Sketchboard

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1 Basic concepts

1.1 With extra capital internal index on embbeding fields

Promote embedding fields X^μ to have an internal group index with D values

$$X^\mu \to X^{\mu I} \ , \quad I = 0, ..., d$$

$$g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{IJ} \sim f D_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu}$$

$$D_a X^{\mu I} = \partial_a X^{\mu I} + \omega_{aJ}^I X^{\mu J}$$

$$e_a^i e_b^j \eta_{ij} = g_{ab} \to e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ} \ , \quad i,j = 0,1$$

$$g = \det(g_{ab}) \to g = \det(\text{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.2 With 1 extra small internal index on embedding fields

Promote embbeding fields to have a internal group index with 2 values

$$X^\mu \to X^{\mu i} \ , \quad i = 0,1$$

$$g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{ij} \sim f D_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu}$$

$$D_a X^{\mu i} = \partial_a X^{\mu i} + \omega_{aj}^i X^{\mu j}$$

$$e_a^i e_b^j \eta_{ij} = g_{ab} \to e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij}$$

$$g = \det(g_{ab}) \to g = \det(\text{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.3 With 2 extra small indices

Promote embbeding fields X^μ to have two internal indices with 2 values

$$X^\mu \to X^{\mu ij} \ , \quad i,j = 0,1$$

1.4 Without extra index on embbeding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$e_a^i e_b^j \eta_{ij} = g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$g = \det(g_{ab}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int \mathrm{d}\tau \wedge \mathrm{d}\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \ ,$$

2.1 With capital internal index

... and promote partial derivative ∂_a to covariant derivative D_a, giving us our first attempt at modified Polyakov action

$$S_{MP1} = -\frac{T_0}{2} \int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}$$

$$\updownarrow$$

$$\mathscr{L}_{MP1} = -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}$$

2.2 With small internal index

2.2.1 Without extra field, 1 internal index

... and promote derivatives to covariant D_a, giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

$$\updownarrow$$

$$\mathscr{L}_{MP2} = -\frac{T_0}{2} \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

2.2.2 With extra field

..., promote partial derivative to covariant and add extra internal WS field v^i leading us to

$$S_{MP3} = -\frac{T}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

$$\updownarrow$$

$$\mathscr{L}_{MP3} = -\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

2.2.3 Without extra field, 2 internal indices

... and promote derivatives to covariant D_a,

$$S_{MP4} = -\frac{T}{2} \int d\tau \wedge d\sigma e e_i^a e^{bi} D_a X^{\mu j j'} \eta_{jj'} \eta_{kk'} D_b X^{\nu k k'} E_\mu^I E_{\nu I}$$

$$\mathscr{L}_{MP4} = -\frac{T}{2} e e_i^a e_j^b D_a X^{\mu i k} \eta_{kl} D_b X^{\nu l j} E_\mu^I E_{\nu I}$$

2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) \ .$$

$$\updownarrow$$

$$\mathscr{L}_{DP} = -\frac{T_0}{2} \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) \ .$$

2.4 Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of D-dimensional gamma matrices and bulk spinors

$$S_{LP} = -\frac{T}{2} \int d\tau \wedge d\sigma \overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

$$\updownarrow$$

$$\mathscr{L}_{LP} = -\frac{T}{2} \overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

3 EoMs

Start by writing $\det(e) = \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n$ and $g_{ab} = g_{ab}^{ij} \eta_{ij}$

3.1 With small internal index

3.1.1 Without extra field, w.r.t e

$$\frac{\delta S_{MP2}}{\delta e^e_l} = \frac{D\mathscr{L}_{MP2}}{De^e_l} = \left(\frac{\partial}{\partial e^e_l} - \partial_f \frac{\partial}{\partial(\partial_f e^e_l)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e^m_c e^n_d e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I}\right) =$$

$$= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(\eta^{ml}g_{ce}e^n_d e^a_i e^b_j + e^m_c \eta^{nl}g_{de}e^a_i e^b_j +$$

$$+ e^m_c e^n_d \delta^a_e \delta^l_i e^b_j + e^m_c e^n_d e^a_i \delta^b_e \delta^l_j)D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} =$$

$$= -\frac{T}{2}\left(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}g_{ce}e^n_d e^a_i e^b_j + 2\det(e)e^a_i \delta^b_e \delta^l_j\right)D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} =$$

$$= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}e^k_c e_{ek}e^n_d e^a_i e^b_j + 2\det(e)e^a_i \delta^b_e \delta^l_j)D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} =$$

$$= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}e^m_c e^n_d e^l_e e^a_i e^b_j + 2\det(e)e^a_i \delta^b_e \delta^l_j)D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} =$$

$$= -T(-\det(e)e^l_e e^a_i e^b_j + \det(e)e^a_i \delta^b_e \delta^l_j)D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} \overset{!}{=} 0$$

$$\downarrow$$

$$T^l_e := E^I_\mu E_{\nu I}(e^a_i D_a X^{\mu i} D_e X^{\nu l} - e^l_e e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j}) = 0$$

$$\Downarrow$$

$$e^l_e = f e^a_i D_a X^{\mu i} D_e X^{\nu l} E^I_\mu E_{\nu I} ,$$

$$\frac{1}{f} = e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I}$$

$$\downarrow$$

$$e^l_e e_{fl} = (f e^a_i D_a X^{\mu i} D_e X^{\nu l} E^I_\mu E_{\nu I})(f e^{a'}_{i'} D_{a'} X^{\mu' i'} D_f X^{\nu'}_l E^{I'}_{\mu'} E_{\nu' I'}) =$$

$$= f^2 D_e X^{\nu l} D_f X^{\nu'}_l E^{I'}_\nu E_{\nu' I'}(e^a_i e^{a'}_{i'} D_a X^{\mu i} D_{a'} X^{\mu' i'} E^I_\mu E_{\mu' I}) =$$

$$= f D_e X^{\mu i} D_f X^\nu_i E^I_\mu E_{\nu I}$$

$$\Downarrow$$

$$D_e X^{\mu i} D_f X^\nu_i E^I_\mu E_{\nu I} = 2\partial_e X^\mu \partial_f X^\nu G_{\mu\nu}$$

$$e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\Downarrow$$

$$D_a X^{\mu 0} = -i\partial_a X^\mu$$

$$D_a X^{\mu 1} = \partial_a X^\mu$$

$$\Downarrow$$

$$X^{\mu 0} = -i X^{\mu 1}$$

3.1.2 With extra field, w.r.t e

$$\frac{\delta S_{MP3}}{\delta e_o^e} = \frac{D\mathscr{L}_{MP3}}{De_o^e} = \left(\frac{\partial}{\partial e_o^e} - \partial_f \frac{\partial}{\partial(\partial_f e_o^e)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e_c^m e_d^n \eta^{ij}e_i^a e_j^b D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ}\right) =$$

$$= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(g_{ce}\eta^{mo}e_d^n \eta^{ij}e_i^a e_j^b + e_c^m g_{de}\eta^{no}\eta^{ij}e_i^a e_j^b +$$

$$+ e_c^m e_d^n \eta^{ij}\delta_e^a \delta_i^o e_j^b + e_c^m e_d^n \eta^{ij}e_i^a \delta_e^b \delta_j^o)D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} =$$

$$= -\frac{T}{2}\varepsilon^{cd}\varepsilon_{mn}(e_c^p e_{ep}e_d^n \eta^{mo}\eta^{ij}e_i^a e_j^b + 2\det(e)e^{ao}\delta_e^b)D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} =$$

$$= -T(-\det(e)e_e^o \eta^{ij}e_i^a e_j^b + \det(e)e^{ao}\delta_e^b)D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{T_e^o := (e^{ao}D_a(X_k^\mu v^k)D_e(X_l^\nu v^l) - e_e^o \eta^{ij}e_i^a e_j^b D_a(X_k^\mu v^k)D_b(X_l^\nu v^l))E_\mu^I E_\nu^J \eta_{IJ} = 0}$$

$$\Downarrow$$

$$e_e^o = f\,e^{ao}D_a(X_k^\mu v^k)D_e(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ}\ ,$$

$$\frac{1}{f} = \eta^{ij}e_i^a e_j^b D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ}$$

3.1.3 With extra field, w.r.t ω

$$\frac{\delta S_{MP3}}{\delta \omega_c^{mn}} = \frac{D\mathscr{L}_{MP3}}{D\omega_c^{mn}} = \left(\frac{\partial}{\partial \omega_c^{mn}} - \partial_d \frac{\partial}{\partial(\partial_d \omega_c^{mn})}\right)\left(-\frac{T}{2}\det(e)\eta^{ij}e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}\right) =$$

$$= -\frac{T}{4}eg^{ab}((-\delta_a^c \delta_{[m}^{k'}\eta_{n]k}X_{k'}^\mu v^k)D_b X_l^\nu v^l + D_a X_k^\mu v^k(-\delta_b^c \delta_{[m}^{l'}\eta_{n]l}X_{l'}^\nu v^l))G_{\mu\nu} =$$

$$= \frac{T}{4}e(g^{ca}X_{[m}^\mu v_{n]}D_a X_k^\nu v^k + g^{ac}D_a X_k^\mu v^k X_{[m}^\nu v_{n]})G_{\mu\nu} =$$

$$= \frac{T}{2}eg^{ac}D_a X_k^\mu v^k X_{[m}^\nu v_{n]}G_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{\mathcal{T}_{ab}^i := e^{ci}D_c X_k^\mu v^k X_{[m}^\nu v_{n]}e_a^m e_b^n G_{\mu\nu} = 0}$$

3.1.4 With extra field, w.r.t v

$$\frac{\delta S_{MP3}}{\delta v^l} = \frac{D\mathscr{L}_{MP3}}{Dv^l} = \left(\frac{\partial}{\partial v^l} - \partial_c \frac{\partial}{\partial(\partial_c v^l)}\right)\left(-\frac{T}{2}ee_i^a e^{bi}D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I}\right) =$$

$$= -\frac{T}{2}g^{ab}(D_a X_j^\mu \delta_l^j D_b X_k^\nu v^k + D_a X_j^\mu v^j D_b X_k^\nu \delta_l^k)G_{\mu\nu} =$$

$$= -Tg^{ab}D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{g^{ab}D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} = 0}$$

3.1.5 With extra field, w.r.t X

$$\frac{\delta S_{MP3}}{\delta X_l^\lambda} = \frac{D\mathscr{L}_{MP3}}{DX_l^\lambda} = \left(\frac{\partial}{\partial X_l^\lambda} - \partial_c \frac{\partial}{\partial(\partial_c X_l^\lambda)}\right)\left(-\frac{T}{2}ee_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I}\right) =$$

$$= -\frac{T}{2}\left(eg^{ab}\left((-\omega_{aj}^{j'}\delta_\lambda^\mu \delta_{j'}^l v^j D_b X_k^\nu v^k - D_a X_j^\mu v^j \omega_{bk}^{k'}\delta_\lambda^\nu \delta_{k'}^l v^k)E_\mu^I E_{\nu I} + \right.\right.$$

$$+ eg^{ab}D_a X_j^\mu v^j D_b X_k^\nu v^k \left(\frac{\partial E_\mu^I}{\partial X_l^\lambda}E_{\nu I} + E_\mu^I \frac{\partial E_{\nu I}}{\partial X_l^\lambda}\right)\right) -$$

$$\left. - \partial_c(eg^{ab}(\delta_a^c \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k + D_a X_j^\mu v^j \delta_b^c \delta_\lambda^\nu \delta_k^l v^k)G_{\mu\nu})\right) =$$

$$= -T\left(-eg^{ab}D_a X_j^\mu v^j \omega_{bk}^l v^k G_{\mu\lambda} + eg^{ab}D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda}E_{\nu I} - \partial_c(eg^{ac}v^l D_a X_j^\mu v^j G_{\mu\lambda})\right)$$

$$= -T\left(eg^{ab}D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda}E_{\nu I} - D_b(eg^{ab}v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I})\right) \stackrel{!}{=} 0$$

$$\downarrow$$

$$\boxed{D_b(ee_i^a e^{bi}v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) = ee_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda}E_{\nu I}}$$

3.2 Without extra index

3.2.1 w.r.t e

$$\frac{\delta S_{DP}}{\delta e_l^e} = \frac{D\mathscr{L}_{DP}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)}\right)\left(-\frac{T_0}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e_c^m e_d^n \eta^{ij}e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I E_\nu^J \eta_{IJ}\right) =$$

$$= -\frac{T_0}{4}$$

3.3 Linear action

3.3.1 w.r.t e

$$\frac{\delta S_{LP}}{\delta\delta e_j^b} = \frac{D\mathscr{L}_{LP}}{De_j^b} = \left(\frac{\partial}{\partial e_j^b} - \partial_e \frac{\partial}{\partial(\partial_e e_j^b)}\right)\left(-\frac{T}{2}\overline{\psi}ee_i^a D_a X^{\mu i}E_\mu^I \gamma_I \psi\right) =$$

$$= -\frac{T}{4}\overline{\psi}\varepsilon^{cd}\varepsilon_{mn}((g_{cb}\eta^{mj}e_d^n + e_c^m g_{db}\eta^{nj})e_i^a + e_c^m e_d^n \delta_b^a \delta_i^j)D_a X^{\mu i}E_\mu^I \gamma_I \psi =$$

$$= -\frac{T}{2}\overline{\psi}(-ee_b^j e_i^a D_a X^{\mu i}E_\mu^I \gamma_I + eD_b X^{\mu j}E_\mu^I \gamma_I)\psi \stackrel{!}{=} 0$$

$$\downarrow$$

$$T_b^j := \overline{\psi}(D_b X^{\mu j}E_\mu^I \gamma_I - e_b^j e_i^a D_a X^{\mu i}E_\mu^I \gamma_I)\psi = 0$$

$$\downarrow$$

$$e_b^j = F\overline{\psi}D_b X^{\mu j}E_\mu^I \gamma_I \psi,$$

$$\frac{1}{F} = \overline{\psi}e_i^a D_a X^{\mu i}E_\mu^I \gamma_I \psi$$

$$\Downarrow$$

$$\overline{\psi}\psi = 1, \ \psi\overline{\psi} = \mathbb{1}$$

$$e_i^a e_j^b D_a X^{\mu i}D_b X^{\nu j}E_\mu^I E_\nu^J \eta_{IJ} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

4 Inverse Area Action

4.1 Nambu-Goto

Start from Nambu-Goto action

$$S_{NG} = -T \int \mathrm{d}^2 x \sqrt{-h},$$

and make quantum geometry correction

$$\sqrt{-h} \to \sqrt{-(h + g\Delta)} \approx \sqrt{-h}\left(1 - \frac{g\Delta}{2(-h)} + \mathcal{O}\left(\frac{g^2}{h^2}\right)\right),$$

leading to modified NG action

$$S_{MNG} = -T \int \mathrm{d}^2 x \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}}\right) = S_{NG} + S_{IA}$$

$$\downarrow$$

$$\mathscr{L}_{MNG} = -T \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}}\right)$$

