

Sketchboard

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1 Basic concepts

1.1 With extra capital internal index on embedding fields

Promote embedding fields X^μ to have an internal group index with D values

$$X^\mu \rightarrow X^{\mu I}, \quad I = 0, \dots, d$$

$$\begin{aligned} g_{ab} &= 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{IJ} \sim f D_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu} \\ D_a X^{\mu I} &= \partial_a X^{\mu I} + \omega_{aJ}^I X^{\mu J} \\ e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ}, \quad i, j = 0, 1 \\ g &= \det(g_{ab}) \rightarrow g = \det(\text{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e) \end{aligned}$$

1.2 With 1 extra small internal index on embedding fields

Promote embedding fields to have a internal group index with 2 values

$$\begin{aligned} X^\mu &\rightarrow X^{\mu i}, \quad i = 0, 1 \\ g_{ab} &= 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{ij} \sim f D_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu} \\ D_a X^{\mu i} &= \partial_a X^{\mu i} + \omega_{aj}^i X^{\mu j} \\ e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij} \\ g &= \det(g_{ab}) \rightarrow g = \det(\text{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e) \end{aligned}$$

1.3 With 2 extra small indices

Promote embedding fields X^μ to have two internal indices with 2 values

$$X^\mu \rightarrow X^{\mu ij}, \quad i, j = 0, 1$$

1.4 Without extra index on embedding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$\begin{aligned} e_a^i e_b^j \eta_{ij} &= g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\ g &= \det(g_{ab}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e) \end{aligned}$$

2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int d\tau \wedge d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

2.1 With capital internal index

... and promote partial derivative ∂_a to covariant derivative D_a , giving us our first attempt at modified Polyakov action

$$\begin{aligned} S_{MP1} &= -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ} \\ &\quad \uparrow \\ \mathcal{L}_{MP1} &= -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ} \end{aligned}$$

2.2 With small internal index

2.2.1 Without extra field, 1 internal index

... and promote derivatives to covariant D_a , giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

↑

$$\mathcal{L}_{MP2} = -\frac{T_0}{2} \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

2.2.2 With extra field

..., promote partial derivative to covariant and add extra internal WS field v^i leading us to

$$S_{MP3} = -\frac{T}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

↑

$$\mathcal{L}_{MP3} = -\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

2.2.3 Without extra field, 2 internal indices

... and promote derivatives to covariant D_a ,

$$S_{MP4} = -\frac{T}{2} \int d\tau \wedge d\sigma e e_i^a e_j^b D_a X^{\mu jj'} \eta_{jj'} \eta_{kk'} D_b X^{\nu kk'} E_\mu^I E_\nu^I$$

$$\mathcal{L}_{MP4} = -\frac{T}{2} e e_i^a e_j^b D_a X^{\mu ik} \eta_{kl} D_b X^{\nu lj} E_\mu^I E_\nu^I$$

2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X).$$

↑

$$\mathcal{L}_{DP} = -\frac{T_0}{2} \det(e) e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X).$$

2.4 Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of D -dimensional gamma matrices and bulk spinors

$$S_{LP} = -\frac{T}{2} \int d\tau \wedge d\sigma \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

↑

$$\mathcal{L}_{LP} = -\frac{T}{2} \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

3 EoMs

Start by writing $\det(e) = \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n$ and $g_{ab} = g_{ab}^{ij} \eta_{ij}$

