Sketchboard

June 1, 2024

1 Basic concepts

1.1 With extra capital internal index on embbeding fields

Promote embedding fields X^μ to have an internal group index with D values

$$X^\mu \to X^{\mu I} \ , \quad I = 0, ..., d$$

$$g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{IJ} \sim fD_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu}$$

$$D_a X^{\mu I} = \partial_a X^{\mu I} + \omega_{aJ}^I X^{\mu J}$$

$$e_a^i e_b^j \eta_{ij} = g_{ab} \to e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ} \ , \quad i, j = 0, 1$$

$$g = \det(g_{ab}) \to g = \det(\text{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.2 With 1 extra small internal index on embedding fields

Promote embbeding fields to have a internal group index with 2 values

$$X^\mu \to X^{\mu i} \ , \quad i = 0, 1$$

$$g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{ij} \sim fD_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu}$$

$$D_a X^{\mu i} = \partial_a X^{\mu i} + \omega_{aj}^i X^{\mu j}$$

$$e_a^i e_b^j \eta_{ij} = g_{ab} \to e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij}$$

$$g = \det(g_{ab}) \to g = \det(\text{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.3 With 2 extra small indices

Promote embbeding fields X^μ to have two internal indices with 2 values

$$X^\mu \to X^{\mu ij} \ , \quad i, j = 0, 1$$

1.4 Without extra index on embbeding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$e_a^i e_b^j \eta_{ij} = g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$g = \det(g_{ab}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int d\tau \wedge d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \ ,$$

2.1 With capital internal index

... and promote partial derivative ∂_a to covariant derivative D_a, giving us our first attempt at modified Polyakov action

$$S_{MP1} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}$$

$$\updownarrow$$

$$\mathscr{L}_{MP1} = -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}$$

2.2 With small internal index

2.2.1 Without extra field, 1 internal index

... and promote derivatives to covariant D_a, giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

$$\updownarrow$$

$$\mathscr{L}_{MP2} = -\frac{T_0}{2} \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

2.2.2 With extra field

..., promote partial derivative to covariant and add extra internal WS field v^i leading us to

$$S_{MP3} = -\frac{T}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

$$\updownarrow$$

$$\mathscr{L}_{MP3} = -\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

2.2.3 Without extra field, 2 internal indices

... and promote derivatives to covariant D_a,

$$S_{MP4} = -\frac{T}{2} \int d\tau \wedge d\sigma e e_i^a e^{bi} D_a X^{\mu j j'} \eta_{jj'} \eta_{kk'} D_b X^{\nu kk'} E_\mu^I E_{\nu I}$$

$$\mathscr{L}_{MP4} = -\frac{T}{2} e e_i^a e_j^b D_a X^{\mu i k} \eta_{kl} D_b X^{\nu l j} E_\mu^I E_{\nu I}$$

2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) \ .$$

$$\updownarrow$$

$$\mathscr{L}_{DP} = -\frac{T_0}{2} \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) \ .$$

2.4 Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of D-dimensional gamma matrices and bulk spinors

$$S_{LP} = -\frac{T}{2} \int d\tau \wedge d\sigma \overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

$$\updownarrow$$

$$\mathscr{L}_{LP} = -\frac{T}{2} \overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

3 EoMs

Start by writing $\det(e) = \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n$ and $g_{ab} = g_{ab}^{ij} \eta_{ij}$

3.1 With small internal index

3.1.1 Without extra field, w.r.t e

$$\frac{\delta S_{MP2}}{\delta e_l^e} = \frac{D\mathscr{L}_{MP2}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e_c^m e_d^n e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I}\right) =$$

$$= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(\eta^{ml}g_{ce}e_d^n e_i^a e_j^b + e_c^m \eta^{nl}g_{de}e_i^a e_j^b +$$

$$+ e_c^m e_d^n \delta_e^a \delta_i^l e_j^b + e_c^m e_d^n e_i^a \delta_e^b \delta_j^l)D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} =$$

$$= -\frac{T}{2}\left(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}g_{ce}e_d^n e_i^a e_j^b + 2\det(e)e_i^a \delta_e^b \delta_j^l\right)D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} =$$

$$= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}e_c^k e_{ek}e_d^n e_i^a e_j^b + 2\det(e)e_i^a \delta_e^b \delta_j^l)D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} =$$

$$= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}e_c^m e_d^n e_e^l e_i^a e_j^b + 2\det(e)e_i^a \delta_e^b \delta_j^l)D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} =$$

$$= -T(-\det(e)e_e^l e_i^a e_j^b + \det(e)e_i^a \delta_e^b \delta_j^l)D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \overset{!}{=} 0$$

$$\downarrow$$

$$T_e^l := E_\mu^I E_{\nu I}(e_i^a D_a X^{\mu i} D_e X^{\nu l} - e_e^l e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j}) = 0$$

$$\Downarrow$$

$$e_e^l = f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I} \ ,$$

$$\frac{1}{f} = e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I}$$

$$\downarrow$$

$$e_e^l e_{fl} = (f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I})(f e_{i'}^{a'} D_{a'} X^{\mu' i'} D_f X_l^{\nu'} E_{\mu'}^{I'} E_{\nu' I'}) =$$

$$= f^2 D_e X^{\nu l} D_f X_l^{\nu'} E_\nu^{I'} E_{\nu' I'}(e_i^a e_{i'}^{a'} D_a X^{\mu i} D_{a'} X^{\mu' i'} E_\mu^I E_{\mu' I}) =$$

$$= f D_e X^{\mu i} D_f X_i^\nu E_\mu^I E_{\nu I}$$

$$\Downarrow$$

$$D_e X^{\mu i} D_f X_i^\nu E_\mu^I E_{\nu I} = 2\partial_e X^\mu \partial_f X^\nu G_{\mu\nu}$$

$$e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\Downarrow$$

$$D_a X^{\mu 0} = -i\partial_a X^\mu$$

$$D_a X^{\mu 1} = \partial_a X^\mu$$

$$\Downarrow$$

$$X^{\mu 0} = -iX^{\mu 1}$$

3.1.2 With extra field, w.r.t e

$$\frac{\delta S_{MP3}}{\delta e_o^e} = \frac{D\mathscr{L}_{MP3}}{De_o^e} = \left(\frac{\partial}{\partial e_o^e} - \partial_f \frac{\partial}{\partial(\partial_f e_o^e)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e_c^m e_d^n \eta^{ij}e_i^a e_j^b D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ}\right) =$$

$$= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(g_{ce}\eta^{mo}e_d^n \eta^{ij}e_i^a e_j^b + e_c^m g_{de}\eta^{no}\eta^{ij}e_i^a e_j^b +$$

$$+ e_c^m e_d^n \eta^{ij}\delta_e^a \delta_i^o e_j^b + e_c^m e_d^n \eta^{ij}e_i^a \delta_e^b \delta_j^o)D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} =$$

$$= -\frac{T}{2}\varepsilon^{cd}\varepsilon_{mn}(e_c^p e_{ep}e_d^n \eta^{mo}\eta^{ij}e_i^a e_j^b + 2\det(e)e^{ao}\delta_e^b)D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} =$$

$$= -T(-\det(e)e_e^o \eta^{ij}e_i^a e_j^b + \det(e)e^{ao}\delta_e^b)D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ} \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{T_e^o := (e^{ao}D_a(X_k^\mu v^k)D_e(X_l^\nu v^l) - e_e^o \eta^{ij}e_i^a e_j^b D_a(X_k^\mu v^k)D_b(X_l^\nu v^l))E_\mu^I E_\nu^J \eta_{IJ} = 0}$$

$$\Downarrow$$

$$e_e^o = f\, e^{ao}D_a(X_k^\mu v^k)D_e(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ}\ ,$$

$$\frac{1}{f} = \eta^{ij}e_i^a e_j^b D_a(X_k^\mu v^k)D_b(X_l^\nu v^l)E_\mu^I E_\nu^J \eta_{IJ}$$

3.1.3 With extra field, w.r.t ω

$$\frac{\delta S_{MP3}}{\delta\omega_c^{mn}} = \frac{D\mathscr{L}_{MP3}}{D\omega_c^{mn}} = \left(\frac{\partial}{\partial\omega_c^{mn}} - \partial_d \frac{\partial}{\partial(\partial_d \omega_c^{mn})}\right)\left(-\frac{T}{2}\det(e)\eta^{ij}e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}\right) =$$

$$= -\frac{T}{4}eg^{ab}((-\delta_a^c \delta_{[m}^{k'}\eta_{n]k}X_{k'}^\mu v^k)D_b X_l^\nu v^l + D_a X_k^\mu v^k(-\delta_b^c \delta_{[m}^{l'}\eta_{n]l}X_{l'}^\nu v^l))G_{\mu\nu} =$$

$$= \frac{T}{4}e(g^{ca}X_{[m}^\mu v_{n]}D_a X_k^\nu v^k + g^{ac}D_a X_k^\mu v^k X_{[m}^\nu v_{n]})G_{\mu\nu} =$$

$$= \frac{T}{2}eg^{ac}D_a X_k^\mu v^k X_{[m}^\nu v_{n]}G_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{\mathcal{T}_{ab}^i := e^{ci}D_c X_k^\mu v^k X_{[m}^\nu v_{n]}e_a^m e_b^n G_{\mu\nu} = 0}$$

3.1.4 With extra field, w.r.t v

$$\frac{\delta S_{MP3}}{\delta v^l} = \frac{D\mathscr{L}_{MP3}}{Dv^l} = \left(\frac{\partial}{\partial v^l} - \partial_c \frac{\partial}{\partial(\partial_c v^l)}\right)\left(-\frac{T}{2}ee_i^a e^{bi}D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I}\right) =$$

$$= -\frac{T}{2}g^{ab}(D_a X_j^\mu \delta_l^j D_b X_k^\nu v^k + D_a X_j^\mu v^j D_b X_k^\nu \delta_l^k)G_{\mu\nu} =$$

$$= -Tg^{ab}D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{g^{ab}D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} = 0}$$

3.1.5 With extra field, w.r.t X

$$\frac{\delta S_{MP3}}{\delta X_l^\lambda} = \frac{D\mathscr{L}_{MP3}}{DX_l^\lambda} = \left(\frac{\partial}{\partial X_l^\lambda} - \partial_c \frac{\partial}{\partial(\partial_c X_l^\lambda)}\right)\left(-\frac{T}{2}ee_i^a e^{bi}D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I}\right) =$$

$$= -\frac{T}{2}\left(eg^{ab}\left((-\omega_{aj}^{j'}\delta_\lambda^\mu \delta_{j'}^l v^j D_b X_k^\nu v^k - D_a X_j^\mu v^j \omega_{bk}^{k'}\delta_\lambda^\nu \delta_{k'}^l v^k)E_\mu^I E_{\nu I}+\right.\right.$$

$$\left.+ eg^{ab}D_a X_j^\mu v^j D_b X_k^\nu v^k\left(\frac{\partial E_\mu^I}{\partial X_l^\lambda}E_{\nu I} + E_\mu^I \frac{\partial E_{\nu I}}{\partial X_l^\lambda}\right)\right)-$$

$$\left.- \partial_c(eg^{ab}(\delta_a^c \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k + D_a X_j^\mu v^j \delta_b^c \delta_\lambda^\nu \delta_k^l v^k)G_{\mu\nu})\right) =$$

$$= -T\left(-eg^{ab}D_a X_j^\mu v^j \omega_{bk}^l v^k G_{\mu\lambda} + eg^{ab}D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda}E_{\nu I} - \partial_c(eg^{ac}v^l D_a X_j^\mu v^j G_{\mu\lambda})\right)$$

$$= -T\left(eg^{ab}D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda}E_{\nu I} - D_b(eg^{ab}v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I})\right) \overset{!}{=} 0$$

$$\downarrow$$

$$\boxed{D_b(ee_i^a e^{bi}v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) = ee_i^a e^{bi}D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda}E_{\nu I}}$$

3.2 Without extra index

3.2.1 w.r.t e

$$\frac{\delta S_{DP}}{\delta e_l^e} = \frac{D\mathscr{L}_{DP}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)}\right)\left(-\frac{T_0}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e_c^m e_d^n \eta^{ij}e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I E_\nu^J \eta_{IJ}\right) =$$

$$= -\frac{T_0}{4}$$

3.3 Linear action

3.3.1 w.r.t e

$$\frac{\delta S_{LP}}{\delta \delta e_j^b} = \frac{D\mathscr{L}_{LP}}{De_j^b} = \left(\frac{\partial}{\partial e_j^b} - \partial_e \frac{\partial}{\partial(\partial_e e_j^b)}\right)\left(-\frac{T}{2}\overline{\psi}ee_i^a D_a X^{\mu i}E_\mu^I \gamma_I \psi\right) =$$

$$= -\frac{T}{4}\overline{\psi}\varepsilon^{cd}\varepsilon_{mn}((g_{cb}\eta^{mj}e_d^n + e_c^m g_{db}\eta^{nj})e_i^a + e_c^m e_d^n \delta_b^a \delta_i^j)D_a X^{\mu i}E_\mu^I \gamma_I \psi =$$

$$= -\frac{T}{2}\overline{\psi}(-ee_b^j e_i^a D_a X^{\mu i}E_\mu^I \gamma_I + eD_b X^{\mu j}E_\mu^I \gamma_I)\psi \overset{!}{=} 0$$

$$\downarrow$$

$$T_b^j := \overline{\psi}(D_b X^{\mu j}E_\mu^I \gamma_I - e_b^j e_i^a D_a X^{\mu i}E_\mu^I \gamma_I)\psi = 0$$

$$\downarrow$$

$$e_b^j = F\overline{\psi}D_b X^{\mu j}E_\mu^I \gamma_I \psi,$$

$$\frac{1}{F} = \overline{\psi}e_i^a D_a X^{\mu i}E_\mu^I \gamma_I \psi$$

$$\Downarrow$$

$$\overline{\psi}\psi = 1, \ \psi\overline{\psi} = \mathbb{1}$$

$$e_i^a e_j^b D_a X^{\mu i}D_b X^{\nu j}E_\mu^I E_\nu^J \eta_{IJ} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

4 Inverse Area Action

4.1 Nambu-Goto

Start from Nambu-Goto action

$$S_{NG} = -T \int \mathrm{d}^2 x \sqrt{-h},$$

and make quantum geometry correction

$$\sqrt{-h} \to \sqrt{-(h + g\Delta)} \approx \sqrt{-h} \left(1 - \frac{g\Delta}{2(-h)} + \mathcal{O}\left(\frac{g^2}{h^2}\right) \right),$$

leading to modified NG action

$$S_{MNG} = -T \int \mathrm{d}^2 x \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}} \right) = S_{NG} + S_{IA}$$

$$\downarrow$$

$$\mathscr{L}_{MNG} = -T \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}} \right)$$

4.1.1 EoMs

Let $\mathcal{P}_\mu^\tau \equiv \partial \mathscr{L}_{MNG}/\partial \dot{X}^\mu$ and $\mathcal{P}_\mu^\sigma \equiv \partial \mathscr{L}_{MNG}/\partial X'^\mu$

$$\delta S_{MNG} = \int \mathrm{d}^2 x \left(\frac{\partial \mathscr{L}_{MNG}}{\partial \dot{X}^\mu} \delta \dot{X}^\mu + \frac{\partial \mathscr{L}_{MNG}}{\partial X'^\mu} \delta X'^\mu \right) =$$

$$= -\int \mathrm{d}^2 x \left(\partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma \right) \delta X^\mu + \int \mathrm{d}\tau \mathcal{P}_\mu^\sigma \delta X^\mu \big|_{\sigma=0}^{\sigma=\sigma_1} \overset{!}{=} 0$$

$$\downarrow$$

$$\text{EoM} : \partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma = 0$$

$$\text{B.C.} : \mathcal{P}_\mu^\sigma \delta X^\mu \big|_{\sigma=0}^{\sigma=\sigma_1} = 0,$$

where

$$\mathcal{P}_\mu^\tau = \frac{\partial \mathscr{L}_{MNG}}{\partial \dot{X}^\mu} = -T \left(\frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} + \frac{g\Delta}{2} \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{(-h)^{3/2}} \right) =$$

$$= -T \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} \left(1 + \frac{g\Delta}{2(-h)} \right) = \mathcal{P}_{\mu(NG)}^\tau \left(1 + \frac{g\Delta}{2(-h)} \right)$$

$$\mathcal{P}_\mu^\sigma = \frac{\partial \mathscr{L}_{MNG}}{\partial X'^\mu} = -T \left(\frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} + \frac{g\Delta}{2} \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{(-h)^{3/2}} \right) =$$

$$= -T \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} \left(1 + \frac{g\Delta}{2(-h)} \right) = \mathcal{P}_{\mu(NG)}^\sigma \left(1 + \frac{g\Delta}{2(-h)} \right)$$

gauge fixing static gauge $\tau = t$ and transverse gauge $\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial s} \frac{\mathrm{d}s}{\mathrm{d}\sigma} = 0$ ($s = $ length along string)

$$\downarrow$$

$$\mathcal{P}_{(NG)}^{\tau\mu} = T \frac{\mathrm{d}s}{\mathrm{d}\sigma} \gamma_{v_\perp} \frac{\partial X^\mu}{\partial t}$$

$$\mathcal{P}_{(NG)}^{\sigma\mu} = \frac{T}{\gamma_{v_\perp}} \frac{\partial X^\mu}{\partial s}$$

$$-h = \frac{1}{\gamma_{v_\perp}^2} \left(\frac{\mathrm{d}s}{\mathrm{d}\sigma} \right)^2$$

string energy get's redefined to

$$\frac{\partial}{\partial t}\left(T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\right)=0,$$

and for the spatial part we have

$$\partial_\tau\vec{\mathcal{P}}^\tau+\partial_\sigma\vec{\mathcal{P}}^\sigma=0$$

$$\downarrow$$

$$\frac{\partial}{\partial t}\left[T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\frac{\partial\vec{X}}{\partial t}\left(1+\frac{g\Delta}{2(-h)}\right)\right]+\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{\partial}{\partial s}\left[-\frac{T}{\gamma_{v_\perp}}\frac{\partial\vec{X}}{\partial s}\left(1+\frac{g\Delta}{2(-h)}\right)\right]=0$$

$$\downarrow$$

$$\mu\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial}{\partial s}\left[\frac{T}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right)\frac{\partial\vec{X}}{\partial s}\right]=0$$