4.1.1 EoMs

Let $\mathcal{P}^\tau_\mu \equiv \partial \mathscr{L}_{MNG}/\partial \dot{X}^\mu$ and $\mathcal{P}^\sigma_\mu \equiv \partial \mathscr{L}_{MNG}/\partial X'^\mu$

$$\delta S_{MNG} = \int \mathrm{d}^2 x \left(\frac{\partial \mathscr{L}_{MNG}}{\partial \dot{X}^\mu}\delta \dot{X}^\mu + \frac{\partial \mathscr{L}_{MNG}}{\partial X'^\mu}\delta X'^\mu\right) =$$

$$= -\int \mathrm{d}^2 x \left(\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu\right)\delta X^\mu + \int \mathrm{d}\tau \mathcal{P}^\sigma_\mu \delta X^\mu\big|_{\sigma=0}^{\sigma=\sigma_1} \overset{!}{=} 0$$

$$\downarrow$$

$$\text{EoM} : \partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0$$

$$\text{B.C.} : \mathcal{P}^\sigma_\mu \delta X^\mu\big|_{\sigma=0}^{\sigma=\sigma_1} = 0,$$

where

$$\mathcal{P}^\tau_\mu = \frac{\partial \mathscr{L}_{MNG}}{\partial \dot{X}^\mu} = -T\left(\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{-h}} + \frac{g\Delta}{2}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{(-h)^{3/2}}\right) =$$

$$= -T\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{-h}}\left(1 + \frac{g\Delta}{2(-h)}\right) = \mathcal{P}^\tau_{\mu(NG)}\left(1 + \frac{g\Delta}{2(-h)}\right)$$

$$\mathcal{P}^\sigma_\mu = \frac{\partial \mathscr{L}_{MNG}}{\partial X'^\mu} = -T\left(\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} + \frac{g\Delta}{2}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{(-h)^{3/2}}\right) =$$

$$= -T\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}}\left(1 + \frac{g\Delta}{2(-h)}\right) = \mathcal{P}^\sigma_{\mu(NG)}\left(1 + \frac{g\Delta}{2(-h)}\right)$$

gauge fixing static gauge $\tau = t$ and transverse gauge $\frac{\partial X}{\partial \tau}\cdot\frac{\partial X}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}\sigma} = 0$ ($s = $ length along string)

$$\downarrow$$

$$\mathcal{P}^{\tau\mu}_{(NG)} = T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\frac{\partial X^\mu}{\partial t}$$

$$\mathcal{P}^{\sigma\mu}_{(NG)} = \frac{T}{\gamma_{v_\perp}}\frac{\partial X^\mu}{\partial s}$$

$$-h = \frac{1}{\gamma_{v_\perp}^2}\left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2$$

string energy get's redefined to

$$\frac{\partial}{\partial t}\left(T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\right)=0,$$

and for the spatial part we have

$$\partial_\tau\vec{\mathcal{P}}^\tau+\partial_\sigma\vec{\mathcal{P}}^\sigma=0$$

$$\downarrow$$

$$\frac{\partial}{\partial t}\left[T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\frac{\partial\vec{X}}{\partial t}\left(1+\frac{g\Delta}{2(-h)}\right)\right]+\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{\partial}{\partial s}\left[-\frac{T}{\gamma_{v_\perp}}\frac{\partial\vec{X}}{\partial s}\left(1+\frac{g\Delta}{2(-h)}\right)\right]=0$$

$$\downarrow$$

$$\mu\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial}{\partial s}\left[\frac{T}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right)\frac{\partial\vec{X}}{\partial s}\right]=0$$

$\implies$ effective mass density becomes $\mu_{eff}=\mu\gamma_{v_\perp}\left(1-\frac{g\Delta}{2(-h)}\right)$ and effective tension becomes $T_{eff}=\frac{T}{\gamma_{v_\perp}}\left(1-\frac{g\Delta}{2(-h)}\right)$

$$\downarrow$$

$$\mu_{eff}\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial}{\partial s}\left[T_{eff}\frac{\partial\vec{X}}{\partial s}\right]=0$$

$$\mathscr{H}=\dot{\vec{X}}\cdot\vec{\pi}-\mathscr{L}=$$
$$=\vec{v}_\perp\cdot\left(T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\vec{v}_\perp\right)-\left(-T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{1}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right)\right)=$$
$$=T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\left(1+\frac{g\Delta}{2(-h)}\right)\left(\gamma_{v_\perp}v_\perp^2+\frac{1}{\gamma_{v_\perp}}\right)=$$
$$=T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)$$

let $\left(1+\frac{g\Delta}{2(-h)}\right)=F$

$$\frac{\partial^2\vec{X}}{\partial t^2}-\frac{1}{F\gamma_{v_\perp}}\frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\partial}{\partial\sigma}\left[\frac{1}{\gamma_{v_\perp}}F\frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\partial\vec{X}}{\partial\sigma}\right]=0$$

$$\downarrow$$

$$A(\sigma)=\frac{\gamma_{v_\perp}}{F}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\overset{!}{=}1$$

$$\downarrow$$

$$\mathrm{d}\sigma=\frac{\gamma_{v_\perp}}{F}\mathrm{d}s=\frac{1}{TF^2}\mathrm{d}E\implies\sigma(q)=\frac{1}{T}\int_0^q\frac{1}{F^2}\mathrm{d}E$$

$$\downarrow$$

$$F^2\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial^2\vec{X}}{\partial\sigma^2}=0$$

$\implies$ speed of wave on the string gets modified by correction factor $v=c/F$.

4.2 Translation and Lorentz symmetry

Starting with global translations, define the transformation by

$$X^\mu\mapsto X^\mu+\epsilon^\mu.$$

This does not change the action since it only depends on derivatives of X^μ and so $\delta(\partial_aX^\mu)=\partial_a(\delta X^\mu)=\partial_a\epsilon^\mu=0$.

As for Lorentz transformations, the infinitesimal for is

$$X^\mu\mapsto X^\mu+\epsilon^{\mu\nu}X_\nu,$$

with $\epsilon^{\mu\nu}$ anti-symmetric.

4.3 Verifying if there is longitudinal velocity

$$-h = \frac{1}{\gamma_{v_\perp}^2} \left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2 = \frac{F^2}{\gamma_{v_\perp}^4} = F^2 \left(1 - v_\perp^2\right)^2$$

$$\downarrow$$

$$-h = \left(1 + \frac{g\Delta}{2(-h)}\right)^2 \left(1 - v_\perp^2\right)^2$$

$$\downarrow$$

$$\frac{-h}{\left(1 + \frac{g\Delta}{2(-h)}\right)^2} = \left(1 - v_\perp^2\right)^2$$

$$\downarrow \text{(WolframAlpha)}$$

$$-h = \frac{(1 - v_\perp^2)^2}{3} - f_1 - f_2,$$

where (let $a = \frac{g\Delta}{2}$ and $b = (1 - v_\perp^2)^2$)

$$f_1 = \frac{\sqrt[3]{-27a^2b + 3\sqrt{3}\sqrt{27a^4b^2 + 4a^3b^3 - 18ab^2 - 2b^3}}}{3\sqrt[3]{2}}$$

$$f_2 = \frac{(6ab + b^2)}{6f_1}$$

$$\downarrow$$

$$F = 1 + \frac{g\Delta}{2(-h)} = 1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}$$

$$\downarrow$$

$$v = \frac{1}{F} = \frac{1}{1 + \dfrac{g\Delta}{2\left(\dfrac{b}{3} - f_1 - f_2\right)}}$$

5 Bimetric Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2} \int \mathrm{d}^2 x \sqrt{-g}\, g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

and promote to bimetric action

$$\downarrow$$

$$S_{BP} = -\frac{T}{2} \int \mathrm{d}^2 x \sqrt{-g}\, g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{T'}{2} \int \mathrm{d}^2 x \sqrt{-h}\, h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

5.1 EoMs

$$\frac{\delta S_{BP}}{\delta g^{cd}} = -\frac{T}{2} \left(\frac{\partial \sqrt{-g}}{\partial g^{cd}} g^{ab} + \sqrt{-g} \frac{\partial g^{ab}}{\partial g^{cd}} \right) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + 0 =$$

$$= -\frac{T}{2} \left(-\frac{1}{2} \sqrt{-g}\, g_{cd} g^{ab} + \sqrt{-g}\, \delta^a_{(c} \delta^b_{d)} \right) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \overset{!}{=} 0$$

$$T^{(G)}_{cd} := \left(\partial_c X^\mu \partial_d X^\nu - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \right) G_{\mu\nu} = 0$$

$$\Downarrow$$

$$g_{cd} = 2 f^{(G)} \partial_c X^\mu \partial_d X^\nu G_{\mu\nu},$$

$$\frac{1}{f^{(G)}} = g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\frac{\delta S_{BP}}{\delta h^{cd}} = ... =$$

$$= -\frac{T'}{2} \left(-\frac{1}{2} \sqrt{-h}\, h_{cd} h^{ab} + \sqrt{-h}\, \delta^a_{(c} \delta^b_{d)} \right) \partial_a X^\mu \partial_b X^\nu H_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$T^{(H)}_{cd} := \left(\partial_c X^\mu \partial_d X^\mu - \frac{1}{2} h_{cd} h^{ab} \partial_a X^\mu \partial_b X^\nu \right) H_{\mu\nu} = 0$$

$$\Downarrow$$

$$h_{cd} = 2 f^{(H)} \partial_c X^\mu \partial_d X^\nu H_{\mu\nu},$$

$$\frac{1}{f^{(H)}} = h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

If the auxiliary metrics g and h concide with the induced metrics from G and H, then $f^{(G)} = f^{(H)} = 1/2$ and the action reduces to

$$S = -T \int \mathrm{d}^2 x \sqrt{-g} - T' \int \mathrm{d}^2 x \sqrt{-h},$$

where if $T' = -Tk\Delta/2$ and $h = g^{-1}$ we recover the NG area corrected action

$$S = -T \int \mathrm{d}^2 x \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right).$$

The condition $h = g^{-1}$ can be achieved by

$$h_{ab} = S_{ac} g^{cd} T_{db},$$

where $S, T \in \mathrm{SL}(2, \mathbb{C})$, so we have a gauge freedom in choosing h_{ab}. Since we require $[h_{ab}] = [h_{ab}]^\dagger$ (h_{ab} must be symmetric and real), this means $[T_{db}] = [S_{db}]^\dagger$ and also that $[S_{ab}]^\dagger = [S_{ab}]$, thus

$$h_{ab} = S_{ac} g^{cd} S_{db}.$$

This restriction constrains our gauge group to just the boosts part of $\mathrm{SL}(2,\mathbb{C})$,

$$S = \mathrm{e}^{-n^i \sigma_i \beta}.$$

This also implies

$$\partial_a X^\mu \partial_b X^\nu H_{\mu\nu} = S_{ac} g^{cd} S_{db} =$$

$$= S_{ac} \left(\frac{1}{2g} \varepsilon^{cc'} \varepsilon^{dd'} g_{c'd'} \right) S_{db} =$$

$$\delta_a^{c'} \delta_b^{d'} \partial_{c'} X^\mu \partial_{d'} X^\nu H_{\mu\nu} = S_{ac} \left(\frac{1}{2g} \varepsilon^{cc'} \varepsilon^{dd'} \partial_{c'} X^\mu \partial_{d'} X^\nu G_{\mu\nu} \right) S_{db}$$

$$\delta_a^{c'} \delta_b^{d'} H_{\mu\nu} = \frac{1}{2g} S_{ac} \varepsilon^{cc'} S_{db} \varepsilon^{dd'} G_{\mu\nu}$$

$$H_{\mu\nu} = -\frac{1}{8g} S_{ac} \varepsilon^{ca} S_{db} \varepsilon^{bd} G_{\mu\nu}$$

$$H_{\mu\nu} = \frac{1}{8(-g)} \mathrm{Tr}(S\varepsilon) \mathrm{Tr}(S\varepsilon) G_{\mu\nu}$$

$$H_{\mu\nu} = \frac{1}{8(-g)} \mathrm{Tr}(S\varepsilon)^2 G_{\mu\nu}.$$

This restricts S to necessarily being not symmetric, otherwise the trace $\mathrm{Tr}(S\varepsilon)$ vanishes and we would get $H_{\mu\nu} = 0$.

$$\frac{\delta S_{BP}}{\delta X^\lambda} = \left(-\frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu -$$

$$- \partial_c \left(-\frac{T}{2} \sqrt{-g} g^{ab} G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} H_{\mu\nu} \right) (\delta_a^c \delta_\lambda^\mu \partial_b X^\nu + \partial_a X^\mu \delta_b^c \delta_\lambda^\nu) =$$

$$= \partial_a \left(T \sqrt{-g} g^{ab} G_{\mu\lambda} + T' \sqrt{-h} h^{ab} H_{\mu\lambda} \right) \partial_b X^\mu -$$

$$- \left(\frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} + \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu \overset{!}{=} 0$$

$$\downarrow$$

$$\partial_a \left(F_{\mu\lambda}^{ab} \partial_b X^\mu \right) = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu,$$

$$F_{\mu\nu}^{ab} = T \sqrt{-g} g^{ab} G_{\mu\nu} + T' \sqrt{-h} h^{ab} H_{\mu\nu}.$$

Assuming the condition for $h = g^{-1}$ and flat background $G_{\mu\nu} = \eta_{\mu\nu}$, the EoM reduces to

$$\partial_a \left(\left(\sqrt{-g} g^{ab} - \frac{k\Delta}{2(-g)} \sqrt{-g}^{-1} (S^{-1})^{ac} g_{cd} (S^{-1})^{db} \right) \partial_b X^\mu \right) = 0,$$

which in conformal gauge $g_{ab} = \phi \eta_{ab}$ with $(S^{-1})^{ac} \eta_{cd} (S^{-1})^{db} = \eta^{ab}$ reduces further

$$\partial_a \left(\left(1 - \frac{k\Delta}{2\phi^2} \right) \eta^{ab} \partial_b X^\mu \right) = 0$$

$$\eta^{ab} \partial_a \left(F^- \partial_b X^\mu \right) = 0$$

$$F^- \eta^{ab} \partial_a \partial_b X^\mu + \eta^{ab} \partial_a F^- \partial_b X^\mu = 0$$

$$\eta^{ab} \partial_a \partial_b X^\mu + \eta^{ab} \partial_a \ln(F^-) \partial_b X^\mu = 0.$$

This is the exact same equation derived in the next section which leads to the Klein-Gordon eqn. The requirement that $(S^{-1})^{ac} \eta_{cd} (S^{-1})^{db} = \eta^{ab}$ implies

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix},$$

$$S = \begin{bmatrix} \cosh\beta - n^3\sinh\beta & -(n^1 - in^2)\sinh\beta \\ -(n^1 + in^2)\sinh\beta & \cosh\beta + n^3\sinh\beta \end{bmatrix},$$

$$S^{-1} = \begin{bmatrix} \cosh\beta + n^3\sinh\beta & +(n^1 - in^2)\sinh\beta \\ +(n^1 + in^2)\sinh\beta & \cosh\beta - n^3\sinh\beta \end{bmatrix}$$

so $n^3 = 0$. Since $(n^1)^2 + (n^2)^2 = 1$, we can parametrize as $n^1 = \cos\theta$ and $n^2 = \sin\theta$, thus S has the form

$$S = \begin{bmatrix} \cosh\beta & -e^{-i\theta}\sinh\beta \\ -e^{i\theta}\sinh\beta & \cosh\beta \end{bmatrix},$$

and $\mathrm{Tr}(S\varepsilon) = -2i\sin\theta\sinh\beta$. Let's try to find more restrictions on S:

$$h^{ab} = (S^{-1})^{ac}g_{cd}(S^{-1})^{db}$$

$$h^{ab} = \frac{1}{2g^{-1}}\varepsilon^{aa'}\varepsilon^{bb'}\partial_{a'}X^\mu\partial_{b'}X^\nu\left(\frac{1}{8(-g)}\mathrm{Tr}(S\varepsilon)^2 G_{\mu\nu}\right)$$

$$\Downarrow$$

$$(S^{-1})^{ac}g_{cd}(S^{-1})^{db} = \frac{1}{2g^{-1}}\varepsilon^{aa'}\varepsilon^{bb'}\frac{1}{8(-g)}\mathrm{Tr}(S\varepsilon)^2\partial_{a'}X^\mu\partial_{b'}X^\nu G_{\mu\nu}$$

$$(S^{-1})^{ac}g_{cd}(S^{-1})^{db} = -\frac{1}{16}\mathrm{Tr}(S\varepsilon)^2\varepsilon^{ac}\varepsilon^{bd}g_{cd}$$

$$(S^{-1})^{ac}g_{cd}(S^{-1})^{db} = \left(\frac{i}{4}\mathrm{Tr}(S\varepsilon)\varepsilon^{ac}\right)g_{cd}\left(\frac{i}{4}\mathrm{Tr}(S\varepsilon)\varepsilon^{db}\right)$$

$$g_{cd} = S_{ca}\left(\frac{i}{4}\mathrm{Tr}(S\varepsilon)^*\varepsilon^{ac'}\right)g_{c'd'}\left(\frac{i}{4}\mathrm{Tr}(S\varepsilon)\varepsilon^{d'b}\right)S_{bd}$$

$$\Downarrow$$

$$S_{ca}\left(\frac{i}{4}\mathrm{Tr}(S\varepsilon)^*\varepsilon^{ac'}\right) = \delta_c^{c'}$$

$$\left(\frac{i}{4}\mathrm{Tr}(S\varepsilon)\varepsilon^{d'b}\right)S_{bd} = \delta_d^{d'}$$

$$\Downarrow$$

$$(S^{-1})^{ab} = \left(\frac{i}{4}\mathrm{Tr}(S\varepsilon)\varepsilon^{ab}\right),$$

thus we have that

$$\begin{bmatrix} \cosh\beta & e^{-i\theta}\sinh\beta \\ e^{i\theta}\sinh\beta & \cosh\beta \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\sin\theta\sinh\beta \\ -\frac{1}{2}\sin\theta\sinh\beta & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\theta = \frac{n\pi}{2}$$

$$\beta = i\frac{m\pi}{2}$$

6 Inverse Area Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2}\int \mathrm{d}^2 x \sqrt{-g}\, g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

and make quantum geometry correction

$$\sqrt{-g} \to \sqrt{-(g+k\Delta)} \approx \sqrt{-g}\left(1 + \frac{k\Delta}{2g} + \mathcal{O}\left(\frac{k^2}{g^2}\right)\right)$$

leading to

$$S_{IAP} = -\frac{T}{2}\int \mathrm{d}^2 x \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

