Sketchboard

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1 Basic concepts

1.1 With extra capital internal index on embbeding fields

Promote embedding fields X^μ to have an internal group index with D values

$$X^\mu \to X^{\mu I} \ , \quad I = 0, ..., d$$

$$g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{IJ} \sim f D_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu}$$
$$D_a X^{\mu I} = \partial_a X^{\mu I} + \omega_{aJ}^I X^{\mu J}$$
$$e_a^i e_b^j \eta_{ij} = g_{ab} \to e_a^i e_b^j \eta_{ij} = \mathrm{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ} \ , \quad i, j = 0, 1$$
$$g = \det(g_{ab}) \to g = \det(\mathrm{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.2 With 1 extra small internal index on embedding fields

Promote embbeding fields to have a internal group index with 2 values

$$X^\mu \to X^{\mu i} \ , \quad i = 0, 1$$

$$g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{ij} \sim f D_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu}$$
$$D_a X^{\mu i} = \partial_a X^{\mu i} + \omega_{aj}^i X^{\mu j}$$
$$e_a^i e_b^j \eta_{ij} = g_{ab} \to e_a^i e_b^j \eta_{ij} = \mathrm{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij}$$
$$g = \det(g_{ab}) \to g = \det(\mathrm{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.3 With 2 extra small indices

Promote embbeding fields X^μ to have two internal indices with 2 values

$$X^\mu \to X^{\mu ij} \ , \quad i, j = 0, 1$$

1.4 Without extra index on embbeding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$e_a^i e_b^j \eta_{ij} = g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$g = \det(g_{ab}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int \mathrm{d}\tau \wedge \mathrm{d}\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \ ,$$

2.1 With capital internal index

... and promote partial derivative ∂_a to covariant derivative D_a, giving us our first attempt at modified Polyakov action

$$S_{MP1} = -\frac{T_0}{2} \int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}$$
$$\updownarrow$$
$$\mathscr{L}_{MP1} = -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}$$

2.2 With small internal index

2.2.1 Without extra field, 1 internal index

... and promote derivatives to covariant D_a, giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

$$\updownarrow$$

$$\mathscr{L}_{MP2} = -\frac{T_0}{2}\det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

2.2.2 With extra field

..., promote partial derivative to covariant and add extra internal WS field v^i leading us to

$$S_{MP3} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e)\eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

$$\updownarrow$$

$$\mathscr{L}_{MP3} = -\frac{T}{2}\det(e)\eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

2.2.3 Without extra field, 2 internal indices

... and promote derivatives to covariant D_a,

$$S_{MP4} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma e e_i^a e^{bi} D_a X^{\mu j j'} \eta_{jj'} \eta_{kk'} D_b X^{\nu k k'} E_\mu^I E_{\nu I}$$

$$\mathscr{L}_{MP4} = -\frac{T}{2} e e_i^a e_j^b D_a X^{\mu ik} \eta_{kl} D_b X^{\nu lj} E_\mu^I E_{\nu I}$$

2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) \ .$$

$$\updownarrow$$

$$\mathscr{L}_{DP} = -\frac{T_0}{2}\det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) \ .$$

2.4 Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of D-dimensional gamma matrices and bulk spinors

$$S_{LP} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

$$\updownarrow$$

$$\mathscr{L}_{LP} = -\frac{T}{2}\overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

3 EoMs

Start by writing $\det(e) = \frac{1}{2}\varepsilon^{cd}\varepsilon_{mn} e_c^m e_d^n$ and $g_{ab} = g_{ab}^{ij}\eta_{ij}$

3.1 With small internal index

3.1.1 Without extra field, w.r.t e

$$\begin{aligned}
\frac{\delta S_{MP2}}{\delta e^e_l} = \frac{D\mathscr{L}_{MP2}}{De^e_l} &= \left(\frac{\partial}{\partial e^e_l} - \partial_f \frac{\partial}{\partial(\partial_f e^e_l)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e^m_c e^n_d e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I}\right) = \\
&= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(\eta^{ml}g_{ce}e^n_d e^a_i e^b_j + e^m_c \eta^{nl} g_{de} e^a_i e^b_j + \\
&+ e^m_c e^n_d \delta^a_e \delta^l_i e^b_j + e^m_c e^n_d e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = \\
&= -\frac{T}{2}\left(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}g_{ce}e^n_d e^a_i e^b_j + 2\det(e) e^a_i \delta^b_e \delta^l_j\right) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = \\
&= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}e^k_c e_{ek} e^n_d e^a_i e^b_j + 2\det(e) e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = \\
&= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}e^m_c e^n_d e^l_e e^a_i e^b_j + 2\det(e) e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = \\
&= -T(-\det(e) e^l_e e^a_i e^b_j + \det(e) e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} \stackrel{!}{=} 0
\end{aligned}$$

$$\downarrow$$

$$T^l_e := E^I_\mu E_{\nu I}(e^a_i D_a X^{\mu i} D_e X^{\nu l} - e^l_e e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j}) = 0$$

$$\Downarrow$$

$$e^l_e = f e^a_i D_a X^{\mu i} D_e X^{\nu l} E^I_\mu E_{\nu I}\ ,$$

$$\frac{1}{f} = e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I}$$

$$\downarrow$$

$$\begin{aligned}
e^l_e e_{fl} &= (f e^a_i D_a X^{\mu i} D_e X^{\nu l} E^I_\mu E_{\nu I})(f e^{a'}_{i'} D_{a'} X^{\mu' i'} D_f X^{\nu'}_l E^{I'}_{\mu'} E_{\nu' I'}) = \\
&= f^2 D_e X^{\nu l} D_f X^{\nu'}_l E^{I'}_\nu E_{\nu' I'} (e^a_i e^{a'}_{i'} D_a X^{\mu i} D_{a'} X^{\mu' i'} E^I_\mu E_{\mu' I}) = \\
&= f D_e X^{\mu i} D_f X^\nu_i E^I_\mu E_{\nu I}
\end{aligned}$$

$$\Downarrow$$

$$D_e X^{\mu i} D_f X^\nu_i E^I_\mu E_{\nu I} = 2\partial_e X^\mu \partial_f X^\nu G_{\mu\nu}$$

$$e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\Downarrow$$

$$D_a X^{\mu 0} = -i\partial_a X^\mu$$

$$D_a X^{\mu 1} = \partial_a X^\mu$$

$$\Downarrow$$

$$X^{\mu 0} = -i X^{\mu 1}$$

3.1.2 With extra field, w.r.t e

$$
\begin{aligned}
\frac{\delta S_{MP3}}{\delta e_o^e} &= \frac{D\mathscr{L}_{MP3}}{De_o^e} = \left(\frac{\partial}{\partial e_o^e} - \partial_f \frac{\partial}{\partial(\partial_f e_o^e)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e_c^m e_d^n \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}\right) = \\
&= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(g_{ce}\eta^{mo}e_d^n\eta^{ij}e_i^a e_j^b + e_c^m g_{de}\eta^{no}\eta^{ij}e_i^a e_j^b + \\
&+ e_c^m e_d^n \eta^{ij}\delta_e^a\delta_i^o e_j^b + e_c^m e_d^n \eta^{ij} e_i^a \delta_e^b \delta_j^o) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -\frac{T}{2}\varepsilon^{cd}\varepsilon_{mn}(e_c^p e_{ep} e_d^n \eta^{mo}\eta^{ij} e_i^a e_j^b + 2\det(e) e^{ao}\delta_e^b) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -T(-\det(e) e_e^o \eta^{ij} e_i^a e_j^b + \det(e) e^{ao}\delta_e^b) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \overset{!}{=} 0
\end{aligned}
$$

$$\downarrow$$

$$\boxed{T_e^o := (e^{ao} D_a(X_k^\mu v^k) D_e(X_l^\nu v^l) - e_e^o \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l)) E_\mu^I E_\nu^J \eta_{IJ} = 0}$$

$$\Downarrow$$

$$
\begin{aligned}
e_e^o &= f\, e^{ao} D_a(X_k^\mu v^k) D_e(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}\ , \\
\frac{1}{f} &= \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}
\end{aligned}
$$

