}{2(-h)}\right)\right]=0$$

$$\downarrow$$

$$\mu\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial}{\partial s}\left[\frac{T}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right)\frac{\partial\vec{X}}{\partial s}\right]=0$$

$\implies$ effective mass density becomes $\mu_{eff}=\mu\gamma_{v_\perp}\left(1-\frac{g\Delta}{2(-h)}\right)$ and effective tension becomes $T_{eff}=\frac{T}{\gamma_{v_\perp}}\left(1-\frac{g\Delta}{2(-h)}\right)$

$$\downarrow$$

$$\mu_{eff}\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial}{\partial s}\left[T_{eff}\frac{\partial\vec{X}}{\partial s}\right]=0$$

$$\begin{aligned}
\mathscr{H}&=\dot{\vec{X}}\cdot\vec{\pi}-\mathscr{L}=\\
&=\vec{v}_\perp\cdot\left(T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\vec{v}_\perp\right)-\left(-T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{1}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right)\right)=\\
&=T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\left(1+\frac{g\Delta}{2(-h)}\right)\left(\gamma_{v_\perp}v_\perp^2+\frac{1}{\gamma_{v_\perp}}\right)=\\
&=T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)
\end{aligned}$$

let $\left(1+\frac{g\Delta}{2(-h)}\right)=F$

$$\frac{\partial^2\vec{X}}{\partial t^2}-\frac{1}{F\gamma_{v_\perp}}\frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\partial}{\partial\sigma}\left[\frac{1}{\gamma_{v_\perp}}F\frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\partial\vec{X}}{\partial\sigma}\right]=0$$

$$\downarrow$$

$$A(\sigma)=\frac{\gamma_{v_\perp}}{F}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\overset{!}{=}1$$

$$\downarrow$$

$$\mathrm{d}\sigma=\frac{\gamma_{v_\perp}}{F}\mathrm{d}s=\frac{1}{TF^2}\mathrm{d}E\implies\sigma(q)=\frac{1}{T}\int_0^q\frac{1}{F^2}\mathrm{d}E$$

$$\downarrow$$

$$F^2\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial^2\vec{X}}{\partial\sigma^2}=0$$

$\implies$ speed of wave on the string gets modified by correction factor $v=c/F$.

4.2 Translation and Lorentz symmetry

Starting with global translations, define the transformation by

$$X^\mu\mapsto X^\mu+\epsilon^\mu.$$

This does not change the action since it only depends on derivatives of X^μ and so $\delta(\partial_a X^\mu)=\partial_a(\delta X^\mu)=\partial_a\epsilon^\mu=0$.

As for Lorentz transformations, the infinitesimal for is

$$X^\mu\mapsto X^\mu+\epsilon^{\mu\nu}X_\nu,$$

with $\epsilon^{\mu\nu}$ anti-symmetric.

4.3 Verifying if there is longitudinal velocity

$$-h = \frac{1}{\gamma_{v_\perp}^2}\left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2 = \frac{F^2}{\gamma_{v_\perp}^4} = F^2\left(1 - v_\perp^2\right)^2$$

$$\downarrow$$

$$-h = \left(1 + \frac{g\Delta}{2(-h)}\right)^2\left(1 - v_\perp^2\right)^2$$

$$\downarrow$$

$$\frac{-h}{\left(1 + \frac{g\Delta}{2(-h)}\right)^2} = \left(1 - v_\perp^2\right)^2$$

$$\downarrow \text{(WolframAlpha)}$$

$$-h = \frac{(1 - v_\perp^2)^2}{3} - f_1 - f_2,$$

where (let $a = \frac{g\Delta}{2}$ and $b = (1 - v_\perp^2)^2$)

$$f_1 = \frac{\sqrt[3]{-27a^2b + 3\sqrt{3}\sqrt{27a^4b^2 + 4a^3b^3} - 18ab^2 - 2b^3}}{3\sqrt[3]{2}}$$

$$f_2 = \frac{(6ab + b^2)}{6f_1}$$

$$\downarrow$$

$$F = 1 + \frac{g\Delta}{2(-h)} = 1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}$$

$$\downarrow$$

$$v = \frac{1}{F} = \frac{1}{1 + \dfrac{g\Delta}{2\left(\dfrac{b}{3} - f_1 - f_2\right)}}$$

5 Bimetric Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

and promote to bimetric action

$$\downarrow$$

$$S_{BP} = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{T'}{2}\int \mathrm{d}^2x\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

5.1 EoMs

$$\begin{aligned}
\frac{\delta S_{BP}}{\delta g^{cd}} &= -\frac{T}{2}\left(\frac{\partial\sqrt{-g}}{\partial g^{cd}}g^{ab} + \sqrt{-g}\frac{\partial g^{ab}}{\partial g^{cd}}\right)\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + 0 = \\
&= -\frac{T}{2}\left(-\frac{1}{2}\sqrt{-g}g_{cd}g^{ab} + \sqrt{-g}\delta^a_{(c}\delta^b_{d)}\right)\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \overset{!}{=} 0
\end{aligned}$$

$$T^{(G)}_{cd} := \left(\partial_c X^\mu \partial_d X^\nu - \frac{1}{2}g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu\right)G_{\mu\nu} = 0$$

$$\Downarrow$$

$$\begin{aligned}
g_{cd} &= 2f^{(G)}\partial_c X^\mu \partial_d X^\nu G_{\mu\nu}, \\
\frac{1}{f^{(G)}} &= g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
\frac{\delta S_{BP}}{\delta h^{cd}} &= \ldots = \\
&= -\frac{T'}{2}\left(-\frac{1}{2}\sqrt{-h}h_{cd}h^{ab} + \sqrt{-h}\delta^a_{(c}\delta^b_{d)}\right)\partial_a X^\mu \partial_b X^\nu H_{\mu\nu} \overset{!}{=} 0
\end{aligned}$$

$$\downarrow$$

$$T^{(H)}_{cd} := \left(\partial_c X^\mu \partial_d X^\mu - \frac{1}{2}h_{cd}h^{ab}\partial_a X^\mu \partial_b X^\nu\right)H_{\mu\nu} = 0$$

$$\Downarrow$$

$$\begin{aligned}
h_{cd} &= 2f^{(H)}\partial_c X^\mu \partial_d X^\nu H_{\mu\nu}, \\
\frac{1}{f^{(H)}} &= h^{ab}\partial_a X^\mu \partial_b X^\nu H_{\mu\nu}
\end{aligned}$$

If the auxiliary metrics g and h concide with the induced metrics from G and H, then $f^{(G)} = f^{(H)} = 1/2$ and the action reduces to

$$S = -T\int \mathrm{d}^2x\sqrt{-g} - T'\int \mathrm{d}^2x\sqrt{-h},$$

where if $T' = -Tk\Delta/2$ and $h = g^{-1}$ we recover the NG area corrected action

$$S = -T\int \mathrm{d}^2x\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right).$$

The condition $h = g^{-1}$ can be achieved by

$$h_{ab} = S_{ac}g^{cd}T_{db},$$

where $S, T \in \mathrm{SL}(2,\mathbb{C})$, so we have a gauge freedom in choosing h_{ab}. Since we require $[h_{ab}] = [h_{ab}]^\dagger$ (h_{ab} must be symmetric and real), this means $[T_{db}] = [S_{db}]^\dagger$ and also that $[S_{ab}]^\dagger = [S_{ab}]$, thus

$$h_{ab} = S_{ac}g^{cd}S_{db}.$$

This restriction constrains our gauge group to just the boosts part of SL$(2,\mathbb{C})$,

$$S = \mathrm{e}^{-n^i\sigma_i\beta}.$$

This also implies

$$\begin{aligned}
\partial_a X^\mu \partial_b X^\nu H_{\mu\nu} &= S_{ac} g^{cd} S_{db} = \\
&= S_{ac}\left(\frac{1}{2g}\varepsilon^{cc'}\varepsilon^{dd'}g_{c'd'}\right)S_{db} = \\
\delta_a^{c'}\delta_b^{d'}\partial_{c'}X^\mu\partial_{d'}X^\nu H_{\mu\nu} &= S_{ac}\left(\frac{1}{2g}\varepsilon^{cc'}\varepsilon^{dd'}\partial_{c'}X^\mu\partial_{d'}X^\nu G_{\mu\nu}\right)S_{db}
\end{aligned}$$

$$\begin{aligned}
\delta_a^{c'}\delta_b^{d'}H_{\mu\nu} &= \frac{1}{2g}S_{ac}\varepsilon^{cc'}S_{db}\varepsilon^{dd'}G_{\mu\nu} \\
H_{\mu\nu} &= -\frac{1}{8g}S_{ac}\varepsilon^{ca}S_{db}\varepsilon^{bd}G_{\mu\nu} \\
H_{\mu\nu} &= \frac{1}{8(-g)}\mathrm{Tr}(S\varepsilon)\mathrm{Tr}(S\varepsilon)G_{\mu\nu} \\
H_{\mu\nu} &= \frac{1}{8(-g)}\mathrm{Tr}(S\varepsilon)^2 G_{\mu\nu}.
\end{aligned}$$

This restricts S to necessarily being not symmetric, otherwise the trace $\mathrm{Tr}(S\varepsilon)$ vanishes and we would get $H_{\mu\nu}=0$.

$$\begin{aligned}
\frac{\delta S_{BP}}{\delta X^\lambda} &= \left(-\frac{T}{2}\sqrt{-g}g^{ab}\partial_\lambda G_{\mu\nu} - \frac{T'}{2}\sqrt{-h}h^{ab}\partial_\lambda H_{\mu\nu}\right)\partial_a X^\mu \partial_b X^\nu - \\
&- \partial_c\left(-\frac{T}{2}\sqrt{-g}g^{ab}G_{\mu\nu} - \frac{T'}{2}\sqrt{-h}h^{ab}H_{\mu\nu}\right)(\delta_a^c\delta_\lambda^\mu\partial_b X^\nu + \partial_a X^\mu \delta_b^c\delta_\lambda^\nu) = \\
&= \partial_a\left(T\sqrt{-g}g^{ab}G_{\mu\lambda} + T'\sqrt{-h}h^{ab}H_{\mu\lambda}\right)\partial_b X^\mu - \\
&- \left(\frac{T}{2}\sqrt{-g}g^{ab}\partial_\lambda G_{\mu\nu} + \frac{T'}{2}\sqrt{-h}h^{ab}\partial_\lambda H_{\mu\nu}\right)\partial_a X^\mu\partial_b X^\nu \stackrel{!}{=} 0
\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}
\partial_a\left(F^{ab}_{\mu\lambda}\partial_b X^\mu\right) &= \frac{1}{2}\partial_\lambda F^{ab}_{\mu\nu}\partial_a X^\mu \partial_b X^\nu, \\
F^{ab}_{\mu\nu} &= T\sqrt{-g}g^{ab}G_{\mu\nu} + T'\sqrt{-h}h^{ab}H_{\mu\nu}.
\end{aligned}$$

Assuming the condition for $h = g^{-1}$ and flat background $G_{\mu\nu} = \eta_{\mu\nu}$, the EoM reduces to

$$\partial_a\left(\left(\sqrt{-g}g^{ab} - \frac{k\Delta}{2(-g)}\sqrt{-g}^{-1}(S^{-1})^{ac}g_{cd}(S^{-1})^{db}\right)\partial_b X^\mu\right) = 0,$$

which in conformal gauge $g_{ab} = \phi\eta_{ab}$ with $(S^{-1})^{ac}\eta_{cd}(S^{-1})^{db} = \eta^{ab}$ reduces further

$$\begin{aligned}
\partial_a\left(\left(1 - \frac{k\Delta}{2\phi^2}\right)\eta^{ab}\partial_b X^\mu\right) &= 0 \\
\eta^{ab}\partial_a\left(F^-\partial_b X^\mu\right) &= 0 \\
F^-\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\partial_a F^-\partial_b X^\mu &= 0 \\
\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\partial_a \ln(F^-)\partial_b X^\mu &= 0.
\end{aligned}$$

This is the exact same equation derived in the next section which leads to the Klein-Gordon eqn. The requirement that $(S^{-1})^{ac}\eta_{cd}(S^{-1})^{db} = \eta^{ab}$ implies

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix},$$

$$S = \begin{bmatrix} \cosh\beta - n^3 \sinh\beta & -(n^1 - in^2)\sinh\beta \\ -(n^1 + in^2)\sinh\beta & \cosh\beta + n^3\sinh\beta \end{bmatrix},$$

$$S^{-1} = \begin{bmatrix} \cosh\beta + n^3 \sinh\beta & +(n^1 - in^2)\sinh\beta \\ +(n^1 + in^2)\sinh\beta & \cosh\beta - n^3\sinh\beta \end{bmatrix}$$

so $n^3 = 0$. Since $(n^1)^2 + (n^2)^2 = 1$, we can parametrize as $n^1 = \cos\theta$ and $n^2 = \sin\theta$, thus S has the form

$$S = \begin{bmatrix} \cosh\beta & -\mathrm{e}^{-i\theta}\sinh\beta \\ -\mathrm{e}^{i\theta}\sinh\beta & \cosh\beta \end{bmatrix},$$

and $\mathrm{Tr}(S\varepsilon) = -2i\sin\theta\sinh\beta$. Let's try to find more restrictions on S:

$$h^{ab} = (S^{-1})^{ac} g_{cd} (S^{-1})^{db}$$
$$h^{ab} = \frac{1}{2g^{-1}} \varepsilon^{aa'} \varepsilon^{bb'} \partial_{a'} X^\mu \partial_{b'} X^\nu \left(\frac{1}{8(-g)} \mathrm{Tr}(S\varepsilon)^2 G_{\mu\nu} \right)$$

$$\Downarrow$$

$$(S^{-1})^{ac} g_{cd} (S^{-1})^{db} = \frac{1}{2g^{-1}} \varepsilon^{aa'} \varepsilon^{bb'} \frac{1}{8(-g)} \mathrm{Tr}(S\varepsilon)^2 \partial_{a'} X^\mu \partial_{b'} X^\nu G_{\mu\nu}$$
$$(S^{-1})^{ac} g_{cd} (S^{-1})^{db} = -\frac{1}{16} \mathrm{Tr}(S\varepsilon)^2 \varepsilon^{ac} \varepsilon^{bd} g_{cd}$$
$$(S^{-1})^{ac} g_{cd} (S^{-1})^{db} = \left(\frac{i}{4} \mathrm{Tr}(S\varepsilon) \varepsilon^{ac} \right) g_{cd} \left(\frac{i}{4} \mathrm{Tr}(S\varepsilon) \varepsilon^{db} \right)$$

$$g_{cd} = S_{ca} \left(\frac{i}{4} \mathrm{Tr}(S\varepsilon)^* \varepsilon^{ac'} \right) g_{c'd'} \left(\frac{i}{4} \mathrm{Tr}(S\varepsilon) \varepsilon^{d'b} \right) S_{bd}$$

$$\Downarrow$$

$$S_{ca} \left(\frac{i}{4} \mathrm{Tr}(S\varepsilon)^* \varepsilon^{ac'} \right) = \delta_c^{c'}$$
$$\left(\frac{i}{4} \mathrm{Tr}(S\varepsilon) \varepsilon^{d'b} \right) S_{bd} = \delta_d^{d'}$$

$$\Downarrow$$

$$(S^{-1})^{ab} = \left(\frac{i}{4} \mathrm{Tr}(S\varepsilon) \varepsilon^{ab} \right),$$

thus we have that

$$\begin{bmatrix} \cosh\beta & \mathrm{e}^{-i\theta}\sinh\beta \\ e^{i\theta}\sinh\beta & \cosh\beta \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\sin\theta\sinh\beta \\ -\frac{1}{2}\sin\theta\sinh\beta & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\theta = \frac{n\pi}{2}$$
$$\beta = i\frac{m\pi}{2}$$

