Sketchboard

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1 Basic concepts

1.1 With extra capital internal index on embbeding fields

Promote embedding fields X^μ to have an internal group index with D values

$$X^\mu \to X^{\mu I} \ , \quad I = 0, ..., d$$

$$g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{IJ} \sim fD_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu}$$
$$D_a X^{\mu I} = \partial_a X^{\mu I} + \omega_{aJ}^I X^{\mu J}$$
$$e_a^i e_b^j \eta_{ij} = g_{ab} \to e_a^i e_b^j \eta_{ij} = \mathrm{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ} \ , \quad i, j = 0, 1$$
$$g = \det(g_{ab}) \to g = \det(\mathrm{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.2 With 1 extra small internal index on embedding fields

Promote embbeding fields to have a internal group index with 2 values

$$X^\mu \to X^{\mu i} \ , \quad i = 0, 1$$

$$g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \to g_{ab}^{ij} \sim fD_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu}$$
$$D_a X^{\mu i} = \partial_a X^{\mu i} + \omega_{aj}^i X^{\mu j}$$
$$e_a^i e_b^j \eta_{ij} = g_{ab} \to e_a^i e_b^j \eta_{ij} = \mathrm{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij}$$
$$g = \det(g_{ab}) \to g = \det(\mathrm{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

1.3 With 2 extra small indices

Promote embbeding fields X^μ to have two internal indices with 2 values

$$X^\mu \to X^{\mu ij} \ , \quad i, j = 0, 1$$

1.4 Without extra index on embbeding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$e_a^i e_b^j \eta_{ij} = g_{ab} = 2f\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$g = \det(g_{ab}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)$$

2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int \mathrm{d}\tau \wedge \mathrm{d}\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \ ,$$

2.1 With capital internal index

... and promote partial derivative ∂_a to covariant derivative D_a, giving us our first attempt at modified Polyakov action

$$S_{MP1} = -\frac{T_0}{2} \int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}$$

$$\updownarrow$$

$$\mathscr{L}_{MP1} = -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}$$

2.2 With small internal index

2.2.1 Without extra field, 1 internal index

... and promote derivatives to covariant D_a, giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

$$\updownarrow$$

$$\mathscr{L}_{MP2} = -\frac{T_0}{2} \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

2.2.2 With extra field

..., promote partial derivative to covariant and add extra internal WS field v^i leading us to

$$S_{MP3} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

$$\updownarrow$$

$$\mathscr{L}_{MP3} = -\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

2.2.3 Without extra field, 2 internal indices

... and promote derivatives to covariant D_a,

$$S_{MP4} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma e e_i^a e^{bi} D_a X^{\mu j j'} \eta_{jj'} \eta_{kk'} D_b X^{\nu k k'} E_\mu^I E_{\nu I}$$

$$\mathscr{L}_{MP4} = -\frac{T}{2} e e_i^a e_j^b D_a X^{\mu i k} \eta_{kl} D_b X^{\nu l j} E_\mu^I E_{\nu I}$$

2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X)\ .$$

$$\updownarrow$$

$$\mathscr{L}_{DP} = -\frac{T_0}{2} \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X)\ .$$

2.4 Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of D-dimensional gamma matrices and bulk spinors

$$S_{LP} = -\frac{T}{2}\int \mathrm{d}\tau \wedge \mathrm{d}\sigma \overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

$$\updownarrow$$

$$\mathscr{L}_{LP} = -\frac{T}{2} \overline{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

3 EoMs

Start by writing $\det(e) = \frac{1}{2}\varepsilon^{cd}\varepsilon_{mn} e_c^m e_d^n$ and $g_{ab} = g_{ab}^{ij}\eta_{ij}$