3.1 With small internal index

3.1.1 Without extra field, w.r.t e

$$\begin{aligned}
\frac{\delta S_{MP2}}{\delta e_l^e} &= \frac{D\mathcal{L}_{MP2}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial (\partial_f e_l^e)} \right) \left(-\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \right) = \\
&= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (\eta^{ml} g_{ce} e_d^n e_i^a e_j^b + e_c^m \eta^{nl} g_{de} e_i^a e_j^b + \\
&\quad + e_c^m e_d^n \delta_e^a \delta_i^l e_j^b + e_c^m e_d^n e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} g_{ce} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} e_c^k e_{ek} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_e^l e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -T (-\det(e) e_e^l e_i^a e_j^b + \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \stackrel{!}{=} 0 \\
&\quad \downarrow \\
T_e^l &:= E_\mu^I E_{\nu I} (e_i^a D_a X^{\mu i} D_e X^{\nu l} - e_e^l e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j}) = 0 \\
&\quad \Downarrow \\
e_e^l &= f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I}, \\
\frac{1}{f} &= e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \\
&\quad \downarrow \\
e_e^l e_{fl} &= (f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I}) (f e_{i'}^{a'} D_{a'} X^{\mu' i'} D_f X_l^{\nu'} E_{\mu'}^I E_{\nu' I'}) = \\
&= f^2 D_e X^{\nu l} D_f X_l^{\nu'} E_{\nu'}^I E_{\nu' I'} (e_i^a e_{i'}^{a'} D_a X^{\mu i} D_{a'} X^{\mu' i'} E_\mu^I E_{\mu' I}) = \\
&= f D_e X^{\mu i} D_f X_i^\nu E_\mu^I E_{\nu I} \\
&\quad \Downarrow \\
D_e X^{\mu i} D_f X_i^\nu E_\mu^I E_{\nu I} &= 2 \partial_e X^\mu \partial_f X^\nu G_{\mu\nu} \\
e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\
&\quad \Downarrow \\
D_a X^{\mu 0} &= -i \partial_a X^\mu \\
D_a X^{\mu 1} &= \partial_a X^\mu \\
&\quad \Downarrow \\
X^{\mu 0} &= -i X^{\mu 1}
\end{aligned}$$

3.1.2 With extra field, w.r.t e

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta e_o^e} &= \frac{D\mathcal{L}_{MP3}}{De_o^e} = \left(\frac{\partial}{\partial e_o^e} - \partial_f \frac{\partial}{\partial (\partial_f e_o^e)} \right) \left(-\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (g_{ce} \eta^{mo} e_d^n \eta^{ij} e_i^a e_j^b + e_c^m g_{de} \eta^{no} \eta^{ij} e_i^a e_j^b + \\
&\quad + e_c^m e_d^n \eta^{ij} \delta_e^a \delta_e^b + e_c^m e_d^n \eta^{ij} e_i^a \delta_e^b \delta_e^o) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -\frac{T}{2} \varepsilon^{cd} \varepsilon_{mn} (e_c^p e_{ep} e_d^n \eta^{mo} \eta^{ij} e_i^a e_j^b + 2 \det(e) e^{ao} \delta_e^b) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -T(-\det(e) e_e^o \eta^{ij} e_i^a e_j^b + \det(e) e^{ao} \delta_e^b) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \stackrel{!}{=} 0
\end{aligned}$$

↓

$T_e^o := (e^{ao} D_a(X_k^\mu v^k) D_e(X_l^\nu v^l) - e_e^o \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l)) E_\mu^I E_\nu^J \eta_{IJ} = 0$

⇓

$$\begin{aligned}
e_e^o &= f e^{ao} D_a(X_k^\mu v^k) D_e(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}, \\
\frac{1}{f} &= \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}
\end{aligned}$$

3.1.3 With extra field, w.r.t ω

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta \omega_c^{mn}} &= \frac{D\mathcal{L}_{MP3}}{D\omega_c^{mn}} = \left(\frac{\partial}{\partial \omega_c^{mn}} - \partial_d \frac{\partial}{\partial (\partial_d \omega_c^{mn})} \right) \left(-\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T}{4} e g^{ab} ((-\delta_a^c \delta_{[m}^{k'} \eta_{n]k} X_{k'}^\mu v^k) D_b X_l^\nu v^l + D_a X_k^\mu v^k (-\delta_b^c \delta_{[m}^{l'} \eta_{n]l} X_{l'}^\nu v^l)) G_{\mu\nu} = \\
&= \frac{T}{4} e (g^{ca} X_{[m}^\mu v_{n]} D_a X_k^\nu v^k + g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_{n]}) G_{\mu\nu} = \\
&= \frac{T}{2} e g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_{n]} G_{\mu\nu} \stackrel{!}{=} 0
\end{aligned}$$

↓

$\mathcal{T}_{ab}^i := e^{ci} D_c X_k^\mu v^k X_{[m}^\nu v_{n]} e_a^m e_b^n G_{\mu\nu} = 0$

