Sketchboard

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1 Basic concepts

1.1 With extra capital internal index on embbeding fields

Promote embedding fields X^μ to have an internal group index with D values

$$X^\mu \to X^{\mu I} \ , \quad I = 0, ..., d$$

$$g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{IJ} \sim f D_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu}$$

$$D_a X^{\mu I} = \partial_a X^{\mu I} + \omega^I_{aJ} X^{\mu J}$$

$$e^i_a e^j_b \eta_{ij} = g_{ab} \to e^i_a e^j_b \eta_{ij} = \mathrm{Tr}(g^{IJ}_{ab} T_I T_J) = g^{IJ}_{ab} \eta_{IJ} \ , \quad i, j = 0, 1$$

$$g = \det(g_{ab}) \to g = \det(\mathrm{Tr}(g^{IJ}_{ab} T_I T_J)) = \det(g^{IJ}_{ab} \eta_{IJ}) = \det(e^i_a e^j_b \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.2 With 1 extra small internal index on embedding fields

Promote embbeding fields to have a internal group index with 2 values

$$X^\mu \to X^{\mu i} \ , \quad i = 0, 1$$

$$g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{ij} \sim f D_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu}$$

$$D_a X^{\mu i} = \partial_a X^{\mu i} + \omega^i_{aj} X^{\mu j}$$

$$e^i_a e^j_b \eta_{ij} = g_{ab} \to e^i_a e^j_b \eta_{ij} = \mathrm{Tr}(g^{ij}_{ab} T_i T_j) = g^{ij}_{ab} \eta_{ij} \implies e^i_a e^j_b = g^{ij}_{ab}$$

$$g = \det(g_{ab}) \to g = \det(\mathrm{Tr}(g^{ij}_{ab} T_i T_j)) = \det(g^{ij}_{ab} \eta_{ij}) = \det(e^i_a e^j_b \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.3 With 2 extra small indices

Promote embbeding fields X^μ to have two internal indices with 2 values

$$X^\mu \to X^{\mu ij} \ , \quad i, j = 0, 1$$

1.4 Without extra index on embbeding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$e^i_a e^j_b \eta_{ij} = g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$g = \det(g_{ab}) = \det(e^i_a e^j_b \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int \mathrm{d}\tau \wedge \mathrm{d}\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \ ,$$

2.1 With capital internal index

... and promote partial derivative ∂_a to covariant derivative D_a, giving us our first attempt at modified Polyakov action

$$S_{MP1} = -\frac{T_0}{2} \int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) \eta^{ij} e^a_i e^b_j D_a X^{\mu I} D_b X^{\nu J} E^K_\mu E_{\nu K} \eta_{IJ}$$

$$\updownarrow$$

$$\mathscr{L}_{MP1} = -\frac{T_0}{2} \det(e) \eta^{ij} e^a_i e^b_j D_a X^{\mu I} D_b X^{\nu J} E^K_\mu E_{\nu K} \eta_{IJ}$$

2.2 With small internal index

2.2.1 Without extra field, 1 internal index

... and promote derivatives to covariant D_a, giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

$$\updownarrow$$

$$\mathscr{L}_{MP2} = -\frac{T_0}{2}\det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

2.2.2 With extra field

..., promote partial derivative to covariant and add extra internal WS field v^i leading us to

$$S_{MP3} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

$$\updownarrow$$

$$\mathscr{L}_{MP3} = -\frac{T}{2}\det(e) \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

2.2.3 Without extra field, 2 internal indices

... and promote derivatives to covariant D_a,

$$S_{MP4} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma e e_i^a e^{bi} D_a X^{\mu jj'} \eta_{jj'} \eta_{kk'} D_b X^{\nu kk'} E_\mu^I E_{\nu I}$$

$$\mathscr{L}_{MP4} = -\frac{T}{2} e e_i^a e_j^b D_a X^{\mu ik} \eta_{kl} D_b X^{\nu lj} E_\mu^I E_{\nu I}$$