3.1 With small internal index

3.1.1 Without extra field, w.r.t e

$$\frac{\delta S_{MP2}}{\delta e^e_l} = \frac{D\mathscr{L}_{MP2}}{De^e_l} = \left(\frac{\partial}{\partial e^e_l} - \partial_f \frac{\partial}{\partial(\partial_f e^e_l)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e^m_c e^n_d e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I}\right) =$$
$$= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(\eta^{ml}g_{ce}e^n_d e^a_i e^b_j + e^m_c \eta^{nl} g_{de} e^a_i e^b_j +$$
$$+ e^m_c e^n_d \delta^a_e \delta^l_i e^b_j + e^m_c e^n_d e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} =$$
$$= -\frac{T}{2}\left(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}g_{ce}e^n_d e^a_i e^b_j + 2\det(e) e^a_i \delta^b_e \delta^l_j\right) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} =$$
$$= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}\eta^{ml}e^k_c e_{ek} e^n_d e^a_i e^b_j + 2\det(e) e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} =$$
$$= -\frac{T}{2}(\varepsilon^{cd}\varepsilon_{mn}e^m_c e^n_d e^l_e e^a_i e^b_j + 2\det(e) e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} =$$
$$= -T(-\det(e) e^l_e e^a_i e^b_j + \det(e) e^a_i \delta^b_e \delta^l_j) D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} \stackrel{!}{=} 0$$

$$\downarrow$$

$$T^l_e := E^I_\mu E_{\nu I}(e^a_i D_a X^{\mu i} D_e X^{\nu l} - e^l_e e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j}) = 0$$

$$\Downarrow$$

$$e^l_e = f e^a_i D_a X^{\mu i} D_e X^{\nu l} E^I_\mu E_{\nu I}\ ,$$
$$\frac{1}{f} = e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I}$$

$$\downarrow$$

$$e^l_e e_{fl} = (f e^a_i D_a X^{\mu i} D_e X^{\nu l} E^I_\mu E_{\nu I})(f e^{a'}_{i'} D_{a'} X^{\mu' i'} D_f X^{\nu'}_l E^{I'}_{\mu'} E_{\nu' I'}) =$$
$$= f^2 D_e X^{\nu l} D_f X^{\nu'}_l E^{I'}_\nu E_{\nu' I'}(e^a_i e^{a'}_{i'} D_a X^{\mu i} D_{a'} X^{\mu' i'} E^I_\mu E_{\mu' I}) =$$
$$= f D_e X^{\mu i} D_f X^\nu_i E^I_\mu E_{\nu I}$$

$$\Downarrow$$

$$D_e X^{\mu i} D_f X^\nu_i E^I_\mu E_{\nu I} = 2\partial_e X^\mu \partial_f X^\nu G_{\mu\nu}$$
$$e^a_i e^b_j D_a X^{\mu i} D_b X^{\nu j} E^I_\mu E_{\nu I} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\Downarrow$$

$$D_a X^{\mu 0} = -i\partial_a X^\mu$$
$$D_a X^{\mu 1} = \partial_a X^\mu$$

3.1.2 With extra field, w.r.t e

$$\frac{\delta S_{MP3}}{\delta e^e_o} = \frac{D\mathscr{L}_{MP3}}{De^e_o} = \left(\frac{\partial}{\partial e^e_o} - \partial_f \frac{\partial}{\partial(\partial_f e^e_o)}\right)\left(-\frac{T}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn}e^m_c e^n_d \eta^{ij} e^a_i e^b_j D_a(X^\mu_k v^k) D_b(X^\nu_l v^l) E^I_\mu E^J_\nu \eta_{IJ}\right) =$$
$$= -\frac{T}{4}\varepsilon^{cd}\varepsilon_{mn}(g_{ce}\eta^{mo}e^n_d \eta^{ij} e^a_i e^b_j + e^m_c g_{de}\eta^{no}\eta^{ij} e^a_i e^b_j +$$
$$+ e^m_c e^n_d \eta^{ij}\delta^a_e \delta^o_i e^b_j + e^m_c e^n_d \eta^{ij} e^a_i \delta^b_e \delta^o_j) D_a(X^\mu_k v^k) D_b(X^\nu_l v^l) E^I_\mu E^J_\nu \eta_{IJ} =$$
$$= -\frac{T}{2}\varepsilon^{cd}\varepsilon_{mn}(e^p_c e_{ep} e^n_d \eta^{mo}\eta^{ij} e^a_i e^b_j + 2\det(e) e^{ao}\delta^b_e) D_a(X^\mu_k v^k) D_b(X^\nu_l v^l) E^I_\mu E^J_\nu \eta_{IJ} =$$
$$= -T(-\det(e) e^o_e \eta^{ij} e^a_i e^b_j + \det(e) e^{ao}\delta^b_e) D_a(X^\mu_k v^k) D_b(X^\nu_l v^l) E^I_\mu E^J_\nu \eta_{IJ} \stackrel{!}{=} 0$$