6.1 EoMs

$$\frac{\delta S_{IAP}}{\delta g^{cd}} = \frac{\partial}{\partial g^{cd}}\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right)\delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} =$$

$$= \left(-\frac{1}{2}\sqrt{-g}g_{cd} + \frac{k\Delta}{4\sqrt{-g}}g_{cd}\right) g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right)\partial_c X^\mu \partial_d X^\nu G_{\mu\nu} =$$

$$= -\frac{1}{2}\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g_{cd} g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right)\partial_c X^\mu \partial_d X^\nu G_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} - \frac{1}{2}g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = 0$$

$$\downarrow$$

$$g_{cd} = 2f \partial_c X^\mu \partial_d X^\nu G_{\mu\nu},$$

$$\frac{1}{f} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\frac{\delta S_{IAP}}{\delta X^\lambda} = \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - \partial_c\left(\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right)2g^{ab}\delta_a^c \delta_\lambda^\mu \partial_b X^\nu G_{\mu\nu}\right) =$$

$$= \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\partial_a\partial_b X^\nu G_{\lambda\nu} -$$

$$- 2\partial_a\left(\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\right)\partial_b X^\nu G_{\lambda\nu} =$$

$$= \left(1 - \frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2\left(1 - \frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_a\partial_b X^\nu G_{\lambda\nu} -$$

$$- 2\partial_a\left(\left(1 - \frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\right)\partial_b X^\nu G_{\lambda\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$\left(1 - \frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_a\partial_b X^\nu G_{\lambda\nu} + \partial_a\left(\left(1 - \frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\right)\partial_b X^\nu G_{\lambda\nu} =$$

$$\frac{1}{2}\left(1 - \frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

imposing conformal symmetry $g^{ab} = (\phi(x))^{-1}\eta^{ab}$ and let $F' = \left(1 - \frac{k\Delta}{2\phi^2}\right)$ leads to

$$\eta^{ab}\partial_a\partial_b X^\nu G_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F' \partial_b X^\nu G_{\lambda\nu} = \eta^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

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assuming flat background $G_{\mu\nu} = \eta_{\mu\nu}$

$$\eta^{ab}\partial_a\partial_b X^\nu \eta_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F'\partial_b X^\nu \eta_{\lambda\nu} = 0$$

$$\downarrow$$

$$\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\partial_a \ln(F')\partial_b X^\mu = 0,$$

use plane-wave ansatz $X^\mu = X_0^\mu e^{-i(E\tau - p\sigma)}$

$$(E^2 - p^2)X^\mu + \partial_\tau \ln(F')iEX^\mu - \partial_\sigma \ln(F')ipX^\mu = 0$$

$$\downarrow$$

$$(E^2 - p^2)X^\mu + i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F'))X^\mu = 0$$

$$\downarrow$$

$$i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F')) = -m^2,$$
$$-m^2 = -E^2 + p^2$$

$$\Downarrow$$

$$\partial_\tau \ln(F') = iE, \ \partial_\sigma \ln(F') = ip$$

$$\Downarrow$$

$$\ln(F') = i(E\tau + p\sigma) + a, \ a \in \mathbb{C}$$

$$\Downarrow$$

$$F' = Ae^{i(E\tau + p\sigma + \phi)}, \ A > 0$$

$$\Downarrow$$

$$\phi(\tau, \sigma) = \sqrt{\frac{k\Delta}{2\left(1 - Ae^{i(E\tau + p\sigma + \phi)}\right)}}$$

$$\downarrow$$

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

7 Quantum geometry Polyakov

Since quantum geometry is expected to violate Lorentz invariance at the Planck scale by modifying the dispersion relation as in

$$E^2 = m^2 + p^2 + \sum_{n \geq 3} c_n \frac{p^n}{E_{pl}^{n-2}},$$

where E_{pl} is the Planck energy, we make a correction to Polyakov action in the form

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2 x \sqrt{-g} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right)$$

where T is the temperature of target-space, (in SI units) μ_0^2 has units of inverse area (perhaps $\mu_0^2 \sim$ (quantum of area)$^{-1}$? Maybe $\mu_0^2 = \mu^2/(M_{pl}^2 A_j)$, where A_j is the quantum of area with spin j) and $m_0 >> m_1 >> ... >> m_d$. This action has manifest WS reparameterisation invariance. As for background Poincaré symmetry and WS Weyl invariance, those hold approximately for the fields with $m_\mu >> T$. The masses m_μ are constrained by the dispersion relations

$$\widetilde{E}_\alpha^2 = \widetilde{p}_\alpha^2 + \sum_{n=3}^d \frac{m_n}{T} \frac{\widetilde{p}_\alpha^n}{E_{pl}^{n-2}}$$

where $\alpha = 3, 4, ..., d$ and $\widetilde{E}_\alpha^2$ ($\widetilde{p}_\alpha^2$) does not mean $\widetilde{E}^\alpha \widetilde{E}_\alpha$ ($\widetilde{p}^\alpha \widetilde{p}_\alpha$), rather it is $\widetilde{E}_3^2(\widetilde{p}_3^2), \widetilde{E}_4^2(\widetilde{p}_4^2), ..., \widetilde{E}_d^2(\widetilde{p}_d^2)$, that is, the squared WS energy (momentum) of each field X^μ. Since m_0, m_1 and m_2 remain unconstrained after quantum geometry is considered, this might mean that the true quantum gravitational field is actually $(2+1)$-dimensional, since precisely the masses of X^0, X^1 and X^2 remain unconstrained, such that those may be regarded as "fundamental" dimensions, while the other dimensions have their masses constrained by the modified dispersion relations.

7.1 EoMs

$$\frac{\delta S}{\delta g^{cd}} \propto \frac{\partial \sqrt{-g}}{\partial g^{cd}} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right) + \sqrt{-g} \delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} =$$

$$= -\frac{1}{2} \sqrt{-g} g_{cd} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right) + \sqrt{-g} \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = 0$$

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right) = 0$$

$$g_{cd} = 2f \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu},$$

$$\frac{1}{f} = \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right).$$

From this, it is clear that $\mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} = 0$, thus the base mass μ_0^2 is a Lagrange multiplier, the auxiliary metric reduces to the induced metric and the action simplify back to regular NG action.

$$\frac{\delta S}{\delta X^\lambda} \propto +2\sqrt{-g} \mu_0^2 \mathrm{e}^{-m_\mu/T} \delta_\lambda^\mu X^\nu \eta_{\mu\nu} - 2\partial_c \left(\sqrt{-g} g^{ab} \delta_a^c \delta_\lambda^\mu \partial_b X^\nu \eta_{\mu\nu} \right) =$$

$$= -2 \left(\partial_a \left(\sqrt{-g} g^{ab} \partial_b X^\nu \eta_{\lambda\nu} \right) - \sqrt{-g} \mu_0^2 \mathrm{e}^{-m_\lambda/T} X^\nu \eta_{\lambda\nu} \right) = 0$$

$$\partial_a \left(\sqrt{-g} g^{ab} \partial_b X^\mu \right) - \sqrt{-g} \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu = 0$$

$$\frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g} g^{ab} \partial_b X^\mu \right) - \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu = 0$$

$$g^{ab} \nabla_a \nabla_b X^\mu - \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu = 0.$$

From the X EoM before simplifying, we can use conformal gauge $g_{ab} = \mathrm{e}^{\phi(x)} \eta_{ab}$ to get a variable-coefficient KG eqn

$$\eta^{ab} \partial_a \partial_b X^\mu - \mathrm{e}^{\phi(x)} \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu = 0$$

$$((\partial_\sigma)^2 - (\partial_\tau)^2) X^\mu - \mu_0^2 \mathrm{e}^{\phi_\mu(x)} X^\mu = 0,$$

where $\phi_\mu(x) \equiv \phi(x) - m_\mu/T$.

8   Relating NG and Polyakov analysis

From NG analysis

$$(F^+)^2 \frac{\partial^2 X^\mu}{\partial t^2} - \frac{\partial^2 X^\mu}{\partial x^2} = 0,$$

make change of variables $\tau = \tau(t, x)$ and $\sigma = \sigma(t, x)$

$$(F^+)^2 \left(\frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial t} \frac{\partial}{\partial \sigma} \right) \left(\frac{\partial \tau}{\partial t} \frac{\partial X^\mu}{\partial \tau} + \frac{\partial \sigma}{\partial t} \frac{\partial X^\mu}{\partial \sigma} \right) - \left(\frac{\partial \tau}{\partial x} \frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial x} \frac{\partial}{\partial \sigma} \right) \left(\frac{\partial \tau}{\partial x} \frac{\partial X^\mu}{\partial \tau} + \frac{\partial \sigma}{\partial x} \frac{\partial X^\mu}{\partial \sigma} \right) = 0$$

$$(F^+)^2 \left(\left(\frac{\partial \tau}{\partial t} \right)^2 \frac{\partial^2 X^\mu}{\partial \tau^2} + 2 \frac{\partial \tau}{\partial t} \frac{\partial \sigma}{\partial t} \frac{\partial^2 X^\mu}{\partial \tau \partial \sigma} + \left(\frac{\partial \sigma}{\partial t} \right)^2 \frac{\partial^2 X^\mu}{\partial \sigma^2} \right) -$$

$$- \left(\left(\frac{\partial \tau}{\partial x} \right)^2 \frac{\partial^2 X^\mu}{\partial \tau^2} + 2 \frac{\partial \tau}{\partial x} \frac{\partial \sigma}{\partial x} \frac{\partial^2 X^\mu}{\partial \tau \partial \sigma} + \left(\frac{\partial \sigma}{\partial x} \right)^2 \frac{\partial^2 X^\mu}{\partial \sigma^2} \right) +$$

$$+ \left((F^+)^2 \frac{\partial^2 \tau}{\partial t^2} - \frac{\partial^2 \tau}{\partial x^2} \right) \frac{\partial X^\mu}{\partial \tau} + \left((F^+)^2 \frac{\partial^2 \sigma}{\partial t^2} - \frac{\partial^2 \sigma}{\partial x^2} \right) \frac{\partial X^\mu}{\partial \sigma} = 0$$

$$\left((F^+)^2 (\partial_t \tau)^2 - (\partial_x \tau)^2 \right) \partial_\tau^2 X^\mu + \left((F^+)^2 (\partial_t \sigma)^2 - (\partial_x \sigma)^2 \right) \partial_\sigma^2 X^\mu +$$
$$+ 2 \left((F^+)^2 \partial_t \tau \partial_t \sigma - \partial_x \tau \partial_x \sigma \right) \partial_\tau \partial_\sigma X^\mu + \left((F^+)^2 \partial_t^2 \tau - \partial_x^2 \tau \right) \partial_\tau X^\mu + \left((F^+)^2 \partial_t^2 \sigma - \partial_x^2 \sigma \right) \partial_\sigma X^\mu = 0$$

Without loss of generality, we can set (just rescale/rotate the coordinates)

$$(F^+)^2 (\partial_t \sigma)^2 - (\partial_x \sigma)^2 = (\partial_x \tau)^2 - (F^+)^2 (\partial_t \tau)^2$$
$$(F^+)^2 \partial_t \tau \partial_t \sigma = \partial_x \tau \partial_x \sigma,$$

or more compactly

$$\partial_t \tau = v \partial_x \sigma$$
$$\partial_t \sigma = v \partial_x \tau,$$

where $v = 1/F^+$, which is equivalent to

$$\partial_t^2 \tau = \partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau + v^2 \partial_x^2 \tau$$
$$\partial_t^2 \sigma = \partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma + v^2 \partial_x^2 \sigma,$$

reducing the big equation to

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu + \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\tau X^\mu + \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\sigma X^\mu = 0.$$

Next introduce $X^\mu = \kappa(\tau, \sigma) Y^\mu(\tau, \sigma)$ s.t terms proportional to $\partial_\tau Y^\mu$ and $\partial_\sigma Y^\mu$ vanish:

$$\partial_\tau X^\mu = \partial_\tau \kappa Y^\mu + \kappa \partial_\tau Y^\mu$$
$$\partial_\sigma X^\mu = \partial_\sigma \kappa Y^\mu + \kappa \partial_\sigma Y^\mu$$
$$\partial_\tau^2 X^\mu = \partial_\tau^2 \kappa Y^\mu + 2 \partial_\tau \kappa \partial_\tau Y^\mu + \kappa \partial_\tau^2 Y^\mu$$
$$\partial_\sigma^2 X^\mu = \partial_\sigma^2 \kappa Y^\mu + 2 \partial_\sigma \kappa \partial_\sigma Y^\mu + \kappa \partial_\sigma^2 Y^\mu$$

$$\kappa \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} + 2 \partial_\tau \kappa \overset{!}{=} 0$$
$$\kappa \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} - 2 \partial_\sigma \kappa \overset{!}{=} 0$$

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$$\partial_\tau \kappa = -\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}$$

$$\partial_\sigma \kappa = \frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)},$$

undoing chain rule for t by multiplying first eq by $\partial_t \tau$ and second by $\partial_t \sigma$ and summing gives

$$\partial_t \kappa = \partial_t \tau \partial_\tau \kappa + \partial_t \sigma \partial_\sigma \kappa =$$

$$= \partial_t \tau \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}\right) + \partial_t \sigma \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}\right) =$$

$$= v \partial_x \sigma \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}\right) + v \partial_x \tau \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}\right) =$$

$$= \frac{v \kappa \partial_t v \left((\partial_x \tau)^2 - (\partial_x \sigma)^2\right)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)} =$$

$$= \frac{v \kappa \partial_t v \left((\partial_x \tau)^2 - (\partial_t \tau/v)^2\right)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)} =$$

$$= -\frac{\kappa \partial_t v}{2v},$$

or simply

$$\frac{\partial_t \kappa}{\kappa} = -\frac{1}{2}\frac{\partial_t v}{v}$$

$$\partial_t \ln(\kappa) = -\frac{1}{2}\partial_t \ln(v)$$

$$\ln(\kappa) = -\frac{1}{2}\ln(v) + f(x).$$

To determine $f(x)$, we undo the chain rule for x now:

$$\partial_x \kappa = \partial_x \tau \partial_\tau \kappa + \partial_x \sigma \partial_\sigma \kappa =$$

$$= \partial_x \tau \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}\right) + \partial_x \sigma \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}\right) =$$

$$= \frac{\kappa v \partial_x v \left((\partial_x \sigma)^2 - (\partial_x \tau)^2\right)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)} =$$

$$= \frac{\kappa v \partial_x v \left((\partial_t \tau/v)^2 - (\partial_x \tau)^2\right)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)} =$$

$$= \frac{\kappa \partial_x v}{2v}$$

$$\frac{\partial_x \kappa}{\kappa} = \frac{1}{2}\frac{\partial_x v}{v}$$

$$\partial_x \ln(\kappa) = \frac{1}{2}\partial_x \ln(v)$$

$$\ln(\kappa) = \frac{1}{2}\ln(v),$$

$$\partial_x \left(-\frac{1}{2}\ln(v) + f(x)\right) = \frac{1}{2}\partial_x \ln(v)$$

$f'(x) = \partial_x \ln(v)$ (perhaps this means somewhere in the composition of functions time dependence is lost?)

$$f(x) = \ln(v)$$

giving us

$$\kappa = v^{\frac{1}{2}}, \quad v = v(x).$$

Substituting this into the earlier equation yields

$$\partial_\tau^2 Y^\mu - \partial_\sigma^2 Y^\mu + m^2 Y^\mu = 0,$$

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with

$$m^2 = \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)\partial_\tau \kappa + (\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)\partial_\sigma \kappa}{\kappa \left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} =$$

$$= \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{\partial_t v (\partial_x \sigma \partial_\tau \kappa + \partial_x \tau \partial_\sigma \kappa) + v \partial_x v \partial_x \kappa}{\kappa \left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)} =$$

$$= \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{(\partial_x v)^2}{\left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)},$$

where by use of chain rule we have that

$$\partial_t^2 \kappa - v^2 \partial_x^2 \kappa - v \partial_x v \partial_x \kappa = ((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)(\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa),$$

where since $\kappa = v^{\frac{1}{2}}$, m^2 reduces to

$$m^2 = \frac{(\partial_x v)^2 - 2v \partial_x^2 v}{4 \left((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2\right)},$$

with τ satisfying

$$\partial_t \tau = v \partial_x \sigma$$
$$v \partial_x \tau = \partial_t \sigma$$

such that $-m^2$ is indeed constant.