2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X)\ .$$

$$\updownarrow$$

$$\mathscr{L}_{DP} = -\frac{T_0}{2}\det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X)\ .$$

2.4 Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of D-dimensional gamma matrices and bulk spinors

$$S_{LP} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

$$\updownarrow$$

$$\mathscr{L}_{LP} = -\frac{T}{2}\overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

3 EoMs

Start by writing $\det(e) = \frac{1}{2}\varepsilon^{cd}\varepsilon_{mn} e_c^m e_d^n$ and $g_{ab} = g_{ab}^{ij}\eta_{ij}$

3.1 With small internal index

3.1.1 Without extra field, w.r.t e

$$
\begin{aligned}
\frac{\delta S_{MP2}}{\delta e^e_l} &= \frac{D\mathscr{L}_{MP2}}{De^e_l} = \left(\frac{\partial}{\partial e^e_l} - \partial_f \frac{\partial}{\partial(\partial_f e^e_l)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e^m_c e^n_d e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I}\right) = \\
&= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(\eta^{ml}g_{ce}e^n_d e^a_i e^b_j + e^m_c \eta^{nl} g_{de} e^a_i e^b_j + \\
&+ e^m_c e^n_d \delta^a_e \delta^l_i e^b_j + e^m_c e^n_d e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = \\
&= -\frac{T}{2}\left(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}g_{ce}e^n_d e^a_i e^b_j + 2\det(e) e^a_i \delta^b_e \delta^l_j\right) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = \\
&= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}e^k_c e_{ek} e^n_d e^a_i e^b_j + 2\det(e) e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = \\
&= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}e^m_c e^n_d e^l_e e^a_i e^b_j + 2\det(e) e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = \\
&= -T(-\det(e) e^l_e e^a_i e^b_j + \det(e) e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} \stackrel{!}{=} 0
\end{aligned}
$$

$$\downarrow$$

$$T^l_e := E^I_\mu E_{\nu I}(e^a_i D_a X^{\mu i} D_e X^{\nu l} - e^l_e e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j}) = 0$$

$$\Downarrow$$

$$e^l_e = f e^a_i D_a X^{\mu i} D_e X^{\nu l} E^I_\mu E_{\nu I} \ ,$$

$$\frac{1}{f} = e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I}$$

$$\downarrow$$

$$
\begin{aligned}
e^l_e e_{fl} &= (f e^a_i D_a X^{\mu i} D_e X^{\nu l} E^I_\mu E_{\nu I})(f e^{a'}_{i'} D_{a'} X^{\mu' i'} D_f X^{\nu'}_l E^{I'}_{\mu'} E_{\nu' I'}) = \\
&= f^2 D_e X^{\nu l} D_f X^{\nu'}_l E^{I'}_\nu E_{\nu' I'}(e^a_i e^{a'}_{i'} D_a X^{\mu i} D_{a'} X^{\mu' i'} E^I_\mu E_{\mu' I}) = \\
&= f D_e X^{\mu i} D_f X^\nu_i E^I_\mu E_{\nu I}
\end{aligned}
$$

$$\Downarrow$$

$$D_e X^{\mu i} D_f X^\nu_i E^I_\mu E_{\nu I} = 2\partial_e X^\mu \partial_f X^\nu G_{\mu\nu}$$

$$e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\Downarrow$$

$$D_a X^{\mu 0} = -i\partial_a X^\mu$$

$$D_a X^{\mu 1} = \partial_a X^\mu$$

$$\Downarrow$$

$$X^{\mu 0} = -i X^{\mu 1}$$

3.1.2 With extra field, w.r.t e

$$\frac{\delta S_{MP3}}{\delta e_o^e} = \frac{D\mathscr{L}_{MP3}}{De_o^e} = \left(\frac{\partial}{\partial e_o^e} - \partial_f \frac{\partial}{\partial(\partial_f e_o^e)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e_c^m e_d^n \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ}\right) =$$