6 Inverse Area Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

and make quantum geometry correction

$$\sqrt{-g} \to \sqrt{-(g+k\Delta)} \approx \sqrt{-g}\left(1+\frac{k\Delta}{2g}+\mathcal{O}\left(\frac{k^2}{g^2}\right)\right)$$

leading to

$$S_{IAP} = -\frac{T}{2}\int \mathrm{d}^2x\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

6.1 EoMs

$$\begin{aligned}
\frac{\delta S_{IAP}}{\delta g^{cd}} &= \frac{\partial}{\partial g^{cd}}\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)\delta^a_c\delta^b_d\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = \\
&= \left(-\frac{1}{2}\sqrt{-g}g_{cd}+\frac{k\Delta}{4\sqrt{-g}}g_{cd}\right)g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)\partial_c X^\mu \partial_d X^\nu G_{\mu\nu} = \\
&= -\frac{1}{2}\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)\partial_c X^\mu \partial_d X^\nu G_{\mu\nu} \overset{!}{=} 0
\end{aligned}$$

$$\downarrow$$

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} - \frac{1}{2}g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = 0$$

$$\downarrow$$

$$\begin{aligned}
g_{cd} &= 2f\partial_c X^\mu \partial_d X^\nu G_{\mu\nu}, \\
\frac{1}{f} &= g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
\frac{\delta S_{IAP}}{\delta X^\lambda} &= \left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - \partial_c\left(\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)2g^{ab}\delta^c_a\delta^\mu_\lambda\partial_b X^\nu G_{\mu\nu}\right) = \\
&= \left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\partial_a\partial_b X^\nu G_{\lambda\nu} - \\
&\quad - 2\partial_a\left(\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\right)\partial_b X^\nu G_{\lambda\nu} = \\
&= \left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2\left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_a\partial_b X^\nu G_{\lambda\nu} - \\
&\quad - 2\partial_a\left(\left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\right)\partial_b X^\nu G_{\lambda\nu} \overset{!}{=} 0
\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}
&\left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_a\partial_b X^\nu G_{\lambda\nu} + \partial_a\left(\left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\right)\partial_b X^\nu G_{\lambda\nu} = \\
&\qquad\qquad \frac{1}{2}\left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},
\end{aligned}$$

imposing conformal symmetry $g^{ab} = (\phi(x))^{-1}\eta^{ab}$ and let $F' = \left(1-\frac{k\Delta}{2\phi^2}\right)$ leads to

$$\eta^{ab}\partial_a\partial_b X^\nu G_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F'\partial_b X^\nu G_{\lambda\nu} = \eta^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

assuming flat background $G_{\mu\nu} = \eta_{\mu\nu}$

$$\eta^{ab}\partial_a\partial_b X^\nu \eta_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F' \partial_b X^\nu \eta_{\lambda\nu} = 0$$

$$\downarrow$$

$$\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\partial_a \ln(F')\partial_b X^\mu = 0,$$

use plane-wave ansatz $X^\mu = X_0^\mu \mathrm{e}^{-i(E\tau - p\sigma)}$

$$(E^2 - p^2)X^\mu + \partial_\tau \ln(F')iEX^\mu - \partial_\sigma \ln(F')ipX^\mu = 0$$

$$\downarrow$$

$$(E^2 - p^2)X^\mu + i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F'))X^\mu = 0$$

$$\downarrow$$

$$i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F')) = -m^2,$$
$$-m^2 = -E^2 + p^2$$

$$\Downarrow$$

$$\partial_\tau \ln(F') = iE,\ \ \partial_\sigma \ln(F') = ip$$

$$\Downarrow$$

$$\ln(F') = i(E\tau + p\sigma) + a,\ \ a \in \mathbb{C}$$

$$\Downarrow$$

$$F' = A\mathrm{e}^{i(E\tau + p\sigma + \phi)},\ \ A > 0$$

$$\Downarrow$$

$$\phi(\tau, \sigma) = \sqrt{\frac{k\Delta}{2\left(1 - A\mathrm{e}^{i(E\tau + p\sigma + \phi)}\right)}}$$

$$\downarrow$$

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

7 Quantum geometry Polyakov

Since quantum geometry is expected to violate Lorentz invariance at the Planck scale by modifying the dispersion relation as in

$$E^2 = m^2 + p^2 + \sum_{n\geq 3} c_n \frac{p^n}{E_{pl}^{n-2}},$$

where E_{pl} is the Planck energy, we make a correction to Polyakov action in the form

$$S = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}\left(g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu}\right)$$

where T is the temperature of target-space, μ_0 has units of mass (perhaps $\mu_0 \sim M_{pl}$ or $\mu_0^2 \sim$ (quantum of area)$^{-1}$?) and $m_0 >> m_1 >> ... >> m_d$. The masses m_μ are constrained by the dispersion relation

$$E_\mu^2 = p_\mu^2 + \sum_{n=0}^{D} m_n^2 \frac{p^n}{E_{pl}^n}$$

where E_μ^2 does not mean $E^\mu E_\mu$, rather it is $(E_0)^2, (E_1)^2, ..., (E_d)^2$, that is, the squared WS energy (and momentum) of each field X^μ. This action has manifest Lorentz invariance and WS reparameterisation and Weyl invariance. As for translational symmetry, it holds approximately for the fields with $m_\mu >> T$.

7.1 EoMs

$$\begin{aligned}
\frac{\delta S}{\delta g^{cd}} &\propto \frac{\partial\sqrt{-g}}{\partial g^{cd}}\left(g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu}\right) + \sqrt{-g}\delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = \\
&= -\frac{1}{2}\sqrt{-g} g_{cd}\left(g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu}\right) + \sqrt{-g}\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = 0
\end{aligned}$$

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd}\left(g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu}\right) = 0$$
$$g_{cd} = 2f\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu},$$
$$\frac{1}{f} = \left(g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu}\right).$$

From this, it is clear that $\mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} = 0$, thus the base mass μ_0^2 is a Lagrange multiplier, the auxiliary metric reduces to the induced metric and the action simplify back to regular NG action.

$$\begin{aligned}
\frac{\delta S}{\delta X^\lambda} &\propto -2\sqrt{-g}\mu_0^2 \mathrm{e}^{-m_\mu/T}\delta_\lambda^\mu X^\nu \eta_{\mu\nu} - 2\partial_c\left(\sqrt{-g} g^{ab}\delta_a^c \delta_\lambda^\mu \partial_b X^\nu \eta_{\mu\nu}\right) = \\
&= -2\left(\partial_a\left(\sqrt{-g} g^{ab}\partial_b X^\nu \eta_{\lambda\nu}\right) + \sqrt{-g}\mu_0^2 \mathrm{e}^{-m_\lambda/T} X^\nu \eta_{\lambda\nu}\right) = 0
\end{aligned}$$

$$\begin{aligned}
\partial_a\left(\sqrt{-g} g^{ab}\partial_b X^\mu\right) + \sqrt{-g}\mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu &= 0 \\
\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g} g^{ab}\partial_b X^\mu\right) + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu &= 0 \\
g^{ab}\nabla_a \nabla_b X^\mu + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu &= 0.
\end{aligned}$$

We see that 2d gravitational field of the WS appears naturally in metric-Christoffel symbols formulation. Perhaps THIS is THE true quantum gravitational field since in high enough energies only $m_0, m_1 >> T$ and thus only X^0 and X^1 remain massless, while the others are massive and thus violate Poincaré invariance. To this end, we may postulate that $m_2 \sim E_{pl}$, such that at $T = T_{pl}$ we have massive $X^2, X^3, ..., X^d$ with (approximately) massless X^0 and X^1. From that and the modified WS dispersion relations for each X^μ we can get the other masses.

8 Relating NG and Polyakov analysis

From NG analysis

$$(F^+)^2\frac{\partial^2 X^\mu}{\partial t^2} - \frac{\partial^2 X^\mu}{\partial x^2} = 0,$$

make change of variables $\tau = \tau(t, x)$ and $\sigma = \sigma(t, x)$

$$(F^+)^2\left(\frac{\partial\tau}{\partial t}\frac{\partial}{\partial\tau} + \frac{\partial\sigma}{\partial t}\frac{\partial}{\partial\sigma}\right)\left(\frac{\partial\tau}{\partial t}\frac{\partial X^\mu}{\partial\tau} + \frac{\partial\sigma}{\partial t}\frac{\partial X^\mu}{\partial\sigma}\right) - \left(\frac{\partial\tau}{\partial x}\frac{\partial}{\partial\tau} + \frac{\partial\sigma}{\partial x}\frac{\partial}{\partial\sigma}\right)\left(\frac{\partial\tau}{\partial x}\frac{\partial X^\mu}{\partial\tau} + \frac{\partial\sigma}{\partial x}\frac{\partial X^\mu}{\partial\sigma}\right) = 0$$

$$\begin{aligned}(F^+)^2\left(\left(\frac{\partial\tau}{\partial t}\right)^2\frac{\partial^2 X^\mu}{\partial\tau^2} + 2\frac{\partial\tau}{\partial t}\frac{\partial\sigma}{\partial t}\frac{\partial^2 X^\mu}{\partial\tau\partial\sigma} + \left(\frac{\partial\sigma}{\partial t}\right)^2\frac{\partial^2 X^\mu}{\partial\sigma^2}\right) - \\ - \left(\left(\frac{\partial\tau}{\partial x}\right)^2\frac{\partial^2 X^\mu}{\partial\tau^2} + 2\frac{\partial\tau}{\partial x}\frac{\partial\sigma}{\partial x}\frac{\partial^2 X^\mu}{\partial\tau\partial\sigma} + \left(\frac{\partial\sigma}{\partial x}\right)^2\frac{\partial^2 X^\mu}{\partial\sigma^2}\right) + \\ + \left((F^+)^2\frac{\partial^2\tau}{\partial t^2} - \frac{\partial^2\tau}{\partial x^2}\right)\frac{\partial X^\mu}{\partial\tau} + \left((F^+)^2\frac{\partial^2\sigma}{\partial t^2} - \frac{\partial^2\sigma}{\partial x^2}\right)\frac{\partial X^\mu}{\partial\sigma} = 0\end{aligned}$$

$$\begin{aligned}\left((F^+)^2(\partial_t\tau)^2 - (\partial_x\tau)^2\right)\partial_\tau^2 X^\mu + \left((F^+)^2(\partial_t\sigma)^2 - (\partial_x\sigma)^2\right)\partial_\sigma^2 X^\mu + \\ + 2\left((F^+)^2\partial_t\tau\partial_t\sigma - \partial_x\tau\partial_x\sigma\right)\partial_\tau\partial_\sigma X^\mu + \left((F^+)^2\partial_t^2\tau - \partial_x^2\tau\right)\partial_\tau X^\mu + \left((F^+)^2\partial_t^2\sigma - \partial_x^2\sigma\right)\partial_\sigma X^\mu = 0\end{aligned}$$

Without loss of generality, we can set (just rescale/rotate the coordinates)

$$\begin{aligned}(F^+)^2(\partial_t\sigma)^2 - (\partial_x\sigma)^2 &= (\partial_x\tau)^2 - (F^+)^2(\partial_t\tau)^2 \\ (F^+)^2\partial_t\tau\partial_t\sigma &= \partial_x\tau\partial_x\sigma,\end{aligned}$$

or more compactly

$$\begin{aligned}\partial_t\tau &= v\partial_x\sigma \\ \partial_t\sigma &= v\partial_x\tau,\end{aligned}$$

where $v = 1/F^+$, which is equivalent to

$$\begin{aligned}\partial_t^2\tau &= \partial_t v\partial_x\sigma + v\partial_x v\partial_x\tau + v^2\partial_x^2\tau \\ \partial_t^2\sigma &= \partial_t v\partial_x\tau + v\partial_x v\partial_x\sigma + v^2\partial_x^2\sigma,\end{aligned}$$

reducing the big equation to

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu + \frac{\partial_t v\partial_x\sigma + v\partial_x v\partial_x\tau}{((\partial_t\tau)^2 - v^2(\partial_x\tau)^2)}\partial_\tau X^\mu + \frac{\partial_t v\partial_x\tau + v\partial_x v\partial_x\sigma}{((\partial_t\tau)^2 - v^2(\partial_x\tau)^2)}\partial_\sigma X^\mu = 0.$$

Next introduce $X^\mu = \kappa(\tau, \sigma)Y^\mu(\tau, \sigma)$ s.t terms proportional to $\partial_\tau Y^\mu$ and $\partial_\sigma Y^\mu$ vanish:

$$\begin{aligned}\partial_\tau X^\mu &= \partial_\tau\kappa Y^\mu + \kappa\partial_\tau Y^\mu \\ \partial_\sigma X^\mu &= \partial_\sigma\kappa Y^\mu + \kappa\partial_\sigma Y^\mu \\ \partial_\tau^2 X^\mu &= \partial_\tau^2\kappa Y^\mu + 2\partial_\tau\kappa\partial_\tau Y^\mu + \kappa\partial_\tau^2 Y^\mu \\ \partial_\sigma^2 X^\mu &= \partial_\sigma^2\kappa Y^\mu + 2\partial_\sigma\kappa\partial_\sigma Y^\mu + \kappa\partial_\sigma^2 Y^\mu\end{aligned}$$

$$\begin{aligned}\kappa\frac{\partial_t v\partial_x\sigma + v\partial_x v\partial_x\tau}{((\partial_t\tau)^2 - v^2(\partial_x\tau)^2)} + 2\partial_\tau\kappa &\overset{!}{=} 0 \\ \kappa\frac{\partial_t v\partial_x\tau + v\partial_x v\partial_x\sigma}{((\partial_t\tau)^2 - v^2(\partial_x\tau)^2)} - 2\partial_\sigma\kappa &\overset{!}{=} 0\end{aligned}$$

$$
\begin{aligned}
\partial_\tau \kappa &= -\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} \\
\partial_\sigma \kappa &= \frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)},
\end{aligned}
$$

undoing chain rule for t by multiplying first eq by $\partial_t \tau$ and second by $\partial_t \sigma$ and summing gives

$$
\begin{aligned}
\partial_t \kappa = \partial_t \tau \partial_\tau \kappa + \partial_t \sigma \partial_\sigma \kappa &= \\
&= \partial_t \tau \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} \right) + \partial_t \sigma \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} \right) = \\
&= v \partial_x \sigma \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} \right) + v \partial_x \tau \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} \right) = \\
&= \frac{v \kappa \partial_t v \left((\partial_x \tau)^2 - (\partial_x \sigma)^2\right)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} = \\
&= \frac{v \kappa \partial_t v \left((\partial_x \tau)^2 - (\partial_t \tau / v)^2\right)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} = \\
&= -\frac{\kappa \partial_t v}{2v},
\end{aligned}
$$

or simply

$$
\begin{aligned}
\frac{\partial_t \kappa}{\kappa} &= -\frac{1}{2} \frac{\partial_t v}{v} \\
\partial_t \ln(\kappa) &= -\frac{1}{2} \partial_t \ln(v) \\
\ln(\kappa) &= -\frac{1}{2} \ln(v) + f(x).
\end{aligned}
$$

