

Sketchboard

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1 Basic concepts

1.1 With extra capital internal index on embedding fields

Promote embedding fields X^μ to have an internal group index with D values

$$\begin{aligned}
X^\mu &\rightarrow X^{\mu I} , \quad I = 0, \dots, d \\
g_{ab} &= 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{IJ} \sim f D_a X^{\mu I} D_b X^{\nu J} G_{\mu\nu} \\
D_a X^{\mu I} &= \partial_a X^{\mu I} + \omega_{aJ}^I X^{\mu J} \\
e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{IJ} T_I T_J) = g_{ab}^{IJ} \eta_{IJ} , \quad i, j = 0, 1 \\
g = \det(g_{ab}) &\rightarrow g = \det(\text{Tr}(g_{ab}^{IJ} T_I T_J)) = \det(g_{ab}^{IJ} \eta_{IJ}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

1.2 With 1 extra small internal index on embedding fields

Promote embedding fields to have a internal group index with 2 values

$$\begin{aligned}
X^\mu &\rightarrow X^{\mu i} , \quad i = 0, 1 \\
g_{ab} &= 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \rightarrow g_{ab}^{ij} \sim f D_a X^{\mu i} D_b X^{\nu j} G_{\mu\nu} \\
D_a X^{\mu i} &= \partial_a X^{\mu i} + \omega_{aj}^i X^{\mu j} \\
e_a^i e_b^j \eta_{ij} &= g_{ab} \rightarrow e_a^i e_b^j \eta_{ij} = \text{Tr}(g_{ab}^{ij} T_i T_j) = g_{ab}^{ij} \eta_{ij} \implies e_a^i e_b^j = g_{ab}^{ij} \\
g = \det(g_{ab}) &\rightarrow g = \det(\text{Tr}(g_{ab}^{ij} T_i T_j)) = \det(g_{ab}^{ij} \eta_{ij}) = \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

1.3 With 2 extra small indices

Promote embedding fields X^μ to have two internal indices with 2 values

$$X^\mu \rightarrow X^{\mu ij} , \quad i, j = 0, 1$$

1.4 Without extra index on embedding fields

Change from WS metric to WS zweibein and connection (which vanishes since in 2d metric is conformally flat)

$$\begin{aligned}
e_a^i e_b^j \eta_{ij} &= g_{ab} = 2f \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\
g = \det(g_{ab}) &= \det(e_a^i e_b^j \eta_{ij}) = -\det(e)^2 \implies \sqrt{-g} = \det(e)
\end{aligned}$$

2 Building an Action

Start with Polyakov action in curved space-time

$$S_P = -\frac{T_0}{2} \int d\tau \wedge d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} ,$$

2.1 With capital internal index

... and promote partial derivative ∂_a to covariant derivative D_a , giving us our first attempt at modified Polyakov action

$$\begin{aligned}
S_{MP1} &= -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ} \\
&\quad \updownarrow \\
\mathcal{L}_{MP1} &= -\frac{T_0}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X^{\mu I} D_b X^{\nu J} E_\mu^K E_{\nu K} \eta_{IJ}
\end{aligned}$$

2.2 With small internal index

2.2.1 Without extra field, 1 internal index

... and promote derivatives to covariant D_a , giving us another modified Polyakov action

$$S_{MP2} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

$$\updownarrow$$

$$\mathcal{L}_{MP2} = -\frac{T_0}{2} \det(e) e_i^a e_j^b D_a X^{\mu i} D_b X^{\mu j} E_\mu^I(X) E_{\nu I}(X)$$

2.2.2 With extra field

..., promote partial derivative to covariant and add extra internal WS field v^i leading us to

$$S_{MP3} = -\frac{T}{2} \int d\tau \wedge d\sigma \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ}$$

$$\updownarrow$$

$$\mathcal{L}_{MP3} = -\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}$$

2.2.3 Without extra field, 2 internal indices

... and promote derivatives to covariant D_a ,

$$S_{MP4} = -\frac{T}{2} \int d\tau \wedge d\sigma e e_i^a e^{bi} D_a X^{\mu jj'} \eta_{jj'} \eta_{kk'} D_b X^{\nu kk'} E_\mu^I E_{\nu I}$$

$$\mathcal{L}_{MP4} = -\frac{T}{2} e e_i^a e_j^b D_a X^{\mu ik} \eta_{kl} D_b X^{\nu lj} E_\mu^I E_{\nu I}$$

2.3 Without extra index

... and swap to new set of variables giving us the dyad-Polyakov action

$$S_{DP} = -\frac{T_0}{2} \int d\tau \wedge d\sigma \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) .$$

$$\updownarrow$$

$$\mathcal{L}_{DP} = -\frac{T_0}{2} \det(e) e_i^a e^{bi} \partial_a X^\mu \partial_b X^\nu E_\mu^I(X) E_{\nu I}(X) .$$

2.4 Linear Polyakov action

..., promote partial derivative to covariant derivative and build a linear action with inclusion of new bulk field

$$S_{LP} = -\frac{T}{2} \int d\tau \wedge d\sigma e e_i^a D_a X^{\mu i} E_\mu^I V_I$$

