

# Sketchboard

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# 1 Basic concepts

## 1.1 With extra capital internal index on embedding fields

Promote embedding fields  $X^\mu$  to have an internal group index with  $D$  values

$$\begin{aligned} X^\mu &\rightarrow X^{\mu I}, \quad I = 0, \dots, d \\ g_{ab} &= 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{IJ} \sim f D_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu} \\ D_a X^{\mu I} &= \partial_a X^{\mu I} + \omega_{aJ}^I X^{\mu J} \\ e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ}, \quad i, j = 0, 1 \\ g &= \det(g_{ab}) \rightarrow g = \det(\text{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e) \end{aligned}$$

## 1.2 With 1 extra small internal index on embedding fields

Promote embedding fields to have a internal group index with 2 values

$$\begin{aligned} X^\mu &\rightarrow X^{\mu i}, \quad i = 0, 1 \\ g_{ab} &= 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{ij} \sim f D_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu} \\ D_a X^{\mu i} &= \partial_a X^{\mu i} + \omega_{aj}^i X^{\mu j} \\ e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij} \\ g &= \det(g_{ab}) \rightarrow g = \det(\text{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e) \end{aligned}$$

## 1.3 With 2 extra small indices

Promote embedding fields  $X^\mu$  to have two internal indices with 2 values

$$X^\mu \rightarrow X^{\mu ij}, \quad i, j = 0, 1$$

## 1.4 Without extra index on embedding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$\begin{aligned} e_a^i e_b^j \eta_{ij} &= g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\ g &= \det(g_{ab}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e) \end{aligned}$$

# 2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int d\tau \wedge d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

## 2.1 With capital internal index

... and promote partial derivative  $\partial_a$  to covariant derivative  $D_a$ , giving us our first attempt at modified Polyakov action

$$\begin{aligned} S_{MP1} &= -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ} \\ &\quad \uparrow \\ \mathcal{L}_{MP1} &= -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ} \end{aligned}$$

## 2.2 With small internal index

### 2.2.1 Without extra field, 1 internal index

... and promote derivatives to covariant  $D_a$ , giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

↑

$$\mathcal{L}_{MP2} = -\frac{T_0}{2} \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

### 2.2.2 With extra field

..., promote partial derivative to covariant and add extra internal WS field  $v^i$  leading us to

$$S_{MP3} = -\frac{T}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

↑

$$\mathcal{L}_{MP3} = -\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

### 2.2.3 Without extra field, 2 internal indices

... and promote derivatives to covariant  $D_a$ ,

$$S_{MP4} = -\frac{T}{2} \int d\tau \wedge d\sigma e e_i^a e_j^b D_a X^{\mu jj'} \eta_{jj'} \eta_{kk'} D_b X^{\nu kk'} E_\mu^I E_\nu^I$$

$$\mathcal{L}_{MP4} = -\frac{T}{2} e e_i^a e_j^b D_a X^{\mu ik} \eta_{kl} D_b X^{\nu lj} E_\mu^I E_\nu^I$$

## 2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X).$$

↑

$$\mathcal{L}_{DP} = -\frac{T_0}{2} \det(e) e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X).$$

## 2.4 Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of  $D$ -dimensional gamma matrices and bulk spinors

$$S_{LP} = -\frac{T}{2} \int d\tau \wedge d\sigma \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

↑

$$\mathcal{L}_{LP} = -\frac{T}{2} \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

## 3 EoMs

Start by writing  $\det(e) = \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n$  and  $g_{ab} = g_{ab}^{ij} \eta_{ij}$

### 3.1 With small internal index

#### 3.1.1 Without extra field, w.r.t $e$

$$\begin{aligned}
\frac{\delta S_{MP2}}{\delta e_l^e} &= \frac{D\mathcal{L}_{MP2}}{De_l^e} = \left( \frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial (\partial_f e_l^e)} \right) \left( -\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \right) = \\
&= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (\eta^{ml} g_{ce} e_d^n e_i^a e_j^b + e_c^m \eta^{nl} g_{de} e_i^a e_j^b + \\
&\quad + e_c^m e_d^n \delta_e^a \delta_i^l e_j^b + e_c^m e_d^n e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} g_{ce} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} e_c^k e_{ek} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_e^l e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -T (-\det(e) e_e^l e_i^a e_j^b + \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \stackrel{!}{=} 0 \\
&\quad \downarrow \\
T_e^l &:= E_\mu^I E_{\nu I} (e_i^a D_a X^{\mu i} D_e X^{\nu l} - e_e^l e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j}) = 0 \\
&\quad \Downarrow \\
e_e^l &= f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I}, \\
\frac{1}{f} &= e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \\
&\quad \downarrow \\
e_e^l e_{fl} &= (f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I}) (f e_{i'}^a D_{a'} X^{\mu' i'} D_f X_l^{\nu'} E_{\mu'}^I E_{\nu' I'}) = \\
&= f^2 D_e X^{\nu l} D_f X_l^{\nu'} E_{\nu'}^I E_{\nu' I'} (e_i^a e_{i'}^a D_a X^{\mu i} D_{a'} X^{\mu' i'} E_\mu^I E_{\mu' I}) = \\
&= f D_e X^{\mu i} D_f X_l^{\nu'} E_{\mu'}^I E_{\nu' I} \\
&\quad \Downarrow \\
D_e X^{\mu i} D_f X_l^{\nu'} E_{\mu'}^I E_{\nu' I} &= 2 \partial_e X^\mu \partial_f X^\nu G_{\mu\nu} \\
e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\
&\quad \Downarrow \\
D_a X^{\mu 0} &= -i \partial_a X^\mu \\
D_a X^{\mu 1} &= \partial_a X^\mu \\
&\quad \Downarrow \\
X^{\mu 0} &= -i X^{\mu 1}
\end{aligned}$$

### 3.1.2 With extra field, w.r.t $e$

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta e_o^e} &= \frac{D\mathcal{L}_{MP3}}{De_o^e} = \left( \frac{\partial}{\partial e_o^e} - \partial_f \frac{\partial}{\partial (\partial_f e_o^e)} \right) \left( -\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (g_{ce} \eta^{mo} e_d^n \eta^{ij} e_i^a e_j^b + e_c^m g_{de} \eta^{no} \eta^{ij} e_i^a e_j^b + \\
&\quad + e_c^m e_d^n \eta^{ij} \delta_e^a \delta_e^b + e_c^m e_d^n \eta^{ij} e_i^a \delta_e^b \delta_e^o) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -\frac{T}{2} \varepsilon^{cd} \varepsilon_{mn} (e_c^p e_{ep} e_d^n \eta^{mo} \eta^{ij} e_i^a e_j^b + 2 \det(e) e^{ao} \delta_e^b) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -T(-\det(e) e_e^o \eta^{ij} e_i^a e_j^b + \det(e) e^{ao} \delta_e^b) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \stackrel{!}{=} 0
\end{aligned}$$

↓

$T_e^o := (e^{ao} D_a(X_k^\mu v^k) D_e(X_l^\nu v^l) - e_e^o \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l)) E_\mu^I E_\nu^J \eta_{IJ} = 0$

⇓

$$\begin{aligned}
e_e^o &= f e^{ao} D_a(X_k^\mu v^k) D_e(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}, \\
\frac{1}{f} &= \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}
\end{aligned}$$

### 3.1.3 With extra field, w.r.t $\omega$

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta \omega_c^{mn}} &= \frac{D\mathcal{L}_{MP3}}{D\omega_c^{mn}} = \left( \frac{\partial}{\partial \omega_c^{mn}} - \partial_d \frac{\partial}{\partial (\partial_d \omega_c^{mn})} \right) \left( -\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T}{4} e g^{ab} ((-\delta_a^c \delta_{[m}^{k'} \eta_{n]k} X_{k'}^\mu v^k) D_b X_l^\nu v^l + D_a X_k^\mu v^k (-\delta_b^c \delta_{[m}^{l'} \eta_{n]l} X_{l'}^\nu v^l)) G_{\mu\nu} = \\
&= \frac{T}{4} e (g^{ca} X_{[m}^\mu v_{n]} D_a X_k^\nu v^k + g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_{n]}) G_{\mu\nu} = \\
&= \frac{T}{2} e g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_{n]} G_{\mu\nu} \stackrel{!}{=} 0
\end{aligned}$$

↓

$\mathcal{T}_{ab}^i := e^{ci} D_c X_k^\mu v^k X_{[m}^\nu v_{n]} e_a^m e_b^n G_{\mu\nu} = 0$

### 3.1.4 With extra field, w.r.t $v$

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta v^l} &= \frac{D\mathcal{L}_{MP3}}{Dv^l} = \left( \frac{\partial}{\partial v^l} - \partial_c \frac{\partial}{\partial (\partial_c v^l)} \right) \left( -\frac{T}{2} e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_\nu^I \right) = \\
&= -\frac{T}{2} g^{ab} (D_a X_j^\mu \delta_l^j D_b X_k^\nu v^k + D_a X_j^\mu v^j D_b X_k^\nu \delta_l^k) G_{\mu\nu} = \\
&= -T g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} \stackrel{!}{=} 0
\end{aligned}$$

↓

$g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} = 0$

### 3.1.5 With extra field, w.r.t $X$

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta X_l^\lambda} &= \frac{D\mathcal{L}_{MP3}}{DX_l^\lambda} = \left( \frac{\partial}{\partial X_l^\lambda} - \partial_c \frac{\partial}{\partial(\partial_c X_l^\lambda)} \right) \left( -\frac{T}{2} ee_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I} \right) = \\
&= -\frac{T}{2} \left( eg^{ab} \left( (-\omega_{aj}^{j'} \delta_\lambda^\mu \delta_{j'}^l v^j D_b X_k^\nu v^k - D_a X_j^\mu v^j \omega_{bk}^{k'} \delta_\lambda^\nu \delta_{k'}^l v^k) E_\mu^I E_{\nu I} + \right. \right. \\
&\quad \left. \left. + eg^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \left( \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} + E_\mu^I \frac{\partial E_{\nu I}}{\partial X_l^\lambda} \right) \right) \right. - \\
&\quad \left. - \partial_c (eg^{ab} (\delta_a^c \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k + D_a X_j^\mu v^j \delta_b^c \delta_\lambda^\nu \delta_k^l v^k) G_{\mu\nu}) \right) = \\
&= -T \left( -eg^{ab} D_a X_j^\mu v^j \omega_{bk}^l v^k G_{\mu\lambda} + eg^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - \partial_c (eg^{ac} v^l D_a X_j^\mu v^j G_{\mu\lambda}) \right) \\
&= -T \left( eg^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - D_b (eg^{ab} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) \right) \stackrel{!}{=} 0 \\
&\quad \downarrow \\
&\boxed{D_b (ee_i^a e^{bi} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) = ee_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I}}
\end{aligned}$$

## 3.2 Without extra index

### 3.2.1 w.r.t $e$

$$\begin{aligned}
\frac{\delta S_{DP}}{\delta e_l^e} &= \frac{D\mathcal{L}_{DP}}{De_l^e} = \left( \frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)} \right) \left( -\frac{T_0}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T_0}{4}
\end{aligned}$$

## 3.3 Linear action

### 3.3.1 w.r.t $e$

$$\begin{aligned}
\frac{\delta S_{LP}}{\delta \delta e_j^b} &= \frac{D\mathcal{L}_{LP}}{De_j^b} = \left( \frac{\partial}{\partial e_j^b} - \partial_e \frac{\partial}{\partial(\partial_e e_j^b)} \right) \left( -\frac{T}{2} \bar{\psi} ee_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi \right) = \\
&= -\frac{T}{4} \bar{\psi} \varepsilon^{cd} \varepsilon_{mn} ((g_{cb} \eta^{mj} e_d^n + e_c^m g_{db} \eta^{nj}) e_i^a + e_c^m e_d^n \delta_b^a \delta_i^j) D_a X^{\mu i} E_\mu^I \gamma_I \psi = \\
&= -\frac{T}{2} \bar{\psi} (-ee_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I + e D_b X^{\mu j} E_\mu^I \gamma_I) \psi \stackrel{!}{=} 0 \\
&\quad \downarrow \\
T_b^j &:= \bar{\psi} (D_b X^{\mu j} E_\mu^I \gamma_I - e_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I) \psi = 0 \\
&\quad \downarrow \\
e_b^j &= F \bar{\psi} D_b X^{\mu j} E_\mu^I \gamma_I \psi, \\
\frac{1}{F} &= \bar{\psi} e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi \\
&\quad \Downarrow \\
\bar{\psi} \psi &= 1, \quad \psi \bar{\psi} = \mathbb{1} \\
e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_\nu^J \eta_{IJ} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}
\end{aligned}$$

## 4 Inverse Area Action

### 4.1 Nambu-Goto

Start from Nambu-Goto action

$$S_{NG} = -T \int d^2x \sqrt{-h},$$

and make quantum geometry correction

$$\sqrt{-h} \rightarrow \sqrt{-(h + g\Delta)} \approx \sqrt{-h} \left( 1 - \frac{g\Delta}{2(-h)} + \mathcal{O}\left(\frac{g^2}{h^2}\right) \right),$$

leading to modified NG action

$$\begin{aligned} S_{MNG} &= -T \int d^2x \left( \sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}} \right) = S_{NG} + S_{IA} \\ &\quad \downarrow \\ \mathcal{L}_{MNG} &= -T \left( \sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}} \right) \end{aligned}$$

#### 4.1.1 EoMs

Let  $\mathcal{P}_\mu^\tau \equiv \partial \mathcal{L}_{MNG} / \partial \dot{X}^\mu$  and  $\mathcal{P}_\mu^\sigma \equiv \partial \mathcal{L}_{MNG} / \partial X'^\mu$

$$\begin{aligned} \delta S_{MNG} &= \int d^2x \left( \frac{\partial \mathcal{L}_{MNG}}{\partial \dot{X}^\mu} \delta \dot{X}^\mu + \frac{\partial \mathcal{L}_{MNG}}{\partial X'^\mu} \delta X'^\mu \right) = \\ &= - \int d^2x (\partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma) \delta X^\mu + \int d\tau \mathcal{P}_\mu^\sigma \delta X^\mu \Big|_{\sigma=0}^{\sigma=\sigma_1} \stackrel{!}{=} 0 \\ &\quad \downarrow \\ \text{EoM} : \partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma &= 0 \\ \text{B.C.} : \mathcal{P}_\mu^\sigma \delta X^\mu \Big|_{\sigma=0}^{\sigma=\sigma_1} &= 0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}_\mu^\tau &= \frac{\partial \mathcal{L}_{MNG}}{\partial \dot{X}^\mu} = -T \left( \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} + \frac{g\Delta}{2} \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{(-h)^{3/2}} \right) = \\ &= -T \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} \left( 1 + \frac{g\Delta}{2(-h)} \right) = \mathcal{P}_{\mu(NG)}^\tau \left( 1 + \frac{g\Delta}{2(-h)} \right) \\ \mathcal{P}_\mu^\sigma &= \frac{\partial \mathcal{L}_{MNG}}{\partial X'^\mu} = -T \left( \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} + \frac{g\Delta}{2} \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{(-h)^{3/2}} \right) = \\ &= -T \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} \left( 1 + \frac{g\Delta}{2(-h)} \right) = \mathcal{P}_{\mu(NG)}^\sigma \left( 1 + \frac{g\Delta}{2(-h)} \right) \end{aligned}$$

gauge fixing static gauge  $\tau = t$  and transverse gauge  $\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial s} \frac{ds}{d\sigma} = 0$  ( $s$  = length along string)

$$\begin{aligned} &\downarrow \\ \mathcal{P}_{(NG)}^{\tau\mu} &= T \frac{ds}{d\sigma} \gamma_{v_\perp} \frac{\partial X^\mu}{\partial t} \\ \mathcal{P}_{(NG)}^{\sigma\mu} &= \frac{T}{\gamma_{v_\perp}} \frac{\partial X^\mu}{\partial s} \\ -h &= \frac{1}{\gamma_{v_\perp}^2} \left( \frac{ds}{d\sigma} \right)^2 \end{aligned}$$

string energy get's redefined to

$$\frac{\partial}{\partial t} \left( T \frac{ds}{d\sigma} \gamma_{v_\perp} \left( 1 + \frac{g\Delta}{2(-h)} \right) \right) = 0,$$

and for the spatial part we have

$$\begin{aligned} & \partial_\tau \vec{P}^\tau + \partial_\sigma \vec{P}^\sigma = 0 \\ & \downarrow \\ & \frac{\partial}{\partial t} \left[ T \frac{ds}{d\sigma} \gamma_{v_\perp} \frac{\partial \vec{X}}{\partial t} \left( 1 + \frac{g\Delta}{2(-h)} \right) \right] + \frac{ds}{d\sigma} \frac{\partial}{\partial s} \left[ -\frac{T}{\gamma_{v_\perp}} \frac{\partial \vec{X}}{\partial s} \left( 1 + \frac{g\Delta}{2(-h)} \right) \right] = 0 \\ & \downarrow \\ & \mu \gamma_{v_\perp} \left( 1 + \frac{g\Delta}{2(-h)} \right) \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial}{\partial s} \left[ \frac{T}{\gamma_{v_\perp}} \left( 1 + \frac{g\Delta}{2(-h)} \right) \frac{\partial \vec{X}}{\partial s} \right] = 0 \\ \implies & \text{effective mass density becomes } \mu_{eff} = \mu \gamma_{v_\perp} \left( 1 - \frac{g\Delta}{2(-h)} \right) \text{ and effective tension becomes } T_{eff} = \frac{T}{\gamma_{v_\perp}} \left( 1 - \frac{g\Delta}{2(-h)} \right) \\ & \downarrow \\ & \mu_{eff} \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial}{\partial s} \left[ T_{eff} \frac{\partial \vec{X}}{\partial s} \right] = 0 \end{aligned}$$

$$\begin{aligned} \mathcal{H} &= \dot{\vec{X}} \cdot \vec{\pi} - \mathcal{L} = \\ &= \vec{v}_\perp \cdot \left( T \frac{ds}{d\sigma} \gamma_{v_\perp} \left( 1 + \frac{g\Delta}{2(-h)} \right) \vec{v}_\perp \right) - \left( -T \frac{ds}{d\sigma} \frac{1}{\gamma_{v_\perp}} \left( 1 + \frac{g\Delta}{2(-h)} \right) \right) = \\ &= T \frac{ds}{d\sigma} \left( 1 + \frac{g\Delta}{2(-h)} \right) \left( \gamma_{v_\perp} v_\perp^2 + \frac{1}{\gamma_{v_\perp}} \right) = \\ &= T \frac{ds}{d\sigma} \gamma_{v_\perp} \left( 1 + \frac{g\Delta}{2(-h)} \right) \end{aligned}$$

let  $\left( 1 + \frac{g\Delta}{2(-h)} \right) = F$

$$\begin{aligned} & \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{1}{F \gamma_{v_\perp}} \frac{ds}{d\sigma} \frac{\partial}{\partial \sigma} \left[ \frac{1}{\gamma_{v_\perp}} F \frac{ds}{d\sigma} \frac{\partial \vec{X}}{\partial \sigma} \right] = 0 \\ & \downarrow \\ & A(\sigma) = \frac{\gamma_{v_\perp}}{F} \frac{ds}{d\sigma} \stackrel{!}{=} 1 \\ & \downarrow \\ & d\sigma = \frac{\gamma_{v_\perp}}{F} ds = \frac{1}{TF^2} dE \implies \sigma(q) = \frac{1}{T} \int_0^q \frac{1}{F^2} dE \\ & \downarrow \\ & F^2 \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial^2 \vec{X}}{\partial \sigma^2} = 0 \end{aligned}$$