$$= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(g_{ce}\eta^{mo}e_d^n \eta^{ij}e_i^a e_j^b + e_c^m g_{de}\eta^{no}\eta^{ij}e_i^a e_j^b +$$

$$+ e_c^m e_d^n \eta^{ij}\delta_e^a \delta_i^o e_j^b + e_c^m e_d^n \eta^{ij} e_i^a \delta_e^b \delta_j^o) D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} =$$

$$= -\frac{T}{2}\varepsilon^{cd}\varepsilon_{mn}(e_c^p e_{ep} e_d^n \eta^{mo}\eta^{ij}e_i^a e_j^b + 2\det(e)e^{ao}\delta_e^b)D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} =$$

$$= -T(-\det(e)e_e^o \eta^{ij}e_i^a e_j^b + \det(e)e^{ao}\delta_e^b)D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} \stackrel{!}{=} 0$$

$$\downarrow$$

$$\boxed{T_e^o := (e^{ao}D_a(X_k^\mu v^k)D_e(X_l^\nu v^l) - e_e^o \eta^{ij}e_i^a e_j^b D_a(X_k^\mu v^k)D_b(X_l^\nu v^l))E_\mu^I E_\nu^J \eta_{IJ} = 0}$$

$$\Downarrow$$

$$e_e^o = f\, e^{ao}D_a(X_k^\mu v^k)D_e(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ}\ ,$$

$$\frac{1}{f} = \eta^{ij}e_i^a e_j^b D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ}$$

3.1.3 With extra field, w.r.t ω

$$\frac{\delta S_{MP3}}{\delta \omega_c^{mn}} = \frac{D\mathscr{L}_{MP3}}{D\omega_c^{mn}} = \left(\frac{\partial}{\partial \omega_c^{mn}} - \partial_d \frac{\partial}{\partial(\partial_d \omega_c^{mn})}\right)\left(-\frac{T}{2}\det(e)\eta^{ij}e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}\right) =$$

$$= -\frac{T}{4}eg^{ab}((-\delta_a^c \delta_{[m}^{k'}\eta_{n]k}X_{k'}^\mu v^k)D_b X_l^\nu v^l + D_a X_k^\mu v^k(-\delta_b^c \delta_{[m}^{l'}\eta_{n]l}X_{l'}^\nu v^l))G_{\mu\nu} =$$

$$= \frac{T}{4}e(g^{ca}X_{[m}^\mu v_{n]}D_a X_k^\nu v^k + g^{ac}D_a X_k^\mu v^k X_{[m}^\nu v_{n]})G_{\mu\nu} =$$

$$= \frac{T}{2}eg^{ac}D_a X_k^\mu v^k X_{[m}^\nu v_{n]}G_{\mu\nu} \stackrel{!}{=} 0$$

$$\downarrow$$

$$\boxed{\mathcal{T}_{ab}^i := e^{ci}D_c X_k^\mu v^k X_{[m}^\nu v_{n]}e_a^m e_b^n G_{\mu\nu} = 0}$$

3.1.4 With extra field, w.r.t v

$$\frac{\delta S_{MP3}}{\delta v^l} = \frac{D\mathscr{L}_{MP3}}{Dv^l} = \left(\frac{\partial}{\partial v^l} - \partial_c \frac{\partial}{\partial(\partial_c v^l)}\right)\left(-\frac{T}{2}ee_i^a e^{bi}D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I}\right) =$$

$$= -\frac{T}{2}g^{ab}(D_a X_j^\mu \delta_l^j D_b X_k^\nu v^k + D_a X_j^\mu v^j D_b X_k^\nu \delta_l^k)G_{\mu\nu} =$$

$$= -Tg^{ab}D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} \stackrel{!}{=} 0$$