To determine $f(x)$, we undo the chain rule for x now:

$$
\begin{aligned}
\partial_x \kappa = \partial_x \tau \partial_\tau \kappa + \partial_x \sigma \partial_\sigma \kappa &= \\
&= \partial_x \tau \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} \right) + \partial_x \sigma \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} \right) = \\
&= \frac{\kappa v \partial_x v \left((\partial_x \sigma)^2 - (\partial_x \tau)^2\right)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} = \\
&= \frac{\kappa v \partial_x v \left((\partial_t \tau / v)^2 - (\partial_x \tau)^2\right)}{2\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} = \\
&= \frac{\kappa \partial_x v}{2v}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial_x \kappa}{\kappa} &= \frac{1}{2} \frac{\partial_x v}{v} \\
\partial_x \ln(\kappa) &= \frac{1}{2} \partial_x \ln(v) \\
\ln(\kappa) &= \frac{1}{2} \ln(v), \\
\partial_x \left(-\frac{1}{2} \ln(v) + f(x) \right) &= \frac{1}{2} \partial_x \ln(v)
\end{aligned}
$$

$f'(x) = \partial_x \ln(v)$ (perhaps this means somewhere in the composition of functions time dependence is lost?)

$$
f(x) = \ln(v)
$$

giving us

$$
\kappa = v^{\frac{1}{2}}, \ v = v(x).
$$

Substituting this into the earlier equation yields

$$
\partial_\tau^2 Y^\mu - \partial_\sigma^2 Y^\mu + m^2 Y^\mu = 0,
$$

with

$$
\begin{aligned}
m^2 &= \frac{\partial_\tau^2\kappa - \partial_\sigma^2\kappa}{\kappa} + \frac{(\partial_t v\partial_x\sigma + v\partial_x v\partial_x\tau)\partial_\tau\kappa + (\partial_t v\partial_x\tau + v\partial_x v\partial_x\sigma)\partial_\sigma\kappa}{\kappa\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)} = \\
&= \frac{\partial_\tau^2\kappa - \partial_\sigma^2\kappa}{\kappa} + \frac{\partial_t v(\partial_x\sigma\partial_\tau\kappa + \partial_x\tau\partial_\sigma\kappa) + v\partial_x v\partial_x\kappa}{\kappa\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)} = \\
&= \frac{\partial_\tau^2\kappa - \partial_\sigma^2\kappa}{\kappa} + \frac{(\partial_x v)^2}{\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)},
\end{aligned}
$$

where by use of chain rule we have that

$$
\partial_t^2\kappa - v^2\partial_x^2\kappa - v\partial_x v\partial_x\kappa = ((\partial_t\tau)^2 - v^2(\partial_x\tau)^2)(\partial_\tau^2\kappa - \partial_\sigma^2\kappa),
$$

where since $\kappa = v^{\frac{1}{2}}$, m^2 reduces to

$$
m^2 = \frac{(\partial_x v)^2 - 2v\partial_x^2 v}{4\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)},
$$

with τ satisfying

$$
\begin{aligned}
\partial_t\tau &= v\partial_x\sigma \\
v\partial_x\tau &= \partial_t\sigma
\end{aligned}
$$

such that $-m^2$ is indeed constant.

9 Solutions for the Polyakov KG eqn (WRONG ONE)

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi\eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2}\eta_{cd}\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\begin{aligned}
\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2}\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\
\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{1}{2}\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\
\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0.
\end{aligned}$$

These simplify further by expanding the summation on a, b:

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2}\left(-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}\right)$$

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}.$$

The EoM is just a one-dimensional Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi}\int \mathrm{d}p \frac{1}{2E(p)}\left(a^\mu(p)\mathrm{e}^{-i(-E\tau+p\sigma)} + b^\mu(p)\mathrm{e}^{i(-E\tau+p\sigma)}\right),$$

where $E(p) = \sqrt{p^2+m^2}$. The derivatives are

$$\begin{aligned}
\partial_\tau X^\mu &= \frac{i}{4\pi}\int \mathrm{d}p\left(a^\mu(p)\mathrm{e}^{-i(-E\tau+p\sigma)} - b^\mu(p)\mathrm{e}^{i(-E\tau+p\sigma)}\right) \\
\partial_\sigma X^\mu &= \frac{i}{4\pi}\int \mathrm{d}p\frac{p}{\sqrt{p^2+m^2}}\left(-a^\mu(p)\mathrm{e}^{-i(-E\tau+p\sigma)} + b^\mu(p)\mathrm{e}^{i(-E\tau+p\sigma)}\right).
\end{aligned}$$

Starting with the mixed constraint,

$$\begin{aligned}
\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = -\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\left(a^\mu(p)\mathrm{e}^{-i(-E(p)\tau+p\sigma)} - b^\mu(p)\mathrm{e}^{i(-E(p)\tau+p\sigma)}\right)\times \\
\times\int \mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\left(-a^\nu(p')\mathrm{e}^{-i(-E(p')\tau+p'\sigma)} + b^\nu(p')\mathrm{e}^{i(-E(p')\tau+p'\sigma)}\right) =
\end{aligned}$$

$$\begin{aligned}
= -\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\left(a^\mu(p)\mathrm{e}^{-i(-E(p)\tau+p\sigma)} - b^\mu(p)\mathrm{e}^{i(-E(p)\tau+p\sigma)}\right)\times \\
\times\left(-a^\nu(p')\mathrm{e}^{-i(-E(p')\tau+p'\sigma)} + b^\nu(p')\mathrm{e}^{i(-E(p')\tau+p'\sigma)}\right) =
\end{aligned}$$

$$\begin{aligned}
= -\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\mathrm{e}^{-i(p+p')\sigma} + a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\mathrm{e}^{-i(p-p')\sigma} + \\
+ b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\mathrm{e}^{i(p-p')\sigma} - b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\mathrm{e}^{i(p+p')\sigma}\Big) = 0.
\end{aligned}$$

If we integrate this expression over σ we obtain

$$-\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\int \mathrm{d}\sigma \mathrm{e}^{-i(p+p')\sigma}+ + a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\int \mathrm{d}\sigma \mathrm{e}^{-i(p-p')\sigma} + b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\int \mathrm{d}\sigma \mathrm{e}^{i(p-p')\sigma}- - b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\int \mathrm{d}\sigma \mathrm{e}^{i(p+p')\sigma}\Big) =$$

$$= -\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p') + a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p')+ + b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p') - b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p')\Big) =$$

$$= -\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\frac{p}{\sqrt{p^2+m^2}}\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} + a^\mu(p)b^\nu(p) + b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big) = 0$$

which implies the whole expression inside the integral must vanish, thus we have (leaving $(a^\mu b^\nu + b^\mu a^\nu)\eta_{\mu\nu}$ as is since eventually these will not be numbers and may not commute, thus will not be equal to $2a^\mu b^\nu \eta_{\mu\nu}$)

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} + a^\mu(p)b^\nu(p) + b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu} = 0.$$

As for the other part of the constraint, let's start with the L.H.S

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)\mathrm{e}^{-i(-E(p)\tau+p\sigma)} - b^\mu(p)\mathrm{e}^{i(-E(p)\tau+p\sigma)}\Big)\times \times\Big(a^\nu(p')\mathrm{e}^{-i(-E(p')\tau+p'\sigma)} - b^\nu(p')\mathrm{e}^{i(-E(p')\tau+p'\sigma)}\Big) =$$

$$= -\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\mathrm{e}^{-i(p+p')\sigma} - a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\mathrm{e}^{-i(p-p')\sigma}- - b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\mathrm{e}^{i(p-p')\sigma} + b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\mathrm{e}^{i(p+p')\sigma}\Big),$$

which we integrate over σ to get

$$-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p') - a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p')- - b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p') + b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p')\Big) =$$

$$= -\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big),$$

and now for the R.H.S

$$-\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{p}{\sqrt{p^2+m^2}}\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)\mathrm{e}^{-i(-E\tau+p\sigma)} + b^\mu(p)\mathrm{e}^{i(-E\tau+p\sigma)}\Big)\times \times\Big(-a^\nu(p')\mathrm{e}^{-i(-E(p')\tau+p'\sigma)} + b^\nu(p')\mathrm{e}^{i(-E(p')\tau+p'\sigma)}\Big) =$$

$$= \frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{pp'}{E(p)E(p')}\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\mathrm{e}^{-i(p+p')\sigma} - a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\mathrm{e}^{-i(p-p')\sigma}- - b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\mathrm{e}^{i(p-p')\sigma} + b^\mu(p)b^\nu(p)\mathrm{e}^{-i(E(p)+E(p'))\tau}\mathrm{e}^{i(p+p')\sigma}\Big),$$

which once again we integrate over σ,

$$\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\mathrm{d}p' \frac{pp'}{E(p)E(p')}\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p') - a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p') -$$
$$- b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p') + b^\mu(p)b^\nu(p)\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p')\Big) =$$
$$= \frac{1}{8\pi}\int \mathrm{d}p \frac{p^2}{p^2+m^2}\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu},$$

which when equated to the L.H.S yields

$$\int \mathrm{d}p\left(\frac{p^2}{p^2+m^2}+1\right)\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu} = 0,$$

which implies

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu} = 0.$$

We can sum and subtract this with the previous condition to simplify and obtain

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu} = 0$$
$$\Big(a^\mu(p)b^\nu(p) + b^\mu(p)a^\nu(p)\Big)\eta_{\mu\nu} = 0,$$

and for while a^μ and b^ν are number-valued, the second condition can be further simplified to

$$a^\mu(p)b^\nu(p)\eta_{\mu\nu} = 0,$$

meaning a^μ and b^ν are orthogonal. In general, this condition states that a^μ and b^ν anti-commute w.r.t Lorentz inner product. The first condition must hold for all values of τ, and since the exponentials are linearly independent, we thus have

$$a^\mu(p)a^\nu(-p)\eta_{\mu\nu} = 0$$
$$b^\mu(p)b^\nu(-p)\eta_{\mu\nu} = 0,$$

meaning that reflecting the argument of a^μ and b^ν creates orthogonal vectors. We can thus write

$$a^\mu(p) = \Lambda^\mu{}_\nu\left(\frac{p}{4}\right)a^\nu(0)$$
$$b^\mu(p) = \Upsilon^\mu{}_\nu\left(\frac{p}{4}\right)b^\nu(0),$$

where $\Lambda, \Upsilon \in \mathrm{SO}^+(1,d)$. Also. since the orthogonality under reflection must hold for all values of p, in particular we have

$$a^\mu(0)a^\nu(0)\eta_{\mu\nu} = 0$$
$$b^\mu(0)b^\nu(0)\eta_{\mu\nu} = 0,$$

meaning that the initial $a^\mu(0)$ and $b^\nu(0)$ are null vectors. This means that $a^\mu(p)$ and $b^\nu(p)$ are also null vectors since they are related to the initial values by Lorentz transformation. With this, the solution becomes

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi}\int \mathrm{d}p\frac{1}{2E(p)}\left(\Lambda^\mu{}_\nu\left(\frac{p}{4}\right)a^\nu(0)\mathrm{e}^{-i(-E\tau+p\sigma)} + \Upsilon^\mu{}_\nu\left(\frac{p}{4}\right)b^\nu(0)\mathrm{e}^{i(-E\tau+p\sigma)}\right)$$

with $(a^\mu b^\nu + b^\nu a^\mu)\eta_{\mu\nu} = 0$. This condition implies

$$\left(\Lambda^\mu{}_\gamma\left(\frac{p}{4}\right)a^\gamma(0)\Upsilon^\nu{}_\rho\left(\frac{p}{4}\right)b^\rho(0)\right)\eta_{\mu\nu} = 0$$

$$\Upsilon^\mu{}_\nu\left(\frac{p}{4}\right) = \Lambda_\nu{}^\mu\left(\frac{p}{4}\right) = \Lambda^\mu{}_\nu\left(-\frac{p}{4}\right),$$

thus turning the solution into

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi}\int \mathrm{d}p \frac{1}{2E(p)}\left(\Lambda^\mu{}_\nu\left(\frac{p}{4}\right)a^\nu(0)\mathrm{e}^{-i(-E\tau+p\sigma)} + \Lambda^\mu{}_\nu\left(-\frac{p}{4}\right)b^\nu(0)\mathrm{e}^{i(-E\tau+p\sigma)}\right).$$

Next, we choose background coordinates s.t $a^\mu(0) = (a_0, a_0, 0, ..., 0) = a_0 x^+$ and $b^\nu(0) = (b_0, b_0, 0, ..., 0) = b_0 x^+$, simplifying the solution to

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi}\int \mathrm{d}p \frac{1}{2E(p)}\left(a_0\Lambda^\mu{}_+\left(\frac{p}{4}\right)\mathrm{e}^{-i(-E\tau+p\sigma)} + b_0\Lambda^\mu{}_+\left(-\frac{p}{4}\right)\mathrm{e}^{i(-E\tau+p\sigma)}\right)x^+$$

10 Solutions for the Polyakov KG eqn

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi \eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta_{cd} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\begin{aligned}
\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\
\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\
\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0.
\end{aligned}$$

These simplify further by expanding the summation on a, b:

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2} \left(-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}\right)$$

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = 0.$$

These can be combined by multiplying the previous one by 2 and summing, leaving us with

$$(\partial_\tau X^\mu + \partial_\sigma X^\mu)(\partial_\tau X^\nu + \partial_\sigma X^\nu)\eta_{\mu\nu} = 0.$$

The EoM is just a one-dimensional finite-space Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n\in\mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu \mathrm{e}^{i(-E_n\tau + n\sigma)} + b_n^\mu \mathrm{e}^{i(E_n\tau + n\sigma)}\right),$$

with $E_n = \sqrt{n^2 + m^2}$. The derivatives are

$$\begin{aligned}
\partial_\tau X^\mu &= \frac{i}{4\pi\alpha'} \sum_{n\in\mathbb{Z}} \left(-a_n^\mu \mathrm{e}^{i(-E_n\tau + n\sigma)} + b_n^\mu \mathrm{e}^{i(E_n\tau + n\sigma)}\right) \\
\partial_\sigma X^\mu &= \frac{i}{4\pi\alpha'} \sum_{n\in\mathbb{Z}} \frac{n}{E_n} \left(a_n^\mu \mathrm{e}^{i(-E_n\tau + n\sigma)} + b_n^\mu \mathrm{e}^{i(E_n\tau + n\sigma)}\right).
\end{aligned}$$

Reality of X^μ implies

$$\begin{aligned}
(X^\mu)^* = \left(\frac{1}{2\pi\alpha'} \sum_{n\in\mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu \mathrm{e}^{i(-E_n\tau + n\sigma)} + b_n^\mu \mathrm{e}^{i(E_n\tau + n\sigma)}\right)\right)^* &= \\
= \frac{1}{2\pi\alpha'} \sum_{n\in\mathbb{Z}} \frac{1}{2E_n} \left((a_n^\mu)^* \mathrm{e}^{-i(-E_n\tau + n\sigma)} + (b_n^\mu)^* \mathrm{e}^{-i(E_n\tau + n\sigma)}\right) &\equiv X^\mu,
\end{aligned}$$

giving us

$$\begin{aligned}
(a_n^\mu)^* &= b_{-n}^\mu \\
(b_n^\mu)^* &= a_{-n}^\mu,
\end{aligned}$$

so the solution becomes for closed strings

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n\in\mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu \mathrm{e}^{i(-E_n\tau + n\sigma)} + (a_{-n}^\mu)^* \mathrm{e}^{i(E_n\tau + n\sigma)}\right).$$