9 Solutions for the Polyakov KG eqn (WRONG ONE)

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2}g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi\eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2}\eta_{cd}\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2}\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \frac{1}{2}\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0.$$

These simplify further by expanding the summation on a, b:

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2}\left(-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}\right)$$

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}.$$

The EoM is just a one-dimensional Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi}\int dp \frac{1}{2E(p)}\left(a^\mu(p)e^{-i(-E\tau+p\sigma)} + b^\mu(p)e^{i(-E\tau+p\sigma)}\right),$$

where $E(p) = \sqrt{p^2 + m^2}$. The derivatives are

$$\partial_\tau X^\mu = \frac{i}{4\pi}\int dp \left(a^\mu(p)e^{-i(-E\tau+p\sigma)} - b^\mu(p)e^{i(-E\tau+p\sigma)}\right)$$

$$\partial_\sigma X^\mu = \frac{i}{4\pi}\int dp \frac{p}{\sqrt{p^2+m^2}}\left(-a^\mu(p)e^{-i(-E\tau+p\sigma)} + b^\mu(p)e^{i(-E\tau+p\sigma)}\right).$$

Starting with the mixed constraint,

$$\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = -\frac{\eta_{\mu\nu}}{(4\pi)^2}\int dp \left(a^\mu(p)e^{-i(-E(p)\tau+p\sigma)} - b^\mu(p)e^{i(-E(p)\tau+p\sigma)}\right)\times$$

$$\times \int dp' \frac{p'}{\sqrt{p'^2+m^2}}\left(-a^\nu(p')e^{-i(-E(p')\tau+p'\sigma)} + b^\nu(p')e^{i(-E(p')\tau+p'\sigma)}\right) =$$

$$= -\frac{\eta_{\mu\nu}}{(4\pi)^2}\int dp dp' \frac{p'}{\sqrt{p'^2+m^2}}\left(a^\mu(p)e^{-i(-E(p)\tau+p\sigma)} - b^\mu(p)e^{i(-E(p)\tau+p\sigma)}\right)\times$$

$$\times \left(-a^\nu(p')e^{-i(-E(p')\tau+p'\sigma)} + b^\nu(p')e^{i(-E(p')\tau+p'\sigma)}\right) =$$

$$= -\frac{\eta_{\mu\nu}}{(4\pi)^2}\int dp dp' \frac{p'}{\sqrt{p'^2+m^2}}\left(-a^\mu(p)a^\nu(p')e^{i(E(p)+E(p'))\tau}e^{-i(p+p')\sigma} + a^\mu(p)b^\nu(p')e^{i(E(p)-E(p'))\tau}e^{-i(p-p')\sigma}+\right.$$

$$\left.+ b^\mu(p)a^\nu(p')e^{-i(E(p)-E(p'))\tau}e^{i(p-p')\sigma} - b^\mu(p)b^\nu(p')e^{-i(E(p)+E(p'))\tau}e^{i(p+p')\sigma}\right) = 0.$$

If we integrate this expression over σ we obtain

$$-\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\int \mathrm{d}\sigma\mathrm{e}^{-i(p+p')\sigma}+$$

$$+a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\int \mathrm{d}\sigma\mathrm{e}^{-i(p-p')\sigma}+b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\int \mathrm{d}\sigma\mathrm{e}^{i(p-p')\sigma}-$$

$$-b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\int \mathrm{d}\sigma\mathrm{e}^{i(p+p')\sigma}\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p')+a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p')+$$

$$+b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p')-b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p')\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\frac{p}{\sqrt{p^2+m^2}}\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}+a^\mu(p)b^\nu(p)+b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)=0$$

which implies the whole expression inside the integral must vanish, thus we have (leaving $(a^\mu b^\nu+b^\mu a^\nu)\eta_{\mu\nu}$ as is since eventually these will not be numbers and may not commute, thus will not be equal to $2a^\mu b^\nu\eta_{\mu\nu}$)

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}+a^\mu(p)b^\nu(p)+b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu}=0.$$

As for the other part of the constraint, let's start with the L.H.S

$$\partial_\tau X^\mu\partial_\tau X^\nu\eta_{\mu\nu}=-\frac{1}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)\mathrm{e}^{-i(-E(p)\tau+p\sigma)}-b^\mu(p)\mathrm{e}^{i(-E(p)\tau+p\sigma)}\Big)\times$$

$$\times\Big(a^\nu(p')\mathrm{e}^{-i(-E(p')\tau+p'\sigma)}-b^\nu(p')\mathrm{e}^{i(-E(p')\tau+p'\sigma)}\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\mathrm{e}^{-i(p+p')\sigma}-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\mathrm{e}^{-i(p-p')\sigma}-$$

$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\mathrm{e}^{i(p-p')\sigma}+b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\mathrm{e}^{i(p+p')\sigma}\Big),$$

which we integrate over σ to get

$$-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p')-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p')-$$

$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p')+b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p')\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}-a^\mu(p)b^\nu(p)-b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big),$$

and now for the R.H.S

$$-\partial_\sigma X^\mu\partial_\sigma X^\nu\eta_{\mu\nu}=\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{p}{\sqrt{p^2+m^2}}\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)\mathrm{e}^{-i(-E\tau+p\sigma)}+b^\mu(p)\mathrm{e}^{i(-E\tau+p\sigma)}\Big)\times$$

$$\times\Big(-a^\nu(p')\mathrm{e}^{-i(-E(p')\tau+p'\sigma)}+b^\nu(p')\mathrm{e}^{i(-E(p')\tau+p'\sigma)}\Big)=$$

$$=\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{pp'}{E(p)E(p')}\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\mathrm{e}^{-i(p+p')\sigma}-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\mathrm{e}^{-i(p-p')\sigma}-$$

$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\mathrm{e}^{i(p-p')\sigma}+b^\mu(p)b^\nu(p)\mathrm{e}^{-i(E(p)+E(p'))\tau}\mathrm{e}^{i(p+p')\sigma}\Big),$$

which once again we integrate over σ,

$$\frac{\eta_{\mu\nu}}{8\pi} \int \mathrm{d}p\mathrm{d}p' \frac{pp'}{E(p)E(p')} \Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p') - a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p') -$$

$$- b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p') + b^\mu(p)b^\nu(p)\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p') \Big) =$$

$$= \frac{1}{8\pi} \int \mathrm{d}p \frac{p^2}{p^2+m^2} \Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau} \Big) \eta_{\mu\nu},$$

which when equated to the L.H.S yields

$$\int \mathrm{d}p \left(\frac{p^2}{p^2+m^2} + 1 \right) \Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau} \Big) \eta_{\mu\nu} = 0,$$

which implies

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau} \Big) \eta_{\mu\nu} = 0.$$

We can sum and subtract this with the previous condition to simplify and obtain

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau} \Big) \eta_{\mu\nu} = 0$$

$$\Big(a^\mu(p)b^\nu(p) + b^\mu(p)a^\nu(p) \Big) \eta_{\mu\nu} = 0,$$

and for while a^μ and b^ν are number-valued, the second condition can be further simplified to

$$a^\mu(p)b^\nu(p)\eta_{\mu\nu} = 0,$$

meaning a^μ and b^ν are orthogonal. In general, this condition states that a^μ and b^ν anti-commute w.r.t Lorentz inner product. The first condition must hold for all values of τ, and since the exponentials are linearly independent, we thus have

$$a^\mu(p)a^\nu(-p)\eta_{\mu\nu} = 0$$

$$b^\mu(p)b^\nu(-p)\eta_{\mu\nu} = 0,$$

meaning that reflecting the argument of a^μ and b^ν creates orthogonal vectors. We can thus write

$$a^\mu(p) = \Lambda^\mu{}_\nu \left(\frac{p}{4} \right) a^\nu(0)$$

$$b^\mu(p) = \Upsilon^\mu{}_\nu \left(\frac{p}{4} \right) b^\nu(0),$$

where $\Lambda, \Upsilon \in \mathrm{SO}^+(1,d)$. Also. since the orthogonality under reflection must hold for all values of p, in particular we have

$$a^\mu(0)a^\nu(0)\eta_{\mu\nu} = 0$$

$$b^\mu(0)b^\nu(0)\eta_{\mu\nu} = 0,$$

meaning that the initial $a^\mu(0)$ and $b^\nu(0)$ are null vectors. This means that $a^\mu(p)$ and $b^\nu(p)$ are also null vectors since they are related to the initial values by Lorentz transformation. With this, the solution becomes

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi} \int \mathrm{d}p \frac{1}{2E(p)} \left(\Lambda^\mu{}_\nu \left(\frac{p}{4} \right) a^\nu(0)\mathrm{e}^{-i(-E\tau+p\sigma)} + \Upsilon^\mu{}_\nu \left(\frac{p}{4} \right) b^\nu(0)\mathrm{e}^{i(-E\tau+p\sigma)} \right)$$

with $(a^\mu b^\nu + b^\nu a^\mu)\eta_{\mu\nu} = 0$. This condition implies

$$\left(\Lambda^\mu{}_\gamma \left(\frac{p}{4} \right) a^\gamma(0) \Upsilon^\nu{}_\rho \left(\frac{p}{4} \right) b^\rho(0) \right) \eta_{\mu\nu} = 0$$

$$\Upsilon^\mu{}_\nu \left(\frac{p}{4} \right) = \Lambda_\nu{}^\mu \left(\frac{p}{4} \right) = \Lambda^\mu{}_\nu \left(-\frac{p}{4} \right),$$

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thus turning the solution into

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi} \int \mathrm{d}p \frac{1}{2E(p)} \left(\Lambda^\mu{}_\nu \left(\frac{p}{4}\right) a^\nu(0) \mathrm{e}^{-i(-E\tau+p\sigma)} + \Lambda^\mu{}_\nu \left(-\frac{p}{4}\right) b^\nu(0) \mathrm{e}^{i(-E\tau+p\sigma)} \right).$$

Next, we choose background coordinates s.t $a^\mu(0) = (a_0, a_0, 0, ..., 0) = a_0 x^+$ and $b^\nu(0) = (b_0, b_0, 0, ..., 0) = b_0 x^+$, simplifying the solution to

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi} \int \mathrm{d}p \frac{1}{2E(p)} \left(a_0 \Lambda^\mu{}_+ \left(\frac{p}{4}\right) \mathrm{e}^{-i(-E\tau+p\sigma)} + b_0 \Lambda^\mu{}_+ \left(-\frac{p}{4}\right) \mathrm{e}^{i(-E\tau+p\sigma)} \right) x^+$$

10   Solutions for the Polyakov KG eqn

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2}g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi\eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2}\eta_{cd}\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2}\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \frac{1}{2}\eta^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0.$$

These simplify further by expanding the summation on a,b:

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2}\left(-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}\right)$$

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = 0.$$

These can be combined by multiplying the previous one by 2 and summing/subtracting, leaving us with

$$(\partial_\tau X \pm \partial_\sigma X)^2 = 0.$$

The EoM is just a one-dimensional finite-space Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}}\frac{1}{2E_n}\left(a_n^\mu e^{i(E_n\tau - n\sigma)} + b_n^\mu e^{-i(E_n\tau - n\sigma)}\right),$$

with $E_n = \sqrt{n^2 + m^2}$.

Reality of X^μ implies

$$(X^\mu)^* = \left(\sum_{n\in\mathbb{Z}}\frac{1}{2E_n}\left(a_n^\mu e^{i(E_n\tau - n\sigma)} + b_n^\mu e^{-i(E_n\tau - n\sigma)}\right)\right)^* =$$

$$= \sum_{n\in\mathbb{Z}}\frac{1}{2E_n}\left((a_n^\mu)^* e^{-i(E_n\tau - n\sigma)} + (b_n^\mu)^* e^{i(E_n\tau - n\sigma)}\right) \equiv X^\mu,$$

giving us

$$(a_n^\mu)^* = b_n^\mu$$
$$(b_n^\mu)^* = a_n^\mu,$$

thus the solution for closed string becomes

$$X^\mu = \sum_n \frac{1}{2E_n}\left(a_n^\mu e^{i(E_n\tau - n\sigma)} + (a_n^\mu)^* e^{-i(E_n\tau - n\sigma)}\right).$$

Analysing the constraints, for the closed string we have the derivatives

$$\partial_\tau X^\mu = \frac{i}{2}\sum_n\left(a_n^\mu e^{i(E_n\tau - n\sigma)} - (a_n^\mu)^* e^{-i(E_n\tau - n\sigma)}\right)$$

$$\partial_\sigma X^\mu = -\frac{i}{2}\sum_n\frac{n}{E_n}\left(a_n^\mu e^{i(E_n\tau - n\sigma)} - (a_n^\mu)^* e^{-i(E_n\tau - n\sigma)}\right),$$

$$\partial_\tau X^\mu \pm \partial_\sigma X^\mu = \frac{i}{2} \sum_n \left(1 \mp \frac{n}{E_n}\right) \left(a_n^\mu e^{i(E_n\tau - n\sigma)} - (a_n^\mu)^* e^{-i(E_n\tau - n\sigma)}\right)$$

so the constraint reads

$$(\partial_\tau X \pm \partial_\sigma X)^2 = -\frac{\eta_{\mu\nu}}{4} \sum_n \sum_p \left(1 \mp \frac{n}{E_n}\right)\left(1 \mp \frac{p}{E_p}\right)\left(a_n^\mu a_p^\nu e^{i(E_n + E_p)\tau} e^{-i(n+p)\sigma} - \right.$$

$$\left. - a_n^\mu (a_p^\nu)^* e^{i(E_n - E_p)\tau} e^{-i(n-p)\sigma} - (a_n^\mu)^* a_p^\nu e^{-i(E_n - E_p)\tau} e^{i(n-p)\sigma} + (a_n^\mu)^*(a_p^\nu)^* e^{-i(E_n + E_p)\tau} e^{i(n+p)\sigma}\right) \equiv 0$$

Since the exponentials are all linearly independent and we can't decouple the τ exponential from any of the sums, we have 2 cases to investigate: $n = p$ and $n \neq p$. For $n \neq p$, we get

$$L^1_{n,p} := a_n^\mu a_p^\nu \eta_{\mu\nu} = 0$$
$$L^2_{n,p} := a_n^\mu (a_p^\nu)^* \eta_{\mu\nu} = 0$$
$$L^3_{n,p} := (a_n^\mu)^* a_p^\nu \eta_{\mu\nu} = 0$$
$$L^4_{n,p} := (a_n^\mu)^*(a_p^\nu)^* \eta_{\mu\nu} = 0,$$

where $L^2_{n,p}$ and $L^3_{n,p}$ are just complex conjugates of each other, so they are redundant. The same is true for $L^1_{n,p}$ and $L^4_{n,p}$, so in the end we have

$$L_{n,p} := a_n^\mu a_p^\nu \eta_{\mu\nu} = 0$$
$$\widetilde{L}_{n,p} := a_n^\mu (a_p^\nu)^* \eta_{\mu\nu} = 0.$$

For $n = p$, the first and last terms are complex conjugates of each other, while the exponentials in the 2 middle terms are reduced to 1, so we get

$$a_n^\mu a_n^\nu \eta_{\mu\nu} = 0$$

$$\sum_n \left(\left(1 \mp \frac{n}{E_n}\right)^2 a_n^\mu (a_n^\nu)^* \eta_{\mu\nu}\right) = 0.$$

Collecting everything we have

$$L_{n,p} := a_n^\mu a_p^\nu \eta_{\mu\nu} = 0, \ \forall \ n, p \in \mathbb{Z},$$
$$\widetilde{L}_{n,p} := a_n^\mu (a_p^\nu)^* \eta_{\mu\nu} = 0, \ n \neq p,$$
$$\sum_n \left(\left(1 \mp \frac{n}{E_n}\right)^2 a_n^\mu (a_n^\nu)^* \eta_{\mu\nu}\right) = 0.$$