$$\downarrow$$

$$\boxed{g^{ab}D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} = 0}$$

3.1.5 With extra field, w.r.t X

$$\frac{\delta S_{MP3}}{\delta X^\lambda_l} = \frac{D\mathscr{L}_{MP3}}{DX^\lambda_l} = \left(\frac{\partial}{\partial X^\lambda_l} - \partial_c \frac{\partial}{\partial(\partial_c X^\lambda_l)}\right)\left(-\frac{T}{2} e e^a_i e^{bi} D_a X^\mu_j v^j D_b X^\nu_k v^k E^I_\mu E_{\nu I}\right) =$$

$$= -\frac{T}{2}\Bigg(eg^{ab}\Bigg((-\omega^{j'}_{aj}\delta^\mu_\lambda \delta^l_{j'} v^j D_b X^\nu_k v^k - D_a X^\mu_j v^j \omega^{k'}_{bk}\delta^\nu_\lambda \delta^l_{k'} v^k) E^I_\mu E_{\nu I} +$$

$$+ eg^{ab} D_a X^\mu_j v^j D_b X^\nu_k v^k \left(\frac{\partial E^I_\mu}{\partial X^\lambda_l} E_{\nu I} + E^I_\mu \frac{\partial E_{\nu I}}{\partial X^\lambda_l}\right)\Bigg) -$$

$$- \partial_c (eg^{ab}(\delta^c_a \delta^\mu_\lambda \delta^l_j v^j D_b X^\nu_k v^k + D_a X^\mu_j v^j \delta^c_b \delta^\nu_\lambda \delta^l_k v^k) G_{\mu\nu})\Bigg) =$$

$$= -T\left(- eg^{ab} D_a X^\mu_j v^j \omega^l_{bk} v^k G_{\mu\lambda} + eg^{ab} D_a X^\mu_j v^j D_b X^\nu_k v^k \frac{\partial E^I_\mu}{\partial X^\lambda_l} E_{\nu I} - \partial_c (eg^{ac} v^l D_a X^\mu_j v^j G_{\mu\lambda})\right)$$

$$= -T\left(eg^{ab} D_a X^\mu_j v^j D_b X^\nu_k v^k \frac{\partial E^I_\mu}{\partial X^\lambda_l} E_{\nu I} - D_b (eg^{ab} v^l D_a X^\mu_j v^j E^I_\mu E_{\lambda I})\right) \stackrel{!}{=} 0$$

$$\downarrow$$

$$\boxed{D_b(ee^a_i e^{bi} v^l D_a X^\mu_j v^j E^I_\mu E_{\lambda I}) = ee^a_i e^{bi} D_a X^\mu_j v^j D_b X^\nu_k v^k \frac{\partial E^I_\mu}{\partial X^\lambda_l} E_{\nu I}}$$

3.2 Without extra index

3.2.1 w.r.t e

$$\frac{\delta S_{DP}}{\delta e^e_l} = \frac{D\mathscr{L}_{DP}}{De^e_l} = \left(\frac{\partial}{\partial e^e_l} - \partial_f \frac{\partial}{\partial(\partial_f e^e_l)}\right)\left(-\frac{T_0}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn} e^m_c e^n_d \eta^{ij} e^a_i e^b_j \partial_a X^\mu \partial_b X^\nu E^I_\mu E^J_\nu \eta_{IJ}\right) =$$

$$= -\frac{T_0}{4}$$

3.3 Linear action

3.3.1 w.r.t e

$$\frac{\delta S_{LP}}{\delta\delta e^b_j} = \frac{D\mathscr{L}_{LP}}{De^b_j} = \left(\frac{\partial}{\partial e^b_j} - \partial_e \frac{\partial}{\partial(\partial_e e^b_j)}\right)\left(-\frac{T}{2}\overline{\psi} e e^a_i D_a X^{\mu i} E^I_\mu \gamma_I \psi\right) =$$

$$= -\frac{T}{4}\overline{\psi}\varepsilon^{cd}\varepsilon_{mn}((g_{cb}\eta^{mj} e^n_d + e^m_c g_{db}\eta^{nj}) e^a_i + e^m_c e^n_d \delta^a_b \delta^j_i) D_a X^{\mu i} E^I_\mu \gamma_I \psi =$$