For open strings with free endpoints B.C we have general solution

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi\alpha'}\sum_{n\in\mathbb{Z}}\frac{1}{2E_n}\left(a_n^\mu \mathrm{e}^{-iE_n\tau} + b_n^\mu \mathrm{e}^{iE_n\tau}\right)\cos(n\sigma),$$

which reality condition implies $b_n^\mu = (a_n^\mu)^*$, thus open string with free endpoint solution becomes

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi\alpha'}\sum_{n\in\mathbb{Z}}\frac{1}{2E_n}\left(a_n^\mu \mathrm{e}^{-iE_n\tau} + (a_n^\mu)^* \mathrm{e}^{iE_n\tau}\right)\cos(n\sigma).$$

Analysing the constraints, for the closed string we have that

$$\partial_\tau X^\mu + \partial_\sigma X^\mu = \frac{i}{4\pi\alpha'}\sum_{n\in\mathbb{Z}}\left(\left(\frac{n}{E_n}-1\right)a_n^\mu \mathrm{e}^{i(-E_n\tau+n\sigma)} + \left(\frac{n}{E_n}+1\right)b_n^\mu \mathrm{e}^{i(E_n\tau+n\sigma)}\right),$$

so the constraint reads

$$\begin{aligned}(\partial_\tau X + \partial_\sigma X)^2 = -\frac{\eta_{\mu\nu}}{(4\pi\alpha')^2}\sum_n\sum_p\Bigg(&\left(\frac{n}{E_n}-1\right)\left(\frac{p}{E_p}-1\right)a_n^\mu a_p^\nu \mathrm{e}^{i(-(E_n+E_p)\tau+(n+p)\sigma)}+\\ &+\left(\frac{n}{E_n}-1\right)\left(\frac{p}{E_p}+1\right)a_n^\mu (a_{-p}^\nu)^* \mathrm{e}^{i(-(E_n-E_p)\tau+(n+p)\sigma)}+\left(\frac{n}{E_n}+1\right)\left(\frac{p}{E_p}-1\right)(a_{-n}^\mu)^* a_p^\nu \mathrm{e}^{i((E_n-E_p)\tau+(n+p)\sigma)}+\\ &+\left(\frac{n}{E_n}+1\right)\left(\frac{p}{E_p}+1\right)(a_{-n}^\mu)^*(a_{-p}^\nu)^* \mathrm{e}^{i((E_n+E_p)\tau+(n+p)\sigma)}\Bigg)\equiv 0.\end{aligned}$$

Since the exponentials are all linearly independent and we can't decouple the τ exponential from any of the sums, we have 3 cases to investigate: $n=p$, $n=-p$ and $n\neq\pm p$. For $n\neq\pm p$, we get

$$\begin{aligned}L_{n,p}^1 &:= \left(\frac{n}{E_n}-1\right)\left(\frac{p}{E_p}-1\right)a_n^\mu a_p^\nu\eta_{\mu\nu} = 0\\ L_{n,p}^2 &:= \left(\frac{n}{E_n}-1\right)\left(\frac{p}{E_p}+1\right)a_n^\mu (a_{-p}^\nu)^*\eta_{\mu\nu} = 0\\ L_{n,p}^3 &:= \left(\frac{n}{E_n}+1\right)\left(\frac{p}{E_p}-1\right)(a_{-n}^\mu)^* a_p^\nu\eta_{\mu\nu} = 0\\ L_{n,p}^4 &:= \left(\frac{n}{E_n}+1\right)\left(\frac{p}{E_p}+1\right)(a_{-n}^\mu)^*(a_{-p}^\nu)^*\eta_{\mu\nu} = 0,\end{aligned}$$

where $L_{n,p}^2$ and $L_{n,p}^3$ are just complex conjugates of each other with $n\mapsto -n$ and $p\mapsto -p$, so they are redundant. The same is true for $L_{n,p}^1$ and $L_{n,p}^4$, so in the end we have

$$\begin{aligned}L_{n,p} &:= \left(\frac{n}{E_n}-1\right)\left(\frac{p}{E_p}-1\right)a_n^\mu a_p^\nu\eta_{\mu\nu} = 0\\ \widetilde{L}_{n,p} &:= \left(\frac{n}{E_n}-1\right)\left(\frac{p}{E_p}+1\right)a_n^\mu (a_{-p}^\nu)^*\eta_{\mu\nu} = 0.\end{aligned}$$

The coefficients of both expressions are actually not necessary, since for them to be 0 we would need $n\to\pm\infty$ or $p\to\pm\infty$, so we can redefine

$$\begin{aligned}L_{n,p} &:= a_n^\mu a_p^\nu\eta_{\mu\nu} = 0\\ \widetilde{L}_{n,p} &:= a_n^\mu (a_{-p}^\nu)^*\eta_{\mu\nu} = 0, n\neq\pm p.\end{aligned}$$

For $n=-p$, the first and last terms are complex conjugates of each other with the relabeling $n\mapsto -n$ and $p\mapsto -p$, while the exponentials in the 2 middle terms are reduced to 1, so we get

$$\left(\frac{n}{E_n}-1\right)^2 a_n^\mu a_{-n}^\nu\eta_{\mu\nu} = 0$$

$$\sum_n\left(\left(\left(\frac{n}{E_n}\right)^2-1\right)a_n^\mu (a_n^\nu)^*\eta_{\mu\nu}\right) = 0,$$

where in the first equation we can ignore the first factor once more. Finally for $n=p$, we get

$$\left(\frac{n}{E_n}-1\right)^2 a_n^\mu a_n^\nu \eta_{\mu\nu}=0$$
$$\left(\left(\frac{n}{E_n}\right)^2-1\right) a_n^\mu (a_{-n}^\nu)^* \eta_{\mu\nu}=0,$$

where we again ignore the first factor because it's only 0 at infinity, thus collecting everything we have

$$L_{n,p}:=a_n^\mu a_p^\nu \eta_{\mu\nu}=0,\ \forall\, n,p\in\mathbb{Z},$$
$$\widetilde{L}_{n,p}:=a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu}=0,\ n\neq -p,$$
$$\sum_n\left(\left(\left(\frac{n}{E_n}\right)^2-1\right)\widetilde{L}_{n,-n}\right)=0.$$

One last thing before quantizing: the $n=0$ term in the solution reads

$$\frac{1}{2\pi\alpha'}\frac{1}{2m}\left(a_0^\mu \mathrm{e}^{-im\tau}+(a_0^\mu)^*\mathrm{e}^{im\tau}\right),$$

which is divergent in the $m\to 0$ limit unless $\mathfrak{Re}(a_0^\mu)\sim m$. Keeping the analogy to the $m\to 0$ regime, we conclude $a_0^\mu=2\pi\alpha' m x^\mu+4\pi i(\alpha')^2 p^\mu$. Since a_0^μ includes p^μ, we have that

$$M^2=-p^\mu p_\mu=\frac{1}{16\pi^2(\alpha')^4}\left((2\pi\alpha')^2 x^\mu x^\nu-a_0^\mu (a_0^\nu)^*\right)\eta_{\mu\nu},$$

which after analysing closer the $n=-p$ condition

$$\sum_n\left(\left(\left(\frac{n}{E_n}\right)^2-1\right)\widetilde{L}_{n,-n}\right)=0$$
$$\sum_{n\neq 0}\left(\left(\left(\frac{n}{E_n}\right)^2-1\right)\widetilde{L}_{n,-n}\right)-\widetilde{L}_{0,0}=0$$
$$\sum_{n\neq 0}\left(\left(\left(\frac{n}{E_n}\right)^2-1\right)\widetilde{L}_{n,-n}\right)=a_0^\mu (a_0^\nu)^*\eta_{\mu\nu}\equiv(4\pi(\alpha')^2)^2 p^\mu p_\mu+(2\pi\alpha' m)^2 x^\mu x_\mu,$$

we have the string mass in terms of vibrational modes... (and center of mass position?!?)

$$M^2=\frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha' m)^2 x^\mu x_\mu-\sum_{n\neq 0}\left(\left(\left(\frac{n}{E_n}\right)^2-1\right)\widetilde{L}_{n,-n}\right)\right)=$$
$$=\frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha' m)^2 x^\mu x_\mu-\sum_{n\neq 0}\left(\left(\left(\frac{n}{E_n}\right)^2-1\right)a_n^\mu (a_n^\nu)^*\eta_{\mu\nu}\right)\right).$$

As for the open string, its derivatives are

$$\partial_\tau X^\mu=\frac{i}{4\pi\alpha'}\sum_n\left(-a_n^\mu \mathrm{e}^{-iE_n\tau}+(a_n^\mu)^*\mathrm{e}^{iE_n\tau}\right)\cos(n\sigma)$$
$$\partial_\sigma X^\mu=-\frac{1}{4\pi\alpha'}\sum_n\frac{n}{E_n}\left(a_n^\mu \mathrm{e}^{-iE_n\tau}+(a_n^\mu)^*\mathrm{e}^{iE_n\tau}\right)\sin(n\sigma)$$

and the constraint reads

$$\partial_\tau X^\mu+\partial_\sigma X^\mu=\frac{1}{4\pi\alpha'}\sum_n\left(-a_n^\mu \mathrm{e}^{-iE_n\tau}\left(i\cos(n\sigma)+\frac{n}{E_n}\sin(n\sigma)\right)+(a_n^\mu)^*\mathrm{e}^{iE_n\tau}\left(i\cos(n\sigma)-\frac{n}{E_n}\sin(n\sigma)\right)\right)$$

$$
\begin{aligned}
(\partial_\tau X+\partial_\sigma X)^2 = \frac{\eta_{\mu\nu}}{(4\pi\alpha')^2}\sum_n\sum_p \Bigg(& a_n^\mu a_p^\nu \mathrm{e}^{-i(E_n+E_p)\tau}\left(i\cos(n\sigma)+\frac{n}{E_n}\sin(n\sigma)\right)\left(i\cos(p\sigma)+\frac{p}{E_p}\sin(p\sigma)\right)- \\
& - a_n^\mu (a_p^\nu)^* \mathrm{e}^{-i(E_n-E_p)\tau}\left(i\cos(n\sigma)+\frac{n}{E_n}\sin(n\sigma)\right)\left(i\cos(p\sigma)-\frac{p}{E_p}\sin(p\sigma)\right)- \\
& - (a_n^\mu)^* a_p^\nu \mathrm{e}^{i(E_n-E_p)\tau}\left(i\cos(n\sigma)-\frac{n}{E_n}\sin(n\sigma)\right)\left(i\cos(p\sigma)+\frac{p}{E_p}\sin(p\sigma)\right)+ \\
& + (a_n^\mu)^* (a_p^\nu)^* \mathrm{e}^{i(E_n+E_p)\tau}\left(i\cos(n\sigma)-\frac{n}{E_n}\sin(n\sigma)\right)\left(i\cos(p\sigma)-\frac{p}{E_p}\sin(p\sigma)\right)\Bigg)=0.
\end{aligned}
$$

Once more, we see that the middle 2 terms and outer 2 terms are complex conjugate pairs. Unlike the closed string, here all cases lead to

$$
\begin{aligned}
L_{n,p} &:= a_n^\mu a_p^\nu \eta_{\mu\nu} = 0 \\
\widetilde{L}_{n,p} &:= a_n^\mu (a_p^\nu)^* \eta_{\mu\nu} = 0,
\end{aligned}
$$

where again to be more in line with ST literature we remap $p \mapsto p-n$

$$
\begin{aligned}
L_{n,p} &= a_n^\mu a_{p-n}^\nu \eta_{\mu\nu} = 0 \\
\widetilde{L}_{n,p} &= a_n^\mu (a_{p-n}^\nu)^* \eta_{\mu\nu} = 0.
\end{aligned}
$$

Since we don't have a summed constraint, the mass of the open string can't be written in terms of vibrational modes, only in terms of the 0-th mode

$$
\begin{aligned}
M^2 = -p^\mu p_\mu &= \frac{1}{(4\pi(\alpha')^2)^2} a_0^\mu a_0^\nu \eta_{\mu\nu} = \\
&= \frac{1}{(4\pi(\alpha')^2)^2} L_{0,0} \equiv 0
\end{aligned}
$$

10.1 The Expansion Modes Algebra

10.1.1 Closed String

Before starting this analysis, let's rename the vibrational modes to α_n^μ to be more in line with the literature Start by finding the canonical momentum

$$
\begin{aligned}
\Pi_\lambda = \frac{\partial \mathscr{L}}{\partial(\partial_\tau X^\lambda)} &= -\frac{T}{2}\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\delta_a^\tau \delta_\lambda^\mu \partial_b X^\nu \eta_{\mu\nu} = \\
&= -\frac{1}{4\pi\alpha'}\left(1-\frac{k\Delta}{2\phi^2}\right)\eta^{\tau b}\partial_b X^\nu \eta_{\lambda\nu} = \\
= \frac{1}{4\pi\alpha'}F^- \partial_\tau X^\nu \eta_{\lambda\nu} &= F^- \frac{i}{4\pi\alpha'}\sum_{n\in\mathbb{Z}}\left(-\alpha_n^\nu \mathrm{e}^{i(-E_n\tau+n\sigma)}+(\alpha_{-n}^\nu)^* \mathrm{e}^{i(E_n\tau+n\sigma)}\right)\eta_{\nu\lambda}.
\end{aligned}
$$