3.1.3 With extra field, w.r.t ω

$$
\begin{aligned}
\frac{\delta S_{MP3}}{\delta \omega_c^{mn}} &= \frac{D\mathscr{L}_{MP3}}{D\omega_c^{mn}} = \left(\frac{\partial}{\partial \omega_c^{mn}} - \partial_d \frac{\partial}{\partial(\partial_d \omega_c^{mn})}\right)\left(-\frac{T}{2}\det(e)\eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}\right) = \\
&= -\frac{T}{4} e g^{ab}((-\delta_a^c \delta_{[m}^{k'}\eta_{n]k} X_{k'}^\mu v^k) D_b X_l^\nu v^l + D_a X_k^\mu v^k (-\delta_b^c \delta_{[m}^{l'}\eta_{n]l} X_{l'}^\nu v^l)) G_{\mu\nu} = \\
&= \frac{T}{4} e (g^{ca} X_{[m}^\mu v_{n]} D_a X_k^\nu v^k + g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_{n]}) G_{\mu\nu} = \\
&= \frac{T}{2} e g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_{n]} G_{\mu\nu} \overset{!}{=} 0
\end{aligned}
$$

$$\downarrow$$

$$\boxed{\mathcal{T}_{ab}^i := e^{ci} D_c X_k^\mu v^k X_{[m}^\nu v_{n]} e_a^m e_b^n G_{\mu\nu} = 0}$$

3.1.4 With extra field, w.r.t v

$$
\begin{aligned}
\frac{\delta S_{MP3}}{\delta v^l} &= \frac{D\mathscr{L}_{MP3}}{Dv^l} = \left(\frac{\partial}{\partial v^l} - \partial_c \frac{\partial}{\partial(\partial_c v^l)}\right)\left(-\frac{T}{2} e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I}\right) = \\
&= -\frac{T}{2} g^{ab}(D_a X_j^\mu \delta_l^j D_b X_k^\nu v^k + D_a X_j^\mu v^j D_b X_k^\nu \delta_l^k) G_{\mu\nu} = \\
&= -T g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} \overset{!}{=} 0
\end{aligned}
$$

$$\downarrow$$

$$\boxed{g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} = 0}$$

3.1.5 With extra field, w.r.t X

$$\frac{\delta S_{MP3}}{\delta X_l^\lambda} = \frac{D\mathscr{L}_{MP3}}{DX_l^\lambda} = \left(\frac{\partial}{\partial X_l^\lambda} - \partial_c \frac{\partial}{\partial(\partial_c X_l^\lambda)}\right)\left(-\frac{T}{2} e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I}\right) =$$

$$= -\frac{T}{2}\left(eg^{ab}\left((-\omega_{aj}^{j'}\delta_\lambda^\mu \delta_{j'}^l v^j D_b X_k^\nu v^k - D_a X_j^\mu v^j \omega_{bk}^{k'}\delta_\lambda^\nu \delta_{k'}^l v^k) E_\mu^I E_{\nu I} + \right.\right.$$

$$\left. + eg^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \left(\frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} + E_\mu^I \frac{\partial E_{\nu I}}{\partial X_l^\lambda}\right)\right) -$$

$$\left. - \partial_c (eg^{ab}(\delta_a^c \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k + D_a X_j^\mu v^j \delta_b^c \delta_\lambda^\nu \delta_k^l v^k) G_{\mu\nu})\right) =$$

$$= -T\left(- eg^{ab} D_a X_j^\mu v^j \omega_{bk}^l v^k G_{\mu\lambda} + eg^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - \partial_c (eg^{ac} v^l D_a X_j^\mu v^j G_{\mu\lambda})\right)$$

$$= -T\left(eg^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - D_b (eg^{ab} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I})\right) \stackrel{!}{=} 0$$

$$\downarrow$$

$$\boxed{D_b (e e_i^a e^{bi} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) = e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I}}$$

3.2 Without extra index

3.2.1 w.r.t e

$$\frac{\delta S_{DP}}{\delta e_l^e} = \frac{D\mathscr{L}_{DP}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)}\right)\left(-\frac{T_0}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I E_\nu^J \eta_{IJ}\right) =$$

$$= -\frac{T_0}{4}$$

3.3 Linear action

3.3.1 w.r.t e

$$\frac{\delta S_{LP}}{\delta\delta e_j^b} = \frac{D\mathscr{L}_{LP}}{De_j^b} = \left(\frac{\partial}{\partial e_j^b} - \partial_e \frac{\partial}{\partial(\partial_e e_j^b)}\right)\left(-\frac{T}{2}\overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi\right) =$$

$$= -\frac{T}{4}\overline{\psi}\varepsilon^{cd}\varepsilon_{mn}((g_{cb}\eta^{mj} e_d^n + e_c^m g_{db}\eta^{nj}) e_i^a + e_c^m e_d^n \delta_b^a \delta_i^j) D_a X^{\mu i} E_\mu^I \gamma_I \psi =$$

$$= -\frac{T}{2}\overline{\psi}(-e e_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I + e D_b X^{\mu j} E_\mu^I \gamma_I)\psi \stackrel{!}{=} 0$$

$$\downarrow$$

$$T_b^j := \overline{\psi}(D_b X^{\mu j} E_\mu^I \gamma_I - e_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I)\psi = 0$$

$$\downarrow$$

$$e_b^j = F\overline{\psi} D_b X^{\mu j} E_\mu^I \gamma_I \psi,$$

$$\frac{1}{F} = \overline{\psi} e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

$$\Downarrow$$

$$\overline{\psi}\psi = 1,\ \psi\overline{\psi} = \mathbb{1}$$

$$e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_\nu^J \eta_{IJ} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

4 Inverse Area Action

4.1 Nambu-Goto

Start from Nambu-Goto action

$$S_{NG} = -T \int \mathrm{d}^2 x \sqrt{-h},$$

and make quantum geometry correction

$$\sqrt{-h} \to \sqrt{-(h+g\Delta)} \approx \sqrt{-h}\left(1 - \frac{g\Delta}{2(-h)} + \mathcal{O}\left(\frac{g^2}{h^2}\right)\right),$$

leading to modified NG action

$$S_{MNG} = -T \int \mathrm{d}^2 x \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}}\right) = S_{NG} + S_{IA}$$

$$\downarrow$$

$$\mathscr{L}_{MNG} = -T\left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}}\right)$$