$$\downarrow$$

$$\boxed{T^o_e := (e^{ao}D_a(X^\mu_k v^k)D_e(X^\nu_l v^l) - e^o_e\eta^{ij}e^a_i e^b_j D_a(X^\mu_k v^k)D_b(X^\nu_l v^l))E^I_\mu E^J_\nu \eta_{IJ} = 0}$$

$$\Downarrow$$

$$\begin{aligned} e^o_e &= f\, e^{ao}D_a(X^\mu_k v^k)D_e(X^\nu_l v^l)E^I_\mu E^J_\nu \eta_{IJ}\ , \\ \frac{1}{f} &= \eta^{ij}e^a_i e^b_j D_a(X^\mu_k v^k)D_b(X^\nu_l v^l)E^I_\mu E^J_\nu \eta_{IJ} \end{aligned}$$

3.1.3 With extra field, w.r.t ω

$$\begin{aligned} \frac{\delta S_{MP3}}{\delta\omega^{mn}_c} = \frac{D\mathscr{L}_{MP3}}{D\omega^{mn}_c} &= \left(\frac{\partial}{\partial\omega^{mn}_c} - \partial_d\frac{\partial}{\partial(\partial_d\omega^{mn}_c)}\right)\left(-\frac{T}{2}\det(e)\eta^{ij}e^a_i e^b_j D_a X^\mu_k v^k D_b X^\nu_l v^l E^I_\mu E^J_\nu \eta_{IJ}\right) = \\ &= -\frac{T}{4}eg^{ab}((-\delta^c_a\delta^{k'}_{[m}\eta_{n]k}X^\mu_{k'}v^k)D_b X^\nu_l v^l + D_a X^\mu_k v^k(-\delta^c_b\delta^{l'}_{[m}\eta_{n]l}X^\nu_{l'}v^l))G_{\mu\nu} = \\ &= \frac{T}{4}e(g^{ca}X^\mu_{[m}v_{n]}D_a X^\nu_k v^k + g^{ac}D_a X^\mu_k v^k X^\nu_{[m}v_{n]})G_{\mu\nu} = \\ &= \frac{T}{2}eg^{ac}D_a X^\mu_k v^k X^\nu_{[m}v_{n]}G_{\mu\nu} \overset{!}{=} 0 \end{aligned}$$

$$\downarrow$$

$$\boxed{\mathcal{T}^i_{ab} := e^{ci}D_c X^\mu_k v^k X^\nu_{[m}v_{n]}e^m_a e^n_b G_{\mu\nu} = 0}$$

3.1.4 With extra field, w.r.t v

$$\begin{aligned} \frac{\delta S_{MP3}}{\delta v^l} = \frac{D\mathscr{L}_{MP3}}{Dv^l} &= \left(\frac{\partial}{\partial v^l} - \partial_c\frac{\partial}{\partial(\partial_c v^l)}\right)\left(-\frac{T}{2}ee^a_i e^{bi}D_a X^\mu_j v^j D_b X^\nu_k v^k E^I_\mu E_{\nu I}\right) = \\ &= -\frac{T}{2}g^{ab}(D_a X^\mu_j \delta^j_l D_b X^\nu_k v^k + D_a X^\mu_j v^j D_b X^\nu_k \delta^k_l)G_{\mu\nu} = \\ &= -Tg^{ab}D_a X^\mu_l D_b X^\nu_j v^j G_{\mu\nu} \overset{!}{=} 0 \end{aligned}$$