We then proceed to the cannonical Poisson bracket relations

$$
\begin{aligned}
\{X^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')\} &= \delta(\sigma-\sigma')\eta^{\mu\nu} \\
\{X^\mu(\tau,\sigma),X^\nu(\tau,\sigma')\} &= \{\Pi^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')\} = 0,
\end{aligned}
$$

where by expanding the first relation we get

$$
\begin{aligned}
\{X^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')\} = \Bigg\{ & \frac{1}{2\pi\alpha'}\sum_{n\in\mathbb{Z}}\frac{1}{2E_n}\left(\alpha_n^\mu \mathrm{e}^{i(-E_n\tau+n\sigma)}+(\alpha_{-n}^\mu)^* \mathrm{e}^{i(E_n\tau+n\sigma)}\right), \\
& F^- \frac{i}{4\pi\alpha'}\sum_{p\in\mathbb{Z}}\left(-\alpha_p^\nu \mathrm{e}^{i(-E_p\tau+p\sigma')}+(\alpha_{-p}^\nu)^* \mathrm{e}^{i(E_p\tau+p\sigma')}\right)\Bigg\} =
\end{aligned}
$$

$$= \frac{i}{8\pi^2(\alpha')^2}\sum_n\sum_p \frac{1}{2E_n}\bigg(- F^-\left\{\alpha_n^\mu, \alpha_p^\nu\right\}\mathrm{e}^{-i(E_n+E_p)\tau}\mathrm{e}^{i(n\sigma+p\sigma')} + F^-\left\{\alpha_n^\mu, (\alpha_{-p}^\nu)^*\right\}\mathrm{e}^{-i(E_n-E_p)\tau}\mathrm{e}^{i(n\sigma+p\sigma')} -$$

$$- F^-\left\{(\alpha_{-n}^\mu)^*, \alpha_p^\nu\right\}\mathrm{e}^{i(E_n-E_p)\tau}\mathrm{e}^{i(n\sigma+p\sigma')} + F^-\left\{(\alpha_{-n}^\mu)^*, (\alpha_{-p}^\nu)^*\right\}\mathrm{e}^{i(E_n+E_p)\tau}\mathrm{e}^{i(n\sigma+p\sigma')}\bigg) = \delta(\sigma-\sigma')\eta^{\mu\nu}$$

$$\Big\downarrow \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma \mathrm{e}^{-ik\sigma}$$

$$\frac{i}{8\pi^2(\alpha')^2}\sum_n\sum_p \frac{1}{2E_n}\bigg(- F^-\left\{\alpha_n^\mu, \alpha_p^\nu\right\}\mathrm{e}^{-i(E_n+E_p)\tau}\delta_{n,k}\mathrm{e}^{ip\sigma'} + F^-\left\{\alpha_n^\mu, (\alpha_{-p}^\nu)^*\right\}\mathrm{e}^{-i(E_n-E_p)\tau}\delta_{n,k}\mathrm{e}^{ip\sigma'} -$$

$$- F^-\left\{(\alpha_{-n}^\mu)^*, \alpha_p^\nu\right\}\mathrm{e}^{i(E_n-E_p)\tau}\delta_{n,k}\mathrm{e}^{ip\sigma'} + F^-\left\{(\alpha_{-n}^\mu)^*, (\alpha_{-p}^\nu)^*\right\}\mathrm{e}^{i(E_n+E_p)\tau}\delta_{n,k}\mathrm{e}^{ip\sigma'}\bigg) = \frac{\eta^{\mu\nu}}{2\pi}\mathrm{e}^{-ik\sigma'}$$

$$\frac{i}{8\pi^2(\alpha')^2}\sum_p \frac{1}{2E_k}\bigg(- F^-\left\{\alpha_k^\mu, \alpha_p^\nu\right\}\mathrm{e}^{-i(E_k+E_p)\tau}\mathrm{e}^{ip\sigma'} + F^-\left\{\alpha_k^\mu, (\alpha_{-p}^\nu)^*\right\}\mathrm{e}^{-i(E_k-E_p)\tau}\mathrm{e}^{ip\sigma'} -$$

$$- F^-\left\{(\alpha_{-k}^\mu)^*, \alpha_p^\nu\right\}\mathrm{e}^{i(E_k-E_p)\tau}\mathrm{e}^{ip\sigma'} + F^-\left\{(\alpha_{-k}^\mu)^*, (\alpha_{-p}^\nu)^*\right\}\mathrm{e}^{i(E_k+E_p)\tau}\mathrm{e}^{ip\sigma'}\bigg) = \frac{\eta^{\mu\nu}}{2\pi}\mathrm{e}^{-ik\sigma'}$$

$$\Big\downarrow \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma' \mathrm{e}^{im\sigma'}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma' F^-\mathrm{e}^{i(p+m)\sigma'},\ F^- = 1 - \frac{k\Delta}{2\phi^2},\ \text{assume } \phi = \mathit{constant}$$

$$\frac{i}{8\pi^2(\alpha')^2}\frac{F^-}{2E_k}\bigg(- \left\{\alpha_k^\mu, \alpha_{-m}^\nu\right\}\mathrm{e}^{-i(E_k+E_m)\tau} + \left\{\alpha_k^\mu, (\alpha_m^\nu)^*\right\}\mathrm{e}^{-i(E_k-E_m)\tau} -$$

$$- \left\{(\alpha_{-k}^\mu)^*, \alpha_{-m}^\nu\right\}\mathrm{e}^{i(E_k-E_m)\tau} + \left\{(\alpha_{-k}^\mu)^*, (\alpha_m^\nu)^*\right\}\mathrm{e}^{i(E_k+E_m)\tau}\bigg) = \frac{\eta^{\mu\nu}}{2\pi}\delta_{m,k}.$$

The only way for this to hold is

$$\left\{\alpha_k^\mu, \alpha_{-m}^\nu\right\} = \left\{(\alpha_{-k}^\mu)^*, (\alpha_m^\nu)^*\right\} = 0$$

$$\left\{\alpha_k^\mu, (\alpha_m^\nu)^*\right\} = -\left\{(\alpha_{-k}^\mu)^*, \alpha_{-m}^\nu\right\} = -i\frac{4\pi(\alpha')^2}{F^-}E_k\eta^{\mu\nu}\delta_{k,m},$$

thus rescaling we get harmonic oscillator bracket relations

$$a_k^\mu := \frac{1}{2\alpha'}\sqrt{\frac{F^-}{\pi}}\frac{1}{\sqrt{E_k}}\alpha_k^\mu$$

$$(a_k^\mu)^* := \frac{1}{2\alpha'}\sqrt{\frac{F^-}{\pi}}\frac{1}{\sqrt{E_k}}(\alpha_k^\mu)^*$$

$$\Downarrow$$

$$\left\{a_k^\mu, (a_p^\nu)^*\right\} = -i\eta^{\mu\nu}\delta_{k,p}$$

$$\left\{a_k^\mu, a_p^\nu\right\} = \left\{(a_k^\mu)^*, (a_p^\nu)^*\right\} = 0.$$

For the $k = p = 0$ case, we also get the expected relations for x^μ and p^ν:

$$\{\alpha_0^\mu, (\alpha_0^\nu)^*\} = -i\frac{4\pi(\alpha')^2}{F^-}E_0\eta^{\mu\nu}$$

$$\left\{2\pi\alpha'\mu x^\mu + 4\pi i(\alpha')^2p^\mu, 2\pi\alpha'\mu x^\nu - 4\pi i(\alpha')^2p^\nu\right\} = -i\frac{4\pi(\alpha')^2}{F^-}\mu\eta^{\mu\nu}$$

$$(2\pi\alpha')^2\left(\mu^2\{x^\mu,x^\nu\}-2i\alpha'\mu\{x^\mu,p^\nu\}+2i\alpha'\mu\{p^\mu,x^\nu\}+(2\alpha')\{p^\mu,p^\nu\}\right)=-i\frac{4\pi(\alpha')^2}{F^-}\mu\eta^{\mu\nu}$$

$$\Downarrow$$

$$\{x^\mu,x^\nu\}=\{p^\mu,p^\nu\}=0$$
$$\{x^\mu,p^\nu\}=\frac{1}{4\pi\alpha' F^-}\eta^{\mu\nu}.$$

Since $F^-=1-k\Delta/(2\phi^2)$, if we choose

$$\phi=\sqrt{\frac{k\Delta}{2\left(1-\dfrac{1}{4\pi\alpha'}\right)}}$$

then the bracket relation simplifies to the expected

$$\{x^\mu,p^\nu\}=\eta^{\mu\nu}.$$

10.2 The Constraint Algebra

With the bracket relations between the vibrational modes in hand, we can now calculate the algebra generated by the constraints. For simplicity, since $E_n=\sqrt{n^2+\mu^2}\neq 0$, we can switch the constraints to be between the normalized harmonic oscillator modes a_n^μ intead of the α_n^μ.

$$\{L_{n,m},L_{k,p}\}=\left\{a_n^\mu a_m^\nu\eta_{\mu\nu},a_k^{\mu'}a_p^{\nu'}\eta_{\mu'\nu'}\right\}=0$$

$$\begin{aligned}\left\{L_{n,m},\widetilde{L}_{k,p}\right\}&=\left\{a_n^\mu a_m^\nu\eta_{\mu\nu},a_k^{\mu'}(a_{-p}^{\nu'})^*\eta_{\mu'\nu'}\right\}=\\&=\eta_{\mu\nu}\left(a_m^\nu\left\{a_n^\mu,(a_{-p}^{\nu'})^*\right\}a_k^{\mu'}+a_n^\mu\left\{a_m^\nu,(a_{-p}^{\nu'})^*\right\}a_k^{\mu'}\right)\eta_{\mu'\nu'}=\\&=\eta_{\mu\nu}\left(a_m^\nu(-i\eta^{\mu\nu'}\delta_{n,-p})a_k^{\mu'}+a_n^\mu(-i\eta^{\nu\nu'}\delta_{m,-p})a_k^{\mu'}\right)\eta_{\mu'\nu'}=\\&=-i\left(a_m^\mu a_k^\nu\eta_{\mu\nu}\delta_{n,-p}+a_n^\mu a_k^\nu\eta_{\mu\nu}\delta_{m,-p}\right)=\\&=-i\left(L_{m,k}\delta_{n,-p}+L_{n,k}\delta_{m,-p}\right)\end{aligned}$$

$$\begin{aligned}\left\{\widetilde{L}_{n,m},\widetilde{L}_{k,p}\right\}&=\left\{a_n^\mu(a_{-m}^\nu)^*\eta_{\mu\nu},a_k^{\mu'}(a_{-p}^{\nu'})^*\eta_{\mu'\nu'}\right\}=\\&=\eta_{\mu\nu}\left((a_{-m}^\nu)^*\left\{a_n^\mu,(a_{-p}^{\nu'})^*\right\}a_k^{\mu'}+a_n^\mu\left\{(a_{-m}^\nu)^*,a_k^{\mu'}\right\}(a_{-p}^{\nu'})^*\right)\eta_{\mu'\nu'}=\\&=\eta_{\mu\nu}\left((a_{-m}^\nu)^*(-i\eta^{\mu\nu'}\delta_{n,-p})a_k^{\mu'}-a_n^\mu(-i\eta^{\nu\mu'}\delta_{-m,k})(a_{-p}^{\nu'})^*\right)\eta_{\mu'\nu'}=\\&=-i\left(a_k^\mu(a_{-m}^\nu)^*\eta_{\mu\nu}\delta_{n,-p}-a_n^\mu(a_{-p}^\nu)^*\eta_{\mu\nu}\delta_{-m,k}\right)=\\&=-i\left(\widetilde{L}_{k,m}\delta_{n,-p}-\widetilde{L}_{n,p}\delta_{-m,k}\right).\end{aligned}$$

11 Into the Quantum Realm

We now proceed to quantize the KG string through canonical quantization. The string X^μ and it's momentum Π^ν are promoted to operators $\hat{X}^\mu$ and $\hat{\Pi}^\nu$, and we turn the Poisson bracket $\{.,.\}$ into commutator between operators $i\hbar[.,.]$. The equal time bracket relations turn into equal time commutation relations

$$\begin{aligned}
\left[\hat{X}^\mu(\tau,\sigma), \hat{\Pi}^\nu(\tau,\sigma')\right] &= i\delta(\sigma-\sigma')\eta^{\mu\nu} \\
\left[\hat{X}^\mu, \hat{X}^\nu\right] &= \left[\hat{\Pi}^\mu, \hat{\Pi}^\nu\right] = 0 \\
[\hat{a}^\mu_n, \hat{a}^\nu_m] &= \left[(\hat{a}^\mu_n)^\dagger, (\hat{a}^\nu_m)^\dagger\right] = 0 \\
\left[\hat{a}^\mu_n, (\hat{a}^\nu_m)^\dagger\right] &= \eta^{\mu\nu}\delta_{n,m} \\
[\hat{x}^\mu, \hat{x}^\nu] &= [\hat{p}^\mu, \hat{p}^\nu] = 0 \\
[\hat{x}^\mu, \hat{p}^\nu] &= i\eta^{\mu\nu}.
\end{aligned}$$

Thus, the rescaled vibrational mode operators $\hat{a}^\mu_n$ and $(\hat{a}^\nu_m)^\dagger$ are annihilation and creations operators, respectively.

We define a vacuum state of the string to obey

$$\hat{a}^\mu_n|0\rangle = 0, \text{ for } n \neq 0.$$

For $n = 0$, we have the center of mass position and momentum operators, so the vacuum also obeys

$$\begin{aligned}
\hat{x}^\mu|0;x\rangle &= x^\mu|0;x\rangle \\
\hat{p}_\mu|0;x\rangle &= -i\frac{\partial}{\partial x^\mu}|0;x\rangle
\end{aligned}$$

in position representation or alternatively

$$\begin{aligned}
\hat{x}^\mu|0;p\rangle &= -i\frac{\partial}{\partial p_\mu}|0;p\rangle \\
\hat{p}_\mu|0;p\rangle &= p_\mu|0;p\rangle
\end{aligned}$$

in momentum representation.

A generic state arises from a sequence of creation operators on the vacuum

$$((a^{\mu_1}_1)^\dagger)^{n_{\mu_1}}((a^{\mu_2}_2)^\dagger)^{n_{\mu_2}}...((a^{\nu_1}_{-1})^\dagger)^{n_{\nu_1}}((a^{\mu_2}_{-2})^\dagger)^{n_{\nu_2}}...|0\rangle.$$

This (should) give rise to particles.

As in regular ST, we have ghosts arising from the Minkowski metric

$$\left[\hat{a}^\mu_n, (\hat{a}^\nu_m)^\dagger\right] = \eta^{\mu\nu}\delta_{nm}.$$

For constraints, classically we have

$$\begin{aligned}
L_{n,p} &= a^\mu_n a^\nu_p \eta_{\mu\nu} = 0 \\
\widetilde{L}_{n,p} &= a^\mu_n (a^\nu_{-p})^* \eta_{\mu\nu} = 0
\end{aligned}$$

which because of the existence of ghosts we require to have vanishing matrix elements when sandwiched between physical states

$$\langle \text{phys}'|\hat{L}_{n,p}|\text{phys}\rangle = 0 = \langle \text{phys}'|\hat{\widetilde{L}}_{n,p}|\text{phys}\rangle.$$

When translating the constraints into quantum operators, in $L_{n,p}$ we have no ambiguity of ordering since $\hat{a}$ commutes with itself. As for $\widetilde{L}_{n,p}$, there is an ambiguity for $p = -n \neq 0$, so we pick normal ordering with the annihilation operators moved to the right

$$\hat{\widetilde{L}}_{n,-n} = (\hat{\alpha}^\mu_n)^\dagger \hat{\alpha}^\nu_n \eta_{\mu\nu}.$$

The ambiguity manifests in the imposition of this constraint as

$$\langle \text{phys}'| \left(\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1 \right) \widetilde{L}_{n,-n} \right) - c \right) |\text{phys}\rangle = 0,$$

for some costant c. Since classically

$$\begin{aligned} M^2 &= \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2 x^\mu x_\mu - \sum_{n\neq 0}\left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right)\widetilde{L}_{n,-n}\right)\right) = \\ &= \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2 x^\mu x_\mu - \sum_{n\neq 0}\left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right)a_n^\mu (a_n^\nu)^* \eta_{\mu\nu}\right)\right). \end{aligned}$$

we see that the string mass spectrum will be affected by this costant

$$\hat{M}^2 = \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2 \hat{x}^\mu \hat{x}_\mu - \left(\sum_{n\neq 0}\left(\left(\frac{n}{E_n}\right)^2 - 1\right)(\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu} - c\right)\right)$$

The commutation relations between the L's are inherited from the bracket relations

$$\begin{aligned} \left[\hat{L}_{n,m}, \hat{L}_{k,p}\right] &= 0 \\ \left[\hat{L}_{n,m}, \hat{\tilde{L}}_{k,p}\right] &= \hat{L}_{m,k}\delta_{n,-p} + \hat{L}_{n,k}\delta_{m,-p} \\ \left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}\right] &= \hat{\tilde{L}}_{k,m}\delta_{n,-p} - \hat{\tilde{L}}_{n,p}\delta_{-m,k}. \end{aligned}$$

Since $[\hat{L}_{n,m}, \hat{\tilde{L}}_{p,k}]$ only has annihilation operators, it has no ordering ambiguities. $[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{p,k}]$ have ordering ambiguities for $m = -k$ and/or $p = -n$, so we add the anomalous terms

$$\left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}\right] = \hat{\tilde{L}}_{k,m}\delta_{n,-p} - \hat{\tilde{L}}_{n,p}\delta_{-m,k} + C_n\delta_{n,-p} + D_k\delta_{-m,k}$$