$\implies$ effective mass density becomes $\mu_{eff}=\mu\gamma_{v_\perp}\left(1-\frac{g\Delta}{2(-h)}\right)$ and effective tension becomes $T_{eff}=\frac{T}{\gamma_{v_\perp}}\left(1-\frac{g\Delta}{2(-h)}\right)$

$$\downarrow$$

$$\mu_{eff}\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial}{\partial s}\left[T_{eff}\frac{\partial\vec{X}}{\partial s}\right]=0$$

$$\mathcal{H}=\dot{\vec{X}}\cdot\vec{\pi}-\mathcal{L}=$$

$$=\vec{v}_\perp\cdot\left(T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)\vec{v}_\perp\right)-\left(-T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\frac{1}{\gamma_{v_\perp}}\left(1+\frac{g\Delta}{2(-h)}\right)\right)=$$

$$=T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\left(1+\frac{g\Delta}{2(-h)}\right)\left(\gamma_{v_\perp}v_\perp^2+\frac{1}{\gamma_{v_\perp}}\right)=$$

$$=T\frac{\mathrm{d}s}{\mathrm{d}\sigma}\gamma_{v_\perp}\left(1+\frac{g\Delta}{2(-h)}\right)$$

let $\left(1+\frac{g\Delta}{2(-h)}\right)=F$

$$\frac{\partial^2\vec{X}}{\partial t^2}-\frac{1}{F\gamma_{v_\perp}}\frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\partial}{\partial\sigma}\left[\frac{1}{\gamma_{v_\perp}}F\frac{\mathrm{d}\sigma}{\mathrm{d}s}\frac{\partial\vec{X}}{\partial\sigma}\right]=0$$

$$\downarrow$$

$$A(\sigma)=\frac{\gamma_{v_\perp}}{F}\frac{\mathrm{d}s}{\mathrm{d}\sigma}\overset{!}{=}1$$

$$\downarrow$$

$$\mathrm{d}\sigma=\frac{\gamma_{v_\perp}}{F}\mathrm{d}s=\frac{1}{TF^2}\mathrm{d}E\implies\sigma(q)=\frac{1}{T}\int_0^q\frac{1}{F^2}\mathrm{d}E$$

$$\downarrow$$

$$F^2\frac{\partial^2\vec{X}}{\partial t^2}-\frac{\partial^2\vec{X}}{\partial\sigma^2}=0$$

$\implies$ speed of wave on the string gets modified by correction factor $v=c/F$.

4.2 Translation and Lorentz symmetry

Starting with global translations, define the transformation by

$$X^\mu\mapsto X^\mu+\epsilon^\mu.$$

This does not change the action since it only depends on derivatives of X^μ and so $\delta(\partial_aX^\mu)=\partial_a(\delta X^\mu)=\partial_a\epsilon^\mu=0$.

As for Lorentz transformations, the infinitesimal for is

$$X^\mu\mapsto X^\mu+\epsilon^{\mu\nu}X_\nu,$$

with $\epsilon^{\mu\nu}$ anti-symmetric.

4.3 Verifying if there is longitudinal velocity

$$-h = \frac{1}{\gamma_{v_\perp}^2}\left(\frac{\mathrm{d}s}{\mathrm{d}\sigma}\right)^2 = \frac{F^2}{\gamma_{v_\perp}^4} = F^2\left(1 - v_\perp^2\right)^2$$

$$\downarrow$$

$$-h = \left(1 + \frac{g\Delta}{2(-h)}\right)^2\left(1 - v_\perp^2\right)^2$$

$$\downarrow$$

$$\frac{-h}{\left(1 + \frac{g\Delta}{2(-h)}\right)^2} = \left(1 - v_\perp^2\right)^2$$

$$\downarrow \text{(WolframAlpha)}$$

$$-h = \frac{\left(1 - v_\perp^2\right)^2}{3} - f_1 - f_2,$$

where (let $a = \frac{g\Delta}{2}$ and $b = (1 - v_\perp^2)^2$)

$$f_1 = \frac{\sqrt[3]{-27a^2b + 3\sqrt{3}\sqrt{27a^4b^2 + 4a^3b^3 - 18ab^2 - 2b^3}}}{3\sqrt[3]{2}}$$

$$f_2 = \frac{(6ab + b^2)}{6f_1}$$

$$\downarrow$$

$$F = 1 + \frac{g\Delta}{2(-h)} = 1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}$$

$$\downarrow$$

$$v = \frac{1}{F} = \frac{1}{1 + \dfrac{g\Delta}{2\left(\dfrac{b}{3} - f_1 - f_2\right)}}$$

5 Bimetric Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

and promote to bimetric action

$$\downarrow$$

$$S_{BP} = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{T'}{2} \int d^2x \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

5.1 EoMs

$$\frac{\delta S_{BP}}{\delta g^{cd}} = -\frac{T}{2} \left(\frac{\partial \sqrt{-g}}{\partial g^{cd}} g^{ab} + \sqrt{-g} \frac{\partial g^{ab}}{\partial g^{cd}} \right) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + 0 =$$

$$= -\frac{T}{2} \left(-\frac{1}{2} \sqrt{-g} g_{cd} g^{ab} + \sqrt{-g} \delta^a_{(c} \delta^b_{d)} \right) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \overset{!}{=} 0$$

$$T^{(G)}_{cd} := \left(\partial_c X^\mu \partial_d X^\nu - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \right) G_{\mu\nu} = 0$$

$$\Downarrow$$

$$g_{cd} = 2 f^{(G)} \partial_c X^\mu \partial_d X^\nu G_{\mu\nu},$$

$$\frac{1}{f^{(G)}} = g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\frac{\delta S_{BP}}{\delta h^{cd}} = ... =$$

$$= -\frac{T'}{2} \left(-\frac{1}{2} \sqrt{-h} h_{cd} h^{ab} + \sqrt{-h} \delta^a_{(c} \delta^b_{d)} \right) \partial_a X^\mu \partial_b X^\nu H_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$T^{(H)}_{cd} := \left(\partial_c X^\mu \partial_d X^\mu - \frac{1}{2} h_{cd} h^{ab} \partial_a X^\mu \partial_b X^\nu \right) H_{\mu\nu} = 0$$

$$\Downarrow$$

$$h_{cd} = 2 f^{(H)} \partial_c X^\mu \partial_d X^\nu H_{\mu\nu},$$

$$\frac{1}{f^{(H)}} = h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

If the auxiliary metrics g and h concide with the induced metrics from G and H, then $f^{(G)} = f^{(H)} = 1/2$ and the action reduces to

$$S = -T \int d^2x \sqrt{-g} - T' \int d^2x \sqrt{-h},$$

where if $T' = -Tk\Delta/2$ and $h = g^{-1}$ we recover the NG area corrected action

$$S = -T \int d^2x \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right).$$

The condition $h = g^{-1}$ can be achieved by

$$h_{ab} = S_{ac} g^{cd} T_{db},$$

where $S, T \in \mathrm{SL}(2, \mathbb{C})$, so we have a gauge freedom in choosing h_{ab}. Since we require $[h_{ab}] = [h_{ab}]^\dagger$ (h_{ab} must be symmetric and real), this means $[T_{db}] = [S_{db}]^\dagger$ and also that $[S_{ab}]^\dagger = [S_{ab}]$, thus

$$h_{ab} = S_{ac} g^{cd} S_{db}.$$

This restriction constrains our gauge group to just the boosts part of $\mathrm{SL}(2, \mathbb{C})$,

$$S = \mathrm{e}^{-n^i \sigma_i \beta}.$$

This also implies

$$\partial_a X^\mu \partial_b X^\nu H_{\mu\nu} = S_{ac} g^{cd} S_{db} =$$

$$= S_{ac} \left(\frac{1}{2g} \varepsilon^{cc'} \varepsilon^{dd'} g_{c'd'} \right) S_{db} =$$

$$\delta_a^{c'} \delta_b^{d'} \partial_{c'} X^\mu \partial_{d'} X^\nu H_{\mu\nu} = S_{ac} \left(\frac{1}{2g} \varepsilon^{cc'} \varepsilon^{dd'} \partial_{c'} X^\mu \partial_{d'} X^\nu G_{\mu\nu} \right) S_{db}$$

$$\left(\delta_a^{c'} \delta_b^{d'} H_{\mu\nu} - \frac{1}{2g} S_{ac} \varepsilon^{cc'} S_{db} \varepsilon^{dd'} G_{\mu\nu} \right) \partial_{c'} X^\mu \partial_{d'} X^\nu = 0.$$

One way this can hold is if the expression in parenthesis is equal to 0, which can be achieved if we choose $S = i\varepsilon$, leaving us with

$$\delta_a^{c'} \delta_b^{d'} H_{\mu\nu} - \frac{1}{2g} \delta_a^{c'} \delta_b^{d'} G_{\mu\nu} = 0$$

$$H_{\mu\nu} = \frac{1}{2g} G_{\mu\nu}.$$

$$\frac{\delta S_{BP}}{\delta X^\lambda} = \left(-\frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu -$$

$$- \partial_c \left(-\frac{T}{2} \sqrt{-g} g^{ab} G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} H_{\mu\nu} \right) (\delta_a^c \delta_\lambda^\mu \partial_b X^\nu + \partial_a X^\mu \delta_b^c \delta_\lambda^\nu) =$$

$$= \partial_a \left(T \sqrt{-g} g^{ab} G_{\mu\lambda} + T' \sqrt{-h} h^{ab} H_{\mu\lambda} \right) \partial_b X^\mu -$$

$$- \left(\frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} + \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu \overset{!}{=} 0$$

$$\downarrow$$

$$\partial_a \left(F_{\mu\lambda}^{ab} \partial_b X^\mu \right) = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu,$$

$$F_{\mu\nu}^{ab} = T \sqrt{-g} g^{ab} G_{\mu\nu} + T' \sqrt{-h} h^{ab} H_{\mu\nu}.$$

Assuming the condition for $h = g^{-1}$ and flat background $G_{\mu\nu} = \eta_{\mu\nu}$, the EoM reduces to

$$\partial_a \left(\left(\sqrt{-g} g^{ab} - \frac{k\Delta}{4g} \sqrt{-g}^{-1} (i\varepsilon^{ac}) g_{cd} (i\varepsilon^{db}) \right) \partial_b X^\mu \right) = 0$$

$$\partial_a \left(\left(\sqrt{-g} g^{ab} - \frac{k\Delta}{2\sqrt{-g}} g^{ab} \right) \partial_b X^\mu \right) = 0$$

$$\partial_a \left(\left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_b X^\mu \right) = 0,$$

which in conformal gauge $g_{ab} = \phi \eta_{ab}$ reduces further

$$\partial_a \left(\left(1 - \frac{k\Delta}{2\phi^2} \right) \eta^{ab} \partial_b X^\mu \right) = 0$$

$$\eta^{ab} \partial_a \left(F^- \partial_b X^\mu \right) = 0$$

$$F^- \eta^{ab} \partial_a \partial_b X^\mu + \eta^{ab} \partial_a F^- \partial_b X^\mu = 0$$

$$\eta^{ab} \partial_a \partial_b X^\mu + \eta^{ab} \partial_a \ln(F^-) \partial_b X^\mu = 0.$$

11

This is the exact same equation derived in the next section which leads to the Klein-Gordon eqn. In fact, plugging the condition for $h = g^{-1}$ and $T' = -Tk\Delta/2$ in the action reproduces the action in next section:

$$S = -\frac{T}{2} \int \mathrm{d}^2 x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \frac{Tk\Delta}{4} \int \mathrm{d}^2 x \sqrt{-g}^{-1} (i\varepsilon^{ac}) g_{cd} (i\varepsilon^{db}) \partial_a X^\mu \partial_b X^\nu \frac{G_{\mu\nu}}{2g} =$$

$$= -\frac{T}{2} \int \mathrm{d}^2 x \left(\sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{k\Delta}{2\sqrt{-g}} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \right) =$$

$$= -\frac{T}{2} \int \mathrm{d}^2 x \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

so the area corrected Polyakov action of next section is just a special case of the bimetric Polyakov action.

6 Inverse Area Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2} \int \mathrm{d}^2 x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

and make quantum geometry correction

$$\sqrt{-g} \rightarrow \sqrt{-(g+k\Delta)} \approx \sqrt{-g} \left(1 + \frac{k\Delta}{2g} + \mathcal{O}\left(\frac{k^2}{g^2}\right) \right)$$

leading to

$$S_{IAP} = -\frac{T}{2} \int \mathrm{d}^2 x \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

6.1 EoMs

$$\frac{\delta S_{IAP}}{\delta g^{cd}} = \frac{\partial}{\partial g^{cd}} \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} =$$

$$= \left(-\frac{1}{2}\sqrt{-g} g_{cd} + \frac{k\Delta}{4\sqrt{-g}} g_{cd} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} =$$

$$= -\frac{1}{2} \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = 0$$

$$\downarrow$$

$$g_{cd} = 2f \partial_c X^\mu \partial_d X^\nu G_{\mu\nu},$$

$$\frac{1}{f} = g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\frac{\delta S_{IAP}}{\delta X^\lambda} = \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - \partial_c \left(\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) 2 g^{ab} \delta_a^c \delta_\lambda^\mu \partial_b X^\nu G_{\mu\nu} \right) =$$

$$= \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2 \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} -$$

$$- 2\partial_a \left(\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} =$$

$$= \left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2 \left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} -$$

$$- 2\partial_a \left(\left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} \overset{!}{=} 0$$

$$\downarrow$$

$$\left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} + \partial_a \left(\left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} =$$

$$\frac{1}{2} \left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

imposing conformal symmetry $g^{ab} = \mathrm{e}^{-\phi(x)} \eta^{ab}$ and let $F' = \left(1 - \frac{k\Delta}{2\mathrm{e}^{2\phi}} \right)$ leads to

$$\eta^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} + \frac{1}{F'} \eta^{ab} \partial_a F' \partial_b X^\nu G_{\lambda\nu} = \eta^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

13

assuming flat background $G_{\mu\nu} = \eta_{\mu\nu}$

$$\eta^{ab}\partial_a\partial_b X^\nu \eta_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F'\partial_b X^\nu \eta_{\lambda\nu} = 0$$

$$\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\frac{k\Delta e^{-2\phi(x)}\partial_a\phi(x)}{1 - \dfrac{k\Delta e^{-2\phi(x)}}{2}}\partial_b X^\mu = 0$$

$$\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\frac{k\Delta\partial_a\phi(x)}{e^{2\phi(x)} - \dfrac{k\Delta}{2}}\partial_b X^\mu = 0$$

$$\eta^{ab}\partial_a\partial_b X^\mu + 2\eta^{ab}\frac{\partial_a\phi(x)}{\dfrac{2e^{2\phi(x)}}{k\Delta} - 1}\partial_b X^\mu = 0.$$

This reveals a wave eqn sourced by a coupling between the conformal factor and the tangent vectors to the string, which could hint towards a connection between the conformal factor and the dilaton field... perhaps $\phi(x)$ can be viewed as a "WS dilaton"? Perhaps, relating to the analysis of emergent dimensions in next section, the conformal factor is a WS dilaton which sets the string scale, which in turn dictates how the extra massive dimensions are hidden since they are inaccessible until they become effectively massless, but the WS still sees them all.