$\implies$  speed of wave on the string gets modified by correction factor  $v = c/F$ .

## 4.2 Translation and Lorentz symmetry

Starting with global translations, define the transformation by

$$X^\mu \mapsto X^\mu + \epsilon^\mu.$$

This does not change the action since it only depends on derivatives of  $X^\mu$  and so  $\delta(\partial_a X^\mu) = \partial_a(\delta X^\mu) = \partial_a \epsilon^\mu = 0$ .

As for Lorentz transformations, the infinitesimal for is

$$X^\mu \mapsto X^\mu + \epsilon^{\mu\nu} X_\nu,$$

with  $\epsilon^{\mu\nu}$  anti-symmetric.

### 4.3 Verifying if there is longitudinal velocity

$$\begin{aligned}
 -h &= \frac{1}{\gamma_{v_\perp}^2} \left( \frac{ds}{d\sigma} \right)^2 = \frac{F^2}{\gamma_{v_\perp}^4} = F^2 (1 - v_\perp^2)^2 \\
 &\quad \downarrow \\
 -h &= \left( 1 + \frac{g\Delta}{2(-h)} \right)^2 (1 - v_\perp^2)^2 \\
 &\quad \downarrow \\
 \frac{-h}{\left( 1 + \frac{g\Delta}{2(-h)} \right)^2} &= (1 - v_\perp^2)^2 \\
 &\quad \downarrow (\text{WolframAlpha}) \\
 -h &= \frac{(1 - v_\perp^2)^2}{3} - f_1 - f_2,
 \end{aligned}$$

where (let  $a = \frac{g\Delta}{2}$  and  $b = (1 - v_\perp^2)^2$ )

$$f_1 = \frac{\sqrt[3]{-27a^2b + 3\sqrt{3}\sqrt{27a^4b^2 + 4a^3b^3 - 18ab^2 - 2b^3}}}{3\sqrt[3]{2}}$$

$$\begin{aligned}
 f_2 &= \frac{(6ab + b^2)}{6f_1} \\
 &\quad \downarrow \\
 F &= 1 + \frac{g\Delta}{2(-h)} = 1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)} \\
 &\quad \downarrow \\
 v &= \frac{1}{F} = \frac{1}{1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}}
 \end{aligned}$$

## 5 Bimetric Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

and promote to bimetric action

$$\downarrow$$

$$S_{BP} = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{T'}{2} \int d^2x \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

### 5.1 EoMs

$$\begin{aligned} \frac{\delta S_{BP}}{\delta g^{cd}} &= -\frac{T}{2} \left( \frac{\partial \sqrt{-g}}{\partial g^{cd}} g^{ab} + \sqrt{-g} \frac{\partial g^{ab}}{\partial g^{cd}} \right) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + 0 = \\ &= -\frac{T}{2} \left( -\frac{1}{2} \sqrt{-g} g_{cd} g^{ab} + \sqrt{-g} \delta_{(c}^a \delta_{d)}^b \right) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \stackrel{!}{=} 0 \end{aligned}$$

$$\begin{aligned} T_{cd}^{(G)} &:= \left( \partial_c X^\mu \partial_d X^\nu - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \right) G_{\mu\nu} = 0 \\ &\Downarrow \\ g_{cd} &= 2f^{(G)} \partial_c X^\mu \partial_d X^\nu G_{\mu\nu}, \\ \frac{1}{f^{(G)}} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \end{aligned}$$

$$\begin{aligned} \frac{\delta S_{BP}}{\delta h^{cd}} &= \dots = \\ &= -\frac{T'}{2} \left( -\frac{1}{2} \sqrt{-h} h_{cd} h^{ab} + \sqrt{-h} \delta_{(c}^a \delta_{d)}^b \right) \partial_a X^\mu \partial_b X^\nu H_{\mu\nu} \stackrel{!}{=} 0 \end{aligned}$$

$$\begin{aligned} &\Downarrow \\ T_{cd}^{(H)} &:= \left( \partial_c X^\mu \partial_d X^\mu - \frac{1}{2} h_{cd} h^{ab} \partial_a X^\mu \partial_b X^\nu \right) H_{\mu\nu} = 0 \\ &\Downarrow \\ h_{cd} &= 2f^{(H)} \partial_c X^\mu \partial_d X^\nu H_{\mu\nu}, \\ \frac{1}{f^{(H)}} &= h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu} \end{aligned}$$

If the auxiliary metrics  $g$  and  $h$  coincide with the induced metrics from  $G$  and  $H$ , then  $f^{(G)} = f^{(H)} = 1/2$  and the action reduces to

$$S = -T \int d^2x \sqrt{-g} - T' \int d^2x \sqrt{-h},$$

where if  $T' = -Tk\Delta/2$  and  $h = g^{-1}$  we recover the NG area corrected action

$$S = -T \int d^2x \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right).$$

The condition  $h = g^{-1}$  can be achieved by

$$h_{ab} = S_{ac} g^{cd} T_{db},$$

where  $S, T \in \text{SL}(2, \mathbb{C})$ , so we have a gauge freedom in choosing  $h_{ab}$ . Since we require  $[h_{ab}] = [h_{ab}]^\dagger$  ( $h_{ab}$  must be symmetric and real), this means  $[T_{db}] = [S_{db}]^\dagger$  and also that  $[S_{ab}]^\dagger = [S_{ab}]$ , thus

$$h_{ab} = S_{ac} g^{cd} S_{db}.$$

This restriction constrains our gauge group to just the boosts part of  $\text{SL}(2, \mathbb{C})$ ,

$$S = e^{-n^i \sigma_i \beta}.$$

This also implies

$$\begin{aligned} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu} &= S_{ac} g^{cd} S_{db} = \\ &= S_{ac} \left( \frac{1}{2g} \varepsilon^{cc'} \varepsilon^{dd'} g_{c'd'} \right) S_{db} = \\ \delta_a^{c'} \delta_b^{d'} \partial_{c'} X^\mu \partial_{d'} X^\nu H_{\mu\nu} &= S_{ac} \left( \frac{1}{2g} \varepsilon^{cc'} \varepsilon^{dd'} \partial_{c'} X^\mu \partial_{d'} X^\nu G_{\mu\nu} \right) S_{db} \\ \delta_a^{c'} \delta_b^{d'} H_{\mu\nu} &= \frac{1}{2g} S_{ac} \varepsilon^{cc'} S_{db} \varepsilon^{dd'} G_{\mu\nu} \\ H_{\mu\nu} &= -\frac{1}{8g} S_{ac} \varepsilon^{ca} S_{db} \varepsilon^{bd} G_{\mu\nu} \\ H_{\mu\nu} &= \frac{1}{8(-g)} \text{Tr}(S\varepsilon) \text{Tr}(S\varepsilon) G_{\mu\nu} \\ H_{\mu\nu} &= \frac{1}{8(-g)} \text{Tr}(S\varepsilon)^2 G_{\mu\nu}. \end{aligned}$$

This restricts  $S$  to necessarily being not symmetric, otherwise the trace  $\text{Tr}(S\varepsilon)$  vanishes and we would get  $H_{\mu\nu} = 0$ .

$$\begin{aligned} \frac{\delta S_{BP}}{\delta X^\lambda} &= \left( -\frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu - \\ &\quad - \partial_c \left( -\frac{T}{2} \sqrt{-g} g^{ab} G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} H_{\mu\nu} \right) (\delta_a^c \delta_\lambda^\mu \partial_b X^\nu + \partial_a X^\mu \delta_b^c \delta_\lambda^\nu) = \\ &= \partial_a \left( T \sqrt{-g} g^{ab} G_{\mu\lambda} + T' \sqrt{-h} h^{ab} H_{\mu\lambda} \right) \partial_b X^\mu - \\ &\quad - \left( \frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} + \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu \stackrel{!}{=} 0 \\ &\quad \downarrow \\ \partial_a (F_{\mu\lambda}^{ab} \partial_b X^\mu) &= \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu, \\ F_{\mu\nu}^{ab} &= T \sqrt{-g} g^{ab} G_{\mu\nu} + T' \sqrt{-h} h^{ab} H_{\mu\nu}. \end{aligned}$$

Assuming the condition for  $h = g^{-1}$  and flat background  $G_{\mu\nu} = \eta_{\mu\nu}$ , the EoM reduces to

$$\partial_a \left( \left( \sqrt{-g} g^{ab} - \frac{k\Delta}{2(-g)} \sqrt{-g}^{-1} (S^{-1})^{ac} g_{cd} (S^{-1})^{db} \right) \partial_b X^\mu \right) = 0,$$

which in conformal gauge  $g_{ab} = \phi \eta_{ab}$  with  $(S^{-1})^{ac} \eta_{cd} (S^{-1})^{db} = \eta^{ab}$  reduces further

$$\begin{aligned} \partial_a \left( \left( 1 - \frac{k\Delta}{2\phi^2} \right) \eta^{ab} \partial_b X^\mu \right) &= 0 \\ \eta^{ab} \partial_a (F^- \partial_b X^\mu) &= 0 \\ F^- \eta^{ab} \partial_a \partial_b X^\mu + \eta^{ab} \partial_a F^- \partial_b X^\mu &= 0 \\ \eta^{ab} \partial_a \partial_b X^\mu + \eta^{ab} \partial_a \ln(F^-) \partial_b X^\mu &= 0. \end{aligned}$$

This is the exact same equation derived in the next section which leads to the Klein-Gordon eqn. The requirement that  $(S^{-1})^{ac} \eta_{cd} (S^{-1})^{db} = \eta^{ab}$  implies

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix},$$

$$S = \begin{bmatrix} \cosh \beta - n^3 \sinh \beta & -(n^1 - in^2) \sinh \beta \\ -(n^1 + in^2) \sinh \beta & \cosh \beta + n^3 \sinh \beta \end{bmatrix},$$

$$S^{-1} = \begin{bmatrix} \cosh \beta + n^3 \sinh \beta & +(n^1 - in^2) \sinh \beta \\ +(n^1 + in^2) \sinh \beta & \cosh \beta - n^3 \sinh \beta \end{bmatrix}$$

so  $n^3 = 0$ . Since  $(n^1)^2 + (n^2)^2 = 1$ , we can parametrize as  $n^1 = \cos \theta$  and  $n^2 = \sin \theta$ , thus  $S$  has the form

$$S = \begin{bmatrix} \cosh \beta & -e^{-i\theta} \sinh \beta \\ -e^{i\theta} \sinh \beta & \cosh \beta \end{bmatrix},$$

and  $\text{Tr}(S\varepsilon) = -2i \sin \theta \sinh \beta$ . Let's try to find more restrictions on  $S$ :

$$\begin{aligned} h^{ab} &= (S^{-1})^{ac} g_{cd} (S^{-1})^{db} \\ h^{ab} &= \frac{1}{2g^{-1}} \varepsilon^{aa'} \varepsilon^{bb'} \partial_{a'} X^\mu \partial_{b'} X^\nu \left( \frac{1}{8(-g)} \text{Tr}(S\varepsilon)^2 G_{\mu\nu} \right) \\ &\Downarrow \\ (S^{-1})^{ac} g_{cd} (S^{-1})^{db} &= \frac{1}{2g^{-1}} \varepsilon^{aa'} \varepsilon^{bb'} \frac{1}{8(-g)} \text{Tr}(S\varepsilon)^2 \partial_{a'} X^\mu \partial_{b'} X^\nu G_{\mu\nu} \\ (S^{-1})^{ac} g_{cd} (S^{-1})^{db} &= -\frac{1}{16} \text{Tr}(S\varepsilon)^2 \varepsilon^{ac} \varepsilon^{bd} g_{cd} \\ (S^{-1})^{ac} g_{cd} (S^{-1})^{db} &= \left( \frac{i}{4} \text{Tr}(S\varepsilon) \varepsilon^{ac} \right) g_{cd} \left( \frac{i}{4} \text{Tr}(S\varepsilon) \varepsilon^{db} \right) \\ g_{cd} &= S_{ca} \left( \frac{i}{4} \text{Tr}(S\varepsilon)^* \varepsilon^{ac'} \right) g_{c'd'} \left( \frac{i}{4} \text{Tr}(S\varepsilon) \varepsilon^{d'b} \right) S_{bd} \\ &\Downarrow \\ S_{ca} \left( \frac{i}{4} \text{Tr}(S\varepsilon)^* \varepsilon^{ac'} \right) &= \delta_c^{c'} \\ \left( \frac{i}{4} \text{Tr}(S\varepsilon) \varepsilon^{d'b} \right) S_{bd} &= \delta_d^{d'} \\ &\Downarrow \\ (S^{-1})^{ab} &= \left( \frac{i}{4} \text{Tr}(S\varepsilon) \varepsilon^{ab} \right), \end{aligned}$$

thus we have that

$$\begin{bmatrix} \cosh \beta & e^{-i\theta} \sinh \beta \\ e^{i\theta} \sinh \beta & \cosh \beta \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \sin \theta \sinh \beta \\ -\frac{1}{2} \sin \theta \sinh \beta & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\theta = \frac{n\pi}{2}$$

$$\beta = i \frac{m\pi}{2}$$

## 6 Inverse Area Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

and make quantum geometry correction

$$\sqrt{-g} \rightarrow \sqrt{-(g + k\Delta)} \approx \sqrt{-g} \left( 1 + \frac{k\Delta}{2g} + \mathcal{O}\left(\frac{k^2}{g^2}\right) \right)$$

leading to

$$S_{IAP} = -\frac{T}{2} \int d^2x \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

### 6.1 EoMs

$$\begin{aligned} \frac{\delta S_{IAP}}{\delta g^{cd}} &= \frac{\partial}{\partial g^{cd}} \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = \\ &= \left( -\frac{1}{2} \sqrt{-g} g_{cd} + \frac{k\Delta}{4\sqrt{-g}} g_{cd} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} = \\ &= -\frac{1}{2} \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} \stackrel{!}{=} 0 \\ &\quad \downarrow \\ T_{cd} := \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} &= 0 \\ &\quad \downarrow \\ g_{cd} &= 2f \partial_c X^\mu \partial_d X^\nu G_{\mu\nu}, \\ \frac{1}{f} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\ \\ \frac{\delta S_{IAP}}{\delta X^\lambda} &= \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - \partial_c \left( \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) 2g^{ab} \delta_a^\lambda \delta_b^\mu \partial_b X^\nu G_{\mu\nu} \right) = \\ &= \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2 \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} - \\ &\quad - 2\partial_a \left( \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} = \\ &= \left( 1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2 \left( 1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} - \\ &\quad - 2\partial_a \left( \left( 1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} \stackrel{!}{=} 0 \\ &\quad \downarrow \\ \left( 1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} &+ \partial_a \left( \left( 1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} = \\ &\quad \frac{1}{2} \left( 1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu}, \end{aligned}$$

imposing conformal symmetry  $g^{ab} = (\phi(x))^{-1} \eta^{ab}$  and let  $F' = \left( 1 - \frac{k\Delta}{2\phi^2} \right)$  leads to

$$\eta^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} + \frac{1}{F'} \eta^{ab} \partial_a F' \partial_b X^\nu G_{\lambda\nu} = \eta^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

assuming flat background  $G_{\mu\nu} = \eta_{\mu\nu}$

$$\eta^{ab}\partial_a\partial_b X^\nu \eta_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F' \partial_b X^\nu \eta_{\lambda\nu} = 0$$