Clearly, $C_0 = 0 = D_0$, since in these cases the equivalent terms are composed by $\hat{\tilde{L}}_{0,0} \sim \hat{x}^\mu\hat{x}_\mu + \hat{p}^\mu\hat{p}_\mu$. Also, because of the δ, we also have that $C_{-n} = -C_n$ and $D_{-k} = -D_k$. We get from the jacobi identity

$$\left[\hat{\tilde{L}}_{n,m}, \left[\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{r,s}\right]\right] + \left[\hat{\tilde{L}}_{r,s}, \left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}\right]\right] + \left[\hat{\tilde{L}}_{k,p}, \left[\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{n,m}\right]\right] = 0$$

$$\begin{aligned} &\left[\hat{\tilde{L}}_{n,m}, \left(\hat{\tilde{L}}_{r,p}\delta_{k,-s} - \hat{\tilde{L}}_{k,s}\delta_{-p,r} + C_k\delta_{k,-s} + D_r\delta_{-p,r}\right)\right] + \\ &\qquad + \left[\hat{\tilde{L}}_{r,s}, \left(\hat{\tilde{L}}_{k,m}\delta_{n,-p} - \hat{\tilde{L}}_{n,p}\delta_{-m,k} + C_n\delta_{n,-p} + D_k\delta_{-m,k}\right)\right] + \\ &\qquad\qquad + \left[\hat{\tilde{L}}_{k,p}, \left(\hat{\tilde{L}}_{n,s}\delta_{r,-m} - \hat{\tilde{L}}_{r,m}\delta_{-s,n} + C_r\delta_{r,-m} + D_n\delta_{-s,n}\right)\right] = 0 \end{aligned}$$

$$\begin{aligned} &\left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{r,p}\right]\delta_{k,-s} - \left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,s}\right]\delta_{-p,r} + \left[\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{k,m}\right]\delta_{n,-p} - \\ &\qquad - \left[\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{n,p}\right]\delta_{-m,k} + \left[\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{n,s}\right]\delta_{r,-m} - \left[\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{r,m}\right]\delta_{-s,n} = 0 \end{aligned}$$

$$\begin{aligned} &\left(\hat{\tilde{L}}_{r,m}\delta_{n,-p} - \hat{\tilde{L}}_{n,p}\delta_{-m,r} + C_n\delta_{n,-p} + D_r\delta_{-m,r}\right)\delta_{k,-s} - \left(\hat{\tilde{L}}_{k,m}\delta_{n,-s} - \hat{\tilde{L}}_{n,s}\delta_{-m,k} + C_n\delta_{n,-s} + D_k\delta_{-m,k}\right)\delta_{-p,r} + \\ &+ \left(\hat{\tilde{L}}_{k,s}\delta_{r,-m} - \hat{\tilde{L}}_{r,m}\delta_{-s,k} + C_r\delta_{r,-m} + D_k\delta_{-s,k}\right)\delta_{n,-p} - \left(\hat{\tilde{L}}_{n,s}\delta_{r,-p} - \hat{\tilde{L}}_{r,p}\delta_{-s,n} + C_r\delta_{r,-p} + D_n\delta_{-s,n}\right)\delta_{-m,k} + \\ &+ \left(\hat{\tilde{L}}_{n,p}\delta_{k,-s} - \hat{\tilde{L}}_{k,s}\delta_{-p,n} + C_k\delta_{k,-s} + D_n\delta_{-p,n}\right)\delta_{r,-m} - \left(\hat{\tilde{L}}_{r,p}\delta_{k,-m} - \hat{\tilde{L}}_{k,m}\delta_{-p,r} + C_k\delta_{k,-m} + D_r\delta_{-p,r}\right)\delta_{-s,n} = 0. \end{aligned}$$

For $k = -s$, $n = -p$ and $r = -m$

$$\begin{aligned} &\left(\hat{\tilde{L}}_{r,-r} - \hat{\tilde{L}}_{n,-n} + C_n + D_r\right) - \left(\hat{\tilde{L}}_{k,-r}\delta_{p,s} - \hat{\tilde{L}}_{n,-k}\delta_{r,k} + C_n\delta_{n,k} + D_k\delta_{r,k}\right)\delta_{n,m} + \\ &\qquad + \left(\hat{\tilde{L}}_{k,-k} - \hat{\tilde{L}}_{r,-r} + C_r + D_k\right) - \left(\hat{\tilde{L}}_{n,-k}\delta_{r,n} - \hat{\tilde{L}}_{r,-n}\delta_{k,n} + C_r\delta_{r,n} + D_n\delta_{k,n}\right)\delta_{r,k} + \\ &\qquad\qquad + \left(\hat{\tilde{L}}_{n,-n} - \hat{\tilde{L}}_{k,-k} + C_k + D_n\right) - \left(\hat{\tilde{L}}_{r,-n}\delta_{k,r} - \hat{\tilde{L}}_{k,-r}\delta_{n,r} + C_k\delta_{k,r} + D_r\delta_{n,r}\right)\delta_{k,n} = 0. \end{aligned}$$

Assuming $n \neq m$, $r \neq k$ and $k \neq n$ leaves us with

$$C_n + C_r + C_k + D_r + D_k + D_n = 0,$$

which for $n + k + r = 0$ becomes

$$C_n + C_k + C_{-n-k} + D_n + D_k + D_{-n-k} = 0$$

$$C_n + C_k - C_{n+k} + D_n + D_k - D_{n+k} = 0,$$

and now setting $k = 1$ gives

$$C_n - C_{n+1} + C_1 + D_n - D_{n+1} + D_1 = 0$$
$$C_{n+1} - C_n - C_1 = -\left(D_{n+1} - D_n - D_1\right),$$

meaning C_n and D_n are negatives of each other differing by some constant β, so it suffices to find C_n and later determine the constant β. It is not difficult to see that the solution to this difference eqn is $C_n = An$, so $D_n = -An + \beta$. Since we know $D_{-n} = -D_n$, then $\beta = 0$, and $A = C_1$. We now need to determine the value of C_1. By now, the commutator of $\hat{\tilde{L}}$ is

$$\left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}\right] = \hat{\tilde{L}}_{k,m}\delta_{n,-p} - \hat{\tilde{L}}_{n,p}\delta_{-m,k} + C_1 n\delta_{n,-p} - C_1 k\delta_{-m,k}.$$

Let's calculate the VEV of this commutator for $m = k \neq 0$ with $n = -p = 1$

$$\langle 0| \left[\hat{\tilde{L}}_{1,k}, \hat{\tilde{L}}_{k,-1}\right] |0\rangle = \langle 0|\hat{\tilde{L}}_{1,k}\hat{\tilde{L}}_{k,-1}|0\rangle$$

$$\langle 0|\hat{\tilde{L}}_{k,k} + C_1|0\rangle = \langle 0|(a^\mu_{-k})^\dagger a^\nu_1 \eta_{\mu\nu}(a^{\mu'}_1)^\dagger a^{\nu'}_k \eta_{\mu'\nu'}|0\rangle$$
$$\langle 0|C_1|0\rangle = 0 \iff C_1 = 0,$$

so in actuallity the constraint algebra has no anomalies.

Let us now denote the ground state of momentum p^μ as $|0;p\rangle$. The mass-shell condition

$$\hat{M}^2 = \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2\hat{x}^\mu\hat{x}_\mu - \left(\sum_{n\neq 0}\left(\left(\frac{n}{E_n}\right)^2 - 1\right)(\hat{a}^\mu_n)^\dagger\hat{a}^\nu_n\eta_{\mu\nu} - c\right)\right)$$

implies that

$$\langle 0;p|\hat{M}^2|0;p\rangle = \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2\langle 0;p|\hat{x}^\mu\hat{x}_\mu|0;p\rangle - \left(\sum_{n\neq 0}\left(\left(\frac{n}{E_n}\right)^2 - 1\right)\langle 0;p|(\hat{a}^\mu_n)^\dagger\hat{a}^\nu_n\eta_{\mu\nu}|0;p\rangle - \langle 0;p|c|0;p\rangle\right)\right)$$

$$M^2 = \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2\frac{\partial\psi^*(p)}{\partial p_\mu}\frac{\partial\psi(p)}{\partial p^\mu} + c\right)$$
$$(4\pi(\alpha')^2)^2 p^2 = -\left((2\pi\alpha'\mu)^2\left|\frac{\partial\psi(p)}{\partial p}\right|^2 + c\right)$$

Now looking at the first excited state $\zeta_\mu(\hat{a}^\mu_1)^\dagger|0;p\rangle$ with $\zeta_\mu = \zeta_\mu(p)$ being the polarization covector, the mass-shell now reads

$$\langle 0;p|\hat{a}^\mu_1\zeta_\mu\hat{M}^2\zeta_\nu(\hat{a}^\nu_1)^\dagger|0;p\rangle = \frac{1}{(4\pi(\alpha')^2)^2}\Bigg((2\pi\alpha'\mu)^2\langle 0;p|\hat{a}^\mu_1\zeta_\mu\hat{x}^{\mu'}\hat{x}_{\mu'}\zeta_\nu(\hat{a}^\nu_1)^\dagger|0;p\rangle -$$
$$-\left(\sum_{n\neq 0}\left(\left(\frac{n}{E_n}\right)^2 - 1\right)\langle 0;p|\hat{a}^\mu_1\zeta_\mu(\hat{a}^{\mu'}_n)^\dagger\hat{a}^{\nu'}_n\eta_{\mu'\nu'}\zeta_\nu(\hat{a}^\nu_1)^\dagger|0;p\rangle - \langle 0;p|\hat{a}^\mu_1\zeta_\mu c\zeta_\nu(\hat{a}^\nu_1)^\dagger|0;p\rangle\right)\Bigg)$$

$$M^2\zeta^\mu\zeta_\mu = \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2\left|\frac{\partial\psi(p)}{\partial p}\right|^2\zeta^\mu\zeta_\mu - \left(\frac{1}{1+\mu^2} - 1\right)\zeta^\mu\zeta_\mu + c\zeta^\mu\zeta_\mu\right)$$

$$(4\pi(\alpha')^2)^2p^2 = -\left((2\pi\alpha'\mu)^2\left|\frac{\partial\psi(p)}{\partial p}\right|^2 - \left(\frac{1}{1+\mu^2}-1\right)+c\right).$$

The auxiliary $\hat{\widetilde{L}}_{1,0}\zeta_{\mu'}(\hat{a}_1^{\mu'})^\dagger|0;p\rangle = 0$ condition implies

$$\hat{\widetilde{L}}_{1,0}\zeta_{\mu'}(\hat{a}_1^{\mu'})^\dagger|0;p\rangle = 0$$

$$(\hat{a}_0^{\mu})^\dagger\hat{a}_1^{\nu}\eta_{\mu\nu}\zeta_{\mu'}(\hat{a}_1^{\mu'})^\dagger|0;p\rangle = 0$$

$$\zeta_\mu(\hat{a}_0^{\mu})^\dagger|0;p\rangle = 0$$

$$\zeta_\mu\left(2\pi\alpha'\mu\hat{x}^\mu - 4\pi(\alpha')^2 i\hat{p}^\mu\right)|0;p\rangle = 0$$

$$-2\pi\alpha'\mu i\zeta_\mu\frac{\partial\psi(p)}{\partial p_\mu} = 4\pi(\alpha')^2 i\zeta_\mu p^\mu$$

$$\zeta_\mu p^\mu = -\frac{\mu}{2\alpha'}\zeta_\mu\frac{\partial\psi(p)}{\partial p_\mu}.$$

If the momentum wave function is chosen to be (need to justify this)

$$\psi(p) = \frac{\alpha'}{\mu}p^2,$$

we get that $\zeta_\mu p^\mu = 0$. Using this in the mass-shell condition for the first excited state yields

$$(4\pi(\alpha')^2)^2p^2 = -\left((2\pi\alpha'\mu)^2 4\frac{(\alpha')^2}{\mu^2}p^2 - \left(\frac{1}{1+\mu^2}-1\right)+c\right)$$

$$2(4\pi(\alpha')^2)^2p^2 = \frac{1}{1+\mu^2} - 1 - c,$$

and for the ground state

$$2(4\pi(\alpha')^2)^2p^2 = -c.$$

In general, the n-th excited state will have momentum given by

$$2(4\pi(\alpha')^2)^2p^2 = \frac{n^2}{n^2+\mu^2} - 1 - c.$$

The norm of the states is given by $\zeta_\mu\zeta^\mu$, thus if we choose p to lie on the $(0,1)$-plane, then the $D-2$ spacelike polarizations normal to that plane clearly have positive norm. If we choose c such that the $n = d$, $d = (D-1)$ (we choose this because the solution only has $n = 0, 1, 2, ..., d$ terms by the $\widetilde{L}$ constraints) excited state is a tachyon with $p^2 > 0$, then p can be taken to have no time component and thus ζ is time-like with negative norm. If $p^2 < 0$, p can be chosen to ONLY have a time component, leaving ζ spacelike with positive norm again. At last, if $p^2 = 0$, then ζ is proportional to p with zero norm. Thus, for the absence of ghosts we require

$$c \geq \frac{d^2}{d^2+\mu^2} - 1.$$

In the boundary case, the 3rd excited vector state is massless and the rest (including the scalar ground state) are tachyons... This analysis, however, is to get spin 1 particles from the closed string. The actual massless particle we want from closed strings is a symmetric spin 2 particle, so let's consider now the state $\zeta_{\mu\nu}(\hat{a}_d^{\mu})^\dagger(\hat{a}_d^{\nu})^\dagger|0;p\rangle$ where $\zeta_{\mu\nu} = \zeta_{\nu\mu}$ is the polarization tensor. The mass-shell is

$$\langle 0;p|\hat{a}_d^{\alpha}\hat{a}_d^{\beta}\zeta_{\alpha\beta}\hat{M}^2\zeta_{\mu\nu}(\hat{a}_d^{\mu})^\dagger(\hat{a}_d^{\nu})^\dagger|0;p\rangle = \frac{1}{(4\pi(\alpha')^2)^2}\Bigg((2\pi\alpha'\mu)^2\langle 0;p|\hat{a}_d^{\alpha}\hat{a}_d^{\beta}\zeta_{\alpha\beta}\hat{x}^{\mu'}\hat{x}_{\mu'}\zeta_{\mu\nu}(\hat{a}_d^{\mu})^\dagger(\hat{a}_d^{\nu})^\dagger|0;p\rangle -$$

$$-\left(\sum_{n\neq 0}\left(\left(\frac{n}{E_n}\right)^2 - 1\right)\langle 0;p|\hat{a}_d^{\alpha}\hat{a}_d^{\beta}\zeta_{\alpha\beta}(\hat{a}_n^{\mu'})^\dagger\hat{a}_n^{\nu'}\eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^{\mu})^\dagger(\hat{a}_d^{\nu})^\dagger|0;p\rangle - \langle 0;p|\hat{a}_d^{\alpha}\hat{a}_d^{\beta}\zeta_{\alpha\beta}c\zeta_{\mu\nu}(\hat{a}_d^{\mu})^\dagger(\hat{a}_d^{\nu})^\dagger|0;p\rangle\right)\Bigg)$$

$$M^2\zeta_{\mu\nu}\zeta^{\mu\nu} = \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2\left|\frac{\partial\psi(p)}{\partial p}\right|^2\zeta_{\mu\nu}\zeta^{\mu\nu} - \left(2\left(\frac{d^2}{d^2+\mu^2}-1\right)\zeta_{\mu\nu}\zeta^{\mu\nu} - c\zeta_{\mu\nu}\zeta^{\mu\nu}\right)\right)$$
$$(4\pi(\alpha')^2)^2p^2 = -\left((2\pi\alpha'\mu)^2\left|\frac{\partial\psi(p)}{\partial p}\right|^2 - \left(2\left(\frac{d^2}{d^2+\mu^2}-1\right) - c\right)\right)$$
$$2(4\pi(\alpha')^2)^2p^2 = 2\left(\frac{d^2}{d^2+\mu^2}-1\right) - c.$$

