

Sketchboard

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1 Basic concepts

1.1 With extra capital internal index on embedding fields

Promote embedding fields X^μ to have an internal group index with D values

$$\begin{aligned}
X^\mu &\rightarrow X^{\mu I} , \quad I = 0, \dots, d \\
g_{ab} &= 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{IJ} \sim f D_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu} \\
D_a X^{\mu I} &= \partial_a X^{\mu I} + \omega_{aJ}^I X^{\mu J} \\
e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ} , \quad i, j = 0, 1 \\
g = \det(g_{ab}) &\rightarrow g = \det(\text{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

1.2 With 1 extra small internal index on embedding fields

Promote embedding fields to have a internal group index with 2 values

$$\begin{aligned}
X^\mu &\rightarrow X^{\mu i} , \quad i = 0, 1 \\
g_{ab} &= 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{ij} \sim f D_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu} \\
D_a X^{\mu i} &= \partial_a X^{\mu i} + \omega_{aj}^i X^{\mu j} \\
e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij} \\
g = \det(g_{ab}) &\rightarrow g = \det(\text{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

1.3 With 2 extra small indices

Promote embedding fields X^μ to have two internal indices with 2 values

$$X^\mu \rightarrow X^{\mu ij} , \quad i, j = 0, 1$$

1.4 Without extra index on embedding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$\begin{aligned}
e_a^i e_b^j \eta_{ij} &= g_{ab} = 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\
g = \det(g_{ab}) &= \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int d\tau \wedge d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} ,$$

2.1 With capital internal index

... and promote partial derivative ∂_a to covariant derivative D_a , giving us our first attempt at modified Polyakov action

$$\begin{aligned}
S_{MP1} &= -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ} \\
&\quad \updownarrow \\
\mathcal{L}_{MP1} &= -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}
\end{aligned}$$

2.2 With small internal index

2.2.1 Without extra field, 1 internal index

... and promote derivatives to covariant D_a , giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

$$\updownarrow$$

$$\mathcal{L}_{MP2} = -\frac{T_0}{2} \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

2.2.2 With extra field

..., promote partial derivative to covariant and add extra internal WS field v^i leading us to

$$S_{MP3} = -\frac{T}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

$$\updownarrow$$

$$\mathcal{L}_{MP3} = -\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

2.2.3 Without extra field, 2 internal indices

... and promote derivatives to covariant D_a ,

$$S_{MP4} = -\frac{T}{2} \int d\tau \wedge d\sigma e e_i^a e^{bi} D_a X^{\mu jj'} \eta_{jj'} \eta_{kk'} D_b X^{\nu kk'} E_\mu^I E_{\nu I}$$

$$\mathcal{L}_{MP4} = -\frac{T}{2} e e_i^a e_j^b D_a X^{\mu ik} \eta_{kl} D_b X^{\nu lj} E_\mu^I E_{\nu I}$$

2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) .$$

$$\updownarrow$$

$$\mathcal{L}_{DP} = -\frac{T_0}{2} \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) .$$

2.4 Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of D -dimensional gamma matrices and bulk spinors

$$S_{LP} = -\frac{T}{2} \int d\tau \wedge d\sigma \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

$$\updownarrow$$

$$\mathcal{L}_{LP} = -\frac{T}{2} \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi$$

3 EoMs

Start by writing $\det(e) = \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n$ and $g_{ab} = g_{ab}^{ij} \eta_{ij}$

3.1 With small internal index

3.1.1 Without extra field, w.r.t e

$$\begin{aligned}
\frac{\delta S_{MP2}}{\delta e_l^e} &= \frac{D\mathcal{L}_{MP2}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)} \right) \left(-\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \right) = \\
&= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (\eta^{ml} g_{ce} e_d^n e_i^a e_j^b + e_c^m \eta^{nl} g_{de} e_i^a e_j^b + \\
&+ e_c^m e_d^n \delta_e^a \delta_i^l e_j^b + e_c^m e_d^n e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} g_{ce} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} e_c^k e_{ek} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_i^l e_j^a e_k^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -T (-\det(e) e_e^l e_i^a e_j^b + \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \stackrel{!}{=} 0 \\
&\quad \downarrow \\
T_e^l &:= E_\mu^I E_{\nu I} (e_i^a D_a X^{\mu i} D_e X^{\nu l} - e_e^l e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j}) = 0 \\
&\quad \Downarrow \\
e_e^l &= f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I} , \\
\frac{1}{f} &= e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \\
&\quad \downarrow \\
e_e^l e_{fl} &= (f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I}) (f e_{i'}^{a'} D_{a'} X^{\mu' i'} D_f X_l^{\nu'} E_{\mu'}^{I'} E_{\nu'}^{I'}) = \\
&= f^2 D_e X^{\nu l} D_f X_l^{\nu'} E_{\nu'}^{I'} E_{\nu'}^{I'} (e_i^a e_{i'}^{a'} D_a X^{\mu i} D_{a'} X^{\mu' i'} E_\mu^I E_{\mu'}^{I'}) = \\
&= f D_e X^{\mu i} D_f X_i^\nu E_\mu^I E_{\nu I} \\
&\quad \Downarrow \\
D_e X^{\mu i} D_f X_i^\nu E_\mu^I E_{\nu I} &= 2 \partial_e X^\mu \partial_f X^\nu G_{\mu\nu} \\
e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\
&\quad \Downarrow \\
D_a X^{\mu 0} &= -i \partial_a X^\mu \\
D_a X^{\mu 1} &= \partial_a X^\mu \\
&\quad \Downarrow \\
X^{\mu 0} &= -i X^{\mu 1}
\end{aligned}$$

3.1.2 With extra field, w.r.t e

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta e_o^e} &= \frac{D\mathcal{L}_{MP3}}{De_o^e} = \left(\frac{\partial}{\partial e_o^e} - \partial_f \frac{\partial}{\partial (\partial_f e_o^e)} \right) \left(-\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (g_{ce} \eta^{mo} e_d^n \eta^{ij} e_i^a e_j^b + e_c^m g_{de} \eta^{no} \eta^{ij} e_i^a e_j^b + \\
&+ e_c^m e_d^n \eta^{ij} \delta_e^a \delta_i^o e_j^b + e_c^m e_d^n \eta^{ij} e_i^a \delta_e^b \delta_j^o) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -\frac{T}{2} \varepsilon^{cd} \varepsilon_{mn} (e_c^p e_{ep} e_d^n \eta^{mo} \eta^{ij} e_i^a e_j^b + 2 \det(e) e^{ao} \delta_e^b) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -T(-\det(e) e_e^o \eta^{ij} e_i^a e_j^b + \det(e) e^{ao} \delta_e^b) D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \stackrel{!}{=} 0
\end{aligned}$$

↓

$$\boxed{T_e^o := (e^{ao} D_a(X_k^\mu v^k) D_e(X_l^\nu v^l) - e_e^o \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l)) E_\mu^I E_\nu^J \eta_{IJ} = 0}$$

⇓

$$\begin{aligned}
e_e^o &= f e^{ao} D_a(X_k^\mu v^k) D_e(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} , \\
\frac{1}{f} &= \eta^{ij} e_i^a e_j^b D_a(X_k^\mu v^k) D_b(X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}
\end{aligned}$$

3.1.3 With extra field, w.r.t ω

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta \omega_c^{mn}} &= \frac{D\mathcal{L}_{MP3}}{D\omega_c^{mn}} = \left(\frac{\partial}{\partial \omega_c^{mn}} - \partial_d \frac{\partial}{\partial (\partial_d \omega_c^{mn})} \right) \left(-\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T}{4} e g^{ab} ((-\delta_a^c \delta_{[m}^{k'} \eta_{n]k} X_k^\mu v^k) D_b X_l^\nu v^l + D_a X_k^\mu v^k (-\delta_b^c \delta_{[m}^{l'} \eta_{n]l} X_l^\nu v^l)) G_{\mu\nu} = \\
&= \frac{T}{4} e (g^{ca} X_{[m}^\mu v_{n]} D_a X_k^\nu v^k + g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_{n]}) G_{\mu\nu} = \\
&= \frac{T}{2} e g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_{n]} G_{\mu\nu} \stackrel{!}{=} 0
\end{aligned}$$

↓

$$\boxed{\mathcal{T}_{ab}^i := e^{ci} D_c X_k^\mu v^k X_{[m}^\nu v_{n]} e_a^m e_b^n G_{\mu\nu} = 0}$$

3.1.4 With extra field, w.r.t v

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta v^l} &= \frac{D\mathcal{L}_{MP3}}{Dv^l} = \left(\frac{\partial}{\partial v^l} - \partial_c \frac{\partial}{\partial (\partial_c v^l)} \right) \left(-\frac{T}{2} e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_\nu^I \right) = \\
&= -\frac{T}{2} g^{ab} (D_a X_j^\mu \delta_l^j D_b X_k^\nu v^k + D_a X_j^\mu v^j D_b X_k^\nu \delta_l^k) G_{\mu\nu} = \\
&= -T g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} \stackrel{!}{=} 0
\end{aligned}$$