$\downarrow$

$$\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\partial_a \ln(F') \partial_b X^\mu = 0,$$

use plane-wave ansatz  $X^\mu = X_0^\mu e^{-i(E\tau - p\sigma)}$

$$(E^2 - p^2)X^\mu + \partial_\tau \ln(F')iEX^\mu - \partial_\sigma \ln(F')ipX^\mu = 0$$

$\downarrow$

$$(E^2 - p^2)X^\mu + i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F'))X^\mu = 0$$

$\downarrow$

$$i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F')) = -m^2,$$

$$-m^2 = -E^2 + p^2$$

$\Downarrow$

$$\partial_\tau \ln(F') = iE, \quad \partial_\sigma \ln(F') = ip$$

$\Downarrow$

$$\ln(F') = i(E\tau + p\sigma) + a, \quad a \in \mathbb{C}$$

$\Downarrow$

$$F' = A e^{i(E\tau + p\sigma + \phi)}, \quad A > 0$$

$\Downarrow$

$$\phi(\tau, \sigma) = \sqrt{\frac{k\Delta}{2(1 - A e^{i(E\tau + p\sigma + \phi)})}}$$

$\downarrow$

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

## 7 Quantum geometry Polyakov

Since quantum geometry is expected to violate Lorentz invariance at the Planck scale by modifying the dispersion relation as in

$$E^2 = m^2 + p^2 + \sum_{n \geq 3} c_n \frac{p^n}{E_{pl}^{n-2}},$$

where  $E_{pl}$  is the Planck energy, we make a correction to Polyakov action in the form

$$S = -\frac{T}{2} \int d^2x \sqrt{-g} \left( g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 e^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right)$$

where  $T$  is the temperature of target-space,  $\mu_0$  has units of mass (perhaps  $\mu_0 \sim M_{pl}$  or  $\mu_0^2 \sim (\text{quantum of area})^{-1}$ ) and  $m_0 >> m_1 >> \dots >> m_d$ . This action has manifest WS reparameterisation invariance. As for background Poincaré symmetry and WS Weyl invariance, those hold approximately for the fields with  $m_\mu >> T$ . The masses  $m_\mu$  are constrained by the dispersion relations

$$\tilde{E}_\mu^2 = \tilde{p}_\mu^2 + \sum_{n=0}^d m_n^2 \frac{\tilde{p}_\mu^n}{E_{pl}^n}$$

where  $\tilde{E}_\mu^2$  does not mean  $\tilde{E}^\mu \tilde{E}_\mu$ , rather it is  $(\tilde{E}_0)^2, (\tilde{E}_1)^2, \dots, (\tilde{E}_d)^2$ , that is, the squared WS energy (and momentum) of each field  $X^\mu$ .

### 7.1 EoMs

$$\begin{aligned} \frac{\delta S}{\delta g^{cd}} &\propto \frac{\partial \sqrt{-g}}{\partial g^{cd}} \left( g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 e^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right) + \sqrt{-g} \delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = \\ &= -\frac{1}{2} \sqrt{-g} g_{cd} \left( g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 e^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right) + \sqrt{-g} \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = 0 \\ T_{cd} &:= \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} \left( g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 e^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right) = 0 \\ g_{cd} &= 2f \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu}, \\ \frac{1}{f} &= \left( g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \mu_0^2 e^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right). \end{aligned}$$

From this, it is clear that  $e^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} = 0$ , thus the base mass  $\mu_0^2$  is a Lagrange multiplier, the auxiliary metric reduces to the induced metric and the action simplify back to regular NG action.

$$\begin{aligned} \frac{\delta S}{\delta X^\lambda} &\propto -2\sqrt{-g} \mu_0^2 e^{-m_\mu/T} \delta_\lambda^\mu X^\nu \eta_{\mu\nu} - 2\partial_c \left( \sqrt{-g} g^{ab} \delta_a^c \delta_\lambda^b \partial_b X^\nu \eta_{\mu\nu} \right) = \\ &= -2 \left( \partial_a \left( \sqrt{-g} g^{ab} \partial_b X^\nu \eta_{\lambda\nu} \right) + \sqrt{-g} \mu_0^2 e^{-m_\lambda/T} X^\nu \eta_{\lambda\nu} \right) = 0 \\ \partial_a \left( \sqrt{-g} g^{ab} \partial_b X^\mu \right) &+ \sqrt{-g} \mu_0^2 e^{-m_\mu/T} X^\mu = 0 \\ \frac{1}{\sqrt{-g}} \partial_a \left( \sqrt{-g} g^{ab} \partial_b X^\mu \right) &+ \mu_0^2 e^{-m_\mu/T} X^\mu = 0 \\ g^{ab} \nabla_a \nabla_b X^\mu &+ \mu_0^2 e^{-m_\mu/T} X^\mu = 0. \end{aligned}$$

We see that 2d gravitational field of the WS appears naturally in metric-Christoffel symbols formulation. Perhaps THIS is THE true quantum gravitational field since in high enough energies only  $m_0, m_1 >> T$  and thus only  $X^0$  and  $X^1$  remain massless, while the others are massive and thus violate Poincaré invariance. To this end, we may postulate that  $m_2 \sim E_{pl}$ , such that at  $T = T_{pl}$  we have massive  $X^2, X^3, \dots, X^d$  with (approximately) massless  $X^0$  and  $X^1$ . From that and the modified WS dispersion relations for each  $X^\mu$  we can get the other masses.

From the  $X$  EoM before simplifying, we can use conformal gauge  $g_{ab} = e^{\phi(x)} \eta_{ab}$  to get a variable-coefficient KG eqn

$$\begin{aligned} \eta^{ab} \partial_a \partial_b X^\mu + e^{\phi(x)} \mu_0^2 e^{-m_\mu/T} X^\mu &= 0 \\ ((\partial_\sigma)^2 - (\partial_\tau)^2) X^\mu + \mu_0^2 e^{\phi_\mu(x)} X^\mu &= 0, \end{aligned}$$

where  $\phi_\mu(x) \equiv \phi(x) - m_\mu/T$

## 8 Relating NG and Polyakov analysis

From NG analysis

$$(F^+)^2 \frac{\partial^2 X^\mu}{\partial t^2} - \frac{\partial^2 X^\mu}{\partial x^2} = 0,$$

make change of variables  $\tau = \tau(t, x)$  and  $\sigma = \sigma(t, x)$

$$(F^+)^2 \left( \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial t} \frac{\partial}{\partial \sigma} \right) \left( \frac{\partial \tau}{\partial t} \frac{\partial X^\mu}{\partial \tau} + \frac{\partial \sigma}{\partial t} \frac{\partial X^\mu}{\partial \sigma} \right) - \left( \frac{\partial \tau}{\partial x} \frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial x} \frac{\partial}{\partial \sigma} \right) \left( \frac{\partial \tau}{\partial x} \frac{\partial X^\mu}{\partial \tau} + \frac{\partial \sigma}{\partial x} \frac{\partial X^\mu}{\partial \sigma} \right) = 0$$

$$\begin{aligned} (F^+)^2 & \left( \left( \frac{\partial \tau}{\partial t} \right)^2 \frac{\partial^2 X^\mu}{\partial \tau^2} + 2 \frac{\partial \tau}{\partial t} \frac{\partial \sigma}{\partial t} \frac{\partial^2 X^\mu}{\partial \tau \partial \sigma} + \left( \frac{\partial \sigma}{\partial t} \right)^2 \frac{\partial^2 X^\mu}{\partial \sigma^2} \right) - \\ & - \left( \left( \frac{\partial \tau}{\partial x} \right)^2 \frac{\partial^2 X^\mu}{\partial \tau^2} + 2 \frac{\partial \tau}{\partial x} \frac{\partial \sigma}{\partial x} \frac{\partial^2 X^\mu}{\partial \tau \partial \sigma} + \left( \frac{\partial \sigma}{\partial x} \right)^2 \frac{\partial^2 X^\mu}{\partial \sigma^2} \right) + \\ & + \left( (F^+)^2 \frac{\partial^2 \tau}{\partial t^2} - \frac{\partial^2 \tau}{\partial x^2} \right) \frac{\partial X^\mu}{\partial \tau} + \left( (F^+)^2 \frac{\partial^2 \sigma}{\partial t^2} - \frac{\partial^2 \sigma}{\partial x^2} \right) \frac{\partial X^\mu}{\partial \sigma} = 0 \end{aligned}$$

$$\begin{aligned} ((F^+)^2 (\partial_t \tau)^2 - (\partial_x \tau)^2) \partial_\tau^2 X^\mu + ((F^+)^2 (\partial_t \sigma)^2 - (\partial_x \sigma)^2) \partial_\sigma^2 X^\mu + \\ + 2 ((F^+)^2 \partial_t \tau \partial_t \sigma - \partial_x \tau \partial_x \sigma) \partial_\tau \partial_\sigma X^\mu + ((F^+)^2 \partial_t^2 \tau - \partial_x^2 \tau) \partial_\tau X^\mu + ((F^+)^2 \partial_t^2 \sigma - \partial_x^2 \sigma) \partial_\sigma X^\mu = 0 \end{aligned}$$

Without loss of generality, we can set (just rescale/rotate the coordinates)

$$\begin{aligned} (F^+)^2 (\partial_t \sigma)^2 - (\partial_x \sigma)^2 &= (\partial_x \tau)^2 - (F^+)^2 (\partial_t \tau)^2 \\ (F^+)^2 \partial_t \tau \partial_t \sigma &= \partial_x \tau \partial_x \sigma, \end{aligned}$$

or more compactly

$$\begin{aligned} \partial_t \tau &= v \partial_x \sigma \\ \partial_t \sigma &= v \partial_x \tau, \end{aligned}$$

where  $v = 1/F^+$ , which is equivalent to

$$\begin{aligned} \partial_t^2 \tau &= \partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau + v^2 \partial_x^2 \tau \\ \partial_t^2 \sigma &= \partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma + v^2 \partial_x^2 \sigma, \end{aligned}$$

reducing the big equation to

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu + \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\tau X^\mu + \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\sigma X^\mu = 0.$$

Next introduce  $X^\mu = \kappa(\tau, \sigma) Y^\mu(\tau, \sigma)$  s.t terms proportional to  $\partial_\tau Y^\mu$  and  $\partial_\sigma Y^\mu$  vanish:

$$\begin{aligned} \partial_\tau X^\mu &= \partial_\tau \kappa Y^\mu + \kappa \partial_\tau Y^\mu \\ \partial_\sigma X^\mu &= \partial_\sigma \kappa Y^\mu + \kappa \partial_\sigma Y^\mu \\ \partial_\tau^2 X^\mu &= \partial_\tau^2 \kappa Y^\mu + 2 \partial_\tau \kappa \partial_\tau Y^\mu + \kappa \partial_\tau^2 Y^\mu \\ \partial_\sigma^2 X^\mu &= \partial_\sigma^2 \kappa Y^\mu + 2 \partial_\sigma \kappa \partial_\sigma Y^\mu + \kappa \partial_\sigma^2 Y^\mu \end{aligned}$$

$$\begin{aligned} \kappa \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} + 2 \partial_\tau \kappa &\stackrel{!}{=} 0 \\ \kappa \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} - 2 \partial_\sigma \kappa &\stackrel{!}{=} 0 \end{aligned}$$

$$\begin{aligned}\partial_\tau \kappa &= -\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \\ \partial_\sigma \kappa &= \frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)},\end{aligned}$$

undoing chain rule for  $t$  by multiplying first eq by  $\partial_t \tau$  and second by  $\partial_t \sigma$  and summing gives

$$\begin{aligned}\partial_t \kappa &= \partial_t \tau \partial_\tau \kappa + \partial_t \sigma \partial_\sigma \kappa = \\ &= \partial_t \tau \left( -\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) + \partial_t \sigma \left( \frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) = \\ &= v \partial_x \sigma \left( -\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) + v \partial_x \tau \left( \frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) = \\ &= \frac{v \kappa \partial_t v ((\partial_x \tau)^2 - (\partial_x \sigma)^2)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} = \\ &= \frac{v \kappa \partial_t v ((\partial_x \tau)^2 - (\partial_t \tau/v)^2)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} = \\ &= -\frac{\kappa \partial_t v}{2v},\end{aligned}$$

or simply

$$\begin{aligned}\frac{\partial_t \kappa}{\kappa} &= -\frac{1}{2} \frac{\partial_t v}{v} \\ \partial_t \ln(\kappa) &= -\frac{1}{2} \partial_t \ln(v) \\ \ln(\kappa) &= -\frac{1}{2} \ln(v) + f(x).\end{aligned}$$

To determine  $f(x)$ , we undo the chain rule for  $x$  now:

$$\begin{aligned}\partial_x \kappa &= \partial_x \tau \partial_\tau \kappa + \partial_x \sigma \partial_\sigma \kappa = \\ &= \partial_x \tau \left( -\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) + \partial_x \sigma \left( \frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) = \\ &= \frac{\kappa v \partial_x v ((\partial_x \sigma)^2 - (\partial_x \tau)^2)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} = \\ &= \frac{\kappa v \partial_x v ((\partial_t \tau/v)^2 - (\partial_x \tau)^2)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} = \\ &= \frac{\kappa \partial_x v}{2v} \\ \frac{\partial_x \kappa}{\kappa} &= \frac{1}{2} \frac{\partial_x v}{v} \\ \partial_x \ln(\kappa) &= \frac{1}{2} \partial_x \ln(v) \\ \ln(\kappa) &= \frac{1}{2} \ln(v), \\ \partial_x \left( -\frac{1}{2} \ln(v) + f(x) \right) &= \frac{1}{2} \partial_x \ln(v)\end{aligned}$$

$f'(x) = \partial_x \ln(v)$  (perhaps this means somewhere in the composition of functions time dependence is lost?)

$$f(x) = \ln(v)$$

giving us

$$\kappa = v^{\frac{1}{2}}, \quad v = v(x).$$

Substituting this into the earlier equation yields

$$\partial_\tau^2 Y^\mu - \partial_\sigma^2 Y^\mu + m^2 Y^\mu = 0,$$

with

$$\begin{aligned}
m^2 &= \frac{\partial_t^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau) \partial_\tau \kappa + (\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma) \partial_\sigma \kappa}{\kappa ((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} = \\
&= \frac{\partial_t^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{\partial_t v (\partial_x \sigma \partial_\tau \kappa + \partial_x \tau \partial_\sigma \kappa) + v \partial_x v \partial_x \kappa}{\kappa ((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} = \\
&= \frac{\partial_t^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{(\partial_x v)^2}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)},
\end{aligned}$$

where by use of chain rule we have that

$$\partial_t^2 \kappa - v^2 \partial_x^2 \kappa - v \partial_x v \partial_x \kappa = ((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2) (\partial_t^2 \kappa - \partial_\sigma^2 \kappa),$$

where since  $\kappa = v^{\frac{1}{2}}$ ,  $m^2$  reduces to

$$m^2 = \frac{(\partial_x v)^2 - 2v \partial_x^2 v}{4 ((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)},$$

with  $\tau$  satisfying

$$\begin{aligned}
\partial_t \tau &= v \partial_x \sigma \\
v \partial_x \tau &= \partial_t \sigma
\end{aligned}$$

such that  $-m^2$  is indeed constant.