Using the auxiliary $\hat{\tilde{L}}_{d,0}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle = 0$ condition implies

$$\hat{\tilde{L}}_{d,0}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle = 0$$
$$(\hat{a}_0^{\mu'})^\dagger\hat{a}_d^{\nu'}\eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle = 0$$
$$\left((\hat{a}_0^{\mu'})^\dagger\eta_{\mu'\nu'}\zeta_{\mu\nu}\eta^{\nu'\mu}(a_d^\nu)^\dagger + (\hat{a}_0^{\mu'})^\dagger\eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger\hat{a}_d^{\nu'}(\hat{a}_d^\nu)^\dagger\right)|0;p\rangle = 0$$
$$\left((\hat{a}_0^{\mu'})^\dagger\eta_{\mu'\nu'}\zeta_{\mu\nu}\eta^{\nu'\mu}(a_d^\nu)^\dagger + (\hat{a}_0^{\mu'})^\dagger\eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger\eta^{\nu'\nu}\right)|0;p\rangle = 0$$
$$2\zeta_{\mu\nu}(\hat{a}_0^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle = 0$$
$$\zeta_{\mu\nu}(\hat{a}_d^\nu)^\dagger\left(2\pi\alpha'\mu\hat{x}^\mu - 4\pi(\alpha')^2i\hat{p}^\mu\right)|0;p\rangle = 0$$
$$\zeta_{\mu\nu}\left(-2\pi\alpha'\mu i\frac{\partial\psi(p)}{\partial p_\mu} - 4\pi(\alpha')^2ip^\mu\right)(\hat{a}_d^\nu)^\dagger|0\rangle = 0$$
$$\zeta_{\mu\nu}p^\mu(\hat{a}_d^\nu)^\dagger|0\rangle = 0,$$

implying that

$$\zeta_{\mu\nu}p^\mu = \zeta_{\mu\nu}p^\nu = 0.$$

Since the norm of the spin-2 states is given by $\zeta^{\mu\nu}\zeta_{\mu\nu}$, we need those to not be negative. Since ζ is symmetric and we work in $(-,+,+,...)$ signature, $\zeta^{\mu\nu}\zeta_{\mu\nu} \geq 0$ always hold, so we just need to choose c such that we get ate leats 1 massless spin-2 state. For that, we see that

$$c = 2\left(\frac{n^2}{n^2+\mu^2}-1\right)$$

does the trick for the n-th state, and since for the non-existence of ghosts in the spin-1 states we required

$$c \geq \frac{d^2}{d^2+\mu^2} - 1,$$

which is negative, we can thus choose

$$c \equiv \frac{2d^2}{d^2+\mu^2} - 2,$$

leaving the d-th spin-2 state massless, the rest being tachyons... This, however, is if we impose that one spin-2 state have to be massless. Since granular space-time affects wave propagation speed, in the effective limit gravitons might be effectively massive, thus we can get rid of the tachyons by letting $c > 0$ if need be.

The fact that $\zeta_{\mu\nu} = \zeta_{\nu\mu}$ leaves $\zeta_{\mu\nu}$ with $D(D+1)/2$ independent components. The traceless condition $\eta^{\mu\nu}\zeta_{\mu\nu} = 0$ takes out 1 from that, leaving $(D(D+1)/2) - 1$ allowed polarizations. Finally, the condition $\zeta_{\mu\nu}p^\mu = 0$ takes D out of those, leaving $(D(D-1)/2) - 1$ allowed polarizations.

$$\frac{D(D-1)}{2} - 1 = \frac{D^2-D-2}{2} = \frac{(D-2)(D+1)}{2}.$$

This leaves 5 polarizations in $D = 4$.

Let's now analyse spurious states to see the conditions to maximize 0-norm states. A physical state $|\phi\rangle$ is a state satisfying the constraints

$$\hat{L}_{n,m}|\phi\rangle = 0,\ \forall n, m \in \mathbb{Z}$$

$$\hat{\tilde{L}}_{n,m}|\phi\rangle = 0,\ n \neq -m$$

$$\left(\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right)\hat{\tilde{L}}_{n,-n}\right) - c\right)|\phi\rangle = 0.$$

A spurious state $|\psi\rangle$ is a state satisfying the last constraint and orthogonal to all physical states, i.e,

$$\left(\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right)\hat{\tilde{L}}_{n,-n}\right) - c\right)|\psi\rangle = 0,$$

$$\langle\phi|\psi\rangle = 0.$$

Spurious states can be written in the from

$$|\psi\rangle = \sum_{n\neq -m} \hat{\tilde{L}}_{n,m}|\chi_{n,m}\rangle$$

with $|\chi_{n,m}\rangle$ satisfying

$$\left(\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right)\hat{\tilde{L}}_{n,-n}\right) - c - k - p\right)|\chi_{k,p}\rangle = 0.$$

To see that this state is indeed orthogonal to $|\phi\rangle$, we perform the inner product

$$\langle\phi|\psi\rangle = \sum_{n\neq -m} \langle\phi|\hat{\tilde{L}}_{n,m}|\chi_{n,m}\rangle = \sum_{n\neq -m} \langle\chi_{n,m}|\hat{\tilde{L}}_{-m,-n}|\phi\rangle^* \equiv 0.$$

The interesting case is when $|\psi\rangle$ is both physical AND spurious, since by construction those states have 0-norm, being orthogonal to every physical state, including themselves. Such states can be constructed by considering spurious states of the form

$$|\psi_{1,n}\rangle = \sum_m \hat{\tilde{L}}_{n,m}|\widetilde{\chi}_{1,m}\rangle$$

where $|\widetilde{\chi}_{1,m}\rangle$ satisfies

$$\hat{\tilde{L}}_{k,p}|\widetilde{\chi}_{1,m}\rangle = 0,\ k \neq -p$$

$$\left(\sum_{n'} \left(\left(\left(\frac{n'}{E_{n'}}\right)^2 - 1\right)\hat{\tilde{L}}_{n',-n'}\right) - c - 1 - m\right)|\widetilde{\chi}_{1,m}\rangle = 0.$$

These states are annihilated by $\hat{\tilde{L}}_{k,p}$, for $p \neq -n$, and for this case we have that

$$\sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right)\hat{\tilde{L}}_{2,-n}|\psi_{1,n}\rangle\right) =$$

$$= \sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right)\hat{\tilde{L}}_{2,-n}\sum_m \hat{\tilde{L}}_{n,m}|\widetilde{\chi}_{1,m}\rangle\right) =$$

$$= \sum_n \left(\left(\left(\frac{n}{E_n}\right)^2 - 1\right)\hat{\tilde{L}}_{n,-n}\right)|\widetilde{\chi}_{1,-2}\rangle,$$

which vanish if $c = 1$.

12 Polyakov Action in terms of LQG Variables

Turn embbeding fields X^μ into vector in the spin-1 representation of $\mathbb{R}^{d+1} \rtimes \mathrm{Spin}(d,1)$, X^I, and promote partial derivative to covariant derivative $\partial_a \mapsto \mathcal{D}_a$ acting as

$$\mathcal{D}_a X^I = \partial_a X^I + k\mathcal{A}^I_{aJ} X^J,$$

where the WS $\mathbb{R}^{d+1} \rtimes \mathrm{Spin}(d,1)$ connection $\mathcal{A}_a$ with $I, J = 0, ..., d, d+1$ is given by

$$\mathcal{A}_a X = \begin{bmatrix} 0 & \beta^{01}_a & \beta^{02}_a & ... & \beta^{0d}_a & p^0_a/l \\ \beta^{01}_a & 0 & -\theta^{12}_a & ... & -\theta^{1d}_a & p^1_a/l \\ \beta^{02}_a & \theta^{12}_a & 0 & ... & -\theta^{2d}_a & p^2_a/l \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta^{0d}_a & \theta^{1d}_a & \theta^{2d}_a & ... & 0 & p^d_a/l \\ -\epsilon p^0_a/l & \epsilon p^1_a/l & \epsilon p^2_a/l & ... & \epsilon p^d_a/l & 0 \end{bmatrix} \begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ \vdots \\ X^d \\ 1 \end{bmatrix} = \left(\beta^{0\alpha}_a B_{0\alpha} + \frac{1}{2}\theta^{\alpha\beta}_a J_{\alpha\beta} + \frac{p^\mu_a}{l} P_\mu \right) X,$$

where $\epsilon = \mathrm{sgn}(\Lambda)$, i.e the sign of the background cosmological constant with $\alpha, \beta = 1, ..., d$ and $\mu = 0, ..., d$, so the action becomes

$$S = -\frac{T}{2} \int \mathrm{d}^2 x \sqrt{-g} g^{ab} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ},$$

where the metrics can be recast in terms of auxiliary and bulk vielbein fields as in

$$S = -\frac{T}{2} \int \mathrm{d}^2 x e e^a_{I'} e^b_{J'} \eta^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ},$$

where

$$e := \sqrt{-\det(e^I_a e^J_b \eta_{IJ})} = \sqrt{-\frac{1}{2} \varepsilon^{ff'} \varepsilon^{gg'} (e^M_f e^N_g \eta_{MN})(e^{M'}_{f'} e^{N'}_{g'} \eta_{M'N'})}.$$

12.1 Checking the derivative

Since the connection is non-abelian, it should transform as

$$\mathcal{A}'_a = \mathcal{P}\mathcal{A}_a \mathcal{P}^{-1} - \frac{1}{k}(\partial_a \mathcal{P})\mathcal{P}^{-1},$$

where $\mathcal{P} = \mathrm{e}^{\beta^{0\alpha} B_{0\alpha} + \theta^{\alpha\beta} J_{\alpha\beta}/2 + p^\mu P_\mu / l}$. Putting this into a gauge transformed derivative we have

$$\begin{aligned} \mathcal{D}'_a X' &= \partial_a X' + k\mathcal{A}'_a X' = \\ &= \partial_a(\mathcal{P}X) + k(\mathcal{P}\mathcal{A}_a\mathcal{P}^{-1} - \frac{1}{k}(\partial_a\mathcal{P})\mathcal{P}^{-1})(\mathcal{P}X) = \\ &= \partial_a \mathcal{P} X + \mathcal{P}\partial_a X + k\mathcal{P}\mathcal{A}_a X - \partial_a \mathcal{P} X \equiv \mathcal{P}\mathcal{D}_a X, \end{aligned}$$

so this is a proper covariant derivative for the Poincaré group.

Since under gauge transformations $\mathcal{D}_a X$ transforms covariantly, the action is not invariant under diffeomorphisms. In fact, it transforms as

$$\begin{aligned} S' = -\frac{T}{2} \int \mathrm{d}^2 x e' e'^a_{I'} e'^b_{J'} \eta^{I'J'} \mathcal{D}'_a X'^I \mathcal{D}'_b X'^J \eta_{IJ} = \\ = -\frac{T}{2} \int \mathrm{d}^2 x e e^a_{K'} (\mathcal{P}^{-1})^{K'}{}_{I'} e^b_{L'} (\mathcal{P}^{-1})^{L'}{}_{J'} \eta^{I'J'} \mathcal{P}^I{}_K \mathcal{D}_a X^K \mathcal{P}^J{}_L \mathcal{D}_b X^L \eta_{IJ}, \end{aligned}$$

so we promote the killing forms η_{IJ} and $\eta^{I'J'}$ to doubly co(ntra)-variant forms K_{IJ} and $K^{I'J'}$ such that $K'_{IJ} = (\mathcal{P}^{-1})^M{}_I (\mathcal{P}^{-1})^N{}_J K_{MN}$ and $K'^{I'J'} = \mathcal{P}^{I'}_{M'} \mathcal{P}^{J'}_{N'} K^{M'N'}$, so the new action is

$$S = -\frac{T}{2} \int \mathrm{d}^2 x e e^a_{I'} e^b_{J'} K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ}.$$

For consistency, let's assume K is a background field $K = K(X)$ like the metric G.

12.2 EoMs

12.2.1 W.r.t e

$$\begin{aligned}
\frac{\delta S}{\delta e^c_K} &\propto \left(\frac{\partial e}{\partial e^c_K} e^a_{I'} e^b_{J'} K^{I'J'} + e \frac{\partial}{\partial e^c_K}(e^a_{I'} e^b_{J'} K^{I'J'})\right) \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} = \\
&= \left(-e e^K_c e^a_{I'} e^b_{J'} K^{I'J'} + 2e \delta^a_c \delta^K_{I'} e^b_{J'} K^{I'J'}\right) \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} = \\
&= -e e^K_c e^a_{I'} e^b_{J'} K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} + 2e e^b_{J'} K^{KJ'} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ} = 0
\end{aligned}$$

$$\begin{aligned}
T^K_c &:= 2 e^b_{J'} K^{KJ'} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ} - e^K_c e^a_{I'} e^b_{J'} K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} = 0 \\
e^K_c &= 2 f e^{bK} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ}, \\
\frac{1}{f} &= e^a_{I'} e^b_{J'} K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ}
\end{aligned}$$

12.2.2 W.r.t A

$$\begin{aligned}
\frac{\delta S}{\delta \mathcal{A}^{KL}_c} &= \frac{T}{2}\left(e e^a_{I'} e^b_{J'} K^{I'J'} \frac{\partial}{\partial \mathcal{A}^{KL}_c} \left(\mathcal{D}_a X^I \mathcal{D}_b X^J\right) K_{IJ}\right) = \\
&= T\left(e e^a_{I'} e^b_{J'} K^{I'J'} \delta^c_a \delta^I_{[K} \eta_{L]I'} X^{I'} \mathcal{D}_b X^J K_{IJ}\right) = \\
&= T e e^c_{I'} e^b_{J'} K^{I'J'} \mathcal{D}_b X_{[K} X_{L]} \overset{!}{=} 0
\end{aligned}$$

$$\mathcal{T}^{IJ}_a = \mathcal{D}_a X^{[I} X^{J]} = 0$$

12.2.3 W.r.t X

$$\begin{aligned}
\frac{\delta S}{\delta X^K} = -\frac{T}{2}\Bigg(&\partial_c\left(e e^a_{I'} e^b_{J'} K^{I'J'} \frac{\partial}{\partial(\partial_c X^K)}\left(\mathcal{D}_a X^I \mathcal{D}_b X^J\right) K_{IJ}\right) - e e^a_{I'} e^b_{J'} K^{I'J'} \frac{\partial}{\partial X^K}\left(\mathcal{D}_a X^I \mathcal{D}_b X^J\right) K_{IJ} - \\
&- e e^a_{I'} e^b_{J'} \frac{\partial K^{I'J'}}{\partial X^K} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} - e e^a_{I'} e^b_{J'} K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \frac{\partial K_{IJ}}{\partial X^K}\Bigg) = \\
= -T\Bigg(&\left(\partial_c\left(e e^a_{I'} e^b_{J'} K^{I'J'} \delta^c_a \delta^I_K \mathcal{D}_b X^J K_{IJ}\right) - e e^a_{I'} e^b_{J'} K^{I'J'} k \mathcal{A}^I_{aL} \delta^{I'}_L \mathcal{D}_b X^J K_{IJ}\right) - \\
&- e e^a_{I'} e^b_{J'} \partial_K K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} - e e^a_{I'} e^b_{J'} K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \partial_K K_{IJ}\Bigg) = \\
= -T&\left(\mathcal{D}_a\left(e e^a_{I'} e^b_{J'} K^{I'J'} \mathcal{D}_b X^I K_{IK}\right) - e e^a_{I'} e^b_{J'}\left(\partial_K K^{I'J'} K_{IJ} + K^{I'J'} \partial_K K_{IJ}\right) \mathcal{D}_a X^I \mathcal{D}_b X^J\right) \overset{!}{=} 0
\end{aligned}$$

$$\mathcal{D}_a\left(e e^a_{I'} e^b_{J'} K^{I'J'} \mathcal{D}_b X^I K_{IK}\right) = e e^a_{I'} e^b_{J'}\left(\partial_K K^{I'J'} K_{IJ} + K^{I'J'} \partial_K K_{IJ}\right) \mathcal{D}_a X^I \mathcal{D}_b X^J$$