3.1.4 With extra field, w.r.t v

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta v^l} &= \frac{D\mathcal{L}_{MP3}}{Dv^l} = \left(\frac{\partial}{\partial v^l} - \partial_c \frac{\partial}{\partial (\partial_c v^l)} \right) \left(-\frac{T}{2} e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_\nu^I \right) = \\
&= -\frac{T}{2} g^{ab} (D_a X_j^\mu \delta_l^j D_b X_k^\nu v^k + D_a X_j^\mu v^j D_b X_k^\nu \delta_l^k) G_{\mu\nu} = \\
&= -T g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} \stackrel{!}{=} 0
\end{aligned}$$

↓

$g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} = 0$

3.1.5 With extra field, w.r.t X

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta X_l^\lambda} &= \frac{D\mathcal{L}_{MP3}}{DX_l^\lambda} = \left(\frac{\partial}{\partial X_l^\lambda} - \partial_c \frac{\partial}{\partial(\partial_c X_l^\lambda)} \right) \left(-\frac{T}{2} ee_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I} \right) = \\
&= -\frac{T}{2} \left(eg^{ab} \left((-\omega_{aj}^{j'} \delta_\lambda^\mu \delta_{j'}^l v^j D_b X_k^\nu v^k - D_a X_j^\mu v^j \omega_{bk}^{k'} \delta_\lambda^\nu \delta_{k'}^l v^k) E_\mu^I E_{\nu I} + \right. \right. \\
&\quad \left. \left. + eg^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \left(\frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} + E_\mu^I \frac{\partial E_{\nu I}}{\partial X_l^\lambda} \right) \right) \right. - \\
&\quad \left. - \partial_c (eg^{ab} (\delta_a^c \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k + D_a X_j^\mu v^j \delta_b^c \delta_\lambda^\nu \delta_k^l v^k) G_{\mu\nu}) \right) = \\
&= -T \left(-eg^{ab} D_a X_j^\mu v^j \omega_{bk}^l v^k G_{\mu\lambda} + eg^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - \partial_c (eg^{ac} v^l D_a X_j^\mu v^j G_{\mu\lambda}) \right) \\
&= -T \left(eg^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - D_b (eg^{ab} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) \right) \stackrel{!}{=} 0 \\
&\quad \downarrow \\
&\boxed{D_b (ee_i^a e^{bi} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) = ee_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I}}
\end{aligned}$$

3.2 Without extra index

3.2.1 w.r.t e

$$\begin{aligned}
\frac{\delta S_{DP}}{\delta e_l^e} &= \frac{D\mathcal{L}_{DP}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)} \right) \left(-\frac{T_0}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T_0}{4}
\end{aligned}$$

3.3 Linear action

3.3.1 w.r.t e

$$\begin{aligned}
\frac{\delta S_{LP}}{\delta \delta e_j^b} &= \frac{D\mathcal{L}_{LP}}{De_j^b} = \left(\frac{\partial}{\partial e_j^b} - \partial_e \frac{\partial}{\partial(\partial_e e_j^b)} \right) \left(-\frac{T}{2} \bar{\psi} ee_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi \right) = \\
&= -\frac{T}{4} \bar{\psi} \varepsilon^{cd} \varepsilon_{mn} ((g_{cb} \eta^{mj} e_d^n + e_c^m g_{db} \eta^{nj}) e_i^a + e_c^m e_d^n \delta_b^a \delta_i^j) D_a X^{\mu i} E_\mu^I \gamma_I \psi = \\
&= -\frac{T}{2} \bar{\psi} (-ee_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I + e D_b X^{\mu j} E_\mu^I \gamma_I) \psi \stackrel{!}{=} 0 \\
&\quad \downarrow \\
T_b^j &:= \bar{\psi} (D_b X^{\mu j} E_\mu^I \gamma_I - e_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I) \psi = 0 \\
&\quad \downarrow \\
e_b^j &= F \bar{\psi} D_b X^{\mu j} E_\mu^I \gamma_I \psi, \\
\frac{1}{F} &= \bar{\psi} e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi \\
&\quad \Downarrow \\
\bar{\psi} \psi &= 1, \quad \psi \bar{\psi} = \mathbb{1} \\
e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_\nu^J \eta_{IJ} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}
\end{aligned}$$