Going back to the equation before plugging in the definition of F', we can simplify it as

$$\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\partial_a \ln(F')\partial_b X^\mu = 0,$$

then use plane-wave ansatz $X^\mu = X_0^\mu e^{-i(E\tau - p\sigma)}$

$$(E^2 - p^2)X^\mu + \partial_\tau \ln(F')iEX^\mu - \partial_\sigma \ln(F')ipX^\mu = 0$$

$$\downarrow$$

$$(E^2 - p^2)X^\mu + i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F'))X^\mu = 0$$

$$\downarrow$$

$$i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F')) = -m^2,$$

$$-m^2 = -E^2 + p^2$$

$$\Downarrow$$

$$\partial_\tau \ln(F') = iE, \ \partial_\sigma \ln(F') = ip$$

$$\Downarrow$$

$$\ln(F') = i(E\tau + p\sigma) + a, \ a \in \mathbb{C}$$

$$\Downarrow$$

$$F' = Ae^{i(E\tau + p\sigma)}, \ A \in \mathbb{C}$$

$$\Downarrow$$

$$e^{\phi(x)} = \sqrt{\frac{k\Delta}{2\left(1 - Ae^{i(E\tau + p\sigma)}\right)}}$$

$$\downarrow$$

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0.$$

The imposition

$$i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F')) = -m^2$$

gives us a way to relate the mass m directly to the discrete geometry parameter $k\Delta$:

$$\frac{i}{F'}(E\partial_\tau F' - p\partial_\sigma F') = -m^2$$

14

$$\frac{ik\Delta \mathrm{e}^{-2\phi}}{1 - \dfrac{k\Delta \mathrm{e}^{-2\phi}}{2}}(E\partial_\tau \phi - p\partial_\sigma \phi) = -m^2$$

$$\frac{2i}{\dfrac{2\mathrm{e}^{2\phi}}{k\Delta} - 1}(E\partial_\tau \phi - p\partial_\sigma \phi) = -m^2$$

$$m^2 = \frac{2i}{\dfrac{2\mathrm{e}^{2\phi}}{k\Delta} - 1}(p\partial_\sigma \phi - E\partial_\tau \phi).$$

7 Emergent dimensions through LQG-strings

Since quantum geometry is expected to violate Lorentz invariance at the Planck scale by modifying the dispersion relation as in

$$E^2 = m^2 + p^2 + \sum_{n \geq 3} c_n \frac{p^n}{E_{pl}^{n-2}},$$

where E_{pl} is the Planck energy, we make a correction to Polyakov action in the form

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2x \sqrt{-g} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right)$$

where T is the temperature of target-space, (in SI units) μ_0^2 has units of inverse area (perhaps $\mu_0^2 \sim$ (quantum of area)$^{-1}$? Maybe $\mu_0^2 = \mu^2/(M_{pl}^2 A_j)$, where A_j is the quantum of area with spin j) and $m_0 >> m_1 >> ... >> m_d$. This action has manifest WS reparameterisation invariance. As for background Poincaré symmetry and WS Weyl invariance, those hold approximately for the fields with $m_\mu >> T$. The masses m_μ are constrained by the dispersion relations

$$\widetilde{E}_\alpha^2 = \widetilde{p}_\alpha^2 + \sum_{n=3}^{d} \frac{m_n}{T} \frac{\widetilde{p}_\alpha^n}{E_{pl}^{n-2}}$$

where $\alpha = 3, 4, ..., d$ and $\widetilde{E}_\alpha^2$ ($\widetilde{p}_\alpha^2$) does not mean $\widetilde{E}^\alpha \widetilde{E}_\alpha$ ($\widetilde{p}^\alpha \widetilde{p}_\alpha$), rather it is $\widetilde{E}_3^2(\widetilde{p}_3^2), \widetilde{E}_4^2(\widetilde{p}_4^2), ..., \widetilde{E}_d^2(\widetilde{p}_d^2)$, that is, the squared WS energy (momentum) of each field X^μ. Since m_0, m_1 and m_2 remain unconstrained after quantum geometry is considered, this might mean that the true quantum gravitational field is actually $(2+1)$-dimensional, since precisely the masses of X^0, X^1 and X^2 remain unconstrained, such that those may be regarded as "fundamental" dimensions, while the other dimensions have their masses constrained by the modified dispersion relations.

7.1 EoMs

$$\frac{\delta S}{\delta g^{cd}} \propto \frac{\partial \sqrt{-g}}{\partial g^{cd}} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right) + \sqrt{-g} \delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} =$$

$$= -\frac{1}{2} \sqrt{-g} g_{cd} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right) + \sqrt{-g} \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = 0$$

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right) = 0$$

$$g_{cd} = 2f \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu},$$

$$\frac{1}{f} = \left(g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} \right).$$

From this, it is clear that $\mathrm{e}^{-m_\mu/T} X^\mu X^\nu \eta_{\mu\nu} = 0$, thus the base mass μ_0^2 is a Lagrange multiplier, the auxiliary metric reduces to the induced metric and the action simplify back to regular NG action.

$$\frac{\delta S}{\delta X^\lambda} \propto +2\sqrt{-g} \mu_0^2 \mathrm{e}^{-m_\mu/T} \delta_\lambda^\mu X^\nu \eta_{\mu\nu} - 2\partial_c \left(\sqrt{-g} g^{ab} \delta_a^c \delta_\lambda^\mu \partial_b X^\nu \eta_{\mu\nu} \right) =$$

$$= -2 \left(\partial_a \left(\sqrt{-g} g^{ab} \partial_b X^\nu \eta_{\lambda\nu} \right) - \sqrt{-g} \mu_0^2 \mathrm{e}^{-m_\lambda/T} X^\nu \eta_{\lambda\nu} \right) = 0$$

$$\partial_a \left(\sqrt{-g} g^{ab} \partial_b X^\mu \right) - \sqrt{-g} \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu = 0$$

$$\frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g} g^{ab} \partial_b X^\mu \right) - \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu = 0$$

$$g^{ab} \nabla_a \nabla_b X^\mu - \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu = 0.$$

From the X EoM before simplifying, we can use conformal gauge $g_{ab} = \mathrm{e}^{\phi(x)} \eta_{ab}$ to get a variable-coefficient KG eqn

$$\eta^{ab} \partial_a \partial_b X^\mu - \mathrm{e}^{\phi(x)} \mu_0^2 \mathrm{e}^{-m_\mu/T} X^\mu = 0$$

$$((\partial_\sigma)^2 - (\partial_\tau)^2) X^\mu - \mu_0^2 \mathrm{e}^{\phi_\mu(x)} X^\mu = 0,$$

where $\phi_\mu(x) \equiv \phi(x) - m_\mu/T$.

8 Relating NG and Polyakov analysis

From NG analysis

$$(F^+)^2 \frac{\partial^2 X^\mu}{\partial t^2} - \frac{\partial^2 X^\mu}{\partial x^2} = 0,$$

make change of variables $\tau = \tau(t, x)$ and $\sigma = \sigma(t, x)$

$$(F^+)^2 \left(\frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial t} \frac{\partial}{\partial \sigma} \right) \left(\frac{\partial \tau}{\partial t} \frac{\partial X^\mu}{\partial \tau} + \frac{\partial \sigma}{\partial t} \frac{\partial X^\mu}{\partial \sigma} \right) - \left(\frac{\partial \tau}{\partial x} \frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial x} \frac{\partial}{\partial \sigma} \right) \left(\frac{\partial \tau}{\partial x} \frac{\partial X^\mu}{\partial \tau} + \frac{\partial \sigma}{\partial x} \frac{\partial X^\mu}{\partial \sigma} \right) = 0$$

$$(F^+)^2 \left(\left(\frac{\partial \tau}{\partial t} \right)^2 \frac{\partial^2 X^\mu}{\partial \tau^2} + 2 \frac{\partial \tau}{\partial t} \frac{\partial \sigma}{\partial t} \frac{\partial^2 X^\mu}{\partial \tau \partial \sigma} + \left(\frac{\partial \sigma}{\partial t} \right)^2 \frac{\partial^2 X^\mu}{\partial \sigma^2} \right) -$$
$$- \left(\left(\frac{\partial \tau}{\partial x} \right)^2 \frac{\partial^2 X^\mu}{\partial \tau^2} + 2 \frac{\partial \tau}{\partial x} \frac{\partial \sigma}{\partial x} \frac{\partial^2 X^\mu}{\partial \tau \partial \sigma} + \left(\frac{\partial \sigma}{\partial x} \right)^2 \frac{\partial^2 X^\mu}{\partial \sigma^2} \right) +$$
$$+ \left((F^+)^2 \frac{\partial^2 \tau}{\partial t^2} - \frac{\partial^2 \tau}{\partial x^2} \right) \frac{\partial X^\mu}{\partial \tau} + \left((F^+)^2 \frac{\partial^2 \sigma}{\partial t^2} - \frac{\partial^2 \sigma}{\partial x^2} \right) \frac{\partial X^\mu}{\partial \sigma} = 0$$

$$\left((F^+)^2 (\partial_t \tau)^2 - (\partial_x \tau)^2 \right) \partial_\tau^2 X^\mu + \left((F^+)^2 (\partial_t \sigma)^2 - (\partial_x \sigma)^2 \right) \partial_\sigma^2 X^\mu +$$
$$+ 2 \left((F^+)^2 \partial_t \tau \partial_t \sigma - \partial_x \tau \partial_x \sigma \right) \partial_\tau \partial_\sigma X^\mu + \left((F^+)^2 \partial_t^2 \tau - \partial_x^2 \tau \right) \partial_\tau X^\mu + \left((F^+)^2 \partial_t^2 \sigma - \partial_x^2 \sigma \right) \partial_\sigma X^\mu = 0$$

Without loss of generality, we can set (just rescale/rotate the coordinates)

$$(F^+)^2 (\partial_t \sigma)^2 - (\partial_x \sigma)^2 = (\partial_x \tau)^2 - (F^+)^2 (\partial_t \tau)^2$$
$$(F^+)^2 \partial_t \tau \partial_t \sigma = \partial_x \tau \partial_x \sigma,$$

or more compactly

$$\partial_t \tau = v \partial_x \sigma$$
$$\partial_t \sigma = v \partial_x \tau,$$

where $v = 1/F^+$, which is equivalent to

$$\partial_t^2 \tau = \partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau + v^2 \partial_x^2 \tau$$
$$\partial_t^2 \sigma = \partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma + v^2 \partial_x^2 \sigma,$$

reducing the big equation to

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu + \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\tau X^\mu + \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\sigma X^\mu = 0.$$

Next introduce $X^\mu = \kappa(\tau, \sigma) Y^\mu(\tau, \sigma)$ s.t terms proportional to $\partial_\tau Y^\mu$ and $\partial_\sigma Y^\mu$ vanish:

$$\partial_\tau X^\mu = \partial_\tau \kappa Y^\mu + \kappa \partial_\tau Y^\mu$$
$$\partial_\sigma X^\mu = \partial_\sigma \kappa Y^\mu + \kappa \partial_\sigma Y^\mu$$
$$\partial_\tau^2 X^\mu = \partial_\tau^2 \kappa Y^\mu + 2 \partial_\tau \kappa \partial_\tau Y^\mu + \kappa \partial_\tau^2 Y^\mu$$
$$\partial_\sigma^2 X^\mu = \partial_\sigma^2 \kappa Y^\mu + 2 \partial_\sigma \kappa \partial_\sigma Y^\mu + \kappa \partial_\sigma^2 Y^\mu$$

$$\kappa \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} + 2 \partial_\tau \kappa \overset{!}{=} 0$$
$$\kappa \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} - 2 \partial_\sigma \kappa \overset{!}{=} 0$$

17

$$\partial_\tau \kappa = -\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}$$

$$\partial_\sigma \kappa = \frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)},$$

undoing chain rule for t by multiplying first eq by $\partial_t \tau$ and second by $\partial_t \sigma$ and summing gives

$$\partial_t \kappa = \partial_t \tau \partial_\tau \kappa + \partial_t \sigma \partial_\sigma \kappa =$$

$$= \partial_t \tau \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}\right) + \partial_t \sigma \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}\right) =$$

$$= v \partial_x \sigma \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}\right) + v \partial_x \tau \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}\right) =$$

$$= \frac{v \kappa \partial_t v \left((\partial_x \tau)^2 - (\partial_x \sigma)^2\right)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)} =$$

$$= \frac{v \kappa \partial_t v \left((\partial_x \tau)^2 - (\partial_t \tau/v)^2\right)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)} =$$

$$= -\frac{\kappa \partial_t v}{2v},$$

or simply

$$\frac{\partial_t \kappa}{\kappa} = -\frac{1}{2}\frac{\partial_t v}{v}$$

$$\partial_t \ln(\kappa) = -\frac{1}{2}\partial_t \ln(v)$$

$$\ln(\kappa) = -\frac{1}{2}\ln(v) + f(x).$$

To determine $f(x)$, we undo the chain rule for x now:

$$\partial_x \kappa = \partial_x \tau \partial_\tau \kappa + \partial_x \sigma \partial_\sigma \kappa =$$

$$= \partial_x \tau \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}\right) + \partial_x \sigma \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)}\right) =$$

$$= \frac{\kappa v \partial_x v \left((\partial_x \sigma)^2 - (\partial_x \tau)^2\right)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)} =$$

$$= \frac{\kappa v \partial_x v \left((\partial_t \tau/v)^2 - (\partial_x \tau)^2\right)}{2\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)} =$$

$$= \frac{\kappa \partial_x v}{2v}$$

$$\frac{\partial_x \kappa}{\kappa} = \frac{1}{2}\frac{\partial_x v}{v}$$

$$\partial_x \ln(\kappa) = \frac{1}{2}\partial_x \ln(v)$$

$$\ln(\kappa) = \frac{1}{2}\ln(v),$$

$$\partial_x \left(-\frac{1}{2}\ln(v) + f(x)\right) = \frac{1}{2}\partial_x \ln(v)$$

$$f'(x) = \partial_x \ln(v) \text{ (perhaps this means somewhere in the composition of functions time dependence is lost?)}$$

$$f(x) = \ln(v)$$

giving us

$$\kappa = v^{\frac{1}{2}}, \quad v = v(x).$$

Substituting this into the earlier equation yields

$$\partial_\tau^2 Y^\mu - \partial_\sigma^2 Y^\mu + m^2 Y^\mu = 0,$$

18

with

$$m^2 = \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)\partial_\tau \kappa + (\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)\partial_\sigma \kappa}{\kappa \left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)} =$$

$$= \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{\partial_t v(\partial_x \sigma \partial_\tau \kappa + \partial_x \tau \partial_\sigma \kappa) + v \partial_x v \partial_x \kappa}{\kappa \left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)} =$$

$$= \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{(\partial_x v)^2}{\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)},$$

where by use of chain rule we have that

$$\partial_t^2 \kappa - v^2 \partial_x^2 \kappa - v \partial_x v \partial_x \kappa = ((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)(\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa),$$

where since $\kappa = v^{\frac{1}{2}}$, m^2 reduces to

$$m^2 = \frac{(\partial_x v)^2 - 2v\partial_x^2 v}{4\left((\partial_t \tau)^2 - v^2(\partial_x \tau)^2\right)},$$

with τ satisfying

$$\partial_t \tau = v \partial_x \sigma$$
$$v \partial_x \tau = \partial_t \sigma$$

such that $-m^2$ is indeed constant.