## 9 Solutions for the Polyakov KG eqn (WRONG ONE)

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge  $g_{ab} = \phi \eta_{ab}$  becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta_{cd} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\begin{aligned} \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ \partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0. \end{aligned}$$

These simplify further by expanding the summation on  $a, b$ :

$$\begin{aligned} \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} (-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}) \\ \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}. \end{aligned}$$

The EoM is just a one-dimensional Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left( a^\mu(p) e^{-i(-E\tau+p\sigma)} + b^\mu(p) e^{i(-E\tau+p\sigma)} \right),$$

where  $E(p) = \sqrt{p^2 + m^2}$ . The derivatives are

$$\begin{aligned} \partial_\tau X^\mu &= \frac{i}{4\pi} \int dp \left( a^\mu(p) e^{-i(-E\tau+p\sigma)} - b^\mu(p) e^{i(-E\tau+p\sigma)} \right) \\ \partial_\sigma X^\mu &= \frac{i}{4\pi} \int dp \frac{p}{\sqrt{p^2 + m^2}} \left( -a^\mu(p) e^{-i(-E\tau+p\sigma)} + b^\mu(p) e^{i(-E\tau+p\sigma)} \right). \end{aligned}$$

Starting with the mixed constraint,

$$\begin{aligned} \partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp \left( a^\mu(p) e^{-i(-E(p)\tau+p\sigma)} - b^\mu(p) e^{i(-E(p)\tau+p\sigma)} \right) \times \\ &\quad \times \int dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left( -a^\nu(p') e^{-i(-E(p')\tau+p'\sigma)} + b^\nu(p') e^{i(-E(p')\tau+p'\sigma)} \right) = \\ &= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left( a^\mu(p) e^{-i(-E(p)\tau+p\sigma)} - b^\mu(p) e^{i(-E(p)\tau+p\sigma)} \right) \times \\ &\quad \times \left( -a^\nu(p') e^{-i(-E(p')\tau+p'\sigma)} + b^\nu(p') e^{i(-E(p')\tau+p'\sigma)} \right) = \\ &= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left( -a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} e^{-i(p+p')\sigma} + a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} e^{-i(p-p')\sigma} + \right. \\ &\quad \left. + b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} e^{i(p-p')\sigma} - b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} e^{i(p+p')\sigma} \right) = 0. \end{aligned}$$

If we integrate this expression over  $\sigma$  we obtain

$$\begin{aligned}
& -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left( -a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} \int d\sigma e^{-i(p+p')\sigma} + \right. \\
& \quad + a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} \int d\sigma e^{-i(p-p')\sigma} + b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} \int d\sigma e^{i(p-p')\sigma} - \\
& \quad \left. - b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} \int d\sigma e^{i(p+p')\sigma} \right) = \\
& = -\frac{\eta_{\mu\nu}}{8\pi} \int dp dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left( -a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} \delta(p+p') + a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} \delta(p-p') + \right. \\
& \quad \left. + b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} \delta(p-p') - b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} \delta(p+p') \right) = \\
& = -\frac{\eta_{\mu\nu}}{8\pi} \int dp \frac{p}{\sqrt{p^2 + m^2}} \left( a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} + a^\mu(p) b^\nu(p) + b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) = 0
\end{aligned}$$

which implies the whole expression inside the integral must vanish, thus we have (leaving  $(a^\mu b^\nu + b^\mu a^\nu) \eta_{\mu\nu}$  as is since eventually these will not be numbers and may not commute, thus will not be equal to  $2a^\mu b^\nu \eta_{\mu\nu}$ )

$$(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} + a^\mu(p) b^\nu(p) + b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau}) \eta_{\mu\nu} = 0.$$

As for the other part of the constraint, let's start with the L.H.S

$$\begin{aligned}
\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{(4\pi)^2} \int dp dp' \left( a^\mu(p) e^{-i(-E(p)\tau+p\sigma)} - b^\mu(p) e^{i(-E(p)\tau+p\sigma)} \right) \times \\
&\quad \times \left( a^\nu(p') e^{-i(-E(p')\tau+p'\sigma)} - b^\nu(p') e^{i(-E(p')\tau+p'\sigma)} \right) = \\
&= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \left( a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} e^{-i(p+p')\sigma} - a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} e^{-i(p-p')\sigma} - \right. \\
&\quad \left. - b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} e^{i(p-p')\sigma} + b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} e^{i(p+p')\sigma} \right),
\end{aligned}$$

which we integrate over  $\sigma$  to get

$$\begin{aligned}
& -\frac{\eta_{\mu\nu}}{8\pi} \int dp dp' \left( a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} \delta(p+p') - a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} \delta(p-p') - \right. \\
& \quad \left. - b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} \delta(p-p') + b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} \delta(p+p') \right) = \\
& = -\frac{\eta_{\mu\nu}}{8\pi} \int dp \left( a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} - a^\mu(p) b^\nu(p) - b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right),
\end{aligned}$$

and now for the R.H.S

$$\begin{aligned}
-\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{p}{\sqrt{p^2 + m^2}} \frac{p'}{\sqrt{p'^2 + m^2}} \left( -a^\mu(p) e^{-i(-E\tau+p\sigma)} + b^\mu(p) e^{i(-E\tau+p\sigma)} \right) \times \\
&\quad \times \left( -a^\nu(p') e^{-i(-E(p')\tau+p'\sigma)} + b^\nu(p') e^{i(-E(p')\tau+p'\sigma)} \right) = \\
&= \frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{pp'}{E(p)E(p')} \left( a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} e^{-i(p+p')\sigma} - a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} e^{-i(p-p')\sigma} - \right. \\
&\quad \left. - b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} e^{i(p-p')\sigma} + b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} e^{i(p+p')\sigma} \right),
\end{aligned}$$

which once again we integrate over  $\sigma$ ,

$$\begin{aligned} \frac{\eta_{\mu\nu}}{8\pi} \int dp dp' \frac{pp'}{E(p)E(p')} & \left( a^\mu(p)a^\nu(p') e^{i(E(p)+E(p'))\tau} \delta(p+p') - a^\mu(p)b^\nu(p') e^{i(E(p)-E(p'))\tau} \delta(p-p') - \right. \\ & \left. - b^\mu(p)a^\nu(p') e^{-i(E(p)-E(p'))\tau} \delta(p-p') + b^\mu(p)b^\nu(p) e^{-i(E(p)+E(p'))\tau} \delta(p+p') \right) = \\ & = \frac{1}{8\pi} \int dp \frac{p^2}{p^2+m^2} \left( a^\mu(p)a^\nu(-p) e^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu}, \end{aligned}$$

which when equated to the L.H.S yields

$$\int dp \left( \frac{p^2}{p^2+m^2} + 1 \right) \left( a^\mu(p)a^\nu(-p) e^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu} = 0,$$

which implies

$$\left( a^\mu(p)a^\nu(-p) e^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu} = 0.$$

We can sum and subtract this with the previous condition to simplify and obtain

$$\begin{aligned} \left( a^\mu(p)a^\nu(-p) e^{2iE(p)\tau} + b^\mu(p)b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu} &= 0 \\ \left( a^\mu(p)b^\nu(p) + b^\mu(p)a^\nu(p) \right) \eta_{\mu\nu} &= 0, \end{aligned}$$

and for while  $a^\mu$  and  $b^\nu$  are number-valued, the second condition can be further simplified to

$$a^\mu(p)b^\nu(p)\eta_{\mu\nu} = 0,$$

meaning  $a^\mu$  and  $b^\nu$  are orthogonal. In general, this condition states that  $a^\mu$  and  $b^\nu$  anti-commute w.r.t Lorentz inner product. The first condition must hold for all values of  $\tau$ , and since the exponentials are linearly independent, we thus have

$$\begin{aligned} a^\mu(p)a^\nu(-p)\eta_{\mu\nu} &= 0 \\ b^\mu(p)b^\nu(-p)\eta_{\mu\nu} &= 0, \end{aligned}$$

meaning that reflecting the argument of  $a^\mu$  and  $b^\nu$  creates orthogonal vectors. We can thus write

$$\begin{aligned} a^\mu(p) &= \Lambda^\mu_\nu \left( \frac{p}{4} \right) a^\nu(0) \\ b^\mu(p) &= \Upsilon^\mu_\nu \left( \frac{p}{4} \right) b^\nu(0), \end{aligned}$$

where  $\Lambda, \Upsilon \in \text{SO}^+(1, d)$ . Also. since the orthogonality under reflection must hold for all values of  $p$ , in particular we have

$$\begin{aligned} a^\mu(0)a^\nu(0)\eta_{\mu\nu} &= 0 \\ b^\mu(0)b^\nu(0)\eta_{\mu\nu} &= 0, \end{aligned}$$

meaning that the initial  $a^\mu(0)$  and  $b^\nu(0)$  are null vectors. This means that  $a^\mu(p)$  and  $b^\nu(p)$  are also null vectors since they are related to the initial values by Lorentz transformation. With this, the solution becomes

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left( \Lambda^\mu_\nu \left( \frac{p}{4} \right) a^\nu(0) e^{-i(-E\tau+p\sigma)} + \Upsilon^\mu_\nu \left( \frac{p}{4} \right) b^\nu(0) e^{i(-E\tau+p\sigma)} \right)$$

with  $(a^\mu b^\nu + b^\mu a^\nu)\eta_{\mu\nu} = 0$ . This condition implies

$$\begin{aligned} \left( \Lambda^\mu_\gamma \left( \frac{p}{4} \right) a^\gamma(0) \Upsilon^\nu_\rho \left( \frac{p}{4} \right) b^\rho(0) \right) \eta_{\mu\nu} &= 0 \\ \Upsilon^\mu_\nu \left( \frac{p}{4} \right) &= \Lambda^\mu_\nu \left( \frac{p}{4} \right) = \Lambda^\mu_\nu \left( -\frac{p}{4} \right), \end{aligned}$$

thus turning the solution into

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left( \Lambda_\nu^\mu \left( \frac{p}{4} \right) a^\nu(0) e^{-i(-E\tau+p\sigma)} + \Lambda_\nu^\mu \left( -\frac{p}{4} \right) b^\nu(0) e^{i(-E\tau+p\sigma)} \right).$$

Next, we choose background coordinates s.t  $a^\mu(0) = (a_0, a_0, 0, \dots, 0) = a_0 x^+$  and  $b^\nu(0) = (b_0, b_0, 0, \dots, 0) = b_0 x^+$ , simplifying the solution to

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left( a_0 \Lambda_+^\mu \left( \frac{p}{4} \right) e^{-i(-E\tau+p\sigma)} + b_0 \Lambda_+^\mu \left( -\frac{p}{4} \right) e^{i(-E\tau+p\sigma)} \right) x^+$$

## 10 Solutions for the Polyakov KG eqn

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge  $g_{ab} = \phi \eta_{ab}$  becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta_{cd} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\begin{aligned} \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ \partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0. \end{aligned}$$

These simplify further by expanding the summation on  $a, b$ :

$$\begin{aligned} \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} (-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}) \\ \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= 0. \end{aligned}$$

These can be combined by multiplying the previous one by 2 and summing, leaving us with

$$(\partial_\tau X^\mu + \partial_\sigma X^\mu)(\partial_\tau X^\nu + \partial_\sigma X^\nu) \eta_{\mu\nu} = 0.$$

The EoM is just a one-dimensional finite-space Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left( a_n^\mu e^{i(-E_n \tau + n\sigma)} + b_n^\mu e^{i(E_n \tau + n\sigma)} \right),$$

with  $E_n = \sqrt{n^2 + m^2}$ . The derivatives are

$$\begin{aligned} \partial_\tau X^\mu &= \frac{i}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \left( -a_n^\mu e^{i(-E_n \tau + n\sigma)} + b_n^\mu e^{i(E_n \tau + n\sigma)} \right) \\ \partial_\sigma X^\mu &= \frac{i}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{n}{E_n} \left( a_n^\mu e^{i(-E_n \tau + n\sigma)} + b_n^\mu e^{i(E_n \tau + n\sigma)} \right). \end{aligned}$$

Reality of  $X^\mu$  implies

$$\begin{aligned} (X^\mu)^* &= \left( \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left( a_n^\mu e^{i(-E_n \tau + n\sigma)} + b_n^\mu e^{i(E_n \tau + n\sigma)} \right) \right)^* = \\ &= \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left( (a_n^\mu)^* e^{-i(-E_n \tau + n\sigma)} + (b_n^\mu)^* e^{-i(E_n \tau + n\sigma)} \right) \equiv X^\mu, \end{aligned}$$

giving us

$$\begin{aligned} (a_n^\mu)^* &= b_{-n}^\mu \\ (b_n^\mu)^* &= a_{-n}^\mu, \end{aligned}$$

so the solution becomes for closed strings

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left( a_n^\mu e^{i(-E_n \tau + n\sigma)} + (a_{-n}^\mu)^* e^{i(E_n \tau + n\sigma)} \right).$$

For open strings with free endpoints B.C we have general solution

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} (a_n^\mu e^{-iE_n\tau} + b_n^\mu e^{iE_n\tau}) \cos(n\sigma),$$

which reality condition implies  $b_n^\mu = (a_n^\mu)^*$ , thus open string with free endpoint solution becomes

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} (a_n^\mu e^{-iE_n\tau} + (a_n^\mu)^* e^{iE_n\tau}) \cos(n\sigma).$$

Analysing the constraints, for the closed string we have that

$$\partial_\tau X^\mu + \partial_\sigma X^\mu = \frac{i}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \left( \left( \frac{n}{E_n} - 1 \right) a_n^\mu e^{i(-E_n\tau+n\sigma)} + \left( \frac{n}{E_n} + 1 \right) b_n^\mu e^{i(E_n\tau+n\sigma)} \right),$$

so the constraint reads

$$\begin{aligned} (\partial_\tau X + \partial_\sigma X)^2 &= -\frac{\eta_{\mu\nu}}{(4\pi\alpha')^2} \sum_n \sum_p \left( \left( \frac{n}{E_n} - 1 \right) \left( \frac{p}{E_p} - 1 \right) a_n^\mu a_p^\nu e^{i(-(E_n+E_p)\tau+(n+p)\sigma)} + \right. \\ &\quad + \left( \frac{n}{E_n} - 1 \right) \left( \frac{p}{E_p} + 1 \right) a_n^\mu (a_{-p}^\nu)^* e^{i(-(E_n-E_p)\tau+(n+p)\sigma)} + \left( \frac{n}{E_n} + 1 \right) \left( \frac{p}{E_p} - 1 \right) (a_{-n}^\mu)^* a_p^\nu e^{i((E_n-E_p)\tau+(n+p)\sigma)} + \\ &\quad \left. + \left( \frac{n}{E_n} + 1 \right) \left( \frac{p}{E_p} + 1 \right) (a_{-n}^\mu)^* (a_{-p}^\nu)^* e^{i((E_n+E_p)\tau+(n+p)\sigma)} \right) \equiv 0. \end{aligned}$$

Since the exponentials are all linearly independent and we can't decouple the  $\tau$  exponential from any of the sums, we have 3 cases to investigate:  $n = p$ ,  $n = -p$  and  $n \neq \pm p$ . For  $n \neq \pm p$ , we get

$$\begin{aligned} L_{n,p}^1 &:= \left( \frac{n}{E_n} - 1 \right) \left( \frac{p}{E_p} - 1 \right) a_n^\mu a_p^\nu \eta_{\mu\nu} = 0 \\ L_{n,p}^2 &:= \left( \frac{n}{E_n} - 1 \right) \left( \frac{p}{E_p} + 1 \right) a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0 \\ L_{n,p}^3 &:= \left( \frac{n}{E_n} + 1 \right) \left( \frac{p}{E_p} - 1 \right) (a_{-n}^\mu)^* a_p^\nu \eta_{\mu\nu} = 0 \\ L_{n,p}^4 &:= \left( \frac{n}{E_n} + 1 \right) \left( \frac{p}{E_p} + 1 \right) (a_{-n}^\mu)^* (a_{-p}^\nu)^* \eta_{\mu\nu} = 0, \end{aligned}$$

where  $L_{n,p}^2$  and  $L_{n,p}^3$  are just complex conjugates of each other with  $n \mapsto -n$  and  $p \mapsto -p$ , so they are redundant. The same is true for  $L_{n,p}^1$  and  $L_{n,p}^4$ , so in the end we have

$$\begin{aligned} L_{n,p} &:= \left( \frac{n}{E_n} - 1 \right) \left( \frac{p}{E_p} - 1 \right) a_n^\mu a_p^\nu \eta_{\mu\nu} = 0 \\ \tilde{L}_{n,p} &:= \left( \frac{n}{E_n} - 1 \right) \left( \frac{p}{E_p} + 1 \right) a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0. \end{aligned}$$

The coefficients of both expressions are actually not necessary, since for them to be 0 we would need  $n \rightarrow \pm\infty$  or  $p \rightarrow \pm\infty$ , so we can redefine

$$\begin{aligned} L_{n,p} &:= a_n^\mu a_p^\nu \eta_{\mu\nu} = 0 \\ \tilde{L}_{n,p} &:= a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0, n \neq \pm p. \end{aligned}$$

For  $n = -p$ , the first and last terms are complex conjugates of each other with the relabeling  $n \mapsto -n$  and  $p \mapsto -p$ , while the exponentials in the 2 middle terms are reduced to 1, so we get

$$\begin{aligned} &\left( \frac{n}{E_n} - 1 \right)^2 a_n^\mu a_{-n}^\nu \eta_{\mu\nu} = 0 \\ &\sum_n \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) a_n^\mu (a_n^\nu)^* \eta_{\mu\nu} \right) = 0, \end{aligned}$$

where in the first equation we can ignore the first factor once more. Finally for  $n = p$ , we get

$$\begin{aligned} \left(\frac{n}{E_n} - 1\right)^2 a_n^\mu a_n^\nu \eta_{\mu\nu} &= 0 \\ \left(\left(\frac{n}{E_n}\right)^2 - 1\right) a_n^\mu (a_{-n}^\nu)^* \eta_{\mu\nu} &= 0, \end{aligned}$$

where we again ignore the first factor because it's only 0 at infinity, thus collecting everything we have

$$\begin{aligned} L_{n,p} &:= a_n^\mu a_p^\nu \eta_{\mu\nu} = 0, \quad \forall n, p \in \mathbb{Z}, \\ \tilde{L}_{n,p} &:= a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0, \quad n \neq -p, \\ \sum_n \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) &= 0. \end{aligned}$$