4.1.1 EoMs

Let $\mathcal{P}^\tau_\mu \equiv \partial \mathscr{L}_{MNG}/\partial \dot{X}^\mu$ and $\mathcal{P}^\sigma_\mu \equiv \partial \mathscr{L}_{MNG}/\partial X'^\mu$

$$\begin{aligned}\delta S_{MNG} &= \int \mathrm{d}^2 x \left(\frac{\partial \mathscr{L}_{MNG}}{\partial \dot{X}^\mu}\delta \dot{X}^\mu + \frac{\partial \mathscr{L}_{MNG}}{\partial X'^\mu}\delta X'^\mu\right) = \\ &= -\int \mathrm{d}^2 x \left(\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu\right)\delta X^\mu + \int \mathrm{d}\tau \mathcal{P}^\sigma_\mu \delta X^\mu \Big|_{\sigma=0}^{\sigma=\sigma_1} \stackrel{!}{=} 0\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}\text{EoM :}&\partial_\tau \mathcal{P}^\tau_\mu + \partial_\sigma \mathcal{P}^\sigma_\mu = 0 \\ \text{B.C. :}&\mathcal{P}^\sigma_\mu \delta X^\mu\Big|_{\sigma=0}^{\sigma=\sigma_1} = 0,\end{aligned}$$

where

$$\begin{aligned}\mathcal{P}^\tau_\mu = \frac{\partial \mathscr{L}_{MNG}}{\partial \dot{X}^\mu} &= -T\left(\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{-h}} + \frac{g\Delta}{2}\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{(-h)^{3/2}}\right) = \\ &= -T\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{-h}}\left(1 + \frac{g\Delta}{2(-h)}\right) = \mathcal{P}^\tau_{\mu(NG)}\left(1 + \frac{g\Delta}{2(-h)}\right) \\ \mathcal{P}^\sigma_\mu = \frac{\partial \mathscr{L}_{MNG}}{\partial X'^\mu} &= -T\left(\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} + \frac{g\Delta}{2}\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{(-h)^{3/2}}\right) = \\ &= -T\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}}\left(1 + \frac{g\Delta}{2(-h)}\right) = \mathcal{P}^\sigma_{\mu(NG)}\left(1 + \frac{g\Delta}{2(-h)}\right)\end{aligned}$$

gauge fixing static gauge $\tau = t$ and transverse gauge $\frac{\partial X}{\partial \tau}\cdot\frac{\partial X}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}\sigma} = 0$ (s = length along string)

$$\downarrow$$

$$\begin{aligned}\mathcal{P}^{\tau\mu}_{(NG)} &= T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\frac{\partial X^\mu}{\partial t} \\ \mathcal{P}^{\sigma\mu}_{(NG)} &= \frac{T}{\gamma_{v_\perp}}\frac{\partial X^\mu}{\partial s} \\ -h &= \frac{1}{\gamma^2_{v_\perp}}\left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2\end{aligned}$$

string energy get's redefined to

$$\frac{\partial}{\partial t}\left(T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\right)=0,$$

and for the spatial part we have

$$\partial_\tau\vec{\mathcal{P}}^\tau+\partial_\sigma\vec{\mathcal{P}}^\sigma=0$$

$$\downarrow$$

$$\frac{\partial}{\partial t}\left[T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\frac{\partial\vec{X}}{\partial t}\left(1+\frac{g\Delta}{2(-h)}\right)\right]+\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{\partial}{\partial s}\left[-\frac{T}{\gamma_{v_\perp}}\frac{\partial\vec{X}}{\partial s}\left(1+\frac{g\Delta}{2(-h)}\right)\right]=0$$

$$\downarrow$$

$$\mu\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial}{\partial s}\left[\frac{T}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right)\frac{\partial\vec{X}}{\partial s}\right]=0$$

$\implies$ effective mass density becomes $\mu_{eff}=\mu\gamma_{v_\perp}\left(1-\frac{g\Delta}{2(-h)}\right)$ and effective tension becomes $T_{eff}=\frac{T}{\gamma_{v_\perp}}\left(1-\frac{g\Delta}{2(-h)}\right)$

$$\downarrow$$

$$\mu_{eff}\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial}{\partial s}\left[T_{eff}\frac{\partial\vec{X}}{\partial s}\right]=0$$

$$\begin{aligned}\mathscr{H}&=\dot{\vec{X}}\cdot\vec{\pi}-\mathscr{L}=\\&=\vec{v}_\perp\cdot\left(T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\vec{v}_\perp\right)-\left(-T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{1}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right)\right)=\\&=T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\left(1+\frac{g\Delta}{2(-h)}\right)\left(\gamma_{v_\perp}v_\perp^2+\frac{1}{\gamma_{v_\perp}}\right)=\\&=T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\end{aligned}$$

let $\left(1+\frac{g\Delta}{2(-h)}\right)=F$

$$\frac{\partial^2\vec{X}}{\partial t^2}-\frac{1}{F\gamma_{v_\perp}}\frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\partial}{\partial\sigma}\left[\frac{1}{\gamma_{v_\perp}}F\frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\partial\vec{X}}{\partial\sigma}\right]=0$$

$$\downarrow$$

$$A(\sigma)=\frac{\gamma_{v_\perp}}{F}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\overset{!}{=}1$$

$$\downarrow$$

$$\mathrm{d}\sigma=\frac{\gamma_{v_\perp}}{F}\mathrm{d}s=\frac{1}{TF^2}\mathrm{d}E\implies\sigma(q)=\frac{1}{T}\int_0^q\frac{1}{F^2}\mathrm{d}E$$

$$\downarrow$$

$$F^2\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial^2\vec{X}}{\partial\sigma^2}=0$$

$\implies$ speed of wave on the string gets modified by correction factor $v=c/F$

$$-h=\frac{1}{\gamma_{v_\perp}^2}\left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2=\frac{F^2}{\gamma_{v_\perp}^4}=F^2\left(1-v_\perp^2\right)^2$$

$$\downarrow$$

$$-h=\left(1+\frac{g\Delta}{2(-h)}\right)^2\left(1-v_\perp^2\right)^2$$

$$\downarrow$$

$$\frac{-h}{\left(1+\frac{g\Delta}{2(-h)}\right)^2} = \left(1-v_\perp^2\right)^2$$

$$\downarrow \text{(WolframAlpha)}$$

$$-h = \frac{(1-v_\perp^2)^2}{3} - f_1 - f_2,$$

where (let $a = \frac{g\Delta}{2}$ and $b = (1-v_\perp^2)^2$)

$$f_1 = \frac{\sqrt[3]{-27a^2b + 3\sqrt{3}\sqrt{27a^4b^2 + 4a^3b^3} - 18ab^2 - 2b^3}}{3\sqrt[3]{2}}$$

$$f_2 = \frac{(6ab + b^2)}{6f_1}$$

$$\downarrow$$

$$F = 1 + \frac{g\Delta}{2(-h)} = 1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}$$

$$\downarrow$$

$$v = \frac{1}{F} = \frac{1}{1 + \cfrac{g\Delta}{2\left(\cfrac{b}{3} - f_1 - f_2\right)}}$$

5 Bimetric Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

and promote to bimetric action

$$\downarrow$$

$$S_{BP} = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{T'}{2}\int \mathrm{d}^2x\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

5.1 EoMs

$$\begin{aligned}\frac{\delta S_{BP}}{\delta g^{cd}} &= -\frac{T}{2}\left(\frac{\partial\sqrt{-g}}{\partial g^{cd}}g^{ab} + \sqrt{-g}\frac{\partial g^{ab}}{\partial g^{cd}}\right)\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + 0 = \\ &= -\frac{T}{2}\left(-\frac{1}{2}\sqrt{-g}g_{cd}g^{ab} + \sqrt{-g}\delta^a_{(c}\delta^b_{d)}\right)\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \overset{!}{=} 0\end{aligned}$$

$$T^{(G)}_{cd} := \left(\partial_c X^\mu \partial_d X^\nu - \frac{1}{2}g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu\right)G_{\mu\nu} = 0$$

$$\Downarrow$$

$$g_{cd} = 2f^{(G)}\partial_c X^\mu \partial_d X^\nu G_{\mu\nu},$$

$$\frac{1}{f^{(G)}} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\begin{aligned}\frac{\delta S_{BP}}{\delta h^{cd}} &= ... = \\ &= -\frac{T'}{2}\left(-\frac{1}{2}\sqrt{-h}h_{cd}h^{ab} + \sqrt{-h}\delta^a_{(c}\delta^b_{d)}\right)\partial_a X^\mu \partial_b X^\nu H_{\mu\nu} \overset{!}{=} 0\end{aligned}$$

$$\downarrow$$

$$T^{(H)}_{cd} := \left(\partial_c X^\mu \partial_d X^\mu - \frac{1}{2} h_{cd} h^{ab} \partial_a X^\mu \partial_b X^\nu\right) H_{\mu\nu} = 0$$