↓

$$\boxed{g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} = 0}$$

3.1.5 With extra field, w.r.t X

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta X_l^\lambda} &= \frac{D\mathcal{L}_{MP3}}{DX_l^\lambda} = \left(\frac{\partial}{\partial X_l^\lambda} - \partial_c \frac{\partial}{\partial (\partial_c X_l^\lambda)} \right) \left(-\frac{T}{2} e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_{\nu I} \right) = \\
&= -\frac{T}{2} \left(e g^{ab} \left((-\omega_{aj}^{j'} \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k - D_a X_j^\mu v^j \omega_{bk}^{k'} \delta_\lambda^\nu \delta_k^l v^k) E_\mu^I E_{\nu I} + \right. \right. \\
&\quad \left. \left. + e g^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \left(\frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} + E_\mu^I \frac{\partial E_{\nu I}}{\partial X_l^\lambda} \right) \right) - \right. \\
&\quad \left. - \partial_c (e g^{ab} (\delta_a^c \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k + D_a X_j^\mu v^j \delta_b^c \delta_\lambda^\nu \delta_k^l v^k) G_{\mu\nu}) \right) = \\
&= -T \left(-e g^{ab} D_a X_j^\mu v^j \omega_{bk}^l v^k G_{\mu\lambda} + e g^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - \partial_c (e g^{ac} v^l D_a X_j^\mu v^j G_{\mu\lambda}) \right) \\
&= -T \left(e g^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - D_b (e g^{ab} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) \right) \stackrel{!}{=} 0 \\
&\quad \downarrow \\
&\quad \boxed{D_b (e e_i^a e^{bi} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) = e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I}}
\end{aligned}$$

3.2 Without extra index

3.2.1 w.r.t e

$$\begin{aligned}
\frac{\delta S_{DP}}{\delta e_l^e} &= \frac{D\mathcal{L}_{DP}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial (\partial_f e_l^e)} \right) \left(-\frac{T_0}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T_0}{4}
\end{aligned}$$

3.3 Linear action

3.3.1 w.r.t e

$$\begin{aligned}
\frac{\delta S_{LP}}{\delta e_j^b} &= \frac{D\mathcal{L}_{LP}}{De_j^b} = \left(\frac{\partial}{\partial e_j^b} - \partial_e \frac{\partial}{\partial (\partial_e e_j^b)} \right) \left(-\frac{T}{2} \bar{\psi} e e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi \right) = \\
&= -\frac{T}{4} \bar{\psi} \varepsilon^{cd} \varepsilon_{mn} ((g_{cb} \eta^{mj} e_d^n + e_c^m g_{db} \eta^{nj}) e_i^a + e_c^m e_d^n \delta_b^a \delta_i^j) D_a X^{\mu i} E_\mu^I \gamma_I \psi = \\
&= -\frac{T}{2} \bar{\psi} (-e e_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I + e D_b X^{\mu j} E_\mu^I \gamma_I) \psi \stackrel{!}{=} 0 \\
&\quad \downarrow \\
&\quad T_b^j := \bar{\psi} (D_b X^{\mu j} E_\mu^I \gamma_I - e_b^j e_i^a D_a X^{\mu i} E_\mu^I \gamma_I) \psi = 0 \\
&\quad \downarrow \\
&\quad e_b^j = F \bar{\psi} D_b X^{\mu j} E_\mu^I \gamma_I \psi, \\
&\quad \frac{1}{F} = \bar{\psi} e_i^a D_a X^{\mu i} E_\mu^I \gamma_I \psi \\
&\quad \Downarrow \\
&\quad \bar{\psi} \psi = 1, \quad \psi \bar{\psi} = \mathbb{1} \\
&\quad e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_\nu^J \eta_{IJ} = g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}
\end{aligned}$$

4 Inverse Area Action

4.1 Nambu-Goto

Start from Nambu-Goto action

$$S_{NG} = -T \int d^2x \sqrt{-h},$$

and make quantum geometry correction

$$\sqrt{-h} \rightarrow \sqrt{-(h + g\Delta)} \approx \sqrt{-h} \left(1 - \frac{g\Delta}{2(-h)} + \mathcal{O}\left(\frac{g^2}{h^2}\right) \right),$$

leading to modified NG action

$$\begin{aligned} S_{MNG} &= -T \int d^2x \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}} \right) = S_{NG} + S_{IA} \\ &\downarrow \\ \mathcal{L}_{MNG} &= -T \left(\sqrt{-h} - \frac{g\Delta}{2\sqrt{-h}} \right) \end{aligned}$$

4.1.1 EoMs

Let $\mathcal{P}_\mu^\tau \equiv \partial \mathcal{L}_{MNG} / \partial \dot{X}^\mu$ and $\mathcal{P}_\mu^\sigma \equiv \partial \mathcal{L}_{MNG} / \partial X'^\mu$

$$\begin{aligned} \delta S_{MNG} &= \int d^2x \left(\frac{\partial \mathcal{L}_{MNG}}{\partial \dot{X}^\mu} \delta \dot{X}^\mu + \frac{\partial \mathcal{L}_{MNG}}{\partial X'^\mu} \delta X'^\mu \right) = \\ &= - \int d^2x \left(\partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma \right) \delta X^\mu + \int d\tau \mathcal{P}_\mu^\sigma \delta X^\mu \Big|_{\sigma=0}^{\sigma=\sigma_1} \stackrel{!}{=} 0 \\ &\downarrow \\ \text{EoM} : \partial_\tau \mathcal{P}_\mu^\tau + \partial_\sigma \mathcal{P}_\mu^\sigma &= 0 \\ \text{B.C.} : \mathcal{P}_\mu^\sigma \delta X^\mu \Big|_{\sigma=0}^{\sigma=\sigma_1} &= 0, \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}_\mu^\tau &= \frac{\partial \mathcal{L}_{MNG}}{\partial \dot{X}^\mu} = -T \left(\frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} + \frac{g\Delta}{2} \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{(-h)^{3/2}} \right) = \\ &= -T \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{-h}} \left(1 + \frac{g\Delta}{2(-h)} \right) = \mathcal{P}_{\mu(NG)}^\tau \left(1 + \frac{g\Delta}{2(-h)} \right) \\ \mathcal{P}_\mu^\sigma &= \frac{\partial \mathcal{L}_{MNG}}{\partial X'^\mu} = -T \left(\frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} + \frac{g\Delta}{2} \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{(-h)^{3/2}} \right) = \\ &= -T \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 X'_\mu}{\sqrt{-h}} \left(1 + \frac{g\Delta}{2(-h)} \right) = \mathcal{P}_{\mu(NG)}^\sigma \left(1 + \frac{g\Delta}{2(-h)} \right) \end{aligned}$$

gauge fixing static gauge $\tau = t$ and transverse gauge $\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial s} \frac{ds}{d\sigma} = 0$ (s = length along string)

$$\begin{aligned} &\downarrow \\ \mathcal{P}_{(NG)}^{\tau\mu} &= T \frac{ds}{d\sigma} \gamma_{v\perp} \frac{\partial X^\mu}{\partial t} \\ \mathcal{P}_{(NG)}^{\sigma\mu} &= \frac{T}{\gamma_{v\perp}} \frac{\partial X^\mu}{\partial s} \\ -h &= \frac{1}{\gamma_{v\perp}^2} \left(\frac{ds}{d\sigma} \right)^2 \end{aligned}$$

string energy get's redefined to

$$\frac{\partial}{\partial t} \left(T \frac{ds}{d\sigma} \gamma_{v\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \right) = 0,$$

and for the spatial part we have

$$\partial_\tau \vec{\mathcal{P}}^\tau + \partial_\sigma \vec{\mathcal{P}}^\sigma = 0$$

↓

$$\frac{\partial}{\partial t} \left[T \frac{ds}{d\sigma} \gamma_{v\perp} \frac{\partial \vec{X}}{\partial t} \left(1 + \frac{g\Delta}{2(-h)} \right) \right] + \frac{ds}{d\sigma} \frac{\partial}{\partial s} \left[-\frac{T}{\gamma_{v\perp}} \frac{\partial \vec{X}}{\partial s} \left(1 + \frac{g\Delta}{2(-h)} \right) \right] = 0$$

↓

$$\mu \gamma_{v\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial}{\partial s} \left[\frac{T}{\gamma_{v\perp}} \left(1 + \frac{g\Delta}{2(-h)} \right) \frac{\partial \vec{X}}{\partial s} \right] = 0$$

\Rightarrow effective mass density becomes $\mu_{eff} = \mu \gamma_{v\perp} \left(1 - \frac{g\Delta}{2(-h)} \right)$ and effective tension becomes $T_{eff} = \frac{T}{\gamma_{v\perp}} \left(1 - \frac{g\Delta}{2(-h)} \right)$

↓

$$\mu_{eff} \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial}{\partial s} \left[T_{eff} \frac{\partial \vec{X}}{\partial s} \right] = 0$$

$$\mathcal{H} = \dot{\vec{X}} \cdot \vec{\pi} - \mathcal{L} =$$

$$\begin{aligned} &= \vec{v}_\perp \cdot \left(T \frac{ds}{d\sigma} \gamma_{v\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \vec{v}_\perp \right) - \left(-T \frac{ds}{d\sigma} \frac{1}{\gamma_{v\perp}} \left(1 + \frac{g\Delta}{2(-h)} \right) \right) = \\ &= T \frac{ds}{d\sigma} \left(1 + \frac{g\Delta}{2(-h)} \right) \left(\gamma_{v\perp} v_\perp^2 + \frac{1}{\gamma_{v\perp}} \right) = \\ &= T \frac{ds}{d\sigma} \gamma_{v\perp} \left(1 + \frac{g\Delta}{2(-h)} \right) \end{aligned}$$

$$\text{let } \left(1 + \frac{g\Delta}{2(-h)} \right) = F$$

$$\frac{\partial^2 \vec{X}}{\partial t^2} - \frac{1}{F \gamma_{v\perp}} \frac{d\sigma}{ds} \frac{\partial}{\partial \sigma} \left[\frac{1}{\gamma_{v\perp}} F \frac{d\sigma}{ds} \frac{\partial \vec{X}}{\partial \sigma} \right] = 0$$

↓

$$A(\sigma) = \frac{\gamma_{v\perp}}{F} \frac{ds}{d\sigma} \stackrel{!}{=} 1$$

↓

$$d\sigma = \frac{\gamma_{v\perp}}{F} ds = \frac{1}{TF^2} dE \Rightarrow \sigma(q) = \frac{1}{T} \int_0^q \frac{1}{F^2} dE$$