3 EoMs

Start by writing $\det(e) = \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n$ and $g_{ab} = g_{ab}^{ij} \eta_{ij}$

3.1 With small internal index

3.1.1 Without extra field, w.r.t e

$$\begin{aligned}
\frac{\delta S_{MP2}}{\delta e_l^e} &= \frac{D\mathcal{L}_{MP2}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)} \right) \left(-\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \right) = \\
&= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (\eta^{ml} g_{ce} e_d^n e_i^a e_j^b + e_c^m \eta^{nl} g_{de} e_i^a e_j^b + \\
&+ e_c^m e_d^n \delta_e^a \delta_i^l e_j^b + e_c^m e_d^n e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} g_{ce} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} \eta^{ml} e_c^k e_{ek} e_d^n e_i^a e_j^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -\frac{T}{2} (\varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n e_i^l e_j^a e_e^b + 2 \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} = \\
&= -T (-\det(e) e_e^l e_i^a e_j^b + \det(e) e_i^a \delta_e^b \delta_j^l) D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I} \stackrel{!}{=} 0 \\
&\quad \downarrow \\
T_e^l &:= E_\mu^I E_{\nu I} (-e_e^l e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} + e_i^a D_a X^{\mu i} D_e X^{\nu l}) = 0 \\
&\quad \Downarrow \\
e_e^l &= f e_i^a D_a X^{\mu i} D_e X^{\nu l} E_\mu^I E_{\nu I} , \\
\frac{1}{f} &= e_i^a e_j^b D_a X^{\mu i} D_b X^{\nu j} E_\mu^I E_{\nu I}
\end{aligned}$$

3.1.2 With extra field, w.r.t e

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta e_o^e} &= \frac{D\mathcal{L}_{MP3}}{De_o^e} = \left(\frac{\partial}{\partial e_o^e} - \partial_f \frac{\partial}{\partial(\partial_f e_o^e)} \right) \left(-\frac{T}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} (g_{ce} \eta^{mo} e_d^n \eta^{ij} e_i^a e_j^b + e_c^m g_{de} \eta^{no} \eta^{ij} e_i^a e_j^b + \\
&+ e_c^m e_d^n \eta^{ij} \delta_e^a \delta_i^o e_j^b + e_c^m e_d^n \eta^{ij} e_i^a \delta_e^b \delta_j^o) D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -\frac{T}{2} \varepsilon^{cd} \varepsilon_{mn} (e_c^p e_{ep} e_d^n \eta^{mo} \eta^{ij} e_i^a e_j^b + 2 \det(e) e^{ao} \delta_e^b) D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} = \\
&= -T (-\det(e) e_o^e \eta^{ij} e_i^a e_j^b + \det(e) e^{ao} \delta_e^b) D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} \stackrel{!}{=} 0 \\
&\quad \downarrow \\
\boxed{T_e^o} &:= (e^{ao} D_a (X_k^\mu v^k) D_e (X_l^\nu v^l) - e_o^e \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l)) E_\mu^I E_\nu^J \eta_{IJ} = 0 \\
&\quad \Downarrow \\
e_e^o &= f e^{ao} D_a (X_k^\mu v^k) D_e (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ} , \\
\frac{1}{f} &= \eta^{ij} e_i^a e_j^b D_a (X_k^\mu v^k) D_b (X_l^\nu v^l) E_\mu^I E_\nu^J \eta_{IJ}
\end{aligned}$$

3.1.3 With extra field, w.r.t ω

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta \omega_c^{mn}} &= \frac{D\mathcal{L}_{MP3}}{D\omega_c^{mn}} = \left(\frac{\partial}{\partial \omega_c^{mn}} - \partial_d \frac{\partial}{\partial (\partial_d \omega_c^{mn})} \right) \left(-\frac{T}{2} \det(e) \eta^{ij} e_i^a e_j^b D_a X_k^\mu v^k D_b X_l^\nu v^l E_\mu^I E_\nu^J \eta_{IJ} \right) = \\
&= -\frac{T}{4} e g^{ab} ((-\delta_a^c \delta_{[m}^{k'} \eta_{n]k} X_{k'}^\mu v^k) D_b X_l^\nu v^l + D_a X_k^\mu v^k (-\delta_b^c \delta_{[m}^{l'} \eta_{n]l} X_{l'}^\nu v^l)) G_{\mu\nu} = \\
&= \frac{T}{4} e (g^{ca} X_{[m}^\mu v_n] D_a X_k^\nu v^k + g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_n]) G_{\mu\nu} = \\
&= \frac{T}{2} e g^{ac} D_a X_k^\mu v^k X_{[m}^\nu v_n] G_{\mu\nu} \stackrel{!}{=} 0
\end{aligned}$$

↓

$$\mathcal{T}_{ab}^i := e^{ci} D_c X_k^\mu v^k X_{[m}^\nu v_n] e_a^m e_b^n G_{\mu\nu} = 0$$