$$\Downarrow$$

$$h_{cd} = 2 f^{(H)} \partial_c X^\mu \partial_d X^\nu H_{\mu\nu},$$

$$\frac{1}{f^{(H)}} = h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

$$\begin{aligned}\frac{\delta S_{BP}}{\delta X^\lambda} &= \left(-\frac{T}{2}\sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} - \frac{T'}{2}\sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu}\right) \partial_a X^\mu \partial_b X^\nu - \\ &- \partial_c \left(-\frac{T}{2}\sqrt{-g} g^{ab} G_{\mu\nu} - \frac{T'}{2}\sqrt{-h} h^{ab} H_{\mu\nu}\right) (\delta^c_a \delta^\mu_\lambda \partial_b X^\nu + \partial_a X^\mu \delta^c_b \delta^\nu_\lambda) = \\ &= \partial_a \left(T\sqrt{-g} g^{ab} G_{\mu\lambda} + T'\sqrt{-h} h^{ab} H_{\mu\lambda}\right) \partial_b X^\mu - \\ &- \left(\frac{T}{2}\sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} + \frac{T'}{2}\sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu}\right) \partial_a X^\mu \partial_b X^\nu \stackrel{!}{=} 0\end{aligned}$$

$$\downarrow$$

$$\partial_a \left(F^{ab}_{\mu\lambda} \partial_b X^\mu\right) = \frac{1}{2} \partial_\lambda F^{ab}_{\mu\nu} \partial_a X^\mu \partial_b X^\nu,$$

$$F^{ab}_{\mu\nu} = T\sqrt{-g} g^{ab} G_{\mu\nu} + T'\sqrt{-h} h^{ab} H_{\mu\nu}$$

let $T' = Tk\Delta/2$ and $h = g^{-1} (H_{\mu\nu} = G_{\mu\nu})$

$$F^{ab}_{\mu\nu} = T\left(\sqrt{-g} g^{ab} G_{\mu\nu} + \frac{k\Delta}{2}\sqrt{-g^{-1}} U^a{}_{a'} (g^{-1})^{a'b'} U_{b'}{}^b (G^{-1})_{\mu\nu}\right),$$

add new field coupled to derivative A_a transforming as

$$A_a g^{bc} \to \left(A_a g^{bc}\right)' = A_a U^b{}_{b'} g^{b'c'} U_{c'}{}^c - \frac{1}{\alpha} g^{b'c'} (\partial_a U^b{}_{b'} U_{c'}{}^c + U^b{}_{b'} \partial_a U_{c'}{}^c),$$

defining new WS covariant derivative

$$D_a g^{bc} = \partial_a g^{bc} + \alpha A_a g^{bc},$$

such that derivative ignores $SO(1,1)$ gauge freedom on inverse metric. Upgrade EoM to incorporate this derivative:

$$D_a (F^{ab}_{\mu\lambda} \partial_b X^\mu) = \frac{1}{2} \partial_\lambda F^{ab}_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$$

$$\downarrow$$

$$F^{ab}_{\mu\lambda} \partial_a \partial_b X^\mu + D_a F^{ab}_{\mu\lambda} \partial_b X^\mu = \frac{1}{2} \partial_\lambda F^{ab}_{\mu\nu} \partial_a X^\mu \partial_b X^\nu,$$

imposing conformal symmetry $g^{ab} = (\phi(x))^{-1} \eta^{ab} \implies D_a F^{ab}_{\mu\lambda} = 0$

$$F^{ab}_{\mu\lambda} \partial_a \partial_b X^\mu = \frac{1}{2} \partial_\lambda F^{ab}_{\mu\nu} \partial_a X^\mu \partial_b X^\nu,$$

$$F^{ab}_{\mu\lambda} = T\left(G_{\mu\lambda} + \frac{k\Delta}{2} (G^{-1})_{\mu\lambda}\right) \eta^{ab}$$

$$(G^{-1})_{\mu\nu} \stackrel{?}{=} \frac{1}{-g} G_{\mu\nu}$$

6 Inverse Area Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

and make quantum geometry correction

$$\sqrt{-g} \to \sqrt{-(g+k\Delta)} \approx \sqrt{-g}\left(1+\frac{k\Delta}{2g}+\mathcal{O}\left(\frac{k^2}{g^2}\right)\right)$$

leading to

$$S_{IAP} = -\frac{T}{2}\int \mathrm{d}^2x\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

6.1 EoMs

$$\begin{aligned}
\frac{\delta S_{IAP}}{\delta g^{cd}} &= \frac{\partial}{\partial g^{cd}}\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)\delta^a_c\delta^b_d\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = \\
&= \left(-\frac{1}{2}\sqrt{-g}g_{cd}+\frac{k\Delta}{4\sqrt{-g}}g_{cd}\right)g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)\partial_c X^\mu \partial_d X^\nu G_{\mu\nu} = \\
&= -\frac{1}{2}\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)\partial_c X^\mu \partial_d X^\nu G_{\mu\nu} \overset{!}{=} 0
\end{aligned}$$

$$\downarrow$$

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} - \frac{1}{2}g_{cd}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = 0$$

$$\downarrow$$

$$\begin{aligned}
g_{cd} &= 2f\partial_c X^\mu \partial_d X^\nu G_{\mu\nu}, \\
\frac{1}{f} &= g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
\frac{\delta S_{IAP}}{\delta X^\lambda} &= \left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - \partial_c\left(\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)2g^{ab}\delta^c_a\delta^\mu_\lambda\partial_b X^\nu G_{\mu\nu}\right) = \\
&= \left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\partial_a\partial_b X^\nu G_{\lambda\nu} - \\
&\quad - 2\partial_a\left(\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\right)\partial_b X^\nu G_{\lambda\nu} = \\
&= \left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2\left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_a\partial_b X^\nu G_{\lambda\nu} - \\
&\quad - 2\partial_a\left(\left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\right)\partial_b X^\nu G_{\lambda\nu} \overset{!}{=} 0
\end{aligned}$$

$$\downarrow$$

$$\left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_a\partial_b X^\nu G_{\lambda\nu} + \partial_a\left(\left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\right)\partial_b X^\nu G_{\lambda\nu} = \frac{1}{2}\left(1-\frac{k\Delta}{2(-g)}\right)\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

imposing conformal symmetry $g^{ab} = (\phi(x))^{-1}\eta^{ab}$ and let $F' = \left(1-\frac{k\Delta}{2\phi^2}\right)$ leads to

$$\eta^{ab}\partial_a\partial_b X^\nu G_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F'\partial_b X^\nu G_{\lambda\nu} = \eta^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

assuming flat background $G_{\mu\nu} = \eta_{\mu\nu}$

$$\eta^{ab}\partial_a\partial_b X^\nu \eta_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F' \partial_b X^\nu \eta_{\lambda\nu} = 0$$

$$\downarrow$$

$$\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\partial_a \ln(F')\partial_b X^\mu = 0,$$

use plane-wave ansatz $X^\mu = X_0^\mu \mathrm{e}^{-i(E\tau - p\sigma)}$

$$(E^2 - p^2)X^\mu + \partial_\tau \ln(F') iEX^\mu - \partial_\sigma \ln(F') ipX^\mu = 0$$

$$\downarrow$$

$$(E^2 - p^2)X^\mu + i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F'))X^\mu = 0$$

$$\downarrow$$

$$i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F')) = -m^2,$$
$$-m^2 = -E^2 + p^2$$