$$= -\frac{T}{2}\overline{\psi}(-ee^j_b e^a_i D_a X^{\mu i} E^I_\mu \gamma_I + eD_b X^{\mu j} E^I_\mu \gamma_I)\psi \stackrel{!}{=} 0$$

$$\downarrow$$

$$T^j_b := \overline{\psi}(D_b X^{\mu j} E^I_\mu \gamma_I - e^j_b e^a_i D_a X^{\mu i} E^I_\mu \gamma_I)\psi = 0$$

$$\downarrow$$

$$e^j_b = F\overline{\psi} D_b X^{\mu j} E^I_\mu \gamma_I \psi,$$

$$\frac{1}{F} = \overline{\psi} e^a_i D_a X^{\mu i} E^I_\mu \gamma_I \psi$$

$$\Downarrow$$

$$\overline{\psi}\psi = 1,\ \psi\overline{\psi} = \mathbb{1}$$

$$e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E^J_\nu \eta_{IJ} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

4 Inverse Area Action

4.1 Nambu-Goto

Start from Nambu-Goto action

$$S_{NG} = -T \int \mathrm{d}^2 x \sqrt{-h},$$

and make quantum geometry correction

$$\sqrt{-h} \to \sqrt{-(h+g\Delta)} \approx \sqrt{-h}\left(1 - \frac{g\Delta}{2(-h)} + \mathcal{O}\left(\frac{g^2}{h^2}\right)\right),$$

leading to modified NG action

$$S_{MNG} = -T \int \mathrm{d}^2 x \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}}\right) = S_{NG} + S_{IA}$$

$$\downarrow$$

$$\mathscr{L}_{MNG} = -T\left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}}\right)$$

4.1.1 EoMs

Let $\mathcal{P}^\tau_\mu \equiv \partial \mathscr{L}_{MNG}/\partial \dot{X}^\mu$ and $\mathcal{P}^\sigma_\mu \equiv \partial \mathscr{L}_{MNG}/\partial X'^\mu$

$$\begin{aligned}\delta S_{MNG} &= \int \mathrm{d}^2 x \left(\frac{\partial \mathscr{L}_{MNG}}{\partial \dot{X}^\mu}\delta \dot{X}^\mu + \frac{\partial \mathscr{L}_{MNG}}{\partial X'^\mu}\delta X'^\mu\right) = \\ &= -\int \mathrm{d}^2 x \left(\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu\right)\delta X^\mu + \int \mathrm{d}\tau \mathcal{P}^\sigma_\mu \delta X^\mu\Big|_{\sigma=0}^{\sigma=\sigma_1} \stackrel{!}{=} 0\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}\text{EoM} &: \partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0 \\ \text{B.C.} &: \mathcal{P}^\sigma_\mu \delta X^\mu\Big|_{\sigma=0}^{\sigma=\sigma_1} = 0,\end{aligned}$$

where

$$\begin{aligned}\mathcal{P}^\tau_\mu = \frac{\partial \mathscr{L}_{MNG}}{\partial \dot{X}^\mu} &= -T\left(\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{-h}} + \frac{g\Delta}{2}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{(-h)^{3/2}}\right) = \\ &= -T\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{-h}}\left(1 + \frac{g\Delta}{2(-h)}\right) = \mathcal{P}^\tau_{\mu(NG)}\left(1 + \frac{g\Delta}{2(-h)}\right) \\ \mathcal{P}^\sigma_\mu = \frac{\partial \mathscr{L}_{MNG}}{\partial X'^\mu} &= -T\left(\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} + \frac{g\Delta}{2}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{(-h)^{3/2}}\right) = \\ &= -T\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}}\left(1 + \frac{g\Delta}{2(-h)}\right) = \mathcal{P}^\sigma_{\mu(NG)}\left(1 + \frac{g\Delta}{2(-h)}\right)\end{aligned}$$

gauge fixing static gauge $\tau = t$ and transverse gauge $\frac{\partial X}{\partial \tau}\cdot\frac{\partial X}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}\sigma} = 0$ ($s =$ length along string)