9 Solutions for the Polyakov KG eqn (WRONG ONE)

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi \eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta_{cd} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0.$$

These simplify further by expanding the summation on a, b:

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2} \left(-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} \right)$$

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}.$$

The EoM is just a one-dimensional Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int \mathrm{d}p \frac{1}{2E(p)} \left(a^\mu(p) \mathrm{e}^{-i(-E\tau + p\sigma)} + b^\mu(p) \mathrm{e}^{i(-E\tau + p\sigma)} \right),$$

where $E(p) = \sqrt{p^2 + m^2}$. The derivatives are

$$\partial_\tau X^\mu = \frac{i}{4\pi} \int \mathrm{d}p \left(a^\mu(p) \mathrm{e}^{-i(-E\tau + p\sigma)} - b^\mu(p) \mathrm{e}^{i(-E\tau + p\sigma)} \right)$$

$$\partial_\sigma X^\mu = \frac{i}{4\pi} \int \mathrm{d}p \frac{p}{\sqrt{p^2 + m^2}} \left(-a^\mu(p) \mathrm{e}^{-i(-E\tau + p\sigma)} + b^\mu(p) \mathrm{e}^{i(-E\tau + p\sigma)} \right).$$

Starting with the mixed constraint,

$$\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int \mathrm{d}p \left(a^\mu(p) \mathrm{e}^{-i(-E(p)\tau + p\sigma)} - b^\mu(p) \mathrm{e}^{i(-E(p)\tau + p\sigma)} \right) \times$$

$$\times \int \mathrm{d}p' \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\nu(p') \mathrm{e}^{-i(-E(p')\tau + p'\sigma)} + b^\nu(p') \mathrm{e}^{i(-E(p')\tau + p'\sigma)} \right) =$$

$$= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int \mathrm{d}p \mathrm{d}p' \frac{p'}{\sqrt{p'^2 + m^2}} \left(a^\mu(p) \mathrm{e}^{-i(-E(p)\tau + p\sigma)} - b^\mu(p) \mathrm{e}^{i(-E(p)\tau + p\sigma)} \right) \times$$

$$\times \left(-a^\nu(p') \mathrm{e}^{-i(-E(p')\tau + p'\sigma)} + b^\nu(p') \mathrm{e}^{i(-E(p')\tau + p'\sigma)} \right) =$$

$$= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int \mathrm{d}p \mathrm{d}p' \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\mu(p) a^\nu(p') \mathrm{e}^{i(E(p) + E(p'))\tau} \mathrm{e}^{-i(p + p')\sigma} + a^\mu(p) b^\nu(p') \mathrm{e}^{i(E(p) - E(p'))\tau} \mathrm{e}^{-i(p - p')\sigma} + \right.$$

$$\left. + b^\mu(p) a^\nu(p') \mathrm{e}^{-i(E(p) - E(p'))\tau} \mathrm{e}^{i(p - p')\sigma} - b^\mu(p) b^\nu(p') \mathrm{e}^{-i(E(p) + E(p'))\tau} \mathrm{e}^{i(p + p')\sigma} \right) = 0.$$

If we integrate this expression over σ we obtain

$$-\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\int \mathrm{d}\sigma \mathrm{e}^{-i(p+p')\sigma}+$$

$$+a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\int \mathrm{d}\sigma \mathrm{e}^{-i(p-p')\sigma}+b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\int \mathrm{d}\sigma \mathrm{e}^{i(p-p')\sigma}-$$

$$-b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\int \mathrm{d}\sigma \mathrm{e}^{i(p+p')\sigma}\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\mathrm{d}p'\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p')+a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p')+$$

$$+b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p')-b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p')\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\frac{p}{\sqrt{p^2+m^2}}\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}+a^\mu(p)b^\nu(p)+b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)=0$$

which implies the whole expression inside the integral must vanish, thus we have (leaving $(a^\mu b^\nu + b^\mu a^\nu)\eta_{\mu\nu}$ as is since eventually these will not be numbers and may not commute, thus will not be equal to $2a^\mu b^\nu \eta_{\mu\nu}$)

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}+a^\mu(p)b^\nu(p)+b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big)\eta_{\mu\nu}=0.$$

As for the other part of the constraint, let's start with the L.H.S

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu}=-\frac{1}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)\mathrm{e}^{-i(-E(p)\tau+p\sigma)}-b^\mu(p)\mathrm{e}^{i(-E(p)\tau+p\sigma)}\Big)\times$$

$$\times\Big(a^\nu(p')\mathrm{e}^{-i(-E(p')\tau+p'\sigma)}-b^\nu(p')\mathrm{e}^{i(-E(p')\tau+p'\sigma)}\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\mathrm{e}^{-i(p+p')\sigma}-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\mathrm{e}^{-i(p-p')\sigma}-$$

$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\mathrm{e}^{i(p-p')\sigma}+b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\mathrm{e}^{i(p+p')\sigma}\Big),$$

which we integrate over σ to get

$$-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\mathrm{d}p'\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p')-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p')-$$

$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p')+b^\mu(p)b^\nu(p')\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p')\Big)=$$

$$=-\frac{\eta_{\mu\nu}}{8\pi}\int \mathrm{d}p\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau}-a^\mu(p)b^\nu(p)-b^\mu(p)a^\nu(p)+b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau}\Big),$$

and now for the R.H.S

$$-\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}=\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{p}{\sqrt{p^2+m^2}}\frac{p'}{\sqrt{p'^2+m^2}}\Big(-a^\mu(p)\mathrm{e}^{-i(-E\tau+p\sigma)}+b^\mu(p)\mathrm{e}^{i(-E\tau+p\sigma)}\Big)\times$$

$$\times\Big(-a^\nu(p')\mathrm{e}^{-i(-E(p')\tau+p'\sigma)}+b^\nu(p')\mathrm{e}^{i(-E(p')\tau+p'\sigma)}\Big)=$$

$$=\frac{\eta_{\mu\nu}}{(4\pi)^2}\int \mathrm{d}p\mathrm{d}p'\frac{pp'}{E(p)E(p')}\Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\mathrm{e}^{-i(p+p')\sigma}-a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\mathrm{e}^{-i(p-p')\sigma}-$$

$$-b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\mathrm{e}^{i(p-p')\sigma}+b^\mu(p)b^\nu(p)\mathrm{e}^{-i(E(p)+E(p'))\tau}\mathrm{e}^{i(p+p')\sigma}\Big),$$

which once again we integrate over σ,

$$\frac{\eta_{\mu\nu}}{8\pi} \int \mathrm{d}p\mathrm{d}p' \frac{pp'}{E(p)E(p')} \Big(a^\mu(p)a^\nu(p')\mathrm{e}^{i(E(p)+E(p'))\tau}\delta(p+p') - a^\mu(p)b^\nu(p')\mathrm{e}^{i(E(p)-E(p'))\tau}\delta(p-p') -$$

$$- b^\mu(p)a^\nu(p')\mathrm{e}^{-i(E(p)-E(p'))\tau}\delta(p-p') + b^\mu(p)b^\nu(p)\mathrm{e}^{-i(E(p)+E(p'))\tau}\delta(p+p') \Big) =$$

$$= \frac{1}{8\pi} \int \mathrm{d}p \frac{p^2}{p^2+m^2} \Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau} \Big)\eta_{\mu\nu},$$

which when equated to the L.H.S yields

$$\int \mathrm{d}p \left(\frac{p^2}{p^2+m^2} + 1 \right) \Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau} \Big)\eta_{\mu\nu} = 0,$$

which implies

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} - a^\mu(p)b^\nu(p) - b^\mu(p)a^\nu(p) + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau} \Big)\eta_{\mu\nu} = 0.$$

We can sum and subtract this with the previous condition to simplify and obtain

$$\Big(a^\mu(p)a^\nu(-p)\mathrm{e}^{2iE(p)\tau} + b^\mu(p)b^\nu(-p)\mathrm{e}^{-2iE(p)\tau} \Big)\eta_{\mu\nu} = 0$$

$$\Big(a^\mu(p)b^\nu(p) + b^\mu(p)a^\nu(p) \Big)\eta_{\mu\nu} = 0,$$

and for while a^μ and b^ν are number-valued, the second condition can be further simplified to

$$a^\mu(p)b^\nu(p)\eta_{\mu\nu} = 0,$$

meaning a^μ and b^ν are orthogonal. In general, this condition states that a^μ and b^ν anti-commute w.r.t Lorentz inner product. The first condition must hold for all values of τ, and since the exponentials are linearly independent, we thus have

$$a^\mu(p)a^\nu(-p)\eta_{\mu\nu} = 0$$

$$b^\mu(p)b^\nu(-p)\eta_{\mu\nu} = 0,$$

meaning that reflecting the argument of a^μ and b^ν creates orthogonal vectors. We can thus write

$$a^\mu(p) = \Lambda^\mu_{\ \nu}\left(\frac{p}{4}\right) a^\nu(0)$$

$$b^\mu(p) = \Upsilon^\mu_{\ \nu}\left(\frac{p}{4}\right) b^\nu(0),$$

where $\Lambda, \Upsilon \in \mathrm{SO}^+(1,d)$. Also. since the orthogonality under reflection must hold for all values of p, in particular we have

$$a^\mu(0)a^\nu(0)\eta_{\mu\nu} = 0$$

$$b^\mu(0)b^\nu(0)\eta_{\mu\nu} = 0,$$

meaning that the initial $a^\mu(0)$ and $b^\nu(0)$ are null vectors. This means that $a^\mu(p)$ and $b^\nu(p)$ are also null vectors since they are related to the initial values by Lorentz transformation. With this, the solution becomes

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi} \int \mathrm{d}p \frac{1}{2E(p)} \left(\Lambda^\mu_{\ \nu}\left(\frac{p}{4}\right) a^\nu(0)\mathrm{e}^{-i(-E\tau+p\sigma)} + \Upsilon^\mu_{\ \nu}\left(\frac{p}{4}\right) b^\nu(0)\mathrm{e}^{i(-E\tau+p\sigma)} \right)$$

with $(a^\mu b^\nu + b^\nu a^\mu)\eta_{\mu\nu} = 0$. This condition implies

$$\left(\Lambda^\mu_{\ \gamma}\left(\frac{p}{4}\right) a^\gamma(0)\Upsilon^\nu_{\ \rho}\left(\frac{p}{4}\right) b^\rho(0) \right)\eta_{\mu\nu} = 0$$

$$\Upsilon^\mu_{\ \nu}\left(\frac{p}{4}\right) = \Lambda_\nu^{\ \mu}\left(\frac{p}{4}\right) = \Lambda^\mu_{\ \nu}\left(-\frac{p}{4}\right),$$

thus turning the solution into

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left(\Lambda^\mu_{\ \nu}\left(\frac{p}{4}\right) a^\nu(0) e^{-i(-E\tau+p\sigma)} + \Lambda^\mu_{\ \nu}\left(-\frac{p}{4}\right) b^\nu(0) e^{i(-E\tau+p\sigma)} \right).$$

Next, we choose background coordinates s.t $a^\mu(0) = (a_0, a_0, 0, ..., 0) = a_0 x^+$ and $b^\nu(0) = (b_0, b_0, 0, ..., 0) = b_0 x^+$, simplifying the solution to

$$X^\mu(\tau,\sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left(a_0 \Lambda^\mu_{\ +}\left(\frac{p}{4}\right) e^{-i(-E\tau+p\sigma)} + b_0 \Lambda^\mu_{\ +}\left(-\frac{p}{4}\right) e^{i(-E\tau+p\sigma)} \right) x^+$$

10   Solutions for the Polyakov KG eqn

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi \eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta_{cd} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

$$\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0.$$

These simplify further by expanding the summation on a, b:

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = -\frac{1}{2} \left(-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} \right)$$

$$\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = 0.$$

These can be combined by multiplying the previous one by 2 and summing/subtracting, leaving us with

$$(\partial_\tau X \pm \partial_\sigma X)^2 = 0.$$

The EoM is just a one-dimensional finite-space Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau, \sigma) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu e^{i(E_n \tau - n\sigma)} + b_n^\mu e^{-i(E_n \tau - n\sigma)} \right),$$

with $E_n = \sqrt{n^2 + m^2}$.

Reality of X^μ implies

$$(X^\mu)^* = \left(\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu e^{i(E_n \tau - n\sigma)} + b_n^\mu e^{-i(E_n \tau - n\sigma)} \right) \right)^* =$$

$$= \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left((a_n^\mu)^* e^{-i(E_n \tau - n\sigma)} + (b_n^\mu)^* e^{i(E_n \tau - n\sigma)} \right) \equiv X^\mu,$$

giving us

$$(a_n^\mu)^* = b_n^\mu$$

$$(b_n^\mu)^* = a_n^\mu,$$

thus the solution for closed string becomes

$$X^\mu = \sqrt{\frac{\alpha'}{2}} \sum_n \frac{1}{2E_n} \left(a_n^\mu e^{i(E_n \tau - n\sigma)} + (a_n^\mu)^* e^{-i(E_n \tau - n\sigma)} \right).$$

Analysing the constraints, for the closed string we have the derivatives

$$\partial_\tau X^\mu = \frac{i}{2} \sqrt{\frac{\alpha'}{2}} \sum_n \left(a_n^\mu e^{i(E_n \tau - n\sigma)} - (a_n^\mu)^* e^{-i(E_n \tau - n\sigma)} \right)$$

$$\partial_\sigma X^\mu = -\frac{i}{2} \sqrt{\frac{\alpha'}{2}} \sum_n \frac{n}{E_n} \left(a_n^\mu e^{i(E_n \tau - n\sigma)} - (a_n^\mu)^* e^{-i(E_n \tau - n\sigma)} \right),$$

$$\partial_\tau X^\mu \pm \partial_\sigma X^\mu = \frac{i}{2}\sqrt{\frac{\alpha'}{2}}\sum_n \left(1 \mp \frac{n}{E_n}\right)\left(a_n^\mu e^{i(E_n\tau - n\sigma)} - (a_n^\mu)^* e^{-i(E_n\tau - n\sigma)}\right)$$

so the constraint reads

$$(\partial_\tau X \pm \partial_\sigma X)^2 = -\frac{\eta_{\mu\nu}\alpha'}{8}\sum_n\sum_p \left(1 \mp \frac{n}{E_n}\right)\left(1 \mp \frac{p}{E_p}\right)\left(a_n^\mu a_p^\nu e^{i(E_n + E_p)\tau}e^{-i(n+p)\sigma} - \right.$$

$$\left. - a_n^\mu (a_p^\nu)^* e^{i(E_n - E_p)\tau}e^{-i(n-p)\sigma} - (a_n^\mu)^* a_p^\nu e^{-i(E_n - E_p)\tau}e^{i(n-p)\sigma} + (a_n^\mu)^*(a_p^\nu)^* e^{-i(E_n + E_p)\tau}e^{i(n+p)\sigma}\right) \equiv 0$$

Since the exponentials are all linearly independent and we can't decouple the τ exponential from any of the sums, we have 2 cases to investigate: $n = p$ and $n \neq p$. For $n \neq p$, we get

$$L_{n,p}^1 := a_n^\mu a_p^\nu \eta_{\mu\nu} = 0$$
$$L_{n,p}^2 := a_n^\mu (a_p^\nu)^* \eta_{\mu\nu} = 0$$
$$L_{n,p}^3 := (a_n^\mu)^* a_p^\nu \eta_{\mu\nu} = 0$$
$$L_{n,p}^4 := (a_n^\mu)^*(a_p^\nu)^* \eta_{\mu\nu} = 0,$$

where $L_{n,p}^2$ and $L_{n,p}^3$ are just complex conjugates of each other, so they are redundant. The same is true for $L_{n,p}^1$ and $L_{n,p}^4$, so in the end we have

$$L_{n,p} := a_n^\mu a_p^\nu \eta_{\mu\nu} = 0$$
$$\widetilde{L}_{n,p} := a_n^\mu (a_p^\nu)^* \eta_{\mu\nu} = 0.$$

For $n = p$, the first and last terms are complex conjugates of each other, while the exponentials in the 2 middle terms are reduced to 1, so we get

$$a_n^\mu a_n^\nu \eta_{\mu\nu} = 0$$

$$\sum_n \left(\left(1 \mp \frac{n}{E_n}\right)^2 a_n^\mu (a_n^\nu)^* \eta_{\mu\nu}\right) = 0.$$

Collecting everything we have

$$L_{n,p} := a_n^\mu a_p^\nu \eta_{\mu\nu} = 0, \ \forall \ n, p \in \mathbb{Z},$$
$$\widetilde{L}_{n,p} := a_n^\mu (a_p^\nu)^* \eta_{\mu\nu} = 0, \ n \neq p,$$
$$\sum_n \left(\left(1 \mp \frac{n}{E_n}\right)^2 a_n^\mu (a_n^\nu)^* \eta_{\mu\nu}\right) = 0.$$