$$\Downarrow$$

$$\partial_\tau \ln(F') = iE,\ \ \partial_\sigma \ln(F') = ip$$

$$\Downarrow$$

$$\ln(F') = i(E\tau + p\sigma) + a,\ \ a \in \mathbb{C}$$

$$\Downarrow$$

$$F' = A\mathrm{e}^{i(E\tau + p\sigma + \phi)},\ \ A > 0$$

$$\Downarrow$$

$$\phi(\tau, \sigma) = \sqrt{\frac{k\Delta}{2\left(1 - A\mathrm{e}^{i(E\tau + p\sigma + \phi)}\right)}}$$

$$\downarrow$$

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

7 Inverse Area Polyakov v2

Quantum geometry correction

$$\sqrt{-g} \to \sqrt{-(g + k\Delta)} \approx \sqrt{-g}\left(1 + \frac{k\Delta}{2g} + \mathcal{O}\left(\frac{k^2}{g^2}\right)\right),$$

which implies

$$g_{ab} \to \tilde{g}_{ab} = g_{ab} + \sqrt{k\Delta}\delta_{ab},$$
$$g^{ab} \to \tilde{g}^{ab},\ \tilde{g}^{ab}\tilde{g}_{bc} = \delta^{a}{}_{c},$$

leading to

$$S = -\frac{T}{2}\int \mathrm{d}^2x\sqrt{-\tilde{g}}\tilde{g}^{ab}\partial_a X^\mu \partial_b X^\nu \widetilde{G}_{\mu\nu},$$
$$\widetilde{G}_{\mu\nu} = G_{\mu\nu} + \sqrt[D]{k\Theta}\delta_{\mu\nu},$$

where Θ is related to minimal hyper-volume of bulk quantum geometry

7.1 EoMs

$$\frac{\delta S}{\delta \tilde{g}^{cd}} = -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}_{cd}\tilde{g}^{ab}\partial_a X^\mu \partial_b X^\nu \widetilde{G}_{\mu\nu} + \sqrt{-\tilde{g}}\partial_c X^\mu \partial_d X^\nu \widetilde{G}_{\mu\nu} \stackrel{!}{=} 0,$$

giving us

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu \widetilde{G}_{\mu\nu} - \frac{1}{2}\tilde{g}_{cd}\tilde{g}^{ab}\partial_a X^\mu \partial_b X^\nu \widetilde{G}_{\mu\nu} = 0$$

from which we extract the corrected metric

$$\tilde{g}_{cd} = 2f\partial_c X^\mu \partial_d X^\nu \widetilde{G}_{\mu\nu},$$

with

$$\frac{1}{f} = \tilde{g}^{ab}\partial_a X^\mu \partial_b X^\nu \widetilde{G}_{\mu\nu}$$

and we can expand the metrics

$$g_{cd} + \sqrt{k\Delta}\delta_{cd} = \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} + \sqrt[D]{k\Theta}\partial_c X^\mu \partial_d X^\nu \delta_{\mu\nu},$$

giving us the constraint

$$\partial_c X^\mu \partial_d X^\nu \delta_{\mu\nu} = \frac{\sqrt{k\Delta}}{\sqrt[D]{k\Theta}}\delta_{cd}$$

8 Relating NG and Polyakov analysis

From NG analysis

$$(F^+)^2\frac{\partial^2 X^\mu}{\partial t^2}-\frac{\partial^2 X^\mu}{\partial x^2}=0,$$

make change of variables $\tau=\tau(t,x)$ and $\sigma=\sigma(t,x)$

$$(F^+)^2\left(\frac{\partial\tau}{\partial t}\frac{\partial}{\partial\tau}+\frac{\partial\sigma}{\partial t}\frac{\partial}{\partial\sigma}\right)\left(\frac{\partial\tau}{\partial t}\frac{\partial X^\mu}{\partial\tau}+\frac{\partial\sigma}{\partial t}\frac{\partial X^\mu}{\partial\sigma}\right)-\left(\frac{\partial\tau}{\partial x}\frac{\partial}{\partial\tau}+\frac{\partial\sigma}{\partial x}\frac{\partial}{\partial\sigma}\right)\left(\frac{\partial\tau}{\partial x}\frac{\partial X^\mu}{\partial\tau}+\frac{\partial\sigma}{\partial x}\frac{\partial X^\mu}{\partial\sigma}\right)=0$$

$$\begin{aligned}(F^+)^2&\left(\left(\frac{\partial\tau}{\partial t}\right)^2\frac{\partial^2 X^\mu}{\partial\tau^2}+2\frac{\partial\tau}{\partial t}\frac{\partial\sigma}{\partial t}\frac{\partial^2 X^\mu}{\partial\tau\partial\sigma}+\left(\frac{\partial\sigma}{\partial t}\right)^2\frac{\partial^2 X^\mu}{\partial\sigma^2}\right)-\\&-\left(\left(\frac{\partial\tau}{\partial x}\right)^2\frac{\partial^2 X^\mu}{\partial\tau^2}+2\frac{\partial\tau}{\partial x}\frac{\partial\sigma}{\partial x}\frac{\partial^2 X^\mu}{\partial\tau\partial\sigma}+\left(\frac{\partial\sigma}{\partial x}\right)^2\frac{\partial^2 X^\mu}{\partial\sigma^2}\right)+\\&+\left((F^+)^2\frac{\partial^2\tau}{\partial t^2}-\frac{\partial^2\tau}{\partial x^2}\right)\frac{\partial X^\mu}{\partial\tau}+\left((F^+)^2\frac{\partial^2\sigma}{\partial t^2}-\frac{\partial^2\sigma}{\partial x^2}\right)\frac{\partial X^\mu}{\partial\sigma}=0\end{aligned}$$

$$\begin{aligned}&\left((F^+)^2(\partial_t\tau)^2-(\partial_x\tau)^2\right)\partial_\tau^2X^\mu+\left((F^+)^2(\partial_t\sigma)^2-(\partial_x\sigma)^2\right)\partial_\sigma^2X^\mu+\\&+2\left((F^+)^2\partial_t\tau\partial_t\sigma-\partial_x\tau\partial_x\sigma\right)\partial_\tau\partial_\sigma X^\mu+\left((F^+)^2\partial_t^2\tau-\partial_x^2\tau\right)\partial_\tau X^\mu+\left((F^+)^2\partial_t^2\sigma-\partial_x^2\sigma\right)\partial_\sigma X^\mu=0\end{aligned}$$

Without loss of generality, we can set (just rescale/rotate the coordinates)

$$\begin{aligned}(F^+)^2(\partial_t\sigma)^2-(\partial_x\sigma)^2&=(\partial_x\tau)^2-(F^+)^2(\partial_t\tau)^2\\(F^+)^2\partial_t\tau\partial_t\sigma&=\partial_x\tau\partial_x\sigma,\end{aligned}$$

or more compactly

$$\begin{aligned}\partial_t\tau&=v\partial_x\sigma\\\partial_t\sigma&=v\partial_x\tau,\end{aligned}$$

where $v=1/F^+$, which is equivalent to

$$\begin{aligned}\partial_t^2\tau&=\partial_tv\partial_x\sigma+v\partial_xv\partial_x\tau+v^2\partial_x^2\tau\\\partial_t^2\sigma&=\partial_tv\partial_x\tau+v\partial_xv\partial_x\sigma+v^2\partial_x^2\sigma,\end{aligned}$$

reducing the big equation to

$$\partial_\tau^2X^\mu-\partial_\sigma^2X^\mu+\frac{\partial_tv\partial_x\sigma+v\partial_xv\partial_x\tau}{((\partial_t\tau)^2-v^2(\partial_x\tau)^2)}\partial_\tau X^\mu+\frac{\partial_tv\partial_x\tau+v\partial_xv\partial_x\sigma}{((\partial_t\tau)^2-v^2(\partial_x\tau)^2)}\partial_\sigma X^\mu=0.$$