↓

$$F^2 \frac{\partial^2 \vec{X}}{\partial t^2} - \frac{\partial^2 \vec{X}}{\partial \sigma^2} = 0$$

\Rightarrow speed of wave on the string gets modified by correction factor $v = c/F$

$$-h = \frac{1}{\gamma_{v\perp}^2} \left(\frac{ds}{d\sigma} \right)^2 = \frac{F^2}{\gamma_{v\perp}^4} = F^2 (1 - v_\perp^2)^2$$

↓

$$-h = \left(1 + \frac{g\Delta}{2(-h)} \right)^2 (1 - v_\perp^2)^2$$

↓

$$\frac{-h}{\left(1 + \frac{g\Delta}{2(-h)}\right)^2} = (1 - v_\perp^2)^2$$

↓ (WolframAlpha)

$$-h = \frac{(1 - v_\perp^2)^2}{3} - f_1 - f_2,$$

where (let $a = \frac{g\Delta}{2}$ and $b = (1 - v_\perp^2)^2$)

$$f_1 = \frac{\sqrt[3]{-27a^2b + 3\sqrt{3}\sqrt{27a^4b^2 + 4a^3b^3 - 18ab^2 - 2b^3}}}{3\sqrt[3]{2}}$$

$$f_2 = \frac{(6ab + b^2)}{6f_1}$$

↓

$$F = 1 + \frac{g\Delta}{2(-h)} = 1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}$$

↓

$$v = \frac{1}{F} = \frac{1}{1 + \frac{g\Delta}{2\left(\frac{b}{3} - f_1 - f_2\right)}}$$

5 Bimetric Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu},$$

and promote to bimetric action

↓

$$S_{BP} = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} - \frac{T'}{2} \int d^2x \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

5.1 EoMs

$$\begin{aligned} \frac{\delta S_{BP}}{\delta g^{cd}} &= -\frac{T}{2} \left(\frac{\partial \sqrt{-g}}{\partial g^{cd}} g^{ab} + \sqrt{-g} \frac{\partial g^{ab}}{\partial g^{cd}} \right) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + 0 = \\ &= -\frac{T}{2} \left(-\frac{1}{2} \sqrt{-g} g_{cd} g^{ab} + \sqrt{-g} \delta_{(c}^a \delta_{d)}^b \right) \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \stackrel{!}{=} 0 \end{aligned}$$

$$T_{cd}^{(G)} := \left(\partial_c X^\mu \partial_d X^\nu - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \right) G_{\mu\nu} = 0$$

⇓

$$g_{cd} = 2f^{(G)} \partial_c X^\mu \partial_d X^\nu G_{\mu\nu},$$

$$\frac{1}{f^{(G)}} = g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$\begin{aligned} \frac{\delta S_{BP}}{\delta h^{cd}} &= \dots = \\ &= -\frac{T'}{2} \left(-\frac{1}{2} \sqrt{-h} h_{cd} h^{ab} + \sqrt{-h} \delta_{(c}^a \delta_{d)}^b \right) \partial_a X^\mu \partial_b X^\nu H_{\mu\nu} \stackrel{!}{=} 0 \end{aligned}$$

$$\downarrow$$

$$T_{cd}^{(H)} := \left(\partial_c X^\mu \partial_d X^\mu - \frac{1}{2} h_{cd} h^{ab} \partial_a X^\mu \partial_b X^\nu \right) H_{\mu\nu} = 0$$

$$\downarrow$$

$$h_{cd} = 2f^{(H)} \partial_c X^\mu \partial_d X^\nu H_{\mu\nu},$$

$$\frac{1}{f^{(H)}} = h^{ab} \partial_a X^\mu \partial_b X^\nu H_{\mu\nu}$$

$$\begin{aligned} \frac{\delta S_{BP}}{\delta X^\lambda} &= \left(-\frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu - \\ &\quad - \partial_c \left(-\frac{T}{2} \sqrt{-g} g^{ab} G_{\mu\nu} - \frac{T'}{2} \sqrt{-h} h^{ab} H_{\mu\nu} \right) (\delta_a^c \delta_\lambda^\mu \partial_b X^\nu + \partial_a X^\mu \delta_b^c \delta_\lambda^\nu) = \\ &= \partial_a \left(T \sqrt{-g} g^{ab} G_{\mu\lambda} + T' \sqrt{-h} h^{ab} H_{\mu\lambda} \right) \partial_b X^\mu - \\ &\quad - \left(\frac{T}{2} \sqrt{-g} g^{ab} \partial_\lambda G_{\mu\nu} + \frac{T'}{2} \sqrt{-h} h^{ab} \partial_\lambda H_{\mu\nu} \right) \partial_a X^\mu \partial_b X^\nu \stackrel{!}{=} 0 \end{aligned}$$

$$\downarrow$$

$$\partial_a (F_{\mu\lambda}^{ab} \partial_b X^\mu) = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu,$$

$$F_{\mu\nu}^{ab} = T \sqrt{-g} g^{ab} G_{\mu\nu} + T' \sqrt{-h} h^{ab} H_{\mu\nu}$$

let $T' = Tk\Delta/2$ and $h = g^{-1}(H_{\mu\nu} = G_{\mu\nu})$

$$F_{\mu\nu}^{ab} = T \left(\sqrt{-g} g^{ab} G_{\mu\nu} + \frac{k\Delta}{2} \sqrt{-g^{-1}} U_{a'}^a (g^{-1})^{a'b'} U_{b'}^b (G^{-1})_{\mu\nu} \right),$$

add new field coupled to derivative A_a transforming as

$$A_a g^{bc} \rightarrow (A_a g^{bc})' = A_a U_{b'}^b g^{b'c'} U_{c'}^c - \frac{1}{\alpha} g^{b'c'} (\partial_a U_{b'}^b U_{c'}^c + U_{b'}^b \partial_a U_{c'}^c),$$

defining new WS covariant derivative

$$D_a g^{bc} = \partial_a g^{bc} + \alpha A_a g^{bc},$$

such that derivative ignores $\text{SO}(1,1)$ gauge freedom on inverse metric. Upgrade EoM to incorporate this derivative:

$$D_a (F_{\mu\lambda}^{ab} \partial_b X^\mu) = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu$$

$$\downarrow$$

$$F_{\mu\lambda}^{ab} \partial_a \partial_b X^\mu + D_a F_{\mu\lambda}^{ab} \partial_b X^\mu = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu,$$

imposing conformal symmetry $g^{ab} = (\phi(x))^{-1} \eta^{ab} \implies D_a F_{\mu\lambda}^{ab} = 0$

$$F_{\mu\lambda}^{ab} \partial_a \partial_b X^\mu = \frac{1}{2} \partial_\lambda F_{\mu\nu}^{ab} \partial_a X^\mu \partial_b X^\nu,$$

$$F_{\mu\lambda}^{ab} = T \left(G_{\mu\lambda} + \frac{k\Delta}{2} (G^{-1})_{\mu\lambda} \right) \eta^{ab}$$

$$(G^{-1})_{\mu\nu} \stackrel{?}{=} \frac{1}{-g} G_{\mu\nu}$$

6 Inverse Area Polyakov

Start with Polyakov action

$$S_P = -\frac{T}{2} \int d^2x \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

and make quantum geometry correction

$$\sqrt{-g} \rightarrow \sqrt{-(g+k\Delta)} \approx \sqrt{-g} \left(1 + \frac{k\Delta}{2g} + \mathcal{O}\left(\frac{k^2}{g^2}\right) \right)$$

leading to

$$S_{IAP} = -\frac{T}{2} \int d^2x \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

6.1 EoMs

$$\begin{aligned} \frac{\delta S_{IAP}}{\delta g^{cd}} &= \frac{\partial}{\partial g^{cd}} \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \delta_c^a \delta_d^b \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = \\ &= \left(-\frac{1}{2} \sqrt{-g} g_{cd} + \frac{k\Delta}{4\sqrt{-g}} g_{cd} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} = \\ &= -\frac{1}{2} \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} \stackrel{!}{=} 0 \end{aligned}$$

↓

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} = 0$$

↓

$$\begin{aligned} g_{cd} &= 2f \partial_c X^\mu \partial_d X^\nu G_{\mu\nu}, \\ \frac{1}{f} &= g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \end{aligned}$$

$$\begin{aligned} \frac{\delta S_{IAP}}{\delta X^\lambda} &= \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - \partial_c \left(\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) 2g^{ab} \delta_a^c \delta_b^\lambda \partial_b X^\nu G_{\mu\nu} \right) = \\ &= \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2 \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} - \\ &\quad - 2\partial_a \left(\left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} = \\ &= \left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu} - 2 \left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} - \\ &\quad - 2\partial_a \left(\left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} \stackrel{!}{=} 0 \end{aligned}$$

↓

$$\begin{aligned} \left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} + \partial_a \left(\left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \right) \partial_b X^\nu G_{\lambda\nu} = \\ \frac{1}{2} \left(1 - \frac{k\Delta}{2(-g)} \right) \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu}, \end{aligned}$$

imposing conformal symmetry $g^{ab} = (\phi(x))^{-1} \eta^{ab}$ and let $F' = \left(1 - \frac{k\Delta}{2\phi^2} \right)$ leads to

$$\eta^{ab} \partial_a \partial_b X^\nu G_{\lambda\nu} + \frac{1}{F'} \eta^{ab} \partial_a F' \partial_b X^\nu G_{\lambda\nu} = \eta^{ab} \partial_a X^\mu \partial_b X^\nu \partial_\lambda G_{\mu\nu},$$

assuming flat background $G_{\mu\nu} = \eta_{\mu\nu}$

$$\eta^{ab}\partial_a\partial_b X^\nu\eta_{\lambda\nu} + \frac{1}{F'}\eta^{ab}\partial_a F'\partial_b X^\nu\eta_{\lambda\nu} = 0$$