One last thing before quantizing: the  $n = 0$  term in the solution reads

$$\frac{1}{2\pi\alpha'} \frac{1}{2m} (a_0^\mu e^{-im\tau} + (a_0^\mu)^* e^{im\tau}),$$

which is divergent in the  $m \rightarrow 0$  limit unless  $\Re(a_0^\mu) \sim m$ . Keeping the analogy to the  $m \rightarrow 0$  regime, we conclude  $a_0^\mu = 2\pi\alpha' mx^\mu + 4\pi i(\alpha')^2 p^\mu$ . Since  $a_0^\mu$  includes  $p^\mu$ , we have that

$$M^2 = -p^\mu p_\mu = \frac{1}{16\pi^2(\alpha')^4} ((2\pi\alpha')^2 x^\mu x^\nu - a_0^\mu (a_0^\nu)^*) \eta_{\mu\nu},$$

which after analysing closer the  $n = -p$  condition

$$\begin{aligned} \sum_n \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) &= 0 \\ \sum_{n \neq 0} \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) - \tilde{L}_{0,0} &= 0 \\ \sum_{n \neq 0} \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) &= a_0^\mu (a_0^\nu)^* \eta_{\mu\nu} \equiv (4\pi(\alpha')^2)^2 p^\mu p_\mu + (2\pi\alpha' m)^2 x^\mu x_\mu, \end{aligned}$$

we have the string mass in terms of vibrational modes... (and center of mass position?!?)

$$\begin{aligned} M^2 &= \frac{1}{(4\pi(\alpha')^2)^2} \left( (2\pi\alpha' m)^2 x^\mu x_\mu - \sum_{n \neq 0} \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) \right) = \\ &= \frac{1}{(4\pi(\alpha')^2)^2} \left( (2\pi\alpha' m)^2 x^\mu x_\mu - \sum_{n \neq 0} \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) a_n^\mu (a_n^\nu)^* \eta_{\mu\nu} \right) \right). \end{aligned}$$

As for the open string, its derivatives are

$$\begin{aligned} \partial_\tau X^\mu &= \frac{i}{4\pi\alpha'} \sum_n (-a_n^\mu e^{-iE_n\tau} + (a_n^\mu)^* e^{iE_n\tau}) \cos(n\sigma) \\ \partial_\sigma X^\mu &= -\frac{1}{4\pi\alpha'} \sum_n \frac{n}{E_n} (a_n^\mu e^{-iE_n\tau} + (a_n^\mu)^* e^{iE_n\tau}) \sin(n\sigma) \end{aligned}$$

and the constraint reads

$$\partial_\tau X^\mu + \partial_\sigma X^\mu = \frac{1}{4\pi\alpha'} \sum_n \left( -a_n^\mu e^{-iE_n\tau} \left( i \cos(n\sigma) + \frac{n}{E_n} \sin(n\sigma) \right) + (a_n^\mu)^* e^{iE_n\tau} \left( i \cos(n\sigma) - \frac{n}{E_n} \sin(n\sigma) \right) \right)$$

$$\begin{aligned}
(\partial_\tau X + \partial_\sigma X)^2 = & \frac{\eta_{\mu\nu}}{(4\pi\alpha')^2} \sum_n \sum_p \left( a_n^\mu a_p^\nu e^{-i(E_n+E_p)\tau} \left( i \cos(n\sigma) + \frac{n}{E_n} \sin(n\sigma) \right) \left( i \cos(p\sigma) + \frac{p}{E_p} \sin(p\sigma) \right) - \right. \\
& - a_n^\mu (a_p^\nu)^* e^{-i(E_n-E_p)\tau} \left( i \cos(n\sigma) + \frac{n}{E_n} \sin(n\sigma) \right) \left( i \cos(p\sigma) - \frac{p}{E_p} \sin(p\sigma) \right) - \\
& - (a_n^\mu)^* a_p^\nu e^{i(E_n-E_p)\tau} \left( i \cos(n\sigma) - \frac{n}{E_n} \sin(n\sigma) \right) \left( i \cos(p\sigma) + \frac{p}{E_p} \sin(p\sigma) \right) + \\
& \left. + (a_n^\mu)^* (a_p^\nu)^* e^{i(E_n+E_p)\tau} \left( i \cos(n\sigma) - \frac{n}{E_n} \sin(n\sigma) \right) \left( i \cos(p\sigma) - \frac{p}{E_p} \sin(p\sigma) \right) \right) = 0.
\end{aligned}$$

Once more, we see that the middle 2 terms and outer 2 terms are complex conjugate pairs. Unlike the closed string, here all cases lead to

$$\begin{aligned}
L_{n,p} &:= a_n^\mu a_p^\nu \eta_{\mu\nu} = 0 \\
\tilde{L}_{n,p} &:= a_n^\mu (a_p^\nu)^* \eta_{\mu\nu} = 0,
\end{aligned}$$

where again to be more in line with ST literature we remap  $p \mapsto p - n$

$$\begin{aligned}
L_{n,p} &= a_n^\mu a_{p-n}^\nu \eta_{\mu\nu} = 0 \\
\tilde{L}_{n,p} &= a_n^\mu (a_{p-n}^\nu)^* \eta_{\mu\nu} = 0.
\end{aligned}$$

Since we don't have a summed constraint, the mass of the open string can't be written in terms of vibrational modes, only in terms of the 0-th mode

$$\begin{aligned}
M^2 = -p^\mu p_\mu &= \frac{1}{(4\pi(\alpha')^2)^2} a_0^\mu a_0^\nu \eta_{\mu\nu} = \\
&= \frac{1}{(4\pi(\alpha')^2)^2} L_{0,0} \equiv 0
\end{aligned}$$

## 10.1 The Expansion Modes Algebra

### 10.1.1 Closed String

Before starting this analysis, let's rename the vibrational modes to  $\alpha_n^\mu$  to be more in line with the literature. Start by finding the canonical momentum

$$\begin{aligned}
\Pi_\lambda &= \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\lambda)} = -\frac{T}{2} \left( \sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \delta_a^\tau \delta_\lambda^\mu \partial_b X^\nu \eta_{\mu\nu} = \\
&= -\frac{1}{4\pi\alpha'} \left( 1 - \frac{k\Delta}{2\phi^2} \right) \eta^{\tau b} \partial_b X^\nu \eta_{\lambda\nu} = \\
&= \frac{1}{4\pi\alpha'} F^- \partial_\tau X^\nu \eta_{\lambda\nu} = F^- \frac{i}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \left( -\alpha_n^\nu e^{i(-E_n\tau+n\sigma)} + (\alpha_{-n}^\nu)^* e^{i(E_n\tau+n\sigma)} \right) \eta_{\nu\lambda}.
\end{aligned}$$

We then proceed to the canonical Poisson bracket relations

$$\begin{aligned}
\{X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} &= \delta(\sigma - \sigma') \eta^{\mu\nu} \\
\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} &= \{\Pi^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} = 0,
\end{aligned}$$

where by expanding the first relation we get

$$\begin{aligned}
\{X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} &= \left\{ \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left( \alpha_n^\mu e^{i(-E_n\tau+n\sigma)} + (\alpha_{-n}^\mu)^* e^{i(E_n\tau+n\sigma)} \right), \right. \\
&\quad \left. F^- \frac{i}{4\pi\alpha'} \sum_{p \in \mathbb{Z}} \left( -\alpha_p^\nu e^{i(-E_p\tau+p\sigma')} + (\alpha_{-p}^\nu)^* e^{i(E_p\tau+p\sigma')} \right) \right\} =
\end{aligned}$$

$$\begin{aligned}
&= \frac{i}{8\pi^2(\alpha')^2} \sum_n \sum_p \frac{1}{2E_n} \left( -F^- \{ \alpha_n^\mu, \alpha_p^\nu \} e^{-i(E_n+E_p)\tau} e^{i(n\sigma+p\sigma')} + F^- \{ \alpha_n^\mu, (\alpha_{-p}^\nu)^* \} e^{-i(E_n-E_p)\tau} e^{i(n\sigma+p\sigma')} - \right. \\
&\quad \left. - F^- \{ (\alpha_{-n}^\mu)^*, \alpha_p^\nu \} e^{i(E_n-E_p)\tau} e^{i(n\sigma+p\sigma')} + F^- \{ (\alpha_{-n}^\mu)^*, (\alpha_{-p}^\nu)^* \} e^{i(E_n+E_p)\tau} e^{i(n\sigma+p\sigma')} \right) = \delta(\sigma - \sigma') \eta^{\mu\nu} \\
&\quad \downarrow \frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{-ik\sigma} \\
&\frac{i}{8\pi^2(\alpha')^2} \sum_n \sum_p \frac{1}{2E_n} \left( -F^- \{ \alpha_n^\mu, \alpha_p^\nu \} e^{-i(E_n+E_p)\tau} \delta_{n,k} e^{ip\sigma'} + F^- \{ \alpha_n^\mu, (\alpha_{-p}^\nu)^* \} e^{-i(E_n-E_p)\tau} \delta_{n,k} e^{ip\sigma'} - \right. \\
&\quad \left. - F^- \{ (\alpha_{-n}^\mu)^*, \alpha_p^\nu \} e^{i(E_n-E_p)\tau} \delta_{n,k} e^{ip\sigma'} + F^- \{ (\alpha_{-n}^\mu)^*, (\alpha_{-p}^\nu)^* \} e^{i(E_n+E_p)\tau} \delta_{n,k} e^{ip\sigma'} \right) = \frac{\eta^{\mu\nu}}{2\pi} e^{-ik\sigma'} \\
&\quad \downarrow \frac{1}{2\pi} \int_0^{2\pi} d\sigma' e^{im\sigma'} \\
&\frac{1}{2\pi} \int_0^{2\pi} d\sigma' F^- e^{i(p+m)\sigma'}, \quad F^- = 1 - \frac{k\Delta}{2\phi^2}, \text{ assume } \phi = \text{constant} \\
&\frac{i}{8\pi^2(\alpha')^2} \frac{F^-}{2E_k} \left( - \{ \alpha_k^\mu, \alpha_{-m}^\nu \} e^{-i(E_k+E_m)\tau} + \{ \alpha_k^\mu, (\alpha_m^\nu)^* \} e^{-i(E_k-E_m)\tau} - \right. \\
&\quad \left. - \{ (\alpha_{-k}^\mu)^*, \alpha_{-m}^\nu \} e^{i(E_k-E_m)\tau} + \{ (\alpha_{-k}^\mu)^*, (\alpha_m^\nu)^* \} e^{i(E_k+E_m)\tau} \right) = \frac{\eta^{\mu\nu}}{2\pi} \delta_{m,k}.
\end{aligned}$$

The only way for this to hold is

$$\begin{aligned}
\{ \alpha_k^\mu, \alpha_{-m}^\nu \} &= \{ (\alpha_{-k}^\mu)^*, (\alpha_m^\nu)^* \} = 0 \\
\{ \alpha_k^\mu, (\alpha_m^\nu)^* \} &= - \{ (\alpha_{-k}^\mu)^*, \alpha_{-m}^\nu \} = -i \frac{4\pi(\alpha')^2}{F^-} E_k \eta^{\mu\nu} \delta_{k,m},
\end{aligned}$$

thus rescaling we get harmonic oscillator bracket relations

$$\begin{aligned}
a_k^\mu &:= \frac{1}{2\alpha'} \sqrt{\frac{F^-}{\pi}} \frac{1}{\sqrt{E_k}} \alpha_k^\mu \\
(a_k^\mu)^* &:= \frac{1}{2\alpha'} \sqrt{\frac{F^-}{\pi}} \frac{1}{\sqrt{E_k}} (\alpha_k^\mu)^* \\
&\Downarrow
\end{aligned}$$

$$\begin{aligned}
\{ a_k^\mu, (a_p^\nu)^* \} &= -i \eta^{\mu\nu} \delta_{k,p} \\
\{ a_k^\mu, a_p^\nu \} &= \{ (a_k^\mu)^*, (a_p^\nu)^* \} = 0.
\end{aligned}$$

For the  $k = p = 0$  case, we also get the expected relations for  $x^\mu$  and  $p^\nu$ :

$$\begin{aligned}
\{ \alpha_0^\mu, (\alpha_0^\nu)^* \} &= -i \frac{4\pi(\alpha')^2}{F^-} E_0 \eta^{\mu\nu} \\
\{ 2\pi\alpha' \mu x^\mu + 4\pi i(\alpha')^2 p^\mu, 2\pi\alpha' \mu x^\nu - 4\pi i(\alpha')^2 p^\nu \} &= -i \frac{4\pi(\alpha')^2}{F^-} \mu \eta^{\mu\nu}
\end{aligned}$$

$$(2\pi\alpha')^2 \left( \mu^2 \{x^\mu, x^\nu\} - 2i\alpha'\mu \{x^\mu, p^\nu\} + 2i\alpha'\mu \{p^\mu, x^\nu\} + (2\alpha') \{p^\mu, p^\nu\} \right) = -i \frac{4\pi(\alpha')^2}{F^-} \mu \eta^{\mu\nu}$$

⇓

$$\begin{aligned} \{x^\mu, x^\nu\} &= \{p^\mu, p^\nu\} = 0 \\ \{x^\mu, p^\nu\} &= \frac{1}{4\pi\alpha' F^-} \eta^{\mu\nu}. \end{aligned}$$

Since  $F^- = 1 - k\Delta/(2\phi^2)$ , if we choose

$$\phi = \sqrt{\frac{k\Delta}{2\left(1 - \frac{1}{4\pi\alpha'}\right)}}$$

then the bracket relation simplifies to the expected

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu}.$$

## 10.2 The Constraint Algebra

With the bracket relations between the vibrational modes in hand, we can now calculate the algebra generated by the constraints. For simplicity, since  $E_n = \sqrt{n^2 + \mu^2} \neq 0$ , we can switch the constraints to be between the normalized harmonic oscillator modes  $a_n^\mu$  instead of the  $\alpha_n^\mu$ .

$$\{L_{n,m}, L_{k,p}\} = \left\{ a_n^\mu a_m^\nu \eta_{\mu\nu}, a_k^{\mu'} a_p^{\nu'} \eta_{\mu'\nu'} \right\} = 0$$

$$\begin{aligned} \{L_{n,m}, \tilde{L}_{k,p}\} &= \left\{ a_n^\mu a_m^\nu \eta_{\mu\nu}, a_k^{\mu'} (a_{-p}^{\nu'})^* \eta_{\mu'\nu'} \right\} = \\ &= \eta_{\mu\nu} \left( a_m^\nu \left\{ a_n^\mu, (a_{-p}^{\nu'})^* \right\} a_k^{\mu'} + a_n^\mu \left\{ a_m^\nu, (a_{-p}^{\nu'})^* \right\} a_k^{\mu'} \right) \eta_{\mu'\nu'} = \\ &= \eta_{\mu\nu} \left( a_m^\nu (-i\eta^{\mu\nu'} \delta_{n,-p}) a_k^{\mu'} + a_n^\mu (-i\eta^{\nu\nu'} \delta_{m,-p}) a_k^{\mu'} \right) \eta_{\mu'\nu'} = \\ &= -i (a_m^\mu a_k^\nu \eta_{\mu\nu} \delta_{n,-p} + a_n^\mu a_k^\nu \eta_{\mu\nu} \delta_{m,-p}) = \\ &= -i (L_{m,k} \delta_{n,-p} + L_{n,k} \delta_{m,-p}) \end{aligned}$$

$$\begin{aligned} \{\tilde{L}_{n,m}, \tilde{L}_{k,p}\} &= \left\{ a_n^\mu (a_{-m})^* \eta_{\mu\nu}, a_k^{\mu'} (a_{-p}^{\nu'})^* \eta_{\mu'\nu'} \right\} = \\ &= \eta_{\mu\nu} \left( (a_{-m}^\nu)^* \left\{ a_n^\mu, (a_{-p}^{\nu'})^* \right\} a_k^{\mu'} + a_n^\mu \left\{ (a_{-m}^\nu)^*, a_k^{\mu'} \right\} (a_{-p}^{\nu'})^* \right) \eta_{\mu'\nu'} = \\ &= \eta_{\mu\nu} \left( (a_{-m}^\nu)^* (-i\eta^{\mu\nu'} \delta_{n,-p}) a_k^{\mu'} - a_n^\mu (-i\eta^{\nu\nu'} \delta_{-m,k}) (a_{-p}^{\nu'})^* \right) \eta_{\mu'\nu'} = \\ &= -i (a_k^\mu (a_{-m}^\nu)^* \eta_{\mu\nu} \delta_{n,-p} - a_n^\mu (a_{-p}^{\nu'})^* \eta_{\mu\nu} \delta_{-m,k}) = \\ &= -i (\tilde{L}_{k,m} \delta_{n,-p} - \tilde{L}_{n,p} \delta_{-m,k}). \end{aligned}$$

## 11 Into the Quantum Realm

We now proceed to quantize the KG string through canonical quantization. The string  $X^\mu$  and it's momentum  $\Pi^\nu$  are promoted to operators  $\hat{X}^\mu$  and  $\hat{\Pi}^\nu$ , and we turn the Poisson bracket  $\{.,.\}$  into commutator between operators  $i\hbar[.,.]$ . The equal time bracket relations turn into equal time commutation relations

$$\begin{aligned}\left[\hat{X}^\mu(\tau, \sigma), \hat{\Pi}^\nu(\tau, \sigma')\right] &= i\delta(\sigma - \sigma')\eta^{\mu\nu} \\ \left[\hat{X}^\mu, \hat{X}^\nu\right] &= \left[\hat{\Pi}^\mu, \hat{\Pi}^\nu\right] = 0 \\ \left[\hat{a}_n^\mu, \hat{a}_m^\nu\right] &= \left[(\hat{a}_n^\mu)^\dagger, (\hat{a}_m^\nu)^\dagger\right] = 0 \\ \left[\hat{a}_n^\mu, (\hat{a}_m^\nu)^\dagger\right] &= \eta^{\mu\nu}\delta_{n,m} \\ \left[\hat{x}^\mu, \hat{x}^\nu\right] &= [\hat{p}^\mu, \hat{p}^\nu] = 0 \\ \left[\hat{x}^\mu, \hat{p}^\nu\right] &= i\eta^{\mu\nu}.\end{aligned}$$

Thus, the rescaled vibrational mode operators  $\hat{a}_n^\mu$  and  $(\hat{a}_m^\nu)^\dagger$  are annihilation and creation operators, respectively.