One last thing before quantizing: the $n = 0$ term in the solution reads

$$\frac{1}{2\pi\alpha'} \frac{1}{2m} \left(a_0^\mu e^{-im\tau} + (a_0^\mu)^* e^{im\tau}\right),$$

which is divergent in the $m \to 0$ limit unless $\mathfrak{Re}(a_0^\mu) \sim m$. Keeping the analogy to the $m \to 0$ regime, we conclude $a_0^\mu = c_1 m x^\mu + i c_2 p^\mu$, where c_1 and c_2 are to be determined. We also have that

$$M^2 = -p^\mu p_\mu = -\frac{1}{(c_2)^2} (a_0^\mu (a_0^\nu)^* - (c_1 m)^2 x^\mu x^\nu) \eta_{\mu\nu},$$

which after analysing closer the $n = p$ condition

$$\sum_n \left(\left(1 \mp \frac{n}{E_n}\right)^2 \widetilde{L}_{n,n}\right) = 0$$

$$\sum_{n \neq 0} \left(\left(1 \mp \frac{n}{E_n}\right)^2 \widetilde{L}_{n,n}\right) - \widetilde{L}_{0,0} = 0$$

$$\sum_{n \neq 0}\left(\left(1 \mp \frac{n}{E_n}\right)^2 \tilde{L}_{n,n}\right) = a_0^\mu (a_0^\nu)^* \eta_{\mu\nu} \equiv (c_1 m)^2 x^\mu x_\mu + (c_2)^2 p^\mu p_\mu,$$

we have the string mass in terms of vibrational modes and center of mass position

$$M^2 = -\frac{1}{(c_2)^2}\left(\sum_{n \neq 0}\left(\left(1 \mp \frac{n}{E_n}\right)^2 \tilde{L}_{n,n}\right) - (c_1 m)^2 x^\mu x_\mu\right) =$$

$$= -\frac{1}{(c_2)^2}\left(\sum_{n \neq 0}\left(\left(1 \mp \frac{n}{E_n}\right)^2 a_n^\mu (a_n^\nu)^* \eta_{\mu\nu}\right) - (c_1 m)^2 x^\mu x_\mu\right).$$

10.1 The Expansion Modes Algebra

10.1.1 Closed String

Before starting this analysis, let's rename the vibrational modes to α_n^μ to be more in line with the literature
Start by finding the canonical momentum

$$\Pi_\lambda = \frac{\partial \mathscr{L}}{\partial(\partial_\tau X^\lambda)} = -\frac{T}{2}\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}}\right) g^{ab}\delta_a^\tau \delta_\lambda^\mu \partial_b X^\nu \eta_{\mu\nu} =$$

$$= -\frac{1}{4\pi\alpha'}\left(1 - \frac{k\Delta}{2\phi^2}\right)\eta^{\tau b}\partial_b X^\nu \eta_{\lambda\nu} =$$

$$= \frac{1}{4\pi\alpha'}F^-\partial_\tau X^\nu \eta_{\lambda\nu} = F^-\frac{i}{8\pi\alpha'}\sum_{n \in \mathbb{Z}}\left(\alpha_n^\nu e^{i(E_n\tau - n\sigma)} - (\alpha_n^\nu)^* e^{-i(E_n\tau - n\sigma)}\right)\eta_{\nu\lambda},$$

where $F^- = \sum_l(A_l e^{i(E_l\tau + l\sigma)} + (A_l)^* e^{-i(E_l\tau + l\sigma)})$. We then proceed to the equal time canonical Poisson bracket relations

$$\{X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} = \delta(\sigma - \sigma')\eta^{\mu\nu}$$

$$\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = \{\Pi^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} = 0,$$

where by expanding the first relation at $\tau = 0$ we get

$$\{X^\mu(0, \sigma), \Pi^\nu(0, \sigma')\} = \left\{\frac{1}{2}\sum_n \frac{1}{E_n}\left(\alpha_n^\mu e^{-in\sigma} + (\alpha_n^\mu)^* e^{in\sigma}\right),\right.$$

$$\left.\frac{i}{8\pi\alpha'}F^-(0, \sigma')\sum_p\left(\alpha_p^\nu e^{-ip\sigma'} - (\alpha_p^\nu)^* e^{ip\sigma'}\right)\right\} = \delta(\sigma - \sigma')\eta^{\mu\nu}$$

$$\frac{i}{16\pi\alpha'}F^-(0, \sigma')\sum_n \frac{1}{E_n}\sum_p\left(\{\alpha_n^\mu, \alpha_p^\nu\} e^{-in\sigma}e^{-ip\sigma'} - \{\alpha_n^\mu, (\alpha_p^\nu)^*\} e^{-in\sigma}e^{ip\sigma'}+\right.$$

$$\left.+ \{(\alpha_n^\mu)^*, \alpha_p^\nu\} e^{in\sigma}e^{-ip\sigma'} - \{(\alpha_n^\mu)^*, (\alpha_p^\nu)^*\} e^{in\sigma}e^{ip\sigma'}\right) = \delta(\sigma - \sigma')\eta^{\mu\nu}$$

$$\left\downarrow \frac{1}{2\pi}\int_0^{2\pi} d\sigma e^{im\sigma}\right.$$

$$\frac{i}{16\pi\alpha'}F^-(0, \sigma')\frac{1}{E_m}\sum_p\left(\{\alpha_m^\mu, \alpha_p^\nu\} e^{-ip\sigma'} - \{\alpha_m^\mu, (\alpha_p^\nu)^*\} e^{ip\sigma'}+\right.$$

$$\left.+ \{(\alpha_{-m}^\mu)^*, \alpha_p^\nu\} e^{-ip\sigma'} - \{(\alpha_{-m}^\mu)^*, (\alpha_p^\nu)^*\} e^{ip\sigma'}\right) = \frac{1}{2\pi}e^{im\sigma'}\eta^{\mu\nu}$$

$$\frac{i}{8\alpha' E_m} \sum_l \sum_p \left(\left(A_l e^{il\sigma'} + (A_l)^* e^{-il\sigma'} \right) \{\alpha_m^\mu, \alpha_p^\nu\} e^{-ip\sigma'} - \right.$$

$$- \left(A_l e^{il\sigma'} + (A_l)^* e^{-il\sigma'} \right) \{\alpha_m^\mu, (\alpha_p^\nu)^*\} e^{ip\sigma'} +$$

$$+ \left(A_l e^{il\sigma'} + (A_l)^* e^{-il\sigma'} \right) \{(\alpha_{-m}^\mu)^*, \alpha_p^\nu\} e^{-ip\sigma'} -$$

$$\left. - \left(A_l e^{il\sigma'} + (A_l)^* e^{-il\sigma'} \right) \{(\alpha_{-m}^\mu)^*, (\alpha_p^\nu)^*\} e^{ip\sigma'} \right) = e^{im\sigma'} \eta^{\mu\nu}$$

$$\frac{i}{8\alpha' E_m} \sum_l \sum_p \left(A_l \{\alpha_m^\mu, \alpha_p^\nu\} e^{i(l-p)\sigma'} + (A_l)^* \{\alpha_m^\mu, \alpha_p^\nu\} e^{-i(l+p)\sigma'} - \right.$$

$$- A_l \{\alpha_m^\mu, (\alpha_p^\nu)^*\} e^{i(l+p)\sigma'} - (A_l)^* \{\alpha_m^\mu, (\alpha_p^\nu)^*\} e^{i(p-l)\sigma'} +$$

$$+ A_l \{(\alpha_{-m}^\mu)^*, \alpha_p^\nu\} e^{i(l-p)\sigma'} + (A_l)^* \{(\alpha_{-m}^\mu)^*, \alpha_p^\nu\} e^{-i(l+p)\sigma'} -$$

$$\left. - A_l \{(\alpha_{-m}^\mu)^*, (\alpha_p^\nu)^*\} e^{i(l+p)\sigma'} - (A_l)^* \{(\alpha_{-m}^\mu)^*, (\alpha_p^\nu)^*\} e^{i(p-l)\sigma'} \right) = e^{im\sigma'} \eta^{\mu\nu}$$

$$\Bigg\downarrow \frac{1}{2\pi} \int_0^{2\pi} d\sigma'$$

$$\frac{i}{8\alpha' E_m} \sum_l \left(A_l \{\alpha_m^\mu, \alpha_l^\nu\} + (A_l)^* \{\alpha_m^\mu, \alpha_{-l}^\nu\} - A_l \{\alpha_m^\mu, (\alpha_{-l}^\nu)^*\} - (A_l)^* \{\alpha_m^\mu, (\alpha_l^\nu)^*\} + \right.$$

$$\left. + A_l \{(\alpha_{-m}^\mu)^*, \alpha_l^\nu\} + (A_l)^* \{(\alpha_{-m}^\mu)^*, \alpha_{-l}^\nu\} - A_l \{(\alpha_{-m}^\mu)^*, (\alpha_{-l}^\nu)^*\} - (A_l)^* \{(\alpha_{-m}^\mu)^*, (\alpha_l^\nu)^*\} \right) = \delta_{m,0} \eta^{\mu\nu}.$$

If we fix the expansion modes of F^- to satisfy

$$\mathfrak{Re}(A_0) = -2\alpha'$$
$$A_{-m} = (A_m)^*,$$

we get the desired

$$\{\alpha_m^\mu, \alpha_l^\nu\} = 0 = \{(\alpha_m^\mu)^*, (\alpha_l^\nu)^*\}$$
$$\{\alpha_m^\mu, (\alpha_l^\nu)^*\} = -i E_m \delta_{m,l} \eta^{\mu\nu} = -\{(\alpha_m^\mu)^*, \alpha_l^\nu\}, \ m, l \geq 0,$$

where since $\alpha_{-n}^\mu \propto \alpha_n^\mu$ the relations for just positive m, l are sufficient. We can rescale the vibrational modes to get harmonic oscillator bracket relations

$$a_n^\mu := \frac{1}{\sqrt{E_n}} \alpha_n^\mu$$

$$(a_n^\mu)^* := \frac{1}{\sqrt{E_n}} (\alpha_n^\mu)^*$$

$$\{a_n^\mu, a_p^\nu\} = 0 = \{(a_n^\mu)^*, (a_p^\nu)^*\}$$
$$\{a_n^\mu, (a_p^\nu)^*\} = -i \delta_{n,p} \eta^{\mu\nu}.$$

This can be used to calculate the bracket relation between x^μ and p^ν:

$$\{\alpha_0^\mu, (\alpha_0^\nu)^*\} = \{(c_1 \mu x^\mu + i c_2 p^\mu), (c_1 \mu x^\nu - i c_2 p^\nu)\} = -i \mu \eta^{\mu\nu}$$

$$(c_1 \mu)^2 \{x^\mu, x^\nu\} + (c_2)^2 \{p^\mu, p^\nu\} - i c_1 c_2 \mu \left(\{x^\mu, p^\nu\} - \{p^\mu, x^\nu\} \right) = -i \mu \eta^{\mu\nu}$$

$$\{x^\mu, x^\nu\} = 0 = \{p^\mu, p^\nu\}$$

$$\{x^\mu, p^\nu\} = \frac{1}{2 c_1 c_2} \eta^{\mu\nu},$$

so we get that $c_2 = 1/(2c_1)$, and the 0-th mode becomes $\alpha_0^\mu = c_1 \mu x^\mu + i p^\mu/(2c_1)$.

10.2   The Constraint Algebra

With the bracket relations between the vibrational modes in hand, we can now calculate the algebra generated by the constraints. For simplicity, since $E_n = \sqrt{n^2 + \mu^2} > 0$, we can focus instead on the algebra of the normalized harmonic oscillator modes a_n^μ instead of the expansion modes α_n^μ.

$$\{L_{n,m}, L_{k,p}\} = \left\{a_n^\mu a_m^\nu \eta_{\mu\nu}, a_k^{\mu'} a_p^{\nu'} \eta_{\mu'\nu'}\right\} = 0$$

$$\left\{L_{n,m}, \widetilde{L}_{k,p}\right\} = \left\{a_n^\mu a_m^\nu \eta_{\mu\nu}, a_k^{\mu'} (a_p^{\nu'})^* \eta_{\mu'\nu'}\right\} =$$

$$= \eta_{\mu\nu} \left(a_m^\nu \left\{a_n^\mu, (a_p^{\nu'})^*\right\} a_k^{\mu'} + a_n^\mu \left\{a_m^\nu, (a_p^{\nu'})^*\right\} a_k^{\mu'}\right) \eta_{\mu'\nu'} =$$

$$= \eta_{\mu\nu} \left(a_m^\nu (-i\eta^{\mu\nu'} \delta_{n,p}) a_k^{\mu'} + a_n^\mu (-i\eta^{\nu\nu'} \delta_{m,p}) a_k^{\mu'}\right) \eta_{\mu'\nu'} =$$

$$= -i \left(a_m^\mu a_k^\nu \eta_{\mu\nu} \delta_{n,p} + a_n^\mu a_k^\nu \eta_{\mu\nu} \delta_{m,p}\right) =$$

$$= -i \left(L_{m,k} \delta_{n,p} + L_{n,k} \delta_{m,p}\right)$$

$$\left\{\widetilde{L}_{n,m}, \widetilde{L}_{k,p}\right\} = \left\{a_n^\mu (a_m^\nu)^* \eta_{\mu\nu}, a_k^{\mu'} (a_p^{\nu'})^* \eta_{\mu'\nu'}\right\} =$$

$$= \eta_{\mu\nu} \left((a_m^\nu)^* \left\{a_n^\mu, (a_p^{\nu'})^*\right\} a_k^{\mu'} + a_n^\mu \left\{(a_m^\nu)^*, a_k^{\mu'}\right\} (a_p^{\nu'})^*\right) \eta_{\mu'\nu'} =$$

$$= \eta_{\mu\nu} \left((a_m^\nu)^* (-i\eta^{\mu\nu'} \delta_{n,p}) a_k^{\mu'} - a_n^\mu (-i\eta^{\nu\mu'} \delta_{m,k}) (a_p^{\nu'})^*\right) \eta_{\mu'\nu'} =$$

$$= -i \left(a_k^\mu (a_m^\nu)^* \eta_{\mu\nu} \delta_{n,p} - a_n^\mu (a_p^\nu)^* \eta_{\mu\nu} \delta_{m,k}\right) =$$

$$= -i \left(\widetilde{L}_{k,m} \delta_{n,p} - \widetilde{L}_{n,p} \delta_{m,k}\right).$$

These relations reveal that, in fact, $\alpha_{-n}^\mu = \alpha_n^\mu$, since the middle relation is only invariant to $p \mapsto -p$ if the proportionality coeficient is 1.

11 Into the Quantum Realm

We now proceed to quantize the KG string through canonical quantization. The string X^μ and it's momentum Π^ν are promoted to operators $\hat{X}^\mu$ and $\hat{\Pi}^\nu$, and we turn the Poisson bracket $\{.,.\}$ into commutator between operators $[.,.] = i\hbar\{.,.\}$. The equal time bracket relations turn into equal time commutation relations

$$\left[\hat{X}^\mu(\tau,\sigma), \hat{\Pi}^\nu(\tau,\sigma')\right] = i\delta(\sigma - \sigma')\eta^{\mu\nu}$$

$$\left[\hat{X}^\mu, \hat{X}^\nu\right] = \left[\hat{\Pi}^\mu, \hat{\Pi}^\nu\right] = 0$$

$$[\hat{a}_n^\mu, \hat{a}_m^\nu] = [(\hat{a}_n^\mu)^\dagger, (\hat{a}_m^\nu)^\dagger] = 0$$

$$[\hat{a}_n^\mu, (\hat{a}_m^\nu)^\dagger] = \eta^{\mu\nu}\delta_{n,m}$$

$$[\hat{x}^\mu, \hat{x}^\nu] = [\hat{p}^\mu, \hat{p}^\nu] = 0$$

$$[\hat{x}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu}.$$

Thus, the rescaled vibrational mode operators $\hat{a}_n^\mu$ and $(\hat{a}_m^\nu)^\dagger$ are annihilation and creations operators, respectively.

We define a vacuum state of the string to obey

$$\hat{a}_n^\mu|0\rangle = 0, \text{ for } n \neq 0.$$

For $n = 0$, we have the center of mass position and momentum operators, so the vacuum also obeys

$$\hat{x}^\mu|0;x\rangle = x^\mu|0;x\rangle$$

$$\hat{p}_\mu|0;x\rangle = -i\frac{\partial}{\partial x^\mu}|0;x\rangle$$

in position representation or alternatively

$$\hat{x}^\mu|0;p\rangle = -i\frac{\partial}{\partial p_\mu}|0;p\rangle$$

$$\hat{p}_\mu|0;p\rangle = p_\mu|0;p\rangle$$

in momentum representation.

A generic state arises from a sequence of creation operators on the vacuum

$$((\hat{a}_1^{\mu_1})^\dagger)^{n_{\mu_1}}((\hat{a}_2^{\mu_2})^\dagger)^{n_{\mu_2}}...((\hat{a}_{-1}^{\nu_1})^\dagger)^{n_{\nu_1}}((\hat{a}_{-2}^{\nu_2})^\dagger)^{n_{\nu_2}}...|0\rangle.$$

This (should) give rise to particles.