$$\downarrow$$

$$\begin{aligned}\mathcal{P}^{\tau\mu}_{(NG)} &= T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\frac{\partial X^\mu}{\partial t} \\ \mathcal{P}^{\sigma\mu}_{(NG)} &= \frac{T}{\gamma_{v_\perp}}\frac{\partial X^\mu}{\partial s} \\ -h &= \frac{1}{\gamma^2_{v_\perp}}\left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2\end{aligned}$$

string energy get's redefined to

$$\frac{\partial}{\partial t}\left(T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\right)=0,$$

and for the spatial part we have

$$\partial_\tau\vec{\mathcal{P}}^\tau+\partial_\sigma\vec{\mathcal{P}}^\sigma=0$$

$$\downarrow$$

$$\frac{\partial}{\partial t}\left[T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\frac{\partial\vec{X}}{\partial t}\left(1+\frac{g\Delta}{2(-h)}\right)\right]+\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{\partial}{\partial s}\left[-\frac{T}{\gamma_{v_\perp}}\frac{\partial\vec{X}}{\partial s}\left(1+\frac{g\Delta}{2(-h)}\right)\right]=0$$

$$\downarrow$$

$$\mu\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial}{\partial s}\left[\frac{T}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right)\frac{\partial\vec{X}}{\partial s}\right]=0$$

$\implies$ effective mass density becomes $\mu_{eff}=\mu\gamma_{v_\perp}\left(1-\frac{g\Delta}{2(-h)}\right)$ and effective tension becomes $T_{eff}=\frac{T}{\gamma_{v_\perp}}\left(1-\frac{g\Delta}{2(-h)}\right)$

$$\downarrow$$

$$\mu_{eff}\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial}{\partial s}\left[T_{eff}\frac{\partial\vec{X}}{\partial s}\right]=0$$

$$\begin{aligned}\mathscr{H}&=\dot{\vec{X}}\cdot\vec{\pi}-\mathscr{L}=\\&=\vec{v}_\perp\cdot\left(T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\vec{v}_\perp\right)-\left(-T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{1}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right)\right)=\\&=T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\left(1+\frac{g\Delta}{2(-h)}\right)\left(\gamma_{v_\perp}v_\perp^2+\frac{1}{\gamma_{v_\perp}}\right)=\\&=T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\end{aligned}$$

let $\left(1+\frac{g\Delta}{2(-h)}\right)=F$

$$\frac{\partial^2\vec{X}}{\partial t^2}-\frac{1}{F\gamma_{v_\perp}}\frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\partial}{\partial\sigma}\left[\frac{1}{\gamma_{v_\perp}}F\frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\partial\vec{X}}{\partial\sigma}\right]=0$$

$$\downarrow$$

$$A(\sigma)=\frac{\gamma_{v_\perp}}{F}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\overset{!}{=}1$$

$$\downarrow$$

$$\mathrm{d}\sigma=\frac{\gamma_{v_\perp}}{F}\mathrm{d}s=\frac{1}{TF^2}\mathrm{d}E\implies\sigma(q)=\frac{1}{T}\int_0^q\frac{1}{F^2}\mathrm{d}E$$

$$\downarrow$$

$$F^2\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial^2\vec{X}}{\partial\sigma^2}=0$$

$\implies$ speed of wave on the string gets modified by correction factor $v=c/F$

$$-h=\frac{1}{\gamma_{v_\perp}^2}\left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2=\frac{F^2}{\gamma_{v_\perp}^4}=F^2\left(1-v_\perp^2\right)^2$$

$$\downarrow$$

$$-h=\left(1+\frac{g\Delta}{2(-h)}\right)^2\left(1-v_\perp^2\right)^2$$

$$\downarrow$$

$$\frac{-h}{\left(1+\frac{g\Delta}{2(-h)}\right)^2} = \left(1-v_\perp^2\right)^2$$