One last thing before quantizing: the $n = 0$ term in the solution reads

$$\frac{1}{2\pi\alpha'}\frac{1}{2m}\left(a_0^\mu e^{-im\tau} + (a_0^\mu)^* e^{im\tau}\right),$$

which is divergent in the $m \to 0$ limit unless $\mathfrak{Re}(a_0^\mu) \sim m$. Keeping the analogy to the $m \to 0$ regime, we conclude $a_0^\mu = c_1 m x^\mu + i c_2 p^\mu$, where c_1 and c_2 are to be determined. We also have that

$$M^2 = -p^\mu p_\mu = -\frac{1}{(c_2)^2}(a_0^\mu (a_0^\nu)^* - (c_1 m)^2 x^\mu x^\nu)\eta_{\mu\nu},$$

which after analysing closer the $n = p$ condition

$$\sum_n \left(\left(1 \mp \frac{n}{E_n}\right)^2 \widetilde{L}_{n,n}\right) = 0$$

$$\sum_{n \neq 0}\left(\left(1 \mp \frac{n}{E_n}\right)^2 \widetilde{L}_{n,n}\right) - \widetilde{L}_{0,0} = 0$$

$$\sum_{n\neq 0}\left(\left(1\mp\frac{n}{E_n}\right)^2\tilde{L}_{n,n}\right)=a_0^\mu(a_0^\nu)^*\eta_{\mu\nu}\equiv(c_1m)^2x^\mu x_\mu+(c_2)^2p^\mu p_\mu,$$

we have the string mass in terms of vibrational modes and center of mass position

$$M^2=-\frac{1}{(c_2)^2}\left(\sum_{n\neq 0}\left(\left(1\mp\frac{n}{E_n}\right)^2\tilde{L}_{n,n}\right)-(c_1m)^2x^\mu x_\mu\right)=$$

$$=-\frac{1}{(c_2)^2}\left(\sum_{n\neq 0}\left(\left(1\mp\frac{n}{E_n}\right)^2a_n^\mu(a_n^\nu)^*\eta_{\mu\nu}\right)-(c_1m)^2x^\mu x_\mu\right).$$

10.1 The Expansion Modes Algebra

10.1.1 Closed String

Before starting this analysis, let's rename the vibrational modes to α_n^μ to be more in line with the literature
Start by finding the canonical momentum

$$\Pi_\lambda=\frac{\partial\mathscr{L}}{\partial(\partial_\tau X^\lambda)}=-\frac{T}{2}\left(\sqrt{-g}-\frac{k\Delta}{2\sqrt{-g}}\right)g^{ab}\delta_a^\tau\delta_\lambda^\mu\partial_b X^\nu\eta_{\mu\nu}=$$

$$=-\frac{1}{4\pi\alpha'}\left(1-\frac{k\Delta}{2\phi^2}\right)\eta^{\tau b}\partial_b X^\nu\eta_{\lambda\nu}=$$

$$=\frac{1}{4\pi\alpha'}F^-\partial_\tau X^\nu\eta_{\lambda\nu}=F^-\frac{i}{8\pi\alpha'}\sum_{n\in\mathbb{Z}}\left(\alpha_n^\nu e^{i(E_n\tau-n\sigma)}-(\alpha_n^\nu)^*e^{-i(E_n\tau-n\sigma)}\right)\eta_{\nu\lambda},$$

where $F^-=\sum_l(A_le^{i(E_l\tau+l\sigma)}+(A_l)^*e^{-i(E_l\tau+l\sigma)})$. We then proceed to the equal time canonical Poisson bracket relations

$$\{X^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')\}=\delta(\sigma-\sigma')\eta^{\mu\nu}$$

$$\{X^\mu(\tau,\sigma),X^\nu(\tau,\sigma')\}=\{\Pi^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')\}=0,$$

where by expanding the first relation at $\tau=0$ we get

$$\{X^\mu(0,\sigma),\Pi^\nu(0,\sigma')\}=\left\{\frac{1}{2}\sum_n\frac{1}{E_n}\left(\alpha_n^\mu e^{-in\sigma}+(\alpha_n^\mu)^*e^{in\sigma}\right),\right.$$

$$\left.\frac{i}{8\pi\alpha'}F^-(0,\sigma')\sum_p\left(\alpha_p^\nu e^{-ip\sigma'}-(\alpha_p^\nu)^*e^{ip\sigma'}\right)\right\}=\delta(\sigma-\sigma')\eta^{\mu\nu}$$

$$\frac{i}{16\pi\alpha'}F^-(0,\sigma')\sum_n\frac{1}{E_n}\sum_p\left(\{\alpha_n^\mu,\alpha_p^\nu\}e^{-in\sigma}e^{-ip\sigma'}-\{\alpha_n^\mu,(\alpha_p^\nu)^*\}e^{-in\sigma}e^{ip\sigma'}+\right.$$

$$\left.+\{(\alpha_n^\mu)^*,\alpha_p^\nu\}e^{in\sigma}e^{-ip\sigma'}-\{(\alpha_n^\mu)^*,(\alpha_p^\nu)^*\}e^{in\sigma}e^{ip\sigma'}\right)=\delta(\sigma-\sigma')\eta^{\mu\nu}$$

$$\left\downarrow\frac{1}{2\pi}\int_0^{2\pi}d\sigma e^{im\sigma}\right.$$

$$\frac{i}{16\pi\alpha'}F^-(0,\sigma')\frac{1}{E_m}\sum_p\left(\{\alpha_m^\mu,\alpha_p^\nu\}e^{-ip\sigma'}-\{\alpha_m^\mu,(\alpha_p^\nu)^*\}e^{ip\sigma'}+\right.$$

$$\left.+\{(\alpha_{-m}^\mu)^*,\alpha_p^\nu\}e^{-ip\sigma'}-\{(\alpha_{-m}^\mu)^*,(\alpha_p^\nu)^*\}e^{ip\sigma'}\right)=\frac{1}{2\pi}e^{im\sigma'}\eta^{\mu\nu}$$

$$\frac{i}{8\alpha' E_m} \sum_l \sum_p \left(\left(A_l e^{il\sigma'} + (A_l)^* e^{-il\sigma'} \right) \{\alpha_m^\mu, \alpha_p^\nu\} e^{-ip\sigma'} - \right.$$

$$- \left(A_l e^{il\sigma'} + (A_l)^* e^{-il\sigma'} \right) \{\alpha_m^\mu, (\alpha_p^\nu)^*\} e^{ip\sigma'} +$$

$$+ \left(A_l e^{il\sigma'} + (A_l)^* e^{-il\sigma'} \right) \{(\alpha_{-m}^\mu)^*, \alpha_p^\nu\} e^{-ip\sigma'} -$$

$$\left. - \left(A_l e^{il\sigma'} + (A_l)^* e^{-il\sigma'} \right) \{(\alpha_{-m}^\mu)^*, (\alpha_p^\nu)^*\} e^{ip\sigma'} \right) = e^{im\sigma'} \eta^{\mu\nu}$$

$$\frac{i}{8\alpha' E_m} \sum_l \sum_p \left(A_l \{\alpha_m^\mu, \alpha_p^\nu\} e^{i(l-p)\sigma'} + (A_l)^* \{\alpha_m^\mu, \alpha_p^\nu\} e^{-i(l+p)\sigma'} - \right.$$

$$- A_l \{\alpha_m^\mu, (\alpha_p^\nu)^*\} e^{i(l+p)\sigma'} - (A_l)^* \{\alpha_m^\mu, (\alpha_p^\nu)^*\} e^{i(p-l)\sigma'} +$$

$$+ A_l \{(\alpha_{-m}^\mu)^*, \alpha_p^\nu\} e^{i(l-p)\sigma'} + (A_l)^* \{(\alpha_{-m}^\mu)^*, \alpha_p^\nu\} e^{-i(l+p)\sigma'} -$$

$$\left. - A_l \{(\alpha_{-m}^\mu)^*, (\alpha_p^\nu)^*\} e^{i(l+p)\sigma'} - (A_l)^* \{(\alpha_{-m}^\mu)^*, (\alpha_p^\nu)^*\} e^{i(p-l)\sigma'} \right) = e^{im\sigma'} \eta^{\mu\nu}$$

$$\left\downarrow \frac{1}{2\pi} \int_0^{2\pi} d\sigma' \right.$$

$$\frac{i}{8\alpha' E_m} \sum_l \left(A_l \{\alpha_m^\mu, \alpha_l^\nu\} + (A_l)^* \{\alpha_m^\mu, \alpha_{-l}^\nu\} - A_l \{\alpha_m^\mu, (\alpha_{-l}^\nu)^*\} - (A_l)^* \{\alpha_m^\mu, (\alpha_l^\nu)^*\} + \right.$$

$$\left. + A_l \{(\alpha_{-m}^\mu)^*, \alpha_l^\nu\} + (A_l)^* \{(\alpha_{-m}^\mu)^*, \alpha_{-l}^\nu\} - A_l \{(\alpha_{-m}^\mu)^*, (\alpha_{-l}^\nu)^*\} - (A_l)^* \{(\alpha_{-m}^\mu)^*, (\alpha_l^\nu)^*\} \right) = \delta_{m,0} \eta^{\mu\nu}.$$

If we fix the expansion modes of F^- to satisfy

$$\mathfrak{Re}(A_0) = -2\alpha',$$

we get the desired

$$\{\alpha_m^\mu, \alpha_l^\nu\} = 0 = \{(\alpha_m^\mu)^*, (\alpha_l^\nu)^*\}$$
$$\{\alpha_m^\mu, (\alpha_l^\nu)^*\} = -iE_m \delta_{|m|,|l|} \eta^{\mu\nu} = - \{(\alpha_m^\mu)^*, \alpha_l^\nu\}, \ m, l \geq 0,$$

where the absolute value comes from the fact that $\alpha_{-n}^\mu = \alpha_n^\mu$. We can rescale the vibrational modes to get harmonic oscillator bracket relations

$$a_n^\mu := \frac{1}{\sqrt{E_n}} \alpha_n^\mu$$

$$(a_n^\mu)^* := \frac{1}{\sqrt{E_n}} (\alpha_n^\mu)^*$$

$$\{a_n^\mu, a_p^\nu\} = 0 = \{(a_n^\mu)^*, (a_p^\nu)^*\}$$
$$\{a_n^\mu, (a_p^\nu)^*\} = -i\delta_{|n|,|p|} \eta^{\mu\nu}.$$

This can be used to calculate the bracket relation between x^μ and p^ν:

$$\{\alpha_0^\mu, (\alpha_0^\nu)^*\} = \{(c_1 \mu x^\mu + ic_2 p^\mu), (c_1 \mu x^\nu - ic_2 p^\nu)\} = -i\mu \eta^{\mu\nu}$$
$$(c_1 \mu)^2 \{x^\mu, x^\nu\} + (c_2)^2 \{p^\mu, p^\nu\} - ic_1 c_2 \mu (\{x^\mu, p^\nu\} - \{p^\mu, x^\nu\}) = -i\mu \eta^{\mu\nu}$$
$$\{\alpha_0^\mu, \alpha_0^\nu\} = \{(c_1 \mu x^\mu + ic_2 p^\mu), (c_1 \mu x^\nu + ic_2 p^\nu)\} = 0$$
$$(c_1 \mu)^2 \{x^\mu, x^\nu\} - (c_2)^2 \{p^\mu, p^\nu\} + ic_1 c_2 \mu (\{x^\mu, p^\nu\} + \{p^\mu, x^\nu\}) = 0.$$

By summing and subtracting these we get that

$$\{x^\mu, x^\nu\} = 0 = \{p^\mu, p^\nu\}$$

$$\{x^\mu, p^\nu\} = \frac{1}{2c_1 c_2} \eta^{\mu\nu} = - \{p^\mu, x^\nu\}$$

so we get that $c_2 = 1/(2c_1)$, and the 0-th mode becomes $\alpha_0^\mu = c_1 \mu x^\mu + ip^\mu/(2c_1)$. Keeping the analogy to the $\mu \to 0$ limit, we see that $c_1 = \sqrt{2}/\sqrt{\alpha'}$.

10.2 The Constraint Algebra

With the bracket relations between the vibrational modes in hand, we can now calculate the algebra generated by the constraints. For simplicity, since $E_n = \sqrt{n^2 + \mu^2} > 0$, we can focus instead on the algebra of the normalized harmonic oscillator modes a_n^μ instead of the expansion modes α_n^μ.

$$\{L_{n,m}, L_{k,p}\} = \left\{ a_n^\mu a_m^\nu \eta_{\mu\nu}, a_k^{\mu'} a_p^{\nu'} \eta_{\mu'\nu'} \right\} = 0$$

$$\left\{ L_{n,m}, \widetilde{L}_{k,p} \right\} = \left\{ a_n^\mu a_m^\nu \eta_{\mu\nu}, a_k^{\mu'} (a_p^{\nu'})^* \eta_{\mu'\nu'} \right\} =$$

$$= \eta_{\mu\nu} \left(a_m^\nu \left\{ a_n^\mu, (a_p^{\nu'})^* \right\} a_k^{\mu'} + a_n^\mu \left\{ a_m^\nu, (a_p^{\nu'})^* \right\} a_k^{\mu'} \right) \eta_{\mu'\nu'} =$$

$$= \eta_{\mu\nu} \left(a_m^\nu (-i\eta^{\mu\nu'} \delta_{n,p}) a_k^{\mu'} + a_n^\mu (-i\eta^{\nu\nu'} \delta_{m,p}) a_k^{\mu'} \right) \eta_{\mu'\nu'} =$$

$$= -i \left(a_m^\mu a_k^\nu \eta_{\mu\nu} \delta_{n,p} + a_n^\mu a_k^\nu \eta_{\mu\nu} \delta_{m,p} \right) =$$

$$= -i \left(L_{m,k} \delta_{n,p} + L_{n,k} \delta_{m,p} \right)$$

$$\left\{ \widetilde{L}_{n,m}, \widetilde{L}_{k,p} \right\} = \left\{ a_n^\mu (a_m^\nu)^* \eta_{\mu\nu}, a_k^{\mu'} (a_p^{\nu'})^* \eta_{\mu'\nu'} \right\} =$$

$$= \eta_{\mu\nu} \left((a_m^\nu)^* \left\{ a_n^\mu, (a_p^{\nu'})^* \right\} a_k^{\mu'} + a_n^\mu \left\{ (a_m^\nu)^*, a_k^{\mu'} \right\} (a_p^{\nu'})^* \right) \eta_{\mu'\nu'} =$$

$$= \eta_{\mu\nu} \left((a_m^\nu)^* (-i\eta^{\mu\nu'} \delta_{n,p}) a_k^{\mu'} - a_n^\mu (-i\eta^{\nu\mu'} \delta_{m,k}) (a_p^{\nu'})^* \right) \eta_{\mu'\nu'} =$$

$$= -i \left(a_k^\mu (a_m^\nu)^* \eta_{\mu\nu} \delta_{n,p} - a_n^\mu (a_p^\nu)^* \eta_{\mu\nu} \delta_{m,k} \right) =$$

$$= -i \left(\widetilde{L}_{k,m} \delta_{n,p} - \widetilde{L}_{n,p} \delta_{m,k} \right).$$

These relations reveal that $\alpha_{-n}^\mu = \alpha_n^\mu$, since the middle relation is only invariant to $p \mapsto -p$ if the expansion modes are symmetric.

11   Into the Quantum Realm

We now proceed to quantize the KG string through canonical quantization. The string X^μ and it's momentum Π^ν are promoted to operators $\hat{X}^\mu$ and $\hat{\Pi}^\nu$, and we turn the Poisson bracket $\{.,.\}$ into commutator between operators $[.,.] = i\hbar\{.,.\}$. The equal time bracket relations turn into equal time commutation relations

$$\left[\hat{X}^\mu(\tau,\sigma), \hat{\Pi}^\nu(\tau,\sigma')\right] = i\delta(\sigma-\sigma')\eta^{\mu\nu}$$

$$\left[\hat{X}^\mu, \hat{X}^\nu\right] = \left[\hat{\Pi}^\mu, \hat{\Pi}^\nu\right] = 0$$

$$[\hat{a}_n^\mu, \hat{a}_m^\nu] = \left[(\hat{a}_n^\mu)^\dagger, (\hat{a}_m^\nu)^\dagger\right] = 0$$

$$\left[\hat{a}_n^\mu, (\hat{a}_m^\nu)^\dagger\right] = \eta^{\mu\nu}\delta_{n,m}$$

$$[\hat{x}^\mu, \hat{x}^\nu] = [\hat{p}^\mu, \hat{p}^\nu] = 0$$

$$[\hat{x}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu}.$$

Thus, the rescaled vibrational mode operators $\hat{a}_n^\mu$ and $(\hat{a}_m^\nu)^\dagger$ are annihilation and creations operators, respectively.

We define a vacuum state of the string to obey

$$\hat{a}_n^\mu|0\rangle = 0, \text{ for } n \neq 0.$$

For $n = 0$, we have the center of mass position and momentum operators, so the vacuum also obeys

$$\hat{x}^\mu|0;x\rangle = x^\mu|0;x\rangle$$

$$\hat{p}_\mu|0;x\rangle = -i\frac{\partial}{\partial x^\mu}|0;x\rangle$$

in position representation or alternatively

$$\hat{x}^\mu|0;p\rangle = -i\frac{\partial}{\partial p_\mu}|0;p\rangle$$

$$\hat{p}_\mu|0;p\rangle = p_\mu|0;p\rangle$$

in momentum representation.