As in regular ST, we have ghosts arising from the Minkowski metric

$$\left[\hat{a}_n^\mu, (\hat{a}_m^\nu)^\dagger\right] = \eta^{\mu\nu}\delta_{nm}.$$

For constraints, classically we have

$$L_{n,p} = \alpha_n^\mu\alpha_p^\nu\eta_{\mu\nu} = 0$$

$$\widetilde{L}_{n,p} = \alpha_n^\mu(\alpha_p^\nu)^*\eta_{\mu\nu} = 0, \ n \neq p$$

which because of the existence of ghosts we require to have vanishing matrix elements when sandwiched between physical states

$$\langle\text{phys}'|\hat{L}_{n,p}|\text{phys}\rangle = 0 = \langle\text{phys}'|\hat{\widetilde{L}}_{n,p}|\text{phys}\rangle.$$

When translating the constraints into quantum operators, in $L_{n,p}$ we have no ambiguity of ordering since $\hat{a}$ commutes with itself. As for $\widetilde{L}_{n,p}$, there is an ambiguity for $p = n \neq 0$, so we pick normal ordering with the annihilation operators moved to the right

$$\hat{\widetilde{L}}_{n,n} = (\hat{\alpha}_n^\mu)^\dagger\hat{\alpha}_n^\nu\eta_{\mu\nu}.$$

The ambiguity manifests in the imposition of this constraint as

$$\langle\text{phys}'|\left(\sum_n\left(\left(1 \mp \frac{n}{E_n}\right)^2\hat{\widetilde{L}}_{n,n}\right) - c\right)|\text{phys}\rangle = 0,$$

for some costant c. Since classically

$$M^2 = (2c_1)^2 \left((c_1\mu)^2 x^\mu x_\mu - \sum_{n\neq 0} \left(\left(1 \mp \frac{n}{E_n}\right)^2 \tilde{L}_{n,n} \right) \right) =$$

$$= (2c_1)^2 \left((c_1\mu)^2 x^\mu x_\mu - \sum_{n\neq 0} \left(\left(1 \mp \frac{n}{E_n}\right)^2 \alpha_n^\mu (\alpha_n^\nu)^* \eta_{\mu\nu} \right) \right),$$

we see that the string mass spectrum will be affected by this constant

$$\hat{M}^2 = (2c_1)^2 \left((c_1\mu)^2 \hat{x}^\mu \hat{x}_\mu - \left(\sum_{n\neq 0} \left(\left(1 \mp \frac{n}{E_n}\right)^2 E_n (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu} \right) - c \right) \right).$$

The commutation relations between the L's are inherited from the bracket relations

$$\left[\hat{L}_{n,m}, \hat{L}_{k,p} \right] = 0$$

$$\left[\hat{L}_{n,m}, \hat{\tilde{L}}_{k,p} \right] = \hat{L}_{m,k}\delta_{n,p} + \hat{L}_{n,k}\delta_{m,p}$$

$$\left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p} \right] = \hat{\tilde{L}}_{k,m}\delta_{n,p} - \hat{\tilde{L}}_{n,p}\delta_{m,k}.$$

Since $[\hat{L}_{n,m}, \hat{\tilde{L}}_{p,k}]$ only has annihilation operators, it has no ordering ambiguities. $[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}]$ have ordering ambiguities for $m = k$ and/or $p = n$, so we add the anomalous terms

$$\left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p} \right] = \hat{\tilde{L}}_{k,m}\delta_{n,p} - \hat{\tilde{L}}_{n,p}\delta_{m,k} + C_n\delta_{n,p} + D_k\delta_{m,k}$$

Clearly $C_0 = 0 = D_0$, since if all indices are equal the commutator should vanish. Also, by the symmetry of the L's and because of the δ, we also have that $C_{-n} = C_n$ and $D_{-k} = D_k$. We get from the Jacobi identity

$$\left[\hat{\tilde{L}}_{n,m}, \left[\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{r,s} \right] \right] + \left[\hat{\tilde{L}}_{r,s}, \left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p} \right] \right] + \left[\hat{\tilde{L}}_{k,p}, \left[\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{n,m} \right] \right] = 0$$

$$\left[\hat{\tilde{L}}_{n,m}, \left(\hat{\tilde{L}}_{r,p}\delta_{k,s} - \hat{\tilde{L}}_{k,s}\delta_{p,r} + C_k\delta_{k,s} + D_r\delta_{p,r} \right) \right] +$$
$$+ \left[\hat{\tilde{L}}_{r,s}, \left(\hat{\tilde{L}}_{k,m}\delta_{n,p} - \hat{\tilde{L}}_{n,p}\delta_{m,k} + C_n\delta_{n,p} + D_k\delta_{m,k} \right) \right] +$$
$$+ \left[\hat{\tilde{L}}_{k,p}, \left(\hat{\tilde{L}}_{n,s}\delta_{r,m} - \hat{\tilde{L}}_{r,m}\delta_{s,n} + C_r\delta_{r,m} + D_n\delta_{s,n} \right) \right] = 0$$

$$\left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{r,p} \right]\delta_{k,s} - \left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,s} \right]\delta_{p,r} + \left[\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{k,m} \right]\delta_{n,p} -$$
$$- \left[\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{n,p} \right]\delta_{m,k} + \left[\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{n,s} \right]\delta_{r,m} - \left[\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{r,m} \right]\delta_{s,n} = 0$$

$$\left(\hat{\tilde{L}}_{r,m}\delta_{n,p} - \hat{\tilde{L}}_{n,p}\delta_{m,r} + C_n\delta_{n,p} + D_r\delta_{m,r} \right)\delta_{k,s} - \left(\hat{\tilde{L}}_{k,m}\delta_{n,s} - \hat{\tilde{L}}_{n,s}\delta_{m,k} + C_n\delta_{n,s} + D_k\delta_{m,k} \right)\delta_{p,r} +$$
$$+ \left(\hat{\tilde{L}}_{k,s}\delta_{r,m} - \hat{\tilde{L}}_{r,m}\delta_{s,k} + C_r\delta_{r,m} + D_k\delta_{s,k} \right)\delta_{n,p} - \left(\hat{\tilde{L}}_{n,s}\delta_{r,p} - \hat{\tilde{L}}_{r,p}\delta_{s,n} + C_r\delta_{r,p} + D_n\delta_{s,n} \right)\delta_{m,k} +$$
$$+ \left(\hat{\tilde{L}}_{n,p}\delta_{k,s} - \hat{\tilde{L}}_{k,s}\delta_{p,n} + C_k\delta_{k,s} + D_n\delta_{p,n} \right)\delta_{r,m} - \left(\hat{\tilde{L}}_{r,p}\delta_{k,m} - \hat{\tilde{L}}_{k,m}\delta_{p,r} + C_k\delta_{k,m} + D_r\delta_{p,r} \right)\delta_{s,n} = 0.$$

For $k = s$, $n = p$ and $r = m$,

$$\left(\hat{\tilde{L}}_{r,r} - \hat{\tilde{L}}_{n,n} + C_n + D_r \right) - \left(\hat{\tilde{L}}_{k,r}\delta_{p,s} - \hat{\tilde{L}}_{n,k}\delta_{r,k} + C_n\delta_{n,k} + D_k\delta_{r,k} \right)\delta_{n,m} +$$
$$+ \left(\hat{\tilde{L}}_{k,k} - \hat{\tilde{L}}_{r,r} + C_r + D_k \right) - \left(\hat{\tilde{L}}_{n,k}\delta_{r,n} - \hat{\tilde{L}}_{r,n}\delta_{k,n} + C_r\delta_{r,n} + D_n\delta_{k,n} \right)\delta_{r,k} +$$
$$+ \left(\hat{\tilde{L}}_{n,n} - \hat{\tilde{L}}_{k,k} + C_k + D_n \right) - \left(\hat{\tilde{L}}_{r,n}\delta_{k,r} - \hat{\tilde{L}}_{k,r}\delta_{n,r} + C_k\delta_{k,r} + D_r\delta_{n,r} \right)\delta_{k,n} = 0.$$

Assuming $n \neq m$, $r \neq k$ and $k \neq n$ leaves us with

$$C_n + C_r + C_k + D_r + D_k + D_n = 0,$$

which for $n + k + r = 0$ becomes

$$C_n + C_k + C_{-n-k} + D_n + D_k + D_{-n-k} = 0$$

$$C_n + C_k + C_{n+k} + D_n + D_k + D_{n+k} = 0,$$

and now setting $k = 1$ gives

$$C_n + C_{n+1} + C_1 + D_n + D_{n+1} + D_1 = 0.$$

Since this is a linear difference eqn, we can define $F_n = C_n + D_n$ and so we have the eqn

$$F_{n+1} + F_n + F_1 = 0$$

The solution to this difference eqn with $F_0 = 0$ is

$$F_n = \frac{F_1}{2} \cos\left(\frac{n\pi}{2}\right) - \frac{F_1}{2} = \frac{F_1}{2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right).$$

Since $C_0 = D_0 = 0$, we conclude that C_n and D_n are both equal to half F_n

$$C_n = \frac{F_1}{4}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)$$

$$D_n = \frac{F_1}{4}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)$$

By now, the commutator of $\hat{\tilde{L}}$ is

$$\left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}\right] = \hat{\tilde{L}}_{k,m}\delta_{n,p} - \hat{\tilde{L}}_{n,p}\delta_{m,k} + \frac{F_1}{4}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)\delta_{n,p} + \frac{F_1}{4}\left(\cos\left(\frac{k\pi}{2}\right) - 1\right)\delta_{m,k}.$$

Let's calculate the VEV of this commutator for $m = -k \neq 0$ with $n = p = 1$

$$\langle 0|\left[\hat{\tilde{L}}_{1,-k}, \hat{\tilde{L}}_{k,1}\right]|0\rangle = \langle 0|\hat{\tilde{L}}_{1,-k}\hat{\tilde{L}}_{k,1}|0\rangle$$

$$\langle 0|\hat{\tilde{L}}_{k,-k} - \frac{F_1}{4}|0\rangle = \langle 0|(a^\mu_{-k})^\dagger a^\nu_1 \eta_{\mu\nu}(a^{\mu'}_1)^\dagger a^{\nu'}_k \eta_{\mu'\nu'}|0\rangle$$

$$\langle 0| - \frac{F_1}{4}|0\rangle = 0 \iff F_1 = 0,$$

so in actuality, the constraint algebra has no anomalies.

Let us now denote the ground state of momentum p^μ as $|0;p\rangle$. The mass-shell condition

$$\hat{M}^2 = (2c_1)^2\left((c_1\mu)^2\hat{x}^\mu\hat{x}_\mu - \left(\sum_{n\neq 0}\left(\left(1 \mp \frac{n}{E_n}\right)^2 E_n(\hat{a}^\mu_n)^\dagger\hat{a}^\nu_n\eta_{\mu\nu}\right) - c\right)\right)$$

implies that

$$\langle 0;p|\hat{M}^2|0;p\rangle = \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2\langle 0;p|\hat{x}^\mu\hat{x}_\mu|0;p\rangle - \left(\sum_{n\neq 0}\left(\left(\frac{n}{E_n}\right)^2 - 1\right)\langle 0;p|(\hat{a}^\mu_n)^\dagger\hat{a}^\nu_n\eta_{\mu\nu}|0;p\rangle - \langle 0;p|c|0;p\rangle\right)\right)$$

$$M^2 = \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2\frac{\partial\psi^*(p)}{\partial p_\mu}\frac{\partial\psi(p)}{\partial p^\mu} + c\right)$$

$$(4\pi(\alpha')^2)^2 p^2 = -\left((2\pi\alpha'\mu)^2\left|\frac{\partial\psi(p)}{\partial p}\right|^2 + c\right)$$

Now looking at the first excited state $\zeta_\mu(\hat{a}_1^\mu)^\dagger|0;p\rangle$ with $\zeta_\mu = \zeta_\mu(p)$ being the polarization covector, the mass-shell now reads

$$\langle 0;p|\hat{a}_1^\mu\zeta_\mu\hat{M}^2\zeta_\nu(\hat{a}_1^\nu)^\dagger|0;p\rangle = \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2\langle 0;p|\hat{a}_1^\mu\zeta_\mu\hat{x}^{\mu'}\,\hat{x}_{\mu'}\zeta_\nu(\hat{a}_1^\nu)^\dagger|0;p\rangle-\right.$$

$$\left.-\left(\sum_{n\neq 0}\left(\left(\frac{n}{E_n}\right)^2-1\right)\langle 0;p|\hat{a}_1^\mu\zeta_\mu(\hat{a}_n^{\mu'})^\dagger\hat{a}_n^{\nu'}\eta_{\mu'\nu'}\zeta_\nu(\hat{a}_1^\nu)^\dagger|0;p\rangle-\langle 0;p|\hat{a}_1^\mu\zeta_\mu c\zeta_\nu(\hat{a}_1^\nu)^\dagger|0;p\rangle\right)\right)$$

$$M^2\zeta^\mu\zeta_\mu = \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2\left|\frac{\partial\psi(p)}{\partial p}\right|^2\zeta^\mu\zeta_\mu-\left(\frac{1}{1+\mu^2}-1\right)\zeta^\mu\zeta_\mu+c\zeta^\mu\zeta_\mu\right)$$

$$(4\pi(\alpha')^2)^2p^2 = -\left((2\pi\alpha'\mu)^2\left|\frac{\partial\psi(p)}{\partial p}\right|^2-\left(\frac{1}{1+\mu^2}-1\right)+c\right).$$

The auxiliary $\hat{\tilde{L}}_{1,0}\zeta_{\mu'}(\hat{a}_1^{\mu'})^\dagger|0;p\rangle = 0$ condition implies

$$\hat{\tilde{L}}_{1,0}\zeta_{\mu'}(\hat{a}_1^{\mu'})^\dagger|0;p\rangle = 0$$

$$(\hat{a}_0^\mu)^\dagger\hat{a}_1^\nu\eta_{\mu\nu}\zeta_{\mu'}(\hat{a}_1^{\mu'})^\dagger|0;p\rangle = 0$$

$$\zeta_\mu(\hat{a}_0^\mu)^\dagger|0;p\rangle = 0$$

$$\zeta_\mu\left(2\pi\alpha'\mu\hat{x}^\mu - 4\pi(\alpha')^2i\hat{p}^\mu\right)|0;p\rangle = 0$$

$$-2\pi\alpha'\mu i\zeta_\mu\frac{\partial\psi(p)}{\partial p_\mu} = 4\pi(\alpha')^2i\zeta_\mu p^\mu$$

$$\zeta_\mu p^\mu = -\frac{\mu}{2\alpha'}\zeta_\mu\frac{\partial\psi(p)}{\partial p_\mu}.$$