\downarrow

$$\eta^{ab}\partial_a\partial_b X^\mu + \eta^{ab}\partial_a \ln(F')\partial_b X^\mu = 0,$$

use plane-wave ansatz $X^\mu = X_0^\mu e^{-i(E\tau - p\sigma)}$

$$(E^2 - p^2)X^\mu + \partial_\tau \ln(F')iEX^\mu - \partial_\sigma \ln(F')ipX^\mu = 0$$

\downarrow

$$(E^2 - p^2)X^\mu + i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F'))X^\mu = 0$$

\downarrow

$$i(E\partial_\tau \ln(F') - p\partial_\sigma \ln(F')) = -m^2,$$

$$-m^2 = -E^2 + p^2$$

\Downarrow

$$\partial_\tau \ln(F') = iE, \quad \partial_\sigma \ln(F') = ip$$

\Downarrow

$$\ln(F') = i(E\tau + p\sigma) + a, \quad a \in \mathbb{C}$$

\Downarrow

$$F' = Ae^{i(E\tau + p\sigma + \phi)}, \quad A > 0$$

\Downarrow

$$\phi(\tau, \sigma) = \sqrt{\frac{k\Delta}{2(1 - Ae^{i(E\tau + p\sigma + \phi)})}}$$

\downarrow

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2X^\mu = 0$$

7 Inverse Area Polyakov v2

Quantum geometry correction

$$\sqrt{-g} \rightarrow \sqrt{-(g+k\Delta)} \approx \sqrt{-g} \left(1 + \frac{k\Delta}{2g} + \mathcal{O}\left(\frac{k^2}{g^2}\right) \right),$$

which implies

$$\begin{aligned} g_{ab} &\rightarrow \tilde{g}_{ab} = g_{ab} + \sqrt{k\Delta}\delta_{ab}, \\ g^{ab} &\rightarrow \tilde{g}^{ab}, \quad \tilde{g}^{ab}\tilde{g}_{bc} = \delta^a_c, \end{aligned}$$

leading to

$$\begin{aligned} S &= -\frac{T}{2} \int d^2x \sqrt{-\tilde{g}} \tilde{g}^{ab} \partial_a X^\mu \partial_b X^\nu \tilde{G}_{\mu\nu}, \\ \tilde{G}_{\mu\nu} &= G_{\mu\nu} + \sqrt[p]{k\Theta} \delta_{\mu\nu}, \end{aligned}$$

where Θ is related to minimal hyper-volume of bulk quantum geometry

7.1 EoMs

$$\frac{\delta S}{\delta \tilde{g}^{cd}} = -\frac{1}{2} \sqrt{-\tilde{g}} \tilde{g}_{cd} \tilde{g}^{ab} \partial_a X^\mu \partial_b X^\nu \tilde{G}_{\mu\nu} + \sqrt{-\tilde{g}} \partial_c X^\mu \partial_d X^\nu \tilde{G}_{\mu\nu} \stackrel{!}{=} 0,$$

giving us

$$T_{cd} := \partial_c X^\mu \partial_d X^\nu \tilde{G}_{\mu\nu} - \frac{1}{2} \tilde{g}_{cd} \tilde{g}^{ab} \partial_a X^\mu \partial_b X^\nu \tilde{G}_{\mu\nu} = 0$$

from which we extract the corrected metric

$$\tilde{g}_{cd} = 2f \partial_c X^\mu \partial_d X^\nu \tilde{G}_{\mu\nu},$$

with

$$\frac{1}{f} = \tilde{g}^{ab} \partial_a X^\mu \partial_b X^\nu \tilde{G}_{\mu\nu}$$

and we can expand the metrics

$$g_{cd} + \sqrt{k\Delta}\delta_{cd} = \partial_c X^\mu \partial_d X^\nu G_{\mu\nu} + \sqrt[p]{k\Theta} \partial_c X^\mu \partial_d X^\nu \delta_{\mu\nu},$$

giving us the constraint

$$\partial_c X^\mu \partial_d X^\nu \delta_{\mu\nu} = \frac{\sqrt{k\Delta}}{\sqrt[p]{k\Theta}} \delta_{cd}$$

8 Relating NG and Polyakov analysis

From NG analysis

$$(F^+)^2 \frac{\partial^2 X^\mu}{\partial t^2} - \frac{\partial^2 X^\mu}{\partial x^2} = 0,$$

make change of variables $\tau = \tau(t, x)$ and $\sigma = \sigma(t, x)$

$$\begin{aligned} (F^+)^2 \left(\frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial t} \frac{\partial}{\partial \sigma} \right) \left(\frac{\partial \tau}{\partial t} \frac{\partial X^\mu}{\partial \tau} + \frac{\partial \sigma}{\partial t} \frac{\partial X^\mu}{\partial \sigma} \right) - \left(\frac{\partial \tau}{\partial x} \frac{\partial}{\partial \tau} + \frac{\partial \sigma}{\partial x} \frac{\partial}{\partial \sigma} \right) \left(\frac{\partial \tau}{\partial x} \frac{\partial X^\mu}{\partial \tau} + \frac{\partial \sigma}{\partial x} \frac{\partial X^\mu}{\partial \sigma} \right) &= 0 \\ (F^+)^2 \left(\left(\frac{\partial \tau}{\partial t} \right)^2 \frac{\partial^2 X^\mu}{\partial \tau^2} + 2 \frac{\partial \tau}{\partial t} \frac{\partial \sigma}{\partial t} \frac{\partial^2 X^\mu}{\partial \tau \partial \sigma} + \left(\frac{\partial \sigma}{\partial t} \right)^2 \frac{\partial^2 X^\mu}{\partial \sigma^2} \right) - \\ &- \left(\left(\frac{\partial \tau}{\partial x} \right)^2 \frac{\partial^2 X^\mu}{\partial \tau^2} + 2 \frac{\partial \tau}{\partial x} \frac{\partial \sigma}{\partial x} \frac{\partial^2 X^\mu}{\partial \tau \partial \sigma} + \left(\frac{\partial \sigma}{\partial x} \right)^2 \frac{\partial^2 X^\mu}{\partial \sigma^2} \right) + \\ &+ \left((F^+)^2 \frac{\partial^2 \tau}{\partial t^2} - \frac{\partial^2 \tau}{\partial x^2} \right) \frac{\partial X^\mu}{\partial \tau} + \left((F^+)^2 \frac{\partial^2 \sigma}{\partial t^2} - \frac{\partial^2 \sigma}{\partial x^2} \right) \frac{\partial X^\mu}{\partial \sigma} = 0 \\ ((F^+)^2 (\partial_t \tau)^2 - (\partial_x \tau)^2) \partial_\tau^2 X^\mu + ((F^+)^2 (\partial_t \sigma)^2 - (\partial_x \sigma)^2) \partial_\sigma^2 X^\mu + \\ &+ 2 ((F^+)^2 \partial_t \tau \partial_t \sigma - \partial_x \tau \partial_x \sigma) \partial_\tau \partial_\sigma X^\mu + ((F^+)^2 \partial_t^2 \tau - \partial_x^2 \tau) \partial_\tau X^\mu + ((F^+)^2 \partial_t^2 \sigma - \partial_x^2 \sigma) \partial_\sigma X^\mu = 0 \end{aligned}$$

Without loss of generality, we can set (just rescale/rotate the coordinates)

$$\begin{aligned} (F^+)^2 (\partial_t \sigma)^2 - (\partial_x \sigma)^2 &= (\partial_x \tau)^2 - (F^+)^2 (\partial_t \tau)^2 \\ (F^+)^2 \partial_t \tau \partial_t \sigma &= \partial_x \tau \partial_x \sigma, \end{aligned}$$

or more compactly

$$\begin{aligned} \partial_t \tau &= v \partial_x \sigma \\ \partial_t \sigma &= v \partial_x \tau, \end{aligned}$$

where $v = 1/F^+$, which is equivalent to

$$\begin{aligned} \partial_t^2 \tau &= \partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau + v^2 \partial_x^2 \tau \\ \partial_t^2 \sigma &= \partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma + v^2 \partial_x^2 \sigma, \end{aligned}$$

reducing the big equation to

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu + \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} \partial_\tau X^\mu + \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \sigma)^2 - v^2 (\partial_x \sigma)^2)} \partial_\sigma X^\mu = 0.$$