A generic state arises from a sequence of creation operators on the vacuum

$$((\hat{a}_1^{\mu_1})^\dagger)^{n_{\mu_1}}((\hat{a}_2^{\mu_2})^\dagger)^{n_{\mu_2}}...((\hat{a}_{-1}^{\nu_1})^\dagger)^{n_{\nu_1}}((\hat{a}_{-2}^{\nu_2})^\dagger)^{n_{\nu_2}}...|0\rangle.$$

This (should) give rise to particles.

As in regular ST, we have ghosts arising from the Minkowski metric

$$\left[\hat{a}_n^\mu, (\hat{a}_m^\nu)^\dagger\right] = \eta^{\mu\nu}\delta_{nm}.$$

For constraints, classically we have

$$L_{n,p} = \alpha_n^\mu \alpha_p^\nu \eta_{\mu\nu} = 0$$

$$\widetilde{L}_{n,p} = \alpha_n^\mu (\alpha_p^\nu)^* \eta_{\mu\nu} = 0, \ n \neq p$$

which because of the existence of ghosts we require to have vanishing matrix elements when sandwiched between physical states

$$\langle\text{phys}'|\hat{L}_{n,p}|\text{phys}\rangle = 0 = \langle\text{phys}'|\hat{\widetilde{L}}_{n,p}|\text{phys}\rangle.$$

When translating the constraints into quantum operators, in $L_{n,p}$ we have no ambiguity of ordering since $\hat{a}$ commutes with itself. As for $\widetilde{L}_{n,p}$, there is an ambiguity for $p = n \neq 0$, so we pick normal ordering with the annihilation operators moved to the right

$$\hat{\widetilde{L}}_{n,n} = (\hat{\alpha}_n^\mu)^\dagger \hat{\alpha}_n^\nu \eta_{\mu\nu}.$$

The ambiguity manifests in the imposition of this constraint as

$$\langle\text{phys}'|\left(\sum_n \left(\left(1 \mp \frac{n}{E_n}\right)^2 \hat{\widetilde{L}}_{n,n}\right) - c\right)|\text{phys}\rangle = 0,$$

for some costant c. Since classically

$$M^2 = \frac{8}{\alpha'} \left(\frac{2\mu^2}{\alpha'} x^\mu x_\mu - \sum_{n \neq 0} \left(\left(1 \mp \frac{n}{E_n} \right)^2 \widetilde{L}_{n,n} \right) \right) =$$

$$= \frac{8}{\alpha'} \left(\frac{2\mu^2}{\alpha'} x^\mu x_\mu - \sum_{n \neq 0} \left(\left(1 \mp \frac{n}{E_n} \right)^2 \alpha_n^\mu (\alpha_n^\nu)^* \eta_{\mu\nu} \right) \right),$$

we see that the string mass spectrum will be affected by this constant

$$\hat{M}^2 = \frac{8}{\alpha'} \left(\frac{2\mu^2}{\alpha'} \hat{x}^\mu \hat{x}_\mu - \left(\sum_{n \neq 0} \left(\left(1 \mp \frac{n}{E_n} \right)^2 E_n (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu} \right) - c \right) \right).$$

By enforcing normal-ordering, we can get the value of this constant:

$$\sum_{n \neq 0} \left(\left(1 \mp \frac{n}{E_n} \right)^2 E_n \hat{a}_n^\mu (\hat{a}_n^\nu)^\dagger \eta_{\mu\nu} \right) =$$

$$= \sum_{n \neq 0} \left(\left(1 + \frac{n}{E_n} \right)^2 E_n \left(\eta^{\mu\nu} \delta_{|n|,|n|} - (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \right) \eta_{\mu\nu} \right)$$

$$= - \left(\sum_{n \neq 0} \left(\left(1 + \frac{n}{E_n} \right)^2 E_n (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu} \right) - D \sum_{n \neq 0} \left(1 + \frac{n}{E_n} \right)^2 E_n \right)$$

giving us

$$c = D \sum_{n \neq 0} \left(1 + \frac{n}{E_n} \right)^2 E_n = D \sum_{n > 0} \left(1 + \frac{n^2}{E_n^2} \right) E_n.$$

The commutation relations between the L's are inherited from the bracket relations

$$\left[\hat{L}_{n,m}, \hat{L}_{k,p} \right] = 0$$

$$\left[\hat{L}_{n,m}, \hat{\tilde{L}}_{k,p} \right] = \hat{L}_{m,k} \delta_{n,p} + \hat{L}_{n,k} \delta_{m,p}$$

$$\left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p} \right] = \hat{\tilde{L}}_{k,m} \delta_{n,p} - \hat{\tilde{L}}_{n,p} \delta_{m,k}.$$

Since $[\hat{L}_{n,m}, \hat{\tilde{L}}_{p,k}]$ only has annihilation operators, it has no ordering ambiguities. $[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p}]$ have ordering ambiguities for $m = k$ and/or $p = n$, so we add the anomalous terms

$$\left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p} \right] = \hat{\tilde{L}}_{k,m} \delta_{n,p} - \hat{\tilde{L}}_{n,p} \delta_{m,k} + C_n \delta_{n,p} + D_k \delta_{m,k}$$

Clearly $C_0 = 0 = D_0$, since if all indices are equal the commutator should vanish. Also, by the symmetry of the L's and because of the δ, we also have that $C_{-n} = C_n$ and $D_{-k} = D_k$. We get from the Jacobi identity

$$\left[\hat{\tilde{L}}_{n,m}, \left[\hat{\tilde{L}}_{k,p}, \hat{\tilde{L}}_{r,s} \right] \right] + \left[\hat{\tilde{L}}_{r,s}, \left[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{k,p} \right] \right] + \left[\hat{\tilde{L}}_{k,p}, \left[\hat{\tilde{L}}_{r,s}, \hat{\tilde{L}}_{n,m} \right] \right] = 0$$

$$\left[\hat{\tilde{L}}_{n,m}, \left(\hat{\tilde{L}}_{r,p} \delta_{k,s} - \hat{\tilde{L}}_{k,s} \delta_{p,r} + C_k \delta_{k,s} + D_r \delta_{p,r} \right) \right] +$$

$$+ \left[\hat{\tilde{L}}_{r,s}, \left(\hat{\tilde{L}}_{k,m} \delta_{n,p} - \hat{\tilde{L}}_{n,p} \delta_{m,k} + C_n \delta_{n,p} + D_k \delta_{m,k} \right) \right] +$$

$$+ \left[\hat{\tilde{L}}_{k,p}, \left(\hat{\tilde{L}}_{n,s} \delta_{r,m} - \hat{\tilde{L}}_{r,m} \delta_{s,n} + C_r \delta_{r,m} + D_n \delta_{s,n} \right) \right] = 0$$

$$\left[\hat{\tilde{L}}_{n,m},\hat{\tilde{L}}_{r,p}\right]\delta_{k,s} - \left[\hat{\tilde{L}}_{n,m},\hat{\tilde{L}}_{k,s}\right]\delta_{p,r} + \left[\hat{\tilde{L}}_{r,s},\hat{\tilde{L}}_{k,m}\right]\delta_{n,p}-$$
$$-\left[\hat{\tilde{L}}_{r,s},\hat{\tilde{L}}_{n,p}\right]\delta_{m,k} + \left[\hat{\tilde{L}}_{k,p},\hat{\tilde{L}}_{n,s}\right]\delta_{r,m} - \left[\hat{\tilde{L}}_{k,p},\hat{\tilde{L}}_{r,m}\right]\delta_{s,n} = 0$$

$$\left(\hat{\tilde{L}}_{r,m}\delta_{n,p} - \hat{\tilde{L}}_{n,p}\delta_{m,r} + C_n\delta_{n,p} + D_r\delta_{m,r}\right)\delta_{k,s} - \left(\hat{\tilde{L}}_{k,m}\delta_{n,s} - \hat{\tilde{L}}_{n,s}\delta_{m,k} + C_n\delta_{n,s} + D_k\delta_{m,k}\right)\delta_{p,r}+$$
$$+\left(\hat{\tilde{L}}_{k,s}\delta_{r,m} - \hat{\tilde{L}}_{r,m}\delta_{s,k} + C_r\delta_{r,m} + D_k\delta_{s,k}\right)\delta_{n,p} - \left(\hat{\tilde{L}}_{n,s}\delta_{r,p} - \hat{\tilde{L}}_{r,p}\delta_{s,n} + C_r\delta_{r,p} + D_n\delta_{s,n}\right)\delta_{m,k}+$$
$$+\left(\hat{\tilde{L}}_{n,p}\delta_{k,s} - \hat{\tilde{L}}_{k,s}\delta_{p,n} + C_k\delta_{k,s} + D_n\delta_{p,n}\right)\delta_{r,m} - \left(\hat{\tilde{L}}_{r,p}\delta_{k,m} - \hat{\tilde{L}}_{k,m}\delta_{p,r} + C_k\delta_{k,m} + D_r\delta_{p,r}\right)\delta_{s,n} = 0.$$

For $k = s$, $n = p$ and $r = m$,

$$\left(\hat{\tilde{L}}_{r,r} - \hat{\tilde{L}}_{n,n} + C_n + D_r\right) - \left(\hat{\tilde{L}}_{k,r}\delta_{p,s} - \hat{\tilde{L}}_{n,k}\delta_{r,k} + C_n\delta_{n,k} + D_k\delta_{r,k}\right)\delta_{n,m}+$$
$$+\left(\hat{\tilde{L}}_{k,k} - \hat{\tilde{L}}_{r,r} + C_r + D_k\right) - \left(\hat{\tilde{L}}_{n,k}\delta_{r,n} - \hat{\tilde{L}}_{r,n}\delta_{k,n} + C_r\delta_{r,n} + D_n\delta_{k,n}\right)\delta_{r,k}+$$
$$+\left(\hat{\tilde{L}}_{n,n} - \hat{\tilde{L}}_{k,k} + C_k + D_n\right) - \left(\hat{\tilde{L}}_{r,n}\delta_{k,r} - \hat{\tilde{L}}_{k,r}\delta_{n,r} + C_k\delta_{k,r} + D_r\delta_{n,r}\right)\delta_{k,n} = 0.$$

Assuming $n \neq m$, $r \neq k$ and $k \neq n$ leaves us with

$$C_n + C_r + C_k + D_r + D_k + D_n = 0,$$

which for $n + k + r = 0$ becomes

$$C_n + C_k + C_{-n-k} + D_n + D_k + D_{-n-k} = 0$$
$$C_n + C_k + C_{n+k} + D_n + D_k + D_{n+k} = 0,$$

and now setting $k = 1$ gives

$$C_n + C_{n+1} + C_1 + D_n + D_{n+1} + D_1 = 0.$$

Since this is a linear difference eqn, we can define $F_n = C_n + D_n$ and so we have the eqn

$$F_{n+1} + F_n + F_1 = 0$$

The solution to this difference eqn with $F_0 = 0$ is

$$F_n = \frac{F_1}{2}\cos\left(\frac{n\pi}{2}\right) - \frac{F_1}{2} = \frac{F_1}{2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right).$$

Since $C_0 = D_0 = 0$, we conclude that C_n and D_n are both equal to half F_n

$$C_n = \frac{F_1}{4}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)$$
$$D_n = \frac{F_1}{4}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)$$

By now, the commutator of $\hat{\tilde{L}}$ is

$$\left[\hat{\tilde{L}}_{n,m},\hat{\tilde{L}}_{k,p}\right] = \hat{\tilde{L}}_{k,m}\delta_{n,p} - \hat{\tilde{L}}_{n,p}\delta_{m,k} + \frac{F_1}{4}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)\delta_{n,p} + \frac{F_1}{4}\left(\cos\left(\frac{k\pi}{2}\right) - 1\right)\delta_{m,k}.$$

Let's calculate the VEV of this commutator for $m = -k \neq 0$ with $n = p = 1$

$$\langle 0|\left[\hat{\tilde{L}}_{1,-k},\hat{\tilde{L}}_{k,1}\right]|0\rangle = \langle 0|\hat{\tilde{L}}_{1,-k}\hat{\tilde{L}}_{k,1}|0\rangle$$

$$\langle 0|\hat{\tilde{L}}_{k,-k} - \frac{F_1}{4}|0\rangle = \langle 0|(a^\mu_{-k})^\dagger a^\nu_1 \eta_{\mu\nu}(a^{\mu'}_1)^\dagger a^{\nu'}_k \eta_{\mu'\nu'}|0\rangle$$

$$\langle 0| - \frac{F_1}{4} |0\rangle = 0 \iff F_1 = 0,$$

so in actuality, the constraint algebra has no anomalies.

Let us now denote the ground state of momentum p^μ as $|0; p\rangle$. The mass-shell condition

$$\hat{M}^2 = \frac{8}{\alpha'} \left(\frac{2\mu^2}{\alpha'} \hat{x}^\mu \hat{x}_\mu - \left(\sum_{n \neq 0} \left(\left(1 \mp \frac{n}{E_n} \right)^2 E_n (\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu} \right) - c \right) \right)$$

implies that

$$\langle 0; p|\hat{M}^2|0; p\rangle = \frac{8}{\alpha'} \left(\frac{2\mu^2}{\alpha'} \langle 0; p|\hat{x}^\mu \hat{x}_\mu|0; p\rangle - \left(\sum_{n \neq 0} \left(\left(1 \mp \frac{n}{E_n} \right)^2 E_n \right) \langle 0; p|(\hat{a}_n^\mu)^\dagger \hat{a}_n^\nu \eta_{\mu\nu}|0; p\rangle - \langle 0; p|c|0; p\rangle \right) \right)$$

$$M^2 = \frac{8}{\alpha'} \left(\frac{2\mu^2}{\alpha'} \frac{\partial \psi^*(p)}{\partial p_\mu} \frac{\partial \psi(p)}{\partial p^\mu} + c \right)$$

$$\frac{\alpha'}{8} p^2 = - \left(\frac{2\mu^2}{\alpha'} \left| \frac{\partial \psi(p)}{\partial p} \right|^2 + c \right)$$

Now looking at the first excited state $\zeta_\mu (\hat{a}_1^\mu)^\dagger |0; p\rangle$ with $\zeta_\mu = \zeta_\mu(p)$ being the polarization covector, the mass-shell now reads

$$\langle 0; p|\hat{a}_1^\mu \zeta_\mu \hat{M}^2 \zeta_\nu (\hat{a}_1^\nu)^\dagger |0; p\rangle = \frac{8}{\alpha'} \left(\frac{2\mu^2}{\alpha'} \langle 0; p|\hat{a}_1^\mu \zeta_\mu \hat{x}^{\mu'} \hat{x}_{\mu'} \zeta_\nu (\hat{a}_1^\nu)^\dagger |0; p\rangle - \right.$$

$$\left. - \left(\sum_{n \neq 0} \left(\left(1 \mp \frac{n}{E_n} \right)^2 E_n \right) \langle 0; p|\hat{a}_1^\mu \zeta_\mu (\hat{a}_n^{\mu'})^\dagger \hat{a}_n^{\nu'} \eta_{\mu'\nu'} \zeta_\nu (\hat{a}_1^\nu)^\dagger |0; p\rangle - \langle 0; p|\hat{a}_1^\mu \zeta_\mu c \zeta_\nu (\hat{a}_1^\nu)^\dagger |0; p\rangle \right) \right)$$

$$M^2 \zeta^\mu \zeta_\mu = \frac{8}{\alpha'} \left(\frac{2\mu^2}{\alpha'} \left| \frac{\partial \psi(p)}{\partial p} \right|^2 \zeta^\mu \zeta_\mu - \left(\left(1 - \frac{1}{\sqrt{1+\mu^2}} \right)^2 + \left(1 + \frac{1}{\sqrt{1+\mu^2}} \right)^2 \right) \sqrt{1+\mu^2} \zeta^\mu \zeta_\mu + c \zeta^\mu \zeta_\mu \right)$$

$$M^2 = \frac{8}{\alpha'} \left(\frac{2\mu^2}{\alpha'} \left| \frac{\partial \psi(p)}{\partial p} \right|^2 - \left(1 + \frac{1}{1+\mu^2} \right) \sqrt{1+\mu^2} + c \right)$$