If the momentum wave function is chosen to be (need to justify this)

$$\psi(p) = \frac{\alpha'}{\mu}p^2,$$

we get that $\zeta_\mu p^\mu = 0$. Using this in the mass-shell condition for the first excited state yields

$$(4\pi(\alpha')^2)^2p^2 = -\left((2\pi\alpha'\mu)^24\frac{(\alpha')^2}{\mu^2}p^2-\left(\frac{1}{1+\mu^2}-1\right)+c\right)$$

$$2(4\pi(\alpha')^2)^2p^2 = \frac{1}{1+\mu^2}-1-c,$$

and for the ground state

$$2(4\pi(\alpha')^2)^2p^2 = -c.$$

In general, the n-th excited state will have momentum given by

$$2(4\pi(\alpha')^2)^2p^2 = \frac{n^2}{n^2+\mu^2}-1-c.$$

The norm of the states is given by $\zeta_\mu\zeta^\mu$, thus if we choose p to lie on the $(0,1)$-plane, then the $D-2$ spacelike polarizations normal to that plane clearly have positive norm. If we choose c such that the $n=d$, $d=(D-1)$ (we choose this because the solution only has $n=0,1,2,...,d$ terms by the $\tilde{L}$ constraints) excited state is a tachyon with $p^2>0$, then p can be taken to have no time component and thus ζ is time-like with negative norm. If $p^2<0$, p can be chosen to ONLY have a time component, leaving ζ spacelike with positive norm again. At last, if $p^2=0$, then ζ is proportional to p with zero norm. Thus, for the absence of ghosts we require

$$c \geq \frac{d^2}{d^2+\mu^2}-1.$$

In the boundary case, the 3rd excited vector state is massless and the rest (including the scalar ground state) are tachyons... This analysis, however, is to get spin 1 particles from the closed string. The actual massless particle we want from closed strings is a symmetric spin 2 particle, so let's consider now the state $\zeta_{\mu\nu}(\hat{a}_d^{\mu})^{\dagger}(\hat{a}_d^{\nu})^{\dagger}|0;p\rangle$ where $\zeta_{\mu\nu} = \zeta_{\nu\mu}$ is the polarization tensor. The mass-shell is

$$\langle 0;p|\hat{a}_d^{\alpha}\hat{a}_d^{\beta}\zeta_{\alpha\beta}\hat{M}^2\zeta_{\mu\nu}(\hat{a}_d^{\mu})^{\dagger}(\hat{a}_d^{\nu})^{\dagger}|0;p\rangle = \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2\langle 0;p|\hat{a}_d^{\alpha}\hat{a}_d^{\beta}\zeta_{\alpha\beta}\hat{x}^{\mu'}\hat{x}_{\mu'}\zeta_{\mu\nu}(\hat{a}_d^{\mu})^{\dagger}(\hat{a}_d^{\nu})^{\dagger}|0;p\rangle - \right.$$

$$\left. - \left(\sum_{n\neq 0}\left(\left(\frac{n}{E_n}\right)^2 - 1\right)\langle 0;p|\hat{a}_d^{\alpha}\hat{a}_d^{\beta}\zeta_{\alpha\beta}(\hat{a}_n^{\mu'})^{\dagger}\hat{a}_n^{\nu'}\eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^{\mu})^{\dagger}(\hat{a}_d^{\nu})^{\dagger}|0;p\rangle - \langle 0;p|\hat{a}_d^{\alpha}\hat{a}_d^{\beta}\zeta_{\alpha\beta}c\zeta_{\mu\nu}(\hat{a}_d^{\mu})^{\dagger}(\hat{a}_d^{\nu})^{\dagger}|0;p\rangle\right)\right)$$

$$M^2\zeta_{\mu\nu}\zeta^{\mu\nu} = \frac{1}{(4\pi(\alpha')^2)^2}\left((2\pi\alpha'\mu)^2\left|\frac{\partial\psi(p)}{\partial p}\right|^2\zeta_{\mu\nu}\zeta^{\mu\nu} - \left(2\left(\frac{d^2}{d^2+\mu^2} - 1\right)\zeta_{\mu\nu}\zeta^{\mu\nu} - c\zeta_{\mu\nu}\zeta^{\mu\nu}\right)\right)$$

$$(4\pi(\alpha')^2)^2 p^2 = -\left((2\pi\alpha'\mu)^2\left|\frac{\partial\psi(p)}{\partial p}\right|^2 - \left(2\left(\frac{d^2}{d^2+\mu^2} - 1\right) - c\right)\right)$$

$$2(4\pi(\alpha')^2)^2 p^2 = 2\left(\frac{d^2}{d^2+\mu^2} - 1\right) - c.$$

Using the auxiliary $\hat{\tilde{L}}_{d,0}\zeta_{\mu\nu}(\hat{a}_d^{\mu})^{\dagger}(\hat{a}_d^{\nu})^{\dagger}|0;p\rangle = 0$ condition implies

$$\hat{\tilde{L}}_{d,0}\zeta_{\mu\nu}(\hat{a}_d^{\mu})^{\dagger}(\hat{a}_d^{\nu})^{\dagger}|0;p\rangle = 0$$

$$(\hat{a}_0^{\mu'})^{\dagger}\hat{a}_d^{\nu'}\eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^{\mu})^{\dagger}(\hat{a}_d^{\nu})^{\dagger}|0;p\rangle = 0$$

$$\left((\hat{a}_0^{\mu'})^{\dagger}\eta_{\mu'\nu'}\zeta_{\mu\nu}\eta^{\nu'\mu}(a_d^{\nu})^{\dagger} + (\hat{a}_0^{\mu'})^{\dagger}\eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^{\mu})^{\dagger}\hat{a}_d^{\nu'}(\hat{a}_d^{\nu})^{\dagger}\right)|0;p\rangle = 0$$

$$\left((\hat{a}_0^{\mu'})^{\dagger}\eta_{\mu'\nu'}\zeta_{\mu\nu}\eta^{\nu'\mu}(a_d^{\nu})^{\dagger} + (\hat{a}_0^{\mu'})^{\dagger}\eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^{\mu})^{\dagger}\eta^{\nu'\nu}\right)|0;p\rangle = 0$$

$$2\zeta_{\mu\nu}(\hat{a}_0^{\mu})^{\dagger}(\hat{a}_d^{\nu})^{\dagger}|0;p\rangle = 0$$

$$\zeta_{\mu\nu}(\hat{a}_d^{\nu})^{\dagger}\left(2\pi\alpha'\mu\hat{x}^{\mu} - 4\pi(\alpha')^2 i\hat{p}^{\mu}\right)|0;p\rangle = 0$$

$$\zeta_{\mu\nu}\left(-2\pi\alpha'\mu i\frac{\partial\psi(p)}{\partial p_{\mu}} - 4\pi(\alpha')^2 ip^{\mu}\right)(\hat{a}_d^{\nu})^{\dagger}|0\rangle = 0$$

$$\zeta_{\mu\nu}p^{\mu}(\hat{a}_d^{\nu})^{\dagger}|0\rangle = 0,$$

implying that

$$\zeta_{\mu\nu}p^{\mu} = \zeta_{\mu\nu}p^{\nu} = 0.$$

Since the norm of the spin-2 states is given by $\zeta^{\mu\nu}\zeta_{\mu\nu}$, we need those to not be negative. Since ζ is symmetric and we work in $(-,+,+,...)$ signature, $\zeta^{\mu\nu}\zeta_{\mu\nu} \geq 0$ always hold, so we just need to choose c such that we get ate leats 1 massless spin-2 state. For that, we see that

$$c = 2\left(\frac{n^2}{n^2+\mu^2} - 1\right)$$

does the trick for the n-th state, and since for the non-existence of ghosts in the spin-1 states we required

$$c \geq \frac{d^2}{d^2+\mu^2} - 1,$$

which is negative, we can thus choose

$$c \equiv \frac{2d^2}{d^2+\mu^2} - 2,$$

leaving the d-th spin-2 state massless, the rest being tachyons... This, however, is if we impose that one spin-2 state have to be massless. Since granular space-time affects wave propagation speed, in the effective limit gravitons might be effectively massive, thus we can get rid of the tachyons by letting $c > 0$ if need be.

The fact that $\zeta_{\mu\nu} = \zeta_{\nu\mu}$ leaves $\zeta_{\mu\nu}$ with $D(D+1)/2$ independent components. The traceless condition $\eta^{\mu\nu}\zeta_{\mu\nu} = 0$ takes out 1 from that, leaving $(D(D+1)/2) - 1$ allowed polarizations. Finally, the condition $\zeta_{\mu\nu}p^{\mu} = 0$ takes D out of those, leaving $(D(D-1)/2) - 1$ allowed polarizations.

$$\frac{D(D-1)}{2} - 1 = \frac{D^2 - D - 2}{2} = \frac{(D-2)(D+1)}{2}.$$

This leaves 5 polarizations in $D = 4$.

Let's now analyse spurious states to see the conditions to maximize 0-norm states. A physical state $|\phi\rangle$ is a state satisfying the constraints

$$\hat{L}_{n,m}|\phi\rangle = 0, \ \forall n, m \in \mathbb{Z}$$

$$\hat{\tilde{L}}_{n,m}|\phi\rangle = 0, \ n \neq -m$$

$$\left(\sum_n \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \hat{\tilde{L}}_{n,-n} \right) - c \right) |\phi\rangle = 0.$$

A spurious state $|\psi\rangle$ is a state satisfying the last constraint and orthogonal to all physical states, i.e,

$$\left(\sum_n \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \hat{\tilde{L}}_{n,-n} \right) - c \right) |\psi\rangle = 0,$$

$$\langle \phi | \psi \rangle = 0.$$

Spurious states can be written in the from

$$|\psi\rangle = \sum_{n \neq -m} \hat{\tilde{L}}_{n,m} |\chi_{n,m}\rangle$$

with $|\chi_{n,m}\rangle$ satisfying

$$\left(\sum_n \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \hat{\tilde{L}}_{n,-n} \right) - c - k - p \right) |\chi_{k,p}\rangle = 0.$$

To see that this state is indeed orthogonal to $|\phi\rangle$, we perform the inner product

$$\langle \phi | \psi \rangle = \sum_{n \neq -m} \langle \phi | \hat{\tilde{L}}_{n,m} |\chi_{n,m}\rangle = \sum_{n \neq -m} \langle \chi_{n,m} | \hat{\tilde{L}}_{-m,-n} |\phi\rangle^* \equiv 0.$$

The interesting case is when $|\psi\rangle$ is both physical AND spurious, since by construction those states have 0-norm, being orthogonal to every physical state, including themselves. Such states can be constructed by considering spurious states of the form

$$|\psi_{1,n}\rangle = \sum_m \hat{\tilde{L}}_{n,m} |\widetilde{\chi}_{1,m}\rangle$$

where $|\widetilde{\chi}_{1,m}\rangle$ satisfies

$$\hat{\tilde{L}}_{k,p} |\widetilde{\chi}_{1,m}\rangle = 0, \ k \neq -p$$

$$\left(\sum_{n'} \left(\left(\left(\frac{n'}{E_{n'}} \right)^2 - 1 \right) \hat{\tilde{L}}_{n',-n'} \right) - c - 1 - m \right) |\widetilde{\chi}_{1,m}\rangle = 0.$$

These states are annihilated by $\hat{\tilde{L}}_{k,p}$, for $p \neq -n$, and for this case we have that

$$\sum_n \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \hat{\tilde{L}}_{2,-n} |\psi_{1,n}\rangle \right) =$$

$$= \sum_n \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \hat{\tilde{L}}_{2,-n} \sum_m \hat{\tilde{L}}_{n,m} |\widetilde{\chi}_{1,m}\rangle \right) =$$

$$= \sum_n \left(\left(\left(\frac{n}{E_n} \right)^2 - 1 \right) \hat{\tilde{L}}_{n,-n} \right) |\widetilde{\chi}_{1,-2}\rangle,$$

which vanish if $c = 1$.

12 Area correction Path Integral

Build the path integral of the area corrected Polyakov action

$$Z = \int D[g]D[X]\mathrm{e}^{-iS},$$

$$S = -\frac{1}{2\pi\alpha'}\int \mathrm{d}^2x\, F^-\sqrt{-g}g^{ab}\partial_a X^\mu\partial_b X^\nu\eta_{\mu\nu},$$

$$F^- = 1 - \frac{k\Delta}{2(-g)}.$$

In conformal gauge and choosing the conformal factor such that the EoM becomes a KG eqn,

$$F^- = \sum_l \left(A_l\mathrm{e}^{i(E_l\tau + l\sigma)} + (A_{-l})^*\mathrm{e}^{i(-E_l\tau + l\sigma)}\right).$$

This will be the gauge chosen for performing the path integral.

13 Polyakov Action in terms of LQG Variables

Turn embbeding fields X^μ into vector in the spin-1 representation of $\mathbb{R}^{d+1} \rtimes \mathrm{Spin}(d,1)$, X^I, and promote partial derivative to covariant derivative $\partial_a \mapsto \mathcal{D}_a$ acting as

$$\mathcal{D}_a X^I = \partial_a X^I + k\mathcal{A}_{aJ}^I X^J,$$

where the WS $\mathbb{R}^{d+1} \rtimes \mathrm{Spin}(d,1)$ connection $\mathcal{A}_a$ with $I, J = 0, ..., d, d+1$ is given by

$$\mathcal{A}_a X = \begin{bmatrix} 0 & \beta_a^{01} & \beta_a^{02} & \cdots & \beta_a^{0d} & p_a^0/l \\ \beta_a^{01} & 0 & -\theta_a^{12} & \cdots & -\theta_a^{1d} & p_a^1/l \\ \beta_a^{02} & \theta_a^{12} & 0 & \cdots & -\theta_a^{2d} & p_a^2/l \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_a^{0d} & \theta_a^{1d} & \theta_a^{2d} & \cdots & 0 & p_a^d/l \\ -\epsilon p_a^0/l & \epsilon p_a^1/l & \epsilon p_a^2/l & \cdots & \epsilon p_a^d/l & 0 \end{bmatrix} \begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ \vdots \\ X^d \\ 1 \end{bmatrix} = \left(\beta_a^{0\alpha} B_{0\alpha} + \frac{1}{2}\theta_a^{\alpha\beta} J_{\alpha\beta} + \frac{p_a^\mu}{l} P_\mu \right) X,$$

where $\epsilon = \mathrm{sgn}(\Lambda)$, i.e the sign of the background cosmological constant with $\alpha, \beta = 1, ..., d$ and $\mu = 0, ..., d$, so the action becomes

$$S = -\frac{T}{2} \int \mathrm{d}^2 x \sqrt{-g} g^{ab} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ},$$

where the metrics can be recast in terms of auxiliary and bulk vielbein fields as in

$$S = -\frac{T}{2} \int \mathrm{d}^2 x e e_{I'}^a e_{J'}^b \eta^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ},$$

where

$$e := \sqrt{-\det(e_a^I e_b^J \eta_{IJ})} = \sqrt{-\frac{1}{2}\varepsilon^{ff'}\varepsilon^{gg'}(e_f^M e_g^N \eta_{MN})(e_{f'}^{M'} e_{g'}^{N'} \eta_{M'N'})}.$$

13.1 Checking the derivative

Since the connection is non-abelian, it should transform as

$$\mathcal{A}_a' = \mathcal{P}\mathcal{A}_a \mathcal{P}^{-1} - \frac{1}{k}(\partial_a \mathcal{P})\mathcal{P}^{-1},$$

where $\mathcal{P} = e^{\beta^{0\alpha} B_{0\alpha} + \theta^{\alpha\beta} J_{\alpha\beta}/2 + p^\mu P_\mu/l}$. Putting this into a gauge transformed derivative we have

$$\mathcal{D}_a' X' = \partial_a X' + k\mathcal{A}_a' X' =$$

$$= \partial_a(\mathcal{P}X) + k(\mathcal{P}\mathcal{A}_a \mathcal{P}^{-1} - \frac{1}{k}(\partial_a \mathcal{P})\mathcal{P}^{-1})(\mathcal{P}X) =$$

$$= \partial_a \mathcal{P}X + \mathcal{P}\partial_a X + k\mathcal{P}\mathcal{A}_a X - \partial_a \mathcal{P}X \equiv \mathcal{P}\mathcal{D}_a X,$$

so this is a proper covariant derivative for the Poincaré group.

Since under gauge transformations $\mathcal{D}_a X$ transforms covariantly, the action is not invariant under diffeomorphisms. In fact, it transforms as

$$S' = -\frac{T}{2} \int \mathrm{d}^2 x e' e_{I'}^{'a} e_{J'}^{'b} \eta^{I'J'} \mathcal{D}_a' X'^I \mathcal{D}_b' X'^J \eta_{IJ} =$$

$$= -\frac{T}{2} \int \mathrm{d}^2 x e e_{K'}^a (\mathcal{P}^{-1})^{K'}{}_{I'} e_{L'}^b (\mathcal{P}^{-1})^{L'}{}_{J'} \eta^{I'J'} \mathcal{P}^I{}_K \mathcal{D}_a X^K \mathcal{P}^J{}_L \mathcal{D}_b X^L \eta_{IJ},$$

so we promote the killing forms η_{IJ} and $\eta^{I'J'}$ to doubly co(ntra)-variant forms K_{IJ} and $K^{I'J'}$ such that $K_{IJ}' = (\mathcal{P}^{-1})^M{}_I (\mathcal{P}^{-1})^N{}_J K_{MN}$ and $K'^{I'J'} = \mathcal{P}^{I'}{}_{M'} \mathcal{P}^{J'}{}_{N'} K^{M'N'}$, so the new action is

$$S = -\frac{T}{2} \int \mathrm{d}^2 x e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ}.$$

For consistency, let's assume K is a background field $K = K(X)$ like the metric G.