Next introduce $X^\mu = \kappa(\tau, \sigma) Y^\mu(\tau, \sigma)$ s.t terms proportional to $\partial_\tau Y^\mu$ and $\partial_\sigma Y^\mu$ vanish:

$$\begin{aligned} \partial_\tau X^\mu &= \partial_\tau \kappa Y^\mu + \kappa \partial_\tau Y^\mu \\ \partial_\sigma X^\mu &= \partial_\sigma \kappa Y^\mu + \kappa \partial_\sigma Y^\mu \\ \partial_\tau^2 X^\mu &= \partial_\tau^2 \kappa Y^\mu + 2 \partial_\tau \kappa \partial_\tau Y^\mu + \kappa \partial_\tau^2 Y^\mu \\ \partial_\sigma^2 X^\mu &= \partial_\sigma^2 \kappa Y^\mu + 2 \partial_\sigma \kappa \partial_\sigma Y^\mu + \kappa \partial_\sigma^2 Y^\mu \end{aligned}$$

$$\begin{aligned} \kappa \frac{\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} + 2 \partial_\tau \kappa &\stackrel{!}{=} 0 \\ \kappa \frac{\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma}{((\partial_t \sigma)^2 - v^2 (\partial_x \sigma)^2)} - 2 \partial_\sigma \kappa &\stackrel{!}{=} 0 \end{aligned}$$

$$\begin{aligned}\partial_\tau \kappa &= -\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \\ \partial_\sigma \kappa &= \frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)},\end{aligned}$$

undoing chain rule for t by multiplying first eq by $\partial_t \tau$ and second by $\partial_t \sigma$ and summing gives

$$\begin{aligned}\partial_t \kappa &= \partial_t \tau \partial_\tau \kappa + \partial_t \sigma \partial_\sigma \kappa = \\ &= \partial_t \tau \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) + \partial_t \sigma \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) = \\ &= v \partial_x \sigma \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) + v \partial_x \tau \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) = \\ &= \frac{v \kappa \partial_t v ((\partial_x \tau)^2 - (\partial_x \sigma)^2)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} = \\ &= \frac{v \kappa \partial_t v ((\partial_x \tau)^2 - (\partial_t \tau / v)^2)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} = \\ &= -\frac{\kappa \partial_t v}{2v},\end{aligned}$$

or simply

$$\begin{aligned}\frac{\partial_t \kappa}{\kappa} &= -\frac{1}{2} \frac{\partial_t v}{v} \\ \partial_t \ln(\kappa) &= -\frac{1}{2} \partial_t \ln(v) \\ \ln(\kappa) &= -\frac{1}{2} \ln(v) + f(x).\end{aligned}$$

To determine $f(x)$, we undo the chain rule for x now:

$$\begin{aligned}\partial_x \kappa &= \partial_x \tau \partial_\tau \kappa + \partial_x \sigma \partial_\sigma \kappa = \\ &= \partial_x \tau \left(-\frac{\kappa(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) + \partial_x \sigma \left(\frac{\kappa(\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} \right) = \\ &= \frac{\kappa v \partial_x v ((\partial_x \sigma)^2 - (\partial_x \tau)^2)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} = \\ &= \frac{\kappa v \partial_x v ((\partial_t \tau / v)^2 - (\partial_x \tau)^2)}{2((\partial_t \tau)^2 - v^2(\partial_x \tau)^2)} = \\ &= \frac{\kappa \partial_x v}{2v}\end{aligned}$$

$$\begin{aligned}\frac{\partial_x \kappa}{\kappa} &= \frac{1}{2} \frac{\partial_x v}{v} \\ \partial_x \ln(\kappa) &= \frac{1}{2} \partial_x \ln(v) \\ \ln(\kappa) &= \frac{1}{2} \ln(v), \\ \partial_x \left(-\frac{1}{2} \ln(v) + f(x) \right) &= \frac{1}{2} \partial_x \ln(v)\end{aligned}$$

$f'(x) = \partial_x \ln(v)$ (perhaps this means somewhere in the composition of functions time dependence is lost?)

$$f(x) = \ln(v)$$

giving us

$$\kappa = v^{\frac{1}{2}}, \quad v = v(x).$$

Substituting this into the earlier equation yields

$$\partial_\tau^2 Y^\mu - \partial_\sigma^2 Y^\mu + m^2 Y^\mu = 0,$$

with

$$\begin{aligned}
m^2 &= \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{(\partial_t v \partial_x \sigma + v \partial_x v \partial_x \tau) \partial_\tau \kappa + (\partial_t v \partial_x \tau + v \partial_x v \partial_x \sigma) \partial_\sigma \kappa}{\kappa ((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} = \\
&= \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{\partial_t v (\partial_x \sigma \partial_\tau \kappa + \partial_x \tau \partial_\sigma \kappa) + v \partial_x v \partial_x \kappa}{\kappa ((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)} = \\
&= \frac{\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa}{\kappa} + \frac{(\partial_x v)^2}{((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)},
\end{aligned}$$

where by use of chain rule we have that

$$\partial_t^2 \kappa - v^2 \partial_x^2 \kappa - v \partial_x v \partial_x \kappa = ((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2) (\partial_\tau^2 \kappa - \partial_\sigma^2 \kappa),$$

where since $\kappa = v^{\frac{1}{2}}$, m^2 reduces to

$$m^2 = \frac{(\partial_x v)^2 - 2v \partial_x^2 v}{4((\partial_t \tau)^2 - v^2 (\partial_x \tau)^2)},$$

with τ satisfying

$$\begin{aligned}
\partial_t \tau &= v \partial_x \sigma \\
v \partial_x \tau &= \partial_t \sigma
\end{aligned}$$

such that $-m^2$ is indeed constant.

9 Solutions for the Polyakov KG eqn (WRONG ONE)

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi \eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta_{cd} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\begin{aligned} \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ \partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0. \end{aligned}$$

These simplify further by expanding the summation on a, b :

$$\begin{aligned} \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} (-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}) \\ \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}. \end{aligned}$$

The EoM is just a one-dimensional Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left(a^\mu(p) e^{-i(-E\tau + p\sigma)} + b^\mu(p) e^{i(-E\tau + p\sigma)} \right),$$

where $E(p) = \sqrt{p^2 + m^2}$. The derivatives are

$$\begin{aligned} \partial_\tau X^\mu &= \frac{i}{4\pi} \int dp \left(a^\mu(p) e^{-i(-E\tau + p\sigma)} - b^\mu(p) e^{i(-E\tau + p\sigma)} \right) \\ \partial_\sigma X^\mu &= \frac{i}{4\pi} \int dp \frac{p}{\sqrt{p^2 + m^2}} \left(-a^\mu(p) e^{-i(-E\tau + p\sigma)} + b^\mu(p) e^{i(-E\tau + p\sigma)} \right). \end{aligned}$$

Starting with the mixed constraint,

$$\begin{aligned} \partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp \left(a^\mu(p) e^{-i(-E(p)\tau + p\sigma)} - b^\mu(p) e^{i(-E(p)\tau + p\sigma)} \right) \times \\ &\quad \times \int dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\nu(p') e^{-i(-E(p')\tau + p'\sigma)} + b^\nu(p') e^{i(-E(p')\tau + p'\sigma)} \right) = \\ &= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left(a^\mu(p) e^{-i(-E(p)\tau + p\sigma)} - b^\mu(p) e^{i(-E(p)\tau + p\sigma)} \right) \times \\ &\quad \times \left(-a^\nu(p') e^{-i(-E(p')\tau + p'\sigma)} + b^\nu(p') e^{i(-E(p')\tau + p'\sigma)} \right) = \\ &= -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\mu(p) a^\nu(p') e^{i(E(p) + E(p'))\tau} e^{-i(p + p')\sigma} + a^\mu(p) b^\nu(p') e^{i(E(p) - E(p'))\tau} e^{-i(p - p')\sigma} + \right. \\ &\quad \left. + b^\mu(p) a^\nu(p') e^{-i(E(p) - E(p'))\tau} e^{i(p - p')\sigma} - b^\mu(p) b^\nu(p') e^{-i(E(p) + E(p'))\tau} e^{i(p + p')\sigma} \right) = 0. \end{aligned}$$

If we integrate this expression over σ we obtain

$$\begin{aligned}
& -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} \int d\sigma e^{-i(p+p')\sigma} + \right. \\
& \quad + a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} \int d\sigma e^{-i(p-p')\sigma} + b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} \int d\sigma e^{i(p-p')\sigma} - \\
& \quad \left. - b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} \int d\sigma e^{i(p+p')\sigma} \right) = \\
& = -\frac{\eta_{\mu\nu}}{8\pi} \int dp dp' \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} \delta(p+p') + a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} \delta(p-p') + \right. \\
& \quad \left. + b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} \delta(p-p') - b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} \delta(p+p') \right) = \\
& = -\frac{\eta_{\mu\nu}}{8\pi} \int dp \frac{p}{\sqrt{p^2 + m^2}} \left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} + a^\mu(p) b^\nu(p) + b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) = 0
\end{aligned}$$

which implies the whole expression inside the integral must vanish, thus we have (leaving $(a^\mu b^\nu + b^\mu a^\nu) \eta_{\mu\nu}$ as is since eventually these will not be numbers and may not commute, thus will not be equal to $2a^\mu b^\nu \eta_{\mu\nu}$)

$$\left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} + a^\mu(p) b^\nu(p) + b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu} = 0.$$

As for the other part of the constraint, let's start with the L.H.S

$$\begin{aligned}
\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{(4\pi)^2} \int dp dp' \left(a^\mu(p) e^{-i(-E(p)\tau+p\sigma)} - b^\mu(p) e^{i(-E(p)\tau+p\sigma)} \right) \times \\
& \quad \times \left(a^\nu(p') e^{-i(-E(p')\tau+p'\sigma)} - b^\nu(p') e^{i(-E(p')\tau+p'\sigma)} \right) = \\
& = -\frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \left(a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} e^{-i(p+p')\sigma} - a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} e^{-i(p-p')\sigma} - \right. \\
& \quad \left. - b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} e^{i(p-p')\sigma} + b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} e^{i(p+p')\sigma} \right),
\end{aligned}$$

which we integrate over σ to get

$$\begin{aligned}
& -\frac{\eta_{\mu\nu}}{8\pi} \int dp dp' \left(a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} \delta(p+p') - a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} \delta(p-p') - \right. \\
& \quad \left. - b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} \delta(p-p') + b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} \delta(p+p') \right) = \\
& = -\frac{\eta_{\mu\nu}}{8\pi} \int dp \left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} - a^\mu(p) b^\nu(p) - b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right),
\end{aligned}$$