The auxiliary $\langle 0; p|\hat{\tilde{L}}_{1,0} \zeta_{\mu'} (\hat{a}_1^{\mu'})^\dagger |0; p\rangle = 0$ condition implies

$$\langle 0; p|\hat{\tilde{L}}_{1,0} \zeta_{\mu'} (\hat{a}_1^{\mu'})^\dagger |0; p\rangle = 0$$

$$\langle 0; p|(\hat{a}_0^\mu)^\dagger \hat{a}_1^\nu \eta_{\mu\nu} \zeta_{\mu'} (\hat{a}_1^{\mu'})^\dagger |0; p\rangle = 0$$

$$\langle 0; p|\zeta_\mu (\hat{a}_0^\mu)^\dagger |0; p\rangle = 0$$

$$\langle 0; p|\zeta_\mu \left(\sqrt{\frac{2}{\alpha'}} \mu \hat{x}^\mu - \frac{1}{2} \sqrt{\frac{\alpha'}{2}} i \hat{p}^\mu \right) |0; p\rangle = 0$$

$$-\sqrt{\frac{2}{\alpha'}} \mu i \zeta_\mu \frac{\partial \psi(p)}{\partial p_\mu} = \frac{1}{2} \sqrt{\frac{\alpha'}{2}} i \zeta_\mu p^\mu$$

$$\zeta_\mu p^\mu = -\frac{4\mu}{\alpha'} \zeta_\mu \frac{\partial \psi(p)}{\partial p_\mu}.$$

If the momentum wave function is chosen to be (need to justify this)

$$\psi(p) = -\frac{\alpha'}{8\mu} p^\mu p_\mu,$$

we get that $\zeta_\mu p^\mu = 0$. Using this in the mass-shell condition for the first excited state yields

$$\frac{\alpha'}{8}p^2 = -\left(\frac{2\mu^2}{\alpha'}\frac{(\alpha')^2}{16\mu^2}p^2 - \left(1 + \frac{1}{1+\mu^2}\right)\sqrt{1+\mu^2} + c\right)$$

$$\frac{\alpha'}{4}p^2 = \left(1 + \frac{1}{1+\mu^2}\right)\sqrt{1+\mu^2} - c,$$

and for the ground state

$$\frac{\alpha'}{4}p^2 = -c.$$

In general, the n-th excited state will have momentum given by

$$\frac{\alpha'}{4}p^2 = \left(1 + \frac{n^2}{n^2+\mu^2}\right)\sqrt{n^2+\mu^2} - c$$

$$\frac{\alpha'}{4}p^2 = \left(1 + \frac{n^2}{n^2+\mu^2}\right)\sqrt{n^2+\mu^2} - D\sum_{k>0}\left(1 + \frac{k^2}{E_k^2}\right)E_k$$

The norm of the states is given by $\zeta_\mu\zeta^\mu$, thus if we choose p to lie on the $(0,1)$-plane, then the $D-2$ spacelike polarizations normal to that plane clearly have positive norm. If we choose c such that the $n = d$, $d = (D-1)$ (we choose this because the solution only has $n = 0, 1, 2, ..., d$ terms by the $\widetilde{L}$ constraints) excited state is a tachyon with $p^2 > 0$, then p can be taken to have no time component and thus ζ is time-like with negative norm. If $p^2 < 0$, p can be chosen to ONLY have a time component, leaving ζ spacelike with positive norm again. At last, if $p^2 = 0$, then ζ is proportional to p with zero norm. Thus, for the absence of ghosts we require

$$c \geq \left(1 + \frac{d^2}{d^2+\mu^2}\right)\sqrt{d^2+\mu^2}.$$

In the boundary case, the d-th excited vector state is massless and the rest (including the scalar ground state) are regular massive states (no tachyons!). This analysis, however, is to get spin 1 particles from the closed string. The actual massless particle we want from closed strings is a symmetric spin 2 particle, so let's consider now the state $\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle$ where $\zeta_{\mu\nu} = \zeta_{\nu\mu}$ with $\eta^{\mu\nu}\zeta_{\mu\nu}$ is the polarization tensor. The mass-shell is

$$\langle 0;p|\hat{a}_d^\alpha\hat{a}_d^\beta\zeta_{\alpha\beta}\hat{M}^2\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle = \frac{8}{\alpha'}\left(\frac{2\mu^2}{\alpha'}\langle 0;p|\hat{a}_d^\alpha\hat{a}_d^\beta\zeta_{\alpha\beta}\hat{x}^{\mu'}\hat{x}_{\mu'}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle - \right.$$

$$\left. -\left(\sum_{n\neq 0}\left(\left(1 \mp \frac{n}{E_n}\right)^2 E_n\right)\langle 0;p|\hat{a}_d^\alpha\hat{a}_d^\beta\zeta_{\alpha\beta}(\hat{a}_n^{\mu'})^\dagger\hat{a}_n^{\nu'}\eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle - \langle 0;p|\hat{a}_d^\alpha\hat{a}_d^\beta\zeta_{\alpha\beta}c\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle\right)\right)$$

$$M^2\zeta_{\mu\nu}\zeta^{\mu\nu} = \frac{8}{\alpha'}\left(\frac{2\mu^2}{\alpha'}\left|\frac{\partial\psi(p)}{\partial p}\right|^2\zeta_{\mu\nu}\zeta^{\mu\nu} - \left(4\left(\left(1 - \frac{d}{\sqrt{d^2+\mu^2}}\right)^2\sqrt{d^2+\mu^2}\right)\zeta_{\mu\nu}\zeta^{\mu\nu} - c\zeta_{\mu\nu}\zeta^{\mu\nu}\right)\right)$$

$$\frac{8}{\alpha'}p^2 = -\left(\frac{2\mu^2}{\alpha'}\left|\frac{\partial\psi(p)}{\partial p}\right|^2 - \left(4\left((d^2+\mu^2)^{1/4} - (d^2+\mu^2)^{-1/4}\right) - c\right)\right)$$

$$\frac{\alpha'}{4}p^2 = 4\left((d^2+\mu^2)^{1/4} - (d^2+\mu^2)^{-1/4}\right) - c$$

$$\frac{\alpha'}{4}p^2 = 8\sinh^2\left(\frac{1}{4}\ln(d^2+\mu^2)\right) - c.$$

Using the auxiliary $\langle 0;p|\hat{\widetilde{L}}_{d,0}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle = 0$ condition implies

$$\langle 0;p|\hat{\widetilde{L}}_{d,0}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle = 0$$

$$\langle 0;p|(\hat{a}_0^{\mu'})^\dagger\hat{a}_d^{\nu'}\eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger(\hat{a}_d^\nu)^\dagger|0;p\rangle = 0$$

$$\langle 0;p|\left((\hat{a}_0^{\mu'})^\dagger \eta_{\mu'\nu'}\zeta_{\mu\nu}\eta^{\nu'\mu}(a_d^\nu)^\dagger + (\hat{a}_0^{\mu'})^\dagger \eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger \hat{a}_d^{\nu'}(\hat{a}_d^\nu)^\dagger\right)|0;p\rangle = 0$$

$$\langle 0;p|\left((\hat{a}_0^{\mu'})^\dagger \eta_{\mu'\nu'}\zeta_{\mu\nu}\eta^{\nu'\mu}(a_d^\nu)^\dagger + (\hat{a}_0^{\mu'})^\dagger \eta_{\mu'\nu'}\zeta_{\mu\nu}(\hat{a}_d^\mu)^\dagger \eta^{\nu'\nu}\right)|0;p\rangle = 0$$

$$\langle 0;p|2\zeta_{\mu\nu}(\hat{a}_0^\mu)^\dagger (\hat{a}_d^\nu)^\dagger|0;p\rangle = 0$$

$$\langle 0;p|\zeta_{\mu\nu}(\hat{a}_d^\nu)^\dagger \left(\sqrt{\frac{2}{\alpha'}}\mu\hat{x}^\mu - \frac{1}{2}\sqrt{\frac{\alpha'}{2}}i\hat{p}^\mu\right)|0;p\rangle = 0$$

$$\langle 0|\zeta_{\mu\nu}\left(-\sqrt{\frac{2}{\alpha'}}\mu i\frac{\partial \psi(p)}{\partial p_\mu} - \frac{1}{2}\sqrt{\frac{\alpha'}{2}}ip^\mu\right)(\hat{a}_d^\nu)^\dagger|0\rangle = 0$$

$$\langle 0|\zeta_{\mu\nu}p^\mu(\hat{a}_d^\nu)^\dagger|0\rangle = 0,$$

implying that

$$\zeta_{\mu\nu}p^\mu = \zeta_{\mu\nu}p^\nu = 0.$$

Since the norm of the spin-2 states is given by $\zeta^{\mu\nu}\zeta_{\mu\nu}$, we need those to not be negative. Since ζ is symmetric and we work in $(-,+,+,...)$ signature, $\zeta^{\mu\nu}\zeta_{\mu\nu} \geq 0$ always hold, so we just need to choose c such that we get at leats 1 massless spin-2 state. For that, we see that

$$c = 8\sinh^2\left(\frac{1}{4}\ln(d^2 + \mu^2)\right)$$

does the trick for the d-th state, and since for the non-existence of ghosts in the spin-1 states we required

$$c \geq 4\sinh^2\left(\frac{1}{4}\ln(d^2 + \mu^2)\right),$$

which is smaller than the condition for the spin-2, so we can safely choose

$$c \equiv 8\sinh^2\left(\frac{1}{4}\ln(d^2 + \mu^2)\right),$$

leaving the d-th spin-2 state massless, the rest spin-2 and all spin-1 particles being massive.

The fact that $\zeta_{\mu\nu} = \zeta_{\nu\mu}$ leaves $\zeta_{\mu\nu}$ with $D(D+1)/2$ independent components. The traceless condition $\eta^{\mu\nu}\zeta_{\mu\nu} = 0$ takes out 1 from that, leaving $(D(D+1)/2) - 1$ allowed polarizations. Finally, the condition $\zeta_{\mu\nu}p^\mu = 0$ takes D out of those, leaving $(D(D-1)/2) - 1$ allowed polarizations.

$$\frac{D(D-1)}{2} - 1 = \frac{D^2 - D - 2}{2} = \frac{(D-2)(D+1)}{2}.$$

This leaves 5 polarizations in $D = 4$.

12   Area correction Path Integral

Build the path integral of the area corrected Polyakov action

$$Z = \int D[g]D[X]\mathrm{e}^{-iS},$$

$$S = -\frac{1}{2\pi\alpha'}\int \mathrm{d}^2 x F^- \sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu},$$

$$F^- = 1 - \frac{k\Delta}{2(-g)}.$$

In conformal gauge and choosing the conformal factor such that the EoM becomes a KG eqn,

$$F^- = \sum_l \left(A_l \mathrm{e}^{i(E_l\tau + l\sigma)} + (A_{-l})^* \mathrm{e}^{i(-E_l\tau + l\sigma)} \right).$$

This will be the gauge chosen for performing the path integral.

13 Polyakov Action in terms of LQG Variables

Turn embbeding fields X^μ into vector in the spin-1 representation of $\mathbb{R}^{d+1} \rtimes \mathrm{Spin}(d,1)$, X^I, and promote partial derivative to covariant derivative $\partial_a \mapsto \mathcal{D}_a$ acting as

$$\mathcal{D}_a X^I = \partial_a X^I + k \mathcal{A}^I_{aJ} X^J,$$

where the WS $\mathbb{R}^{d+1} \rtimes \mathrm{Spin}(d,1)$ connection $\mathcal{A}_a$ with $I, J = 0, ..., d, d+1$ is given by

$$\mathcal{A}_a X = \begin{bmatrix} 0 & \beta_a^{01} & \beta_a^{02} & \cdots & \beta_a^{0d} & p_a^0/l \\ \beta_a^{01} & 0 & -\theta_a^{12} & \cdots & -\theta_a^{1d} & p_a^1/l \\ \beta_a^{02} & \theta_a^{12} & 0 & \cdots & -\theta_a^{2d} & p_a^2/l \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_a^{0d} & \theta_a^{1d} & \theta_a^{2d} & \cdots & 0 & p_a^d/l \\ -\epsilon p_a^0/l & \epsilon p_a^1/l & \epsilon p_a^2/l & \cdots & \epsilon p_a^d/l & 0 \end{bmatrix} \begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ \vdots \\ X^d \\ 1 \end{bmatrix} = \left(\beta_a^{0\alpha} B_{0\alpha} + \frac{1}{2}\theta_a^{\alpha\beta} J_{\alpha\beta} + \frac{p_a^\mu}{l} P_\mu \right) X,$$

where $\epsilon = \mathrm{sgn}(\Lambda)$, i.e the sign of the background cosmological constant with $\alpha, \beta = 1, ..., d$ and $\mu = 0, ..., d$, so the action becomes

$$S = -\frac{T}{2} \int \mathrm{d}^2 x \sqrt{-g} g^{ab} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ},$$

where the metrics can be recast in terms of auxiliary and bulk vielbein fields as in

$$S = -\frac{T}{2} \int \mathrm{d}^2 x e e^a_{I'} e^b_{J'} \eta^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \eta_{IJ},$$

where

$$e := \sqrt{-\det(e^I_a e^J_b \eta_{IJ})} = \sqrt{-\frac{1}{2}\varepsilon^{ff'} \varepsilon^{gg'} (e^M_f e^N_g \eta_{MN})(e^{M'}_{f'} e^{N'}_{g'} \eta_{M'N'})}.$$

13.1 Checking the derivative

Since the connection is non-abelian, it should transform as

$$\mathcal{A}'_a = \mathcal{P}\mathcal{A}_a \mathcal{P}^{-1} - \frac{1}{k}(\partial_a \mathcal{P})\mathcal{P}^{-1},$$

where $\mathcal{P} = e^{\beta^{0\alpha} B_{0\alpha} + \theta^{\alpha\beta} J_{\alpha\beta}/2 + p^\mu P_\mu/l}$. Putting this into a gauge transformed derivative we have

$$\mathcal{D}'_a X' = \partial_a X' + k \mathcal{A}'_a X' =$$

$$= \partial_a(\mathcal{P}X) + k(\mathcal{P}\mathcal{A}_a \mathcal{P}^{-1} - \frac{1}{k}(\partial_a \mathcal{P})\mathcal{P}^{-1})(\mathcal{P}X) =$$

$$= \partial_a \mathcal{P}X + \mathcal{P}\partial_a X + k\mathcal{P}\mathcal{A}_a X - \partial_a \mathcal{P}X \equiv \mathcal{P}\mathcal{D}_a X,$$

so this is a proper covariant derivative for the Poincaré group.

Since under gauge transformations $\mathcal{D}_a X$ transforms covariantly, the action is not invariant under diffeomorphisms. In fact, it transforms as

$$S' = -\frac{T}{2} \int \mathrm{d}^2 x e' e'^a_{I'} e'^b_{J'} \eta^{I'J'} \mathcal{D}'_a X'^I \mathcal{D}'_b X'^J \eta_{IJ} =$$

$$= -\frac{T}{2} \int \mathrm{d}^2 x e e^a_{K'} (\mathcal{P}^{-1})^{K'}_{I'} e^b_{L'} (\mathcal{P}^{-1})^{L'}_{J'} \eta^{I'J'} \mathcal{P}^I_K \mathcal{D}_a X^K \mathcal{P}^J_L \mathcal{D}_b X^L \eta_{IJ},$$

so we promote the killing forms η_{IJ} and $\eta^{I'J'}$ to doubly co(ntra)-variant forms K_{IJ} and $K^{I'J'}$ such that $K'_{IJ} = (\mathcal{P}^{-1})^M_I (\mathcal{P}^{-1})^N_J K_{MN}$ and $K'^{I'J'} = \mathcal{P}^{I'}_{M'} \mathcal{P}^{J'}_{N'} K^{M'N'}$, so the new action is

$$S = -\frac{T}{2} \int \mathrm{d}^2 x e e^a_{I'} e^b_{J'} K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ}.$$

For consistency, let's assume K is a background field $K = K(X)$ like the metric G.