We define a vacuum state of the string to obey

$$\hat{a}_n^\mu |0\rangle = 0, \text{ for } n \neq 0.$$

For  $n = 0$ , we have the center of mass position and momentum operators, so the vacuum also obeys

$$\begin{aligned}\hat{x}^\mu |0; x\rangle &= x^\mu |0; x\rangle \\ \hat{p}_\mu |0; x\rangle &= -i\frac{\partial}{\partial x^\mu} |0; x\rangle\end{aligned}$$

in position representation or alternatively

$$\begin{aligned}\hat{x}^\mu |0; p\rangle &= -i\frac{\partial}{\partial p_\mu} |0; p\rangle \\ \hat{p}_\mu |0; p\rangle &= p_\mu |0; p\rangle\end{aligned}$$

in momentum representation.

A generic state arises from a sequence of creation operators on the vacuum

$$((a_1^{\mu_1})^\dagger)^{n_{\mu_1}} ((a_2^{\mu_2})^\dagger)^{n_{\mu_2}} \dots ((a_{-1}^{\nu_1})^\dagger)^{n_{\nu_1}} ((a_{-2}^{\mu_2})^\dagger)^{n_{\nu_2}} \dots |0\rangle.$$

This (should) give rise to particles.

As in regular ST, we have ghosts arising from the Minkowski metric

$$\left[\hat{a}_n^\mu, (\hat{a}_m^\nu)^\dagger\right] = \eta^{\mu\nu}\delta_{nm}.$$

For constraints, classically we have

$$\begin{aligned}L_{n,p} &= a_n^\mu a_p^\nu \eta_{\mu\nu} = 0 \\ \tilde{L}_{n,p} &= a_n^\mu (a_{-p}^\nu)^* \eta_{\mu\nu} = 0\end{aligned}$$

which because of the existence of ghosts we require to have vanishing matrix elements when sandwiched between physical states

$$\langle \text{phys}' | \hat{L}_{n,p} | \text{phys} \rangle = 0 = \langle \text{phys}' | \hat{\tilde{L}}_{n,p} | \text{phys} \rangle.$$

When translating the constraints into quantum operators, in  $L_{n,p}$  we have no ambiguity of ordering since  $\hat{a}$  commutes with itself. As for  $\tilde{L}_{n,p}$ , there is an ambiguity for  $p = -n \neq 0$ , so we pick normal ordering with the annihilation operators moved to the right

$$\hat{\tilde{L}}_{n,-n} = (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu}.$$

The ambiguity manifests in the imposition of this constraint as

$$\langle \text{phys}' | \left( \sum_n \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \hat{L}_{n,-n} \right) - c \right) | \text{phys} \rangle = 0,$$

for some constant  $c$ . Since classically

$$\begin{aligned} M^2 &= \frac{1}{(4\pi(\alpha')^2)^2} \left( (2\pi\alpha'\mu)^2 x^\mu x_\mu - \sum_{n \neq 0} \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \tilde{L}_{n,-n} \right) \right) = \\ &= \frac{1}{(4\pi(\alpha')^2)^2} \left( (2\pi\alpha'\mu)^2 x^\mu x_\mu - \sum_{n \neq 0} \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) a_n^\mu (a_n^\nu)^* \eta_{\mu\nu} \right) \right). \end{aligned}$$

we see that the string mass spectrum will be affected by this constant

$$\hat{M}^2 = \frac{1}{(4\pi(\alpha')^2)^2} \left( (2\pi\alpha'\mu)^2 \hat{x}^\mu \hat{x}_\mu - \left( \sum_{n \neq 0} \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu} - c \right) \right)$$

The commutation relations between the  $L$ 's are inherited from the bracket relations

$$\begin{aligned} [\hat{L}_{n,m}, \hat{L}_{k,p}] &= 0 \\ [\hat{L}_{n,m}, \hat{\tilde{L}}_{k,p}] &= \hat{L}_{m,k} \delta_{n,-p} + \hat{L}_{n,k} \delta_{m,-p} \\ [\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}] &= \hat{\tilde{L}}_{k,m} \delta_{n,-p} - \hat{\tilde{L}}_{n,p} \delta_{-m,k}. \end{aligned}$$

Since  $[\hat{L}_{n,m}, \hat{\tilde{L}}_{p,k}]$  only has annihilation operators, it has no ordering ambiguities.  $[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{p,k}]$  have ordering ambiguities for  $m = -k$  and/or  $p = -n$ , so we add the anomalous terms

$$[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}] = \hat{\tilde{L}}_{k,m} \delta_{n,-p} - \hat{\tilde{L}}_{n,p} \delta_{-m,k} + C_n \delta_{n,-p} + D_k \delta_{-m,k}$$

Clearly,  $C_0 = 0 = D_0$ , since in these cases the equivalent terms are composed by  $\hat{L}_{0,0} \sim \hat{x}^\mu \hat{x}_\mu + \hat{p}^\mu \hat{p}_\mu$ . Also, because of the  $\delta$ , we also have that  $C_{-n} = -C_n$  and  $D_{-k} = -D_k$ . We get from the jacobi identity

$$[\hat{\tilde{L}}_{n,m}, [\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{r,s}]] + [\hat{\tilde{L}}_{r,s}, [\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}]] + [\hat{\tilde{L}}_{k,p}, [\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{n,m}]] = 0$$

$$\begin{aligned} &[\hat{\tilde{L}}_{n,m}, (\hat{\tilde{L}}_{r,p} \delta_{k,-s} - \hat{\tilde{L}}_{k,s} \delta_{-p,r} + C_k \delta_{k,-s} + D_r \delta_{-p,r})] + \\ &+ [\hat{\tilde{L}}_{r,s}, (\hat{\tilde{L}}_{k,m} \delta_{n,-p} - \hat{\tilde{L}}_{n,p} \delta_{-m,k} + C_n \delta_{n,-p} + D_k \delta_{-m,k})] + \\ &+ [\hat{\tilde{L}}_{k,p}, (\hat{\tilde{L}}_{n,s} \delta_{r,-m} - \hat{\tilde{L}}_{r,m} \delta_{-s,n} + C_r \delta_{r,-m} + D_n \delta_{-s,n})] = 0 \end{aligned}$$

$$\begin{aligned} &[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{r,p}] \delta_{k,-s} - [\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,s}] \delta_{-p,r} + [\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{k,m}] \delta_{n,-p} - \\ &- [\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{n,p}] \delta_{-m,k} + [\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{n,s}] \delta_{r,-m} - [\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{r,m}] \delta_{-s,n} = 0 \end{aligned}$$

$$\begin{aligned} &(\hat{\tilde{L}}_{r,m} \delta_{n,-p} - \hat{\tilde{L}}_{n,p} \delta_{-m,r} + C_n \delta_{n,-p} + D_r \delta_{-m,r}) \delta_{k,-s} - (\hat{\tilde{L}}_{k,m} \delta_{n,-s} - \hat{\tilde{L}}_{n,s} \delta_{-m,k} + C_n \delta_{n,-s} + D_k \delta_{-m,k}) \delta_{-p,r} + \\ &+ (\hat{\tilde{L}}_{k,s} \delta_{r,-m} - \hat{\tilde{L}}_{r,m} \delta_{-s,k} + C_r \delta_{r,-m} + D_k \delta_{-s,k}) \delta_{n,-p} - (\hat{\tilde{L}}_{n,s} \delta_{r,-p} - \hat{\tilde{L}}_{r,p} \delta_{-s,n} + C_r \delta_{r,-p} + D_n \delta_{-s,n}) \delta_{-m,k} + \\ &+ (\hat{\tilde{L}}_{n,p} \delta_{k,-s} - \hat{\tilde{L}}_{k,s} \delta_{-p,n} + C_k \delta_{k,-s} + D_n \delta_{-p,n}) \delta_{r,-m} - (\hat{\tilde{L}}_{r,p} \delta_{k,-m} - \hat{\tilde{L}}_{k,m} \delta_{-p,r} + C_k \delta_{k,-m} + D_r \delta_{-p,r}) \delta_{-s,n} = 0. \end{aligned}$$

For  $k = -s$ ,  $n = -p$  and  $r = -m$

$$\begin{aligned} &(\hat{\tilde{L}}_{r,-r} - \hat{\tilde{L}}_{n,-n} + C_n + D_r) - (\hat{\tilde{L}}_{k,-r} \delta_{p,s} - \hat{\tilde{L}}_{n,-k} \delta_{r,k} + C_n \delta_{n,k} + D_k \delta_{r,k}) \delta_{n,m} + \\ &+ (\hat{\tilde{L}}_{k,-k} - \hat{\tilde{L}}_{r,-r} + C_r + D_k) - (\hat{\tilde{L}}_{n,-k} \delta_{r,n} - \hat{\tilde{L}}_{r,-n} \delta_{k,n} + C_r \delta_{r,n} + D_n \delta_{k,n}) \delta_{r,k} + \\ &+ (\hat{\tilde{L}}_{n,-n} - \hat{\tilde{L}}_{k,-k} + C_k + D_n) - (\hat{\tilde{L}}_{r,-n} \delta_{k,r} - \hat{\tilde{L}}_{k,-r} \delta_{n,r} + C_k \delta_{k,r} + D_r \delta_{n,r}) \delta_{k,n} = 0. \end{aligned}$$

Assuming  $n \neq m$ ,  $r \neq k$  and  $k \neq n$  leaves us with

$$C_n + C_r + C_k + D_r + D_k + D_n = 0,$$

which for  $n + k + r = 0$  becomes

$$C_n + C_k + C_{-n-k} + D_n + D_k + D_{-n-k} = 0$$

$$C_n + C_k - C_{n+k} + D_n + D_k - D_{n+k} = 0,$$

and now setting  $k = 1$  gives

$$C_n - C_{n+1} + C_1 + D_n - D_{n+1} + D_1 = 0$$

$$C_{n+1} - C_n - C_1 = -(D_{n+1} - D_n - D_1),$$

meaning  $C_n$  and  $D_n$  are negatives of each other differing by some constant  $\beta$ , so it suffices to find  $C_n$  and later determine the constant  $\beta$ . It is not difficult to see that the solution to this difference eqn is  $C_n = An$ , so  $D_n = -An + \beta$ . Since we know  $D_{-n} = -D_n$ , then  $\beta = 0$ , and  $A = C_1$ . We now need to determine the value of  $C_1$ . By now, the commutator of  $\hat{\tilde{L}}$  is

$$[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}] = \hat{\tilde{L}}_{k,m}\delta_{n,-p} - \hat{\tilde{L}}_{n,p}\delta_{-m,k} + C_1 n\delta_{n,-p} - C_1 k\delta_{-m,k}.$$

Let's calculate the VEV of this commutator for  $m = k \neq 0$  with  $n = -p = 1$

$$\begin{aligned} \langle 0 | [\hat{\tilde{L}}_{1,k}, \hat{\tilde{L}}_{k,-1}] | 0 \rangle &= \langle 0 | \hat{\tilde{L}}_{1,k} \hat{\tilde{L}}_{k,-1} | 0 \rangle \\ \langle 0 | \hat{\tilde{L}}_{k,k} + C_1 | 0 \rangle &= \langle 0 | (a_{-k}^\mu)^\dagger a_1^\nu \eta_{\mu\nu} (a_1^{\mu'})^\dagger a_k^{\nu'} \eta_{\mu'\nu'} | 0 \rangle \\ \langle 0 | C_1 | 0 \rangle &= 0 \iff C_1 = 0, \end{aligned}$$

so in actuality the constraint algebra has no anomalies.

Let us now denote the ground state of momentum  $p^\mu$  as  $|0;p\rangle$ . The mass-shell condition

$$\hat{M}^2 = \frac{1}{(4\pi(\alpha')^2)^2} \left( (2\pi\alpha'\mu)^2 \hat{x}^\mu \hat{x}_\mu - \left( \sum_{n \neq 0} \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu} - c \right) \right)$$

implies that

$$\begin{aligned} \langle 0; p | \hat{M}^2 | 0; p \rangle &= \frac{1}{(4\pi(\alpha')^2)^2} \left( (2\pi\alpha'\mu)^2 \langle 0; p | \hat{x}^\mu \hat{x}_\mu | 0; p \rangle - \left( \sum_{n \neq 0} \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \langle 0; p | (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu} | 0; p \rangle - \langle 0; p | c | 0; p \rangle \right) \right) \\ M^2 &= \frac{1}{(4\pi(\alpha')^2)^2} \left( (2\pi\alpha'\mu)^2 \frac{\partial \psi^*(p)}{\partial p_\mu} \frac{\partial \psi(p)}{\partial p^\mu} + c \right) \\ (4\pi(\alpha')^2)^2 p^2 &= - \left( (2\pi\alpha'\mu)^2 \left| \frac{\partial \psi(p)}{\partial p} \right|^2 + c \right) \end{aligned}$$

Now looking at the first excited state  $\zeta_\mu(\hat{a}_1^\mu)^\dagger |0; p\rangle$  with  $\zeta_\mu = \zeta_\mu(p)$  being the polarization covector, the mass-shell now reads

$$\begin{aligned} \langle 0; p | \hat{a}_1^\mu \zeta_\mu \hat{M}^2 \zeta_\nu(\hat{a}_1^\nu)^\dagger | 0; p \rangle &= \frac{1}{(4\pi(\alpha')^2)^2} \left( (2\pi\alpha'\mu)^2 \langle 0; p | \hat{a}_1^\mu \zeta_\mu \hat{x}^{\mu'} \hat{x}_{\mu'} \zeta_\nu(\hat{a}_1^\nu)^\dagger | 0; p \rangle - \right. \\ &\quad \left. - \left( \sum_{n \neq 0} \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \langle 0; p | \hat{a}_1^\mu \zeta_\mu (\hat{a}_n^{\mu'})^\dagger \hat{a}_n^{\nu'} \eta_{\mu'\nu'} \zeta_\nu(\hat{a}_1^\nu)^\dagger | 0; p \rangle - \langle 0; p | \hat{a}_1^\mu \zeta_\mu c \zeta_\nu(\hat{a}_1^\nu)^\dagger | 0; p \rangle \right) \right) \\ M^2 \zeta^\mu \zeta_\mu &= \frac{1}{(4\pi(\alpha')^2)^2} \left( (2\pi\alpha'\mu)^2 \left| \frac{\partial \psi(p)}{\partial p} \right|^2 \zeta^\mu \zeta_\mu - \left( \frac{1}{1 + \mu^2} - 1 \right) \zeta^\mu \zeta_\mu + c \zeta^\mu \zeta_\mu \right) \end{aligned}$$

$$(4\pi(\alpha')^2)^2 p^2 = - \left( (2\pi\alpha'\mu)^2 \left| \frac{\partial\psi(p)}{\partial p} \right|^2 - \left( \frac{1}{1+\mu^2} - 1 \right) + c \right).$$

The auxiliary  $\hat{L}_{1,0}\zeta_{\mu'}(\hat{a}_1^{\mu'})^\dagger|0;p\rangle = 0$  condition implies

$$\begin{aligned} \hat{L}_{1,0}\zeta_{\mu'}(\hat{a}_1^{\mu'})^\dagger|0;p\rangle &= 0 \\ (\hat{a}_0^\mu)^\dagger \hat{a}_1^\nu \eta_{\mu\nu} \zeta_{\mu'}(\hat{a}_1^{\mu'})^\dagger|0;p\rangle &= 0 \\ \zeta_\mu(\hat{a}_0^\mu)^\dagger|0;p\rangle &= 0 \\ \zeta_\mu (2\pi\alpha'\mu\hat{x}^\mu - 4\pi(\alpha')^2 i\hat{p}^\mu)|0;p\rangle &= 0 \\ -2\pi\alpha'\mu i\zeta_\mu \frac{\partial\psi(p)}{\partial p_\mu} &= 4\pi(\alpha')^2 i\zeta_\mu p^\mu \\ \zeta_\mu p^\mu &= -\frac{\mu}{2\alpha'} \zeta_\mu \frac{\partial\psi(p)}{\partial p_\mu}. \end{aligned}$$