and now for the R.H.S

$$\begin{aligned}
-\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{p}{\sqrt{p^2 + m^2}} \frac{p'}{\sqrt{p'^2 + m^2}} \left(-a^\mu(p) e^{-i(-E\tau+p\sigma)} + b^\mu(p) e^{i(-E\tau+p\sigma)} \right) \times \\
& \quad \times \left(-a^\nu(p') e^{-i(-E(p')\tau+p'\sigma)} + b^\nu(p') e^{i(-E(p')\tau+p'\sigma)} \right) = \\
& = \frac{\eta_{\mu\nu}}{(4\pi)^2} \int dp dp' \frac{pp'}{E(p)E(p')} \left(a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} e^{-i(p+p')\sigma} - a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} e^{-i(p-p')\sigma} - \right. \\
& \quad \left. - b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} e^{i(p-p')\sigma} + b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} e^{i(p+p')\sigma} \right),
\end{aligned}$$

which once again we integrate over σ ,

$$\begin{aligned} \frac{\eta_{\mu\nu}}{8\pi} \int dp dp' \frac{pp'}{E(p)E(p')} & \left(a^\mu(p) a^\nu(p') e^{i(E(p)+E(p'))\tau} \delta(p+p') - a^\mu(p) b^\nu(p') e^{i(E(p)-E(p'))\tau} \delta(p-p') - \right. \\ & \left. - b^\mu(p) a^\nu(p') e^{-i(E(p)-E(p'))\tau} \delta(p-p') + b^\mu(p) b^\nu(p') e^{-i(E(p)+E(p'))\tau} \delta(p+p') \right) = \\ & = \frac{1}{8\pi} \int dp \frac{p^2}{p^2+m^2} \left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} - a^\mu(p) b^\nu(p) - b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu}, \end{aligned}$$

which when equated to the L.H.S yields

$$\int dp \left(\frac{p^2}{p^2+m^2} + 1 \right) \left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} - a^\mu(p) b^\nu(p) - b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu} = 0,$$

which implies

$$\left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} - a^\mu(p) b^\nu(p) - b^\mu(p) a^\nu(p) + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu} = 0.$$

We can sum and subtract this with the previous condition to simplify and obtain

$$\begin{aligned} \left(a^\mu(p) a^\nu(-p) e^{2iE(p)\tau} + b^\mu(p) b^\nu(-p) e^{-2iE(p)\tau} \right) \eta_{\mu\nu} &= 0 \\ \left(a^\mu(p) b^\nu(p) + b^\mu(p) a^\nu(p) \right) \eta_{\mu\nu} &= 0, \end{aligned}$$

and for while a^μ and b^ν are number-valued, the second condition can be further simplified to

$$a^\mu(p) b^\nu(p) \eta_{\mu\nu} = 0,$$

meaning a^μ and b^ν are orthogonal. In general, this condition states that a^μ and b^ν anti-commute w.r.t Lorentz inner product. The first condition must hold for all values of τ , and since the exponentials are linearly independent, we thus have

$$\begin{aligned} a^\mu(p) a^\nu(-p) \eta_{\mu\nu} &= 0 \\ b^\mu(p) b^\nu(-p) \eta_{\mu\nu} &= 0, \end{aligned}$$

meaning that reflecting the argument of a^μ and b^ν creates orthogonal vectors. We can thus write

$$\begin{aligned} a^\mu(p) &= \Lambda^\mu{}_\nu \left(\frac{p}{4} \right) a^\nu(0) \\ b^\mu(p) &= \Upsilon^\mu{}_\nu \left(\frac{p}{4} \right) b^\nu(0), \end{aligned}$$

where $\Lambda, \Upsilon \in \text{SO}^+(1, d)$. Also. since the orthogonality under reflection must hold for all values of p , in particular we have

$$\begin{aligned} a^\mu(0) a^\nu(0) \eta_{\mu\nu} &= 0 \\ b^\mu(0) b^\nu(0) \eta_{\mu\nu} &= 0, \end{aligned}$$

meaning that the initial $a^\mu(0)$ and $b^\nu(0)$ are null vectors. This means that $a^\mu(p)$ and $b^\nu(p)$ are also null vectors since they are related to the initial values by Lorentz transformation. With this, the solution becomes

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left(\Lambda^\mu{}_\nu \left(\frac{p}{4} \right) a^\nu(0) e^{-i(-E\tau+p\sigma)} + \Upsilon^\mu{}_\nu \left(\frac{p}{4} \right) b^\nu(0) e^{i(-E\tau+p\sigma)} \right)$$

with $(a^\mu b^\nu + b^\nu a^\mu) \eta_{\mu\nu} = 0$. This condition implies

$$\begin{aligned} \left(\Lambda^\mu{}_\gamma \left(\frac{p}{4} \right) a^\gamma(0) \Upsilon^\nu{}_\rho \left(\frac{p}{4} \right) b^\rho(0) \right) \eta_{\mu\nu} &= 0 \\ \Upsilon^\mu{}_\nu \left(\frac{p}{4} \right) &= \Lambda^\mu{}_\nu \left(\frac{p}{4} \right) = \Lambda^\mu{}_\nu \left(-\frac{p}{4} \right), \end{aligned}$$

thus turning the solution into

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left(\Lambda^\mu{}_\nu \left(\frac{p}{4} \right) a^\nu(0) e^{-i(-E\tau+p\sigma)} + \Lambda^\mu{}_\nu \left(-\frac{p}{4} \right) b^\nu(0) e^{i(-E\tau+p\sigma)} \right).$$

Next, we choose background coordinates s.t $a^\mu(0) = (a_0, a_0, 0, \dots, 0) = a_0 x^+$ and $b^\nu(0) = (b_0, b_0, 0, \dots, 0) = b_0 x^+$, simplifying the solution to

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi} \int dp \frac{1}{2E(p)} \left(a_0 \Lambda^\mu{}_+ \left(\frac{p}{4} \right) e^{-i(-E\tau+p\sigma)} + b_0 \Lambda^\mu{}_+ \left(-\frac{p}{4} \right) e^{i(-E\tau+p\sigma)} \right) x^+$$

10 Solutions for the Polyakov KG eqn

From the inverse-area corrected Polyakov action we have the equation

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu - m^2 X^\mu = 0$$

subjected to the constraint

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} - \frac{1}{2} g_{cd} g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} = 0,$$

which with conformal gauge $g_{ab} = \phi \eta_{ab}$ becomes

$$\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \frac{1}{2} \eta_{cd} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

or more explicitly,

$$\begin{aligned} \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \frac{1}{2} \eta^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ \partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} &= \partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu} = 0. \end{aligned}$$

These simplify further by expanding the summation on a, b :

$$\begin{aligned} \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\frac{1}{2} (-\partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} + \partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}) \\ \partial_\tau X^\mu \partial_\tau X^\nu \eta_{\mu\nu} &= -\partial_\sigma X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}. \end{aligned}$$

The EoM is just a one-dimensional finite-space Klein-Gordon equation, which has general solution given by Fourier transform

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu e^{i(-E_n\tau+n\sigma)} + b_n^\mu e^{i(E_n\tau+n\sigma)} \right),$$

with $E_n = \sqrt{n^2 + m^2}$. The derivatives are

$$\begin{aligned} \partial_\tau X^\mu &= \frac{i}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \left(-a_n^\mu e^{i(-E_n\tau+n\sigma)} + b_n^\mu e^{i(E_n\tau+n\sigma)} \right) \\ \partial_\sigma X^\mu &= \frac{i}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{n}{E_n} \left(a_n^\mu e^{i(-E_n\tau+n\sigma)} + b_n^\mu e^{i(E_n\tau+n\sigma)} \right). \end{aligned}$$

Reality of X^μ implies

$$\begin{aligned} (X^\mu)^* &= \left(\frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu e^{i(-E_n\tau+n\sigma)} + b_n^\mu e^{i(E_n\tau+n\sigma)} \right) \right)^* = \\ &= \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left((a_n^\mu)^* e^{-i(-E_n\tau+n\sigma)} + (b_n^\mu)^* e^{-i(E_n\tau+n\sigma)} \right) \equiv X^\mu, \end{aligned}$$

giving us

$$\begin{aligned} (a_n^\mu)^* &= b_{-n}^\mu \\ (b_n^\mu)^* &= a_{-n}^\mu, \end{aligned}$$

so the solution becomes

$$X^\mu(\tau, \sigma) = \frac{1}{2\pi\alpha'} \sum_{n \in \mathbb{Z}} \frac{1}{2E_n} \left(a_n^\mu e^{i(-E_n\tau+n\sigma)} + (a_{-n}^\mu)^* e^{i(E_n\tau+n\sigma)} \right).$$