$$\downarrow \text{(WolframAlpha)}$$

$$-h = \frac{(1-v_\perp^2)^2}{3} - f_1 - f_2,$$

where (let $a = \frac{g\Delta}{2}$ and $b = (1-v_\perp^2)^2$)

$$f_1 = \frac{\sqrt[3]{-27a^2b + 3\sqrt{3}\sqrt{27a^4b^2 + 4a^3b^3} - 18ab^2 - 2b^3}}{3\sqrt[3]{2}}$$

$$f_2 = \frac{(6ab + b^2)}{6f_1}$$

$$\downarrow$$

$$F = 1 + \frac{g\Delta}{2(-h)} = 1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}$$

5 Bimetric Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

and promote to bimetric action

$$\downarrow$$

$$S_{BP} = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{T'}{2}\int \mathrm{d}^2x\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

5.1 EoMs

$$\begin{aligned}\frac{\delta S_{BP}}{\delta g^{cd}} &= -\frac{T}{2}\left(\frac{\partial\sqrt{-g}}{\partial g^{cd}}g^{ab} + \sqrt{-g}\frac{\partial g^{ab}}{\partial g^{cd}}\right)\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + 0 = \\ &= -\frac{T}{2}\left(-\frac{1}{2}\sqrt{-g}g_{cd}g^{ab} + \sqrt{-g}\delta^a_{(c}\delta^b_{d)}\right)\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \stackrel{!}{=} 0\end{aligned}$$

$$T^{(G)}_{cd} := \left(\partial_c X^\mu \partial_d X^\nu - \frac{1}{2}g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu\right)G_{\mu\nu} = 0$$

$$\Downarrow$$

$$g_{cd} = 2f^{(G)}\partial_c X^\mu \partial_d X^\nu G_{\mu\nu},$$

$$\frac{1}{f^{(G)}} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\begin{aligned}\frac{\delta S_{BP}}{\delta h^{cd}} &= ... = \\ &= -\frac{T'}{2}\left(-\frac{1}{2}\sqrt{-h}h_{cd}h^{ab} + \sqrt{-h}\delta^a_{(c}\delta^b_{d)}\right)\partial_a X^\mu \partial_b X^\nu H_{\mu\nu} \stackrel{!}{=} 0\end{aligned}$$

$$\downarrow$$

$$T^{(H)}_{cd} := \left(\partial_c X^\mu \partial_d X^\mu - \frac{1}{2}h_{cd}h^{ab}\partial_a X^\mu \partial_b X^\nu\right)H_{\mu\nu} = 0$$

$$\Downarrow$$

$$h_{cd} = 2f^{(H)}\partial_c X^\mu \partial_d X^\nu H_{\mu\nu},$$

$$\frac{1}{f^{(H)}} = h^{ab}\partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

$$\begin{aligned}
\frac{\delta S_{BP}}{\delta X^\lambda} &= \left(-\frac{T}{2}\sqrt{-g}g^{ab}\partial_\lambda G_{\mu\nu} - \frac{T'}{2}\sqrt{-h}h^{ab}\partial_\lambda H_{\mu\nu}\right)\partial_a X^\mu \partial_b X^\nu - \\
&- \partial_c\left(-\frac{T}{2}\sqrt{-g}g^{ab}G_{\mu\nu} - \frac{T'}{2}\sqrt{-h}h^{ab}H_{\mu\nu}\right)(\delta^c_a\delta^\mu_\lambda\partial_b X^\nu + \partial_a X^\mu \delta^c_b\delta^\nu_\lambda) = \\
&= \partial_a\left(T\sqrt{-g}g^{ab}G_{\mu\lambda} + T'\sqrt{-h}h^{ab}H_{\mu\lambda}\right)\partial_b X^\mu - \\
&- \left(\frac{T}{2}\sqrt{-g}g^{ab}\partial_\lambda G_{\mu\nu} + \frac{T'}{2}\sqrt{-h}h^{ab}\partial_\lambda H_{\mu\nu}\right)\partial_a X^\mu \partial_b X^\nu \overset{!}{=} 0
\end{aligned}$$