If the momentum wave function is chosen to be (need to justify this)

$$\psi(p) = \frac{\alpha'}{\mu} p^2,$$

we get that  $\zeta_\mu p^\mu = 0$ . Using this in the mass-shell condition for the first excited state yields

$$\begin{aligned} (4\pi(\alpha')^2)^2 p^2 &= - \left( (2\pi\alpha'\mu)^2 4 \frac{(\alpha')^2}{\mu^2} p^2 - \left( \frac{1}{1+\mu^2} - 1 \right) + c \right) \\ 2(4\pi(\alpha')^2)^2 p^2 &= \frac{1}{1+\mu^2} - 1 - c, \end{aligned}$$

and for the ground state

$$2(4\pi(\alpha')^2)^2 p^2 = -c.$$

In general, the  $n$ -th excited state will have momentum given by

$$2(4\pi(\alpha')^2)^2 p^2 = \frac{n^2}{n^2 + \mu^2} - 1 - c.$$

The norm of the states is given by  $\zeta_\mu \zeta^\mu$ , thus if we choose  $p$  to lie on the  $(0, 1)$ -plane, then the  $D - 2$  spacelike polarizations normal to that plane clearly have positive norm. If we choose  $c$  such that the  $n = d$ ,  $d = (D - 1)$  (we choose this because the solution only has  $n = 0, 1, 2, \dots, d$  terms by the  $\tilde{L}$  constraints) excited state is a tachyon with  $p^2 > 0$ , then  $p$  can be taken to have no time component and thus  $\zeta$  is time-like with negative norm. If  $p^2 < 0$ ,  $p$  can be chosen to ONLY have a time component, leaving  $\zeta$  spacelike with positive norm again. At last, if  $p^2 = 0$ , then  $\zeta$  is proportional to  $p$  with zero norm. Thus, for the absence of ghosts we require

$$c \geq \frac{d^2}{d^2 + \mu^2} - 1.$$

In the boundary case, the 3rd excited vector state is massless and the rest (including the scalar ground state) are tachyons... This analysis, however, is to get spin 1 particles from the closed string. The actual massless particle we want from closed strings is a symmetric spin 2 particle, so let's consider now the state  $\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle$  where  $\zeta_{\mu\nu} = \zeta_{\nu\mu}$  is the polarization tensor. The mass-shell is

$$\begin{aligned} \langle 0; p | \hat{a}_d^\alpha \hat{a}_d^\beta \zeta_{\alpha\beta} \hat{M}^2 \zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger | 0; p \rangle &= \frac{1}{(4\pi(\alpha')^2)^2} \left( (2\pi\alpha'\mu)^2 \langle 0; p | \hat{a}_d^\alpha \hat{a}_d^\beta \zeta_{\alpha\beta} \hat{x}^{\mu'} \hat{x}_{\mu'} \zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger | 0; p \rangle - \right. \\ &\quad \left. - \left( \sum_{n \neq 0} \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \langle 0; p | \hat{a}_d^\alpha \hat{a}_d^\beta \zeta_{\alpha\beta}(\hat{a}_n^{\mu'})^\dagger \hat{a}_n^{\nu'} \eta_{\mu'\nu'} \zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger | 0; p \rangle - \langle 0; p | \hat{a}_d^\alpha \hat{a}_d^\beta \zeta_{\alpha\beta} c \zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger | 0; p \rangle \right) \right) \end{aligned}$$

$$\begin{aligned}
M^2 \zeta_{\mu\nu} \zeta^{\mu\nu} &= \frac{1}{(4\pi(\alpha')^2)^2} \left( (2\pi\alpha' \mu)^2 \left| \frac{\partial \psi(p)}{\partial p} \right|^2 \zeta_{\mu\nu} \zeta^{\mu\nu} - \left( 2 \left( \frac{d^2}{d^2 + \mu^2} - 1 \right) \zeta_{\mu\nu} \zeta^{\mu\nu} - c \zeta_{\mu\nu} \zeta^{\mu\nu} \right) \right) \\
(4\pi(\alpha')^2)^2 p^2 &= - \left( (2\pi\alpha' \mu)^2 \left| \frac{\partial \psi(p)}{\partial p} \right|^2 - \left( 2 \left( \frac{d^2}{d^2 + \mu^2} - 1 \right) - c \right) \right) \\
2(4\pi(\alpha')^2)^2 p^2 &= 2 \left( \frac{d^2}{d^2 + \mu^2} - 1 \right) - c.
\end{aligned}$$

Using the auxiliary  $\hat{L}_{d,0} \zeta_{\mu\nu} (\hat{a}_d^\mu)^\dagger (\hat{a}_d^\nu)^\dagger |0; p\rangle = 0$  condition implies

$$\begin{aligned}
\hat{L}_{d,0} \zeta_{\mu\nu} (\hat{a}_d^\mu)^\dagger (\hat{a}_d^\nu)^\dagger |0; p\rangle &= 0 \\
(\hat{a}_0^{\mu'})^\dagger \hat{a}_d^{\nu'} \eta_{\mu'\nu'} \zeta_{\mu\nu} (\hat{a}_d^\mu)^\dagger (\hat{a}_d^\nu)^\dagger |0; p\rangle &= 0 \\
\left( (\hat{a}_0^{\mu'})^\dagger \eta_{\mu'\nu'} \zeta_{\mu\nu} \eta^{\nu'\mu} (a_d^\nu)^\dagger + (\hat{a}_0^{\mu'})^\dagger \eta_{\mu'\nu'} \zeta_{\mu\nu} (\hat{a}_d^\mu)^\dagger \hat{a}_d^{\nu'} (\hat{a}_d^\nu)^\dagger \right) |0; p\rangle &= 0 \\
\left( (\hat{a}_0^{\mu'})^\dagger \eta_{\mu'\nu'} \zeta_{\mu\nu} \eta^{\nu'\mu} (a_d^\nu)^\dagger + (\hat{a}_0^{\mu'})^\dagger \eta_{\mu'\nu'} \zeta_{\mu\nu} (\hat{a}_d^\mu)^\dagger \eta^{\nu'\nu} \right) |0; p\rangle &= 0 \\
2\zeta_{\mu\nu} (\hat{a}_0^\mu)^\dagger (\hat{a}_d^\nu)^\dagger |0; p\rangle &= 0 \\
\zeta_{\mu\nu} (\hat{a}_d^\nu)^\dagger (2\pi\alpha' \mu \hat{x}^\mu - 4\pi(\alpha')^2 i \hat{p}^\mu) |0; p\rangle &= 0 \\
\zeta_{\mu\nu} \left( -2\pi\alpha' \mu i \frac{\partial \psi(p)}{\partial p_\mu} - 4\pi(\alpha')^2 i p^\mu \right) (\hat{a}_d^\nu)^\dagger |0\rangle &= 0 \\
\zeta_{\mu\nu} p^\mu (\hat{a}_d^\nu)^\dagger |0\rangle &= 0,
\end{aligned}$$

implying that

$$\zeta_{\mu\nu} p^\mu = \zeta_{\mu\nu} p^\nu = 0.$$

Since the norm of the spin-2 states is given by  $\zeta^{\mu\nu} \zeta_{\mu\nu}$ , we need those to not be negative. Since  $\zeta$  is symmetric and we work in  $(-, +, +, \dots)$  signature,  $\zeta^{\mu\nu} \zeta_{\mu\nu} \geq 0$  always hold, so we just need to choose  $c$  such that we get at least 1 massless spin-2 state. For that, we see that

$$c = 2 \left( \frac{n^2}{n^2 + \mu^2} - 1 \right)$$

does the trick for the  $n$ -th state, and since for the non-existence of ghosts in the spin-1 states we required

$$c \geq \frac{d^2}{d^2 + \mu^2} - 1,$$

which is negative, we can thus choose

$$c \equiv \frac{2d^2}{d^2 + \mu^2} - 2,$$

leaving the  $d$ -th spin-2 state massless, the rest being tachyons... This, however, is if we impose that one spin-2 state have to be massless. Since granular space-time affects wave propagation speed, in the effective limit gravitons might be effectively massive, thus we can get rid of the tachyons by letting  $c > 0$  if needed.

The fact that  $\zeta_{\mu\nu} = \zeta_{\nu\mu}$  leaves  $\zeta_{\mu\nu}$  with  $D(D+1)/2$  independent components. The traceless condition  $\eta^{\mu\nu} \zeta_{\mu\nu} = 0$  takes out 1 from that, leaving  $(D(D+1)/2) - 1$  allowed polarizations. Finally, the condition  $\zeta_{\mu\nu} p^\mu = 0$  takes  $D$  out of those, leaving  $(D(D-1)/2) - 1$  allowed polarizations.

$$\frac{D(D-1)}{2} - 1 = \frac{D^2 - D - 2}{2} = \frac{(D-2)(D+1)}{2}.$$

This leaves 5 polarizations in  $D = 4$ .

Let's now analyse spurious states to see the conditions to maximize 0-norm states. A physical state  $|\phi\rangle$  is a state satisfying the constraints

$$\begin{aligned}\hat{L}_{n,m}|\phi\rangle &= 0, \quad \forall n, m \in \mathbb{Z} \\ \hat{\tilde{L}}_{n,m}|\phi\rangle &= 0, \quad n \neq -m \\ \left( \sum_n \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \hat{\tilde{L}}_{n,-n} \right) - c \right) |\phi\rangle &= 0.\end{aligned}$$

A spurious state  $|\psi\rangle$  is a state satisfying the last constraint and orthogonal to all physical states, i.e,

$$\begin{aligned}\left( \sum_n \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \hat{\tilde{L}}_{n,-n} \right) - c \right) |\psi\rangle &= 0, \\ \langle \phi | \psi \rangle &= 0.\end{aligned}$$

Spurious states can be written in the form

$$|\psi\rangle = \sum_{n \neq -m} \hat{\tilde{L}}_{n,m} |\chi_{n,m}\rangle$$

with  $|\chi_{n,m}\rangle$  satisfying

$$\left( \sum_n \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \hat{\tilde{L}}_{n,-n} \right) - c - k - p \right) |\chi_{k,p}\rangle = 0.$$

To see that this state is indeed orthogonal to  $|\phi\rangle$ , we perform the inner product

$$\langle \phi | \psi \rangle = \sum_{n \neq -m} \langle \phi | \hat{\tilde{L}}_{n,m} |\chi_{n,m}\rangle = \sum_{n \neq -m} \langle \chi_{n,m} | \hat{\tilde{L}}_{-m,-n} |\phi\rangle^* \equiv 0.$$

The interesting case is when  $|\psi\rangle$  is both physical AND spurious, since by construction those states have 0-norm, being orthogonal to every physical state, including themselves. Such states can be constructed by considering spurious states of the form

$$|\psi_{1,n}\rangle = \sum_m \hat{\tilde{L}}_{n,m} |\tilde{\chi}_{1,m}\rangle$$

where  $|\tilde{\chi}_{1,m}\rangle$  satisfies

$$\begin{aligned}\hat{\tilde{L}}_{k,p} |\tilde{\chi}_{1,m}\rangle &= 0, \quad k \neq -p \\ \left( \sum_{n'} \left( \left( \left( \frac{n'}{E_{n'}} \right)^2 - 1 \right) \hat{\tilde{L}}_{n',-n'} \right) - c - 1 - m \right) |\tilde{\chi}_{1,m}\rangle &= 0.\end{aligned}$$

These states are annihilated by  $\hat{\tilde{L}}_{k,p}$ , for  $p \neq -n$ , and for this case we have that

$$\begin{aligned}\sum_n \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \hat{\tilde{L}}_{2,-n} |\psi_{1,n}\rangle \right) &= \\ = \sum_n \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \hat{\tilde{L}}_{2,-n} \sum_m \hat{\tilde{L}}_{n,m} |\tilde{\chi}_{1,m}\rangle \right) &= \\ = \sum_n \left( \left( \left( \frac{n}{E_n} \right)^2 - 1 \right) \hat{\tilde{L}}_{n,-n} \right) |\tilde{\chi}_{1,-2}\rangle,\end{aligned}$$

which vanish if  $c = 1$ .

## 12 Polyakov Action in terms of LQG Variables

Turn embedding fields  $X^\mu$  into vector in the spin-1 representation of  $\mathbb{R}^{d+1} \rtimes \text{Spin}(d, 1)$ ,  $X^I$ , and promote partial derivative to covariant derivative  $\partial_a \mapsto \mathcal{D}_a$  acting as

$$\mathcal{D}_a X^I = \partial_a X^I + k \mathcal{A}_{aJ}^I X^J,$$

where the WS  $\mathbb{R}^{d+1} \rtimes \text{Spin}(d, 1)$  connection  $\mathcal{A}_a$  with  $I, J = 0, \dots, d, d+1$  is given by

$$\mathcal{A}_a X = \begin{bmatrix} 0 & \beta_a^{01} & \beta_a^{02} & \dots & \beta_a^{0d} & p_a^0/l \\ \beta_a^{01} & 0 & -\theta_a^{12} & \dots & -\theta_a^{1d} & p_a^1/l \\ \beta_a^{02} & \theta_a^{12} & 0 & \dots & -\theta_a^{2d} & p_a^2/l \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_a^{0d} & \theta_a^{1d} & \theta_a^{2d} & \dots & 0 & p_a^d/l \\ -\epsilon p_a^0/l & \epsilon p_a^1/l & \epsilon p_a^2/l & \dots & \epsilon p_a^d/l & 0 \end{bmatrix} \begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ \vdots \\ X^d \\ 1 \end{bmatrix} = \left( \beta_a^{0\alpha} B_{0\alpha} + \frac{1}{2} \theta_a^{\alpha\beta} J_{\alpha\beta} + \frac{p_a^\mu}{l} P_\mu \right) X,$$

where  $\epsilon = \text{sgn}(\Lambda)$ , i.e the sign of the background cosmological constant with  $\alpha, \beta = 1, \dots, d$  and  $\mu = 0, \dots, d$ , so the action becomes

$$S = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ},$$

where the metrics can be recast in terms of auxiliary and bulk vielbein fields as in

$$S = -\frac{T}{2} \int d^2x e e_I^a e_J^b \eta^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ},$$

where

$$e := \sqrt{-\det(e_a^I e_b^J \eta_{IJ})} = \sqrt{-\frac{1}{2} \varepsilon^{ff'} \varepsilon^{gg'} (e_f^M e_g^N \eta_{MN})(e_{f'}^{M'} e_{g'}^{N'} \eta_{M'N'})}.$$

### 12.1 Checking the derivative

Since the connection is non-abelian, it should transform as

$$\mathcal{A}'_a = \mathcal{P} \mathcal{A}_a \mathcal{P}^{-1} - \frac{1}{k} (\partial_a \mathcal{P}) \mathcal{P}^{-1},$$

where  $\mathcal{P} = e^{\beta^{0\alpha} B_{0\alpha} + \theta^{\alpha\beta} J_{\alpha\beta}/2 + p^\mu P_\mu/l}$ . Putting this into a gauge transformed derivative we have

$$\begin{aligned} \mathcal{D}'_a X' &= \partial_a X' + k \mathcal{A}'_a X' = \\ &= \partial_a (\mathcal{P} X) + k (\mathcal{P} \mathcal{A}_a \mathcal{P}^{-1} - \frac{1}{k} (\partial_a \mathcal{P}) \mathcal{P}^{-1})(\mathcal{P} X) = \\ &= \partial_a \mathcal{P} X + \mathcal{P} \partial_a X + k \mathcal{P} \mathcal{A}_a X - \partial_a \mathcal{P} X \equiv \mathcal{P} \mathcal{D}_a X, \end{aligned}$$

so this is a proper covariant derivative for the Poincaré group.

Since under gauge transformations  $\mathcal{D}_a X$  transforms covariantly, the action is not invariant under diffeomorphisms. In fact, it transforms as

$$\begin{aligned} S' &= -\frac{T}{2} \int d^2x e e'_I e'^a_J \eta^{I'J'} \mathcal{D}'_a X'^I \mathcal{D}'_b X'^J \eta_{IJ} = \\ &= -\frac{T}{2} \int d^2x e e_K^a (\mathcal{P}^{-1})^{K'_I} e_{J'}^b (\mathcal{P}^{-1})^{L'_J} \eta^{I'J'} \mathcal{P}^I_K \mathcal{D}_a X^K \mathcal{P}^J_L \mathcal{D}_b X^L \eta_{IJ}, \end{aligned}$$

so we promote the killing forms  $\eta_{IJ}$  and  $\eta^{I'J'}$  to doubly co(ntra)-variant forms  $K_{IJ}$  and  $K^{I'J'}$  such that  $K'_{IJ} = (\mathcal{P}^{-1})^M_I (\mathcal{P}^{-1})^N_J K_{MN}$  and  $K'^{I'J'} = \mathcal{P}^{I'}_{M'} \mathcal{P}^{J'}_{N'} K^{M'N'}$ , so the new action is

$$S = -\frac{T}{2} \int d^2x e e_I^a e_J^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ}.$$

For consistency, let's assume  $K$  is a background field  $K = K(X)$  like the metric  $G$ .