13.2 EoMs

13.2.1 W.r.t e

$$\frac{\delta S}{\delta e_K^c} \propto \left(\frac{\partial e}{\partial e_K^c} e_{I'}^a e_{J'}^b K^{I'J'} + e \frac{\partial}{\partial e_K^c} (e_{I'}^a e_{J'}^b K^{I'J'}) \right) \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} =$$

$$= \left(-e e_c^K e_{I'}^a e_{J'}^b K^{I'J'} + 2 e \delta_c^a \delta_{I'}^K e_{J'}^b K^{I'J'} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} =$$

$$= -e e_c^K e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} + 2 e e_{J'}^b K^{KJ'} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ} = 0$$

$$T_c^K := 2 e_{J'}^b K^{KJ'} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ} - e_c^K e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} = 0$$

$$e_c^K = 2 f e^{bK} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ},$$

$$\frac{1}{f} = e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ}$$

13.2.2 W.r.t A

$$\frac{\delta S}{\delta \mathcal{A}_c^{KL}} = \frac{T}{2} \left(e e_{I'}^a e_{J'}^b K^{I'J'} \frac{\partial}{\partial \mathcal{A}_c^{KL}} \left(\mathcal{D}_a X^I \mathcal{D}_b X^J \right) K_{IJ} \right) =$$

$$= T \left(e e_{I'}^a e_{J'}^b K^{I'J'} \delta_a^c \delta_{[K}^I \eta_{L]I'} X^{I'} \mathcal{D}_b X^J K_{IJ} \right) =$$

$$= T e e_{I'}^c e_{J'}^b K^{I'J'} \mathcal{D}_b X_{[K} X_{L]} \stackrel{!}{=} 0$$

$$\mathcal{T}_a^{IJ} = \mathcal{D}_a X^{[I} X^{J]} = 0$$

13.2.3 W.r.t X

$$\frac{\delta S}{\delta X^K} = -\frac{T}{2} \left(\partial_c \left(e e_{I'}^a e_{J'}^b K^{I'J'} \frac{\partial}{\partial (\partial_c X^K)} \left(\mathcal{D}_a X^I \mathcal{D}_b X^J \right) K_{IJ} \right) - e e_{I'}^a e_{J'}^b K^{I'J'} \frac{\partial}{\partial X^K} \left(\mathcal{D}_a X^I \mathcal{D}_b X^J \right) K_{IJ} - \right.$$

$$\left. - e e_{I'}^a e_{J'}^b \frac{\partial K^{I'J'}}{\partial X^K} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} - e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \frac{\partial K_{IJ}}{\partial X^K} \right) =$$

$$= -T \left(\left(\partial_c \left(e e_{I'}^a e_{J'}^b K^{I'J'} \delta_a^c \delta_K^I \mathcal{D}_b X^J K_{IJ} \right) - e e_{I'}^a e_{J'}^b K^{I'J'} k \mathcal{A}_{aL}^I \delta_L^{I'} \mathcal{D}_b X^J K_{IJ} \right) - \right.$$

$$\left. - e e_{I'}^a e_{J'}^b \partial_K K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} - e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \partial_K K_{IJ} \right) =$$

$$= -T \left(\mathcal{D}_a \left(e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_b X^I K_{IK} \right) - e e_{I'}^a e_{J'}^b \left(\partial_K K^{I'J'} K_{IJ} + K^{I'J'} \partial_K K_{IJ} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J \right) \stackrel{!}{=} 0$$

$$\mathcal{D}_a \left(e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_b X^I K_{IK} \right) = e e_{I'}^a e_{J'}^b \left(\partial_K K^{I'J'} K_{IJ} + K^{I'J'} \partial_K K_{IJ} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J$$

In "constant gauge" $K_{IJ}(X) = \eta_{IJ}$ and in conformal gauge $e e_{I'}^a e_{J'}^b \eta^{I'J'} = \eta^{ab}$, this reduces to

$$\eta^{ab} \mathcal{D}_a \mathcal{D}_b X^I = 0.$$

More explicitly,

$$\eta^{ab} \mathcal{D}_a \left(\partial_b X^I + k \mathcal{A}_{bJ}^I X^J \right) = 0$$

$$\eta^{ab} \left(\left(\partial_a \partial_b X^I + k \mathcal{A}_{aJ}^I \partial_b X^J \right) + \left(k \partial_a \left(\mathcal{A}_{bJ}^I X^J \right) + k^2 \mathcal{A}_{aI'}^I \mathcal{A}_{bJ}^{I'} X^J \right) \right) = 0$$

$$\eta^{ab} \left(\partial_a \partial_b X^I + 2k \mathcal{A}_{aJ}^I \partial_b X^J + k \partial_a \mathcal{A}_{bJ}^I X^J + k^2 \mathcal{A}_{aI'}^I \mathcal{A}_{bJ}^{I'} X^J \right) = 0$$

$$\eta^{ab}\left(\delta_J^I \partial_a \partial_b + 2k\mathcal{A}_{aJ}^I \partial_b\right) X^J = -\left(k\eta^{ab}\partial_a \mathcal{A}_{bJ}^I X^J + k^2 \eta^{ab} \mathcal{A}_{aI'}^I \mathcal{A}_{bJ}^{I'} X^J\right),$$

which has implicit solution given by

$$X^K(x) = -\int \mathrm{d}^2 x' G_I^K(x,x')\left(k\eta^{a'b'}\partial_{a'}\mathcal{A}_{b'J}^I(x')X^J(x') + k^2\eta^{a'b'}\mathcal{A}_{a'I'}^I(x')\mathcal{A}_{b'J}^{I'}(x')X^J(x')\right),$$

where the Green's function matrix $G_J^K(x,x')$ satisfies

$$\eta^{ab}\left(\delta_K^I \partial_a \partial_b + 2k\mathcal{A}_{aK}^I(x)\partial_b\right) G_J^K(x,x') = \delta_J^I \delta(x,x').$$

Boundary conditions come from

$$\delta S = \int \mathrm{d}^2 x \left(\frac{\partial \mathscr{L}}{\partial X^K}\delta X^K + \frac{\partial \mathscr{L}}{\partial(\partial_c X^K)}\delta(\partial_c X^K) + \ldots\right) =$$

$$= \int \mathrm{d}^2 x \left((\mathrm{EoM})\delta X^K + \partial_c\left(\frac{\partial \mathscr{L}}{\partial(\partial_c X^K)}\delta X^K\right) + \ldots\right) =$$

$$= \int \mathrm{d}^2 x (\mathrm{EoM})\delta X^K + \int \mathrm{d}\tau \left(\frac{\partial \mathscr{L}}{\partial(\partial_\sigma X^K)}\delta X^K\right)\Big|_0^{\sigma_1} + \ldots = 0,$$

thus

$$\left(\frac{\partial \mathscr{L}}{\partial(\partial_\sigma X^K)}\delta X^K\right)\Big|_0^{\sigma_1} = 0$$

$$\left(\left(-Tee_i^\sigma e^{bi}\mathcal{D}_b X_K\right)\delta X^K\right)\Big|_0^{\sigma_1} = 0$$

$$\left((\mathcal{D}_\sigma X_K)\delta X^K\right)\Big|_0^{\sigma_1} = 0,$$

so either $\delta X^K(\tau,\sigma_*) = 0$, $\sigma_* = 0, \sigma_1$ (Dirichlet B.C) or $\mathcal{D}_\sigma X^K(\tau,\sigma_*) = 0$, $\sigma_* = 0, \sigma_1$ (free end-point B.C). For closed strings, $X^K(\tau,\sigma) = X^K(\tau,\sigma+2\pi)$.

When connection is trivial EoM reduces to regular wave eqn, which have the regular string solution

$$X_0^I(\tau,\sigma) = X_{0L}^I(\sigma^+) + X_{0R}^I(\sigma^-),$$

with

$$X_{0L}^I(\sigma^+) = \frac{1}{2}x^I + \frac{1}{2}\alpha' p^I \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^I e^{-in\sigma^+}, \ \sigma^+ = \tau + \sigma$$

$$X_{0R}^I(\sigma^-) = \frac{1}{2}x^I + \frac{1}{2}\alpha' p^I \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I e^{-in\sigma^-}, \ \sigma^- = \tau - \sigma.$$

This, however, does not amount to any motion in actual space-time, since with trivial connection the vector does not change after parallel transport. For a more interesting case, let's consider $A_\tau^{\alpha\beta} = A_\sigma^{\alpha\beta} = a\varepsilon^{\alpha\beta}$, which amounts to flat space-time. The EoM turn into

$$\left(\partial_\sigma^2 - \partial_\tau^2\right) X^I + 2ka(\varepsilon^{\alpha\beta}T_{\alpha\beta})_J^I\left(\partial_\sigma - \partial_\tau\right) X^J = 0$$

$$\left(\partial_\sigma^2 - \partial_\tau^2\right)\begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ \vdots \\ X^d \end{bmatrix} + 4ka\left(\partial_\sigma - \partial_\tau\right)\begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & -1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -1 & \ldots & 0 \end{bmatrix}\begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ \vdots \\ X^d \end{bmatrix} = 0$$

$$\begin{bmatrix} \left(\partial_\sigma^2 - \partial_\tau^2\right)X^0 + 4ka\left(\partial_\sigma - \partial_\tau\right)\sum_{i>0}X^i \\ \left(\partial_\sigma^2 - \partial_\tau^2\right)X^1 + 4ka\left(\partial_\sigma - \partial_\tau\right)\sum_{i\neq 1}X^i \\ \left(\partial_\sigma^2 - \partial_\tau^2\right)X^2 + 4ka\left(\partial_\sigma - \partial_\tau\right)\left(X^0 - X^1 + \sum_{i>2}X^i\right) \\ \vdots \\ \left(\partial_\sigma^2 - \partial_\tau^2\right)X^d + 4ka\left(\partial_\sigma - \partial_\tau\right)\left(X^0 - \sum_{i=1}^{d-1}X^i\right) \end{bmatrix} = 0$$

$$\left(\partial_\sigma^2 - \partial_\tau^2\right) X^I + 4ka \left(\partial_\sigma - \partial_\tau\right) \left((\operatorname{sgn}(I))^2 X^0 + \sum_{i>0} \operatorname{sgn}(i-I) X^i\right) = 0$$

$$\left(\partial_\sigma - \partial_\tau\right) \left(\left(\partial_\sigma + \partial_\tau\right) X^I + 4ka \left((\operatorname{sgn}(I))^2 X^0 + \sum_{i>0} \operatorname{sgn}(i-I) X^i\right)\right) = 0$$

$$\left(\partial_\sigma + \partial_\tau\right) X^I + 4ka \left((\operatorname{sgn}(I))^2 X^0 + \sum_{i>0} \operatorname{sgn}(i-I) X^i\right) = C_1^I.(\tau + \sigma),\ C^I \in \mathbb{C}$$

$$\left(\partial_\sigma + \partial_\tau\right) X^I = C_1^I.(\tau + \sigma) - 4ka \left((\operatorname{sgn}(I))^2 X^0 + \sum_{i>0} \operatorname{sgn}(i-I) X^i\right)$$

$$X^I = \frac{1}{2} C_1^I.\left(\tau^2 + \sigma^2\right) + C_2^I.(\tau - \sigma) - 4ka \int \mathrm{d}^2 x' \left((\operatorname{sgn}(I))^2 X^0 + \sum_{i>0} \operatorname{sgn}(i-I) X^i\right).$$

If $D = 2$, we get a symmetric system of eqns

$$X^0 = \frac{1}{2} C_1^0.\left(\tau^2 + \sigma^2\right) + C_2^0.(\tau - \sigma) - 4ka \int \mathrm{d}^2 x' X^1$$

$$X^1 = \frac{1}{2} C_1^1.\left(\tau^2 + \sigma^2\right) + C_2^1.(\tau - \sigma) - 4ka \int \mathrm{d}^2 x' X^0$$

$$X^0 = \frac{1}{2} C_1^0.\left(\tau^2 + \sigma^2\right) + C_2^0.(\tau - \sigma) - 4ka \int \mathrm{d}^2 x' \left(\frac{1}{2} C_1^1.\left(\tau'^2 + \sigma'^2\right) + C_2^1.(\tau' - \sigma') - 4ka \int \mathrm{d}^2 x'' X^0\right)$$

$$X^0 - 16k^2 a^2 \int \mathrm{d}^2 x' \int \mathrm{d}^2 x'' X^0 = \frac{1}{2} C_1^0.\left(\tau^2 + \sigma^2\right) + C_2^0.(\tau - \sigma) - \frac{1}{2} C_1^1 \left(\frac{\tau^3}{3}\sigma + \tau \frac{\sigma^3}{3}\right) - C_2^1 \left(\frac{\tau^2}{2}\sigma - \tau \frac{\sigma^2}{2}\right).$$

This integral equation is tricky to deal with, but we can turn it into a differential equation by applying $\partial_\sigma \partial_\tau$ twice:

$$\left(\partial_\sigma \partial_\tau\right)^2 X^0 - 16k^2 a^2 X^0 = 0.$$

This has general solution given by sum of plane waves and exponentials

$$X^0 = \sum_{n \neq 0} \frac{1}{n^4} \left(a_n^0 e^{i2\sqrt{ka}n(\tau+\sigma)} + b_n^0 e^{i2\sqrt{ka}n(\tau-\sigma)} + c_n^0 e^{2\sqrt{ka}n(\tau+\sigma)} + d_n^0 e^{2\sqrt{ka}n(\tau-\sigma)}\right).$$

In actuallity, only the $n = 1$ and $n = -1$ terms satisfy the differential equation, so we have just

$$X^0 = a_1^0 e^{i2\sqrt{ka}(\tau+\sigma)} + b_1^0 e^{i2\sqrt{ka}(\tau-\sigma)} + c_1^0 e^{2\sqrt{ka}(\tau+\sigma)} + d_1^0 e^{2\sqrt{ka}(\tau-\sigma)} +$$

$$+ a_{-1}^0 e^{-i2\sqrt{ka}(\tau+\sigma)} + b_{-1}^0 e^{-i2\sqrt{ka}(\tau-\sigma)} + c_{-1}^0 e^{-2\sqrt{ka}(\tau+\sigma)} + d_{-1}^0 e^{-2\sqrt{ka}(\tau-\sigma)}$$

14 Polyakov-LQG V2

Same idea as before, but now contract the dyads directly with the $\mathcal{D}_a X^I$ factors

$$S = -\frac{T}{4} \int \mathrm{d}^2 x\, e e_I^a e_J^b \mathcal{D}_a X^I \mathcal{D}_b X^J.$$

14.1 EoMs

14.1.1 W.r.t e

$$\frac{\delta S}{\delta e_K^c} \propto \left(\frac{\partial e}{\partial e_K^c} e_I^a e_J^b + e \frac{\partial}{\partial e_K^c} (e_I^a e_J^b) \right) \mathcal{D}_a X^I \mathcal{D}_b X^J =$$
$$= \left(-e e_c^K e_I^a e_J^b + 2 e \delta_I^K \delta_c^a e_J^b \right) \mathcal{D}_a X^I \mathcal{D}_b X^J =$$
$$= -e e_c^K e_I^a e_J^b \mathcal{D}_a X^I \mathcal{D}_b X^J + 2 e e_J^b \mathcal{D}_c X^K \mathcal{D}_b X^J = 0$$

$$T_c^K := 2 e_J^b \mathcal{D}_c X^K \mathcal{D}_b X^J - e_c^K e_I^a e_J^b \mathcal{D}_a X^I \mathcal{D}_b X^J = 0,$$
$$e_c^K = 2 f \mathcal{D}_c X^K,$$
$$\frac{1}{f} = e_J^b \mathcal{D}_b X^J.$$

14.1.2 W.r.t A

$$\frac{\delta S}{\delta \mathcal{A}_c^{KL}} \propto e e_I^a e_J^b \frac{\partial}{\partial \mathcal{A}_c^{KL}} (\mathcal{D}_a X^I \mathcal{D}_b X^J) =$$
$$= 2 e e_I^a e_J^b \delta_a^c \delta_{[K}^I \eta_{L]I'} X^{I'} \mathcal{D}_b X^J =$$
$$= 2 e e_{[K}^c X_{L]} e_J^b \mathcal{D}_b X^J = 0$$

$$\mathcal{T}_{KL}^c := e_{[K}^c X_{L]} = 0.$$

14.1.3 W.r.t X

$$\frac{\delta S}{\delta X^K} \propto 2 e e_I^a e_J^b k \mathcal{A}_{aI'}^I \delta_K^{I'} \mathcal{D}_b X^J - 2 \partial_c \left(e e_I^a e_J^b \delta_a^c \delta_K^I \mathcal{D}_b X^J \right) =$$
$$= 2 e e_I^a e_J^b k \mathcal{A}_{aK}^I \mathcal{D}_b X^J - 2 \partial_a \left(e e_K^a e_J^b \mathcal{D}_b X^J \right) = 0$$

$$\mathcal{D}_a \left(e e_K^a e_J^b \mathcal{D}_b X^J \right) = 0.$$

If conformal gauge makes it so that $e e_K^a e_J^b = \eta^{ab} \eta_{KJ}$, this reduces to

$$\eta^{ab} \mathcal{D}_a \mathcal{D}_b X^I = 0$$