$$\downarrow$$

$$\partial_a\left(F^{ab}_{\mu\lambda}\partial_b X^\mu\right) = \frac{1}{2}\partial_\lambda F^{ab}_{\mu\nu}\partial_a X^\mu \partial_b X^\nu,$$

$$F^{ab}_{\mu\nu} = T\sqrt{-g}g^{ab}G_{\mu\nu} + T'\sqrt{-h}h^{ab}H_{\mu\nu}$$

let $T' = Tk\Delta/2$ and $h = g^{-1}$

$$F^{ab}_{\mu\nu} = T\left(\sqrt{-g}g^{ab}G_{\mu\nu} + \frac{k\Delta}{2}\sqrt{-g^{-1}}U^a{}_{a'}(g^{-1})^{a'b'}U_{b'}{}^b(G^{-1})_{\mu\nu}\right),$$

add new field coupled to derivative A_a transforming as

$$A_a g^{bc} \to \left(A_a g^{bc}\right)' = A_a U^b{}_{b'}g^{b'c'}U_{c'}{}^c - \frac{1}{\alpha}g^{b'c'}(\partial_a U^b{}_{b'}U_{c'}{}^c + U^b{}_{b'}\partial_a U_{c'}{}^c),$$

defining new WS covariant derivative

$$D_a g^{bc} = \partial_a g^{bc} + \alpha A_a g^{bc},$$

such that derivative ignores SO$(1,1)$ gauge freedom on inverse metric. Upgrade EoM to incorporate this derivative:

$$D_a(F^{ab}_{\mu\lambda}\partial_b X^\mu) = \frac{1}{2}\partial_\lambda F^{ab}_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$$

$$\downarrow$$

$$F^{ab}_{\mu\lambda}\partial_a\partial_b X^\mu + D_a F^{ab}_{\mu\lambda}\partial_b X^\mu = \frac{1}{2}\partial_\lambda F^{ab}_{\mu\nu}\partial_a X^\mu \partial_b X^\nu,$$

imposing conformal symmetry $g^{ab} = (\phi(x))^{-1}\eta^{ab} \implies D_a F^{ab}_{\mu\lambda} = 0$

$$F^{ab}_{\mu\lambda}\partial_a\partial_b X^\mu = \frac{1}{2}\partial_\lambda F^{ab}_{\mu\nu}\partial_a X^\mu \partial_b X^\nu,$$

$$F^{ab}_{\mu\lambda} = T\left(G_{\mu\lambda} + \frac{k\Delta}{2}(G^{-1})_{\mu\lambda}\right)\eta^{ab}$$

$$(G^{-1})_{\mu\nu} \overset{?}{=} \frac{1}{-g}G_{\mu\nu}$$

$$\partial_a\left(\sqrt{-g^{-1}}(g^{-1})^{bc}\right) \to \partial_a\left(\sqrt{-g^{-1}}U^b{}_{b'}(g^{-1})^{b'c'}U_{c'}{}^c\right) =$$
$$= U^b{}_{b'}\partial_a(\sqrt{-g^{-1}}(g^{-1})^{b'c'})U_{c'}{}^c + \sqrt{-g^{-1}}(g^{-1})^{b'c'}\left(\partial_a U^b{}_{b'}U_{c'}{}^c + U^b{}_{b'}\partial_a U_{c'}{}^c\right)$$
$$\Downarrow$$
$$A'_a U^b{}_{b'}\sqrt{-g^{-1}}(g^{-1})^{b'c'}U_{c'}{}^c = A_a U^b{}_{b'}\sqrt{-g^{-1}}(g^{-1})^{b'c'}U_{c'}{}^c - \sqrt{-g^{-1}}(g^{-1})^{b'c'}(\partial_a U^b{}_{b'}U_{c'}{}^c + U^b{}_{b'}\partial_a U_{c'}{}^c)$$