13.2 EoMs

13.2.1 W.r.t e

$$\frac{\delta S}{\delta e_K^c} \propto \left(\frac{\partial e}{\partial e_K^c} e_I^a e_{J'}^b K^{I'J'} + e \frac{\partial}{\partial e_K^c} (e_{I'}^a e_{J'}^b K^{I'J'}) \right) \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} =$$

$$= \left(-e e_c^K e_{I'}^a e_{J'}^b K^{I'J'} + 2e \delta_c^a \delta_{I'}^K e_{J'}^b K^{I'J'} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} =$$

$$= -e e_c^K e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} + 2e e_{J'}^b K^{KJ'} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ} = 0$$

$$T_c^K := 2 e_{J'}^b K^{KJ'} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ} - e_c^K e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} = 0$$

$$e_c^K = 2 f e^{bK} \mathcal{D}_c X^I \mathcal{D}_b X^J K_{IJ},$$

$$\frac{1}{f} = e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ}$$

13.2.2 W.r.t A

$$\frac{\delta S}{\delta \mathcal{A}_c^{KL}} = \frac{T}{2} \left(e e_{I'}^a e_{J'}^b K^{I'J'} \frac{\partial}{\partial \mathcal{A}_c^{KL}} \left(\mathcal{D}_a X^I \mathcal{D}_b X^J \right) K_{IJ} \right) =$$

$$= T \left(e e_{I'}^a e_{J'}^b K^{I'J'} \delta_a^c \delta_{[K}^I \eta_{L]I'} X^I \mathcal{D}_b X^J K_{IJ} \right) =$$

$$= T e e_{I'}^c e_{J'}^b K^{I'J'} \mathcal{D}_b X_{[K} X_{L]} \overset{!}{=} 0$$

$$\mathcal{T}_a^{IJ} = \mathcal{D}_a X^{[I} X^{J]} = 0$$

13.2.3 W.r.t X

$$\frac{\delta S}{\delta X^K} = -\frac{T}{2} \left(\partial_c \left(e e_{I'}^a e_{J'}^b K^{I'J'} \frac{\partial}{\partial(\partial_c X^K)} \left(\mathcal{D}_a X^I \mathcal{D}_b X^J \right) K_{IJ} \right) - e e_{I'}^a e_{J'}^b K^{I'J'} \frac{\partial}{\partial X^K} \left(\mathcal{D}_a X^I \mathcal{D}_b X^J \right) K_{IJ} - \right.$$

$$\left. - e e_{I'}^a e_{J'}^b \frac{\partial K^{I'J'}}{\partial X^K} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} - e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \frac{\partial K_{IJ}}{\partial X^K} \right) =$$

$$= -T \left(\left(\partial_c \left(e e_{I'}^a e_{J'}^b K^{I'J'} \delta_a^c \delta_K^I \mathcal{D}_b X^J K_{IJ} \right) - e e_{I'}^a e_{J'}^b K^{I'J'} k \mathcal{A}_{aL}^I \delta_L^{I'} \mathcal{D}_b X^J K_{IJ} \right) - \right.$$

$$\left. - e e_{I'}^a e_{J'}^b \partial_K K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J K_{IJ} - e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_a X^I \mathcal{D}_b X^J \partial_K K_{IJ} \right) =$$

$$= -T \left(\mathcal{D}_a \left(e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_b X^I K_{IK} \right) - e e_{I'}^a e_{J'}^b \left(\partial_K K^{I'J'} K_{IJ} + K^{I'J'} \partial_K K_{IJ} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J \right) \overset{!}{=} 0$$

$$\mathcal{D}_a \left(e e_{I'}^a e_{J'}^b K^{I'J'} \mathcal{D}_b X^I K_{IK} \right) = e e_{I'}^a e_{J'}^b \left(\partial_K K^{I'J'} K_{IJ} + K^{I'J'} \partial_K K_{IJ} \right) \mathcal{D}_a X^I \mathcal{D}_b X^J$$

In "constant gauge" $K_{IJ}(X) = \eta_{IJ}$ and in conformal gauge $e e_{I'}^a e_{J'}^b \eta^{I'J'} = \eta^{ab}$, this reduces to

$$\eta^{ab} \mathcal{D}_a \mathcal{D}_b X^I = 0.$$

More explicitly,

$$\eta^{ab} \mathcal{D}_a \left(\partial_b X^I + k \mathcal{A}_{bJ}^I X^J \right) = 0$$

$$\eta^{ab} \left(\left(\partial_a \partial_b X^I + k \mathcal{A}_{aJ}^I \partial_b X^J \right) + \left(k \partial_a \left(\mathcal{A}_{bJ}^I X^J \right) + k^2 \mathcal{A}_{aI'}^I \mathcal{A}_{bJ}^{I'} X^J \right) \right) = 0$$

$$\eta^{ab} \left(\partial_a \partial_b X^I + 2k \mathcal{A}_{aJ}^I \partial_b X^J + k \partial_a \mathcal{A}_{bJ}^I X^J + k^2 \mathcal{A}_{aI'}^I \mathcal{A}_{bJ}^{I'} X^J \right) = 0$$

$$\eta^{ab}\left(\delta_J^I\partial_a\partial_b + 2k\mathcal{A}_{aJ}^I\partial_b\right)X^J = -\left(k\eta^{ab}\partial_a\mathcal{A}_{bJ}^I X^J + k^2\eta^{ab}\mathcal{A}_{aI'}^I\mathcal{A}_{bJ}^{I'}X^J\right),$$

which has implicit solution given by

$$X^K(x) = -\int \mathrm{d}^2x' G_I^K(x,x')\left(k\eta^{a'b'}\partial_{a'}\mathcal{A}_{b'J}^I(x')X^J(x') + k^2\eta^{a'b'}\mathcal{A}_{a'I'}^I(x')\mathcal{A}_{b'J}^{I'}(x')X^J(x')\right),$$

where the Green's function matrix $G_J^K(x,x')$ satisfies

$$\eta^{ab}\left(\delta_K^I\partial_a\partial_b + 2k\mathcal{A}_{aK}^I(x)\partial_b\right)G_J^K(x,x') = \delta_J^I\delta(x,x').$$

Boundary conditions come from

$$\delta S = \int \mathrm{d}^2x\left(\frac{\partial\mathscr{L}}{\partial X^K}\delta X^K + \frac{\partial\mathscr{L}}{\partial(\partial_c X^K)}\delta(\partial_c X^K) + ...\right) =$$

$$= \int \mathrm{d}^2x\left((\text{EoM})\delta X^K + \partial_c\left(\frac{\partial\mathscr{L}}{\partial(\partial_c X^K)}\delta X^K\right) + ...\right) =$$

$$= \int \mathrm{d}^2x(\text{EoM})\delta X^K + \int \mathrm{d}\tau\left(\frac{\partial\mathscr{L}}{\partial(\partial_\sigma X^K)}\delta X^K\right)\Big|_0^{\sigma_1} + ... = 0,$$

thus

$$\left(\frac{\partial\mathscr{L}}{\partial(\partial_\sigma X^K)}\delta X^K\right)\Big|_0^{\sigma_1} = 0$$

$$\left(\left(-Tee_i^\sigma e^{bi}\mathcal{D}_b X_K\right)\delta X^K\right)\Big|_0^{\sigma_1} = 0$$

$$\left((\mathcal{D}_\sigma X_K)\delta X^K\right)\Big|_0^{\sigma_1} = 0,$$

so either $\delta X^K(\tau,\sigma_*) = 0,\ \sigma_* = 0,\sigma_1$ (Dirichlet B.C) or $\mathcal{D}_\sigma X^K(\tau,\sigma_*) = 0,\ \sigma_* = 0,\sigma_1$ (free end-point B.C). For closed strings, $X^K(\tau,\sigma) = X^K(\tau,\sigma + 2\pi)$.

When connection is trivial EoM reduces to regular wave eqn, which have the regular string solution

$$X_0^I(\tau,\sigma) = X_{0L}^I(\sigma^+) + X_{0R}^I(\sigma^-),$$

with

$$X_{0L}^I(\sigma^+) = \frac{1}{2}x^I + \frac{1}{2}\alpha'p^I\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^I e^{-in\sigma^+},\ \sigma^+ = \tau + \sigma$$

$$X_{0R}^I(\sigma^-) = \frac{1}{2}x^I + \frac{1}{2}\alpha'p^I\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I e^{-in\sigma^-},\ \sigma^- = \tau - \sigma.$$

This, however, does not amount to any motion in actual space-time, since with trivial connection the vector does not change after parallel transport. For a more interesting case, let's consider $A_\tau^{\alpha\beta} = A_\sigma^{\alpha\beta} = a\varepsilon^{\alpha\beta}$, which amounts to flat space-time. The EoM turn into

$$\left(\partial_\sigma^2 - \partial_\tau^2\right)X^I + 2ka(\varepsilon^{\alpha\beta}T_{\alpha\beta})^I{}_J\left(\partial_\sigma - \partial_\tau\right)X^J = 0$$

$$\left(\partial_\sigma^2 - \partial_\tau^2\right)\begin{bmatrix}X^0\\X^1\\X^2\\\vdots\\X^d\end{bmatrix} + 4ka\left(\partial_\sigma - \partial_\tau\right)\begin{bmatrix}0 & 1 & 1 & \cdots & 1\\1 & 0 & 1 & \cdots & 1\\1 & -1 & 0 & \cdots & 1\\\vdots & \vdots & \vdots & \ddots & \vdots\\1 & -1 & -1 & \cdots & 0\end{bmatrix}\begin{bmatrix}X^0\\X^1\\X^2\\\vdots\\X^d\end{bmatrix} = 0$$

$$\begin{bmatrix}\left(\partial_\sigma^2 - \partial_\tau^2\right)X^0 + 4ka\left(\partial_\sigma - \partial_\tau\right)\sum_{i>0}X^i\\\left(\partial_\sigma^2 - \partial_\tau^2\right)X^1 + 4ka\left(\partial_\sigma - \partial_\tau\right)\sum_{i\neq 1}X^i\\\left(\partial_\sigma^2 - \partial_\tau^2\right)X^2 + 4ka\left(\partial_\sigma - \partial_\tau\right)\left(X^0 - X^1 + \sum_{i>2}X^i\right)\\\vdots\\\left(\partial_\sigma^2 - \partial_\tau^2\right)X^d + 4ka\left(\partial_\sigma - \partial_\tau\right)\left(X^0 - \sum_{i=1}^{d-1}X^i\right)\end{bmatrix} = 0$$

$$\left(\partial_\sigma^2 - \partial_\tau^2\right) X^I + 4ka \left(\partial_\sigma - \partial_\tau\right) \left((\mathrm{sgn}(I))^2 X^0 + \sum_{i>0} \mathrm{sgn}(i-I) X^i\right) = 0$$

$$\left(\partial_\sigma - \partial_\tau\right) \left(\left(\partial_\sigma + \partial_\tau\right) X^I + 4ka \left((\mathrm{sgn}(I))^2 X^0 + \sum_{i>0} \mathrm{sgn}(i-I) X^i\right)\right) = 0$$

$$\left(\partial_\sigma + \partial_\tau\right) X^I + 4ka \left((\mathrm{sgn}(I))^2 X^0 + \sum_{i>0} \mathrm{sgn}(i-I) X^i\right) = C_1^I.(\tau + \sigma), \ C^I \in \mathbb{C}$$

$$\left(\partial_\sigma + \partial_\tau\right) X^I = C_1^I.(\tau + \sigma) - 4ka \left((\mathrm{sgn}(I))^2 X^0 + \sum_{i>0} \mathrm{sgn}(i-I) X^i\right)$$

$$X^I = \frac{1}{2} C_1^I.\left(\tau^2 + \sigma^2\right) + C_2^I.(\tau - \sigma) - 4ka \int \mathrm{d}^2 x' \left((\mathrm{sgn}(I))^2 X^0 + \sum_{i>0} \mathrm{sgn}(i-I) X^i\right).$$

If $D = 2$, we get a symmetric system of eqns

$$X^0 = \frac{1}{2} C_1^0.\left(\tau^2 + \sigma^2\right) + C_2^0.(\tau - \sigma) - 4ka \int \mathrm{d}^2 x' X^1$$

$$X^1 = \frac{1}{2} C_1^1.\left(\tau^2 + \sigma^2\right) + C_2^1.(\tau - \sigma) - 4ka \int \mathrm{d}^2 x' X^0$$

$$X^0 = \frac{1}{2} C_1^0.\left(\tau^2 + \sigma^2\right) + C_2^0.(\tau - \sigma) - 4ka \int \mathrm{d}^2 x' \left(\frac{1}{2} C_1^1.\left(\tau'^2 + \sigma'^2\right) + C_2^1.(\tau' - \sigma') - 4ka \int \mathrm{d}^2 x'' X^0\right)$$

$$X^0 - 16k^2 a^2 \int \mathrm{d}^2 x' \int \mathrm{d}^2 x'' X^0 = \frac{1}{2} C_1^0.\left(\tau^2 + \sigma^2\right) + C_2^0.(\tau - \sigma) - \frac{1}{2} C_1^1 \left(\frac{\tau^3}{3}\sigma + \tau\frac{\sigma^3}{3}\right) - C_2^1 \left(\frac{\tau^2}{2}\sigma - \tau\frac{\sigma^2}{2}\right).$$

This integral equation is tricky to deal with, but we can turn it into a differential equation by applying $\partial_\sigma \partial_\tau$ twice:

$$\left(\partial_\sigma \partial_\tau\right)^2 X^0 - 16k^2 a^2 X^0 = 0.$$

This has general solution given by sum of plane waves and exponentials

$$X^0 = \sum_{n \neq 0} \frac{1}{n^4} \left(a_n^0 e^{i2\sqrt{ka}n(\tau+\sigma)} + b_n^0 e^{i2\sqrt{ka}n(\tau-\sigma)} + c_n^0 e^{2\sqrt{ka}n(\tau+\sigma)} + d_n^0 e^{2\sqrt{ka}n(\tau-\sigma)}\right).$$

In actuallity, only the $n = 1$ and $n = -1$ terms satisfy the differential equation, so we have just

$$X^0 = a_1^0 e^{i2\sqrt{ka}(\tau+\sigma)} + b_1^0 e^{i2\sqrt{ka}(\tau-\sigma)} + c_1^0 e^{2\sqrt{ka}(\tau+\sigma)} + d_1^0 e^{2\sqrt{ka}(\tau-\sigma)} +$$

$$+ a_{-1}^0 e^{-i2\sqrt{ka}(\tau+\sigma)} + b_{-1}^0 e^{-i2\sqrt{ka}(\tau-\sigma)} + c_{-1}^0 e^{-2\sqrt{ka}(\tau+\sigma)} + d_{-1}^0 e^{-2\sqrt{ka}(\tau-\sigma)}$$

14 Polyakov-LQG V2

Same idea as before, but now contract the dyads directly with the $\mathcal{D}_a X^I$ factors

$$S = -\frac{T}{4} \int \mathrm{d}^2 x \, e e_I^a e_J^b \mathcal{D}_a X^I \mathcal{D}_b X^J.$$

14.1 EoMs

14.1.1 W.r.t e

$$\frac{\delta S}{\delta e_K^c} \propto \left(\frac{\partial e}{\partial e_K^c} e_I^a e_J^b + e \frac{\partial}{\partial e_K^c} (e_I^a e_J^b) \right) \mathcal{D}_a X^I \mathcal{D}_b X^J =$$

$$= \left(-e e_c^K e_I^a e_J^b + 2 e \delta_I^K \delta_c^a e_J^b \right) \mathcal{D}_a X^I \mathcal{D}_b X^J =$$

$$= -e e_c^K e_I^a e_J^b \mathcal{D}_a X^I \mathcal{D}_b X^J + 2 e e_J^b \mathcal{D}_c X^K \mathcal{D}_b X^J = 0$$

$$T_c^K := 2 e_J^b \mathcal{D}_c X^K \mathcal{D}_b X^J - e_c^K e_I^a e_J^b \mathcal{D}_a X^I \mathcal{D}_b X^J = 0,$$

$$e_c^K = 2 f \mathcal{D}_c X^K,$$

$$\frac{1}{f} = e_J^b \mathcal{D}_b X^J.$$

14.1.2 W.r.t A

$$\frac{\delta S}{\delta \mathcal{A}_c^{KL}} \propto e e_I^a e_J^b \frac{\partial}{\partial \mathcal{A}_c^{KL}} (\mathcal{D}_a X^I \mathcal{D}_b X^J) =$$

$$= 2 e e_I^a e_J^b \delta_a^c \delta_{[K}^I \eta_{L]I'} X^{I'} \mathcal{D}_b X^J =$$

$$= 2 e e_{[K}^c X_{L]} e_J^b \mathcal{D}_b X^J = 0$$

$$\mathcal{T}_{KL}^c := e_{[K}^c X_{L]} = 0.$$

14.1.3 W.r.t X

$$\frac{\delta S}{\delta X^K} \propto 2 e e_I^a e_J^b k \mathcal{A}_{aI'}^I \delta_K^{I'} \mathcal{D}_b X^J - 2 \partial_c \left(e e_I^a e_J^b \delta_a^c \delta_K^I \mathcal{D}_b X^J \right) =$$

$$= 2 e e_I^a e_J^b k \mathcal{A}_{aK}^I \mathcal{D}_b X^J - 2 \partial_a \left(e e_K^a e_J^b \mathcal{D}_b X^J \right) = 0$$

$$\mathcal{D}_a \left(e e_K^a e_J^b \mathcal{D}_b X^J \right) = 0.$$

If conformal gauge makes it so that $e e_K^a e_J^b = \eta^{ab} \eta_{KJ}$, this reduces to

$$\eta^{ab} \mathcal{D}_a \mathcal{D}_b X^I = 0$$