Analysing the constraints, starting by the mixed one we get

$$\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu} = -\frac{\eta_{\mu\nu}}{16\pi^2(\alpha')^2} \sum_n \sum_k \frac{k}{E_k} \left(-a_n^\mu e^{i(-E_n\tau+n\sigma)} + (a_{-n}^\mu)^* e^{i(E_n\tau+n\sigma)} \right) \left(a_k^\nu e^{i(-E_k\tau+k\sigma)} + (a_{-k}^\nu)^* e^{i(E_k\tau+k\sigma)} \right) =$$

$$\begin{aligned}
&= -\frac{\eta_{\mu\nu}}{16\pi^2(\alpha')^2} \sum_n \sum_k \frac{k}{E_k} \left(-a_n^\mu a_k^\nu e^{-i(E_n+E_k)\tau} e^{i(n+k)\sigma} - a_n^\mu (a_{-k}^\nu)^* e^{-i(E_n-E_k)\tau} e^{i(n+k)\sigma} + \right. \\
&\quad \left. + (a_{-n}^\mu)^* a_k^\nu e^{i(E_n-E_k)\tau} e^{i(n+k)\sigma} + (a_{-n}^\mu)^* (a_{-k}^\nu)^* e^{i(E_n+E_k)\tau} e^{i(n+k)\sigma} \right) = \\
&= -\frac{\eta_{\mu\nu}}{16\pi^2(\alpha')^2} \sum_n \sum_p \frac{p-n}{E_{p-n}} \left(-a_n^\mu a_{p-n}^\nu e^{-i(E_n+E_{p-n})\tau} e^{ip\sigma} - a_n^\mu (a_{n-p}^\nu)^* e^{-i(E_n-E_{p-n})\tau} e^{ip\sigma} + \right. \\
&\quad \left. + (a_{-n}^\mu)^* a_{p-n}^\nu e^{i(E_n-E_{p-n})\tau} e^{ip\sigma} + (a_{-n}^\mu)^* (a_{n-p}^\nu)^* e^{i(E_n+E_{p-n})\tau} e^{ip\sigma} \right) = 0,
\end{aligned}$$

where since this must hold for all values of τ and σ , each term must vanish separately, giving us

$$\begin{aligned}
&\sum_n \sum_p \frac{p-n}{E_{p-n}} a_n^\mu a_{p-n}^\nu \eta_{\mu\nu} e^{-i(E_n+E_{p-n})\tau} e^{ip\sigma} = 0 \\
&\sum_n \sum_p \frac{p-n}{E_{p-n}} a_n^\mu (a_{n-p}^\nu)^* \eta_{\mu\nu} e^{-i(E_n-E_{p-n})\tau} e^{ip\sigma} = 0 \\
&\sum_n \sum_p \frac{p-n}{E_{p-n}} (a_{-n}^\mu)^* a_{p-n}^\nu \eta_{\mu\nu} e^{i(E_n-E_{p-n})\tau} e^{ip\sigma} = 0 \\
&\sum_n \sum_p \frac{p-n}{E_{p-n}} (a_{-n}^\mu)^* (a_{n-p}^\nu)^* \eta_{\mu\nu} e^{i(E_n+E_{p-n})\tau} e^{ip\sigma} = 0.
\end{aligned}$$

The 2 middle equations are just complex conjugates of each other, so they are redundant. The same is true for the first and last. This leaves us with

$$\begin{aligned}
L_{n,p} &:= \frac{p-n}{E_{p-n}} a_n^\mu a_{p-n}^\nu \eta_{\mu\nu} = 0 \\
\tilde{L}_{n,p} &:= \frac{p-n}{E_{p-n}} a_n^\mu (a_{n-p}^\nu)^* \eta_{\mu\nu} = 0.
\end{aligned}$$

One last thing before quantizing: the $n=0$ term in the solution reads

$$\frac{1}{2\pi\alpha'} \frac{1}{2m} \left(a_0^\mu e^{-im\tau} + (a_0^\mu)^* e^{im\tau} \right),$$

which is divergent in the $m \rightarrow 0$ limit unless $\Re(a_0^\mu) = 0$, thus a_0^μ must be purely imaginary. Keeping the analogy to the $m \rightarrow 0$ regime, we conclude $a_0^\mu = 4\pi i(\alpha')^2 p^\mu$. Since a_0^μ is proportional to p^μ , we have that

$$M^2 = -p^\mu p_\mu = \frac{1}{16\pi^2(\alpha')^4} a_0^\mu a_0^\nu \eta_{\mu\nu}$$

11 Into the Quantum Realm

Start by finding the canonical momentum

$$\begin{aligned}\Pi_\lambda &= \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\lambda)} = -\frac{T}{2} \left(\sqrt{-g} - \frac{k\Delta}{2\sqrt{-g}} \right) g^{ab} \delta_a^\tau \delta_\lambda^\mu \partial_b X^\nu \eta_{\mu\nu} = \\ &= -\frac{1}{4\pi\alpha'} \left(1 - \frac{k\Delta}{2\phi^2} \right) \eta^{\tau b} \partial_b X^\nu \eta_{\lambda\nu} = \\ &= \frac{1}{4\pi\alpha'} F^- \partial_\tau X^\nu \eta_{\lambda\nu} = F^- \frac{i}{(4\pi\alpha')^2} \sum_p \left(-\alpha_p^\nu e^{i(-E_p\tau+p\sigma)} + (\alpha_{-p}^\nu)^* e^{i(E_p\tau+p\sigma)} \right) \eta_{\nu\lambda}.\end{aligned}$$

Translating the equal-time Poisson bracket relations

$$\{X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} = \delta(\sigma - \sigma') \eta^{\mu\nu}$$

$$\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = \{\Pi^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} = 0$$

into equal time commutation relations for the now operators \hat{X}^μ and $\hat{\Pi}^\nu$

$$[\hat{X}^\mu(\tau, \sigma), \hat{\Pi}^\nu(\tau, \sigma')] = i\delta(\sigma - \sigma') \eta^{\mu\nu}$$

$$[\hat{X}^\mu(\tau, \sigma), \hat{X}^\nu(\tau, \sigma')] = [\hat{\Pi}^\mu(\tau, \sigma), \hat{\Pi}^\nu(\tau, \sigma')] = 0,$$

we start by the non-zero one

$$\begin{aligned}[\hat{X}^\mu(\tau, \sigma), \hat{\Pi}^\nu(\tau, \sigma')] &= \left[\frac{1}{2\pi\alpha'} \sum_{p \in \mathbb{Z}} \frac{1}{2E_p} \left(\hat{\alpha}_p^\mu e^{i(-E_p\tau+p\sigma)} + (\hat{\alpha}_{-p}^\mu)^\dagger e^{i(E_p\tau+p\sigma)} \right), \right. \\ &\quad \left. F^- \frac{i}{(4\pi\alpha')^2} \sum_{n \in \mathbb{Z}} \left(-\hat{\alpha}_n^\nu e^{i(-E_n\tau+n\sigma')} + (\hat{\alpha}_{-n}^\nu)^\dagger e^{i(E_n\tau+n\sigma')} \right) \right] = \\ &= F^- \frac{i}{(4\pi\alpha')^3} \sum_p \sum_n \frac{1}{E_p} \left(-[\hat{\alpha}_p^\mu, \hat{\alpha}_n^\nu] e^{-i(E_p+E_n)\tau} e^{i(p\sigma+n\sigma')} + [\hat{\alpha}_p^\mu, (\hat{\alpha}_{-n}^\nu)^\dagger] e^{-i(E_p-E_n)\tau} e^{i(p\sigma+n\sigma')} - \right. \\ &\quad \left. - [(\hat{\alpha}_{-p}^\mu)^\dagger, \hat{\alpha}_n^\nu] e^{i(E_p-E_n)\tau} e^{i(p\sigma+n\sigma')} + [(\hat{\alpha}_{-p}^\mu)^\dagger, (\hat{\alpha}_{-n}^\nu)^\dagger] e^{i(E_p+E_n)\tau} e^{i(p\sigma+n\sigma')} \right) \equiv i\delta(\sigma - \sigma') \eta^{\mu\nu},\end{aligned}$$

from which we conclude that

$$\begin{aligned}[\hat{\alpha}_p^\mu, (\hat{\alpha}_{-n}^\nu)^\dagger] &= \eta^{\mu\nu} \delta_{p, -n} \\ [\hat{\alpha}_p^\mu, \hat{\alpha}_n^\nu] &= 0 = [(\hat{\alpha}_{-p}^\mu)^\dagger, (\hat{\alpha}_{-n}^\nu)^\dagger],\end{aligned}$$

giving us the harmonic oscillator operators

$$\begin{aligned}\hat{a}_n^\mu &:= \hat{\alpha}_n^\mu \\ (\hat{a}_n^\mu)^\dagger &:= (\hat{\alpha}_n^\mu)^\dagger, \quad n \neq 0.\end{aligned}$$

We define a vacuum state of the string to obey

$$\hat{a}_n^\mu |0\rangle = 0, \quad \text{for } n \neq 0.$$

(What does $(\hat{a}_{-n}^\mu)^\dagger |0\rangle$ mean? Also, note how in our case, each field X^μ only gives rise to one tower of creation and annihilation operators...) For $n = 0$, we have the center of mass momentum operator, so the vacuum also obeys

$$\hat{p}^\mu |0\rangle = p^\mu |0\rangle.$$

A generic state arises from a sequence of creation operators on the vacuum

$$((a_1^{\mu_1})^\dagger)^{n_{\mu_1}} ((a_2^{\mu_2})^\dagger)^{n_{\mu_2}} \dots ((a_1^{\nu_1})^\dagger)^{n_{\nu_1}} ((a_2^{\mu_2})^\dagger)^{n_{\nu_2}} \dots |0\rangle.$$

This (should) give rise to particles.