In "constant gauge" $K_{IJ}(X) = \eta_{IJ}$ and in conformal gauge $e e^a_{I'} e^b_{J'} \eta^{I'J'} = \eta^{ab}$, this reduces to

$$\eta^{ab} \mathcal{D}_a \mathcal{D}_b X^I = 0.$$

More explicitly,

$$\begin{aligned}
\eta^{ab} \mathcal{D}_a\left(\partial_b X^I + k \mathcal{A}^I_{bJ} X^J\right) &= 0 \\
\eta^{ab}\left(\left(\partial_a \partial_b X^I + k \mathcal{A}^I_{aJ} \partial_b X^J\right) + \left(k \partial_a\left(\mathcal{A}^I_{bJ} X^J\right) + k^2 \mathcal{A}^I_{aI'} \mathcal{A}^{I'}_{bJ} X^J\right)\right) &= 0 \\
\eta^{ab}\left(\partial_a \partial_b X^I + 2k \mathcal{A}^I_{aJ} \partial_b X^J + k \partial_a \mathcal{A}^I_{bJ} X^J + k^2 \mathcal{A}^I_{aI'} \mathcal{A}^{I'}_{bJ} X^J\right) &= 0
\end{aligned}$$

$$\eta^{ab}\left(\delta^I_J\partial_a\partial_b + 2k\mathcal{A}^I_{aJ}\partial_b\right)X^J = -\left(k\eta^{ab}\partial_a\mathcal{A}^I_{bJ}X^J + k^2\eta^{ab}\mathcal{A}^I_{aI'}\mathcal{A}^{I'}_{bJ}X^J\right),$$

which has implicit solution given by

$$X^K(x) = -\int \mathrm{d}^2x' G^K_{\ I}(x,x')\left(k\eta^{a'b'}\partial_{a'}\mathcal{A}^I_{b'J}(x')X^J(x') + k^2\eta^{a'b'}\mathcal{A}^I_{a'I'}(x')\mathcal{A}^{I'}_{b'J}(x')X^J(x')\right),$$

where the Green's function matrix $G^K_{\ J}(x,x')$ satisfies

$$\eta^{ab}\left(\delta^I_K\partial_a\partial_b + 2k\mathcal{A}^I_{aK}(x)\partial_b\right)G^K_{\ J}(x,x') = \delta^I_J\delta(x,x').$$

Boundary conditions come from

$$\begin{aligned}\delta S &= \int \mathrm{d}^2x\left(\frac{\partial\mathscr{L}}{\partial X^K}\delta X^K + \frac{\partial\mathscr{L}}{\partial(\partial_c X^K)}\delta(\partial_c X^K) + ...\right) = \\ &= \int \mathrm{d}^2x\left((\mathrm{EoM})\delta X^K + \partial_c\left(\frac{\partial\mathscr{L}}{\partial(\partial_c X^K)}\delta X^K\right) + ...\right) = \\ &= \int \mathrm{d}^2x(\mathrm{EoM})\delta X^K + \int \mathrm{d}\tau\left(\frac{\partial\mathscr{L}}{\partial(\partial_\sigma X^K)}\delta X^K\right)\Big|_0^{\sigma_1} + ... = 0,\end{aligned}$$

thus

$$\begin{gathered}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\sigma X^K)}\delta X^K\right)\Big|_0^{\sigma_1} = 0 \\ \left(\left(-Tee^\sigma_i e^{bi}\mathcal{D}_b X_K\right)\delta X^K\right)\big|_0^{\sigma_1} = 0 \\ \left((\mathcal{D}_\sigma X_K)\delta X^K\right)\big|_0^{\sigma_1} = 0,\end{gathered}$$

so either $\delta X^K(\tau,\sigma_*) = 0,\ \sigma_* = 0,\sigma_1$ (Dirichlet B.C) or $\mathcal{D}_\sigma X^K(\tau,\sigma_*) = 0,\ \sigma_* = 0,\sigma_1$ (free end-point B.C). For closed strings, $X^K(\tau,\sigma) = X^K(\tau,\sigma+2\pi)$.

When connection is trivial EoM reduces to regular wave eqn, which have the regular string solution

$$X^I_0(\tau,\sigma) = X^I_{0L}(\sigma^+) + X^I_{0R}(\sigma^-),$$

with

$$X^I_{0L}(\sigma^+) = \frac{1}{2}x^I + \frac{1}{2}\alpha' p^I\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^I_n \mathrm{e}^{-in\sigma^+},\ \sigma^+ = \tau+\sigma$$

$$X^I_{0R}(\sigma^-) = \frac{1}{2}x^I + \frac{1}{2}\alpha' p^I\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^I_n \mathrm{e}^{-in\sigma^-},\ \sigma^- = \tau-\sigma.$$

This, however, does not amount to any motion in actual space-time, since with trivial connection the vector does not change after parallel transport. For a more interesting case, let's consider $A^{\alpha\beta}_\tau = A^{\alpha\beta}_\sigma = a\varepsilon^{\alpha\beta}$, which amounts to flat space-time. The EoM turn into

$$\left(\partial^2_\sigma - \partial^2_\tau\right)X^I + 2ka(\varepsilon^{\alpha\beta}T_{\alpha\beta})^I_{\ J}\left(\partial_\sigma - \partial_\tau\right)X^J = 0$$

$$\left(\partial^2_\sigma - \partial^2_\tau\right)\begin{bmatrix}X^0\\X^1\\X^2\\\vdots\\X^d\end{bmatrix} + 4ka\left(\partial_\sigma - \partial_\tau\right)\begin{bmatrix}0 & 1 & 1 & ... & 1\\1 & 0 & 1 & ... & 1\\1 & -1 & 0 & ... & 1\\\vdots & \vdots & \vdots & \ddots & \vdots\\1 & -1 & -1 & ... & 0\end{bmatrix}\begin{bmatrix}X^0\\X^1\\X^2\\\vdots\\X^d\end{bmatrix} = 0$$

$$\begin{bmatrix}\left(\partial^2_\sigma - \partial^2_\tau\right)X^0 + 4ka\left(\partial_\sigma - \partial_\tau\right)\sum_{i>0}X^i\\ \left(\partial^2_\sigma - \partial^2_\tau\right)X^1 + 4ka\left(\partial_\sigma - \partial_\tau\right)\sum_{i\neq 1}X^i\\ \left(\partial^2_\sigma - \partial^2_\tau\right)X^2 + 4ka\left(\partial_\sigma - \partial_\tau\right)\left(X^0 - X^1 + \sum_{i>2}X^i\right)\\ \vdots\\ \left(\partial^2_\sigma - \partial^2_\tau\right)X^d + 4ka\left(\partial_\sigma - \partial_\tau\right)\left(X^0 - \sum_{i=1}^{d-1}X^i\right)\end{bmatrix} = 0$$

$$\left(\partial_\sigma^2 - \partial_\tau^2\right) X^I + 4ka\left(\partial_\sigma - \partial_\tau\right)\left((\mathrm{sgn}(I))^2 X^0 + \sum_{i>0}\mathrm{sgn}(i-I)X^i\right) = 0$$

$$\left(\partial_\sigma - \partial_\tau\right)\left(\left(\partial_\sigma + \partial_\tau\right) X^I + 4ka\left((\mathrm{sgn}(I))^2 X^0 + \sum_{i>0}\mathrm{sgn}(i-I)X^i\right)\right) = 0$$

$$\left(\partial_\sigma + \partial_\tau\right) X^I + 4ka\left((\mathrm{sgn}(I))^2 X^0 + \sum_{i>0}\mathrm{sgn}(i-I)X^i\right) = C_1^I.(\tau+\sigma),\ C^I \in \mathbb{C}$$

$$\left(\partial_\sigma + \partial_\tau\right) X^I = C_1^I.(\tau+\sigma) - 4ka\left((\mathrm{sgn}(I))^2 X^0 + \sum_{i>0}\mathrm{sgn}(i-I)X^i\right)$$

$$X^I = \frac{1}{2}C_1^I.\left(\tau^2+\sigma^2\right) + C_2^I.(\tau-\sigma) - 4ka\int \mathrm{d}^2x'\left((\mathrm{sgn}(I))^2 X^0 + \sum_{i>0}\mathrm{sgn}(i-I)X^i\right).$$

If $D=2$, we get a symmetric system of eqns

$$X^0 = \frac{1}{2}C_1^0.\left(\tau^2+\sigma^2\right) + C_2^0.(\tau-\sigma) - 4ka\int \mathrm{d}^2x' X^1$$
$$X^1 = \frac{1}{2}C_1^1.\left(\tau^2+\sigma^2\right) + C_2^1.(\tau-\sigma) - 4ka\int \mathrm{d}^2x' X^0$$

$$X^0 = \frac{1}{2}C_1^0.\left(\tau^2+\sigma^2\right) + C_2^0.(\tau-\sigma) - 4ka\int \mathrm{d}^2x'\left(\frac{1}{2}C_1^1.\left(\tau'^2+\sigma'^2\right) + C_2^1.(\tau'-\sigma') - 4ka\int \mathrm{d}^2x'' X^0\right)$$
$$X^0 - 16k^2a^2\int \mathrm{d}^2x'\int \mathrm{d}^2x'' X^0 = \frac{1}{2}C_1^0.\left(\tau^2+\sigma^2\right) + C_2^0.(\tau-\sigma) - \frac{1}{2}C_1^1\left(\frac{\tau^3}{3}\sigma + \tau\frac{\sigma^3}{3}\right) - C_2^1\left(\frac{\tau^2}{2}\sigma - \tau\frac{\sigma^2}{2}\right).$$

This integral equation is tricky to deal with, but we can turn it into a differential equation by applying $\partial_\sigma\partial_\tau$ twice:

$$\left(\partial_\sigma\partial_\tau\right)^2 X^0 - 16k^2a^2 X^0 = 0.$$

This has general solution given by sum of plane waves and exponentials

$$X^0 = \sum_{n\neq 0}\frac{1}{n^4}\left(a_n^0 \mathrm{e}^{i2\sqrt{ka}n(\tau+\sigma)} + b_n^0 \mathrm{e}^{i2\sqrt{ka}n(\tau-\sigma)} + c_n^0 \mathrm{e}^{2\sqrt{ka}n(\tau+\sigma)} + d_n^0 \mathrm{e}^{2\sqrt{ka}n(\tau-\sigma)}\right).$$

In actuallity, only the $n=1$ and $n=-1$ terms satisfy the differential equation, so we have just

$$X^0 = a_1^0 \mathrm{e}^{i2\sqrt{ka}(\tau+\sigma)} + b_1^0 \mathrm{e}^{i2\sqrt{ka}(\tau-\sigma)} + c_1^0 \mathrm{e}^{2\sqrt{ka}(\tau+\sigma)} + d_1^0 \mathrm{e}^{2\sqrt{ka}(\tau-\sigma)} +$$
$$+ a_{-1}^0 \mathrm{e}^{-i2\sqrt{ka}(\tau+\sigma)} + b_{-1}^0 \mathrm{e}^{-i2\sqrt{ka}(\tau-\sigma)} + c_{-1}^0 \mathrm{e}^{-2\sqrt{ka}(\tau+\sigma)} + d_{-1}^0 \mathrm{e}^{-2\sqrt{ka}(\tau-\sigma)}$$

13 Polyakov-LQG V2

Same idea as before, but now contract the dyads directly with the $\mathcal{D}_a X^I$ factors

$$S = -\frac{T}{4}\int \mathrm{d}^2 x e e^a_I e^b_J \mathcal{D}_a X^I \mathcal{D}_b X^J.$$

13.1 EoMs

13.1.1 W.r.t e

$$\begin{aligned}
\frac{\delta S}{\delta e^c_K} &\propto \left(\frac{\partial e}{\partial e^c_K} e^a_I e^b_J + e\frac{\partial}{\partial e^c_K}(e^a_I e^b_J)\right)\mathcal{D}_a X^I \mathcal{D}_b X^J = \\
&= \left(-e e^K_c e^a_I e^b_J + 2e\delta^K_I \delta^a_c e^b_J\right)\mathcal{D}_a X^I \mathcal{D}_b X^J = \\
&= -e e^K_c e^a_I e^b_J \mathcal{D}_a X^I \mathcal{D}_b X^J + 2e e^b_J \mathcal{D}_c X^K \mathcal{D}_b X^J = 0
\end{aligned}$$

$$\begin{aligned}
T^K_c &:= 2e^b_J \mathcal{D}_c X^K \mathcal{D}_b X^J - e^K_c e^a_I e^b_J \mathcal{D}_a X^I \mathcal{D}_b X^J = 0, \\
e^K_c &= 2f\mathcal{D}_c X^K, \\
\frac{1}{f} &= e^b_J \mathcal{D}_b X^J.
\end{aligned}$$

13.1.2 W.r.t A

$$\begin{aligned}
\frac{\delta S}{\delta \mathcal{A}^{KL}_c} &\propto e e^a_I e^b_J \frac{\partial}{\partial \mathcal{A}^{KL}_c}(\mathcal{D}_a X^I \mathcal{D}_b X^J) = \\
&= 2e e^a_I e^b_J \delta^c_a \delta^I_{[K}\eta_{L]I'} X^{I'} \mathcal{D}_b X^J = \\
&= 2e e^c_{[K} X_{L]} e^b_J \mathcal{D}_b X^J = 0
\end{aligned}$$

$$\mathcal{T}^c_{KL} := e^c_{[K} X_{L]} = 0.$$

13.1.3 W.r.t X

$$\begin{aligned}
\frac{\delta S}{\delta X^K} &\propto 2e e^a_I e^b_J k \mathcal{A}^I_{aI'} \delta^{I'}_K \mathcal{D}_b X^J - 2\partial_c\left(e e^a_I e^b_J \delta^c_a \delta^I_K \mathcal{D}_b X^J\right) = \\
&= 2e e^a_I e^b_J k \mathcal{A}^I_{aK} \mathcal{D}_b X^J - 2\partial_a\left(e e^a_K e^b_J \mathcal{D}_b X^J\right) = 0
\end{aligned}$$

$$\mathcal{D}_a\left(e e^a_K e^b_J \mathcal{D}_b X^J\right) = 0.$$

If conformal gauge makes it so that $e e^a_K e^b_J = \eta^{ab}\eta_{KJ}$, this reduces to

$$\eta^{ab}\mathcal{D}_a \mathcal{D}_b X^I = 0$$