4 Inverse Area Action

Start from Nambu-Goto action

$$S_{NG} = -T \int d^2x \sqrt{-h},$$

and make quantum geometry correction

$$\sqrt{-h} \rightarrow \sqrt{-(h + g\Delta)} \approx \sqrt{-h} \left(1 - \frac{g\Delta}{2(-h)} + \mathcal{O}\left(\frac{g^2}{h^2}\right) \right),$$

leading to modified NG action

$$\begin{aligned} S_{MNG} &= -T \int d^2x \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}} \right) = S_{NG} + S_{IA} \\ &\quad \downarrow \\ \mathcal{L}_{MNG} &= -T \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}} \right) \end{aligned}$$

4.1 EoMs

Let $\mathcal{P}_\mu^\tau \equiv \partial \mathcal{L}_{MNG} / \partial \dot{X}^\mu$ and $\mathcal{P}_\mu^\sigma \equiv \partial \mathcal{L}_{MNG} / \partial X'^\mu$

$$\begin{aligned} \delta S_{MNG} &= \int d^2x \left(\frac{\partial \mathcal{L}_{MNG}}{\partial \dot{X}^\mu} \delta \dot{X}^\mu + \frac{\partial \mathcal{L}_{MNG}}{\partial X'^\mu} \delta X'^\mu \right) = \\ &= - \int d^2x (\partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma) \delta X^\mu + \int d\tau \mathcal{P}_\mu^\sigma \delta X^\mu|_{\sigma=0}^{\sigma=\sigma_1} \stackrel{!}{=} 0 \\ &\quad \downarrow \\ \text{EoM} : &\partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma = 0 \\ \text{B.C.} : &\mathcal{P}_\mu^\sigma \delta X^\mu|_{\sigma=0}^{\sigma=\sigma_1} = 0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}_\mu^\tau &= \frac{\partial \mathcal{L}_{MNG}}{\partial \dot{X}^\mu} = -T \left(\frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} + \frac{g\Delta}{2} \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{(-h)^{3/2}} \right) = \\ &= -T \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} \left(1 + \frac{g\Delta}{2(-h)} \right) = \mathcal{P}_{\mu(NG)}^\tau \left(1 + \frac{g\Delta}{2(-h)} \right) \\ \mathcal{P}_\mu^\sigma &= \frac{\partial \mathcal{L}_{MNG}}{\partial X'^\mu} = -T \left(\frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} + \frac{g\Delta}{2} \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{(-h)^{3/2}} \right) = \\ &= -T \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} \left(1 + \frac{g\Delta}{2(-h)} \right) = \mathcal{P}_{\mu(NG)}^\sigma \left(1 + \frac{g\Delta}{2(-h)} \right) \end{aligned}$$

gauge fixing static gauge $\tau = t$ and transverse gauge $\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial s} \frac{ds}{d\sigma} = 0$ (s = length along string)

$$\begin{aligned} &\downarrow \\ \mathcal{P}_{(NG)}^{\tau\mu} &= T \frac{ds}{d\sigma} \gamma_{v_\perp} \frac{\partial X^\mu}{\partial t} \\ \mathcal{P}_{(NG)}^{\sigma\mu} &= \frac{T}{\gamma_{v_\perp}} \frac{\partial X^\mu}{\partial s} \\ -h &= \frac{1}{\gamma_{v_\perp}^2} \left(\frac{ds}{d\sigma} \right)^2 \end{aligned}$$

string energy get's redefined to

$$\frac{\partial}{\partial t} \left(T \frac{ds}{d\sigma} \gamma_{v_\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \right) = 0,$$