Next introduce $X^\mu=\kappa(\tau,\sigma)Y^\mu(\tau,\sigma)$ s.t terms proportional to $\partial_\tau Y^\mu$ and $\partial_\sigma Y^\mu$ vanish:

$$\begin{aligned}\partial_\tau X^\mu&=\partial_\tau\kappa Y^\mu+\kappa\partial_\tau Y^\mu\\\partial_\sigma X^\mu&=\partial_\sigma\kappa Y^\mu+\kappa\partial_\sigma Y^\mu\\\partial_\tau^2X^\mu&=\partial_\tau^2\kappa Y^\mu+2\partial_\tau\kappa\partial_\tau Y^\mu+\kappa\partial_\tau^2Y^\mu\\\partial_\sigma^2X^\mu&=\partial_\sigma^2\kappa Y^\mu+2\partial_\sigma\kappa\partial_\sigma Y^\mu+\kappa\partial_\sigma^2Y^\mu\end{aligned}$$

$$\begin{aligned}\kappa\frac{\partial_tv\partial_x\sigma+v\partial_xv\partial_x\tau}{((\partial_t\tau)^2-v^2(\partial_x\tau)^2)}+2\partial_\tau\kappa&\overset{!}{=}0\\\kappa\frac{\partial_tv\partial_x\tau+v\partial_xv\partial_x\sigma}{((\partial_t\tau)^2-v^2(\partial_x\tau)^2)}-2\partial_\sigma\kappa&\overset{!}{=}0\end{aligned}$$

$$\partial_\tau\kappa = -\frac{\kappa(\partial_t v\partial_x\sigma + v\partial_x v\partial_x\tau)}{2\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)}$$
$$\partial_\sigma\kappa = \frac{\kappa(\partial_t v\partial_x\tau + v\partial_x v\partial_x\sigma)}{2\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)},$$

undoing chain rule for t by multiplying first eq by $\partial_t\tau$ and second by $\partial_t\sigma$ and summing gives

$$\begin{aligned}
\partial_t\kappa &= \partial_t\tau\partial_\tau\kappa + \partial_t\sigma\partial_\sigma\kappa = \\
&= \partial_t\tau\left(-\frac{\kappa(\partial_t v\partial_x\sigma + v\partial_x v\partial_x\tau)}{2\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)}\right) + \partial_t\sigma\left(\frac{\kappa(\partial_t v\partial_x\tau + v\partial_x v\partial_x\sigma)}{2\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)}\right) = \\
&= v\partial_x\sigma\left(-\frac{\kappa(\partial_t v\partial_x\sigma + v\partial_x v\partial_x\tau)}{2\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)}\right) + v\partial_x\tau\left(\frac{\kappa(\partial_t v\partial_x\tau + v\partial_x v\partial_x\sigma)}{2\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)}\right) = \\
&= \frac{v\kappa\partial_t v\left((\partial_x\tau)^2 - (\partial_x\sigma)^2\right)}{2\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)} = \\
&= \frac{v\kappa\partial_t v\left((\partial_x\tau)^2 - (\partial_t\tau/v)^2\right)}{2\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)} = \\
&= -\frac{\kappa\partial_t v}{2v},
\end{aligned}$$

or simply

$$\frac{\partial_t\kappa}{\kappa} = -\frac{1}{2}\frac{\partial_t v}{v}$$
$$\partial_t\ln(\kappa) = -\frac{1}{2}\partial_t\ln(v)$$
$$\ln(\kappa) = -\frac{1}{2}\ln(v) + f(x).$$

To determine $f(x)$, we undo the chain rule for x now:

$$\begin{aligned}
\partial_x\kappa &= \partial_x\tau\partial_\tau\kappa + \partial_x\sigma\partial_\sigma\kappa = \\
&= \partial_x\tau\left(-\frac{\kappa(\partial_t v\partial_x\sigma + v\partial_x v\partial_x\tau)}{2\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)}\right) + \partial_x\sigma\left(\frac{\kappa(\partial_t v\partial_x\tau + v\partial_x v\partial_x\sigma)}{2\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)}\right) = \\
&= \frac{\kappa v\partial_x v\left((\partial_x\sigma)^2 - (\partial_x\tau)^2\right)}{2\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)} = \\
&= \frac{\kappa v\partial_x v\left((\partial_t\tau/v)^2 - (\partial_x\tau)^2\right)}{2\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)} = \\
&= \frac{\kappa\partial_x v}{2v}
\end{aligned}$$

$$\frac{\partial_x\kappa}{\kappa} = \frac{1}{2}\frac{\partial_x v}{v}$$
$$\partial_x\ln(\kappa) = \frac{1}{2}\partial_x\ln(v)$$
$$\ln(\kappa) = \frac{1}{2}\ln(v),$$
$$\partial_x\left(-\frac{1}{2}\ln(v) + f(x)\right) = \frac{1}{2}\partial_x\ln(v)$$

$f'(x) = \partial_x\ln(v)$ (perhaps this means somewhere in the composition of functions time dependence is lost?)

$$f(x) = \ln(v)$$

giving us

$$\kappa = v^{\frac{1}{2}},\ v = v(x).$$

Substituting this into the earlier equation yields

$$\partial_\tau^2 Y^\mu - \partial_\sigma^2 Y^\mu + m^2 Y^\mu = 0,$$

with

$$
\begin{aligned}
m^2 &= \frac{\partial_\tau^2\kappa - \partial_\sigma^2\kappa}{\kappa} + \frac{(\partial_t v\partial_x\sigma + v\partial_x v\partial_x\tau)\partial_\tau\kappa + (\partial_t v\partial_x\tau + v\partial_x v\partial_x\sigma)\partial_\sigma\kappa}{\kappa\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)} = \\
&= \frac{\partial_\tau^2\kappa - \partial_\sigma^2\kappa}{\kappa} + \frac{\partial_t v(\partial_x\sigma\partial_\tau\kappa + \partial_x\tau\partial_\sigma\kappa) + v\partial_x v\partial_x\kappa}{\kappa\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)} = \\
&= \frac{\partial_\tau^2\kappa - \partial_\sigma^2\kappa}{\kappa} + \frac{(\partial_x v)^2}{\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)},
\end{aligned}
$$

where by use of chain rule we have that

$$
\partial_t^2\kappa - v^2\partial_x^2\kappa - v\partial_x v\partial_x\kappa = ((\partial_t\tau)^2 - v^2(\partial_x\tau)^2)(\partial_\tau^2\kappa - \partial_\sigma^2\kappa),
$$

where since $\kappa = v^{\frac{1}{2}}$, m^2 reduces to

$$
m^2 = \frac{(\partial_x v)^2 - 2v\partial_x^2 v}{4\left((\partial_t\tau)^2 - v^2(\partial_x\tau)^2\right)},
$$

with τ satisfying

$$
\begin{aligned}
\partial_t\tau &= v\partial_x\sigma \\
v\partial_x\tau &= \partial_t\sigma
\end{aligned}
$$

such that $-m^2$ is indeed constant.