## 12.2 EoMs

### 12.2.1 W.r.t $e$

$$\begin{aligned}
\frac{\delta S}{\delta e_K^c} &\propto \left( \frac{\partial e}{\partial e_K^c} e_{I'}^a e_{J'}^b K^{I'J'} + e \frac{\partial}{\partial e_K^c} (e_{I'}^a e_{J'}^b K^{I'J'}) \right) \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} = \\
&= \left( -ee_c^K e_{I'}^a e_{J'}^b K^{I'J'} + 2e \delta_c^a \delta_{I'}^K e_{J'}^b K^{I'J'} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} = \\
&= -ee_c^K e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} + 2ee_{J'}^b K^{KJ'} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ} = 0 \\
T_c^K &:= 2e_{J'}^b K^{KJ'} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ} - e_c^K e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} = 0 \\
e_c^K &= 2f e^{bK} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ}, \\
\frac{1}{f} &= e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ}
\end{aligned}$$

### 12.2.2 W.r.t $A$

$$\begin{aligned}
\frac{\delta S}{\delta \mathcal{A}_c^{KL}} &= \frac{T}{2} \left( ee_{I'}^a e_{J'}^b K^{I'J'} \frac{\partial}{\partial \mathcal{A}_c^{KL}} (\mathcal{D}_a X^I \mathcal{D}_b X^J) K_{IJ} \right) = \\
&= T \left( ee_{I'}^a e_{J'}^b K^{I'J'} \delta_a^c \delta_{[K}^L \eta_{L]I'} X^{I'} \mathcal{D}_b X^J K_{IJ} \right) = \\
&= Tee_{I'}^c e_{J'}^b K^{I'J'} \mathcal{D}_b X_{[K} X_{L]} \stackrel{!}{=} 0 \\
\mathcal{T}_a^{IJ} &= \mathcal{D}_a X^{[I} X^{J]} = 0
\end{aligned}$$

### 12.2.3 W.r.t $X$

$$\begin{aligned}
\frac{\delta S}{\delta X^K} &= -\frac{T}{2} \left( \partial_c \left( ee_{I'}^a e_{J'}^b K^{I'J'} \frac{\partial}{\partial (\partial_c X^K)} (\mathcal{D}_a X^I \mathcal{D}_b X^J) K_{IJ} \right) - ee_{I'}^a e_{J'}^b K^{I'J'} \frac{\partial}{\partial X^K} (\mathcal{D}_a X^I \mathcal{D}_b X^J) K_{IJ} - \right. \\
&\quad \left. - ee_{I'}^a e_{J'}^b \frac{\partial K^{I'J'}}{\partial X^K} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} - ee_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \frac{\partial K_{IJ}}{\partial X^K} \right) = \\
&= -T \left( \left( \partial_c \left( ee_{I'}^a e_{J'}^b K^{I'J'} \delta_a^c \delta_K^I \mathcal{D}_b X^J K_{IJ} \right) - ee_{I'}^a e_{J'}^b K^{I'J'} k \mathcal{A}_{aL}^I \delta_L^{I'} \mathcal{D}_b X^J K_{IJ} \right) - \right. \\
&\quad \left. - ee_{I'}^a e_{J'}^b \partial_K K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} - ee_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \partial_K K_{IJ} \right) = \\
&= -T \left( \mathcal{D}_a \left( ee_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_b X^I K_{IK} \right) - ee_{I'}^a e_{J'}^b \left( \partial_K K^{I'J'} K_{IJ} + K^{I'J'} \partial_K K_{IJ} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J \right) \stackrel{!}{=} 0 \\
\mathcal{D}_a \left( ee_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_b X^I K_{IK} \right) &= ee_{I'}^a e_{J'}^b \left( \partial_K K^{I'J'} K_{IJ} + K^{I'J'} \partial_K K_{IJ} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J
\end{aligned}$$

In “constant gauge”  $K_{IJ}(X) = \eta_{IJ}$  and in conformal gauge  $ee_{I'}^a e_{J'}^b \eta^{I'J'} = \eta^{ab}$ , this reduces to

$$\eta^{ab} \mathcal{D}_a \mathcal{D}_b X^I = 0.$$

More explicitly,

$$\begin{aligned}
&\eta^{ab} \mathcal{D}_a (\partial_b X^I + k \mathcal{A}_{bJ}^I X^J) = 0 \\
&\eta^{ab} \left( (\partial_a \partial_b X^I + k \mathcal{A}_{aJ}^I \partial_b X^J) + (k \partial_a (\mathcal{A}_{bJ}^I X^J) + k^2 \mathcal{A}_{aI'}^I \mathcal{A}_{bJ}^{I'} X^J) \right) = 0 \\
&\eta^{ab} \left( \partial_a \partial_b X^I + 2k \mathcal{A}_{aJ}^I \partial_b X^J + k \partial_a \mathcal{A}_{bJ}^I X^J + k^2 \mathcal{A}_{aI'}^I \mathcal{A}_{bJ}^{I'} X^J \right) = 0
\end{aligned}$$

$$\eta^{ab} (\delta_J^I \partial_a \partial_b + 2k \mathcal{A}_{aJ}^I \partial_b) X^J = - \left( k \eta^{ab} \partial_a \mathcal{A}_{bJ}^I X^J + k^2 \eta^{ab} \mathcal{A}_{aI'}^I \mathcal{A}_{bJ}^{I'} X^J \right),$$

which has implicit solution given by

$$X^K(x) = - \int d^2x' G_I^K(x, x') \left( k \eta^{a'b'} \partial_{a'} \mathcal{A}_{b'J}^I(x') X^J(x') + k^2 \eta^{a'b'} \mathcal{A}_{a'I'}^I(x') \mathcal{A}_{b'J}^{I'}(x') X^J(x') \right),$$

where the Green's function matrix  $G_J^K(x, x')$  satisfies

$$\eta^{ab} (\delta_K^I \partial_a \partial_b + 2k \mathcal{A}_{aK}^I(x) \partial_b) G_J^K(x, x') = \delta_J^I \delta(x, x').$$

Boundary conditions come from

$$\begin{aligned} \delta S &= \int d^2x \left( \frac{\partial \mathcal{L}}{\partial X^K} \delta X^K + \frac{\partial \mathcal{L}}{\partial (\partial_c X^K)} \delta (\partial_c X^K) + \dots \right) = \\ &= \int d^2x \left( (\text{EoM}) \delta X^K + \partial_c \left( \frac{\partial \mathcal{L}}{\partial (\partial_c X^K)} \delta X^K \right) + \dots \right) = \\ &= \int d^2x (\text{EoM}) \delta X^K + \int d\tau \left( \frac{\partial \mathcal{L}}{\partial (\partial_\sigma X^K)} \delta X^K \right) \Big|_0^{\sigma_1} + \dots = 0, \end{aligned}$$

thus

$$\begin{aligned} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\sigma X^K)} \delta X^K \right) \Big|_0^{\sigma_1} &= 0 \\ ((-Tee_i^\sigma e^{bi} \mathcal{D}_b X_K) \delta X^K) \Big|_0^{\sigma_1} &= 0 \\ ((\mathcal{D}_\sigma X_K) \delta X^K) \Big|_0^{\sigma_1} &= 0, \end{aligned}$$

so either  $\delta X^K(\tau, \sigma_*) = 0$ ,  $\sigma_* = 0, \sigma_1$  (Dirichlet B.C) or  $\mathcal{D}_\sigma X^K(\tau, \sigma_*) = 0$ ,  $\sigma_* = 0, \sigma_1$  (free end-point B.C). For closed strings,  $X^K(\tau, \sigma) = X^K(\tau, \sigma + 2\pi)$ .

When connection is trivial EoM reduces to regular wave eqn, which have the regular string solution

$$X_0^I(\tau, \sigma) = X_{0L}^I(\sigma^+) + X_{0R}^I(\sigma^-),$$

with

$$\begin{aligned} X_{0L}^I(\sigma^+) &= \frac{1}{2} x^I + \frac{1}{2} \alpha' p^I \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^I e^{-in\sigma^+}, \quad \sigma^+ = \tau + \sigma \\ X_{0R}^I(\sigma^-) &= \frac{1}{2} x^I + \frac{1}{2} \alpha' p^I \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^I e^{-in\sigma^-}, \quad \sigma^- = \tau - \sigma. \end{aligned}$$

This, however, does not amount to any motion in actual space-time, since with trivial connection the vector does not change after parallel transport. For a more interesting case, let's consider  $A_\tau^{\alpha\beta} = A_\sigma^{\alpha\beta} = a\varepsilon^{\alpha\beta}$ , which amounts to flat space-time. The EoM turn into

$$\begin{aligned} &(\partial_\sigma^2 - \partial_\tau^2) X^I + 2ka(\varepsilon^{\alpha\beta} T_{\alpha\beta})^I_J (\partial_\sigma - \partial_\tau) X^J = 0 \\ &(\partial_\sigma^2 - \partial_\tau^2) \begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ \vdots \\ X^d \end{bmatrix} + 4ka(\partial_\sigma - \partial_\tau) \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & -1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & -1 & \dots & 0 \end{bmatrix} \begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ \vdots \\ X^d \end{bmatrix} = 0 \\ &\begin{bmatrix} (\partial_\sigma^2 - \partial_\tau^2) X^0 + 4ka(\partial_\sigma - \partial_\tau) \sum_{i>0} X^i \\ (\partial_\sigma^2 - \partial_\tau^2) X^1 + 4ka(\partial_\sigma - \partial_\tau) \sum_{i\neq 1} X^i \\ (\partial_\sigma^2 - \partial_\tau^2) X^2 + 4ka(\partial_\sigma - \partial_\tau) (X^0 - X^1 + \sum_{i>2} X^i) \\ \vdots \\ (\partial_\sigma^2 - \partial_\tau^2) X^d + 4ka(\partial_\sigma - \partial_\tau) (X^0 - \sum_{i=1}^{d-1} X^i) \end{bmatrix} = 0 \end{aligned}$$

$$\begin{aligned}
& (\partial_\sigma^2 - \partial_\tau^2) X^I + 4ka(\partial_\sigma - \partial_\tau) \left( (\text{sgn}(I))^2 X^0 + \sum_{i>0} \text{sgn}(i-I) X^i \right) = 0 \\
& (\partial_\sigma - \partial_\tau) \left( (\partial_\sigma + \partial_\tau) X^I + 4ka \left( (\text{sgn}(I))^2 X^0 + \sum_{i>0} \text{sgn}(i-I) X^i \right) \right) = 0 \\
& (\partial_\sigma + \partial_\tau) X^I + 4ka \left( (\text{sgn}(I))^2 X^0 + \sum_{i>0} \text{sgn}(i-I) X^i \right) = C_1^I \cdot (\tau + \sigma), \quad C^I \in \mathbb{C} \\
& (\partial_\sigma + \partial_\tau) X^I = C_1^I \cdot (\tau + \sigma) - 4ka \left( (\text{sgn}(I))^2 X^0 + \sum_{i>0} \text{sgn}(i-I) X^i \right) \\
& X^I = \frac{1}{2} C_1^I \cdot (\tau^2 + \sigma^2) + C_2^I \cdot (\tau - \sigma) - 4ka \int d^2x' \left( (\text{sgn}(I))^2 X^0 + \sum_{i>0} \text{sgn}(i-I) X^i \right).
\end{aligned}$$

If  $D = 2$ , we get a symmetric system of eqns

$$\begin{aligned}
X^0 &= \frac{1}{2} C_1^0 \cdot (\tau^2 + \sigma^2) + C_2^0 \cdot (\tau - \sigma) - 4ka \int d^2x' X^1 \\
X^1 &= \frac{1}{2} C_1^1 \cdot (\tau^2 + \sigma^2) + C_2^1 \cdot (\tau - \sigma) - 4ka \int d^2x' X^0 \\
X^0 &= \frac{1}{2} C_1^0 \cdot (\tau^2 + \sigma^2) + C_2^0 \cdot (\tau - \sigma) - 4ka \int d^2x' \left( \frac{1}{2} C_1^1 \cdot (\tau'^2 + \sigma'^2) + C_2^1 \cdot (\tau' - \sigma') - 4ka \int d^2x'' X^0 \right) \\
X^0 - 16k^2 a^2 \int d^2x' \int d^2x'' X^0 &= \frac{1}{2} C_1^0 \cdot (\tau^2 + \sigma^2) + C_2^0 \cdot (\tau - \sigma) - \frac{1}{2} C_1^1 \left( \frac{\tau^3}{3} \sigma + \tau \frac{\sigma^3}{3} \right) - C_2^1 \left( \frac{\tau^2}{2} \sigma - \tau \frac{\sigma^2}{2} \right).
\end{aligned}$$

This integral equation is tricky to deal with, but we can turn it into a differential equation by applying  $\partial_\sigma \partial_\tau$  twice:

$$(\partial_\sigma \partial_\tau)^2 X^0 - 16k^2 a^2 X^0 = 0.$$

This has general solution given by sum of plane waves and exponentials

$$X^0 = \sum_{n \neq 0} \frac{1}{n^4} \left( a_n^0 e^{i2\sqrt{kan}(\tau+\sigma)} + b_n^0 e^{i2\sqrt{kan}(\tau-\sigma)} + c_n^0 e^{2\sqrt{kan}(\tau+\sigma)} + d_n^0 e^{2\sqrt{kan}(\tau-\sigma)} \right).$$

In actuality, only the  $n = 1$  and  $n = -1$  terms satisfy the differential equation, so we have just

$$\begin{aligned}
X^0 &= a_1^0 e^{i2\sqrt{ka}(\tau+\sigma)} + b_1^0 e^{i2\sqrt{ka}(\tau-\sigma)} + c_1^0 e^{2\sqrt{ka}(\tau+\sigma)} + d_1^0 e^{2\sqrt{ka}(\tau-\sigma)} + \\
&\quad + a_{-1}^0 e^{-i2\sqrt{ka}(\tau+\sigma)} + b_{-1}^0 e^{-i2\sqrt{ka}(\tau-\sigma)} + c_{-1}^0 e^{-2\sqrt{ka}(\tau+\sigma)} + d_{-1}^0 e^{-2\sqrt{ka}(\tau-\sigma)}
\end{aligned}$$

## 13 Polyakov-LQG V2

Same idea as before, but now contract the dyads directly with the  $\mathcal{D}_a X^I$  factors

$$S = -\frac{T}{4} \int d^2x e e_I^a e_J^b \mathcal{D}_a X^I \mathcal{D}_b X^J.$$

### 13.1 EoMs

#### 13.1.1 W.r.t $e$

$$\begin{aligned} \frac{\delta S}{\delta e_K^c} &\propto \left( \frac{\partial e}{\partial e_K^c} e_I^a e_J^b + e \frac{\partial}{\partial e_K^c} (e_I^a e_J^b) \right) \mathcal{D}_a X^I \mathcal{D}_b X^J = \\ &= (-e e_c^K e_I^a e_J^b + 2e \delta_I^K \delta_c^a e_J^b) \mathcal{D}_a X^I \mathcal{D}_b X^J = \\ &= -e e_c^K e_I^a e_J^b \mathcal{D}_a X^I \mathcal{D}_b X^J + 2e e_J^b \mathcal{D}_c X^K \mathcal{D}_b X^J = 0 \\ T_c^K &:= 2e_J^b \mathcal{D}_c X^K \mathcal{D}_b X^J - e_c^K e_I^a e_J^b \mathcal{D}_a X^I \mathcal{D}_b X^J = 0, \\ e_c^K &= 2f \mathcal{D}_c X^K, \\ \frac{1}{f} &= e_J^b \mathcal{D}_b X^J. \end{aligned}$$

#### 13.1.2 W.r.t $A$

$$\begin{aligned} \frac{\delta S}{\delta A_c^{KL}} &\propto e e_I^a e_J^b \frac{\partial}{\partial A_c^{KL}} (\mathcal{D}_a X^I \mathcal{D}_b X^J) = \\ &= 2e e_I^a e_J^b \delta_a^L \delta_{[K}^I \eta_{L]I'} X^{I'} \mathcal{D}_b X^J = \\ &= 2e e_{[K}^c X_{L]} e_J^b \mathcal{D}_b X^J = 0 \\ \mathcal{T}_{KL}^c &:= e_{[K}^c X_{L]} = 0. \end{aligned}$$

#### 13.1.3 W.r.t $X$

$$\begin{aligned} \frac{\delta S}{\delta X^K} &\propto 2e e_I^a e_J^b k \mathcal{A}_{aI'}^I \delta_{K'}^I \mathcal{D}_b X^J - 2\partial_c (e e_I^a e_J^b \delta_a^c \delta_K^I \mathcal{D}_b X^J) = \\ &= 2e e_I^a e_J^b k \mathcal{A}_{aK}^I \mathcal{D}_b X^J - 2\partial_a (e e_K^a e_J^b \mathcal{D}_b X^J) = 0 \\ \mathcal{D}_a (e e_K^a e_J^b \mathcal{D}_b X^J) &= 0. \end{aligned}$$

If conformal gauge makes it so that  $e e_K^a e_J^b = \eta^{ab} \eta_{KJ}$ , this reduces to

$$\eta^{ab} \mathcal{D}_a \mathcal{D}_b X^I = 0$$