and for the spatial part we have

$$\begin{aligned} & \partial_\tau \vec{\mathcal{P}}^\tau + \partial_\sigma \vec{\mathcal{P}}^\sigma = 0 \\ & \downarrow \\ & \frac{\partial}{\partial t} \left[T \frac{ds}{d\sigma} \gamma_{v_\perp} \frac{\partial \vec{X}}{\partial t} \left(1 + \frac{g\Delta}{2(-h)} \right) \right] + \frac{ds}{d\sigma} \frac{\partial}{\partial s} \left[-\frac{T}{\gamma_{v_\perp}} \frac{\partial \vec{X}}{\partial s} \left(1 + \frac{g\Delta}{2(-h)} \right) \right] = 0 \\ & \downarrow \\ & \mu \gamma_{v_\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial}{\partial s} \left[\frac{T}{\gamma_{v_\perp}} \left(1 + \frac{g\Delta}{2(-h)} \right) \frac{\partial \vec{X}}{\partial s} \right] = 0 \\ \implies & \text{effective mass density becomes } \mu_{eff} = \mu \gamma_{v_\perp} \left(1 - \frac{g\Delta}{2(-h)} \right) \text{ and effective tension becomes } T_{eff} = \frac{T}{\gamma_{v_\perp}} \left(1 - \frac{g\Delta}{2(-h)} \right) \\ & \downarrow \\ & \mu_{eff} \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial}{\partial s} \left[T_{eff} \frac{\partial \vec{X}}{\partial s} \right] = 0 \end{aligned}$$

$$\begin{aligned} \mathcal{H} &= \dot{\vec{X}} \cdot \vec{\pi} - \mathcal{L} = \\ &= \vec{v}_\perp \cdot \left(T \frac{ds}{d\sigma} \gamma_{v_\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \vec{v}_\perp \right) - \left(-T \frac{ds}{d\sigma} \frac{1}{\gamma_{v_\perp}} \left(1 + \frac{g\Delta}{2(-h)} \right) \right) = \\ &= T \frac{ds}{d\sigma} \left(1 + \frac{g\Delta}{2(-h)} \right) \left(\gamma_{v_\perp} v_\perp^2 + \frac{1}{\gamma_{v_\perp}} \right) = \\ &= T \frac{ds}{d\sigma} \gamma_{v_\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \end{aligned}$$

let $\left(1 + \frac{g\Delta}{2(-h)} \right) = F$

$$\begin{aligned} & \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{1}{F \gamma_{v_\perp}} \frac{ds}{d\sigma} \frac{\partial}{\partial \sigma} \left[\frac{1}{\gamma_{v_\perp}} F \frac{ds}{d\sigma} \frac{\partial \vec{X}}{\partial \sigma} \right] = 0 \\ & \downarrow \\ & A(\sigma) = \frac{\gamma_{v_\perp}}{F} \frac{ds}{d\sigma} \stackrel{!}{=} 1 \\ & \downarrow \\ & ds = \frac{\gamma_{v_\perp}}{F} ds = \frac{1}{TF^2} dE \implies \sigma(q) = \frac{1}{T} \int_0^q \frac{1}{F^2} dE \\ & \downarrow \\ & F^2 \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial^2 \vec{X}}{\partial \sigma^2} = 0 \end{aligned}$$

\implies speed of wave on the string gets modified by correction factor $v = c/F$

$$\begin{aligned} -h &= \frac{1}{\gamma_{v_\perp}^2} \left(\frac{ds}{d\sigma} \right)^2 = \frac{F^2}{\gamma_{v_\perp}^4} = F^2 (1 - v_\perp^2)^2 \\ & \downarrow \\ & -h = \left(1 + \frac{g\Delta}{2(-h)} \right)^2 (1 - v_\perp^2)^2 \\ & \downarrow \end{aligned}$$

$$\frac{-h}{\left(1 + \frac{g\Delta}{2(-h)}\right)^2} = (1 - v_{\perp}^2)^2$$

\downarrow (WolframAlpha)

$$-h = \frac{1}{3\gamma_{v\perp}^4} - f_1 - f_2,$$

where (let $a = \frac{g\Delta}{2}$ and $b = \frac{1}{\gamma_{v\perp}^4}$)

$$f_1 = \frac{\sqrt[3]{-27a^2b + 3\sqrt{3}\sqrt{27a^4b^2 + 4a^3b^3 - 18ab^2 - 2b^3}}}{3\sqrt[3]{2}}$$

$$f_2 = \frac{(6ab + b^2)}{6f_1}$$

\downarrow

$$F = 1 + \frac{g\Delta}{2(-h)} = 1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}$$