

Sketchboard

November 6, 2023

1 Basic concepts

1.1 With extra capital internal index on embedding fields

Promote embedding fields X^μ to have an internal group index with D values

$$\begin{aligned}
X^\mu &\rightarrow X^{\mu I} , \quad I = 0, \dots, d \\
g_{ab} &= 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{IJ} = 2f D_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu} \\
D_a X^{\mu I} &= \partial_a X^{\mu I} + \omega_{aJ}^I X^{\mu J} \\
e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ} , \quad i, j = 0, 1 \\
g = \det(g_{ab}) &\rightarrow g = \det(\text{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

1.2 With extra small internal index on embedding fields

Promote embedding fields to have a internal group index with 2 values

$$\begin{aligned}
X^\mu &\rightarrow X^{\mu i} , \quad i = 0, 1 \\
g_{ab} &= 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{ij} = 2f D_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu} \\
D_a X^{\mu i} &= \partial_a X^{\mu i} + \omega_{aj}^i X^{\mu j} \\
e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij} \\
g = \det(g_{ab}) &\rightarrow g = \det(\text{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

1.3 Without extra index on embedding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$\begin{aligned}
e_a^i e_b^j \eta_{ij} &= g_{ab} = 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\
g = \det(g_{ab}) &= \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int d\tau \wedge d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} ,$$

2.1 With capital internal index

... and promote partial derivative ∂_a to covariant derivative D_a , giving us our first attempt at modified Polyakov action

$$\begin{aligned}
S_{MP1} &= -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ} \\
&\quad \updownarrow \\
\mathcal{L}_{MP1} &= -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}
\end{aligned}$$

2.2 With small internal index

... and promote derivatives to covariant D_a , giving us another modified Polyakov action

$$\begin{aligned}
S_{MP2} &= -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I(X) E_{\nu I}(X) \\
&\quad \updownarrow \\
\mathcal{L}_{MP2} &= -\frac{T_0}{2} \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I(X) E_{\nu I}(X)
\end{aligned}$$

2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) .$$

$$\updownarrow$$

$$\mathcal{L}_{DP} = -\frac{T_0}{2} \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) .$$

3 EoMs

Start by writing $\det(e) = \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n$ and $g_{ab} = g_{ab}^{ij} \eta_{ij}$

3.1 With small internal index

3.1.1 w.r.t e

$$\begin{aligned} \frac{\delta S_{MP2}}{\delta e_l^e} &= \frac{D \mathcal{L}_{MP2}}{D e_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial (\partial_f e_l^e)} \right) \left(-\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \right) = \\ &= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (\eta^{ml} g_{ce} e_d^n e_i^a e_j^b + e_c^m \eta^{nl} g_{de} e_i^a e_j^b + \\ &+ e_c^m e_d^n \delta_e^a \delta_i^l e_j^b + e_c^m e_d^n e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\ &= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} g_{ce} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\ &= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} e_c^k e_{ek} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\ &= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_e^l e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\ &= -T (\det(e) e_e^l e_i^a e_j^b + \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \stackrel{!}{=} 0 \end{aligned}$$

$$\downarrow$$

$$T_e^l := E_\mu^I E_{\nu I} (e_e^l e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} + e_i^a D_a X^{\mu i} D_e X^{\nu l}) = 0$$

$$\Downarrow$$

$$e_e^l = -f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I} ,$$

$$\frac{1}{f} = e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I}$$

3.2 Without extra index

3.2.1 w.r.t e

$$\begin{aligned} \frac{\delta S_{DP}}{\delta e_l^e} &= \frac{D \mathcal{L}_{DP}}{D e_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial (\partial_f e_l^e)} \right) \left(-\frac{T_0}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I E_\nu^J \eta_{IJ} \right) = \\ &= -\frac{T_0}{4} \end{aligned}$$