As in regular ST, we have ghosts arising from the Minkowski metric

$$\left[\hat{\alpha}_n^\mu, (\hat{\alpha}_m^\nu)^\dagger \right] = \eta^{\mu\nu} \delta_{nm}.$$

For constraints, classically we have

$$\begin{aligned} L_{n,p} &= \frac{p-n}{E_{p-n}} \alpha_n^\mu \alpha_{n-p}^\nu \eta_{\mu\nu} = 0 \\ \tilde{L}_{n,p} &= \frac{p-n}{E_{p-n}} \alpha_n^\mu (\alpha_{n-p}^\nu)^* \eta_{\mu\nu} = 0, \end{aligned}$$

which because of the existence of ghosts we require to have vanishing matrix elements when sandwiched between physical states

$$\langle \text{phys}' | \hat{L}_{n,p} | \text{phys} \rangle = 0 = \langle \text{phys}' | \hat{\tilde{L}}_{n,p} | \text{phys} \rangle.$$

When translating the constraints into quantum operators, in $L_{n,p}$ we have no ambiguity of ordering since $\hat{\alpha}$ commutes with itself for different indices. As for $\tilde{L}_{n,p}$, there is an ambiguity for $p = 0$, and so we pick normal ordering with the annihilation operators moved to the right

$$\hat{\tilde{L}}_{n,0} = \frac{-n}{E_n} (\hat{\alpha}_n^\mu)^\dagger \hat{\alpha}_n^\nu \eta_{\mu\nu}.$$

The ambiguity manifests in the imposition of this constraint as

$$\langle \text{phys}' | (\hat{\tilde{L}}_{n,0} - a) | \text{phys} \rangle = 0,$$

for some constant a . Since classically

$$M^2 = -p^\mu p_\mu = -\frac{1}{(4\pi(\alpha')^2)^2} \alpha_0^\mu (\alpha_0^\nu)^* \eta_{\mu\nu} = -\frac{1}{(4\pi(\alpha')^2)^2} \lim_{n \rightarrow 0} \left(-\frac{E_n}{n} \tilde{L}_{n,0} \right),$$

we see that the string mass spectrum will be shifted by

$$\hat{M}^2 = -\frac{1}{(4\pi(\alpha')^2)^2} \left((\hat{\alpha}_0^\mu)^\dagger \hat{\alpha}_0^\nu \eta_{\mu\nu} + \lim_{n \rightarrow 0} \frac{E_n}{n} a \right),$$

so either the mass spectrum of the string diverges, or the constant a isn't so constant and $a = \mu n$, making the limit equal to $\mu E_0 = \mu^2$, the KG "mass" term.

The commutation relations between the L 's are

$$[\hat{L}_{n,m}, \hat{L}_{p,k}] = 0 \text{ (this has only annihilation operators)}$$

$$\begin{aligned} [\hat{L}_{n,m}, \hat{\tilde{L}}_{p,k}] &= \left[\frac{m-n}{E_{m-n}} \hat{\alpha}_n^\mu \hat{\alpha}_{n-m}^\nu \eta_{\mu\nu}, \frac{k-p}{E_{k-p}} (\hat{\alpha}_k^{\mu'})^\dagger \hat{\alpha}_{k-p}^{\nu'} \eta_{\mu'\nu'} \right] = \\ &= \frac{(m-n)}{E_{m-n}} \frac{(k-p)}{E_{k-p}} \left(\eta_{\mu\nu} \left[\hat{\alpha}_n^\mu \hat{\alpha}_{n-m}^\nu, (\hat{\alpha}_k^{\mu'})^\dagger \right] \hat{\alpha}_{k-p}^{\nu'} \eta_{\mu'\nu'} + \eta_{\mu\nu} (\hat{\alpha}_k^{\mu'})^\dagger \left[\hat{\alpha}_n^\mu \hat{\alpha}_{n-m}^\nu, \hat{\alpha}_{k-p}^{\nu'} \right] \eta_{\mu'\nu'} \right) = \\ &= \frac{(m-n)}{E_{m-n}} \frac{(k-p)}{E_{k-p}} \left(\eta_{\mu\nu} \hat{\alpha}_n^\mu \left[\hat{\alpha}_{n-m}^\nu, (\hat{\alpha}_k^{\mu'})^\dagger \right] \hat{\alpha}_{k-p}^{\nu'} \eta_{\mu'\nu'} + \eta_{\mu\nu} \left[\hat{\alpha}_n^\mu, (\hat{\alpha}_k^{\mu'})^\dagger \right] \hat{\alpha}_{n-m}^\nu \hat{\alpha}_{k-p}^{\nu'} \eta_{\mu'\nu'} \right) = \\ &= \frac{(m-n)}{E_{m-n}} \frac{(k-p)}{E_{k-p}} \eta_{\mu\nu} \eta_{\mu'\nu'} \left(\hat{\alpha}_n^\mu \eta^{\nu\mu'} \delta_{n-m,k} \hat{\alpha}_{k-p}^{\nu'} + \eta^{\mu\mu'} \delta_{nk} \hat{\alpha}_{n-m}^\nu \hat{\alpha}_{k-p}^{\nu'} \right) = \\ &= \frac{(m-n)}{E_{m-n}} \frac{(k-p)}{E_{k-p}} \eta_{\mu\nu} \left(\hat{\alpha}_n^\mu \hat{\alpha}_{k-p}^\nu \delta_{n-m,k} + \hat{\alpha}_{n-m}^\mu \hat{\alpha}_{k-p}^\nu \delta_{n,k} \right) = \\ &= \frac{m-n}{E_{m-n}} \left(\hat{L}_{n+k-p,n} \delta_{n-m,k} + \hat{L}_{n-m+k-p,n-m} \delta_{n,k} \right) \end{aligned}$$

$$\begin{aligned}
[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{p,k}] &= \left[\frac{(m-n)}{E_{m-n}} (\hat{\alpha}_n^\mu)^\dagger \hat{\alpha}_{n-m}^\nu \eta_{\mu\nu}, \frac{(k-p)}{E_{k-p}} (\hat{\alpha}_k^{\mu'})^\dagger \hat{\alpha}_{k-p}^{\nu'} \eta_{\mu'\nu'} \right] = \\
&= \frac{(m-n)}{E_{m-n}} \frac{(k-p)}{E_{k-p}} \eta_{\mu\nu} \eta_{\mu'\nu'} \left((\hat{\alpha}_n^\mu)^\dagger [\hat{\alpha}_{n-m}^\nu, (\hat{\alpha}_k^{\mu'})^\dagger] \hat{\alpha}_{k-p}^{\nu'} + (\hat{\alpha}_k^{\mu'})^\dagger [(\hat{\alpha}_n^\mu)^\dagger, \hat{\alpha}_{k-p}^{\nu'}] \hat{\alpha}_{n-m}^\nu \right) = \\
&= \frac{(m-n)}{E_{m-n}} \frac{(k-p)}{E_{k-p}} \eta_{\mu\nu} \eta_{\mu'\nu'} \left((\hat{\alpha}_n^\mu)^\dagger \eta^{\nu\mu'} \delta_{n-m,k} \hat{\alpha}_{k-p}^{\nu'} + (\hat{\alpha}_k^{\mu'})^\dagger \eta^{\mu\nu'} \delta_{n,k-p} \hat{\alpha}_{n-m}^\nu \right) = \\
&= \frac{(m-n)}{E_{m-n}} \frac{(k-p)}{E_{k-p}} \eta_{\mu\nu} \left((\hat{\alpha}_n^\mu)^\dagger \hat{\alpha}_{k-p}^\nu \delta_{n-m,k} + (\hat{\alpha}_k^\mu)^\dagger \hat{\alpha}_{n-m}^\nu \delta_{n,k-p} \right) = \\
&= \frac{(m-n)}{E_{m-n}} \frac{(k-p)}{E_{k-p}} \left(\frac{E_{-n}}{(-n)} \hat{\tilde{L}}_{k-p-n,k-p} \delta_{n-m,k} + \frac{E_{-k}}{(-k)} \hat{\tilde{L}}_{n-m-k,n-m} \delta_{n,k-p} \right).
\end{aligned}$$

Since $[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{p,k}]$ only has annihilation operators, it has no ordering ambiguities. $[\hat{\tilde{L}}_{n,m}, \hat{\tilde{L}}_{p,k}]$ would have ordering ambiguities for $k-p=0$ and $n-m=0$, however in both cases regardless of the order the commutator would vanish because of the $(m-n)(k-p)$ factor multiplying the whole expression, thus our algebra has no anomalous terms.

Considering the ground state $|0; k\rangle$, mass-shell condition

$$-(4\pi(\alpha')^2)^2 \hat{M}^2 + \lim_{n \rightarrow 0} \frac{E_n}{n} \hat{\tilde{L}}_{n,0} = \lim_{n \rightarrow 0} \frac{E_n}{n} c_n$$

implies

$$-(4\pi(\alpha')^2)^2 (M^2 + k^\mu k_\mu) = \lim_{n \rightarrow 0} \frac{E_n}{n} c_n.$$

Now considering states at the first excited level, given by $\zeta_\mu(\hat{\alpha}_1^\mu)^\dagger |0; k\rangle$ with $\zeta_\mu(k)$ being a polarization vector, the mass-shell condition will now read

$$\begin{aligned}
&\left(-(4\pi(\alpha')^2)^2 \hat{M}^2 + \lim_{n \rightarrow 0} \frac{E_n}{n} \hat{\tilde{L}}_{n,0} \right) \zeta_\mu(\hat{\alpha}_1^\mu)^\dagger |0; k\rangle = \lim_{n \rightarrow 0} \frac{E_n}{n} c_n \zeta_\mu(\hat{\alpha}_1^\mu)^\dagger |0; k\rangle \\
&\left(-(4\pi(\alpha')^2)^2 \hat{M}^2 - \lim_{n \rightarrow 0} (\hat{\alpha}_n^\lambda)^\dagger \hat{\alpha}_n^\nu \eta_{\lambda\nu} \right) \zeta_\mu(\hat{\alpha}_1^\mu)^\dagger |0; k\rangle = \lim_{n \rightarrow 0} \frac{E_n}{n} c_n \zeta_\mu(\hat{\alpha}_1^\mu)^\dagger |0; k\rangle \\
&-(4\pi(\alpha')^2)^2 \hat{M}^2 - \lim_{n \rightarrow 0} \eta_{\lambda\nu} \zeta_\mu \left((\hat{\alpha}_n^\lambda)^\dagger (\hat{\alpha}_1^\mu)^\dagger \hat{\alpha}_n^\nu + (\hat{\alpha}_n^\lambda)^\dagger \eta^{\nu\mu} \delta_{n,1} \right) |0; k\rangle = \lim_{n \rightarrow 0} \frac{E_n}{n} c_n \zeta_\mu(\hat{\alpha}_1^\mu)^\dagger |0; k\rangle
\end{aligned}$$