$$\downarrow$$

$$\boxed{g^{ab}D_a X^\mu_l D_b X^\nu_j v^j G_{\mu\nu} = 0}$$

3.1.5 With extra field, w.r.t X

$$\begin{aligned} \frac{\delta S_{MP3}}{\delta X^\lambda_l} = \frac{D\mathscr{L}_{MP3}}{DX^\lambda_l} &= \left(\frac{\partial}{\partial X^\lambda_l} - \partial_c\frac{\partial}{\partial(\partial_c X^\lambda_l)}\right)\left(-\frac{T}{2}ee^a_i e^{bi}D_a X^\mu_j v^j D_b X^\nu_k v^k E^I_\mu E_{\nu I}\right) = \\ &= -\frac{T}{2}\Bigg(eg^{ab}\Bigg((-\omega^{j'}_{aj}\delta^\mu_\lambda\delta^l_{j'}v^j D_b X^\nu_k v^k - D_a X^\mu_j v^j\omega^{k'}_{bk}\delta^\nu_\lambda\delta^l_{k'}v^k)E^I_\mu E_{\nu I} + \\ &+ eg^{ab}D_a X^\mu_j v^j D_b X^\nu_k v^k\left(\frac{\partial E^I_\mu}{\partial X^\lambda_l}E_{\nu I} + E^I_\mu\frac{\partial E_{\nu I}}{\partial X^\lambda_l}\right)\Bigg) - \\ &- \partial_c(eg^{ab}(\delta^c_a\delta^\mu_\lambda\delta^l_j v^j D_b X^\nu_k v^k + D_a X^\mu_j v^j\delta^c_b\delta^\nu_\lambda\delta^l_k v^k)G_{\mu\nu})\Bigg) = \\ &= -T\left(-eg^{ab}D_a X^\mu_j v^j\omega^l_{bk}v^k G_{\mu\lambda} + eg^{ab}D_a X^\mu_j v^j D_b X^\nu_k v^k\frac{\partial E^I_\mu}{\partial X^\lambda_l}E_{\nu I} - \partial_c(eg^{ac}v^l D_a X^\mu_j v^j G_{\mu\lambda})\right) \\ &= -T\left(eg^{ab}D_a X^\mu_j v^j D_b X^\nu_k v^k\frac{\partial E^I_\mu}{\partial X^\lambda_l}E_{\nu I} - D_b(eg^{ab}v^l D_a X^\mu_j v^j E^I_\mu E_{\lambda I})\right) \overset{!}{=} 0 \end{aligned}$$

$$\downarrow$$

$$\boxed{D_b(ee_i^a e^{bi} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) = ee_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I}}$$

3.2 Without extra index

3.2.1 w.r.t e

$$\begin{aligned}\frac{\delta S_{DP}}{\delta e_l^e} &= \frac{D\mathscr{L}_{DP}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)}\right)\left(-\frac{T_0}{2}\frac{1}{2}\varepsilon^{cd}\varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I E_\nu^J \eta_{IJ}\right) = \\ &= -\frac{T_0}{4}\end{aligned}$$

3.3 Linear action

3.3.1 w.r.t e

$$\begin{aligned}\frac{\delta S_{LP}}{\delta\delta e_j^b} &= \frac{D\mathscr{L}_{LP}}{De_j^b} = \left(\frac{\partial}{\partial e_j^b} - \partial_e \frac{\partial}{\partial(\partial_e e_j^b)}\right)\left(-\frac{T}{2}\overline{\psi} ee_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi\right) = \\ &= -\frac{T}{4}\overline{\psi}\varepsilon^{cd}\varepsilon_{mn}((g_{cb}\eta^{mj} e_d^n + e_c^m g_{db}\eta^{nj}) e_i^a + e_c^m e_d^n \delta_b^a \delta_i^j) D_a X^{\mu i} E_\mu^I \gamma_I \psi = \\ &= -\frac{T}{2}\overline{\psi}(-ee_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I + eD_b X^{\mu j} E_\mu^I \gamma_I)\psi \stackrel{!}{=} 0\end{aligned}$$

$$\downarrow$$

$$T_b^j := \overline{\psi}(D_b X^{\mu j} E_\mu^I \gamma_I - e_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I)\psi = 0$$

$$\downarrow$$

$$e_b^j = F\overline{\psi} D_b X^{\mu j} E_\mu^I \gamma_I \psi,$$

$$\frac{1}{F} = \overline{\psi} e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

$$\Downarrow$$

$$\overline{\psi}\psi = 1,\ \psi\overline{\psi} = \mathbb{1}$$

$$e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_\nu^J \eta_{IJ} = g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$