9 Solutions for the Polyakov KG eqn (WRONG ONE)

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi \eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta_{cd} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\begin{aligned}
\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\
\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\
\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0.
\end{aligned}$$

These simplify further by expanding the summation on a, b:

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2} \left(-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} \right)$$

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}.$$

The EoM is just a one-dimensional Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int \mathrm{d}p \frac{1}{2E(p)} \left(a^\mu(p) \mathrm{e}^{-i(-E\tau + p\sigma)} + b^\mu(p) \mathrm{e}^{i(-E\tau + p\sigma)} \right),$$

where $E(p) = \sqrt{p^2 + m^2}$. The derivatives are

$$\begin{aligned}
\partial_\tau X^\mu &= \frac{i}{4\pi} \int \mathrm{d}p \left(a^\mu(p) \mathrm{e}^{-i(-E\tau + p\sigma)} - b^\mu(p) \mathrm{e}^{i(-E\tau + p\sigma)} \right) \\
\partial_\sigma X^\mu &= \frac{i}{4\pi} \int \mathrm{d}p \frac{p}{\sqrt{p^2 + m^2}} \left(-a^\mu(p) \mathrm{e}^{-i(-E\tau + p\sigma)} + b^\mu(p) \mathrm{e}^{i(-E\tau + p\sigma)} \right).
\end{aligned}$$

Starting with the mixed constraint,

$$\begin{aligned}
\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int \mathrm{d}p \left(a^\mu(p) \mathrm{e}^{-i(-E(p)\tau + p\sigma)} - b^\mu(p) \mathrm{e}^{i(-E(p)\tau + p\sigma)} \right) \times \\
\times \int \mathrm{d}p' \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\nu(p') \mathrm{e}^{-i(-E(p')\tau + p'\sigma)} + b^\nu(p') \mathrm{e}^{i(-E(p')\tau + p'\sigma)} \right) =
\end{aligned}$$

$$\begin{aligned}
= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int \mathrm{d}p\mathrm{d}p' \frac{p'}{\sqrt{p'^2 + m^2}} \left(a^\mu(p) \mathrm{e}^{-i(-E(p)\tau + p\sigma)} - b^\mu(p) \mathrm{e}^{i(-E(p)\tau + p\sigma)} \right) \times \\
\times \left(-a^\nu(p') \mathrm{e}^{-i(-E(p')\tau + p'\sigma)} + b^\nu(p') \mathrm{e}^{i(-E(p')\tau + p'\sigma)} \right) =
\end{aligned}$$

$$\begin{aligned}
= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int \mathrm{d}p\mathrm{d}p' \frac{p'}{\sqrt{p'^2 + m^2}} \Big(- a^\mu(p) a^\nu(p') \mathrm{e}^{i(E(p) + E(p'))\tau} \mathrm{e}^{-i(p + p')\sigma} + a^\mu(p) b^\nu(p') \mathrm{e}^{i(E(p) - E(p'))\tau} \mathrm{e}^{-i(p - p')\sigma} + \\
+ b^\mu(p) a^\nu(p') \mathrm{e}^{-i(E(p) - E(p'))\tau} \mathrm{e}^{i(p - p')\sigma} - b^\mu(p) b^\nu(p') \mathrm{e}^{-i(E(p) + E(p'))\tau} \mathrm{e}^{i(p + p')\sigma} \Big) = 0.
\end{aligned}$$

If we integrate this expression over σ we obtain

$$-\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\int \mathrm{d}\sigma \mathrm{e}^{-i(p+p')\sigma}+$$
$$+a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\int \mathrm{d}\sigma \mathrm{e}^{-i(p-p')\sigma}+b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\int \mathrm{d}\sigma \mathrm{e}^{i(p-p')\sigma}-$$
$$-b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\int \mathrm{d}\sigma \mathrm{e}^{i(p+p')\sigma}\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p')+a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p')+$$
$$+b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p')-b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p')\Big)=$$
$$=-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\frac{p}{\sqrt{p^2+m^2}}\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}+a^\mu(p)b^\nu(p)+b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)=0$$

which implies the whole expression inside the integral must vanish, thus we have (leaving $(a^\mu b^\nu+b^\mu a^\nu)\eta_{\mu\nu}$ as is since eventually these will not be numbers and may not commute, thus will not be equal to $2a^\mu b^\nu\eta_{\mu\nu}$)

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}+a^\mu(p)b^\nu(p)+b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu}=0.$$

As for the other part of the constraint, let's start with the L.H.S

$$\partial_\tau X^\mu\partial_\tau X^\nu\eta_{\mu\nu}=-\frac{1}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)\mathrm{e}^{-i(-E(p)\tau+p\sigma)}-b^\mu(p)\mathrm{e}^{i(-E(p)\tau+p\sigma)}\Big)\times$$
$$\times\Big(a^\nu(p')\mathrm{e}^{-i(-E(p')\tau+p'\sigma)}-b^\nu(p')\mathrm{e}^{i(-E(p')\tau+p'\sigma)}\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\mathrm{e}^{-i(p+p')\sigma}-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\mathrm{e}^{-i(p-p')\sigma}-$$
$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\mathrm{e}^{i(p-p')\sigma}+b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\mathrm{e}^{i(p+p')\sigma}\Big),$$

which we integrate over σ to get

$$-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p')-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p')-$$
$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p')+b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p')\Big)=$$
$$=-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}-a^\mu(p)b^\nu(p)-b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big),$$

and now for the R.H.S

$$-\partial_\sigma X^\mu\partial_\sigma X^\nu\eta_{\mu\nu}=\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{p}{\sqrt{p^2+m^2}}\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)\mathrm{e}^{-i(-E\tau+p\sigma)}+b^\mu(p)\mathrm{e}^{i(-E\tau+p\sigma)}\Big)\times$$
$$\times\Big(-a^\nu(p')\mathrm{e}^{-i(-E(p')\tau+p'\sigma)}+b^\nu(p')\mathrm{e}^{i(-E(p')\tau+p'\sigma)}\Big)=$$

$$=\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{pp'}{E(p)E(p')}\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\mathrm{e}^{-i(p+p')\sigma}-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\mathrm{e}^{-i(p-p')\sigma}-$$
$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\mathrm{e}^{i(p-p')\sigma}+b^\mu(p)b^\nu(p)\mathrm{e}^{-i(E(p)+E(p'))\tau}\mathrm{e}^{i(p+p')\sigma}\Big),$$

which once again we integrate over σ,

$$\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\mathrm{d}p'\frac{pp'}{E(p)E(p')}\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p')-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p')-$$
$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p')+b^\mu(p)b^\nu(p)\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p')\Big)=$$

$$=\frac{1}{8\pi}\int \mathrm{d}p\frac{p^2}{p^2+m^2}\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}-a^\mu(p)b^\nu(p)-b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu},$$

which when equated to the L.H.S yields

$$\int \mathrm{d}p\left(\frac{p^2}{p^2+m^2}+1\right)\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}-a^\mu(p)b^\nu(p)-b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu}=0,$$

which implies

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}-a^\mu(p)b^\nu(p)-b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu}=0.$$

We can sum and subtract this with the previous condition to simplify and obtain

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu}=0$$
$$\Big(a^\mu(p)b^\nu(p)+b^\mu(p)a^\nu(p)\Big)\eta_{\mu\nu}=0,$$

and for while a^μ and b^ν are number-valued, the second condition can be further simplified to

$$a^\mu(p)b^\nu(p)\eta_{\mu\nu}=0,$$

meaning a^μ and b^ν are orthogonal. In general, this condition states that a^μ and b^ν anti-commute w.r.t Lorentz inner product. The first condition must hold for all values of τ, and since the exponentials are linearly independent, we thus have

$$a^\mu(p)a^\nu(-p)\eta_{\mu\nu}=0$$
$$b^\mu(p)b^\nu(-p)\eta_{\mu\nu}=0,$$

meaning that reflecting the argument of a^μ and b^ν creates orthogonal vectors. We can thus write

$$a^\mu(p)=\Lambda^\mu{}_\nu\left(\frac{p}{4}\right)a^\nu(0)$$
$$b^\mu(p)=\Upsilon^\mu{}_\nu\left(\frac{p}{4}\right)b^\nu(0),$$

where $\Lambda,\Upsilon\in\mathrm{SO}^+(1,d)$. Also. since the orthogonality under reflection must hold for all values of p, in particular we have

$$a^\mu(0)a^\nu(0)\eta_{\mu\nu}=0$$
$$b^\mu(0)b^\nu(0)\eta_{\mu\nu}=0,$$

meaning that the initial $a^\mu(0)$ and $b^\nu(0)$ are null vectors. This means that $a^\mu(p)$ and $b^\nu(p)$ are also null vectors since they are related to the initial values by Lorentz transformation. With this, the solution becomes