3.1.4 With extra field, w.r.t v

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta v^l} &= \frac{D\mathcal{L}_{MP3}}{Dv^l} = \left(\frac{\partial}{\partial v^l} - \partial_c \frac{\partial}{\partial (\partial_c v^l)} \right) \left(-\frac{T}{2} e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_\nu^I \right) = \\
&= -\frac{T}{2} g^{ab} (D_a X_j^\mu \delta_l^j D_b X_k^\nu v^k + D_a X_j^\mu v^j D_b X_k^\nu \delta_l^k) G_{\mu\nu} = \\
&= -T g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} \stackrel{!}{=} 0
\end{aligned}$$

↓

$$g^{ab} D_a X_l^\mu D_b X_j^\nu v^j G_{\mu\nu} = 0$$

3.1.5 With extra field, w.r.t X

$$\begin{aligned}
\frac{\delta S_{MP3}}{\delta X_l^\lambda} &= \frac{D\mathcal{L}_{MP3}}{DX_l^\lambda} = \left(\frac{\partial}{\partial X_l^\lambda} - \partial_c \frac{\partial}{\partial (\partial_c X_l^\lambda)} \right) \left(-\frac{T}{2} e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k E_\mu^I E_\nu^I \right) = \\
&= -\frac{T}{2} \left(e g^{ab} \left((-\omega_{aj}^{j'} \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k - D_a X_j^\mu v^j \omega_{bk}^{k'} \delta_\lambda^\nu \delta_k^l v^k) E_\mu^I E_\nu^I + \right. \right. \\
&\quad \left. \left. + e g^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \left(\frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} + E_\mu^I \frac{\partial E_{\nu I}}{\partial X_l^\lambda} \right) \right) - \right. \\
&\quad \left. - \partial_c (e g^{ab} (\delta_a^c \delta_\lambda^\mu \delta_j^l v^j D_b X_k^\nu v^k + D_a X_j^\mu v^j \delta_b^c \delta_\lambda^\nu \delta_k^l v^k) G_{\mu\nu}) \right) = \\
&= -T \left(-e g^{ab} D_a X_j^\mu v^j \omega_{bk}^{k'} v^k G_{\mu\lambda} + e g^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - \partial_c (e g^{ac} v^l D_a X_j^\mu v^j G_{\mu\lambda}) \right) \\
&= -T \left(e g^{ab} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I} - D_b (e g^{ab} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) \right) \stackrel{!}{=} 0
\end{aligned}$$

↓

$$D_b (e e_i^a e^{bi} v^l D_a X_j^\mu v^j E_\mu^I E_{\lambda I}) = e e_i^a e^{bi} D_a X_j^\mu v^j D_b X_k^\nu v^k \frac{\partial E_\mu^I}{\partial X_l^\lambda} E_{\nu I}$$

3.2 Without extra index

3.2.1 w.r.t e

$$\begin{aligned}\frac{\delta S_{DP}}{\delta e_l^e} &= \frac{D\mathcal{L}_{DP}}{De_l^e} = \left(\frac{\partial}{\partial e_l^e} - \partial_f \frac{\partial}{\partial(\partial_f e_l^e)} \right) \left(-\frac{T_0}{2} \frac{1}{2} \varepsilon^{cd} \varepsilon_{mn} e_c^m e_d^n \eta^{ij} e_i^a e_j^b \partial_a X^\mu \partial_b X^\nu E_\mu^I E_\nu^J \eta_{IJ} \right) = \\ &= -\frac{T_0}{4}\end{aligned}$$

3.3 Linear action

3.3.1 w.r.t e

$$\begin{aligned}\frac{\delta S_{LP}}{\delta e_j^b} &= \frac{D\mathcal{L}_{LP}}{De_j^b} = \left(\frac{\partial}{\partial e_j^b} - \partial_e \frac{\partial}{\partial(\partial_e e_j^b)} \right) \left(-\frac{T}{2} e e_i^a D_a X^{\mu i} E_\mu^I V_I \right) = \\ &= -\frac{T}{4} \varepsilon^{cd} \varepsilon_{mn} ((g_{cb} \eta^{mj} e_d^n + e_c^m g_{db} \eta^{nj}) e_i^a + e_c^m e_d^n \delta_b^a \delta_i^j) D_a X^{\mu i} E_\mu^I V_I = \\ &= -\frac{T}{2} (-e e_b^j e_i^a D_a X^{\mu i} E_\mu^I V_I + e D_b X^{\mu j} E_\mu^I V_I) \stackrel{!}{=} 0 \\ &\quad \downarrow \\ T_b^j &:= D_b X^{\mu j} E_\mu^I V_I - e_b^j e_i^a D_a X^{\mu i} E_\mu^I V_I = 0 \\ &\quad \downarrow \\ e_b^j &= F D_b X^{\mu j} E_\mu^I V_I, \\ \frac{1}{F} &= e_i^a D_a X^{\mu i} E_\mu^I V_I\end{aligned}$$