$$X^\mu(\tau,\sigma)=\frac{1}{2\pi}\int \mathrm{d}p\frac{1}{2E(p)}\left(\Lambda^\mu{}_\nu\left(\frac{p}{4}\right)a^\nu(0)\mathrm{e}^{-i(-E\tau+p\sigma)}+\Upsilon^\mu{}_\nu\left(\frac{p}{4}\right)b^\nu(0)\mathrm{e}^{i(-E\tau+p\sigma)}\right)$$

with $(a^\mu b^\nu+b^\nu a^\mu)\eta_{\mu\nu}=0$. This condition implies

$$\left(\Lambda^\mu{}_\gamma\left(\frac{p}{4}\right)a^\gamma(0)\Upsilon^\nu{}_\rho\left(\frac{p}{4}\right)b^\rho(0)\right)\eta_{\mu\nu}=0$$

$$\Upsilon^\mu{}_\nu\left(\frac{p}{4}\right)=\Lambda_\nu{}^\mu\left(\frac{p}{4}\right)=\Lambda^\mu{}_\nu\left(-\frac{p}{4}\right),$$

thus turning the solution into

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi}\int \mathrm{d}p \frac{1}{2E(p)}\left(\Lambda^\mu{}_\nu\left(\frac{p}{4}\right)a^\nu(0)\mathrm{e}^{-i(-E\tau+p\sigma)} + \Lambda^\mu{}_\nu\left(-\frac{p}{4}\right)b^\nu(0)\mathrm{e}^{i(-E\tau+p\sigma)}\right).$$

Next, we choose background coordinates s.t $a^\mu(0) = (a_0, a_0, 0, ..., 0) = a_0 x^+$ and $b^\nu(0) = (b_0, b_0, 0, ..., 0) = b_0 x^+$, simplifying the solution to

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi}\int \mathrm{d}p \frac{1}{2E(p)}\left(a_0\Lambda^\mu{}_+\left(\frac{p}{4}\right)\mathrm{e}^{-i(-E\tau+p\sigma)} + b_0\Lambda^\mu{}_+\left(-\frac{p}{4}\right)\mathrm{e}^{i(-E\tau+p\sigma)}\right)x^+$$

10 Solutions for the Polyakov KG eqn

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi\eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta_{cd} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\begin{aligned}
\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\
\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\
\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0.
\end{aligned}$$

These simplify further by expanding the summation on a, b:

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2} \left(-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} \right)$$

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}.$$

The EoM is just a one-dimensional finite-space Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu \mathrm{e}^{-i(-E_n\tau + n\sigma)} + b_n^\mu \mathrm{e}^{i(-E_n\tau+n\sigma)} \right),$$

with $E_n = \sqrt{n^2 + m^2}$. The derivatives are

$$\begin{aligned}
\partial_\tau X^\mu &= \frac{i}{2} \sum_{n\in\mathbb{Z}} \left(a_n^\mu \mathrm{e}^{-i(-E_n\tau+n\sigma)} - b_n^\mu \mathrm{e}^{i(-E_n\tau+n\sigma)} \right) \\
\partial_\sigma X^\mu &= \frac{i}{2} \sum_{n\in\mathbb{Z}} \frac{n}{E_n} \left(-a_n^\mu \mathrm{e}^{-i(-E_n\tau+n\sigma)} + b_n^\mu \mathrm{e}^{i(-E_n\tau+n\sigma)} \right).
\end{aligned}$$

Starting from the mixed constraint, we have

$$\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = -\frac{\eta_{\mu\nu}}{4} \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \frac{k}{E_k} \left(a_n^\mu \mathrm{e}^{-i(-E_n\tau+n\sigma)} - b_n^\mu \mathrm{e}^{i(-E_n\tau+n\sigma)} \right) \left(-a_k^\nu \mathrm{e}^{-i(-E_k\tau+k\sigma)} + b_k^\nu \mathrm{e}^{i(-E_k\tau+k\sigma)} \right) =$$

$$\begin{aligned}
= -\frac{\eta_{\mu\nu}}{4} \sum_n \sum_k \frac{k}{E_k} \Big(&- a_n^\mu a_k^\nu \mathrm{e}^{i(E_n+E_k)\tau} \mathrm{e}^{-i(n+k)\sigma} + a_n^\mu b_k^\nu \mathrm{e}^{i(E_n-E_k)\tau} \mathrm{e}^{-i(n-k)\sigma} + \\
&+ b_n^\mu a_k^\nu \mathrm{e}^{-i(E_n-E_k)\tau} \mathrm{e}^{i(n-k)\sigma} - b_n^\mu b_k^\nu \mathrm{e}^{-i(E_n+E_k)\tau} \mathrm{e}^{i(n+k)\sigma} \Big) = 0,
\end{aligned}$$

for which we have 3 cases to analyse: $n = k$, $n = -k$ and $-k \neq n \neq k$. Starting with $-k \neq n \neq k$, since the equality must hold for all τ and σ and the exponentials are linearly independent, we get that each term must vanish:

$$\begin{aligned}
a_n^\mu a_k^\nu \eta_{\mu\nu} &= 0 \\
a_n^\mu b_k^\nu \eta_{\mu\nu} &= 0 \\
b_n^\mu a_k^\nu \eta_{\mu\nu} &= 0 \\
b_n^\mu b_k^\nu \eta_{\mu\nu} &= 0, \ -k \neq n \neq k.
\end{aligned}$$

For $n = k$, the 2 middle terms become linearly dependent, thus we get

$$\begin{aligned} a_n^\mu a_n^\nu \eta_{\mu\nu} &= 0 \\ (a_n^\mu b_n^\mu + b_n^\mu a_n^\nu)\eta_{\mu\nu} &= 0 \\ b_n^\mu b_n^\nu \eta_{\mu\nu} &= 0, \end{aligned}$$

and finally for $n = -k$, this time the τ exponential still survives since E_n is symmetric, thus we get

$$\begin{aligned} a_n^\mu a_{-n}^\nu \eta_{\mu\nu} &= 0 \\ a_n^\mu b_{-n}^\nu \eta_{\mu\nu} &= 0 \\ b_n^\mu a_{-n}^\nu \eta_{\mu\nu} &= 0 \\ b_n^\mu b_{-n}^\nu \eta_{\mu\nu} &= 0. \end{aligned}$$

The first and last conditions in each group can be combined into

$$\begin{aligned} a_n^\mu a_k^\nu \eta_{\mu\nu} &= 0 \\ b_n^\mu b_k^\nu \eta_{\mu\nu} &= 0, \ \forall \ n, k \in \mathbb{Z}. \end{aligned}$$

The middle conditions can be summed to combine and become

$$(a_n^\mu b_k^\nu + b_n^\nu a_k^\mu)\eta_{\mu\nu} = 0, \ \forall \ n, k \in \mathbb{Z}.$$

Repeating the analysis for the other constraint will give the same results.

So far, we have thus that a and b for different values of n are orthogonal to each other, and in particular when the index are equal. Thus, they are null vectors, and remain always colinear since that's the only way two null vectors can have a 0 dot product. Altogether, this leads to

$$a_n^\mu = a_n a_0^\mu, \ a_0 = 1, \ a_0^\mu a_0^\nu \eta_{\mu\nu} = 0$$

$$b_n^\mu = b_n a_n^\mu = b_n a_n a_0^\mu.$$

Without loss of generality, we can choose background coordinates such that $a_0^\mu = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, ..., 0) = x^+$, giving us

$$X^\mu(\tau, \sigma) = a_0^\mu \sum_n \frac{a_n}{2E_n} \left(\mathrm{e}^{-i(-E_n\tau + n\sigma)} + b_n \mathrm{e}^{i(-E_n\tau + n\sigma)} \right)$$